Mathematical Logic Assignment 2

522030910158 Song Yuanyi

October 2024

1. Suppose $\mathcal{Q}$ is the algorithm to list all members of the set Q. P contains all first element of the pairs in Q. Therefore, we can construct an algorithm to list all members of the set P from $\mathcal{Q}$.

1. for $i := 0, 1, 2, \ldots$
   (a) run $\mathcal{Q}$ $i+1$ times to get a list $\{(a_{11}, a_{12}), \cdots, (a_{i+1,1}, a_{i+1,2})\}$
   (b) print $a_{i+1,1}$
2. halt when $\mathcal{Q}$ halts

So P is effectively enumerable.

2.Assume $\mathcal{A}$ is an algorithm for computing function f which is totally effectively computable. $\mathcal{A}(n)$ outputs $f(n)$ for every $n \in dom(f)$. f is strictly increasing, so the number between two adjacent values $f(i), f(i+1)$ is not in the $rng(f)$. Then we construct the algorithm for deciding the membership in $rng(f)$ as follows.

1. give an input x, a bool flag to mark the existence of x
2. for $i := 0, 1, 2, \ldots$
   (a) run $value := \mathcal{A}(n)$
      i. if $value < x$, continue
      ii. if $value = x$, print 'yes' and break
      iii. if $value > x$, print 'no' and break

So the range of f is effectively decidable.

3.From the definition we know that $B = \{(m,n)|m \in A_n\}$. $A_0, A_1, \dots, A_n, \dots$ is effectively decidable $\Rightarrow$ effectively enumerable, we can enumerate a table which lines are A_i and columns are outputs of enumerating. Define $A_i(j)$ is the j_{th} element in A_i. Suppose B is effectively decidable. Let $S_k = \{a_i|a_i \neq A_i(i), i = 0,1,2,\dots\} \Rightarrow \forall i \in \mathbb{N} : S_k \neq A_i$ so S_k is not effectively decidable. Construct a subset of B, $B' = \{(m,n)|m = A_n(n), n = 0,1,2,\dots\}$. B is effectively decidable so as B'. Therefore, we can construct a decidable algorithm for S_k from B'. Let correspondent deciding algorithm be$\mathcal{B}(m,n) = \mathcal{A}_n(m) = 1$ if $(m,n) \in B, m \in \mathcal{A}_n$ otherwise 0, and $\mathcal{S}(m) = 1 - \mathcal{B}(m,n)$. $A_i(i)$ can be get because of effectively enumerable. Therefore, $S(m) = 1$ means $m \in S_k$ otherwise $0 \Rightarrow \mathcal{S}$ is an deciding algorithm of membership $\Rightarrow S_k$ is effectively decidable, which makes a contradiction. In conclusion, B is not effectively decidable.

4. Basic facts: A_1: Cancer will be cured. A_2: The cause of cancer is determined. A_3: A new drug for cancer is found. Logical relations: $(\neg A_1)$ unless A_2 and A_3 is equal to $(\neg A_1)$ if not A_2 and A_3. Therefore, the well-formed formula is $((\neg(A_2 \wedge A_3)) \rightarrow (\neg A_1))$.

5. Proof by induction:

- Step 1: First considering about simplest situation, suppose α and β are both single sentence symbol (simplest wff) in which $s = 1$ and $c = 0$.
  - $(\alpha \wedge \beta) :\ s = 2$ and $c = 1$
  - $(\alpha \vee \beta) :\ s = 2$ and $c = 1$
  - $(\alpha \rightarrow \beta) :\ s = 2$ and $c = 1$
  - $(\alpha \leftrightarrow \beta) :\ s = 2$ and $c = 1$

  We found that they all satisfy $s = c + 1$.

- Step 2: Then let α and β be two well-formed formula, and check four binary connectives.
  - $(\alpha \wedge \beta) :\ c = c(\alpha) + c(\beta) + 1$ and$s = s(\alpha) + s(\beta) = c(\alpha) + 1 + c(\beta) + 1 = c + 1$
  - $(\alpha \vee \beta)$, $(\alpha \rightarrow \beta)$, $(\alpha \leftrightarrow \beta)$ are similar to process above.

- To sum up, for any wff α, the number of occurrence of sentences symbols in α is 1 greater than the number of binary connectives.

PS: Considering about $\neg\alpha$: $s(\neg\alpha) = s(\alpha), c(\neg\alpha) = c(\alpha)$