Gauge Theory

Li-Na Yan[1*]

[1] *School of Physics, Northwest University, Xi'an 710127, China*

Abstract

We

*E-mail: ccli@nwu.edu.cn

1 Gauge invariance

1.1 U(1) gauge invariance

The Lagrangian of scalar particles is

$$\mathcal{L} = (\partial_\mu \Phi)^* (\partial^\mu \Phi) - m^2 \Phi^* \Phi = \partial_\mu \Phi^* (\partial^\mu \Phi) - m^2 \Phi^* \Phi \tag{1.1}$$

Φ changes as follows (fundamental representation U=$e^{-iq\chi}$)

$$\Phi \to \Phi' = e^{-iq\chi}\Phi = U\Phi \tag{1.2}$$
$$\Phi^* \to \Phi'^* = e^{iq\chi}\Phi^* = U^*\Phi^* \tag{1.3}$$

Then, the Lagrangian transform as

$$\begin{aligned}
\mathcal{L} \to \mathcal{L}' &= \partial_\mu (e^{iq\chi}\Phi^*)\partial^\mu (e^{-iq\chi}\Phi) - m^2 e^{iq\chi}\Phi^* e^{-iq\chi}\Phi \\
&= (iq\partial_\mu \chi e^{iq\chi}\Phi^* + e^{iq\chi}\partial_\mu \Phi^*)(-iq\partial_\mu \chi e^{-iq\chi}\Phi + e^{-iq\chi}\partial_\mu \Phi) - m^2\Phi^*\Phi \\
&= (iq\partial_\mu \chi \Phi^* + \partial_\mu \Phi^*)(-iq\partial_\mu \chi \Phi + \partial_\mu \Phi) - m^2\Phi^*\Phi \\
&= iq(\partial_\mu \chi)\Phi^*\partial^\mu \Phi - iq(\partial_\mu \chi)\Phi\partial^\mu \Phi^* + \partial_\mu \Phi^* \partial^\mu \Phi - m^2\Phi^*\Phi
\end{aligned} \tag{1.4}$$

we find $\mathcal{L} \neq \mathcal{L}'$.
Introduction of gauge field A_μ

$$A^\mu \to A'^\mu = A^\mu - \partial^\mu \chi \tag{1.5}$$
$$D^\mu \Phi \equiv (\partial^\mu - \mathrm{i}qA^\mu)\ \Phi \tag{1.6}$$

$$\begin{aligned}
D_\mu \Phi \to (D_\mu \Phi)' &= (\partial_\mu - \mathrm{i}qA'_\mu)\Phi' \\
&= [\partial_\mu - \mathrm{i}q(A_\mu - \partial_\mu \chi)]\mathrm{e}^{-\mathrm{i}q\chi(x)}\Phi \\
&= (-iq)\partial_\mu \chi \mathrm{e}^{-\mathrm{i}q\chi(x)}\Phi + \mathrm{e}^{-\mathrm{i}q\chi(x)}\partial_\mu \Phi - \mathrm{i}qA_\mu \mathrm{e}^{-\mathrm{i}q\chi(x)}\Phi + iq\partial_\mu \chi \mathrm{e}^{-\mathrm{i}q\chi(x)}\Phi \\
&= \mathrm{e}^{-\mathrm{i}q\chi(x)}\left(\partial_\mu - \mathrm{i}qA_\mu\right)\Phi \\
&= \mathrm{e}^{-\mathrm{i}q\chi(x)} D_\mu \Phi
\end{aligned} \tag{1.7}$$

So we know that D_μ changes as follows:($D_\mu \Phi$ and Φ change in the same way).
$D_\mu \!\to\! D'_\mu = e^{-iq\chi}D_\mu e^{iq\chi}$ =$UD_\mu U^\dagger$ (adjont representation)

$$D'_\mu \Phi' = e^{-iq\chi}D_\mu e^{iq\chi}e^{-iq\chi}\Phi = e^{-iq\chi}D_\mu \Phi \tag{1.8}$$

The Lagrangian is

$$\mathcal{L} = (D_\mu \Phi)^\dagger (D^\mu \Phi) - m^2 \Phi^\dagger \Phi \tag{1.9}$$

The Lagrangian is both Lorentz-invariant and $U(1)$ gauge invariant.
We can construct an antisymmetric tensor:

$$\begin{aligned}
D_\mu D_\nu \Phi &= (\partial_\mu - iqA_\mu)(\partial_\nu - iqA_\nu)\Phi \\
&= \partial_\mu \partial_\nu \Phi + \partial_\mu(-iqA_\nu \Phi) - iqA_\mu \partial_\nu \Phi - q^2 A_\mu A_\nu \Phi \\
&= \partial_\mu \partial_\nu \Phi - iq\partial_\mu A_\nu \Phi - iqA_\nu \partial_\mu \Phi - iqA_\mu \partial_\nu \Phi - q^2 A_\mu A_\nu \Phi
\end{aligned} \tag{1.10}$$

$$D_\nu D_\mu \Phi = \partial_\nu \partial_\mu \Phi - iq\partial_\nu A_\mu \Phi - iqA_\mu \partial_\nu \Phi - iqA_\nu \partial_\mu \Phi - q^2 A_\nu A_\mu \Phi \tag{1.11}$$

Antisymmetry is obtained:

$$\begin{aligned}(D_\mu D_\nu - D_\nu D_\mu)\Phi &= (-iq\partial_\mu A_\nu \Phi - iqA_\nu \partial_\mu \Phi - iqA_\mu \partial_\nu \Phi + iq\partial_\nu A_\mu \Phi + iqA_\mu \partial_\nu \Phi + iqA_\nu \partial_\mu)\Phi \\ &= -iq(\partial_\mu A_\nu - \partial_\nu A_\mu)\Phi \\ &= -iqF_{\mu\nu}\Phi \end{aligned} \tag{1.12}$$

The definition of $F_{\mu\nu}$ is

$$F_{\mu\nu} \equiv [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{1.13}$$

$$F'_{\mu\nu}\Phi' = UF_{\mu\nu}U^\dagger U\Phi = UF_{\mu\nu}\Phi \tag{1.14}$$

$$F'_{\mu\nu}\Phi' = [D'_\mu, D'_\nu]\Phi' = e^{-ig\chi(x)}[D_\mu, D_\nu]\Phi = e^{-ig\chi(x)}F_{\mu\nu}\Phi = F_{\mu\nu}\Phi' \tag{1.15}$$

Therefore,

$$F'_{\mu\nu} = F_{\mu\nu} \tag{1.16}$$

So,the final Lagrangian is

$$\mathcal{L} = (D_\mu\Phi)^\dagger(D^\mu\Phi) - m^2\Phi^\dagger\Phi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{1.17}$$

1.2 SU(2) gauge invariance

$$\Phi = \begin{pmatrix}\phi_1 \\ \phi_2\end{pmatrix} \tag{1.18}$$

The Lagrangian of scalar particles satisfying renormalization is

$$\mathcal{L} = (\partial_\mu\Phi)^*(\partial^\mu\Phi) - m^2\Phi^*\Phi = \partial_\mu\Phi^*(\partial^\mu\Phi) - m^2\Phi^*\Phi \tag{1.19}$$

Φ changes as follows (fundamental representation U=$e^{-ig\alpha_i\sigma^i}$)

$$\Phi \to \Phi' = e^{-ig\alpha_i\sigma^i}\Phi = U\Phi \tag{1.20}$$

$$\Phi^* \to \Phi'^* = e^{ig\alpha_i\sigma^i}\Phi^* = U^*\Phi^* \tag{1.21}$$

$\mathbf{A}$ is an element of a $SU(2)$ Lie algebra:

$$\mathbf{A}_\mu = A^i_\mu(x)\sigma^i \tag{1.22}$$

$$D_\mu\Phi = \partial_\mu\Phi(x) - ig\mathbf{A}_\mu\Phi = \partial_\mu\Phi(x) - igA^i_\mu(x)\sigma^i \tag{1.23}$$

Require D_μ to follow the same changes as :

$$(D_\mu\Phi)' = U(D_\mu\Phi) \tag{1.24}$$

That determines the change in $\mathbf{A}_\mu$:

$$\begin{aligned}(\partial_\mu - ig\mathbf{A}'_\mu)(U\Phi) &= U(\partial_\mu - ig\mathbf{A}_\mu)\Phi \\ (\partial_\mu - ig\mathbf{A}'_\mu)(U\Phi) &= \partial_\mu(U\Phi) - ig\mathbf{A}'_\mu(U\Phi) \\ U(\partial_\mu - ig\mathbf{A}_\mu)\Phi &= U\partial_\mu\Phi - igU\mathbf{A}_\mu\Phi \end{aligned} \tag{1.25}$$

$$\partial_\mu(U\Phi)U^{-1} - ig\mathbf{A}'_\mu(U\Phi)U^{-1} = U\partial_\mu\Phi U^{-1} - igU\mathbf{A}_\mu\Phi U^{-1} \tag{1.26}$$

So,

$$\mathbf{A}'_\mu(U\Phi)U^{-1} = \frac{U\partial_\mu\Phi U^{-1}}{-ig} + U\mathbf{A}_\mu\Phi U^{-1} - \frac{\partial_\mu(U\Phi)U^{-1}}{-ig} \tag{1.27}$$

Thus,

$$\mathbf{A}'_\mu\Phi = \frac{U\partial_\mu\Phi U^{-1}}{-ig} + U\mathbf{A}_\mu\Phi U^{-1} - \frac{(\partial_\mu U)U^{-1}\Phi}{-ig} - \frac{U(\partial_\mu\Phi)U^{-1}}{-ig} \tag{1.28}$$

So we get the change in the gauge field:

$$\mathbf{A}'_\mu = U\mathbf{A}_\mu U^{-1} - \frac{i}{g}(\partial_\mu U)U^{-1} \tag{1.29}$$

A term with two covariant derivations can be written as:

$$\begin{aligned} D_\mu D_\nu\Phi &= (\partial_\mu - ig\mathbf{A}_\mu)(\partial_\nu - ig\mathbf{A}_\nu)\Phi & (1.30)\\ &= \partial_\mu\partial_\nu\Phi + \partial_\mu(-ig\mathbf{A}_\nu\Phi) - ig\mathbf{A}_\mu\partial_\nu\Phi + (ig)^2\mathbf{A}_\mu\mathbf{A}_\nu\Phi \\ &= \partial_\mu\partial_\nu\Phi - ig\partial_\mu\mathbf{A}_\nu\Phi - ig\mathbf{A}_\nu\partial_\mu\Phi - ig\mathbf{A}_\mu\partial_\nu\Phi + (ig)^2\mathbf{A}_\mu\mathbf{A}_\nu\Phi & (1.31)\end{aligned}$$

$$D_\nu D_\mu\Phi = \partial_\nu\partial_\mu\Phi - ig\partial_\nu\mathbf{A}_\mu\Phi - ig\mathbf{A}_\mu\partial_\nu\Phi - ig\mathbf{A}_\nu\partial_\mu\Phi + (ig)^2\mathbf{A}_\nu\mathbf{A}_\mu\Phi \tag{1.32}$$

Antisymmetry is obtained:

$$\begin{aligned} (D_\mu D_\nu - D_\nu D_\mu)\Phi &= -ig(\partial_\mu\mathbf{A}_\nu - \partial_\nu\mathbf{A}_\mu)\Phi + (ig)^2[\mathbf{A}_\mu, \mathbf{A}_\nu]\Phi & (1.33)\\ &= -ig\mathbf{F}_{\mu\nu}\Phi & (1.34)\end{aligned}$$

Then,

$$\mathbf{F}_{\mu\nu} \equiv [\mathbf{D}_\mu, \mathbf{D}_\nu] = \partial_\mu\mathbf{A}_\nu - \partial_\nu\mathbf{A}_\mu - ig[\mathbf{A}_\mu, \mathbf{A}_\nu] \tag{1.35}$$

$\mathbf{F}_{\mu\nu} \rightarrow \mathbf{F}_{\mu\nu}{}' = U\mathbf{F}_{\mu\nu}U^\dagger$
$\mathbf{F}_{\mu\nu}{}'\mathbf{F}^{\mu\nu\prime} = U\mathbf{F}_{\mu\nu}U^\dagger U\mathbf{F}^{\mu\nu}U^\dagger = U\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu}U^\dagger$. We find it is a matrix, so we need to take the trace.

$$\begin{aligned} &tr[\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu}] \rightarrow tr[\mathbf{F}_{\mu\nu}{}'\mathbf{F}^{\mu\nu\prime}] = tr[U\mathbf{F}_{\mu\nu}U^\dagger U\mathbf{F}^{\mu\nu}U^\dagger] \\ &= tr[U\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu}U^\dagger] = tr[U^\dagger U\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu}] = tr[\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu}] \qquad (1.36)\end{aligned}$$

The $tr[\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu}]$ is gauge invariance.
The Lagrangian that is both Lorentz-invariant and norm-invariant is

$$\mathcal{L} = (D_\mu\Phi)^\dagger(D^\mu\Phi) - m^2\Phi^\dagger\Phi - tr[\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu}] \tag{1.37}$$

1.3 examples

1.3.1 scalar particles

$$\begin{aligned} \mathcal{L} &= (D_\mu\Phi)^\dagger(D^\mu\Phi) - m^2\Phi^\dagger\Phi - \frac{\lambda}{4}(\Phi^\dagger\Phi)^2 - \frac{1}{4}tr[\mathbf{W}_{\mu\nu}\mathbf{W}^{\mu\nu}] - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} & (1.38)\\ D_\mu &= \partial_\mu\mathbf{I} - \frac{i}{2}g_1 YA_\mu - \frac{i}{2}g_2\mathbf{W}_\mu{}^i\sigma_i & (1.39)\\ Y &= \frac{1}{2}\mathbf{I} & (1.40)\end{aligned}$$

$$\begin{aligned} F_{\mu\nu} &\equiv [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu & (1.41)\\ \mathbf{W}_{\mu\nu} &\equiv [\mathbf{W}_\mu, \mathbf{W}_\nu] = \partial_\mu\mathbf{W}_\nu - \partial_\nu\mathbf{W}_\mu - ig[\mathbf{W}_\mu, \mathbf{W}_\nu] & (1.42)\end{aligned}$$

1.3.2 Dirac particles

$$\mathcal{L} = \overline{\psi} i\gamma^\mu D_\mu \psi - m\overline{\psi}\psi - \frac{1}{4}tr[\mathbf{W}_{\mu\nu}\mathbf{W}^{\mu\nu}] - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{1.43}$$

$$D_\mu = \partial_\mu \mathbf{I} - \frac{i}{2}g_1 Y A_\mu - \frac{i}{2}g_2 \mathbf{W}_\mu{}^i \sigma_i \tag{1.44}$$

2 The most general Lagrangian

$P_L L_m = \begin{pmatrix} P_L\nu_m \\ P_L\epsilon_m \end{pmatrix}$	(1,2,-$\frac{1}{2}$)
$P_R E_m$	$(1, 1, -1)$
$P_L Q_m = \begin{pmatrix} P_L\mathcal{U}_m \\ P_L\mathcal{D}_m \end{pmatrix}$	$(3, 2, \frac{1}{6})$
$P_R U_m$	$(3, 1, \frac{2}{3})$
$P_R D_m$	$(3, 1, -\frac{1}{3})$

The most general Lagrangian involving these fields is (kinetic terms) (Lorentz and gauge invariance)

$$\begin{aligned}\mathcal{L}_{fg} = &- \frac{1}{4}G^\alpha_{\mu\nu}G^{\alpha\mu\nu} - \frac{1}{4}W^\alpha_{\mu\nu}W^{\alpha\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{g_3^2\Theta_3}{64\pi^2}\epsilon_{\mu\nu\lambda\rho}G^{\alpha\mu\nu}G^{\alpha\lambda\rho} \\ &- \frac{g_2^2\Theta_2}{64\pi^2}\epsilon_{\mu\nu\lambda\rho}W^{\alpha\mu\nu}W^{\alpha\lambda\rho} - \frac{g_1^2\Theta_1}{64\pi^2}\epsilon_{\mu\nu\lambda\rho}B^{\mu\nu}B^{\lambda\rho} - \frac{1}{2}\overline{L}_m \not{D}\, L_m - \overline{D}_L \not{D} L_R - \overline{D}_L m L_R \\ &- \frac{1}{2}\overline{E}_m \not{D}\, E_m - \frac{1}{2}\overline{Q}_m \not{D}\, Q_m - \frac{1}{2}\overline{U}_m \not{D}\, U_m - \frac{1}{2}\overline{D}_m \not{D}\, D_m - \overline{D}_m m D_m \end{aligned} \tag{2.45}$$

But we'll see that the blue term doesn't exist, when they both take their left hand

$$\overline{D}_L \not{D} L_R = (P_L D)^\dagger \gamma_0 \gamma^\mu D_\mu P_R L = D^\dagger P_L D_0 P_R L = 0 \; (\mu = 0) \tag{2.46}$$

$$\overline{D}_L m L_R = (P_L D)^\dagger \gamma_0 m P_R L = D^\dagger P_L \gamma_0 m P_R L = m D^\dagger \gamma_0 P_R P_R L \neq 0$$

$$\overline{D}_L m L_R : \quad U(1): \quad e^{-\frac{1}{3}q\chi} e^{-\frac{1}{2}q\chi} \neq 0 \quad SU(2): \quad \overline{D}_L m L_R = (UD)^\dagger \gamma_0 m U' L \neq 0 \tag{2.47}$$

$$\overline{D}_L m D_L = (P_L D)^\dagger \gamma_0 m P_L D = D^\dagger P_L \gamma_0 m P_L D = m D^\dagger \gamma_0 P_R P_L D = 0 \tag{2.48}$$

in which the gauge field-strengths are given by

$$G^\alpha_{\mu\nu} = \partial_\mu G^\alpha_\nu - \partial_\nu G^\alpha_\mu + g_3 f^\alpha_{\beta\gamma} G^\beta_\mu G^\gamma_\nu \tag{2.49}$$

$$W^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g_3 \epsilon_{abc} W^b_\mu W^c_\nu \tag{2.50}$$

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \tag{2.51}$$

The gauge-covariant derivatives are

$$D_\mu L_m = \partial_\mu L_m + [\frac{i}{2}g_1 B_\mu - \frac{i}{2}g_2 W^a_\mu \tau_a] P_L L_m + [-\frac{i}{2}g_1 B_\mu + \frac{i}{2}g_2 W^a_\mu \tau^*_a] P_R L_m \tag{2.52}$$

$$D_\mu E_m = \partial_\mu E_m - i g_1 B_\mu P_L E_m + i g_1 B_\mu P_R E_m \tag{2.53}$$

$$\begin{aligned} D_\mu Q_m = &\partial_\mu Q_m + [-\frac{i}{6}g_1 B_\mu - \frac{i}{2}g_2 W^a_\mu \tau_a - \frac{i}{2}g_3 G^\alpha_\mu \lambda_\alpha] P_L Q_m \\ &+ [\frac{i}{6}g_1 B_\mu + \frac{i}{2}g_2 W^a_\mu \tau^*_a + \frac{i}{2}g_3 G^\alpha_\mu \lambda^*_\alpha] P_R L_m \end{aligned} \tag{2.54}$$

$$D_\mu U_m = \partial_\mu U_m + [\frac{2i}{3}g_1 B_\mu + \frac{i}{2}g_3 G^\alpha_\mu \lambda^*_\alpha]P_L U_m$$
$$+ [-\frac{2i}{3}g_1 B_\mu - \frac{i}{2}g_3 G^\alpha_\mu \lambda_\alpha]P_R U_m \quad (2.55)$$
$$D_\mu D_m = \partial_\mu D_m + [-\frac{i}{3}g_1 B_\mu + \frac{i}{2}g_3 G^\alpha_\mu \lambda^*_\alpha]P_L D_m$$
$$+ [\frac{i}{3}g_1 B_\mu - \frac{i}{2}g_3 G^\alpha_\mu \lambda_\alpha]P_R D_m \quad (2.56)$$

Their masses may be read off are 0! The scalar field is supposed to produce a mass for the fermions after it develops a v.e.v. The simplest choice is therefore to add a single (complex) scalar doublet, called the Higgs field:

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \quad (2.57)$$

transforming as $(1,\, 2,\, \frac{1}{2})$. Its complex conjugate,

$$\widetilde{\phi} \equiv \begin{pmatrix} \phi^{0*} \\ -\phi^{+*} \end{pmatrix} = \epsilon\phi^* = i\sigma_2\phi^* \quad (2.58)$$

transforming as $(1,\, 2,\, -\frac{1}{2})$ The new terms that may then appear in Lagrangian are

$$\mathcal{L}_{Higgs} = -\ (D_\mu\phi)^+(D^\mu\phi) - V(\phi^+\phi)$$
$$-\ (f_{mn}\overline{L}_m P_R E_n\phi + h_{mn}\overline{Q}_m P_R D_n\phi + g_{mn}\overline{Q}_m P_R U_n\widetilde{\phi} + h.c.) \quad (2.59)$$

$$\overline{L}_m P_R = L^\dagger_m \gamma_0 P_R = L^\dagger_m P_L \gamma_0 = (P_L L_m)^\dagger \gamma_0 \quad (2.60)$$

For the first term,the charge of $\overline{L}_m P_R$ is the complex conjugate of$P_L L_m$

$$\overline{L}_m P_R E_n \phi = \frac{1}{2} - 1 + \frac{1}{2} = 0 \quad (2.61)$$

For the second term,the charge of $\overline{Q}_m P_R$ is the complex conjugate of$P_L Q_m$

$$\overline{Q}_m P_R D_n \phi = -\frac{1}{6} - \frac{1}{3} + \frac{1}{2} = 0 \quad (2.62)$$

For the finally term,the charge of $\overline{Q}_m U_R$ is the complex conjugate of$P_L Q_m$

$$\overline{Q}_m P_R U_n \phi = -\frac{1}{6} + \frac{2}{3} + \frac{1}{2} \neq 0 \quad (2.63)$$

So,we need to consider $\widetilde{\phi}$

$$\overline{Q}_m P_R U_n \widetilde{\phi} = -\frac{1}{6} + \frac{2}{3} - \frac{1}{2} = 0 \quad (2.64)$$

in which

$$V(\phi^\dagger\phi) = \lambda[\phi^\dagger\phi - \frac{\mu^2}{2\lambda}]^2$$
$$= \lambda(\phi^\dagger\phi)^2 - \mu^2\phi^\dagger\phi + \frac{\mu^4}{4\lambda} \quad (2.65)$$

$$D_\mu\phi = \partial_\mu\phi - \frac{i}{2}g_1 B_\mu\phi - \frac{i}{2}g_2 W_\mu^i\tau_i\phi \tag{2.66}$$

Let's write ϕ in the form

$$\phi = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}(v + H(x)) \end{pmatrix} \tag{2.67}$$

v is determined by minimizing the potential in (2.65) and satisfies

$$v^2 = \frac{\mu^2}{\lambda} \tag{2.68}$$

$$D_\mu = \partial_\mu\mathbf{I} - \frac{i}{2}g_1 Y B_\mu - \frac{i}{2}g_2 W_\mu^i\tau_i \tag{2.69}$$

$$Y = \frac{1}{2}\mathbf{I} \tag{2.70}$$

Masses comes from the expansion of $\mathcal{L}_{Higgs}$, the expansion of the scalar-field kinetic term becomes:

$$\begin{aligned}(D_\mu\phi)^\dagger &= \frac{1}{\sqrt{2}}[[\partial_\mu\mathbf{I} - \frac{i}{2}g_2\begin{pmatrix} W_\mu^3 & W_\mu^1 - iW_\mu^2 \\ W_\mu^1 + iW_\mu^2 & -W_\mu^3\end{pmatrix} - i\frac{g_1}{2}B_\mu\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}]\begin{pmatrix}0 \\ v + H(x)\end{pmatrix}]^\dagger \\ &= \begin{pmatrix} -\frac{i}{2}g_2(W_\mu^1 - iW_\mu^2)(v+H) \\ \partial_\mu + \frac{i}{2}g_2 W_\mu^3 - i\frac{g_1}{2}B_\mu(v+H)\end{pmatrix}^\dagger \end{aligned}\tag{2.71}$$

$$\begin{aligned}-(D_\mu\phi)^\dagger(D^\mu\phi) =& -\frac{1}{2}\partial_\mu H\partial^\mu H - \frac{1}{8}(v+H)^2 g_2^2(W_\mu^1 + iW_\mu^2)(W^{1\mu} - iW^{2\mu}) \\ & - \frac{1}{8}(v+H)^2(-g_2W_\mu^3 + g_1B_\mu)(-g_2W^{3\mu} + g_1B^\mu)\end{aligned}\tag{2.72}$$

The scalar potential term contributes

$$\begin{aligned}V(\phi^\dagger\phi) &= \lambda[\phi^\dagger\phi - \frac{\mu^2}{2\lambda}]^2 \\ &= \frac{\lambda}{4}[(v+H)^2 - \frac{\mu^2}{\lambda}]^2 = \frac{\lambda}{4}(v^2 + 2vH + H^2 - \frac{\mu^2}{\lambda})^2 = \frac{\lambda}{4}(2vH + H^2)^2 \\ &= \lambda v^2H^2 + \lambda vH^3 + \frac{\lambda}{4}H^4\end{aligned}\tag{2.73}$$

The Yukawa couplings may be expanded in an identical way:

$$\begin{aligned}\overline{L}_m P_R E_n\phi &= \frac{1}{\sqrt{2}}\begin{pmatrix}\overline{\nu}_m \\ \overline{\varepsilon}_m\end{pmatrix}^{\mathrm{T}} P_R E_n\begin{pmatrix}0 \\ v + H\end{pmatrix} \\ &= \frac{1}{\sqrt{2}}(v+H)\overline{\varepsilon}_m P_R E_n\end{aligned}\tag{2.74}$$

$$\begin{aligned}\overline{Q}_m P_R U_n\widetilde{\phi} &= \frac{1}{\sqrt{2}}\begin{pmatrix}\overline{\mathcal{U}}_m \\ \overline{\mathcal{D}}_m\end{pmatrix}^{\mathrm{T}} P_R U_n\begin{pmatrix}v + H \\ 0\end{pmatrix} \\ &= \frac{1}{\sqrt{2}}(v+H)\overline{\mathcal{U}}_m P_R U_n\end{aligned}\tag{2.75}$$

$$\overline{Q}_m P_R D_n\phi = \frac{1}{\sqrt{2}}\begin{pmatrix}\overline{\mathcal{U}}_m \\ \overline{\mathcal{D}}_m\end{pmatrix}^{\mathrm{T}} P_R D_n\begin{pmatrix}0 \\ v + H\end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}(v+H)\overline{\mathcal{D}}_m P_R D_n \tag{2.76}$$

Combining all of these results gives the expansion of $\mathcal{L}_{Higgs}$to be

$$\begin{aligned}
\mathcal{L}_{Higgs} = &-\frac{1}{2}\partial_\mu H \partial^\mu H - \lambda v^2 H^2 - \lambda v H^3 - \frac{\lambda}{4}H^4 \\
&-\frac{1}{8}(v+H)^2 g_2^2 |W_\mu^1 + iW_\mu^2|^2 \\
&-\frac{1}{8}(v+H)^2(-g_2 W^{3\mu} + g_1 B^\mu)^2 \\
&-\frac{1}{\sqrt{2}}(v+H)[f_{mn}\overline{\varepsilon}_m P_R E_n + h.c.] \\
&-\frac{1}{\sqrt{2}}(v+H)[g_{mn}\overline{\mathcal{U}}_m P_R U_n + h.c.] \\
&-\frac{1}{\sqrt{2}}(v+H)[h_{mn}\overline{\mathcal{D}}_m P_R D_n + h.c.]
\end{aligned} \tag{2.77}$$

2.1 Boson masses

$\mathcal{L}_{Higgs}$ contains all of the mass terms, although some of these are not diagonal.

2.1.1 Spin-zero particles

Comparing the H^2 term of $\mathcal{L}_{Higgs}$ with the standard form, $-\frac{1}{2}m_H^2 H^2$, gives

$$m_H^2 = 2\lambda v^2 = 2\mu^2 \tag{2.78}$$

2.1.2 Spin-one particles

The relevant terms in this case are:

$$-\frac{1}{8}v^2 g_2^2 |W_\mu^1 + iW_\mu^2|^2 - \frac{1}{8}v^2(-g_2 W^{3\mu} + g_1 B^\mu)^2 \tag{2.79}$$

The fields W_μ^1 and W_μ^2 only appear in the combination $W_1^\mu W_{1\mu} + W_2^\mu W_{2\mu}$ and do not mix with any other fields. Comparing this term to

$$-\frac{1}{2}M_1^2 W_\mu^1 W^{1\mu} - \frac{1}{2}M_2^2 W_\mu^2 W^{2\mu} \tag{2.80}$$

gives

$$M_1^2 = M_2^2 = \frac{v^2 g_2^2}{4} \tag{2.81}$$

The mass term B_μ and W_μ^3 are only appear in one particular combination, $g_1 B_\mu - g_2 W_\mu^3$. We may normalize this combination to define the mass eigenstate:

$$\begin{aligned}
Z_\mu &\equiv \frac{-g_1 B_\mu + g_2 W_\mu^3}{\sqrt{g_1^2 + g_2^2}} \\
&\equiv W_\mu^3 \cos\theta_W - B_\mu \sin\theta_W
\end{aligned} \tag{2.82}$$

$$\cos\theta_W = \frac{g_2}{\sqrt{g_1^2 + g_2^2}} \tag{2.83}$$

$$\sin\theta_W = \frac{g_1}{\sqrt{g_1^2+g_2^2}} \tag{2.84}$$

In terms of this field the mass term, (2.77), is $-\frac{1}{8}v^2(g_1^2+g_2^2)Z_\mu Z^\mu$, from which the mass may be read off:

$$M_Z^2 = \frac{1}{4}v^2(g_1^2+g_2^2) \tag{2.85}$$

The final mass eigenstate is the combination of W_μ^3 and B_μ that is orthogonal to Z_μ:

$$A_\mu = W_\mu^3 \sin\theta_W + B_\mu \cos\theta_W = \frac{g_2 B_\mu + g_1 W_\mu^3}{\sqrt{g_1^2+g_2^2}} \tag{2.86}$$

This is massless. (The Lagrangian does not have this term.)

$$\begin{aligned}
W_\mu^3 &= c_\mathrm{W} Z_\mu + s_\mathrm{W} A_\mu & Z_\mu &= c_\mathrm{W} W_\mu^3 - s_\mathrm{W} B_\mu \\
B_\mu &= -s_\mathrm{W} Z_\mu + c_\mathrm{W} A_\mu & A_\mu &= s_\mathrm{W} W_\mu^3 + c_\mathrm{W} B_\mu \\
\\
\sqrt{2}W_\mu^+ &= W_\mu^1 - \mathrm{i}W_\mu^2 & \sqrt{2}W_\mu^1 &= W_\mu^+ + W_\mu^- \\
\sqrt{2}W_\mu^- &= W_\mu^1 + \mathrm{i}W_\mu^2 & \sqrt{2}W_\mu^2 &= \mathrm{i}W_\mu^+ - \mathrm{i}W_\mu^- \\
\\
\sqrt{g_2^2+g_1^2}W_\mu^3 &= g_2 Z_\mu + g_1 A_\mu & \sqrt{g_2^2+g_1^2}Z_\mu &= g_2 W_\mu^3 - g_1 B_\mu \\
\sqrt{g_2^2+g_1^2}B_\mu &= -g_1 Z_\mu + g_2 A_\mu & \sqrt{g_2^2+g_1^2}A_\mu &= g_1 W_\mu^3 + g_2 B_\mu
\end{aligned} \tag{2.87}$$

2.2 Fermion masses

The mass terms induced by the Yukawa couplings of fermions to the Higgs v.e.v. are in general not diagonal.They may be diagonalized following the procedure.

$$\begin{aligned}
P_L\varepsilon_m &= U_{mn}^{(e)} P_L \varepsilon_n' \\
P_R E_m &= V_{mn}^{(e)} P_R E_n' \\
P_L \mathcal{U}_m &= U_{mn}^{(e)} P_L \mathcal{U}_n' \\
P_R U_m &= V_{mn}^{(e)} P_R U_n' \\
P_L \mathcal{D}_m &= U_{mn}^{(e)} P_L \mathcal{D}_n' \\
P_R D_m &= V_{mn}^{(e)} P_R D_n'
\end{aligned} \tag{2.88}$$

where the matrices $U^{(e)}$,$U^{(u)}$,$U^{(d)}$,$V^{(e)}$,$V^{(u)}$,$V^{(d)}$, act on the generation indices and must be unitary in order to preserve the canonical form for the kinetic terms.It is always possible to choose $U^{(e)}$=$V^{(e)*}$,$U^{(u)}$=$V^{(u)*}$,$U^{(d)}$=$V^{(d)*}$and then choose $U^{(e)}$ to ensure that the new mass matrices are diagonal:

$$U^{(e)\dagger} f V^{(e)} = V^{(e)T} f V^{(e)} \tag{2.89}$$

The same may be done for $V^{(u)T} f V^{(u)}$ and $V^{(d)T} f V^{(d)}$. The resulting mass terms then become

$$\mathcal{L} = -\frac{1}{\sqrt{2}} v [f_m \overline{\varepsilon}_m P_R E_m + g_m \overline{\mathcal{U}}_m P_R U_m + h_m \overline{\mathcal{D}}_m P_R D_m + h.c.] \tag{2.90}$$

This has a simple expression in terms of the Dirac spinors, e_m, d_m, and u_m, defined as

$$e_m \equiv P_L \varepsilon_m + P_R E_m$$

$$d_m \equiv P_L \mathcal{D}_m + P_R D_m$$
$$u_m \equiv P_L \mathcal{U}_m + P_R U_m$$

$$\begin{aligned}
\overline{\varepsilon}_m P_R E_m + h.c. &= \overline{\varepsilon}_m P_R E_m + (\overline{\varepsilon}_m P_R E_m)^\dagger \\
&= \overline{\varepsilon}_m P_R E_m + \overline{\varepsilon}_m P_L E_m \qquad (2.91)\\
&= \overline{\varepsilon}_m P_R E_m + (\varepsilon_m^\dagger \gamma_0 P_R E_m)^\dagger \\
&= \overline{\varepsilon}_m P_R E_m + E_m^\dagger \gamma_0 P_L \varepsilon_m \\
&= \overline{\varepsilon}_m P_R E_m + \overline{E}_m P_L \varepsilon_m \\
&= \overline{e}_m P_R e_m + \overline{e}_m P_L e_m \\
&= \overline{e}_m e_m \qquad (2.92)\\
\overline{e}_m P_R e_m &= (P_L \varepsilon_m + P_R E_m)^\dagger \gamma_0 P_R e_m \\
&= \varepsilon_m^\dagger P_L \gamma_0 P_R e_m + E_m^\dagger P_R \gamma_0 P_R e_m \\
&= \varepsilon_m^\dagger \gamma_0 P_R P_R e_m + E_m^\dagger \gamma_0 P_L P_R e_m \\
&= \varepsilon_m^\dagger \gamma_0 P_R e_m \\
&= \overline{\varepsilon_m} P_R e_m = \overline{\varepsilon_m} P_R E_m \qquad (2.93)
\end{aligned}$$

In terms of these Dirac spinors, the final form for the mass terms is

$$\mathcal{L} = -\frac{1}{\sqrt{2}} v[f_m \overline{e}_m e_m + g_m \overline{u}_m u_m + h_m \overline{d}_m d_m] \qquad (2.94)$$

which, when compared to the standard mass term, $-m\overline{\psi}\psi$, gives the fermion masses as

$$m_n^{(e)} = \frac{1}{\sqrt{2}} f_n v, m_n^{(u)} = \frac{1}{\sqrt{2}} g_n v, m_n^{(d)} = \frac{1}{\sqrt{2}} h_n v, \qquad (2.95)$$

2.3 CKM

The only other electroweak interactions in the theory are the couplings between the electroweak bosons and spin-half and spin-zero particles. Since the couplings with the Higgs boson (spin-zero)are given in the previous section, they need not be reconsidered again here. The W_μ^a and B_μ fermion couplings arise from the following kinetic terms,

$$\mathcal{L} = -\frac{1}{2}\overline{L}_m \not{D}\, L_m - \frac{1}{2}\overline{E}_m \not{D}\, E_m - \frac{1}{2}\overline{Q}_m \not{D}\, Q_m - \frac{1}{2}\overline{U}_m \not{D}\, U_m - \frac{1}{2}\overline{D}_m \not{D}\, D_m \qquad (2.96)$$

$$\begin{aligned}
D_\mu L_m &= \partial_\mu L_m + [\frac{i}{2} g_1 B_\mu - \frac{i}{2} g_2 W_\mu^a \tau_a] P_L L_m + [-\frac{i}{2} g_1 B_\mu + \frac{i}{2} g_2 W_\mu^a \tau_a^*] P_R L_m \qquad (2.97)\\
D_\mu E_m &= \partial_\mu E_m - i g_1 B_\mu P_L E_m + i g_1 B_\mu P_R E_m \qquad (2.98)\\
D_\mu Q_m &= \partial_\mu Q_m + [-\frac{i}{6} g_1 B_\mu - \frac{i}{2} g_2 W_\mu^a \tau_a - \frac{i}{2} g_3 G_\mu^\alpha \lambda_\alpha] P_L Q_m \\
&\quad + [\frac{i}{6} g_1 B_\mu + \frac{i}{2} g_2 W_\mu^a \tau_a^* + \frac{i}{2} g_3 G_\mu^\alpha \lambda_\alpha^*] P_R L_m \qquad (2.99)\\
D_\mu U_m &= \partial_\mu U_m + [\frac{2i}{3} g_1 B_\mu + \frac{i}{2} g_3 G_\mu^\alpha \lambda_\alpha^*] P_L U_m \\
&\quad + [-\frac{2i}{3} g_1 B_\mu - \frac{i}{2} g_3 G_\mu^\alpha \lambda_\alpha] P_R U_m \qquad (2.100)\\
D_\mu D_m &= \partial_\mu D_m + [-\frac{i}{3} g_1 B_\mu + \frac{i}{2} g_3 G_\mu^\alpha \lambda_\alpha^*] P_L D_m
\end{aligned}$$

$$+ [\frac{i}{3}g_1 B_\mu - \frac{i}{2}g_3 G^\alpha_\mu \lambda_\alpha] P_R D_m \tag{2.101}$$

Expanding each field in terms of the mass eigenstates gives

$$\begin{aligned}
\mathcal{L}_{ew} = &+ \frac{i}{4}\begin{pmatrix}\overline{\nu}_m \\ \overline{\varepsilon}_m\end{pmatrix}^T \gamma^\mu P_L \begin{pmatrix} -g_1 B_\mu + g_2 W^3_\mu & g_2(W^1_\mu - iW^2_\mu) \\ g_2(W^1_\mu + iW^2_\mu) & -g_1 B_\mu - g_2 W^3_\mu \end{pmatrix}\begin{pmatrix}\nu_m \\ \varepsilon_m\end{pmatrix} \\
&+ \frac{i}{4}\begin{pmatrix}\overline{\mathcal{U}}_m \\ \overline{\mathcal{D}}_m\end{pmatrix}^T \gamma^\mu P_L \begin{pmatrix} \frac{1}{3}g_1 B_\mu + g_2 W^3_\mu & g_2(W^1_\mu - iW^2_\mu) \\ g_2(W^1_\mu + iW^2_\mu) & \frac{1}{3}g_1 B_\mu - g_2 W^3_\mu \end{pmatrix}\begin{pmatrix}\mathcal{U}_m \\ \mathcal{D}_m\end{pmatrix} \\
&+ \frac{i}{3}g_1 B_\mu \overline{U}_m \gamma^\mu P_R U_m - \frac{i}{6}g_1 B_\mu \overline{D}_m \gamma^\mu P_R D_m \\
&- \frac{i}{2}g_1 B_\mu \overline{E}_m \gamma^\mu P_R E_m + h.c. \qquad (2.102) \\
= &+ \frac{i}{4}[\overline{\nu}_m(-g_1 B_\mu + g_2 W^3_\mu)\nu_m + \overline{\varepsilon}_m g_2(W^1_\mu + iW^2_\mu)\nu_m \\
&+ \overline{\nu}_m g_2(W^1_\mu - iW^2_\mu)\varepsilon_m + \overline{\varepsilon}_m(-g_1 B_\mu - g_2 W^3_\mu)\varepsilon_m]\gamma^\mu P_L \\
&+ \frac{i}{4}[\overline{\mathcal{U}}_m(-\frac{1}{3}g_1 B_\mu + g_2 W^3_\mu)\mathcal{U}_m + \overline{\mathcal{D}}_m(g_2(W^1_\mu + iW^2_\mu))\mathcal{U}_m \\
&+ \overline{\mathcal{U}}_m g_2(W^1_\mu - iW^2_\mu)\mathcal{D}_m + \overline{\mathcal{D}}_m(\frac{1}{3}g_1 B_\mu - g_2 W^3_\mu)\mathcal{D}_m]\gamma^\mu P_L \\
&+ \frac{i}{3}g_1 B_\mu \overline{U}_m \gamma^\mu P_R U_m - \frac{i}{6}g_1 B_\mu \overline{D}_m \gamma^\mu P_R D_m \\
&- \frac{i}{2}g_1 B_\mu \overline{E}_m \gamma^\mu P_R E_m + h.c. \qquad (2.103)
\end{aligned}$$

The couplings between fermions and the charged spin-one particle, $W^\pm_\mu$, are called the charged-current interactions. Note that G^α_μ is not involved here because there is no effect on $SU(3)$.

$$\begin{aligned}
\mathcal{L}_{cc} &= \frac{ig_2}{\sqrt{2}}[W^+_\mu(\overline{\nu}_m\gamma^\mu P_L e_m + \overline{u}_m\gamma^\mu P_L d_m) + W^-_\mu(\overline{e}_m\gamma^\mu P_L \nu_m + \overline{d}_m\gamma^\mu P_L u_m)] \qquad (2.104) \\
W^+_\mu &= \sqrt{2}(W^1_\mu - iW^2_\mu)
\end{aligned}$$

To learn what the interactions are in terms of the mass basis, we must perform the same transformations, $e_m = U^{(e)}_{mn} e_n\prime$, $u_m = U^{(e)}_{mn} u_n\prime$, and $d_m = U^{(e)}_{mn} d_n\prime$. Since there is no mass term for neutrinos, we are free to also redefine the neutrino field by $\nu_m = U^{(e)}_{mn}\nu_n\prime$. Defining

$$V_{mn} = (U^{(u)\dagger}U^d)_{mn} \tag{2.105}$$

and introducing $e_W \equiv \frac{g_2}{2\sqrt{2}}$, gives the following expression:

$$\overline{\nu}_m = \nu^\dagger_m\gamma_0 = \nu'^\dagger_n U^{(e)\dagger}_{mn}\gamma_0 = \nu'^\dagger_n\gamma_0\gamma_0 U^{(e)\dagger}_{mn}\gamma_0 = \overline{\nu}'_n U^{(e)\dagger}_{mn} \tag{2.106}$$

$$\overline{\nu}_m\gamma^\mu P_L e_m = \overline{\nu}'_n U^{(e)\dagger}_{mn}\gamma^\mu P_L U^{(e)}_{mn} e'_n = \overline{\nu}'_n\gamma^\mu P_L e'_n U^{(e)\dagger}_{mn} U^{(e)}_{mn} \tag{2.107}$$

$$\begin{aligned}
\mathcal{L}_{cc} = \ i \ \ & e_W[W^+_\mu(\overline{\nu}'_m\gamma^\mu(1+\gamma_5)e'_n + V_{mn}\overline{u}'_m\gamma^\mu(1+\gamma_5)d'_n) \\
&+ W^-_\mu(\overline{e}'_m\gamma^\mu(1+\gamma_5)\nu'_n + (V^\dagger)_{mn}\overline{d}'_m\gamma^\mu(1+\gamma_5)u'_n)] \qquad (2.108)
\end{aligned}$$

V_{mn} is a 3 × 3 unitary matrix called the Kobayashi–Maskawa (KM) – or sometimes the Cabbibo–Kobayashi–Maskawa (CKM)–matrix. It arises due to the necessity to perform different field redefinitions for up- and down-type quarks when diagonalizing masses. Since the matrix V_{mn} is 3×3 and unitary, it is described by 9 real parameters. Not all of these nine parameters can be physically significant, however, because they may be changed by performing a field redefinition which has no other effects on the standard model Lagrangian. The choice of how to use these phase redefinitions to rotate the KM matrix is somewhat arbitrary. The parameterization advocated by the Particle Data Group is:

$$\begin{aligned} V &= \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \\ &= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{13}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{13}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{13}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{13}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{13}} & c_{23}c_{13} \end{pmatrix} \end{aligned} \tag{2.109}$$

in which c_{ij} and s_{ij} are shorthand for $\cos\theta_{ij}$ and $\sin\theta_{ij}$ respectively.

3 neutrino

3.1 neutrino masses

3.1.1 Dirac mass term

By analogy with leptons we will assume that inaddition to the flavor left-handed fields $\nu_{lL}(x)$, three right-handed neutrino fields $\nu_{lR}(x)$ enter into the Lagrangian. In this case the most general neutrino mass term have the form

$$\mathcal{L}^D(x) = -\sum_{l',l} \overline{\nu}_{l'L}(x) M^D_{l'l} \nu_{lR}(x) + h.c. \tag{3.110}$$

Here the indices l and l' takes values e, μ, τ and M^D is a 3×3 complex, nondiagonal matrix. Diagonal matrix M^D.

$$M^D = U^\dagger m V \tag{3.111}$$

Here U and V are unitary matrices. From (3.110) and (3.111) we find

$$\begin{aligned} \mathcal{L}^D(x) &= -\sum_{l',l} \overline{\nu}_{l'L}(x) M^D_{l'l} \nu_{lR}(x) - \overline{\nu}_{lR}(x) M^{D\dagger}_{l'l} \nu_{l'L}(x) \\ &= -\sum_{l',l} \overline{\nu}_{l'L}(x) U^\dagger_{l'} m V_l \nu_{lR}(x) - \overline{\nu}_{lR}(x) (U^\dagger_{l'} m V_l)^\dagger \nu_{l'L}(x) \\ &= -\sum_{l',l} \overline{\nu}_{il}(x) U^\dagger_{li} m_i V_{li} \nu_{iR}(x) - \overline{\nu}_{iR}(x) V^\dagger_{li} m_i U_{li} \nu_{iL}(x) \\ &= -\sum_{i=1}^{3} m_i \overline{\nu}_i(x) \nu_i(x) \end{aligned} \tag{3.112}$$

Thus, $\nu_i(x)$=$\nu_{iL}(x)$+$\nu_{iR}(x)$ is the field of neutrino with mass m_i. Flavor field $\nu_{lL}(x)$ are connected with left-handed components of the fields of neutrinos (ν_{iL})with definite masses by the relations

$$\nu_{lL}(x) = \sum_{i=1}^{3} U_{li} \nu_{iL}(x) \quad (l = e, \mu, \tau) \tag{3.113}$$

Notice that for the right-handed fields ν_{lR} we have

$$\nu_{lR}(x) = \sum_{i=1}^{3} V_{li}\nu_{iR}(x) \quad (l = e, \mu, \tau) \tag{3.114}$$

3.1.1.1 seesaw mechanism Let us consider the following $SU_L(2) \times U_Y(1)$ invariant Lagrangian of the Yukawa interaction of the leptons and the Higgs boson.

$$\mathcal{L}_Y = -\sqrt{2}\sum_{l_1,l_2} \overline{\psi}^{lep}_{l_1L} Y'_{l_1l_2}\nu'_{l_2R}\widetilde{H} + h.c. \tag{3.115}$$

Here ψ^{lep}_{lL} is the left-handed lepton doublet, $\widetilde{H}$=i$\tau_2 H^*$ is the conjugated Higgs doublet,($SU(2)$) ν'_{lR} are right-handed singlets of the $SU_L(2) \times U_Y$ (1) group and $Y'_{ll'}$ are dimensionless, complex Yukawa constants.

$$\psi^{lep}_{eL} = \begin{pmatrix} \nu'_{eL} \\ e'_L \end{pmatrix} \quad \psi^{lep}_{\mu L} = \begin{pmatrix} \nu'_{\mu L} \\ \mu'_L \end{pmatrix} \quad \psi^{lep}_{\tau L} = \begin{pmatrix} \nu'_{\tau L} \\ \tau'_L \end{pmatrix} \tag{3.116}$$

$$\widetilde{H} = \begin{pmatrix} \frac{1}{\sqrt{2}}(v + H) \\ 0 \end{pmatrix} \tag{3.117}$$

After the spontaneous symmetry braking we find

$$\mathcal{L}_Y = -\sum_{l_1,l_2} \overline{\nu}'_{l_1L} Y'_{l_1l_2}\nu'_{l_2R}(v + H) + h.c. = -\overline{\nu}'_L Y'\nu'_R(v + H) + h.c. \tag{3.118}$$

Here

$$\nu'_L = \begin{pmatrix} \nu'_{eL} \\ \nu'_{\mu L} \\ \nu'_{\tau L} \end{pmatrix} \quad \nu'_R = \begin{pmatrix} \nu'_{eR} \\ \nu'_{\mu R} \\ \nu'_{\tau R} \end{pmatrix} \tag{3.119}$$

In terms of the flavor neutrino fields (ν_L=$U_L^\dagger\nu'_L$) the proportional to v term of the Lagrangian (3.120) takes the form

$$\overline{\nu}_L = \nu_L^\dagger\gamma^0 = \nu_L'^\dagger U_L\gamma^0 = \overline{\nu}'_L U_L \tag{3.120}$$

$$\overline{\nu}'_L = \overline{\nu}_L U_L^\dagger \tag{3.121}$$

$$\begin{aligned} \mathcal{L}^D &= -v\overline{\nu}'_L Y'\nu'_R + h.c. \\ &= -v\overline{\nu}_L U_L^\dagger Y'\nu'_R + h.c. \\ &= -v\overline{\nu}_L Y\nu_R + h.c. \end{aligned} \tag{3.122}$$

where Y=$U_L^\dagger Y'$, ν_R=ν'_R. Taking into account the results of the previous chapter we conclude that the Higgs mechanism generates the Dirac neutrino mass term.
Let's diagonalize Y

$$Y = UyV^\dagger \tag{3.123}$$

where U and V are unitary matrices. From (3.124) and (3.125) for the neutrino mass term we find the following standard expression

$$\mathcal{L}^D(x) = -v\overline{\nu}_L Y\nu_R + h.c.$$

$$
\begin{aligned}
&= -v\overline{\nu}_L U y V^\dagger \nu_R + h.c. \\
&= -v\overline{U^\dagger \nu_L} y V^\dagger \nu_R - v\nu_R^\dagger V y \gamma_0 U^\dagger \nu_L \\
&= -v\overline{U^\dagger \nu_L} y V^\dagger \nu_R - v\overline{\nu}_R^\dagger V y U^\dagger \nu_L \qquad (3.124) \\
&= -\sum_{i=1}^{3} m_i \overline{\nu}_i(x)\nu_i(x) \qquad (3.125)
\end{aligned}
$$

The Dirac fields of neutrinos with definite masses ν_i are determined by the relation

$$
\nu_i(x) = U^\dagger \nu_L(x) + V^\dagger \nu_R(x) = \begin{pmatrix} \nu_1(x) \\ \nu_2(x) \\ \nu_3(x) \end{pmatrix} \qquad (3.126)
$$

$$
= \nu_{iL} + \nu_{iR} \qquad (3.127)
$$

and the Dirac neutrino masses are given by the relation

$$
\begin{aligned}
m_i &= y_i v \qquad (3.128) \\
y &= \frac{m}{v} \quad (y \; is \; small) \qquad (3.129)
\end{aligned}
$$

so

$$
\begin{aligned}
\nu_{iL} &= U^\dagger \nu_L \\
\nu_L &= U \nu_{iL} \\
\nu_{lL} &= U_{li} \nu_{iL} \qquad (3.130)
\end{aligned}
$$

For the neutrino mixing we have

$$
\nu_{lL}(x) = \sum_{i=1}^{3} U_{li} \nu_{iL}(x) \qquad (3.131)
$$

3.1.2 Majorana Mass Term

The most general neutrino mass term in which only flavor fields enter has the form

$$
\mathcal{L}^M = -\frac{1}{2} \sum_{l',l=e,\mu,\tau} \overline{\nu}_{l'L} M^M_{l'l} (\nu_{lL})^c + h.c. = -\frac{1}{2} \overline{\nu}_L M^M (\nu_L)^c + h.c. \qquad (3.132)
$$

Here

$$
\nu_L = \begin{pmatrix} \nu_{eL} \\ \nu_{\mu L} \\ \nu_{\tau L} \end{pmatrix} \qquad (3.133)
$$

and M^M is a complex 3×3 nondiagonal matrix. In fact, we have

$$
\overline{\nu}_L M^M (\nu_L)^c = \overline{\nu}_L M^M C \overline{\nu}_L^T = -\overline{\nu}_L (M^M)^T C^T \overline{\nu}_L^T = \overline{\nu}_L (M^M)^T C \overline{\nu}_L^T = \overline{\nu}_L (M^M)^T (\nu_L)^c \qquad (3.134)
$$

So, we know M^M is a symmetrical matrix.

$$
M^M = (M^M)^T \qquad (3.135)
$$

The symmetrical matrix M^M can be presented in the form

$$
M^M = U m U^T \qquad (3.136)
$$

where U is an unitary matrix and m is a diagonal matrix.

$$\begin{aligned}
\mathcal{L}^M &= -\frac{1}{2}\overline{\nu}_L M^M (\nu_L)^c + h.c. = -\frac{1}{2}\overline{\nu}_L U m U^T C \overline{\nu}_L^T + h.c. \qquad (3.137)\\
&= -\frac{1}{2}\overline{U^\dagger \nu_L} m (U^\dagger \nu_L)^c + h.c. = -\frac{1}{2}\overline{U^\dagger \nu_L} m (U^\dagger \nu_L)^c - \frac{1}{2}[(U^\dagger \nu_L)^c]^\dagger m (\overline{U^\dagger \nu_L})^\dagger \\
(U^\dagger \nu_L)^c &= C\overline{(U^\dagger \nu_L)}^T = C[(U^\dagger \nu_L)^\dagger \gamma_0]^T = C[\nu_L^\dagger U \gamma_0]^T = C\gamma_0 U^T (\nu_L^\dagger)^T \\
&= C\gamma_0 U^T \nu_L^* = U^T C \gamma_0 \nu_L^* = U^T C \overline{\nu}_L^T \qquad (3.138)\\
((U^\dagger \nu_L)^c)^\dagger &= (U^T C \overline{\nu}_L^T)^\dagger = [U^T C (\nu_L^\dagger \gamma_0)^T]^\dagger = [U^T C \gamma_0 \nu_L^*]^\dagger = \nu_L^T \gamma_0 C^\dagger U^* \qquad (3.139)\\
(\overline{U^\dagger \nu_L})^\dagger &= [(U^\dagger \nu_L)^\dagger \gamma_0]^\dagger = [\nu_L^\dagger U \gamma_0]^\dagger = \gamma_0 U^\dagger \nu_L \qquad (3.140)
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}^M &= -\frac{1}{2}\overline{U^\dagger \nu_L} m (U^\dagger \nu_L)^c - \frac{1}{2}[(U^\dagger \nu_L)^c]^\dagger m (\overline{U^\dagger \nu_L})^\dagger \\
&= -\frac{1}{2}\overline{U^\dagger \nu_L} m (U^\dagger \nu_L)^c - \frac{1}{2}\nu_L^T \gamma_0 C^\dagger U^* m \gamma_0 U^\dagger \nu_L \\
&= -\frac{1}{2}\overline{\nu}^M m \nu^M \qquad (3.141)\\
\overline{(U^\dagger \nu_L)^c} &= \overline{U^T C \overline{\nu}_L^T} = (U^T C \overline{\nu}_L^T)^\dagger \gamma_0 = [U^T C (\nu_L^\dagger \gamma_0)^T]^\dagger \gamma_0 = [U^T C \gamma_0 \nu_L^*]^\dagger \gamma_0 \\
&= \nu_L^T \gamma_0 C^\dagger U^* \gamma_0 \qquad (3.142)
\end{aligned}$$

Here,

$$\nu^M = U^\dagger \nu_L + (U^\dagger \nu_L)^c = \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}, \quad m = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}, \qquad (3.143)$$

From (3.142) and (3.142) we have

$$\mathcal{L}^M = -\frac{1}{2}\sum_{i=1}^{3} m_i \overline{\nu}_i \nu_i \qquad (3.144)$$

Thus, $\nu_i(x)$ is the field of the neutrino with mass m_i. From (3.143) we obviously have

$$(\nu^M(x))^c = \nu^M(x) \qquad (3.145)$$

Thus, the field of neutrinos with definite mass $\nu_i(x)$ satisfy the Majorana condition

$$\nu_i^c(x) = \nu_i(x) \qquad (3.146)$$

The mass term (3.132) is called the Majorana mass term.

3.1.2.1 seesaw mechanism If the Standard Model neutrinos are massless particles, small neutrino masses are generated by a beyond the Standard Model (BSM) mechanism. A general method is the method of the effective Lagrangian.
In order to build the effective Lagrangian which generate a neutrino mass term let us consider the $SU_L(2) \times U_Y(1)$ scalar

$$(\overline{\psi}_{lL}^{lep} \widetilde{H})$$

where $\overline{\psi}^{lep}_{lL}$ is the lepton doublet (1.2), $\widetilde{H}$=i$\tau_2 H^*$ is the conjugated Higgs doublet. After the spontaneous symmetry breaking (SSB) we have

$$(\overline{\psi}^{lep}_{lL}\widetilde{H}) = \frac{v+H}{\sqrt{2}}\overline{\nu}'_{lL} \tag{3.147}$$

From this expression it is obvious that the only possible $SU_L(2)\times U_Y\ (1)$ invariant effective Lagrangian which generate the neutrino mass term (after SSB) has a form (Weinberg)

$$\begin{aligned}\mathcal{L}^{eff}_I &= -\frac{1}{\Lambda}\sum_{l_1,l_2}(\overline{\psi}^{lep}_{l_1L}\widetilde{H})Y'_{l_1l_2}C((\overline{\psi}^{lep}_{l_2L}\widetilde{H}))^T + h.c.\\ &= -\frac{1}{\Lambda}\sum_{l_1,l_2}(\overline{\psi}^{lep}_{l_1L}\widetilde{H})Y'_{l_1l_2}(\psi^{lep}_{l_2L}\widetilde{H})^c + h.c.\end{aligned} \tag{3.148}$$

The operator in the expression (1.19) has a dimension M^5. Because the Lagrangian must have the dimension M^4 the constant Λ has a dimension of a mass. The Lagrangian (3.150) does not conserve the total lepton number L. Thus, the constant Λ characterizes the scale of a L-violating, beyond the SM physics. After spontaneous symmetry breaking from (3.150) we obtain the following neutrino mass term.

$$\mathcal{L}^M = -\frac{1}{2}(\frac{v^2}{\Lambda})\sum_{l_1,l_2}\overline{\nu}'_{l_1L}Y'_{l_1l_2}C(\overline{\nu}'_{l_2L})^T + h.c. = -\frac{1}{2}(\frac{v^2}{\Lambda})\overline{\nu}'_LY'C(\overline{\nu}'_L)^T + h.c. \tag{3.149}$$

The fields ν'_L are connected with the flavor neutrino fields ν_{lL} by the relation

$$\nu'_L = U_L\nu_L \tag{3.150}$$

where U_L is an unitary matrix. From (3.151) and (3.152) for the neutrino mass term we find the following expression

$$\overline{\nu}'_L = (\nu'_L)^\dagger\gamma^0 = \nu^\dagger_LU^\dagger_L\gamma^0 = \overline{\nu}_LU^\dagger_L \tag{3.151}$$
$$(\overline{\nu}'_L)^T = (\nu'^\dagger_L\gamma^0)^T = \gamma^0(\nu'^\dagger_L)^T = \gamma^0(\nu^\dagger_LU^\dagger_L)^T = (U^\dagger_L)^T\gamma^0(\nu^\dagger_L)^T \tag{3.152}$$
$$(\overline{\nu}_L)^T = (\nu^\dagger_L\gamma^0)^T = \gamma^0(\nu^\dagger_L)^T \tag{3.153}$$

$$\begin{aligned}\mathcal{L}^M &= -\frac{1}{2}(\frac{v^2}{\Lambda})\overline{\nu}'_LY'C(\overline{\nu}'_L)^T + h.c.\\ &= -\frac{1}{2}(\frac{v^2}{\Lambda})\overline{\nu}_LU^\dagger_LY'C(U^\dagger_L)^T\gamma^0(\nu^\dagger_L)^T + h.c.\\ &= -\frac{1}{2}(\frac{v^2}{\Lambda})\overline{\nu}_LU^\dagger_LY'C(U^\dagger_L)^T(\overline{\nu}_L)^T + h.c. && (3.154)\\ &= -\frac{1}{2}(\frac{v^2}{\Lambda})\overline{\nu}_LYC(\overline{\nu}_L)^T + h.c. = -\frac{1}{2}(\frac{v^2}{\Lambda})\sum_{l_1,l_2}\overline{\nu}_{l_1L}Y_{l_1l_2}(\nu_{l_2L})^c + h.c. && (3.155)\\ &= -\frac{1}{2}(\frac{v^2}{\Lambda})\overline{\nu}_LUyU^TC(\overline{\nu}_L)^T + h.c.\\ &= \frac{1}{2}(\frac{v^2}{\Lambda})\overline{U^\dagger\nu_L}y(U^\dagger\nu_L)^c + h.c. && (3.156)\end{aligned}$$

Here $Y = U^\dagger_LY'(U^\dagger_L)^T$ is a symmetrical 3×3 matrix. The matrix Y can be presented in the form

$$Y = UyU^T, \quad U^\dagger U = 1 \tag{3.157}$$

From (3.157) and (3.159) we find the following standard expression for the Majorana neutrino mass term

$$\mathcal{L}^{M} = -\frac{1}{2}\sum_{i=1}^{3} m_i \overline{\nu}_i \nu_i \tag{3.158}$$

$$\nu_i = U^{\dagger}\nu_L + (U^{\dagger}\nu_L)^c \tag{3.159}$$

Here ν_i=ν_i^c is the Majorana field with the mass

$$m_i = \frac{v^2}{\Lambda} y_i = (\frac{v}{\Lambda})(v y_i) \tag{3.160}$$

The flavor fields ν_{lL} are connected with the left-handed components of the fields of neutrinos with definite mass ν_{iL} by the standard mixing relation

$$\nu_{lL} = \sum_{i=1}^{3} U_{li}\nu_{iL} \tag{3.161}$$

3.1.3 Dirac and Majorana Mass Term

The most general Lagrangian is

$$\mathcal{L}^{D+M} = -\frac{1}{2}\overline{\nu}_L M_L^M (\nu_L)^c - \overline{\nu}_L M^D \nu_R - \frac{1}{2}\overline{(\nu_R)^c} M_R^M \nu_R + h.c. \tag{3.162}$$

Here M_L^M and M_R^M are complex non-diagonal symmetrical 3×3 matrices, M^D is a complex non-diagonal 3×3 matrix, ν_L is given by (3.133) and

$$\nu_R = \begin{pmatrix} \nu_{s_1 R} \\ \nu_{s_2 R} \\ \nu_{s_3 R} \end{pmatrix} \tag{3.163}$$

The mass term $\mathcal{L}^{D+M}$ can be written in the following matrix form

$$\mathcal{L}^{D+M} = -\frac{1}{2}\overline{n}_L M^{D+M} (n_L)^c + h.c. \tag{3.164}$$

Here

$$n_L = \begin{pmatrix} \nu_L \\ (\nu_R)^c \end{pmatrix}, \quad M^{D+M} = \begin{pmatrix} M_L^M & M^D \\ (M^D)^T & M_R^M \end{pmatrix} \tag{3.165}$$

Substitute (3.1.3) back into (3.1.3)

$$\begin{aligned} \mathcal{L}^{D+M} &= -\frac{1}{2}\overline{n}_L M^{D+M} (n_L)^c + h.c. \\ &= -\frac{1}{2}\begin{pmatrix} \overline{\nu}_L & \overline{(\nu_R)^c} \end{pmatrix} \begin{pmatrix} M_L^M & M^D \\ (M^D)^T & M_R^M \end{pmatrix} \begin{pmatrix} (\nu_L)^c \\ \nu_R \end{pmatrix} + h.c. \\ &= -\frac{1}{2}(\overline{\nu}_L M_L^M (\nu_L)^c + \overline{(\nu_R)^c}(M^D)^T(\nu_L)^c + \overline{\nu}_L M^D \nu_R + \overline{(\nu_R)^c} M_R^M \nu_R) + h.c. \end{aligned} \tag{3.166}$$

M^{D+M} is a symmetrical 6×6 matrix.

$$\overline{\nu}_L M^D \nu_R = \overline{(\nu_R)^c}(M^D)^T(\nu_L)^c \tag{3.167}$$

The matrix M^{M+D} can be presented in the following diagonal form

$$M^{D+M} = UmU^T \tag{3.168}$$

where U is an unitary 6×6 matrix. So, we have

$$\begin{aligned}
\mathcal{L}^{D+M} &= -\frac{1}{2}\overline{n}_L M^{D+M}(n_L)^c + h.c. \\
&= -\frac{1}{2}\overline{n}_L UmU^T(n_L)^c + h.c. \\
&= -\frac{1}{2}\overline{U^\dagger n_L} m (U^\dagger n_L)^c + h.c. \\
&= -\frac{1}{2}\overline{\nu}^M m \nu^M \qquad (3.169) \\
&= -\frac{1}{2}\sum_{i=1}^{6} m_i \overline{\nu}_i \nu_i \qquad (3.170)
\end{aligned}$$

Here

$$\nu^M = \nu_L^M + (\nu_L^M)^c = \begin{pmatrix} \nu_1 \\ \vdots \\ \nu_6 \end{pmatrix} = U^\dagger n_L + (U^\dagger n_L)^c \tag{3.171}$$

where

$$\nu_L^M = U^\dagger n_L \tag{3.172}$$

From (3.171) we have

$$(\nu^M)^c = \nu^M \quad and \quad \nu_i^c = \nu_i \quad (i = 1, 2, \dots, 6) \tag{3.173}$$

From (3.171) and (3.173) follow that ν_i is the field of Majorana particles with mass m_i.

3.1.3.1 Neutrino Mass Term in the Simplest Case of Two Neutrino Fields

Let us consider the Dirac and Majorana mass term in the case of one generation. We have

$$\begin{aligned}
\mathcal{L}^{D+M} &= -\frac{1}{2}\overline{\nu}_L M_L^M(\nu_L)^c - \overline{\nu}_L M^D \nu_R - \frac{1}{2}\overline{(\nu_R)^c} M_R^M \nu_R + h.c. \\
&= -\frac{1}{2}\overline{n}_L M^{D+M}(n_L)^c + h.c. \qquad (3.174)
\end{aligned}$$

Here m_L, m_D and m_R are real parameters (we assume CP invariance) and

$$M^{D+M} = \begin{pmatrix} m_L & m_D \\ m_D & m_R \end{pmatrix}, \quad n_L = \begin{pmatrix} \nu_L \\ (\nu_R)^c \end{pmatrix} \tag{3.175}$$

It is convenient to present the matrix M^{D+M} in the form

$$M^{D+M} = \frac{1}{2}TrM^{D+M} + M \tag{3.176}$$

where TrM^{D+M}=m_L+m_R and TrM=0. We have

$$M = \begin{pmatrix} -\frac{1}{2}(m_R - m_L) & m_D \\ m_D & \frac{1}{2}(m_R - m_L) \end{pmatrix} \tag{3.177}$$

Substitute (3.177) into (3.176)

$$M^{D+M} = \begin{pmatrix} \frac{1}{2}(m_L+m_R) & 0 \\ 0 & \frac{1}{2}(m_L+m_R) \end{pmatrix} + \begin{pmatrix} -\frac{1}{2}(m_R-m_L) & m_D \\ m_D & \frac{1}{2}(m_R-m_L) \end{pmatrix} = \begin{pmatrix} m_L & m_D \\ m_D & m_R \end{pmatrix} \tag{3.178}$$

The matrix M can be easily diagonalized by the orthogonal transformation. We have

$$M = O\overline{m}O^T \tag{3.179}$$

where

$$O = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad \overline{m} = \begin{pmatrix} \overline{m}_1 & 0 \\ 0 & \overline{m}_2 \end{pmatrix} \tag{3.180}$$

Here

$$\overline{m}_{1,2} = \mp\frac{1}{2}\sqrt{(m_R-m_L)^2+4m_D^2} \tag{3.181}$$

$$\tan 2\theta = \frac{2m_D}{m_R-m_L}, \quad \cos 2\theta = \frac{m_R-m_L}{\sqrt{(m_R-m_L)^2+4m_D^2}} \tag{3.182}$$

So,

$$M^{D+M} = Om'O^T \tag{3.183}$$

where

$$\overline{m}'_{1,2} = \frac{1}{2}(m_R+m_L) \mp \frac{1}{2}\sqrt{(m_R-m_L)^2+4m_D^2} \tag{3.184}$$

are eigenvalues of the matrix M^{M+D} which can be positive or negative. Let us write down

$$m'_i = m_i\eta_i \tag{3.185}$$

where m_i=$|m'_i|$ and η_i=±1. So,

$$M^{D+M} = Om\eta O^T = UmU^T \tag{3.186}$$

where U=O$\eta^{1/2}$ is unitary matrix. (2×2)
From (3.174) and (3.186) we obtain the following expression for the mass term

$$\begin{aligned}\mathcal{L}^{D+M} &= -\frac{1}{2}\overline{n}_L M^{D+M}(n_L)^c + h.c. \\ &= -\frac{1}{2}\overline{\nu}^M m\nu^M \\ &= -\frac{1}{2}\sum_{i=1,2} m_i\overline{\nu}_i\nu_i \end{aligned} \tag{3.187}$$

Here

$$\nu^M = U^\dagger n_L + (U^\dagger n_L)^c = \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \tag{3.188}$$

From this equation we know

$$\nu_i^c = \nu_i \tag{3.189}$$

Thus, ν_1 and ν_2 are fields of Majorana neutrino with masses m_1 and m_2, respectively. Form (3.172),(3.175),(3.180) and (3.188),we obtain the following mixing relations in the case of the Dirac and Majorana mass term for one neutrino family

$$U^\dagger = \eta^{1/2} O^\dagger = \begin{pmatrix} \cos\theta\sqrt{\eta_1} & -\sin\theta\sqrt{\eta_2} \\ \sin\theta\sqrt{\eta_1} & \cos\theta\sqrt{\eta_2} \end{pmatrix} \tag{3.190}$$

According to

$$n_L = U\nu_L^M \tag{3.191}$$

$$\begin{pmatrix} \nu_L \\ (\nu_R)^c \end{pmatrix} = \begin{pmatrix} \cos\theta\sqrt{\eta_1} & \sin\theta\sqrt{\eta_2} \\ -\sin\theta\sqrt{\eta_1} & \cos\theta\sqrt{\eta_2} \end{pmatrix} \begin{pmatrix} \nu_{1L} \\ \nu_{2L} \end{pmatrix} \tag{3.192}$$

$$\begin{aligned} \nu_L &= \cos\theta\sqrt{\eta_1}\nu_{1L} + \sin\theta\sqrt{\eta_2}\nu_{2L} \\ (\nu_R)^c &= -\sin\theta\sqrt{\eta_1}\nu_{1L} + \cos\theta\sqrt{\eta_2}\nu_{2L} \end{aligned} \tag{3.193}$$

The neutrino masses m_1 and m_2 and the mixing angle θ are determined by three real parameters m_L, m_R and m_D (see relations (3.182) and (3.184). The parameter η_i (i = 1, 2) determines the CP parity of the Majorana neutrino ν_i.

3.1.3.2 seesaw mechanism Here we will discuss the seesaw mechanism in the framework of the Dirac and Majorana mass term. In order to expose the main idea of the mechanism let us consider the simplest case of one family. General case of one family was considered in previous section. We assume that

- there is no left-handed Majorana mass term, i.e. $m_L = 0$,
- the Dirac mass term m_D is generated by the Standard Higgs mechanism, i.e. it is of the order of a mass of quark or lepton,
- the lepton number is violated at a scale which is much larger than the electroweak scale, i.e. $m_R \gg m_D$.

According to

$$m'_{1,2} = \frac{1}{2}(m_R + m_L) \mp \frac{1}{2}\sqrt{(m_R - m_L)^2 + 4m_D^2} \tag{3.194}$$

we find in this case

$$\begin{aligned} m_1 &= \frac{1}{2}|m_R - \sqrt{m_R^2 + 4m_D^2}| \cong \frac{1}{2}m_R[1 - (1 + \frac{4m_D^2}{m_R^2})^{1/2}] \cong \frac{1}{2}m_R[1 - (1 + \frac{1}{2}(\frac{2m_D}{m_R})^2) \cong \frac{m_D^2}{m_R} \ll m_D \\ m_2 &= \frac{1}{2}|m_R + \sqrt{m_R^2 + 4m_D^2}| \cong m_R \gg m_D \end{aligned} \tag{3.195}$$

For the mixing angle from

$$\tan 2\theta = \frac{2m_D}{m_R - m_L} \quad 2\theta \cong \frac{2m_D}{m_R} \tag{3.196}$$

we have

$$\theta \cong \frac{m_D}{m_R} \ll 1 \tag{3.197}$$

Let us consider now the case of three families.
The Dirac and Majorana seesaw matrix has the form

$$M = \begin{pmatrix} 0 & m_D \\ m_D^T & M_R \end{pmatrix} \tag{3.198}$$

where m_D and M_R are 3×3 matrices and $M_R = M_R^T$. We will assume that M_R is a diagonal matrix.
Let us introduce the matrix m by the relation

$$V^T M V = m \tag{3.199}$$

where V is a unitary matrix. We will show that the matrix V can be chosen in such a form that the matrix m has a block-diagonal form. In the one family case we have

$$U \cong \begin{pmatrix} 1 & \frac{m_D}{m_R} \\ -\frac{m_D}{m_R} & 1 \end{pmatrix} \quad O = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \quad U = \eta^{1/2} O \tag{3.200}$$

By analogy we will present the unitary matrix V in the form and $VV^\dagger$=1

$$V = \begin{pmatrix} 1 & a^\dagger \\ -a & 1 \end{pmatrix} \tag{3.201}$$

$$VV^\dagger = \begin{pmatrix} 1 & a^\dagger \\ -a & 1 \end{pmatrix} \begin{pmatrix} 1 & -a^\dagger \\ a & 1 \end{pmatrix} = \begin{pmatrix} 1 + aa^\dagger & 0 \\ 0 & 1 + aa^\dagger \end{pmatrix} \tag{3.202}$$

so $aa^\dagger$=0.
where a≪1. From (3.199) and (3.201) follows that the matrix m takes the block-diagonal form if we choose

$$V^T M V = \begin{pmatrix} -am_D^T - a(m_D - aM_R) & m_D - aa^\dagger m_D^T - aM_R \\ m_D^T - a(a^\dagger m_D + M_R) & a^\dagger m_D + a^\dagger m_D^T + M_R \end{pmatrix} \tag{3.203}$$

$$a = M_R^{-1} m_D^T \tag{3.204}$$

We have

$$m \cong \begin{pmatrix} -m_D M_R^{-1} m_D^T & 0 \\ 0 & M_R \end{pmatrix} \tag{3.205}$$

From (3.199) and (3.205) for the Dirac and Majorana mass term we find the following expression

$$\mathcal{L}^{seesaw} = -\frac{1}{2}\overline{\nu}_L m_L (\nu_L)^c - \frac{1}{2}\overline{(\nu_R)^c} M_R \nu_R + h.c. \tag{3.206}$$

where left-handed Majorana matrix is given by the relation

$$m_L = -m_D M_R^{-1} m_D^T \tag{3.207}$$

3.2 PMNS matrix

We will consider here the unitary 3×3 mixing matrix for Dirac neutrinos and introduce the standard parameters which characterizes it: three mixing angles and one phase.

$$\nu_{lL}(x) = \sum_{i=1}^{3} U_{li}\nu_{iL}(x) \tag{3.208}$$

state of the flavor neutrinos $|\,\nu_l>(l = e, \mu, \tau)$ is connected with states $|\,\nu_i>$ of neutrinos with masses m_i (i = 1, 2, 3) and momentum p by the relation

$$|\,\nu_l >= \sum_{i=1}^{3} U^*_{li}|\,\nu_i > \tag{3.209}$$

where

$$< \nu_i|\nu_k >= \delta_{ik} \tag{3.210}$$

In the matrix form the relation (3.209) can be written as follows

$$|\,\nu_f >= U^*|\,\nu > \tag{3.211}$$

where

$$|\,\nu_f >= \begin{pmatrix} |\,\nu_e > \\ |\,\nu_\mu > \\ |\,\nu_\tau > \end{pmatrix}, \quad |\,\nu >= \begin{pmatrix} |\,\nu_1 > \\ |\,\nu_2 > \\ |\,\nu_3 > \end{pmatrix} \tag{3.212}$$

In order to parameterize the matrix U we perform three Euler rotations. The first rotation will be performed at the angle θ_{12} around the vector $|\,\nu_3 >$. The new vectors are

$$|\,\nu >'= R_{12}(\theta_{12})|\,\nu > \tag{3.213}$$

here

$$R_{12}(\theta_{12}) = \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.214}$$

The second rotation will be performed at the angle θ_{13} around the vector $|\,\nu_2 >'$. At this step we will introduce the CP phase δ. We will obtain the following three orthogonal and normalized vectors: (Dirac neutrinos only have 1 phase) In the matrix form we have

$$|\,\nu >''= R^*_{13}(\theta_{13})|\,\nu >'= R^*_{13}(\theta_{13})R_{12}(\theta_{12})|\,\nu > \tag{3.215}$$

here

$$R^*_{13}(\theta_{13}) = \begin{pmatrix} c_{13} & 0 & s_{13}e^{i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{-i\delta} & 0 & c_{13} \end{pmatrix} \tag{3.216}$$

Finally, after the third rotation at the angle θ_{23} around the vector In the matrix form we have

$$|\,\nu_f > = R_{23}(\theta_{23})|\,\nu >''= R_{23}(\theta_{23})R^*_{13}(\theta_{13})|\,\nu >'= R_{23}(\theta_{23})R^*_{13}(\theta_{13})R_{12}(\theta_{12})|\,\nu >$$

$$= U^*|\,\nu > \tag{3.217}$$

here

$$R_{23}(\theta_{23}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \tag{3.218}$$

From (3.219) we find

$$U^* = R_{23}(\theta_{23}) R^*_{13}(\theta_{13}) R_{12}(\theta_{12}) \tag{3.219}$$

Thus,

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.220}$$

From (3.220) we find the following standard expression for the Dirac neutrino mixing matrix

$$U^D = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{13}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{13}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{13}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{13}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{13}} & c_{23}c_{13} \end{pmatrix} \tag{3.221}$$

The 3×3 Majorana mixing matrix has the form

$$U^M = U^D s^M(\overline{\alpha}) \tag{3.222}$$

where the phase matrix $S^M(\overline{\alpha})$ is characterized by two Majorana phases and has the form

$$S^M(\overline{\alpha}) = \begin{pmatrix} e^{i\overline{\alpha}_1} & 0 & 0 \\ 0 & e^{i\overline{\alpha}_2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.223}$$

and the matrix U^D is given by (3.221).

3.3 trimaximal mixing matrix

It is obvious from the previous subsection that if θ_{13}= 0 the general unitary mixing matrix can be obtained by (1,2) and (2,3) Euler rotations. It has the following form ($R^*_{13}(\theta_{13})$=1)

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} c_{12} & s_{12} & 0 \\ -c_{23}s_{12} & c_{23}c_{12} & s_{23} \\ s_{23}s_{12} & -s_{23}c_{12} & c_{23} \end{pmatrix} \tag{3.224}$$

These data were compatible with the simplest assumption

$$\sin^2\theta_{12} = \frac{1}{3}, \quad \sin^2\theta_{23} = \frac{1}{2} \tag{3.225}$$

From (3.221) follows that the neutrino mixing matrix can be chosen in the following form

$$U = U_{TB} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} & 0 \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & -\sqrt{\frac{1}{2}} \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{2}} \end{pmatrix} \tag{3.226}$$

The matrix U_{TB} is called tri-bimaximal mixing matrix.
We will consider the tri-bimaximal mixing from the point of view of the broken A4 symmetry group.
We will assume that neutrino with definite masses are Majorana particles. In this case for the mass matrix M we have

$$M = U^M m (U^M)^T \tag{3.227}$$

where U^M is the mixing matrix. Let us assume that $U^D = U_{TB}$ and $\overline{\alpha}_{1,2} = 0$. In this case the neutrino mass matrix is given by the expression, form (3.222), we know U^M=U^D

$$M_{TB} = U_{TB} m U_{TB}^T \tag{3.228}$$

From (1.45) and (1.48) for the neutrino mixing matrix we easily find the following expression

$$\begin{aligned} M_{TB} &= \begin{pmatrix} \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} & 0 \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & -\sqrt{\frac{1}{2}} \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} m_1 & m_1 & m_1 \\ m_2 & m_2 & m_2 \\ m_3 & m_3 & m_3 \end{pmatrix} \begin{pmatrix} \sqrt{\frac{2}{3}} & -\sqrt{\frac{1}{6}} & -\sqrt{\frac{1}{6}} \\ -\sqrt{\frac{1}{3}} & \sqrt{\frac{1}{3}} & -\sqrt{\frac{1}{3}} \\ 0 & -\sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} \end{pmatrix} \\ &= \begin{pmatrix} (\frac{2}{3}m_1 + \frac{1}{3}m_2) & (-\frac{1}{3}m_1 + \frac{1}{3}m_2 & (-\frac{1}{3}m_1 + \frac{1}{3}m_2) \\ (-\frac{1}{3}m_1 + \frac{1}{3}m_2) & (\frac{1}{6}m_1 + \frac{1}{3}m_2 + \frac{1}{2}m_3) & (\frac{1}{6}m_1 + \frac{1}{3}m_2 - \frac{1}{2}m_3) \\ (-\frac{1}{3}m_1 + \frac{1}{3}m_2) & (\frac{1}{6}m_1 + \frac{1}{3}m_2 - \frac{1}{2}m_3) & (\frac{1}{6}m_1 + \frac{1}{3}m_2 + \frac{1}{2}m_3) \end{pmatrix} \end{aligned} \tag{3.229}$$

The tri-bimaximal mass matrix depends on three parameters. Let us introduce

$$x = \frac{2}{3}m_1 + \frac{1}{3}m_2, \quad y = -\frac{1}{3}m_1 + \frac{1}{3}m_2, \quad z = -\frac{1}{3}m_1 + \frac{1}{3}m_3 \tag{3.230}$$

From (3.229) we have

$$M_{TB} = \begin{pmatrix} x & y & y \\ y & x+v & y-v \\ y & y-v & x+v \end{pmatrix} \tag{3.231}$$

All elements of the $A4$ group are products of two generators S and T which satisfy the relations

$$S^2 = T^3 = (ST)^3 = 1 \tag{3.232}$$

Taking into account the relations (3.232) we see that elements of the A4 group (all possible products of S and T) are given by
1, $T, S, ST, TS, T^2, T^2S, TST, ST^2, STS, T^2ST, TST^2$
The group A4 has four irreducible representations: one triplet and three singlets 3,1,$1', 1''$.
For one-dimensional unitary representations, if the generator satisfys S^2=1, then we have S=±1, if the generator satisfys T^3=1, then we have T=1, $e^{i\frac{2\pi}{3}}, e^{i\frac{4\pi}{3}}$, if the generator satisfys $(ST)^3$=1, then we have S=1, T=1, $e^{i\frac{2\pi}{3}}, e^{i\frac{4\pi}{3}}$. Thus, we have

$$1: S=1 \quad T=1; \quad 1': S=1 \quad T=w^2; \quad 1'': S=1 \quad T=w \tag{3.233}$$

where $w = e^{i\frac{2\pi}{3}}$.
We can choose the three-dimensional representation of the generator T in the form

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}. \tag{3.234}$$

The real unitary matrix S satisfies the condition $S^T S = 1$. Taking into account that $S^2 = 1$, we have S^T = S. It is easy to check that the three-dimensional orthogonal symmetrical matrix

$$S = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}. \tag{3.235}$$

Let us assume that the lepton doublets L_{lL} (l = e, μ, τ) are A4 triplets. In order to generate the neutrino mass term we will further assume that exist a scalar triplet Higgs-like flavon fields ϕ_T , ϕ_S and a singlet field ξ which enter into the A4 invariant Yukawa interactions together with lepton doublets.
If we assume the vacuum alignments

$$<\phi_T>= (v_T, 0, 0), \quad <\phi_S>= (v_S, v_S, v_S), \quad <\xi>= u \tag{3.236}$$

This subscript actually reflects symmetry, for example, $<\phi_T>$ is symmetric with respect to T, ($T<\phi_T>^T$=1), $<\phi_S>$ is symmetric with respect to S, ($S<\phi_S>^T$=1).
On this basis, the change in ν_L

$$\nu_L \to \nu'_L = \rho_3(S)\nu_L \tag{3.237}$$

$$\overline{\nu}_L \to \overline{\nu}'_L = \nu'^{\dagger}_L\gamma_0 = \nu^{\dagger}_L\rho_3(S)^{\dagger}\gamma_0 = \overline{\nu}_L\rho_3(S) \tag{3.238}$$

$$\overline{\nu}^T_L \to \overline{\nu}'^T_L = (\rho_3(S))^T(\overline{\nu}_L)^T \tag{3.239}$$

For neutrinos, the Lagrangian is

$$\mathcal{L}^M = -\frac{1}{2}\overline{\nu}_L M(\nu_L)^c + h.c. = -\frac{1}{2}\overline{\nu}_L MC\overline{\nu}_L + h.c \tag{3.240}$$

$$\begin{aligned}\mathcal{L}^M \to \mathcal{L}^{M\prime} &= -\frac{1}{2}\overline{\nu}'_L MC\overline{\nu}^{T\prime}_L + h.c \\ &= -\frac{1}{2}\overline{\nu}_L\rho_3(S)MC(\rho_3(S))^T(\overline{\nu}_L)^T\end{aligned} \tag{3.241}$$

If the Lagrangian is invariant,then

$$\rho_3(S)M(\rho_3(S))^T = M \tag{3.242}$$

(ρ_3 is a symmetric matrix, that's the S in the book. That's the commutation of S and M.) where S is given by (3.235). From (3.242) we find that the mass matrix M is given by

$$M = \begin{pmatrix} x & y & z \\ y & x+v+y-z & w \\ z & w & x+v \end{pmatrix}. \tag{3.243}$$

This matrix is characterized by four real parameters and does not have the tribimaximal form (3.226).
We will come to the tri-bimaximal mixing if we further assume μ-τ symmetry of the neutrino mixing matrix (accident symmetry)

$$S_{\mu\tau}MS_{\mu\tau} = M \tag{3.244}$$

where

$$S_{\mu\tau} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{3.245}$$

From (3.244) and (4.272) we find that

$$z = y \tag{3.246}$$

Now we come to the tri-bimaximal neutrino mixing matrix (3.226).
In 2012 it was discovered the angle θ_{13} is different from zero. As we saw in previous section, the leptonic charged current is given by the expression

$$j^{\rm CC}_\alpha(x) = 2\bar{\nu}'_L(x)\gamma_\alpha L'_L(x) = 2\sum_{l=e,\mu,\tau}\bar{\nu}'_{lL}(x)\gamma_\alpha l'_L(x). \tag{3.247}$$

After the diagonalization of the charged lepton and neutrino mass terms we have

$$L'_L(x) = U^{\rm lep}_L\, L_L(x), \quad \nu'_L(x) = U^\nu_L \nu_L(x). \tag{3.248}$$

where U^{lep}_L and U^ν_L are unitary 3×3 matrices and

$$L_L(x) = \begin{pmatrix} e_L(x) \\ \mu_L(x) \\ \tau_L(x) \end{pmatrix}, \quad \nu_L(x) = \begin{pmatrix} \nu_{1L}(x) \\ \nu_{2L}(x) \\ \nu_{1L}(x) \end{pmatrix}. \tag{3.249}$$

Here $l(x)$ ($l = e,\ \mu,\ \tau$) is the field of a lepton with mass m_l and ν_i(x) (i = 1, 2, 3) is the field of neutrino with mass m_i.
From (3.247) and (3.248) we have

$$j^{\rm CC}_\alpha(x) = 2\bar{\nu}_L(x)U^{\nu\dagger}_L U^{\rm lep}_L\gamma_\alpha L_L(x) = 2\bar{\nu}_L(x)U^\dagger\gamma_\alpha L_L(x) = 2\sum_{l,i}\bar{\nu}_i(x)U^*_{li}\gamma_\alpha l_L(x). \tag{3.250}$$

Thus the PMNS mixing matrix is given by the relation

$$U = U^{lep\dagger}_L U^\nu_L. \tag{3.251}$$

It was assumed that the matrix U^ν_L is determined by a broken finite flavor symmetry (A4, S4 and others) and $e'3$ element of this matrix is equal to zero. The tri-bimaximal matrix (3.226) is one of the possible examples. In this approach nonzero value of the angle θ_{13} is due to the matrix U^{lep}_L.
So let us assume that the PMNS mixing matrix is given by the expression

$$U = U_{TB}R_{23}(\alpha) \tag{3.252}$$

where U_{TB} is the tri-bimaximal matrix (3.226) and the matrix $R_{23}(\alpha)$ has the form

$$R_{23}(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_\alpha & s_\alpha \\ 0 & -s_\alpha & c_\alpha \end{pmatrix} \tag{3.253}$$

where the parameter α is determined by θ_{13}. The relation (3.252) leads to the following sum rules

$$\sin^2\theta_{23} = \frac{1}{2} - \sqrt{2}\sin\theta_{13}\cos\delta + O(\sin^2\theta_{13}), \quad \sin^2\theta_{12} = \frac{1}{3} - \frac{2}{3}\sin^2\theta_{13} + O(\sin^4\theta_{13}). \tag{3.254}$$

Another possibility is to assume

$$U = U_{TB}R_{13}(\alpha) \tag{3.255}$$

where

$$R^*_{13}(\alpha) = \begin{pmatrix} c_\alpha & 0 & s_\alpha \\ 0 & 1 & 0 \\ -s_\alpha & 0 & c_\alpha \end{pmatrix} \tag{3.256}$$

The relation (3.255) leads to another sum rules

$$\sin^2\theta_{23} = \frac{1}{2} + \frac{1}{\sqrt{2}}\sin\theta_{13}\cos\delta + O\left(\sin^2\theta_{13}\right), \quad \sin^2\theta_{12} = \frac{1}{3} + \frac{1}{3}\sin^2\theta_{13} + O\left(\sin^4\theta_{13}\right). \tag{3.257}$$

4 tri-bimaximal

$$w = e^{i\frac{2\pi i}{3}} \tag{4.258}$$

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & w^2 & 0 \\ 0 & 0 & w \end{pmatrix} \tag{4.259}$$

$$S = \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix} \tag{4.260}$$

$$S_{\mu\tau} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{4.261}$$

The eigenvalues and eigenvectors of T are $-(-1)^{1/3}, (-1)^{2/3}, 1$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{4.262}$$

The eigenvalues and eigenvectors of S are $-1, -1, 1$

$$\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \quad \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{4.263}$$

The third column matrix determines one of the columns of the tri-bimaximal matrix.
The eigenvalues and eigenvectors of $S_{\mu\tau}$ are $-1, 1, 1$

$$\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{4.264}$$

The first column matrix determines one of the columns of the tri-bimaximal matrix.
The eigenvalues and eigenvectors of $SS_{\mu\tau}$ are $-1, 1, 1$

$$\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \quad \begin{pmatrix} \frac{1}{2} \\ 0 \\ 1 \end{pmatrix} \quad \begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix} \tag{4.265}$$

The first column matrix determines one of the columns of the tri-bimaximal matrix.
We find

$$[S, S_{\mu\tau}] = 0 \tag{4.266}$$

So there are common eigenstates. From(3.242), we know they're both commutated with M.
The eigenvector is formed into a complete matrix and normalized

$$U_\nu = \begin{pmatrix} \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} & 0 \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & -\sqrt{\frac{1}{2}} \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{2}} \end{pmatrix} \tag{4.267}$$

This is the tri-bimaximal matrix.
Now we can diagonal the tri-bimaximal mass matrix, with (3.229),

$$m_\nu = U_\nu^T M_{TB} U_\nu = \begin{pmatrix} x-y & 0 & 0 \\ 0 & x+2y & 0 \\ 0 & 0 & 2v+x-y \end{pmatrix} \tag{4.268}$$

If we only consider S matrix, write his eigenvector as a normalized matrix.

$$U_{\nu S} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{3}} \end{pmatrix} \tag{4.269}$$

Only one column of the tri-bimaximal matrix can be determined. Because S has only one non-degenerate eigenvalue.
If we think about perturbations

$$R = \begin{pmatrix} c_\theta & s_\theta & 0 \\ -s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.270}$$

$$U_{\nu SR} = U_{\nu S} R = \begin{pmatrix} \frac{-\cos\theta+\sin\theta}{\sqrt{2}} & -\frac{\cos\theta+\sin\theta}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{-\sin\theta}{\sqrt{2}} & \frac{\cos\theta}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{\cos\theta}{\sqrt{2}} & \frac{\sin\theta}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix} \tag{4.271}$$

Add

$$Rp = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{4.272}$$

$$U_\nu Spp = U_{\nu SR} Rp = \begin{pmatrix} \frac{-\cos\theta+\sin\theta}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{\cos\theta+\sin\theta}{\sqrt{2}} \\ \frac{-\sin\theta}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{\cos\theta}{\sqrt{2}} \\ \frac{\cos\theta}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{\sin\theta}{\sqrt{2}} \end{pmatrix} \tag{4.273}$$

We find that we can only determine the second column of the U_ν matrix.
Similarly, If we only consider $S_{\mu\tau}$ matrix, write his eigenvector as a normalized matrix.

$$U_{\mu\tau} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \tag{4.274}$$

Only one column of the tri-bimaximal matrix can be determined. Because $S_{\mu\tau}$ has only one non-degenerate eigenvalue.
If we think about perturbations

$$R2 = \begin{pmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{pmatrix} \tag{4.275}$$

$$Rp1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.276}$$

$$U_{\nu Spp2} = U_{\mu\tau} R2 Rp1 = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ -\frac{\sin\theta}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{\cos\theta}{\sqrt{2}} \\ -\frac{\sin\theta}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{\cos\theta}{\sqrt{2}} \end{pmatrix} \tag{4.277}$$

We find that we can only determine the third column of the U_ν matrix.