Appendix A: Bloch-basis and length gauge

$$
\begin{aligned}
< n\mathbf{k}|\hat{\mathbf{x}}|n''\mathbf{k}'' > &= < n\mathbf{k}|\{-i\nabla_{\mathbf{k}''}e^{i\mathbf{k}''\cdot\hat{\mathbf{x}}}\}|u_{n''\mathbf{k}''} > \\
&= -i\nabla_{\mathbf{k}''} < n\mathbf{k}|n''\mathbf{k}'' > +i < u_{n\mathbf{k}}|e^{-i(\mathbf{k}-\mathbf{k}'')\cdot\hat{\mathbf{x}}}|\nabla_{\mathbf{k}''}u_{n''\mathbf{k}''} > \\
&= -i\nabla_{\mathbf{k}''} < n\mathbf{k}|n''\mathbf{k}'' > +\delta(\mathbf{k}-\mathbf{k}'')\mathbf{d}_{nn''}(\mathbf{k}) \qquad \text{(A1)} \\
< n''\mathbf{k}''|\hat{\mathbf{x}}|n'\mathbf{k}' > &= < u_{n''\mathbf{k}''}|\{i\nabla_{\mathbf{k}''}e^{-i\mathbf{k}''\cdot\hat{\mathbf{x}}}\}|n'\mathbf{k}' > \\
&= i\nabla_{\mathbf{k}''} < n''\mathbf{k}''|n'\mathbf{k}' > -i < \nabla_{\mathbf{k}''}u_{n''\mathbf{k}''}|e^{-i(\mathbf{k}''-\mathbf{k}')\cdot\hat{\mathbf{x}}}|u_{n'\mathbf{k}'} > \\
&= i\nabla_{\mathbf{k}''} < n''\mathbf{k}''|n'\mathbf{k}' > +\delta(\mathbf{k}''-\mathbf{k}')\mathbf{d}_{n''n'}(\mathbf{k}') \qquad \text{(A2)}
\end{aligned}
$$

$$
\begin{aligned}
i < u_{n\mathbf{k}}|e^{-i(\mathbf{k}-\mathbf{k}'')\cdot\hat{\mathbf{x}}}|\nabla_{\mathbf{k}''}u_{n''\mathbf{k}''} > &= i\int d\mathbf{x} < u_{n\mathbf{k}}|\mathbf{x}>< \mathbf{x}|\nabla_{\mathbf{k}''}u_{n''\mathbf{k}''} > e^{-i(\mathbf{k}-\mathbf{k}'')\cdot\mathbf{x}} \\
&= i\sum_{\mathbf{R}} e^{-i(\mathbf{k}-\mathbf{k}'')\cdot\mathbf{R}}\int_{BL} d\mathbf{y} < u_{n\mathbf{k}}|\mathbf{y}>< \mathbf{y}|\nabla_{\mathbf{k}''}u_{n''\mathbf{k}''} > e^{-i(\mathbf{k}-\mathbf{k}'')\cdot\mathbf{y}} \\
&= \delta(\mathbf{k}-\mathbf{k}'')(i\int_{BL} d\mathbf{y} < u_{n\mathbf{k}}|\mathbf{y}>< \mathbf{y}|\nabla_{\mathbf{k}''}u_{n''\mathbf{k}''} >) \\
&\equiv \delta(\mathbf{k}-\mathbf{k}'')\mathbf{d}_{nn''}(\mathbf{k}) \qquad \text{(A3)} \\
i < \nabla_{\mathbf{k}''}u_{n''\mathbf{k}''}|e^{-i(\mathbf{k}''-\mathbf{k}')\cdot\hat{\mathbf{x}}}|u_{n'\mathbf{k}'} > &= i\int d\mathbf{x} < \nabla_{\mathbf{k}''}u_{n''\mathbf{k}''}|\mathbf{x}>< \mathbf{x}|u_{n'\mathbf{k}'} > e^{-i(\mathbf{k}''-\mathbf{k}')\cdot\mathbf{x}} \\
&= i\sum_{\mathbf{R}} e^{-i(\mathbf{k}''-\mathbf{k}')\cdot\mathbf{R}}\int_{BL} d\mathbf{y} < \nabla_{\mathbf{k}''}u_{n''\mathbf{k}''}|\mathbf{y}>< \mathbf{y}|u_{n'\mathbf{k}'} > e^{-i(\mathbf{k}''-\mathbf{k}')\cdot\mathbf{y}} \\
&= \delta(\mathbf{k}''-\mathbf{k}')(i\int_{BL} d\mathbf{y} < \nabla_{\mathbf{k}''}u_{n''\mathbf{k}''}|\mathbf{y}>< \mathbf{y}|u_{n'\mathbf{k}'} >) \\
&= -\delta(\mathbf{k}''-\mathbf{k}')(i\int_{BL} d\mathbf{y} < u_{n'\mathbf{k}'}|\mathbf{y}>< \mathbf{y}|\nabla_{\mathbf{k}''}u_{n''\mathbf{k}''} >)^* \\
&= -\delta(\mathbf{k}''-\mathbf{k}')\mathbf{d}_{n''n'}(\mathbf{k}') \qquad \text{(A4)}
\end{aligned}
$$

$$
\begin{aligned}
0 &= i\nabla_{\mathbf{k}} < u_{n\mathbf{k}}|u_{n'\mathbf{k}} > \\
&= i < \nabla_{\mathbf{k}}u_{n\mathbf{k}}|u_{n'\mathbf{k}} > +i < u_{n\mathbf{k}}|\nabla_{\mathbf{k}}u_{n'\mathbf{k}} > \\
&= -\mathbf{d}^*_{n'n}(\mathbf{k}) + \mathbf{d}_{nn'}(\mathbf{k}). \qquad \text{(A5)}
\end{aligned}
$$

$$
\begin{aligned}
&i\hbar\frac{\partial}{\partial t}< n\mathbf{k}|\hat{\rho}|n'\mathbf{k}' > \\
&\quad =\sum_{n''}\int_{BZ} dk''(< n\mathbf{k}|\hat{H}_0 - e\mathbf{E}\cdot\hat{\mathbf{x}}|n''\mathbf{k}'' >< n''\mathbf{k}''|\hat{\rho}|n'\mathbf{k}' > \\
&\qquad - < n\mathbf{k}|\hat{\rho}|n''\mathbf{k}'' >< n''\mathbf{k}''|\hat{H}_0 - e\mathbf{E}\cdot\hat{\mathbf{x}}|n'\mathbf{k}' >) \\
&\quad =(\epsilon_{n\mathbf{k}} - \epsilon_{n'\mathbf{k}'}) < n\mathbf{k}|\hat{\rho}|n'\mathbf{k}' >
\end{aligned}
$$

$$
\begin{aligned}
&-e\mathbf{E}\cdot\sum_{n''}\int_{BZ}dk''(<n\mathbf{k}|\hat{\mathbf{x}}|n''\mathbf{k}''><n''\mathbf{k}''|\hat{\rho}|n'\mathbf{k}'>-<n\mathbf{k}|\hat{\rho}|n''\mathbf{k}''><n''\mathbf{k}''|\hat{\mathbf{x}}|n'\mathbf{k}'>)\\
=&(\epsilon_{n\mathbf{k}}-\epsilon_{n'\mathbf{k}'})<n\mathbf{k}|\hat{\rho}|n'\mathbf{k}'>\\
&-e\mathbf{E}\cdot[i(\nabla_{\mathbf{k}}+\nabla_{\mathbf{k}'})<n\mathbf{k}|\hat{\rho}|n'\mathbf{k}'>+\sum_{n''}\{\mathbf{d}_{nn''}(\mathbf{k})<n''\mathbf{k}|\hat{\rho}|n'\mathbf{k}'>-\mathbf{d}_{n''n'}(\mathbf{k}')<n\mathbf{k}|\hat{\rho}|n''\mathbf{k}'>\}]
\end{aligned}
\tag{A6}
$$

$$
\begin{aligned}
&i\hbar\frac{\partial}{\partial t}<v\mathbf{k}|\hat{\rho}|v\mathbf{k}>\\
&\quad=-e\mathbf{E}\cdot[i\nabla_{\mathbf{k}}<v\mathbf{k}|\hat{\rho}|v\mathbf{k}>+\{\mathbf{d}_{vc}(\mathbf{k})<c\mathbf{k}|\hat{\rho}|v\mathbf{k}>-\mathbf{d}_{cv}(\mathbf{k})<v\mathbf{k}|\hat{\rho}|c\mathbf{k}>\}]\\
&\quad=-e\mathbf{E}\cdot[i\nabla_{\mathbf{k}}<v\mathbf{k}|\hat{\rho}|v\mathbf{k}>-2i\mathbf{Im}\{\mathbf{d}^*_{vc}(\mathbf{k})<v\mathbf{k}|\hat{\rho}|c\mathbf{k}>\}] \qquad \text{(A7)}\\
&i\hbar\frac{\partial}{\partial t}<c\mathbf{k}|\hat{\rho}|c\mathbf{k}>\\
&\quad=-e\mathbf{E}\cdot[i\nabla_{\mathbf{k}}<c\mathbf{k}|\hat{\rho}|c\mathbf{k}>+\{\mathbf{d}_{cv}(\mathbf{k})<v\mathbf{k}|\hat{\rho}|c\mathbf{k}>-\mathbf{d}_{vc}(\mathbf{k})<c\mathbf{k}|\hat{\rho}|v\mathbf{k}>\}]\\
&\quad=-e\mathbf{E}\cdot[i\nabla_{\mathbf{k}}<c\mathbf{k}|\hat{\rho}|c\mathbf{k}>+2i\mathbf{Im}\{\mathbf{d}^*_{vc}(\mathbf{k})<v\mathbf{k}|\hat{\rho}|c\mathbf{k}>\}] \qquad \text{(A8)}\\
&i\hbar\frac{\partial}{\partial t}<v\mathbf{k}|\hat{\rho}|c\mathbf{k}>\\
&\quad=(\epsilon_{v\mathbf{k}}-\epsilon_{c\mathbf{k}})<v\mathbf{k}|\hat{\rho}|c\mathbf{k}>\\
&\qquad-e\mathbf{E}\cdot[i\nabla_{\mathbf{k}}<v\mathbf{k}|\hat{\rho}|c\mathbf{k}>+\sum_{n''}\{\mathbf{d}_{vn''}(\mathbf{k})<n''\mathbf{k}|\hat{\rho}|c\mathbf{k}>-\mathbf{d}_{n''c}(\mathbf{k})<v\mathbf{k}|\hat{\rho}|n''\mathbf{k}>\}]\\
&\quad=-(\epsilon_{c\mathbf{k}}-\epsilon_{v\mathbf{k}})<v\mathbf{k}|\hat{\rho}|c\mathbf{k}>-ie\mathbf{E}\cdot\nabla_{\mathbf{k}}<v\mathbf{k}|\hat{\rho}|c\mathbf{k}>\\
&\qquad-e\mathbf{E}\cdot[\{\mathbf{d}_{vv}(\mathbf{k})-\mathbf{d}_{cc}(\mathbf{k})\}<v\mathbf{k}|\hat{\rho}|c\mathbf{k}>-\mathbf{d}_{vc}(\mathbf{k})\{<c\mathbf{k}|\hat{\rho}|c\mathbf{k}>-<v\mathbf{k}|\hat{\rho}|v\mathbf{k}>\}] \qquad \text{(A9)}
\end{aligned}
$$

$$
\begin{aligned}
H_L&=tr\{\hat{\rho}(\hat{H}_0-e\mathbf{E}\cdot\hat{\mathbf{x}})\}\\
&=\sum_n\int d\mathbf{k}<n\mathbf{k}|\hat{\rho}|n\mathbf{k}>\epsilon_{n\mathbf{k}}-e\mathbf{E}\cdot\{\sum_{nn'}\int d\mathbf{k}\int d\mathbf{k}'<n\mathbf{k}|\hat{\rho}|n'\mathbf{k}'><n'\mathbf{k}'|\hat{\mathbf{x}}|n\mathbf{k}>\}\\
&=\sum_n\int d\mathbf{k}<n\mathbf{k}|\hat{\rho}|n\mathbf{k}>\epsilon_{n\mathbf{k}}\\
&\quad-e\mathbf{E}\cdot[\sum_{nn'}\int d\mathbf{k}\int d\mathbf{k}'<n\mathbf{k}|\hat{\rho}|n'\mathbf{k}'>\{\frac{1}{2i}(\nabla_{\mathbf{k}}-\nabla_{\mathbf{k}'})<n'\mathbf{k}'|n\mathbf{k}>+\delta(\mathbf{k}'-\mathbf{k})\mathbf{d}_{n'n}(\mathbf{k})\}]\\
&=\sum_n\int d\mathbf{k}<n\mathbf{k}|\hat{\rho}|n\mathbf{k}>\epsilon_{n\mathbf{k}}\\
&\quad-e\mathbf{E}\cdot\{\sum_n\int d\mathbf{k}\,\mathbf{Im}(\nabla_{\mathbf{k}}<n\mathbf{k}|\hat{\rho}|n\mathbf{k}>)+\sum_{nn'}\int d\mathbf{k}<n\mathbf{k}|\hat{\rho}|n'\mathbf{k}>\mathbf{d}_{n'n}(\mathbf{k})\}\\
&=\sum_n\int d\mathbf{k}<n\mathbf{k}|\hat{\rho}|n\mathbf{k}>\epsilon_{n\mathbf{k}}-e\mathbf{E}\cdot\{\sum_{nn'}\int d\mathbf{k}<n\mathbf{k}|\hat{\rho}|n'\mathbf{k}>\mathbf{d}_{n'n}(\mathbf{k})\}
\end{aligned}
\tag{A10}
$$

$$
\begin{aligned}
H_L^2 =& \sum_n \int d\mathbf{k} < n\mathbf{k}|\hat{\rho}|n\mathbf{k} > \epsilon_{n\mathbf{k}} - e\mathbf{E}\cdot\{\sum_{nn'}\int d\mathbf{k} < n\mathbf{k}|\hat{\rho}|n'\mathbf{k} > \mathbf{d}_{n'n}(\mathbf{k})\} \\
=& \int d\mathbf{k}(< v\mathbf{k}|\hat{\rho}|v\mathbf{k} > \epsilon_{v\mathbf{k}} + < c\mathbf{k}|\hat{\rho}|c\mathbf{k} > \epsilon_{c\mathbf{k}}) \\
& -e\mathbf{E}\cdot\int d\mathbf{k}[< v\mathbf{k}|\hat{\rho}|v\mathbf{k} > \mathbf{d}_{vv}(\mathbf{k}) + < c\mathbf{k}|\hat{\rho}|c\mathbf{k} > \mathbf{d}_{cc}(\mathbf{k}) + 2\mathbf{Re}\{< v\mathbf{k}|\hat{\rho}|c\mathbf{k} > \mathbf{d}^*_{vc}(\mathbf{k})\}]
\end{aligned}
\tag{A11}
$$

Appendix B: Bloch-basis and velocity gauge

$$
[\hat{\mathbf{x}}, \hat{H}_0] = [\hat{\mathbf{x}}, \frac{\hat{\mathbf{p}}}{2m}] = \frac{i\hbar}{m}\hat{\mathbf{p}} \tag{B1}
$$

$$
\begin{aligned}
< n\mathbf{k}|\hat{\mathbf{p}}|n'\mathbf{k}' > &= i\frac{m(\epsilon_{n\mathbf{k}} - \epsilon_{n'\mathbf{k}'})}{\hbar} < n\mathbf{k}|\hat{\mathbf{x}}|n'\mathbf{k}' > \\
&= \begin{cases} \frac{m}{\hbar}(\epsilon_{n\mathbf{k}} - \epsilon_{n'\mathbf{k}'})\{-\nabla_{\mathbf{k}} < n\mathbf{k}|n'\mathbf{k}' > + i\delta(\mathbf{k}-\mathbf{k}')\mathbf{d}_{nn'}(\mathbf{k})\} \\ \frac{m}{\hbar}(\epsilon_{n\mathbf{k}} - \epsilon_{n'\mathbf{k}'})\{\nabla_{\mathbf{k}'} < n\mathbf{k}|n'\mathbf{k}' > + i\delta(\mathbf{k}-\mathbf{k}')\mathbf{d}_{nn'}(\mathbf{k})\} \end{cases}
\end{aligned}
\tag{B2}
$$

$$
\begin{aligned}
< n\mathbf{k}|\hat{\mathbf{p}}|n\mathbf{k} > &= \frac{m}{i\hbar} < n\mathbf{k}|[\hat{\mathbf{x}}, \hat{H}_0]|n\mathbf{k} > \\
&= \frac{m}{i\hbar}\{< u_{n\mathbf{k}}|(i\nabla_{\mathbf{k}}e^{-i\mathbf{k}\cdot\hat{\mathbf{x}}})\hat{H}_0|n\mathbf{k} > - < n\mathbf{k}|\hat{H}_0(-i\nabla_{\mathbf{k}}e^{i\mathbf{k}\cdot\hat{\mathbf{x}}})|u_{n\mathbf{k}} >\} \\
&= \frac{m}{\hbar}\{\nabla_{\mathbf{k}} < n\mathbf{k}|\hat{H}_0|n\mathbf{k} > - < \nabla_{\mathbf{k}}u_{n\mathbf{k}}|e^{-i\mathbf{k}\cdot\hat{\mathbf{x}}}\hat{H}_0|n\mathbf{k} > - < n\mathbf{k}|\hat{H}_0 e^{i\mathbf{k}\cdot\hat{\mathbf{x}}}|\nabla_{\mathbf{k}}u_{n\mathbf{k}} >\} \\
&= \frac{m}{\hbar}\{\nabla_{\mathbf{k}}\epsilon_{n\mathbf{k}} - \epsilon_{n\mathbf{k}}\nabla_{\mathbf{k}} < u_{n\mathbf{k}}|u_{n\mathbf{k}} >\} \\
&= \frac{m}{\hbar}\nabla_{\mathbf{k}}\epsilon_{n\mathbf{k}}
\end{aligned}
\tag{B3}
$$

$$
\begin{aligned}
& i\hbar\frac{\partial}{\partial t} < n\mathbf{k}|\hat{\rho}|n'\mathbf{k}' > \\
&= \sum_{n''}\int d\mathbf{k}''(< n\mathbf{k}|\hat{H}_0 - \frac{e\mathbf{A}\cdot\hat{\mathbf{p}}}{m}|n''\mathbf{k}'' >< n''\mathbf{k}''|\hat{\rho}|n'\mathbf{k}' > \\
& \quad - < n\mathbf{k}|\hat{\rho}|n''\mathbf{k}'' >< n''\mathbf{k}''|\hat{H}_0 - \frac{e\mathbf{A}\cdot\hat{\mathbf{p}}}{m}|n'\mathbf{k}' >) \\
&= (\epsilon_{n\mathbf{k}} - \epsilon_{n'\mathbf{k}'}) < n\mathbf{k}|\hat{\rho}|n'\mathbf{k}' >
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{e\mathbf{A}}{m}\cdot\sum_{n''}\int d\mathbf{k}''\{< n\mathbf{k}|\hat{\mathbf{p}}|n''\mathbf{k}'' >< n''\mathbf{k}''|\hat{\rho}|n'\mathbf{k}' > - < n\mathbf{k}|\hat{\rho}|n''\mathbf{k}'' >< n''\mathbf{k}''|\hat{\mathbf{p}}|n'\mathbf{k}' >\} \\
=&(\epsilon_{n\mathbf{k}} - \epsilon_{n'\mathbf{k}'}) < n\mathbf{k}|\hat{\rho}|n'\mathbf{k}' > \\
&-\frac{e\mathbf{A}}{\hbar}\cdot\sum_{n''}\int d\mathbf{k}''[(\epsilon_{n\mathbf{k}} - \epsilon_{n''\mathbf{k}''})\{\nabla_{\mathbf{k}''} < n\mathbf{k}|n''\mathbf{k}'' > + i\delta(\mathbf{k}-\mathbf{k}'')\mathbf{d}_{nn''}(\mathbf{k})\} < n''\mathbf{k}''|\hat{\rho}|n'\mathbf{k}' > \\
&- < n\mathbf{k}|\hat{\rho}|n''\mathbf{k}'' > (\epsilon_{n''\mathbf{k}''} - \epsilon_{n'\mathbf{k}'})\{-\nabla_{\mathbf{k}''} < n''\mathbf{k}''|n'\mathbf{k}' > + i\delta(\mathbf{k}''-\mathbf{k}')\mathbf{d}_{n''n'}(\mathbf{k}')\}] \\
=&(\epsilon_{n\mathbf{k}} - \epsilon_{n'\mathbf{k}'}) < n\mathbf{k}|\hat{\rho}|n'\mathbf{k}' > \\
&-\frac{e\mathbf{A}}{\hbar}\cdot(\nabla_{\mathbf{k}}\epsilon_{n\mathbf{k}} - \nabla_{\mathbf{k}'}\epsilon_{n'\mathbf{k}'}) < n\mathbf{k}|\hat{\rho}|n'\mathbf{k}' > \\
&-\frac{e\mathbf{A}}{\hbar}\cdot\sum_{n''}[(\epsilon_{n\mathbf{k}} - \epsilon_{n''\mathbf{k}})\mathbf{d}_{nn''}(\mathbf{k}') < n''\mathbf{k}|\hat{\rho}|n'\mathbf{k}' > - (\epsilon_{n''\mathbf{k}'} - \epsilon_{n'\mathbf{k}'}) < n\mathbf{k}|\hat{\rho}|n''\mathbf{k}' > \mathbf{d}_{n''n'}(\mathbf{k}')]
\end{aligned} \tag{B4}
$$

$$
\begin{aligned}
&i\hbar\frac{\partial}{\partial t} < v\mathbf{k}|\hat{\rho}|v\mathbf{k} > \\
&\quad = -\frac{e\mathbf{A}}{\hbar}\cdot\{(\epsilon_{v\mathbf{k}} - \epsilon_{c\mathbf{k}})\mathbf{d}_{vc}(\mathbf{k}) < c\mathbf{k}|\hat{\rho}|v\mathbf{k} > - (\epsilon_{c\mathbf{k}} - \epsilon_{v\mathbf{k}}) < v\mathbf{k}|\hat{\rho}|c\mathbf{k} > \mathbf{d}_{cv}(\mathbf{k})\} \\
&\quad = +\frac{e\mathbf{A}}{\hbar}\cdot[2(\epsilon_{c\mathbf{k}} - \epsilon_{v\mathbf{k}})\mathbf{Re}\{\mathbf{d}^*_{vc}(\mathbf{k}) < v\mathbf{k}|\hat{\rho}|c\mathbf{k} >\}]
\end{aligned} \tag{B5}
$$

$$
\begin{aligned}
&i\hbar\frac{\partial}{\partial t} < c\mathbf{k}|\hat{\rho}|c\mathbf{k} > \\
&\quad = -\frac{e\mathbf{A}}{\hbar}\cdot\{(\epsilon_{c\mathbf{k}} - \epsilon_{v\mathbf{k}})\mathbf{d}_{cv}(\mathbf{k}) < v\mathbf{k}|\hat{\rho}|c\mathbf{k} > - (\epsilon_{v\mathbf{k}} - \epsilon_{c\mathbf{k}}) < c\mathbf{k}|\hat{\rho}|v\mathbf{k} > \mathbf{d}_{vc}(\mathbf{k})\} \\
&\quad = -\frac{e\mathbf{A}}{\hbar}\cdot[2(\epsilon_{c\mathbf{k}} - \epsilon_{v\mathbf{k}})\mathbf{Re}\{\mathbf{d}^*_{vc}(\mathbf{k}) < v\mathbf{k}|\hat{\rho}|c\mathbf{k} >\}]
\end{aligned} \tag{B6}
$$

$$
\begin{aligned}
&i\hbar\frac{\partial}{\partial t} < v\mathbf{k}|\hat{\rho}|c\mathbf{k} > \\
&\quad = (\epsilon_{v\mathbf{k}} - \epsilon_{c\mathbf{k}}) < v\mathbf{k}|\hat{\rho}|c\mathbf{k} > - \frac{e\mathbf{A}}{\hbar}\cdot\{(\nabla_{\mathbf{k}}\epsilon_{v\mathbf{k}} - \nabla_{\mathbf{k}}\epsilon_{c\mathbf{k}}) < v\mathbf{k}|\hat{\rho}|c\mathbf{k} >\} \\
&\qquad -\frac{e\mathbf{A}}{\hbar}\cdot\{(\epsilon_{v\mathbf{k}} - \epsilon_{c\mathbf{k}})\mathbf{d}_{vc}(\mathbf{k}) < c\mathbf{k}|\hat{\rho}|c\mathbf{k} > - (\epsilon_{v\mathbf{k}} - \epsilon_{c\mathbf{k}}) < v\mathbf{k}|\hat{\rho}|v\mathbf{k} > \mathbf{d}_{vc}(\mathbf{k})\} \\
&\quad = -(\epsilon_{c\mathbf{k}} - \epsilon_{v\mathbf{k}}) < v\mathbf{k}|\hat{\rho}|c\mathbf{k} > + \frac{e\mathbf{A}}{\hbar}\cdot\{(\nabla_{\mathbf{k}}\epsilon_{c\mathbf{k}} - \nabla_{\mathbf{k}}\epsilon_{v\mathbf{k}}) < v\mathbf{k}|\hat{\rho}|c\mathbf{k} >\} \\
&\qquad +\frac{e\mathbf{A}}{\hbar}\cdot\{(\epsilon_{c\mathbf{k}} - \epsilon_{v\mathbf{k}})\mathbf{d}_{vc}(\mathbf{k})(< c\mathbf{k}|\hat{\rho}|c\mathbf{k} > - < v\mathbf{k}|\hat{\rho}|v\mathbf{k} >)\}
\end{aligned} \tag{B7}
$$

$$
\begin{aligned}
H_V &= tr\{\hat{\rho}(\hat{H}_0 - \frac{e\mathbf{A}\cdot\hat{\mathbf{p}}}{m})\} \\
&= \sum_n \int d\mathbf{k} < n\mathbf{k}|\hat{\rho}|n\mathbf{k} > \epsilon_{n\mathbf{k}} - \frac{e\mathbf{A}}{m}\cdot\{\sum_{nn'}\int d\mathbf{k}\int d\mathbf{k}' < n\mathbf{k}|\hat{\rho}|n'\mathbf{k}' >< n'\mathbf{k}'|\hat{\mathbf{p}}|n\mathbf{k} >\}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_n \int d\mathbf{k} < n\mathbf{k}|\hat{\rho}|n\mathbf{k} > \epsilon_{n\mathbf{k}} && \text{(B8)}\\
&\quad -\frac{e\mathbf{A}}{\hbar}\cdot [\sum_{nn'} \int d\mathbf{k} \int d\mathbf{k}'(\epsilon_{n'\mathbf{k}'} - \epsilon_{n\mathbf{k}}) < n\mathbf{k}|\hat{\rho}|n'\mathbf{k}' > \{\nabla_{\mathbf{k}} < n'\mathbf{k}'|n\mathbf{k} > + i\delta(\mathbf{k}' - \mathbf{k})\mathbf{d}_{n'n}(\mathbf{k})\}] \\
&= \sum_n \int d\mathbf{k} < n\mathbf{k}|\hat{\rho}|n\mathbf{k} > \epsilon_{n\mathbf{k}} && \text{(B9)}\\
&\quad -\frac{e\mathbf{A}}{\hbar}\cdot [\sum_{n} \int d\mathbf{k} \nabla_{\mathbf{k}}\epsilon_{n\mathbf{k}} < n\mathbf{k}|\hat{\rho}|n\mathbf{k} > + i\sum_{nn'} \int d\mathbf{k}(\epsilon_{n'\mathbf{k}'} - \epsilon_{n\mathbf{k}}) < n\mathbf{k}|\hat{\rho}|n'\mathbf{k} > \mathbf{d}_{n'n}(\mathbf{k})\}] && \text{(B10)}
\end{aligned}
$$

$$
\begin{aligned}
H_V^2 = &\int d\mathbf{k}(< v\mathbf{k}|\hat{\rho}|v\mathbf{k} > \epsilon_{v\mathbf{k}} + < c\mathbf{k}|\hat{\rho}|c\mathbf{k} > \epsilon_{c\mathbf{k}}) \\
&-\frac{e\mathbf{A}}{\hbar}\cdot \int d\mathbf{k}(< v\mathbf{k}|\hat{\rho}|v\mathbf{k} > \nabla_{\mathbf{k}}\epsilon_{v\mathbf{k}} + < c\mathbf{k}|\hat{\rho}|c\mathbf{k} > \nabla_{\mathbf{k}}\epsilon_{c\mathbf{k}}) \\
&+\frac{2e\mathbf{A}}{\hbar}\cdot \sum_{nn'} \int d\mathbf{k}(\epsilon_{c\mathbf{k}} - \epsilon_{v\mathbf{k}})\mathbf{Im}\{< v\mathbf{k}|\hat{\rho}|c\mathbf{k} > \mathbf{d}^*_{vc}(\mathbf{k})\}
\end{aligned} \tag{B11}
$$

Appendix C: velocity/current and position

$$
\begin{aligned}
J &= -e\frac{\partial}{\partial t} tr(\hat{\rho}\hat{\mathbf{x}}) \\
&= -e\, tr(\frac{\partial \hat{\rho}}{\partial t}\hat{\mathbf{x}}) \\
&= -\frac{e}{i\hbar} tr([\hat{H}, \hat{\rho}]\hat{\mathbf{x}}) \\
&= -\frac{e}{i\hbar} tr\{\hat{\rho}(\hat{\mathbf{x}}\hat{H} - \hat{H}\hat{\mathbf{x}})\} \\
&= -\frac{e}{i\hbar} tr\{\hat{\rho}[\hat{\mathbf{x}}, \hat{H}]\} \\
&= -\frac{e}{m} tr\{\hat{\rho}(\hat{\mathbf{p}} - e\mathbf{A})\} \\
&= -\frac{e}{m}\sum_{nn'} \int d\mathbf{k} \int d\mathbf{k}' < n\mathbf{k}|\hat{\rho}|n'\mathbf{k}' >< n'\mathbf{k}'|\hat{\mathbf{p}}|n\mathbf{k} > + \frac{e^2\mathbf{A}}{m} tr(\hat{\rho}) \\
&= -\frac{e}{\hbar}\sum_{nn'} \int d\mathbf{k} \int d\mathbf{k}' < n\mathbf{k}|\hat{\rho}|n'\mathbf{k}' > (\epsilon_{n'\mathbf{k}'} - \epsilon_{n\mathbf{k}})\{\nabla_{\mathbf{k}} < n'\mathbf{k}'|n\mathbf{k} > + i\delta(\mathbf{k}' - \mathbf{k})\mathbf{d}_{n'n}(\mathbf{k})\} \\
&\quad + \frac{e^2\mathbf{A}}{m} tr(\hat{\rho}) \\
&= +\frac{e}{\hbar}\sum_{nn'} \int d\mathbf{k} \int d\mathbf{k}' \nabla_{\mathbf{k}}\{< n\mathbf{k}|\hat{\rho}|n'\mathbf{k}' > (\epsilon_{n'\mathbf{k}'} - \epsilon_{n\mathbf{k}})\} < n'\mathbf{k}'|n\mathbf{k} >
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{e}{\hbar}\sum_{nn'}\int d\mathbf{k} < n\mathbf{k}|\hat{\rho}|n'\mathbf{k} > (\epsilon_{n'\mathbf{k}} - \epsilon_{n\mathbf{k}})\{i\mathbf{d}_{n'n}(\mathbf{k})\} + \frac{e^2\mathbf{A}}{m}tr(\hat{\rho}) \\
&= -\frac{e}{\hbar}\sum_{n}\int d\mathbf{k} < n\mathbf{k}|\hat{\rho}|n\mathbf{k} > \nabla_{\mathbf{k}}\epsilon_{n\mathbf{k}} - \frac{e}{\hbar}\sum_{nn'}\int d\mathbf{k} < n\mathbf{k}|\hat{\rho}|n'\mathbf{k} > (\epsilon_{n'\mathbf{k}} - \epsilon_{n\mathbf{k}})\{i\mathbf{d}_{n'n}(\mathbf{k})\} + \frac{e^2\mathbf{A}}{m}tr(\hat{\rho})
\end{aligned} \tag{C1}
$$

$$
\begin{aligned}
J^2 &= -\frac{e}{\hbar}\int d\mathbf{k}(< v\mathbf{k}|\hat{\rho}|v\mathbf{k} > \nabla_{\mathbf{k}}\epsilon_{v\mathbf{k}} + < c\mathbf{k}|\hat{\rho}|c\mathbf{k} > \nabla_{\mathbf{k}}\epsilon_{c\mathbf{k}}) + \frac{e^2\mathbf{A}}{m}tr(\hat{\rho}) \\
&\quad -\frac{e}{\hbar}\int d\mathbf{k}[< v\mathbf{k}|\hat{\rho}|c\mathbf{k} > (\epsilon_{c\mathbf{k}} - \epsilon_{v\mathbf{k}})\{i\mathbf{d}_{cv}(\mathbf{k})\} + < c\mathbf{k}|\hat{\rho}|v\mathbf{k} > (\epsilon_{v\mathbf{k}} - \epsilon_{c\mathbf{k}})\{i\mathbf{d}_{vc}(\mathbf{k})\}] \\
&= -\frac{e}{\hbar}\int d\mathbf{k}(< v\mathbf{k}|\hat{\rho}|v\mathbf{k} > \nabla_{\mathbf{k}}\epsilon_{v\mathbf{k}} + < c\mathbf{k}|\hat{\rho}|c\mathbf{k} > \nabla_{\mathbf{k}}\epsilon_{c\mathbf{k}}) + \frac{e^2\mathbf{A}}{m}tr(\hat{\rho}) \\
&\quad +\frac{e}{\hbar}\int d\mathbf{k}(\epsilon_{c\mathbf{k}} - \epsilon_{v\mathbf{k}})\mathbf{Im}[< v\mathbf{k}|\hat{\rho}|c\mathbf{k} > \mathbf{d}_{vc}]
\end{aligned} \tag{C2}
$$

$$
\begin{aligned}
\mathbf{x} &= tr(\hat{\rho}\hat{\mathbf{x}}) \\
&= \sum_{nn'}\int d\mathbf{k}\int d\mathbf{k}' < n\mathbf{k}|\hat{\rho}|n'\mathbf{k}' >< n'\mathbf{k}'|\hat{\mathbf{x}}|n\mathbf{k} > \\
&= \sum_{nn'}\int d\mathbf{k}\int d\mathbf{k}' < n\mathbf{k}|\hat{\rho}|n'\mathbf{k}' > \{\frac{1}{2i}(\nabla_{\mathbf{k}} - \nabla_{\mathbf{k}'}) < n'\mathbf{k}'|n\mathbf{k} > + \delta(\mathbf{k}' - \mathbf{k})\mathbf{d}_{n'n}(\mathbf{k})\} \\
&= \sum_{nn'}\int d\mathbf{k} < n\mathbf{k}|\hat{\rho}|n'\mathbf{k} > \mathbf{d}_{n'n}(\mathbf{k})
\end{aligned} \tag{C3}
$$

$$
\mathbf{x}^2 = \int d\mathbf{k}\{< v\mathbf{k}|\hat{\rho}|v\mathbf{k} > \mathbf{d}_{vv}(\mathbf{k}) + < c\mathbf{k}|\hat{\rho}|c\mathbf{k} > \mathbf{d}_{cc}(\mathbf{k}) + 2\mathbf{Re}\{< v\mathbf{k}|\hat{\rho}|c\mathbf{k} > \mathbf{d}^*_{vc}(\mathbf{k})\}] \tag{C4}
$$