Appendix A: Bloch-basis and length gauge

$$
\begin{aligned}
< n\mathbf{k}|\hat{\mathbf{x}}|n''\mathbf{k}'' > &= < n\mathbf{k}|\{-i\nabla_{\mathbf{k}''}e^{i\mathbf{k}''\cdot\hat{\mathbf{x}}}\}|u_{n''\mathbf{k}''} > \\
&= -i\nabla_{\mathbf{k}''} < n\mathbf{k}|n''\mathbf{k}'' > +i < u_{n\mathbf{k}}|e^{-i(\mathbf{k}-\mathbf{k}'')\cdot\hat{\mathbf{x}}}|\nabla_{\mathbf{k}''}u_{n''\mathbf{k}''} > \\
&= -i\nabla_{\mathbf{k}''} < n\mathbf{k}|n''\mathbf{k}'' > +\delta(\mathbf{k}-\mathbf{k}'')\mathbf{d}_{nn''}(\mathbf{k}) \qquad \text{(A1)} \\
< n''\mathbf{k}''|\hat{\mathbf{x}}|n'\mathbf{k}' > &= < u_{n''\mathbf{k}''}|\{i\nabla_{\mathbf{k}''}e^{-i\mathbf{k}''\cdot\hat{\mathbf{x}}}\}|n'\mathbf{k}' > \\
&= i\nabla_{\mathbf{k}''} < n''\mathbf{k}''|n'\mathbf{k}' > -i < \nabla_{\mathbf{k}"}u_{n''\mathbf{k}''}|e^{-i(\mathbf{k}''-\mathbf{k}')\cdot\hat{\mathbf{x}}}|u_{n'\mathbf{k}'} > \\
&= i\nabla_{\mathbf{k}''} < n''\mathbf{k}''|n'\mathbf{k}' > +\delta(\mathbf{k}''-\mathbf{k}')\mathbf{d}_{n''n'}(\mathbf{k}') \qquad \text{(A2)}
\end{aligned}
$$

$$
\begin{aligned}
i < u_{n\mathbf{k}}|e^{-i(\mathbf{k}-\mathbf{k}'')\cdot\hat{\mathbf{x}}}|\nabla_{\mathbf{k}''}u_{n''\mathbf{k}''} > &= i\int d\mathbf{x} < u_{n\mathbf{k}}|\mathbf{x} >< \mathbf{x}|\nabla_{\mathbf{k}''}u_{n''\mathbf{k}''} > e^{-i(\mathbf{k}-\mathbf{k}'')\cdot\mathbf{x}} \\
&= i\sum_{\mathbf{R}} e^{-i(\mathbf{k}-\mathbf{k}'')\cdot\mathbf{R}}\int_{BL} d\mathbf{y} < u_{n\mathbf{k}}|\mathbf{y} >< \mathbf{y}|\nabla_{\mathbf{k}''}u_{n''\mathbf{k}''} > e^{-i(\mathbf{k}-\mathbf{k}'')\cdot\mathbf{y}} \\
&= \delta(\mathbf{k}-\mathbf{k}'')(i\int_{BL} d\mathbf{y} < u_{n\mathbf{k}}|\mathbf{y} >< \mathbf{y}|\nabla_{\mathbf{k}''}u_{n''\mathbf{k}''} >) \\
&\equiv \delta(\mathbf{k}-\mathbf{k}'')\mathbf{d}_{nn''}(\mathbf{k}) \qquad \text{(A3)} \\
i < \nabla_{\mathbf{k}"}u_{n''\mathbf{k}''}|e^{-i(\mathbf{k}''-\mathbf{k}')\cdot\hat{\mathbf{x}}}|u_{n'\mathbf{k}'} > &= i\int d\mathbf{x} < \nabla_{\mathbf{k}"}u_{n''\mathbf{k}''}|\mathbf{x} >< \mathbf{x}|u_{n'\mathbf{k}'} > e^{-i(\mathbf{k}''-\mathbf{k}')\cdot\mathbf{x}} \\
&= i\sum_{\mathbf{R}} e^{-i(\mathbf{k}''-\mathbf{k}')\cdot\mathbf{R}}\int_{BL} d\mathbf{y} < \nabla_{\mathbf{k}"}u_{n''\mathbf{k}''}|\mathbf{y} >< \mathbf{y}|u_{n'\mathbf{k}'} > e^{-i(\mathbf{k}''-\mathbf{k}')\cdot\mathbf{y}} \\
&= \delta(\mathbf{k}''-\mathbf{k}')(i\int_{BL} d\mathbf{y} < \nabla_{\mathbf{k}"}u_{n''\mathbf{k}''}|\mathbf{y} >< \mathbf{y}|u_{n'\mathbf{k}'} >) \\
&= -\delta(\mathbf{k}''-\mathbf{k}')(i\int_{BL} d\mathbf{y} < u_{n'\mathbf{k}'}|\mathbf{y} >< \mathbf{y}|\nabla_{\mathbf{k}"}u_{n''\mathbf{k}''} >)^* \\
&= -\delta(\mathbf{k}''-\mathbf{k}')\mathbf{d}_{n''n'}(\mathbf{k}') \qquad \text{(A4)}
\end{aligned}
$$

$$
\begin{aligned}
0 &= i\nabla_{\mathbf{k}} < u_{n\mathbf{k}}|u_{n'\mathbf{k}} > \\
&= i < \nabla_{\mathbf{k}}u_{n\mathbf{k}}|u_{n'\mathbf{k}} > +i < u_{n\mathbf{k}}|\nabla_{\mathbf{k}}u_{n'\mathbf{k}} > \\
&= -\mathbf{d}^*_{n'n}(\mathbf{k}) + \mathbf{d}_{nn'}(\mathbf{k}). \qquad \text{(A5)}
\end{aligned}
$$

$$
\begin{aligned}
&i\hbar\frac{\partial}{\partial t} < n\mathbf{k}|\hat{\rho}|n'\mathbf{k}' > \\
&= \sum_{n''}\int_{BZ} dk''(< n\mathbf{k}|\hat{H}_0 - e\mathbf{E}\cdot\hat{\mathbf{x}}|n''\mathbf{k}'' >< n''\mathbf{k}''|\hat{\rho}|n'\mathbf{k}' > \\
&\quad - < n\mathbf{k}|\hat{\rho}|n''\mathbf{k}'' >< n''\mathbf{k}''|\hat{H}_0 - e\mathbf{E}\cdot\hat{\mathbf{x}}|n'\mathbf{k}' >) \\
&= (\epsilon_{n\mathbf{k}} - \epsilon_{n'\mathbf{k}'}) < n\mathbf{k}|\hat{\rho}|n'\mathbf{k}' >
\end{aligned}
$$

$$
\begin{aligned}
&-e\mathbf{E}\cdot\sum_{n''}\int_{BZ}dk''(<n\mathbf{k}|\hat{\mathbf{x}}|n''\mathbf{k}''><n''\mathbf{k}''|\hat{\rho}|n'\mathbf{k}'>-<n\mathbf{k}|\hat{\rho}|n''\mathbf{k}''><n''\mathbf{k}''|\hat{\mathbf{x}}|n'\mathbf{k}'>)\\
=&(\epsilon_{n\mathbf{k}}-\epsilon_{n'\mathbf{k}'})<n\mathbf{k}|\hat{\rho}|n'\mathbf{k}'>\\
&-e\mathbf{E}\cdot[i(\nabla_{\mathbf{k}}+\nabla_{\mathbf{k}'})<n\mathbf{k}|\hat{\rho}|n'\mathbf{k}'>+\sum_{n''}\{\mathbf{d}_{nn''}(\mathbf{k})<n''\mathbf{k}|\hat{\rho}|n'\mathbf{k}'>-\mathbf{d}_{n''n'}(\mathbf{k}')<n\mathbf{k}|\hat{\rho}|n''\mathbf{k}'>\}]
\end{aligned}
\tag{A6}
$$

$$
\begin{aligned}
&i\hbar\frac{\partial}{\partial t}<v\mathbf{k}|\hat{\rho}|v\mathbf{k}>\\
&\quad=-e\mathbf{E}\cdot[i\nabla_{\mathbf{k}}<v\mathbf{k}|\hat{\rho}|v\mathbf{k}>+\{\mathbf{d}_{vc}(\mathbf{k})<c\mathbf{k}|\hat{\rho}|v\mathbf{k}>-\mathbf{d}_{cv}(\mathbf{k})<v\mathbf{k}|\hat{\rho}|c\mathbf{k}>\}]\\
&\quad=-e\mathbf{E}\cdot[i\nabla_{\mathbf{k}}<v\mathbf{k}|\hat{\rho}|v\mathbf{k}>-2i\mathbf{Im}\{\mathbf{d}^*_{vc}(\mathbf{k})<v\mathbf{k}|\hat{\rho}|c\mathbf{k}>\}]
\end{aligned}
\tag{A7}
$$

$$
\begin{aligned}
&i\hbar\frac{\partial}{\partial t}<c\mathbf{k}|\hat{\rho}|c\mathbf{k}>\\
&\quad=-e\mathbf{E}\cdot[i\nabla_{\mathbf{k}}<c\mathbf{k}|\hat{\rho}|c\mathbf{k}>+\{\mathbf{d}_{cv}(\mathbf{k})<v\mathbf{k}|\hat{\rho}|c\mathbf{k}>-\mathbf{d}_{vc}(\mathbf{k})<c\mathbf{k}|\hat{\rho}|v\mathbf{k}>\}]\\
&\quad=-e\mathbf{E}\cdot[i\nabla_{\mathbf{k}}<c\mathbf{k}|\hat{\rho}|c\mathbf{k}>+2i\mathbf{Im}\{\mathbf{d}^*_{vc}(\mathbf{k})<v\mathbf{k}|\hat{\rho}|c\mathbf{k}>\}]
\end{aligned}
\tag{A8}
$$

$$
\begin{aligned}
&i\hbar\frac{\partial}{\partial t}<v\mathbf{k}|\hat{\rho}|c\mathbf{k}>\\
&\quad=(\epsilon_{v\mathbf{k}}-\epsilon_{c\mathbf{k}})<v\mathbf{k}|\hat{\rho}|c\mathbf{k}>\\
&\qquad-e\mathbf{E}\cdot[i\nabla_{\mathbf{k}}<v\mathbf{k}|\hat{\rho}|c\mathbf{k}>+\sum_{n''}\{\mathbf{d}_{vn''}(\mathbf{k})<n''\mathbf{k}|\hat{\rho}|c\mathbf{k}>-\mathbf{d}_{n''c}(\mathbf{k})<v\mathbf{k}|\hat{\rho}|n''\mathbf{k}>\}]\\
&\quad=-(\epsilon_{c\mathbf{k}}-\epsilon_{v\mathbf{k}})<v\mathbf{k}|\hat{\rho}|c\mathbf{k}>-ie\mathbf{E}\cdot\nabla_{\mathbf{k}}<v\mathbf{k}|\hat{\rho}|c\mathbf{k}>\\
&\qquad-e\mathbf{E}\cdot[\{\mathbf{d}_{vv}(\mathbf{k})-\mathbf{d}_{cc}(\mathbf{k})\}<v\mathbf{k}|\hat{\rho}|c\mathbf{k}>-\mathbf{d}_{vc}(\mathbf{k})\{<c\mathbf{k}|\hat{\rho}|c\mathbf{k}>-<v\mathbf{k}|\hat{\rho}|v\mathbf{k}>\}]
\end{aligned}
\tag{A9}
$$

Appendix B: Bloch-basis and velocity gauge

$$
[\hat{\mathbf{x}},\hat{H}_0]=[\hat{\mathbf{x}},\frac{\hat{\mathbf{p}}}{2m}]=\frac{i\hbar}{m}\hat{\mathbf{p}}
\tag{B1}
$$

$$
\begin{aligned}
<n\mathbf{k}|\hat{\mathbf{p}}|n'\mathbf{k}'>&=i\frac{m(\epsilon_{n\mathbf{k}}-\epsilon_{n'\mathbf{k}'})}{\hbar}<n\mathbf{k}|\hat{\mathbf{x}}|n'\mathbf{k}'>\\
&=\begin{cases}\frac{m}{\hbar}(\epsilon_{n\mathbf{k}}-\epsilon_{n'\mathbf{k}'})\{-\nabla_{\mathbf{k}}<n\mathbf{k}|n'\mathbf{k}'>+i\delta(\mathbf{k}-\mathbf{k}')\mathbf{d}_{nn'}(\mathbf{k})\}\\ \frac{m}{\hbar}(\epsilon_{n\mathbf{k}}-\epsilon_{n'\mathbf{k}'})\{\nabla_{\mathbf{k}'}<n\mathbf{k}|n'\mathbf{k}'>+i\delta(\mathbf{k}-\mathbf{k}')\mathbf{d}_{nn'}(\mathbf{k})\}\end{cases}
\end{aligned}
\tag{B2}
$$

$$
\begin{aligned}
< n\mathbf{k}|\hat{\mathbf{p}}|n\mathbf{k} > &= \frac{m}{i\hbar} < n\mathbf{k}|[\hat{\mathbf{x}}, \hat{H}_0]|n\mathbf{k} > \\
&= \frac{m}{i\hbar}\{< u_{n\mathbf{k}}|(i\nabla_{\mathbf{k}} e^{-i\mathbf{k}\cdot\hat{\mathbf{x}}})\hat{H}_0|n\mathbf{k} > - < n\mathbf{k}|\hat{H}_0(-i\nabla_{\mathbf{k}} e^{i\mathbf{k}\cdot\hat{\mathbf{x}}})|u_{n\mathbf{k}} >\} \\
&= \frac{m}{\hbar}\{\nabla_{\mathbf{k}} < n\mathbf{k}|\hat{H}_0|n\mathbf{k} > - < \nabla_{\mathbf{k}} u_{n\mathbf{k}}|e^{-i\mathbf{k}\cdot\hat{\mathbf{x}}}\hat{H}_0|n\mathbf{k} > - < n\mathbf{k}|\hat{H}_0 e^{i\mathbf{k}\cdot\hat{\mathbf{x}}}|\nabla_{\mathbf{k}} u_{n\mathbf{k}} >\} \\
&= \frac{m}{\hbar}\{\nabla_{\mathbf{k}}\epsilon_{n\mathbf{k}} - \epsilon_{n\mathbf{k}}\nabla_{\mathbf{k}} < u_{n\mathbf{k}}|u_{n\mathbf{k}} >\} \\
&= \frac{m}{\hbar}\nabla_{\mathbf{k}}\epsilon_{n\mathbf{k}} \qquad \text{(B3)}
\end{aligned}
$$

$$
\begin{aligned}
&i\hbar\frac{\partial}{\partial t}< n\mathbf{k}|\hat{\rho}|n'\mathbf{k}' > \\
&=\sum_{n''}\int d\mathbf{k}''(< n\mathbf{k}|\hat{H}_0 - \frac{e\mathbf{A}\cdot\hat{\mathbf{p}}}{m}|n''\mathbf{k}'' >< n''\mathbf{k}''|\hat{\rho}|n'\mathbf{k}' > \\
&\quad - < n\mathbf{k}|\hat{\rho}|n''\mathbf{k}'' >< n''\mathbf{k}''|\hat{H}_0 - \frac{e\mathbf{A}\cdot\hat{\mathbf{p}}}{m}|n'\mathbf{k}' >) \\
&=(\epsilon_{n\mathbf{k}} - \epsilon_{n'\mathbf{k}'}) < n\mathbf{k}|\hat{\rho}|n'\mathbf{k}' > \\
&\quad -\frac{e\mathbf{A}}{m}\cdot\sum_{n''}\int d\mathbf{k}''\{< n\mathbf{k}|\hat{\mathbf{p}}|n''\mathbf{k}'' >< n''\mathbf{k}''|\hat{\rho}|n'\mathbf{k}' > - < n\mathbf{k}|\hat{\rho}|n''\mathbf{k}'' >< n''\mathbf{k}''|\hat{\mathbf{p}}|n'\mathbf{k}' >\} \\
&=(\epsilon_{n\mathbf{k}} - \epsilon_{n'\mathbf{k}'}) < n\mathbf{k}|\hat{\rho}|n'\mathbf{k}' > \\
&\quad -\frac{e\mathbf{A}}{\hbar}\cdot\sum_{n''}\int d\mathbf{k}''[(\epsilon_{n\mathbf{k}} - \epsilon_{n''\mathbf{k}''})\{\nabla_{\mathbf{k}''} < n\mathbf{k}|n''\mathbf{k}'' > +i\delta(\mathbf{k}-\mathbf{k}'')\mathbf{d}_{nn''}(\mathbf{k})\} < n''\mathbf{k}''|\hat{\rho}|n'\mathbf{k}' > \\
&\quad - < n\mathbf{k}|\hat{\rho}|n''\mathbf{k}'' > (\epsilon_{n''\mathbf{k}''} - \epsilon_{n'\mathbf{k}'})\{-\nabla_{\mathbf{k}''} < n''\mathbf{k}''|n'\mathbf{k}' > +i\delta(\mathbf{k}''-\mathbf{k}')\mathbf{d}_{n''n'}(\mathbf{k}')\}] \\
&=(\epsilon_{n\mathbf{k}} - \epsilon_{n'\mathbf{k}'}) < n\mathbf{k}|\hat{\rho}|n'\mathbf{k}' > \\
&\quad -\frac{e\mathbf{A}}{\hbar}\cdot(\nabla_{\mathbf{k}}\epsilon_{n\mathbf{k}} - \nabla_{\mathbf{k}'}\epsilon_{n'\mathbf{k}'}) < n\mathbf{k}|\hat{\rho}|n'\mathbf{k}' > \\
&\quad -\frac{e\mathbf{A}}{\hbar}\cdot\sum_{n''}[(\epsilon_{n\mathbf{k}} - \epsilon_{n''\mathbf{k}})\mathbf{d}_{nn''}(\mathbf{k}') < n''\mathbf{k}|\hat{\rho}|n'\mathbf{k}' > -(\epsilon_{n''\mathbf{k}'} - \epsilon_{n'\mathbf{k}'}) < n\mathbf{k}|\hat{\rho}|n''\mathbf{k}' > \mathbf{d}_{n''n'}(\mathbf{k}')] \\
&\qquad\qquad \text{(B4)}
\end{aligned}
$$

$$
\begin{aligned}
&i\hbar\frac{\partial}{\partial t}< v\mathbf{k}|\hat{\rho}|v\mathbf{k} > \\
&\quad =-\frac{e\mathbf{A}}{\hbar}\cdot\{(\epsilon_{v\mathbf{k}} - \epsilon_{c\mathbf{k}})\mathbf{d}_{vc}(\mathbf{k}) < c\mathbf{k}|\hat{\rho}|v\mathbf{k} > -(\epsilon_{c\mathbf{k}} - \epsilon_{v\mathbf{k}}) < v\mathbf{k}|\hat{\rho}|c\mathbf{k} > \mathbf{d}_{cv}(\mathbf{k})\} \\
&\quad =+\frac{e\mathbf{A}}{\hbar}\cdot[2(\epsilon_{c\mathbf{k}} - \epsilon_{v\mathbf{k}})\mathbf{Re}\{\mathbf{d}^*_{vc}(\mathbf{k}) < v\mathbf{k}|\hat{\rho}|c\mathbf{k} >\}] \qquad \text{(B5)} \\
&i\hbar\frac{\partial}{\partial t}< c\mathbf{k}|\hat{\rho}|c\mathbf{k} >
\end{aligned}
$$

$$
\begin{aligned}
&=-\frac{e\mathbf{A}}{\hbar}\cdot\{(\epsilon_{c\mathbf{k}}-\epsilon_{v\mathbf{k}})\mathbf{d}_{cv}(\mathbf{k})<v\mathbf{k}|\hat{\rho}|c\mathbf{k}>-(\epsilon_{v\mathbf{k}}-\epsilon_{c\mathbf{k}})<c\mathbf{k}|\hat{\rho}|v\mathbf{k}>\mathbf{d}_{vc}(\mathbf{k})\}\\
&=-\frac{e\mathbf{A}}{\hbar}\cdot[2(\epsilon_{c\mathbf{k}}-\epsilon_{v\mathbf{k}})\mathbf{Re}\{\mathbf{d}^*_{vc}(\mathbf{k})<v\mathbf{k}|\hat{\rho}|c\mathbf{k}>\}] \qquad \text{(B6)}\\
&i\hbar\frac{\partial}{\partial t}<v\mathbf{k}|\hat{\rho}|c\mathbf{k}>\\
&=(\epsilon_{v\mathbf{k}}-\epsilon_{c\mathbf{k}})<v\mathbf{k}|\hat{\rho}|c\mathbf{k}>-\frac{e\mathbf{A}}{\hbar}\cdot\{(\nabla_{\mathbf{k}}\epsilon_{v\mathbf{k}}-\nabla_{\mathbf{k}}\epsilon_{c\mathbf{k}})<v\mathbf{k}|\hat{\rho}|c\mathbf{k}>\}\\
&\quad-\frac{e\mathbf{A}}{\hbar}\cdot\{(\epsilon_{v\mathbf{k}}-\epsilon_{c\mathbf{k}})\mathbf{d}_{vc}(\mathbf{k})<c\mathbf{k}|\hat{\rho}|c\mathbf{k}>-(\epsilon_{v\mathbf{k}}-\epsilon_{c\mathbf{k}})<v\mathbf{k}|\hat{\rho}|v\mathbf{k}>\mathbf{d}_{vc}(\mathbf{k})\}\\
&=-(\epsilon_{c\mathbf{k}}-\epsilon_{v\mathbf{k}})<v\mathbf{k}|\hat{\rho}|c\mathbf{k}>+\frac{e\mathbf{A}}{\hbar}\cdot\{(\nabla_{\mathbf{k}}\epsilon_{c\mathbf{k}}-\nabla_{\mathbf{k}}\epsilon_{v\mathbf{k}})<v\mathbf{k}|\hat{\rho}|c\mathbf{k}>\}\\
&\quad+\frac{e\mathbf{A}}{\hbar}\cdot\{(\epsilon_{c\mathbf{k}}-\epsilon_{v\mathbf{k}})\mathbf{d}_{vc}(\mathbf{k})(<c\mathbf{k}|\hat{\rho}|c\mathbf{k}>-<v\mathbf{k}|\hat{\rho}|v\mathbf{k}>)\} \qquad \text{(B7)}
\end{aligned}
$$

Appendix C: Intraband, Interband, and anomalous current

$$
\begin{aligned}
J&=-e\frac{\partial}{\partial t}tr(\hat{\rho}\hat{\mathbf{x}})\\
&=-e\,tr(\frac{\partial\hat{\rho}}{\partial t}\hat{\mathbf{x}})\\
&=-\frac{e}{i\hbar}tr([\hat{H},\hat{\rho}]\hat{\mathbf{x}})\\
&=-\frac{e}{i\hbar}tr\{\hat{\rho}(\hat{\mathbf{x}}\hat{H}-\hat{H}\hat{\mathbf{x}})\}\\
&=-\frac{e}{i\hbar}tr\{\hat{\rho}[\hat{\mathbf{x}},\hat{H}]\}\\
&=-\frac{e}{m}tr\{\hat{\rho}(\hat{\mathbf{p}}-e\mathbf{A})\}\\
&=-\frac{e}{m}\sum_{nn'}\int d\mathbf{k}\int d\mathbf{k}'<n\mathbf{k}|\hat{\rho}|n'\mathbf{k}'><n'\mathbf{k}'|\hat{\mathbf{p}}|n\mathbf{k}>+\frac{e^2\mathbf{A}}{m}tr(\hat{\rho})\\
&=-\frac{e}{\hbar}\sum_{nn'}\int d\mathbf{k}\int d\mathbf{k}'<n\mathbf{k}|\hat{\rho}|n'\mathbf{k}'>(\epsilon_{n'\mathbf{k}'}-\epsilon_{n\mathbf{k}})\{\nabla_{\mathbf{k}}<n'\mathbf{k}'|n\mathbf{k}>+i\delta(\mathbf{k}'-\mathbf{k})\mathbf{d}_{n'n}(\mathbf{k})\}\\
&\quad+\frac{e^2\mathbf{A}}{m}tr(\hat{\rho})\\
&=+\frac{e}{\hbar}\sum_{nn'}\int d\mathbf{k}\int d\mathbf{k}'\nabla_{\mathbf{k}}\{<n\mathbf{k}|\hat{\rho}|n'\mathbf{k}'>(\epsilon_{n'\mathbf{k}'}-\epsilon_{n\mathbf{k}})\}<n'\mathbf{k}'|n\mathbf{k}>\\
&\quad-\frac{e}{\hbar}\sum_{nn'}\int d\mathbf{k}<n\mathbf{k}|\hat{\rho}|n'\mathbf{k}>(\epsilon_{n'\mathbf{k}}-\epsilon_{n\mathbf{k}})\{i\mathbf{d}_{n'n}(\mathbf{k})\}+\frac{e^2\mathbf{A}}{m}tr(\hat{\rho})\\
&=-\frac{e}{\hbar}\sum_{n}\int d\mathbf{k}<n\mathbf{k}|\hat{\rho}|n\mathbf{k}>\nabla_{\mathbf{k}}\epsilon_{n\mathbf{k}}-\frac{e}{\hbar}\sum_{nn'}\int d\mathbf{k}<n\mathbf{k}|\hat{\rho}|n'\mathbf{k}>(\epsilon_{n'\mathbf{k}}-\epsilon_{n\mathbf{k}})\{i\mathbf{d}_{n'n}(\mathbf{k})\}+\frac{e^2\mathbf{A}}{m}tr(\hat{\rho})
\end{aligned}
\qquad \text{(C1)}
$$

$$
\begin{aligned}
J_2 &= -\frac{e}{\hbar}\int d\mathbf{k}(< v\mathbf{k}|\hat{\rho}|v\mathbf{k} > \nabla_{\mathbf{k}}\epsilon_{v\mathbf{k}} + < c\mathbf{k}|\hat{\rho}|c\mathbf{k} > \nabla_{\mathbf{k}}\epsilon_{c\mathbf{k}}) + \frac{e^2\mathbf{A}}{m}tr(\hat{\rho}) \\
&\quad -\frac{e}{\hbar}\int d\mathbf{k}[< v\mathbf{k}|\hat{\rho}|c\mathbf{k} > (\epsilon_{c\mathbf{k}} - \epsilon_{v\mathbf{k}})\{i\mathbf{d}_{cv}(\mathbf{k})\} + < c\mathbf{k}|\hat{\rho}|v\mathbf{k} > (\epsilon_{v\mathbf{k}} - \epsilon_{c\mathbf{k}})\{i\mathbf{d}_{vc}(\mathbf{k})\}] \\
&= -\frac{e}{\hbar}\int d\mathbf{k}(< v\mathbf{k}|\hat{\rho}|v\mathbf{k} > \nabla_{\mathbf{k}}\epsilon_{v\mathbf{k}} + < c\mathbf{k}|\hat{\rho}|c\mathbf{k} > \nabla_{\mathbf{k}}\epsilon_{c\mathbf{k}}) + \frac{e^2\mathbf{A}}{m}tr(\hat{\rho}) \\
&\quad +\frac{e}{\hbar}\int d\mathbf{k}(\epsilon_{c\mathbf{k}} - \epsilon_{v\mathbf{k}})\mathbf{Im}[< v\mathbf{k}|\hat{\rho}|c\mathbf{k} > \mathbf{d}_{vc}]
\end{aligned}
\tag{C2}
$$