Report Problems for the lecture in solid state physics 4

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Problem 1

In the Heisenberg representation, the total derivative of the eigenvector (Bloch vector) respect to time is $\frac{d}{dt}|\psi_{n,k}>=\dot{k}_j|\frac{\partial}{\partial k_j}\psi_{n,k}>$. Assuming the Libniz's rule $\frac{d}{dt}(<\phi|A|\psi>)=<\phi|(\frac{d}{dt}A)|\psi>+(\frac{d}{dt}<\phi|)A|\psi>+<\phi|A(\frac{d}{dt}|\psi>)$ which generally holds in the Hilbert space, and using the Einstein's convention, we have[1]

$$
\begin{aligned}
v^i_{n,k} &= \frac{d}{dt}(<\psi_{n,k}|\hat{x}^i|\psi_{n,k}>) \\
&= <\psi_{n,k}|(\frac{d}{dt}\hat{x}^i)|\psi_{n,k}>+\dot{k}_j<\frac{\partial}{\partial k_j}\psi_{n,k}|\hat{x}^i|\psi_{n,k}>+\dot{k}_j<\psi_{n,k}|\hat{x}^i|\frac{\partial}{\partial k_j}\psi_{n,k}> \\
&= -\frac{i}{\hbar}<\psi_{n,k}|[\hat{x}^i,H]|\psi_{n,k}> \\
&\quad +\dot{k}_j(<\frac{\partial}{\partial k_j}u_{n,k}|-i<u_{nk}|\hat{x}^j)\hat{x}^i|u_{n,k}> \\
&\quad +\dot{k}_j<u_{n,k}|\hat{x}^i(|\frac{\partial}{\partial k_j}u_{n,k}>+i\hat{x}^j|u_{nk}>) \\
&= \frac{1}{\hbar}\{<u_{n,k}|(\frac{\partial}{\partial k_i}e^{-ik\hat{x}})H|\psi_{n,k}>+<\psi_{n,k}|H(\frac{\partial}{\partial k_i}e^{ik\hat{x}})|u_{n,k}>\} \\
&\quad +\dot{k}_j(<\frac{\partial}{\partial k_j}u_{n,k}|\hat{x}^i|u_{n,k}>+<u_{n,k}|\hat{x}^i|\frac{\partial}{\partial k_j}u_{n,k}>) \\
&= \frac{1}{\hbar}\{\frac{\partial}{\partial k_i}(<\psi_{n,k}|H|\psi_{n,k}>)-<\frac{\partial}{\partial k_i}u_{n,k}|e^{-ik\hat{x}}H|\psi_{n,k}>-<\psi_{n,k}|He^{ik\hat{x}}|\frac{\partial}{\partial k_i}u_{n,k}>\} \\
&\quad +\dot{k}_j\frac{\partial}{\partial k_j}(<u_{n,k}|\hat{x}^i|u_{n,k}>) \\
&= \frac{1}{\hbar}\{\frac{\partial}{\partial k_i}\epsilon_{n,k}-\epsilon_{n,k}\frac{\partial}{\partial k_i}<u_{n,k}|u_{n,k}>\} \\
&\quad +\dot{k}_j\frac{\partial}{\partial k_j}\{i<u_{n,k}|\frac{\partial}{\partial k_i}u_{n,k}>-i<\psi_{n,k}|\frac{\partial}{\partial k_i}\psi_{n,k}>\} \\
&= \frac{1}{\hbar}\frac{\partial}{\partial k_i}\epsilon_{n,k}+\dot{k}_j\frac{\partial}{\partial k_j}\mathcal{A}^i_{nk}-i\dot{k}_j\frac{\partial}{\partial k_j}<\psi_{n,k}|\frac{\partial}{\partial k_i}\psi_{n,k}>) \\
&= \frac{1}{\hbar}\frac{\partial}{\partial k_i}\epsilon_{n,k}+\dot{k}_j\frac{\partial}{\partial k_j}\mathcal{A}^i_{nk}-i\dot{k}_j\frac{\partial}{\partial k_i}<\psi_{n,k}|\frac{\partial}{\partial k_j}\psi_{n,k}>) \\
&= \frac{1}{\hbar}\frac{\partial}{\partial k_i}\epsilon_{n,k}+\dot{k}_j(\frac{\partial}{\partial k_j}\mathcal{A}^i_{nk}-\frac{\partial}{\partial k_i}\mathcal{A}^j_{nk}) \\
&= \frac{1}{\hbar}\frac{\partial}{\partial k_i}\epsilon_{n,k}+\epsilon^{ijk}\dot{k}_j\Omega_k
\end{aligned} \tag{1}
$$

where $\mathcal{A}^i_{nk} = i < u_{n,k}|\frac{\partial}{\partial k_i} u_{n,k} >$ is the Berry connection. In the last expression, the relation $\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} = \epsilon_{ijk}\epsilon_{lmk}$ is used. From the last expression, the equation (9) is easily deduced.

Problem 2

The fine structure constant α is [2]

$$\begin{aligned} \alpha &= \frac{1}{4\pi\epsilon_0}\frac{e^2}{\hbar c} \\ &= \frac{1}{2\epsilon_0 c} R_H^{-1} \\ &= \frac{c\mu_0}{2} R_H^{-1} \\ &= \frac{c\mu_0}{8} R_H'^{-1} \end{aligned} \tag{2}$$

where $R_H = h/e^2$ is the von Klitzing constant and $R'_H = h/4e^2$.

Therefore,

$$\begin{aligned} R_H &= \frac{\alpha^{-1}}{2\epsilon_0 c} \\ &= \frac{\alpha^{-1}\mu_0 c}{2}. \end{aligned} \tag{3}$$

References

[1] Ganesh Sundaram and Qian Niu. Wave-packet dynamics in slowly perturbed crystals: Gradient corrections and berry-phase effects. *Phys. Rev. B*, Vol. 59, pp. 14915–14925, Jun 1999.

[2] K. v. Klitzing, G. Dorda, and M. Pepper. New method for high-accuracy determination of the fine-structure constant based on quantized hall resistance. *Phys. Rev. Lett.*, Vol. 45, pp. 494–497, Aug 1980.