## 3.2 Exponentially Hard Games

A simple example of a full information game is *Linear Chess*, played on a finite linear board. Each piece has a 1-byte type, including *loyalty* to one of two sides: W (weak) or S (shy), *gender* M/F and a 6-bit *rank*. All cells of the board are filled and all W's are always on the left of all S's. Changes occur only at the *active* border where W and S meet (and fight). The winner of a fight is determined by the following Gender Rules:

- 1. If S and W are of the same sex, W (being weaker) loses.
- 2. If S and W are of different sexes, S gets confused and loses.

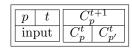
The party of a winning piece A replaces the loser's piece B by its own piece C. The choice of C is restricted by the table of rules listing all allowed triples (ABC). We will see that this game cannot be solved in a subexponential time. We first prove that (see [Chandra, Kozen, Stockmeyer 81]) for an artificial game. Then we reduce this  $Halting\ Game$  to Linear Chess showing that any fast algorithm to solve Linear Chess, could be used to solve Halting Game, thus requiring exponential time. For Exp-Time Completeness of regular (but  $n \times n$ ) Chess, Go, Checkers see: [Fraenkel, Lichtenstein 81, Robson 83, 84].

## **Exptime Complete Halting Game**

We use a universal Turing Machine u (defined as 1-pointer cellular automata) which halts only by its head rolling off of the tape's left end, leaving a blank. Bounded Halting Problem BHP(x) determines if u(x) stops (i.e. the leftmost tape cell points left: d=-1) within  $2^{||x||}$  steps. This cannot be determined in  $o(2^{||x||})$  steps.

We now convert BHP into the Halting Game. The players are: W claiming u(x) halts in time (and should have

The players are: W claiming u(x) halts in time (and should have winning strategy iff this is true); His opponent is S. The **board** has four parts: the C diagram, the input x to u, positive integers p (position) and t (time in the execution of u(x)):



The diagram shows the states  $C_p^{t+1}$ ,  $C_p^t$  of cell p at times t+1,t, and  $C_{p'}^t$  of cell p'=p+d',  $d' \in \{\pm 1\}$  at time t. C include present d and previous d' pointers direction;  $C^t$  may be replaced by "?". Some board configurations are illegal: if (1) both  $C_p, C_{p'}$  point away from each other, or (2)  $C^{t+1}$  differs from the result prescribed by the transition rules for  $C^t$ , or (3) t=1, while  $C_p^1 \neq x_p$ . (At t=1, u(x) is just starting, so its tape has the input x at the left starting with the head in the initial state, followed by blanks at the right.)

Here are the **Game Rules:** The game starts in the configuration at the right. W, in its moves, replaces the ?s with symbols claiming to reflect the state of cells p', p at step t of u(x). S then chooses  $s \in \{0, 1\}$ , replaces p with p+sd', moves  $C_p$  to top C box, fills lower C boxes with ?s, and decrements t:

$p=1 \mid t=2^{  x  }$	+	_ ]
input x	?	?

Note that W may lie (i.e fill in "?" distorting the actual computation of u(x)), as long as he is consistent with the above "local" rules. All S can do is to check the two consecutive board configurations. He cannot refer to past moves or to actual computation of u(x) as an evidence of W's violation.

**Strategy:** If u(x) does indeed halt within  $2^{\|x\|}$  steps, then the initial configuration is true to the computation of u(x). Then W has an obvious (though hard to compute) winning strategy: just tell truly (and thus always consistently) what actually happens in the computation. S will lose when t=1 and cannot decrease any more. If the initial configuration is a lie, S can force W to lie all the way down to t=1. How?

If the upper box  $C_p^{t+1}$  of a legal configuration is false then the lower boxes  $C_{p'}^t$   $C_p^t$  cannot both be true, since the rules of u determine  $C_p^{t+1}$  uniquely from them. If S correctly points the false C and brings it to the top on his move, then W is forced to keep on lying. At time t=1 the lie is exposed: the configuration doesn't match the actual input string x, i.e. is illegal.

Solving this game amounts to deciding correctness of the initial configuration, i.e. u(x) halting in  $2^{\|x\|}$  steps: impossible in time  $o(2^{\|x\|})$ . This Halting Game is artificial, still has a BHP flavor, though it does not refer to exponents. We now reduce it to a nicer game (Linear Chess) to prove it exponentially hard, too.