Quantum Harmonic Oscillator Ladder Operators

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Here a is the demotion (annililation, lowering) operator; and a^{\dagger} is the promotion (creation, raising) operator for the quantum-mechanical simple harmonic oscillator.

One Operator

$$a \mid n \rangle = \sqrt{n} \mid n-1 \rangle$$

 $a^{\dagger} \mid n \rangle = \sqrt{n+1} \mid n+1 \rangle$

Two Operators

$$\begin{array}{rcl} aa \mid n \rangle & = & \sqrt{n(n-1)} \mid n-2 \rangle \\ aa^{\dagger} \mid n \rangle & = & (n+1) \mid n \rangle \\ a^{\dagger}a \mid n \rangle & = & n \mid n \rangle \\ a^{\dagger}a^{\dagger} \mid n \rangle & = & \sqrt{(n+1)(n+2)} \mid n+2 \rangle \end{array}$$

Three Operators

$$\begin{array}{rcl} aaa & |n\rangle & = & \sqrt{n(n-1)(n-2)} & |n-3\rangle \\ aaa^{\dagger} & |n\rangle & = & (n+1)\sqrt{n} & |n-1\rangle \\ aa^{\dagger}a & |n\rangle & = & n^{3/2} & |n-1\rangle \\ aa^{\dagger}a^{\dagger} & |n\rangle & = & (n+2)\sqrt{n+1} & |n+1\rangle \\ a^{\dagger}aa & |n\rangle & = & (n-1)\sqrt{n} & |n-1\rangle \\ a^{\dagger}aa^{\dagger} & |n\rangle & = & (n+1)^{3/2} & |n+1\rangle \\ a^{\dagger}a^{\dagger}a & |n\rangle & = & n\sqrt{n+1} & |n+1\rangle \\ a^{\dagger}a^{\dagger}a^{\dagger} & |n\rangle & = & \sqrt{(n+1)(n+2)(n+3)} & |n+3\rangle \end{array}$$

Four Operators

$$aaaa \mid n \rangle = \sqrt{n(n-1)(n-2)(n-3)} \mid n-4 \rangle$$

$$aaaa^{\dagger} \mid n \rangle = (n+1)\sqrt{n(n-1)} \mid n-2 \rangle$$

$$aaa^{\dagger}a \mid n \rangle = n^{3/2}\sqrt{n-1} \mid n-2 \rangle$$

$$aaa^{\dagger}a^{\dagger} \mid n \rangle = (n+1)(n+2) \mid n \rangle$$

$$aa^{\dagger}aa \mid n \rangle = (n-1)^{3/2}\sqrt{n} \mid n-2 \rangle$$

$$aa^{\dagger}aa^{\dagger} \mid n \rangle = (n+1)^{2} \mid n \rangle$$

$$aa^{\dagger}a^{\dagger}a \mid n \rangle = n(n+1) \mid n \rangle$$

$$aa^{\dagger}a^{\dagger}a^{\dagger}a \mid n \rangle = (n+3)\sqrt{(n+1)(n+2)} \mid n+2 \rangle$$

$$a^{\dagger}aaa \mid n \rangle = (n-2)\sqrt{n(n-1)} \mid n-2 \rangle$$

$$a^{\dagger}aaa^{\dagger}a \mid n \rangle = n(n+1) \mid n \rangle$$

$$a^{\dagger}aa^{\dagger}a \mid n \rangle = n^{2} \mid n \rangle$$

$$a^{\dagger}aa^{\dagger}a \mid n \rangle = (n+2)^{3/2}\sqrt{n+1} \mid n+2 \rangle$$

$$a^{\dagger}a^{\dagger}aa \mid n \rangle = n(n-1) \mid n \rangle$$

$$a^{\dagger}a^{\dagger}aa^{\dagger}a \mid n \rangle = (n+1)^{3/2}\sqrt{n+2} \mid n+2 \rangle$$

$$a^{\dagger}a^{\dagger}a^{\dagger}a \mid n \rangle = n\sqrt{(n+1)(n+2)} \mid n+2 \rangle$$

$$a^{\dagger}a^{\dagger}a^{\dagger}a \mid n \rangle = n\sqrt{(n+1)(n+2)} \mid n+2 \rangle$$

$$a^{\dagger}a^{\dagger}a^{\dagger}a^{\dagger}a \mid n \rangle = \sqrt{(n+1)(n+2)(n+3)(n+4)} \mid n+4 \rangle$$

Five Operators

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aaaaa |n\rangle = \sqrt{n(n-1)(n-2)(n-3)(n-4)} |n-5\rangle
      aaaaa^{\dagger} |n\rangle = (n+1)\sqrt{n(n-1)(n-2)} |n-3\rangle
      aaaa^{\dagger}a |n\rangle = n^{3/2}\sqrt{(n-1)(n-2)} |n-3\rangle
    aaaa^{\dagger}a^{\dagger} |n\rangle = (n+1)(n+2)\sqrt{n} |n-1\rangle
      aaa^{\dagger}aa |n\rangle = (n-1)^{3/2} \sqrt{n(n-2)} |n-3\rangle
    aaa^{\dagger}aa^{\dagger} |n\rangle = (n+1)^2 \sqrt{n} |n-1\rangle
    aaa^{\dagger}a^{\dagger}a |n\rangle = (n+1)n^{3/2} |n-1\rangle
   aaa^{\dagger}a^{\dagger}a^{\dagger} |n\rangle = (n+2)(n+3)\sqrt{n+1} |n+1\rangle
      aa^{\dagger}aaa |n\rangle = (n-2)^{3/2} \sqrt{n(n-1)} |n-3\rangle
    aa^{\dagger}aaa^{\dagger} |n\rangle = (n+1)n^{3/2} |n-1\rangle
    aa^{\dagger}aa^{\dagger}a \mid n \rangle = n^{5/2} \mid n \rangle
   aa^{\dagger}aa^{\dagger}a^{\dagger} |n\rangle = (n+2)^2 \sqrt{n+1} |n+1\rangle
    aa^{\dagger}a^{\dagger}aa |n\rangle = (n-1)n^{3/2} |n-1\rangle
   aa^{\dagger}a^{\dagger}aa^{\dagger} |n\rangle = (n+2)(n+1)^{3/2} |n+1\rangle
   aa^{\dagger}a^{\dagger}a^{\dagger}a \mid n \rangle = n(n+2)\sqrt{(n+1)} \mid n+1 \rangle
 aa^{\dagger}a^{\dagger}a^{\dagger}a^{\dagger}|n\rangle = (n+4)\sqrt{(n+1)(n+2)(n+3)}|n+3\rangle
      a^{\dagger}aaaa |n\rangle = (n-3)\sqrt{n(n-1)(n-2)} |n-3\rangle
    a^{\dagger}aaaa^{\dagger} |n\rangle = (n-1)(n+1)\sqrt{n} |n-1\rangle
    a^{\dagger}aaa^{\dagger}a |n\rangle = (n-1)n^{3/2} |n-1\rangle
   a^{\dagger}aaa^{\dagger}a^{\dagger} |n\rangle = (n+2)(n+1)^{3/2} |n+1\rangle
    a^{\dagger}aa^{\dagger}aa |n\rangle = (n-1)^2 \sqrt{n} |n-1\rangle
   a^{\dagger}aa^{\dagger}aa^{\dagger} |n\rangle = (n+1)^{5/2} |n+1\rangle
   a^{\dagger}aa^{\dagger}a^{\dagger}a |n\rangle = n(n+1)^{3/2} |n+1\rangle
 a^{\dagger}aa^{\dagger}a^{\dagger}a^{\dagger}|n\rangle = (n+3)^{3/2}\sqrt{(n+1)(n+2)}|n+3\rangle
    a^{\dagger}a^{\dagger}aaa |n\rangle = (n-1)(n-2)\sqrt{n} |n-1\rangle
  a^{\dagger}a^{\dagger}aaa^{\dagger} |n\rangle = n(n+1)^{3/2} |n+1\rangle
   a^{\dagger}a^{\dagger}aa^{\dagger}a \mid n \rangle = n^2 \sqrt{n+1} \mid n+1 \rangle
 a^{\dagger}a^{\dagger}aa^{\dagger}a^{\dagger}|n\rangle = (n+2)^{3/2}\sqrt{(n+1)(n+3)}|n+3\rangle
  a^{\dagger}a^{\dagger}a^{\dagger}aa |n\rangle = n(n-1)\sqrt{n+1}|n+1\rangle
 a^{\dagger}a^{\dagger}a^{\dagger}aa^{\dagger} |n\rangle = (n+1)^{3/2} \sqrt{(n+2)(n+3)} |n+3\rangle
 a^{\dagger}a^{\dagger}a^{\dagger}a^{\dagger}a |n\rangle = n\sqrt{(n+1)(n+2)(n+3)} |n+3\rangle
a^{\dagger}a^{\dagger}a^{\dagger}a^{\dagger}a^{\dagger}|n\rangle = \sqrt{(n+1)(n+2)(n+3)(n+4)(n+5)}|n+5\rangle
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