3.2 Exponentially Hard Games

A simple example of a full information game is ***Linear Chess***, played on a finite linear board. Each piece has a 1-byte type, including *loyalty* to one of two sides: W (weak) or S (shy), *gender* M/F and a 6-bit *rank*. All cells of the board are filled and all W's are always on the left of all S's. Changes occur only at the ***active*** border where W and S meet (and fight). The winner of a fight is determined by the following Gender Rules:

1. If S and W are of the same sex, W (being weaker) loses.
2. If S and W are of different sexes, S gets confused and loses.

The party of a winning piece A replaces the loser's piece B by its own piece C. The choice of C is restricted by the table of rules listing all allowed triples (ABC). We will see that this game *cannot* be solved in a *subexponential* time. We first prove that (see [Chandra, Kozen, Stockmeyer 81]) for an artificial game. Then we reduce this ***Halting Game*** to Linear Chess showing that any fast algorithm to solve Linear Chess, could be used to solve Halting Game, thus requiring exponential time. For Exp-Time Completeness of regular (but $n \times n$) Chess, Go, Checkers see: [Fraenkel, Lichtenstein 81, Robson 83, 84].

Exptime Complete Halting Game

We use a universal Turing Machine u (defined as 1-pointer cellular automata) which halts only by its head rolling off of the tape's left end, leaving a blank. Bounded Halting Problem BHP(x) determines if $u(x)$ stops (i.e. the leftmost tape cell points left: $d{=}-1$) within $2^{\|x\|}$ steps. This cannot be determined in $o(2^{\|x\|})$ steps.

We now convert BHP into the Halting Game.

The players are: W claiming $u(x)$ halts in time (and should have winning strategy iff this is true); His opponent is S. The ***board*** has four parts: the C diagram, the input x to u, positive integers p (position) and t (time in the execution of $u(x)$):

p	t	C_p^{t+1}	
input		C_p^t	$C_{p'}^t$

The diagram shows the states C_p^{t+1}, C_p^t of cell p at times $t{+}1, t$, and $C_{p'}^t$ of cell $p' = p{+}d'$, $d'{\in}\{\pm 1\}$ at time t. C include present d and previous d' pointers direction; C^t may be replaced by "?". Some board configurations are illegal: if (1) both $C_p, C_{p'}$ point away from each other, or (2) C^{t+1} differs from the result prescribed by the transition rules for C^t, or (3) $t{=}1$, while $C_p^1 \neq x_p$. (At $t{=}1$, $u(x)$ is just starting, so its tape has the input x at the left starting with the head in the initial state, followed by blanks at the right.)

Here are the **Game Rules:** The game starts in the configuration at the right. W, in its moves, replaces the ?s with symbols claiming to reflect the state of cells p', p at step t of $u(x)$. S then chooses $s{\in}\{0,1\}$, replaces p with $p{+}sd'$, moves C_p to top C box, fills lower C boxes with ?s, and decrements t:

$p{=}1$	$t{=}2^{\|x\|}$	$\leftarrow$	
input x		?	?

Note that W may lie (i.e fill in "?" distorting the actual computation of $u(x)$), as long as he is consistent with the above "local" rules. All S can do is to check the two consecutive board configurations. He cannot refer to past moves or to actual computation of $u(x)$ as an evidence of W's violation.

Strategy: If $u(x)$ does indeed halt within $2^{\|x\|}$ steps, then the initial configuration is true to the computation of $u(x)$. Then W has an obvious (though hard to compute) winning strategy: just tell truly (and thus always consistently) what actually happens in the computation. S will lose when $t{=}1$ and cannot decrease any more. If the initial configuration is a lie, S can force W to lie all the way down to $t = 1$. How?

If the upper box C_p^{t+1} of a legal configuration is false then the lower boxes $C_{p'}^t$, C_p^t cannot both be true, since the rules of u determine C_p^{t+1} uniquely from them. If S correctly points the false C and brings it to the top on his move, then W is forced to keep on lying. At time $t{=}1$ the lie is exposed: the configuration doesn't match the actual input string x, i.e. is illegal.

Solving this game amounts to deciding correctness of the initial configuration, i.e. $u(x)$ halting in $2^{\|x\|}$ steps: impossible in time $o(2^{\|x\|})$. This Halting Game is artificial, still has a BHP flavor, though it does not refer to exponents. We now reduce it to a nicer game (Linear Chess) to prove it exponentially hard, too.