Part I
Basics

1 Deterministic Models; Polynomial Time & Church's Thesis

Sections 1,2 study deterministic computations. Non-deterministic aspects of computations (inputs, interaction, errors, randomization, etc.) are crucial and challenging in advanced theory and practice. Defining them as an extension of deterministic computations is simple. The latter, however, while simpler conceptually, require elaborate models for definition. These models may be sophisticated if we need a precise measurement of all required resources. However, if we only need to define what is computable and get a very rough magnitude of the needed resources, all reasonable models turn out equivalent, even to the simplest ones. We will pay significant attention to this surprising and important fact. The simplest models are most useful for proving negative results and the strongest ones for positive results.

We start with terminology common to all models, gradually making it more specific to those we actually study. We represent ***computations*** as graphs: the edges reflect various relations between nodes (***events***). Nodes, edges have attributes: labels, states, colors, parameters, etc. (affecting the computation or its analysis). ***Causal*** edges run from each event to all events essential for its occurrence or attributes. They form a directed acyclic graph (though cycles may be added artificially to mark the external input parts of the computation).

We will study only ***synchronous*** computations. Their nodes have a ***time*** parameter. It reflects logical steps, not necessarily a precise value of any physical clock. Causal edges only span short (typically, ≤ 3 moments) time intervals. One event among the causes of a node is called its ***parent***. ***Pointer*** edges connect the parent of each event to all its other possible causes and reflect connections that allow simultaneous events to interact and have a joint effect. Pointers with the same source have different labels. The (labeled) subgraph of events/edges at a given time is an instant memory ***configuration*** of the model.

Each non-terminal configuration has ***active*** nodes/edges around which it may change. The models with only a small active area at any step of the computation are ***sequential***. Others are called ***parallel***.

Complexity. We use the following measures of computing resources of a machine A on input x:

Time: The greatest depth $D_{A(x)}$ of causal chains is the number of computation steps. The volume $V_{A(x)}$ is the combined number of active edges during all steps. Time $T_{A(x)}$ is used (depending on the context) as either depth or volume, which are close for sequential models. Note that time complexity is robust only up to a constant factor: a machine can be modified into a new one with a larger alphabet of labels, representing several locations in one. It would produce identical results in a fraction of time and space (provided that the time limits suffice for transforming the input and output into the other alphabet).

Space: $S_{A(x)}$ or $S_A(x)$ of a synchronous computation is the greatest (over time) size of its configurations. Sometimes excluded are nodes/edges unchanged since the input.

Growth Rates (typically expressed as functions of bit length $n = \|x, y\|$ of input/output x/y):

O, Ω: $f(n) = O(g(n))$[1] $\iff g(n) = \Omega(f(n)) \iff \sup_n \frac{f(n)}{g(n)} < \infty$.

$o, \omega : f(n) = o(g(n)) \iff g(n) = \omega(f(n)) \iff \lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$.

$\Theta : f(n) = \Theta(g(n)) \iff (f(n) = O(g(n)) \text{ and } g(n) = O(f(n)))$.

Here are a few examples of frequently appearing growth rates: negligible $(\log n)^{O(1)}$; moderate $n^{\Theta(1)}$ (called polynomial or P, like in P-time); infeasible: $2^{n^{\Omega(1)}}$, also $n! = (n/e)^n \sqrt{\pi(2n{+}1/3) + \varepsilon/n}$, $\varepsilon{\in}[0, .1]$.[2]

The reason for ruling out exponential (and neglecting logarithmic) rates is that the known Universe is too small to accommodate exponents. Its radius is about 46.5 giga-light-years $\sim 2^{204}$ Plank units. A system of $\gg R^{1.5}$ atoms packed in R Plank Units radius collapses rapidly, be it Universe-sized or a neutron star. So the number of atoms is $< 2^{306} \ll 4^{4^4} \ll 5!!$.

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[1] This is a customary but somewhat misleading notation. The clear notations would be like $f(n) \in O(g(n))$

[2] A rougher estimate follows by computing $\ln n! = t\ln(t/e)|_{t=1.5}^{n+.5} + O(1)$ using that $|\sum_{i=2}^{n} g(i) - \int_{1.5}^{n+.5} g(t)\,\mathbf{d}t|$ is bounded by the total variation v of $g'/8$. So for monotone $g'(t) = \ln'(t) = 1/t$ the $O(1)$ is $< v < 1/12$.