# Review of Concepts

EE 20 Spring 2014
University of California, Berkeley

# 1 Introduction

In this note, we will discuss many of the central topics in EE 20 and how they relate to one another. Note that this is an unofficial review which should be ignored in favor of anything presented by the official course staff.

## 1.1 Outline

- Fourier representations
  - Fourier series expansions
  - Fourier transforms
- Time-frequency uncertainty
- Multi-rate operations
  - Decimation
  - Interpolation

# 2 Fourier Representations

## 2.1 Fourier Series Expansions

## 2.2 Fourier Transforms

# 3 Multi-Rate Operations

In digital signal processing (DSP), *multi-rate operations* are operations which can be used to change the sampling rate of a DT signal. We refer to the sampling rate $f_s = \frac{1}{T}$ of a DT signal $x[n]$ as the rate at which the CT signal $x(t)$ was sampled to yield $x[n] = x(nT)$, where $T$ is the sampling period.

## 3.1 Decimation

The act of *decreasing the sampling rate* by an integer factor of $M$ is referred to as *decimation*. If $y[n]$ is the signal produced via decimation of $x[n]$ by a factor of $M$, then $y[n]$ is equivalent to the DT signal that would result from sampling $x(t)$ with sampling period $T' = MT$. This corresponds to a decreased sampling rate of

$$\begin{aligned} f_s' &= \frac{1}{T'} \\ &= \frac{1}{MT} \\ &= \frac{1}{M} f_s. \end{aligned}$$

Decimation is realized by first filtering $x[n]$ with an *anti-aliasing (decimation) filter*, then *downsampling* by a factor of $M$.

### 3.1.1 Downsampling

A system which *downsamples* by an integer factor of $M$ creates as output a new signal which contains every $M^{\text{th}}$ sample of its input. Such a system is represented in a block diagram via the following symbol:

$$\mathbf{x[n]} \;—\!\!\!\bigcirc\!\!\!\!\!\!\!\!\!\!\quad {\downarrow}\mathbf{M} \;—\; \mathbf{y[n]}$$

Downsampling is the DT analogy to CT sampling. As in CT sampling, the time domain analysis is very simple, whereas the frequency domain analysis is a bit more involved.

- **Time Domain**
  The time domain representation of a downsampler with input $x[n]$ and output $y[n]$ is simply

  $$\boxed{y[n] = x[nM]}.$$

  As discussed above, the output is the sequence of every $M^{\text{th}}$ sample of its input.

- **Frequency Domain**
  Let $X_c(\omega) \overset{\mathcal{F}}{\leftrightarrow} x(t)$ denote the CTFT of $x(t)$ and $X_d(\omega) \overset{\mathcal{F}}{\leftrightarrow} x[n]$ denote the DTFT of $x[n] = x(nT)$. Recall from lecture the relationship between $X_c(\omega)$ and $X_d(\omega)$ (Theorem 22.1):

  $$X_d(\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c\left(\frac{\omega - 2\pi k}{T}\right)$$

Since $y[n] = x[nM] = x(nMT)$ is simply $x(t)$ sampled with $T' = MT$, we can write

$$\begin{aligned} Y(\omega) &= \frac{1}{T'} \sum_{k=-\infty}^{\infty} X_c\left(\frac{\omega - 2\pi k}{T'}\right) \\ &= \frac{1}{MT} \sum_{k=-\infty}^{\infty} X_c\left(\frac{\omega - 2\pi k}{MT}\right). \end{aligned}$$

Our goal is to find the frequency domain equivalent of $y[n] = x[nM]$, i.e. the relationship between $Y(\omega)$ and $X_d(\omega)$. To do so, we will perform a series of convoluted variable substitutions. First, let $k = l + mM$ and note that summing from $k = -\infty$ to $k = \infty$ is the same as doing a double-sum over $l = 0$ to $M - 1$ and $m = -\infty$ to $\infty$. That is,

$$Y(\omega) = \frac{1}{MT} \sum_{l=0}^{M-1} \sum_{m=-\infty}^{\infty} X_c\left(\frac{\omega - 2\pi(l + mM)}{MT}\right).$$

Rearranging terms, we have

$$Y(\omega) = \frac{1}{M} \sum_{l=0}^{M-1} \frac{1}{T} \sum_{m=-\infty}^{\infty} X_c\left(\frac{\omega}{MT} - \frac{2\pi l}{MT} - \frac{2\pi m}{T}\right),$$

and recombining the terms in $X_c(\cdot)$ yields

$$Y(\omega) = \frac{1}{M} \sum_{l=0}^{M-1} \frac{1}{T} \sum_{m=-\infty}^{\infty} X_c\left(\frac{\left(\frac{\omega - 2\pi l}{M}\right) - 2\pi m}{T}\right).$$

Substituting in Theorem 22.1 and letting $k = l$, we see the desired relationship is

$$\boxed{Y(\omega) = \frac{1}{M} \sum_{k=0}^{M-1} X_d\left(\frac{\omega - 2\pi k}{M}\right)}.$$

### 3.1.2 The Anti-Aliasing (Decimation) Filter

Just as in CT sampling, downsampling a DT signal results in adding up aliased copies of the frequency-stretched (dividing $\omega$ by $M$ stretches the frequency axis) and attenuated (the amplitude is decreased by $M$) spectrum of the input, so if we are to avoid aliasing, we require an *anti-aliasing filter* before performing downsampling. Since we are in DT where frequencies are $2\pi$-periodic, we must be clear as to what we mean by aliasing: we say that aliasing occurs when the periods of the output signal's spectrum overlap as a result of downsampling. Since the spectrum is $2\pi$-periodic, we may look at one period to see what's going on and extrapolate from there. On the interval $\omega \in [-\pi, \pi)$, aliasing occurs if frequency components of the input get stretched outside this interval – they would then overlap with neighboring periods. Since downsampling stretches the frequency axis by $M$, any nonzero frequency components in the input signal at $|\omega| \geq \frac{\pi}{M}$ will result in aliasing after downsampling (since $\left|\frac{\omega}{M}\right| \geq \frac{\pi}{M} \Rightarrow |\omega| \geq \pi$). Thus we define the anti-aliasing filter to be an ideal DT LTI low-pass filter with cutoff frequency $\omega_c = \frac{\pi}{M}$. If we apply this filter before downsampling, there will be no nonzero frequency components at $|\omega| \geq \frac{\pi}{M}$ which would be stretched out to $|\omega| \geq \pi$ after downsampling. Additionally, since upsampling decreases the amplitude of the input spectrum, our filter should have a gain of $M$ at all frequencies to preserve the amplitude of the signal.

#### 3.1.3 System Analysis

Figure 1 shows a decimator in block diagram form:

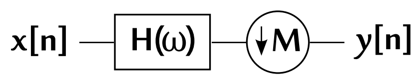


Figure 1: A decimator consisting of an ideal LPF and a downsampler.

As you've seen in class, systems that are *linear* and *time-invariant* (LTI) are of great interest to us since they can be compactly represented by an impulse response, or equivalently, a frequency response which has a very intuitive physical interpretation. Naturally, then, we would like to determine whether or not the decimator is an LTI system. Since it is the cascade of an LTI filter (...which is certainly LTI) and a downsampler, it suffices to determine whether or not the downsampler is LTI:

- **Linearity**
  Suppose $y_1[n]$ and $y_2[n]$ are the outputs of an $M$-downsampler corresponding to inputs $x_1[n]$ and $x_2[n]$, respectively. Let $\hat{y}[n]$ denote the output corresponding to input $\hat{x}[n] = ax_1[n] + bx_2[n]$, with $a, b \in \mathbb{R}$. From the definition of downsampling ($y[n] = x[nM]$), it is clear that $\hat{y}[n] = ay_1[n] + by_2[n]$, and thus we see that **downsampling is linear**. ✓

- **Time-Invariance**
  Suppose $y[n]$ is the output of an $M$-downsampler corresponding to input $x[n]$. Let $\hat{y}[n]$ denote the output corresponding to input $\hat{x}[n] = x[n - n_0]$. From the definition of downsampling ($y[n] = x[nM]$), we see that

  $$\begin{aligned}\hat{y}[n] &= \hat{x}[nM] \\ &= x[nM - n_0]\end{aligned}$$

  and

  $$\begin{aligned}y[n - n_0] &= x[(n - n_0)M] \\ &= x[nM - n_0M].\end{aligned}$$

Since $\hat{y}[n] \neq y[n-n_0]$, we see that **downsampling is not time-invariant**. $\ddot{\frown}$

## 3.2 Interpolation

The act of *increasing the sampling rate* by an integer factor of $L$ is referred to as *interpolation*. If $y[n]$ is the signal produced via interpolation of $x[n]$ by a factor of $L$, then $y[n]$ is equivalent to the DT signal that would result from sampling $x(t)$ with sampling period $T' = \frac{T}{L}$. This corresponds to an increased sampling rate of

$$\begin{aligned} f_s' &= \frac{1}{T'} \\ &= \frac{L}{T} \\ &= Lf_s. \end{aligned}$$

Interpolation is realized by first *upsampling* $x[n]$ by a factor of $L$, then filtering with an *anti-aliasing (interpolation)* filter.

### 3.2.1 Upsampling

A system which *upsamples* by an integer factor of $L$ creates as output a new signal which contains the samples of its input with $L-1$ zeroes between each input sample. Such a system is represented in a block diagram via the following symbol:

$$\mathbf{x[n]} \;—\; \uparrow\mathbf{L} \;—\; \mathbf{y[n]}$$

Upsampling is the DT analogy to $ImpulseGen_T$ in the $DiscToCont_T$ system presented in Chapter 11 of Lee & Varaiya. As upsampling and downsampling perform "opposite" operations, it may make some sense that the frequency domain analysis is now very simple, whereas the time domain analysis is a bit more involved.

- **Time Domain**
  The most natural time domain representation of an upsampler with input $x[n]$ and output $y[n]$ (i.e. that which most obviously conveys the idea of "inserting zeroes between input samples") is

$$y[n] = \begin{cases} x\left[\frac{n}{L}\right], & \text{if } \frac{n}{L} \text{ is an integer} \\ 0, & \text{otherwise} \end{cases}.$$

  However, this representation does not lend itself well to analysis. Fortunately, as seen in a previous homework, we can write the above in a form that is easier to work with:

$$\boxed{y[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-kL]}$$

- **Frequency Domain**
  Let $X(\omega) \overset{\mathcal{F}}{\leftrightarrow} x[n]$ denote the DTFT of $x[n]$ and $Y(\omega)$ denote the DTFT of $y[n]$. To determine the relationship between $Y(\omega)$ and $X(\omega)$, we take the DTFT of both sides of the time domain relationship given above and apply the linearity of the Fourier transform:

$$\mathcal{F}\{y[n]\} = \mathcal{F}\left\{\sum_{k=-\infty}^{\infty} x[k]\delta[n-kL]\right\}$$
$$Y(\omega) = \sum_{k=-\infty}^{\infty} x[k]\mathcal{F}\{\delta[n-kL]\}$$

  Note that $x[k]$ is a constant inside the sum over $k$ which allows us to pull it out of the Fourier transform. We now recall the transform pair for the Kronecker delta

$$\delta[n] \overset{\mathcal{F}}{\leftrightarrow} 1$$

  and time-shift property of the Fourier transform

$$x[n-n_0] \overset{\mathcal{F}}{\leftrightarrow} X(\omega)e^{-i\omega n_0}$$

  to see that $\mathcal{F}\{\delta[n-kL]\} = e^{-i\omega kL}$. Putting it all together yields

$$Y(\omega) = \sum_{k=-\infty}^{\infty} x[k]e^{-i\omega kL}.$$

  Substituting in the definition of the DTFT analysis equation, we see the desired relationship is

$$\boxed{Y(\omega) = X(\omega L)}.$$

#### 3.2.2 The Anti-Aliasing (Interpolation) Filter

Similar to the reconstruction process in CT sampling, upsampling a DT signal results in a repetition of frequency-compressed (multiplying $\omega$ by $L$ compresses the frequency axis) versions of the spectrum of the input. Just as in reconstruction in CT sampling, to interpolate the new samples that result from upsampling, we require an *interpolation filter*.[1] The frequency compression that

[1]In most DSP literature, this filter is also referred to as an *anti-aliasing* filter.

results from upsampling can be viewed as "aliasing" – by limiting our view to the interval $[-\pi, \pi)$, we see that copies of the spectrum that were previously outside this interval are compressed into it. Since upsampling compresses the frequency axis by $L$, any nonzero frequency components at $|\omega| \geq \frac{\pi}{L}$ in the output signal must have been shifted in due to upsampling (since $|\omega L| \geq \pi \Rightarrow |\omega| \geq \frac{\pi}{L}$). Thus we define the anti-aliasing filter to be an ideal DT LTI low-pass filter with cutoff frequency $\omega_C = \frac{\pi}{M}$.

### 3.2.3 System Analysis

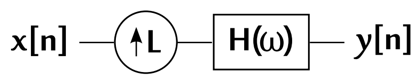


Figure 2: An interpolator consisting of an upsampler and an ideal LPF.

- **Linearity**

- **Time-Invariance**

## 3.3 Example: Fractional Rate Changes

Note that downsampling and upsampling are only defined for integer factors $M$ and $L$, respectively. However, there are situations in which we would like to change the sampling rate by a fractional (non-integer) factor. For example, say you wanted to change the sampling rate from $f_s$ to $f_s' = \frac{L}{M} f_s$, for integers $L$ and $M$. To accomplish this, we simply cascade an $L$-interpolator and an $M$-decimator which share the same anti-aliasing filter having cutoff frequency $\omega_c = \min\left\{\frac{\pi}{M}, \frac{\pi}{L}\right\}$.

# 4 Closing Thoughts