The DFT and Sparsity
Solutions

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1 Introduction

Below are the solutions to the bonus worksheet. Please attempt the exercises by yourself or with a group before reading on!

2 Vector Spaces and the DFT

2.1 Review

2.2 Signals as Vectors

2.2.1 Exercise:

Let $x_1[n]$ and $x_2[n]$ be two arbitrary signals in $\mathbb{C}^N$. To verify that $\mathbb{C}^N$ is a vector space according to the properties given in the worksheet, we must show that it is closed under addition and scalar multiplication (again, note that there are more properties which a vector space must have – refer to Wikipedia for a complete list). In other words, we must show that $x_1[n],\ x_2[n] \in \mathbb{C}^N \Rightarrow x_1[n]+x_2[n] \in \mathbb{C}^N$ and $x[n] \in \mathbb{C}^N,\ \alpha \in \mathbb{C} \Rightarrow \alpha x[n] \in \mathbb{C}^N$. The sum of $x_1[n]$ and $x_2[n]$ is given by

$$\begin{aligned} x_1[n] + x_2[n] &= \begin{bmatrix} x_1[0] \\ \vdots \\ x_1[N-1] \end{bmatrix} + \begin{bmatrix} x_2[0] \\ \vdots \\ x_2[N-1] \end{bmatrix} \\ &= \begin{bmatrix} x_1[0] + x_2[0] \\ \vdots \\ x_1[0] + x_2[N-1] \end{bmatrix}. \end{aligned}$$

Since $x_1[n]$ and $x_2[n]$ are complex valued, their sum is complex valued (a sum of complex numbers is a complex number), and thus the first property is satisfied. Recall that a scalar in $\mathbb{C}^N$ is any $\alpha \in \mathbb{C}$. Then for any $x[n] \in \mathbb{C}^N$ and $\alpha \in \mathbb{C}$, the signal $\alpha x[n]$ is given by

$$\begin{aligned}\alpha x[n] &= \alpha \begin{bmatrix} x[0] \\ \vdots \\ x[N-1] \end{bmatrix} \\ &= \begin{bmatrix} \alpha x[0] \\ \vdots \\ \alpha x[N-1] \end{bmatrix}.\end{aligned}$$

Since $x[n]$ is complex valued and α is a complex number, the signal $\alpha x[n]$ is complex valued (a product of complex numbers is a complex number), and thus the second property is satisfied, and we are convinced that $\mathbb{C}^N$ is indeed a vector space.

2.2.2 Exercise:

Following the same strategy as before, let $x_1[n]$ and $x_2[n]$ be two arbitrary signals in $\ell_2(\mathbb{Z})$. To show that $\ell_2(\mathbb{Z})$ is closed under addition, we must show that $x_1[n] + x_2[n]$ is also in $\ell_2(\mathbb{Z})$, i.e. we must show $x_1[n] + x_2[n]$ has finite energy. Since $x_1[n]$ and $x_2[n]$ are in $\ell_2(\mathbb{Z})$, they must have finite energy, and we will denote their respective norms by $\|x_1[n]\|_2 = \mathcal{E}_1$ and $\|x_2[n]\|_2 = \mathcal{E}_2$. Using the definition of the ℓ_2 norm provided, we have

$$\begin{aligned}\|x_1[n] + x_2[n]\|_2 &\leq \|x_1[n]\|_2 + \|x_2[n]\|_2 \\ &= \mathcal{E}_1 + \mathcal{E}_2 \\ &< \infty,\end{aligned}$$

where the first inequality is the triangle inequality, and the last inequality results from the fact that $x_1[n]$ and $x_2[n]$ are in $\ell_2(\mathbb{Z})$. Since $\|x_1[n]+x_2[n]\|_2$ is finite, the energy $\|x_1[n] + x_2[n]\|_2^2$ must be finite, and thus the first property is satisfied. Recall that a scalar in $\ell_2(\mathbb{Z})$ is any $\alpha \in \mathbb{C}$. Then for any $x[n] \in \ell_2(\mathbb{Z})$ with $\|x[n]\|_2 = \mathcal{E} < \infty$ and $\alpha \in \mathbb{C}$, the ℓ_2 norm is

$$\begin{aligned}\|\alpha x[n]\|_2 &= \left(\sum_{n=-\infty}^{\infty} |\alpha x[n]|^2 \right)^{\frac{1}{2}} \\ &= \left(\sum_{n=-\infty}^{\infty} |\alpha|^2 |x[n]|^2 \right)^{\frac{1}{2}} \\ &= |\alpha| \left(\sum_{n=-\infty}^{\infty} |x[n]|^2 \right)^{\frac{1}{2}} \\ &= |\alpha| \mathcal{E} \\ &< \infty.\end{aligned}$$

Since $\|\alpha x[n]\|_2$ is finite, the energy $\|\alpha x[n]\|_2^2$ must be finite, and thus the second property is satisfied, and we see that $\ell_2(\mathbb{Z})$ is indeed a vector space.

2.3 The DFT as a Change of Basis

2.3.1 Exercise:

To show that the set $\{\phi_k[n]\}_{k=0}^{N-1}$ is an orthogonal basis for $\mathbb{C}^N$, we must show that it is a set of N linearly independent vectors. By construction, the set contains N vectors. So, since orthogonality implies linear independence, it suffices to show that the N vectors satisfy

$$\langle \phi_j[n], \phi_k[n] \rangle = \|\phi_j[n]\|^2 \delta[j-k] = \|\phi_k[n]\|^2 \delta[j-k]$$

for all $j, k \in \{0, \ldots, N-1\}$. Plugging in the definition of $\phi_k[n]$ from the worksheet and using the definition of the inner product in $\mathbb{C}^N$, we have

$$\begin{aligned}
\langle \phi_j[n], \phi_k[n] \rangle &= \sum_{n=0}^{N-1} \phi_j[n] \phi_k^\star[n] \\
&= \frac{1}{N} \sum_{n=0}^{N-1} e^{i\frac{2\pi jn}{N}} e^{-i\frac{2\pi kn}{N}} \\
&= \frac{1}{N} \sum_{n=0}^{N-1} e^{i\frac{2\pi(j-k)n}{N}} \\
&= \frac{1}{N} \cdot \frac{1 - \left(e^{i\frac{2\pi(j-k)}{N}}\right)^N}{1 - e^{i\frac{2\pi(j-k)}{N}}} \\
&= \frac{1}{N} \cdot \frac{1 - e^{i2\pi(j-k)}}{1 - e^{i\frac{2\pi(j-k)}{N}}}.
\end{aligned}$$

This is a pretty messy expression, but keep in mind that all we care about is when $j = k$ and $j \neq k$. Let's define $m = j - k$, then $j = k \Rightarrow m = 0$ and $j \neq k \Rightarrow m \in \mathbb{Z},\ m \neq 0$. Substitution yields

$$\langle \phi_j[n], \phi_k[n] \rangle = \frac{1}{N} \cdot \frac{1 - e^{i2\pi m}}{1 - e^{i\frac{2\pi m}{N}}},$$

and we see that when m is an integer not equal to zero, i.e. when $j \neq k$, $e^{i2\pi m} = 1$ since $2\pi m$ is an integer multiple of 2π. In this case, the expression simplifies to zero:

$$\begin{aligned}
m \neq 0 \Rightarrow \frac{1}{N} \cdot \frac{1 - e^{i2\pi m}}{1 - e^{i\frac{2\pi m}{N}}} &= \frac{1}{N} \cdot \frac{1 - 1}{1 - e^{i\frac{2\pi m}{N}}} \\
&= 0
\end{aligned}$$

When $m = 0$, i.e. when $j = k$, however, we have

$$\begin{aligned} m = 0 \Rightarrow \frac{1}{N} \cdot \frac{1 - e^{i2\pi m}}{1 - e^{\frac{i2\pi m}{N}}} &= \frac{1}{N} \cdot \frac{1 - e^{i\cdot 0}}{1 - e^{i\cdot 0}} \\ &= \frac{1}{N} \cdot \frac{1 - 1}{1 - 1} \\ &= \frac{0}{0}, \end{aligned}$$

which is an indeterminate form. We must then apply L'Hôspital's rule:

$$\begin{aligned} \lim_{m\to 0} \frac{1}{N} \cdot \frac{1 - e^{i2\pi m}}{1 - e^{i\frac{2\pi m}{N}}} &\overset{\text{LH}}{=} \lim_{m\to 0} \frac{1}{N} \cdot \frac{\frac{d}{dm}\left(1 - e^{i2\pi m}\right)}{\frac{d}{dm}\left(1 - e^{i\frac{2\pi m}{N}}\right)} \\ &= \lim_{m\to 0} \frac{1}{N} \cdot \frac{-i2\pi e^{i2\pi m}}{-i\frac{2\pi}{N} e^{i\frac{2\pi m}{N}}} \\ &= \lim_{m\to 0} \frac{1}{N} \cdot \frac{N e^{i2\pi m}}{e^{i\frac{2\pi m}{N}}} \\ &= \frac{N}{N} \cdot \frac{e^{i2\pi\cdot 0}}{e^{i\frac{2\pi\cdot 0}{N}}} \\ &= 1 \end{aligned}$$

And thus, since $m \neq 0 \Rightarrow j \neq k \Rightarrow \langle \phi_j[n], \phi_k[n] \rangle = 0$ and $m = 0 \Rightarrow j = k \Rightarrow \langle \phi_j[n], \phi_k[n] \rangle = 1$, we have

$$\langle \phi_j[n], \phi_k[n] \rangle = \delta[j-k],$$

which implies the set $\{\phi_k[n]\}_{k=0}^{N-1}$ is an orthonormal basis for $\mathbb{C}^N$.

2.3.2 Exercise:

Let $\phi_k[n] = \frac{1}{\sqrt{N}} e^{i\frac{2\pi k n}{N}} = \frac{1}{\sqrt{N}} \begin{bmatrix} e^{i\frac{2\pi k\cdot 0}{N}} & e^{i\frac{2\pi k\cdot 1}{N}} & \cdots & e^{i\frac{2\pi k\cdot (N-1)}{N}} \end{bmatrix}^{\mathsf{T}}$. For an arbitrary $x[n] \in \mathbb{C}^N$, we have

$$\begin{aligned} \alpha_k &= \langle x[n], \phi_k[n] \rangle \\ &= \sum_{n=0}^{N-1} x[n] \phi_k^{\star}[n] \\ &= \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n] \left(e^{i\frac{2\pi k}{N} n}\right)^{\star} \\ &= \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n] e^{-i\frac{2\pi k}{N} n} \end{aligned}$$

and

$$x[n] = \sum_{k=0}^{N-1} \alpha_k \phi_k[n]$$
$$= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \alpha_k e^{i\frac{2\pi k}{N}n}.$$

Note that these are exactly the DFT analysis and synthesis equations (within a constant factor)! And thus we see the DFT as nothing more than a change of basis from the canonical basis to the *Fourier basis* (name given to the set $\{\phi_k[n]\}_{k=0}^{N-1}$). This version of the DFT ($\frac{1}{\sqrt{N}}$ in both equations instead of $\frac{1}{N}$ in just one) is called the *unitary DFT*. [1]

3 Sparse Signal Processing

3.1 Fourier Perspective on Sparsity

3.1.1 Exercise:

The unknown DFT coefficients ($X[0],\ X[1],\ X[2],\ X[3]$) are related to the known samples ($x[0],\ x[15],\ x[30],\ x[45]$) via the DFT synthesis equation

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{i\frac{2\pi k}{N}n},$$

which we can rewrite as (taking into account $N = 60$, $K = 4$, and the known structure of the DFT $X[k]$)

$$x[n] = \frac{1}{60} \sum_{k=0}^{3} X[k] e^{i\frac{2\pi k}{60}n}.$$

[1] `http://en.wikipedia.org/wiki/Discrete_Fourier_transform#The_unitary_DFT`

For the samples of $x[n]$ given, we have the following system of equations:

$$x[0] = \frac{1}{60}\sum_{k=0}^{3} X[k]e^{i\frac{2\pi k}{60}\cdot 0} = \frac{1}{60}\sum_{k=0}^{3} X[k]$$

$$x[15] = \frac{1}{60}\sum_{k=0}^{3} X[k]e^{i\frac{2\pi k}{60}\cdot 15} = \frac{1}{60}\sum_{k=0}^{3} X[k]e^{i\frac{\pi}{2}k} = \frac{1}{60}\sum_{k=0}^{3} X[k]i^k$$

$$x[30] = \frac{1}{60}\sum_{k=0}^{3} X[k]e^{i\frac{2\pi k}{60}\cdot 30} = \frac{1}{60}\sum_{k=0}^{3} X[k]e^{i\pi k} = \frac{1}{60}\sum_{k=0}^{3} X[k](-1)^k$$

$$x[45] = \frac{1}{60}\sum_{k=0}^{3} X[k]e^{i\frac{2\pi k}{60}\cdot 45} = \sum_{k=0}^{3} X[k]e^{i\frac{3\pi}{2}k} = \sum_{k=0}^{3} X[k](-i)^k$$

$$\Rightarrow \begin{bmatrix} x[0] \\ x[15] \\ x[30] \\ x[45] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} \begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \end{bmatrix}$$

Let W denote the matrix relating the vectors of samples and DFT coefficients. if W is invertible, your friend can simply invert it, calculate $\{X[k]\}_{k=0}^{3}$, then compute an inverse DFT to find $\{x[n]\}_{n=0}^{N-1}$. So, is W invertible? Yes! There are a few ways to see this. By inspection, the four equations it generates are distinct, and thus you can solve the system. You could also show that the columns of W are linearly independent. A more general method for showing invertibility of matrices like W will be discussed in a later exercise.

3.1.2 Exercise:

Coming soon...

3.1.3 Exercise:

Let $w = e^{i\frac{2\pi}{60}}$, and let $n_0,\ n_1,\ n_2$, and n_3 be the four time indices which we sample at. Then, as computed in a previous exercise, the system of equations relating the unknown DFT coefficients to the known samples is

$$\begin{bmatrix} x[n_0] \\ x[n_1] \\ x[n_2] \\ x[n_3] \end{bmatrix} = \begin{bmatrix} 1 & w^{n_0} & w^{2\cdot n_0} & w^{3\cdot n_0} \\ 1 & w^{n_1} & w^{2\cdot n_1} & w^{3\cdot n_1} \\ 1 & w^{n_2} & w^{2\cdot n_2} & w^{3\cdot n_2} \\ 1 & w^{n_3} & w^{2\cdot n_3} & w^{3\cdot n_3} \end{bmatrix}.$$

A matrix of this form is called a *Vandermonde matrix*, and it is invertible if and only if $\{w^{n_k}\}_{k=0}^{3}$ are all distinct. As long as $\{n_k\}_{k=0}^{3}$ are distinct, $\{w^{n_k}\}_{k=0}^{3}$ will be distinct, and thus your friend can recover the entire signal $x[n]$ from *any four distinct* samples.