A Sample of Things to Come

EE 20 Spring 2014
University of California, Berkeley

1 Introduction

At the beginning of the course, you learned that "almost" any periodic continuous time signal with period p can be represented as a linear combination of sinusoids. Such a representation of a signal

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{ik\omega_0 t}$$

is called the Fourier Series expansion of $x(t)$, where $\omega_0 = \frac{2\pi}{p}$, and the coefficients X_k are given by

$$X_k = \frac{1}{p}\int_0^p x(t)e^{-ik\omega_0 t}dt.$$

Upon first seeing these equations, some natural questions to ask may be, "Why do we want to write signals as linear combinations of sinusoids? What's the point? And what's so special about sinusoids?" At this point in the course, we now know the answers to these questions.

EE 20 is an introductory course in signal processing, a field which has applications everywhere from music to finance to medicine. Some common operations we perform on signals in these fields include noise reduction and echo cancellation, and as you've seen in class, these operations can be realized via *LTI filters*. The connection between signal processing and Fourier Series is established via the eigenfunction property of LTI systems: For any LTI system H,

$$x(t) \rightarrow \boxed{H} \rightarrow y(t)$$
$$x(t) = e^{i\omega_0 t} \Rightarrow y(t) = H(\omega_0)e^{i\omega_0 t}.$$

This property allows us two write the following relation for an arbitrary $x(t)$:

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{ik\omega_0 t} \rightarrow \boxed{H} \rightarrow y(t) = \sum_{k=-\infty}^{\infty} H(k\omega_0) X_k e^{ik\omega_0 t}$$

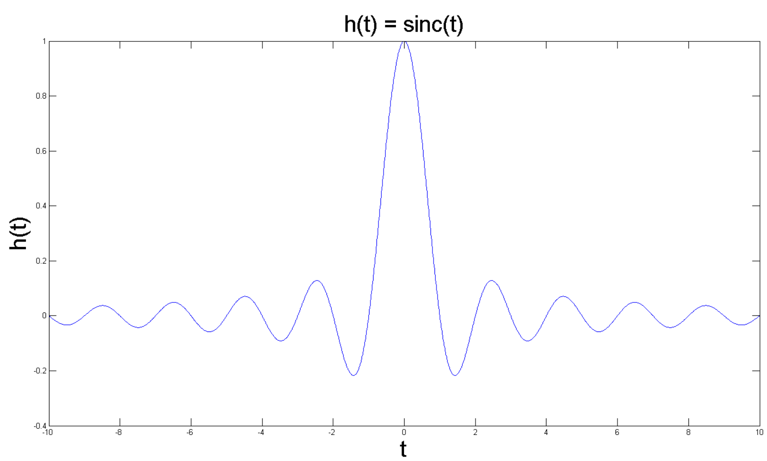
In filtering, we are often interested in suppressing some undesirable frequencies in the input signal (e.g. noise) while preserving or amplifying interesting ones (e.g. those in an EEG signal). With the above relation between input, output, and frequency response, it becomes very clear how a filter effects an input signal, and we can easily design filters by specifying how they behave at certain frequencies.

That being said, filtering isn't *everything* in signal processing. In this exercise, we will introduce a different type of series expansion that serves as the mathematical basis for another fundamental operation in signal processing.

2 The Sinc Function

Consider the continuous time sinc function:

$$\text{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$$



2.1 Exercise:

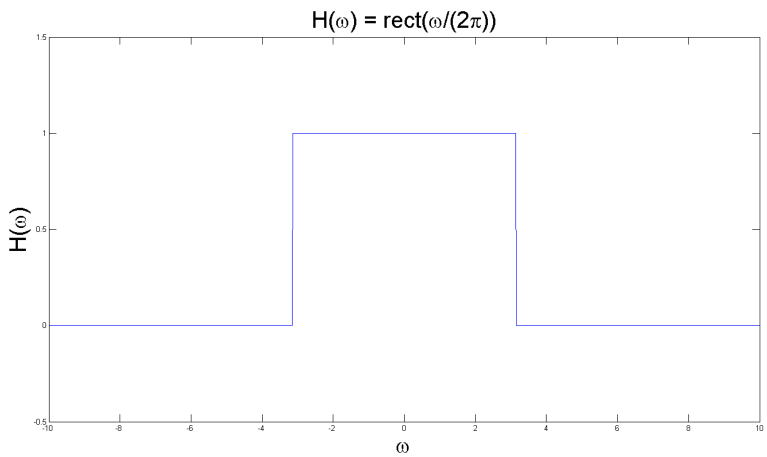
Show that $\text{sinc}(0) = 1$ (Hint: L'Hôspital's rule).

2.2 Exercise:

Recall that a signal $h(t)$ can be synthesized from its Continuous Time Fourier Transform (CTFT) $H(\omega)$ as follows:

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega) e^{i\omega t} d\omega$$

Suppose $H(\omega)$ is the frequency response of an ideal low-pass filter with cutoff frequency $\omega_c = \pi$, i.e. $H(\omega) = \begin{cases} 1, & |\omega| < \pi \\ 0, & \text{otherwise} \end{cases}$.



Find the expression for the impulse response $h(t)$.

3 "Sinc Series" Expansion

Just as we defined the Fourier Series expansion of a periodic signal, let us now define the "Sinc Series" expansion of a signal $x(t)$ which is *bandlimited* to $(-\pi.\pi)$, where bandlimited in this case means $X(\omega) = 0$ for $|\omega| > \pi$:

$$x(t) = \sum_{n=-\infty}^{\infty} X_n \mathrm{sinc}(t-n)$$
$$X_n = \int_{-\infty}^{\infty} x(t) \mathrm{sinc}(t-n) dt$$

Note that, just like in the Fourier Series expansion, $t \in \mathbb{R}$ and $n \in \mathbb{Z}$. [1]

3.1 Exercise:

Given the CTFT pair $h(t) \overset{\mathcal{F}}{\leftrightarrow} H(\omega)$, find an expression for $\hat{H}(\omega)$, the CTFT of $\hat{h}(t) = h(t-n)$.

[1]Note that we did not prove the synthesis equality above – for now, take it as fact, but it can be proven using techniques learned in EE 20.

3.2 Exercise:

Suppose $x(t)$ is real valued and bandlimited to $(-\pi, \pi)$. Find an expression for X_n that involves no integrals or sincs. You will need to use the fact that $x(t)$ is bandlimited, and you may find the following fact useful in simplifying your calculations:

- **Parseval's Relation**
  Given two real valued continuous time signals $x(t)$, $h(t)$ and their CTFTs $X(\omega)$, $H(\omega)$:

$$\int_{-\infty}^{\infty} x(t)h(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)H(\omega)d\omega$$

3.3 Exercise:

Let $X(\omega) = \delta(\omega - \omega_0)$. What is the $x(t)$, the inverse CTFT of $X(\omega)$?

3.4 Exercise:

Let $x(t) = \cos(\frac{\pi}{4}t)$. Show that $x(t)$ is bandlimited to $(-\pi, \pi)$, then, using the expression you found in the previous exercise, sketch $x(t)$ and its "Sinc Series" coefficients X_n on the same plot. How would you qualitatively describe the relationship between the continuous time signal $x(t)$ and the discrete time signal X_n?

4 Closing Thoughts

Consider the consequences of what you have just shown: since a signal $x(t)$ bandlimited on $(-\pi, \pi)$ can be written as

$$x(t) = \sum_{n=-\infty}^{\infty} X_n \text{sinc}(t - n),$$

it must be true that such a signal is completely determined by its "Sinc Series" coefficients X_n. Now suppose we were interested in storing the values of $x(t)$ on a computer. Is it possible to store a continuous time signal in memory? No! Since $t \in \mathbb{R}$, there are *uncountably infinite* values of $x(t)$ between any two values of t. What about X_n? By now you should be convinced that X_n contains *all of the information* contained in $x(t)$, thus it suffices to store X_n instead. Is this possible?