A Sample of Things to Come
Solutions

EE 20 Spring 2014
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1 Introduction

Below are the solutions to the bonus worksheet. Please attempt the exercises by yourself or with a group before reading on!

2 The Sinc Function

2.1 Exercise:

Since $\text{sinc}(0) = \frac{\sin(\pi \cdot 0)}{\pi \cdot 0} = \frac{0}{0}$ is indeterminate, we use L'Hôspital's Rule:

$$\begin{aligned}
\lim_{t \to 0} \text{sinc}(t) &= \lim_{t \to 0} \frac{\sin(\pi t)}{\pi t} \\
&\stackrel{\text{LH}}{=} \lim_{t \to 0} \frac{\frac{d}{dt}\sin(\pi t)}{\frac{d}{dt}\pi t} \\
&= \lim_{t \to 0} \frac{\pi \cos(\pi t)}{\pi} \\
&= \lim_{t \to 0} \cos(\pi t) \\
&= 1
\end{aligned}$$

2.2 Exercise:

Plugging into the CTFT synthesis equation, we have:

$$\begin{aligned}
h(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)e^{i\omega t}d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} 1\cdot e^{i\omega t}d\omega \\
&= \frac{1}{2\pi}\left.\frac{e^{i\omega t}}{it}\right|_{\omega=-\pi}^{\omega=\pi} \\
&= \frac{1}{\pi t}\left(\frac{e^{i\pi t}-e^{-i\pi t}}{2i}\right) \\
&= \frac{\sin \pi t}{\pi t} \\
&= \operatorname{sinc}(t)
\end{aligned}$$

3 "Sinc Series" Expansion

3.1 Exercise:

Plugging into the CTFT analysis equation, we have:

$$\begin{aligned}
\hat{H}(\omega) &= \int_{-\infty}^{\infty} \hat{h}(t)e^{-i\omega t}dt \\
&= \int_{-\infty}^{\infty} h(t-n)e^{-i\omega t}dt
\end{aligned}$$

Let $\tau = t-n \Rightarrow t = \tau + n$, then integrating over t from $-\infty$ to ∞ is the same as integrating over τ from $-\infty$ to ∞:

$$\begin{aligned}
\hat{H}(\omega) &= \int_{-\infty}^{\infty} h(\tau)e^{-i\omega(\tau+n)}d\tau \\
&= \int_{-\infty}^{\infty} h(\tau)e^{-i\omega\tau}e^{-i\omega n}d\tau \\
&= e^{-i\omega n}H(\omega)
\end{aligned}$$

3.2 Exercise:

Trying to evaluate the expression for X_n in the time domain is no good! Parseval's relation lets us write:

$$\begin{aligned}
X_n &= \int_{-\infty}^{\infty} x(t)\mathrm{sinc}(t-n)dt \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{F}\{x(t)\}(\mathcal{F}\{\mathrm{sinc}(t-n)\})^{\star} d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)(e^{-i\omega n}\mathcal{F}\{\mathrm{sinc}(t)\})^{\star} d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{i\omega n}(\mathcal{F}\{\mathrm{sinc}(t)\})^{\star} d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{i\omega n}\cdot 1 \ d\omega
\end{aligned}$$

However, since $x(t)$ is bandlimited to $(-\pi, \pi)$, $X(\omega) = 0$ outside the interval $(-\pi, \pi)$, and we can write:

$$X_n = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{i\omega n} d\omega$$

Now we see that this is simply the CTFT synthesis equation for $x(t)$ evaluated at $t = n$! In other words:

$$X_n = x(t)\big|_{t=n} = x(n)$$

3.3 Exercise:

Plugging into the CTFT synthesis equation, we have:

$$\begin{aligned}
x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{i\omega t} d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega-\omega_0)e^{i\omega t} d\omega \\
&= e^{i\omega_0 t}
\end{aligned}$$

3.4 Exercise:

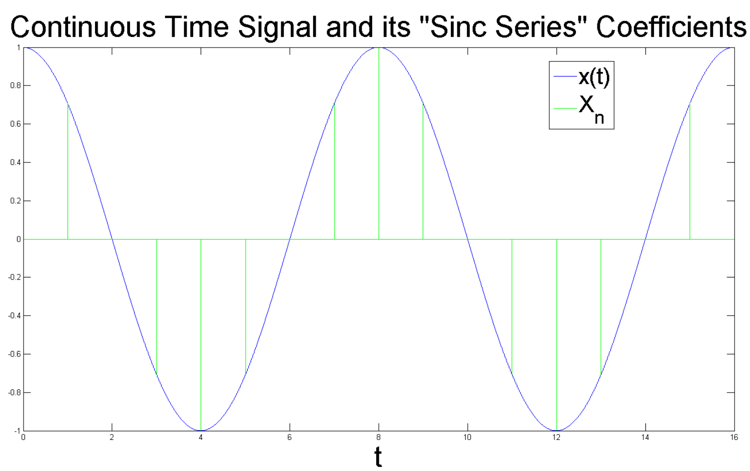
Using the Fourier Series expansion of $\cos(\frac{\pi}{4}t)$ and the linearity of the Fourier Transform, we have:

$$\begin{aligned}
x(t) &= \cos(\frac{\pi}{4}t) \\
&= \frac{1}{2}e^{i\frac{\pi}{4}t} + \frac{1}{2}e^{-i\frac{\pi}{4}t} \\
X(\omega) &= \mathcal{F}\{x(t)\} \\
&= \mathcal{F}\{\frac{1}{2}e^{i\frac{\pi}{4}t} + \frac{1}{2}e^{-i\frac{\pi}{4}t}\} \\
&= \frac{1}{2}\mathcal{F}\{e^{i\frac{\pi}{4}t}\} + \frac{1}{2}\mathcal{F}\{e^{-i\frac{\pi}{4}t}\} \\
&= \frac{1}{2}\delta(\omega - \frac{\pi}{4}) + \frac{1}{2}\delta(\omega + \frac{\pi}{4})
\end{aligned}$$

Since $X(\omega)$ only takes nonzero values at $\omega = \pm\frac{\pi}{4}$, $x(t)$ is clearly bandlimited to (π, π), and thus we can find its "Sinc Series" coefficients:

$$X_n = \cos(\frac{\pi}{4}n), \ n \in \mathbb{Z}$$

In other words, X_n is a *discrete time* signal that is equal to $x(t)$ at integer values of t:



Thus we see X_n as a *sampled* version of $x(t)$!