07/08/2024

# Gravitational Waves (Theory)

## ① Einstein Field Equations

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Einstein tensor (geometry) — Stress-energy tensor (source)

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Ricci tensor — Ricci scalar

## ② Spacetime Metric

$$g_{\mu\nu} = \eta_{\mu\nu} + \underbrace{h_{\mu\nu}}_{\ll 1}$$

Minkowski ("flat")

$$\begin{pmatrix} -1 & & & 0 \\ & +1 & & \\ & & +1 & \\ 0 & & & +1 \end{pmatrix}$$

⟹ **Important convention**: since metric perturbation small, we can raise + lower tensor indices with Minkowski metric.

$$\therefore \quad h^{\mu\nu} = \eta^{\mu\gamma} \eta^{\nu\delta} h_{\gamma\delta}$$

... rather than $h^{\mu\nu} = g^{\mu\gamma} g^{\nu\delta} h_{\gamma\delta}$

ONLY EXCEPTION... $g^{\mu\nu} = (g_{\mu\nu})^{-1} = (\eta_{\mu\nu} + h_{\mu\nu})^{-1}$

$$= \eta_{\mu\nu} - h_{\mu\nu} + \mathcal{O}(h^2)$$

## ③ Linearize the Einstein tensor

↳ tedious, but not difficult.

The Riemann tensor is defined as:

$$R_{\mu\nu\gamma}{}^{\delta} = -\partial_\mu \Gamma^\delta_{\nu\gamma} + \partial_\nu \Gamma^\delta_{\mu\gamma} - \Gamma^\delta_{\mu\alpha}\Gamma^\alpha_{\nu\gamma} + \Gamma^\delta_{\nu\alpha}\Gamma^\alpha_{\mu\gamma}$$

$\boxed{\text{NOTE}}$ : $\partial_\mu := \frac{\partial}{\partial x^\mu}$

$$\Gamma^\gamma_{\mu\nu} = \frac{1}{2} g^{\gamma\delta}\left(\partial_\mu g_{\nu\delta} + \partial_\nu g_{\delta\mu} - \partial_\delta g_{\mu\nu}\right).$$

⇒ LINEARIZE $\Gamma$...

$$\Gamma^\gamma_{\mu\nu} = \frac{1}{2}\eta^{\gamma\delta}\left(\partial_\mu h_{\nu\delta} + \partial_\nu h_{\delta\mu} - \partial_\delta h_{\mu\nu}\right) + \mathcal{O}(h^2)...$$

$\Gamma\Gamma$ terms are $O(h^2)$, so we ignore!

$\Rightarrow$ LINEARIZE Riemann tensor...

$$R_{\mu\nu\gamma\delta} = \eta_{\varepsilon\delta} R_{\mu\nu\gamma}{}^{\varepsilon}$$

$$= \eta_{\varepsilon\delta}\left[ -\partial_\mu \Gamma^{\varepsilon}_{\nu\gamma} + \partial_\nu \Gamma^{\varepsilon}_{\mu\gamma} \right].$$

$$= \frac{\eta_{\varepsilon\delta}}{2}\Big[ -\eta^{\varepsilon\alpha}\left(\partial_\mu\partial_\nu h_{\gamma\alpha} + \partial_\mu\partial_\gamma h_{\nu\alpha} - \partial_\mu\partial_\alpha h_{\nu\gamma}\right) + \eta^{\varepsilon\alpha}\left(\partial_\nu\partial_\mu h_{\gamma\alpha} + \partial_\nu\partial_\gamma h_{\mu\alpha} - \partial_\nu\partial_\alpha h_{\mu\gamma}\right)\Big].$$

$$= \frac{1}{2}\delta^{\alpha}{}_{\delta}\Big[ \cancel{-\partial_\mu\partial_\nu h_{\gamma\alpha}} + \cancel{\partial_\nu\partial_\mu h_{\gamma\alpha}} - \partial_\mu\partial_\gamma h_{\nu\alpha} + \partial_\nu\partial_\gamma h_{\mu\alpha} + \partial_\mu\partial_\alpha h_{\nu\gamma} - \partial_\nu\partial_\alpha h_{\mu\gamma}\Big]$$

$$\boxed{R_{\mu\nu\gamma\delta} = \frac{1}{2}\Big[ -\partial_\mu\partial_\gamma h_{\nu\delta} + \partial_\nu\partial_\gamma h_{\mu\delta} + \partial_\mu\partial_\delta h_{\nu\gamma} - \partial_\nu\partial_\delta h_{\mu\gamma}\Big] + O(h^2)} \ldots$$

PHEW!

OK, so that's the linearized Riemann tensor, but we now need the Ricci tensor and Ricci scalar.

$$R_{\mu\nu} = R_{\mu\gamma\nu}{}^{\gamma}$$ ⇒ take $R_{\mu\gamma\nu\delta}$, raise the "$\delta$" and make it equal to "$\gamma$".

∘ $$R_{\mu\nu} = \frac{1}{2}\Big[-\partial_\mu\partial_\nu h_\gamma{}^\gamma + \partial_\gamma\partial_\nu h_\mu{}^\gamma + \partial_\mu\partial_\gamma h_\nu{}^\gamma - \eta^{\gamma\delta}\partial_\gamma\partial_\delta h_{\mu\nu}\Big].$$

$h = h^\mu{}_\mu$

$$= \frac{1}{2}\Big[-\partial_\mu\partial_\nu h + \partial_\gamma\partial_\nu h_\mu{}^\gamma + \partial_\mu\partial_\gamma h_\nu{}^\gamma - \eta^{\gamma\delta}\partial_\gamma\partial_\delta h_{\mu\nu}\Big].$$

∘ $$R = \eta^{\mu\nu}R_{\mu\nu}$$
$$= \frac{1}{2}\Big[-\eta^{\mu\nu}\partial_\mu\partial_\nu h + \partial_\gamma\partial_\nu h^{\nu\gamma} + \partial_\mu\partial_\gamma h^{\mu\gamma} - \eta^{\gamma\delta}\partial_\gamma\partial_\delta h\Big]$$
$$= \partial_\alpha\partial_\beta h^{\alpha\beta} - \eta^{\alpha\beta}\partial_\alpha\partial_\beta h$$

FINALLY ...

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$$
$$= R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R$$

$$G_{\mu\nu} = \frac{1}{2}\Big[-\partial_\mu\partial_\nu h + \partial_\gamma\partial_\nu h_\mu{}^\gamma + \partial_\mu\partial_\gamma h_\nu{}^\gamma - \eta^{\gamma\delta}\partial_\gamma\partial_\delta h_{\mu\nu} - \eta_{\mu\nu}\partial_\gamma\partial_\delta h^{\gamma\delta} + \eta_{\mu\nu}\eta^{\gamma\delta}\partial_\gamma\partial_\delta h\Big].$$

LINEARIZED EINSTEIN TENSOR

4 We can make this much more compact by defining the trace-reversed metric perturbation.

$$\bar{h}_{\mu\nu} := h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h$$

NOTE: $\bar{h} = \eta^{\mu\nu}\bar{h}_{\mu\nu} = h - \frac{1}{2}\underbrace{\eta^{\mu\nu}\eta_{\mu\nu}}_{=4} h$

$= -h \Leftarrow$ hence, trace reversed.

Substituting, and cancelling terms gives...

$$G_{\mu\nu} = \frac{1}{2}\Big[\partial_\mu\partial_\gamma \bar{h}^\gamma{}_\nu + \partial_\gamma\partial_\nu \bar{h}^\gamma{}_\mu - \eta^{\gamma\delta}\partial_\gamma\partial_\delta \bar{h}_{\mu\nu} - \eta_{\mu\nu}\partial_\gamma\partial_\delta \bar{h}^{\gamma\delta}\Big].$$

The linearized Einstein field equations are:

$$-\boxed{\eta^{\gamma\delta}\partial_\gamma\partial_\delta\bar{h}_{\mu\nu}} - \eta_{\mu\nu}\partial_\gamma\partial_\delta\bar{h}^{\gamma\delta}$$
$$+\partial_\mu\partial_\gamma\bar{h}^\gamma{}_\nu + \partial_\gamma\partial_\nu\bar{h}^\gamma{}_\mu = \frac{16\pi G}{c^4}T_{\mu\nu}$$

$\Rightarrow$ The $\boxed{\phantom{x}}$ term is just $\Box\bar{h}_{\mu\nu}$ where $\Box$ is the spacetime D'Alembertian operator

i.e. $\Box = -\frac{\partial}{\partial t^2} + \frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2} + \frac{\partial}{\partial z^2}$

$\Rightarrow$ Looks awfully like a wave equation... Can we make those other terms disappear?

5 Lorenz Gauge $\Rightarrow$ find a co-ordinate transformation where divergences of $\bar{h}_{\mu\nu} = 0$

$\Rightarrow$ $x'^\mu = x^\mu + \xi^\alpha$

$\Rightarrow$ $g'_{\mu\nu} = g_{\mu\nu} - \partial_\mu\xi_\nu - \partial_\nu\xi_\mu + O(h^2)$

$\Rightarrow$ $h'_{\mu\nu} = h_{\mu\nu} - \partial_\mu\xi_\nu - \partial_\nu\xi_\mu + O(h^2)$

Changes to Riemann tensor are $O(h^2)$ $\therefore \sim 0$.

$\Rightarrow \bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu + \eta_{\mu\nu}\eta^{\gamma\delta}\partial_\gamma \xi_\delta$

Now, we want $\partial'_\gamma \bar{h}'^\gamma{}_\nu = 0$ in new gauge

$$\therefore \partial'_\gamma \bar{h}'^\gamma{}_\nu = 0 = \partial_\gamma \bar{h}^\gamma{}_\nu - \Box \xi_\nu$$

So, we need to find a co-ordinate transformation generated by a vector offset $\xi^\alpha$ that solves...

$$\Box \xi_\nu = \partial_\gamma \bar{h}^\gamma{}_\nu$$

$\Rightarrow$ With 4 equations and 4 unknowns, we can always solve this!

$\Rightarrow$ Change to Lorenz gauge (co-ordinate system) where all other terms on LHS of Einstein equation are 0.

$$\underbrace{\boxed{-\Box \bar{h}_{\mu\nu} = \frac{16\pi G}{c^4} T_{\mu\nu}}}$$

wave equation sourced by RHS. ☺

$\Rightarrow$ In a vacuum ... $\underbrace{\Box \bar{h}_{\mu\nu} = 0}$

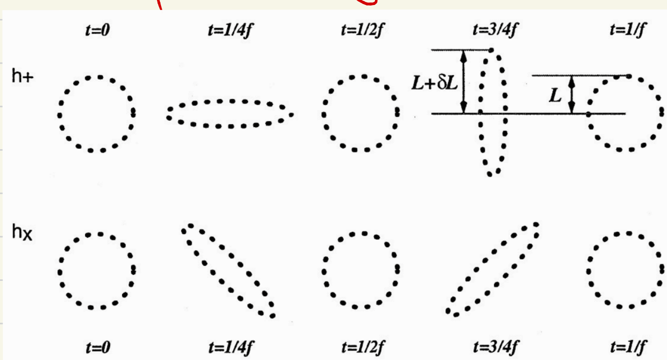
SOLUTION : $\bar{h}_{\mu\nu} = A_{\mu\nu} e^{i k_\sigma x^\sigma}$

We can remove final extraneous degrees of freedom by moving to the transverse traceless gauge

TT gauge : $h^{0\mu} = 0$ ; $h^i_{\ i} = 0$ ; $\partial^i \bar{h}_{ij} = 0$

$$\bar{h}_{\mu\nu}(t,z) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cos[\omega(t-z)]$$

$\Rightarrow$ transverse wave propagating along $+z$ at speed of light.



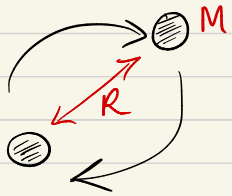

$$T^{\alpha\beta} = \begin{array}{c} \\ t \\ x \\ y \\ z \end{array} \begin{array}{c|ccc} t & x & y & z \\ \hline \text{energy density} & & \text{energy flux} & \\ \hline \text{momentum density} & & \text{stress tensor} & \end{array}$$

# Production of Gravitational Waves

| Moment | Electromagnetism | Gravity |
| --- | --- | --- |
| monopole | $\int \rho_e(\vec{r})\, d^3r = q$ <br> $\left.\frac{dq}{dt} = 0 \right\}$ charge conservation | $\int \rho(\vec{r})\, d^3r = M$ <br> $\left.\frac{dM}{dt} = 0 \right\}$ mass conservation |
| dipole | $\int \rho_e(\vec{r})\, \vec{r}\, d^3r = M \vec{r}_{com}$ <br> $M \frac{d\vec{r}_{com}}{dt} = M \vec{v}_{com} = \vec{P}$ <br> $\Rightarrow$ No conservation law <br> $\hookrightarrow$ dipolar EM waves | $\int \rho(\vec{r})\, \vec{r}\, d^3r = M \vec{r}_{com}$ <br> $M \frac{d\vec{r}_{com}}{dt} = M \vec{v}_{com} = \vec{P}$ <br> $\left.\frac{d\vec{P}}{dt} = 0 \right\}$ momentum conservation <br> $\Rightarrow$ need quadrupole mass moment for GWs. |

$$I^{ij}(t) = \int d^3r \, \rho(t, \vec{r})\, r^i r^j$$

$$\boxed{\bar{h}^{ij}(t, \vec{r}) \underset{r \to \infty}{=} \frac{2}{r} \frac{d^2}{dt^2} I^{ij}(t - r)}$$



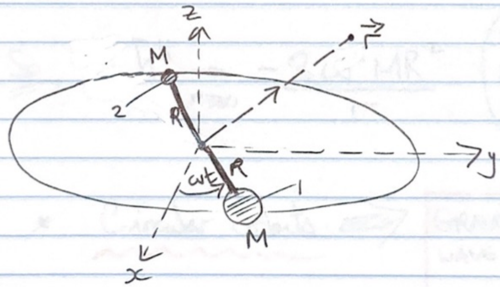
OOM: $h \propto \frac{1}{R} \cdot \frac{1}{P^2} M R^2$

using K3 $\Big\{ \propto \frac{M^{5/3} P^{-2/3}}{R}$

$$\propto \frac{M^{5/3} f_{orb}^{2/3}}{R}$$

$$h \sim 10^{-16} \left( \frac{M}{10^9 M_\odot} \right)^{5/3} \left( \frac{f_{orb}}{10\,\text{nHz}} \right)^{2/3} \left( \frac{100\,\text{Mpc}}{R} \right)$$

- Gravitational Waves From Binaries [FULL CALCULATION].




* Binary with equal mass components, M.
* Circular orbit radius, R.
* Orbital angular frequency, $\omega = 2\pi/P$.
* Interested in GWs emitted toward $\vec{r}$.

$$x_1 = R\cos\omega t \;;\; y_1 = R\sin\omega t \;;\; z_1 = 0$$
$$x_2 = -R\cos\omega t \;;\; y_2 = -R\sin\omega t \;;\; z_2 = 0$$

$$\Rightarrow I^{ij} = \int d^3x \underbrace{\mu(t,\vec{x})}_{\text{mass-distribution here are 2 delta functions centered}} x^i x^j$$

Thus;
$$I^{xx} = M(x_1)^2 + M(x_2)^2$$
$$= 2MR^2\cos^2\omega t = MR^2[1+\cos 2\omega t].$$
$$I^{xy} = Mx_1y_1 + Mx_2y_2$$
$$= 2MR^2\sin\omega t\cos\omega t = MR^2\sin 2\omega t.$$
$$I^{yy} = M(y_1)^2 + M(y_2)^2$$
$$= 2MR^2\sin^2\omega t = MR^2[1-\cos 2\omega t].$$

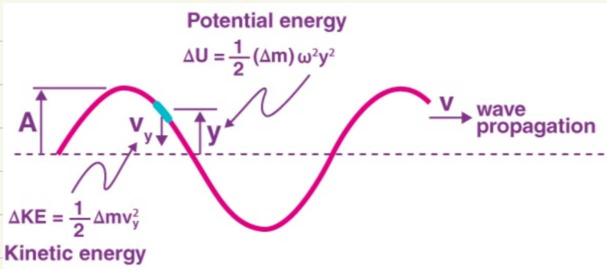
$$\Rightarrow I^{zz}, I^{zx}, I^{zy} = 0$$

* GW frequency = 2 x ORBITAL FREQUENCY

So,
$$\bar{h}^{ij} \xrightarrow[r\to\infty]{} \frac{-8\omega^2MR^2}{r} \cdot \begin{pmatrix} \cos[2\omega(t-r)] & \sin[2\omega(t-r)] & 0 \\ \sin[2\omega(t-r)] & -\cos[2\omega(t-r)] & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

* FOR CIRCULAR ORBITS

# Quadrupole Formula for GW Emission




**Power in a wave**

$$y = A\sin(kx - \omega t)$$

$$\underbrace{\langle P \rangle}_{\text{power averaged over period}} \propto \omega^2 A^2$$

OOM $\Longrightarrow$ (1) Energy flux quadratic in $A$ ($\therefore \dddot{I}^{ij}$)
(2) $\omega \propto 1/\text{time}$ implies a time derivative

$$L_{gw} = \frac{1}{5}\left\langle \dddot{I}_{ij}\,\dddot{I}^{ij} \right\rangle = \frac{128}{5} M^2 R^4 \omega^6$$

$$= \frac{128}{5}\, 4^{1/3}\, \frac{c^5}{G}\left(\frac{\pi G M}{c^3 P}\right)^{10/3}$$

- Consider total (KE + PE) in a binary orbit...

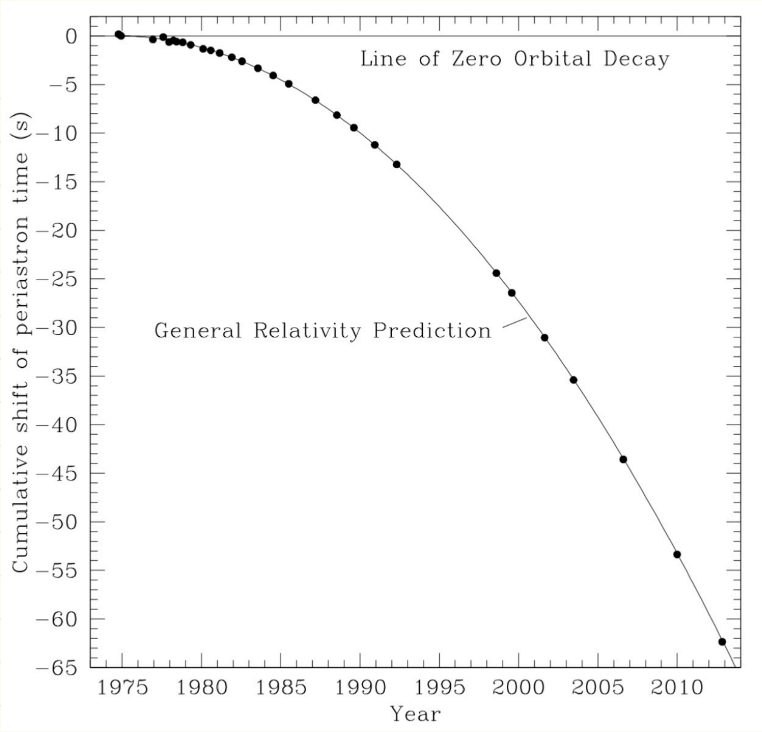
$$E_{\text{Newton}} = -M^2/4R$$

- Equate $\frac{dE_{\text{Newton}}}{dt}$ to $-L_{gw}$ ...

$$\frac{dP}{dt} = -\frac{96}{5}\pi\, 4^{1/3}\left(\frac{2\pi M}{P}\right)^{5/3}$$

negative! GWs cause orbit to shrink!

$$\frac{df_{orb}}{dt} = \frac{96}{5}\pi\, 4^{1/3}(2\pi M)^{5/3} f_{orb}^{11/3}$$

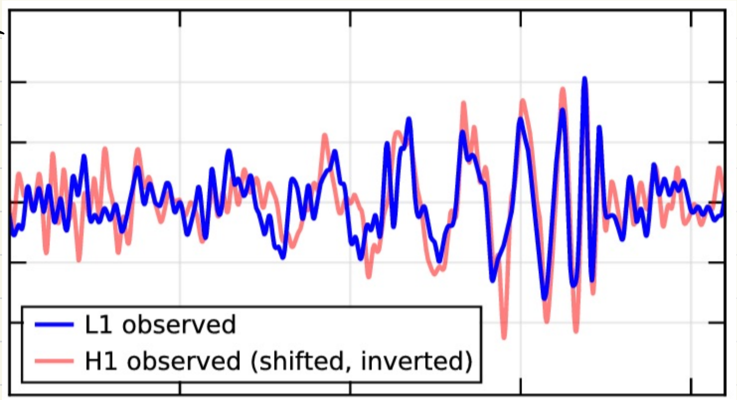



CUMULATIVE SHIFT IN PERIASTRON TIME OF BINARY SYSTEM FROM PULSAR B1913+16

$\Downarrow$

1st indirect evidence for GWs (1993 Nobel).

STRAIN SIGNAL FROM LIGO DETECTORS SHOWING FREQUENCY "CHIRP" FROM GWs




$\Downarrow$

1st direct evidence of GWs (2017 Nobel).