TTK4225 System theory, Autumn 2023

Assignment 7

The expected output is a .pdf written in LaTeX or a Python notebook exported to .pdf, even if photos of your handwritten notes or drawings will work. Every person shall hand in her/his assignment, independently of whether it has been done together with others. When dealing with mathematical derivations, unless otherwise stated, explain how you got your answer (tips: use programming aids like Python, Matlab, Maple, or compendia like Rottmann's to check if you have obtained the right answer).

Question 1

Content units indexing this question:

- free evolution
- state space system
- LTI
- diagonalizability

Characterize and draw qualitatively the free evolution of the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

from a generic initial condition in $\mathbb{R}^2$. All the derivations should be done by hand. For every quantity you use in the derivations, provide its mathematical definition, its geometrical description, plus comment what is the purpose and meaning from control perspectives.

Solution 1:

Content units indexing this solution:

- diagonalizable matrix
- eigenspaces
- modes

The first step is to find the eigenvalues-eigenspaces pairs associated to A. This means finding first the characteristic polynomial $\det(sI - A)$, factorize it, and find in this way the eigenvalues. It comes out that

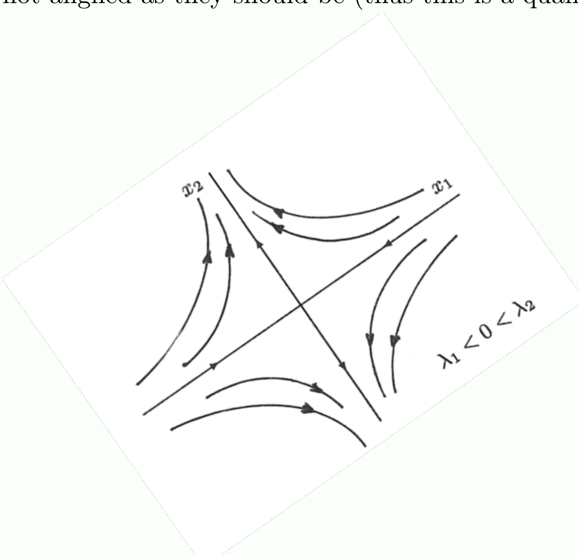
$$\det\left(\begin{bmatrix} s-2 & -1 \\ -1 & s+2 \end{bmatrix}\right) = (s+3)\,(s-3)$$

so that the eigenvalues are $\lambda_{1,2} = \pm 3$.

The second step is to compute the eigenspaces associated to each of such eigenvalues by computing $\ker(\lambda_i I - A)$. Considering that the kernel of a matrix is given by vectors that are orthogonal to its rows, we get the eigenspace associated to $\lambda_1 = +3$ to be spanned by $[-1, 1]^T$ and associated to $\lambda_2 = -3$ to be spanned by $[5, 1]^T$.

Now trajectories starting with initial conditions that are on the eigenspaces (say $\widetilde{y}_0$ units along the eigenspace) lead to trajectories that stay on the eigenspaces, with a position on the eigenspace that evolves in time like $\widetilde{y}_0 e^{\lambda_i t}$.

If instead the initial conditions are in generic position then the whole trajectory combines the two trajectories that would be obtained by oblique-projecting the initial condition on the eigenspaces. Thus something that looks like the following figure, even if it is not to scale and with the eigenspaces not aligned as they should be (thus this is a qualitative figure).



Question 2

Content units indexing this question:

- free evolution
- state space system

- LTI
- diagonalizability

Describe why in your opinion at this point of the course you cannot do the same characterization of the system above when considering

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

from a generic initial condition in $\mathbb{R}^2$, but only from a specific subset (which one, by the way?). What can you though say about the stability properties of the origin?

Solution 1:

Content units indexing this solution:

- Jordan forms
- generalized eigenspaces

A is not diagonalizable, that means that the geometric multiplicity of $\lambda = -2$ is insufficient to find a basis of R^2 that is made of eigenvectors of A. We indeed have that this geometric multiplicity is one, that means that we can find a one-dimensional subspace (actually the x axis) for which we have that starting on that subspace as initial condition the free evolution will remain on that subspace and evolve as $e^{\lambda t}$. At this point of the course we though do not know what happens if starting outside of such eigenspace. As we will see the evolution will combine the two modes $e^{\lambda t}$ and $te^{\lambda t}$. In any case the origin is thus an asymptotically stable equilibrium.

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