TTK4225 System theory, Autumn 2023

Assignment 4

The expected output is a .pdf written in LaTeX or a Python notebook exported to .pdf, even if photos of your handwritten notes or drawings will work. Every person shall hand in her/his assignment, independently of whether it has been done together with others. When dealing with mathematical derivations, unless otherwise stated, explain how you got your answer (tips: use programming aids like Python, Matlab, Maple, or compendia like Rottmann's to check if you have obtained the right answer).

Question 1

Content units indexing this question:

- kernel

Under what conditions does a square matrix have an empty kernel, i.e., there is no $\boldsymbol{v}$ such that $A\boldsymbol{v} = \mathbf{0}$?

Solution 1:

None. Every matrix admits $\mathbf{0}$ in its kernel. So a kernel cannot be empty.

Question 2

Content units indexing this question:

- convergent equilibrium

Design a system, and write it as an ODE (even in discrete time, if you prefer), for which the origin is a convergent equilibrium but **not** a marginally stable equilibrium.

Solution 1:

It certainly can't be a linear system. Then the simplest thing to do is to make a switching system such that initially the equilibrium is unstable, but after the trajectory has left a certain zone the parameters of the system change and the equilibrium becomes a global attractor. . Example:

$$\dot{x} = a(t)x$$

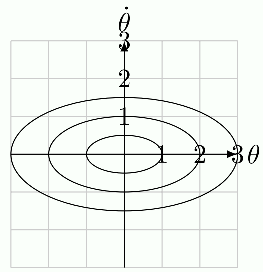
with $a = 1$ if there has never been a $\tau \leq t$ for which $|x(\tau)| > 1$, $a = -1$ otherwise.

Question 3

Content units indexing this question:

- marginal stability

Consider the following phase portrait, corresponding to the trajectories of a pendulum without friction,



and the fact that the origin is a marginally stable equilibrium of the system. Considering the definition of marginal stability based on "ε, δ", what is the largest δ that can be considered by taking $\varepsilon = 2$?

Solution 1:

$\delta_{max} = 1$, because if we start from $0, 1$ as initial condition on the edge of the ball of radius δ we arrive at $2, 0$, i.e., on the edge of the radius ballε.

Question 4

Content units indexing this question:

- Lyapunov stability

The definition for marginal stability goes as follows:

$$\forall\, \epsilon > 0\, \exists\, \delta > 0 \mid \text{if } ||x(0) - x_e|| < \delta \text{ then } ||x(t) - x_e|| < \epsilon\, \forall\, t \geq 0 \tag{1}$$

where x_e is an equilibrium.

a) How does switching ϵ and δ such that the equation definition becomes:

$$\forall\, \delta > 0\, \exists\, \epsilon > 0 \mid \text{if } ||x(0) - x_e|| < \delta \text{ then } ||x(t) - x_e|| < \epsilon\, \forall\, t \geq 0 \tag{2}$$

affect the meaning of (1)?

b) Using the incorrect stability definition (2), how does this change the stability property of the origin in a Van der Pol oscillator?

Solution 1:

a) The meaning now becomes "For every delta bigger than zero there exists an epsiolon bigger than zero such that if you start in delta, you stay in epsilon for all t greater than zero". This implies that one only needs to find one epsilon for which the case holds, rather than requiring it to hold for every epsilon greater than zero.

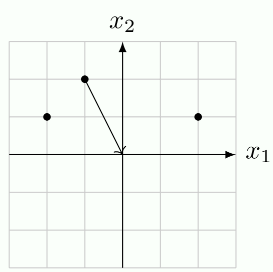
b) From a) we know that we only need to find one epsilon that contains the trajectory for all time greater than zero. Choosing an epsilon that contains the limit cycle will now result in the origin of the van der pol oscillator to be wrongly classified as a mariginally stable equilibrium.

Question 5

Content units indexing this question:

- autonomous linear time invariant systems

Consider an autonomous system of the second order for which, through three distinct experiments, it was discovered that: *a*) the point $(-2, 1)$ is an equilibrium; *b*) also the point $(2, 1)$ is an equilibrium; *c*) the point $(-1, 2)$ is not an equilibrium. Indeed starting from that initial condition the evolved following a straight path that converges to the origin. Can the system be LTI? Why?



Solution 1:

If it were LTI then since 2.1 and -2.1 are on two different subspaces, all $\mathbb{R}^2$ would have to be made up of equilibria. But this is not true, due experiment *c*. So it can't be LTI.

Question 6

Content units indexing this question:

- LTI system
- equilibria
- isolated equilibria

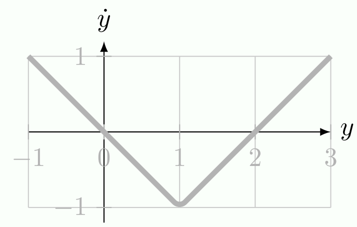
Consider a state space LTI system $\dot{\boldsymbol{x}} = A\boldsymbol{x} + Bu$. Under what condition on A does such a system admit isolated equilibria? And how many isolated equilibria are admissible?

Solution 1:

If $\ker(A) = \{0\}$ there is a single equilibrium (therefore isolated) and that is the origin.

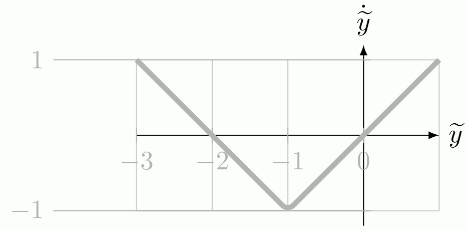
Question 7

Consider the continuous-time autonomous system described as in the figure alongside, and consider the initial condition $y_0 = 1.9$. Where will the trajectory converge? And how long will it take to converge to a 0.1-neighborhood of such convergence point?



Solution 1:

The trajectory converges to 0. We can then think of dividing the movement in two phases. In a first one, we translate the system so that the new coordinate system is as below:



In this new coordinate system, initially the trajectory moves as starting from -0.1 and following the mode e^t until it reaches -1. The time T it takes to perform this movement is given by the relation $\widetilde{y}(T) = -0.1e^T = -1$. This means $T = \ln 10$. Then for the second part of the trajectory we can refer back to the original coordinate system, and consider that at time T the new "initial" condition may be thought as being $y(0) = 1$. Now the trajectory will follow the mode e^{-t} until it reaches 0.1. The time it will take to move from 1 to 0.1 is thus given by an other relationship like the one above, more precisely $y(T') = 1e^{-T'} = 0.1$. This can then be translated into $e^{T'} = 10$, and thus $T = T'$. This in retrospect was expected, since the movement from 1.9 to 1 and the one from 1 to 0.1 in the original coordinate system are somehow symmetric. So in total it takes $2\ln 10$ unit of measure to do the movement asked in the exercise. Seconds? Hours? Months? It depends on how t is defined in the problem - in this question it has not been defined so we cannot say that one unit of measure is more correct than another.

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