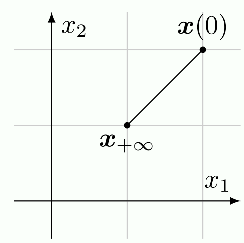
TTK4225 System theory, Autumn 2023

Assignment 6

The expected output is a .pdf written in LaTeX or a Python notebook exported to .pdf, even if photos of your handwritten notes or drawings will work. Every person shall hand in her/his assignment, independently of whether it has been done together with others. When dealing with mathematical derivations, unless otherwise stated, explain how you got your answer (tips: use programming aids like Python, Matlab, Maple, or compendia like Rottmann's to check if you have obtained the right answer).

Question 1

Consider the following trajectory, which starts from $\boldsymbol{x}(0)$ and asymptotically reaches $\boldsymbol{x}_{+\infty}$ (i.e., the system converges for $t \to +\infty$ to that point). Could this trajectory correspond to a free evolution of a linear time invariant state space system of dimension 2? Motivate the answer.



Solution 1:

For who knows linear algebra: if it moves along a line it means that it is moving along an eigenspace. So it should move following a time evolution of the type $e^{\lambda t}\boldsymbol{x}_0$. But then the trajectory either diverges, or converges to zero, or stays where it begins. So no, it cannot be a free evolution of an LTI as defined in the exercise.
Alternatively, one may think at the fact that that convergent point is an equilibrium that is along a subspace that contains that trajectory. This is impossible: if a non-null point is an equilibrium for an LTI then the whole subspace spanned by that equilibrium is made of equilibria.

Question 2

The rank-nullity theorem is a central theorem in linear algebra, and states the following:

$$\text{rank}(A) + \dim(\ker(\text{A})) = y$$

where y is the number of columns in A.

Given a square matrix $A \in \mathrm{R}^{2\times 2}$ and its eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 2$. What is its rank?

Solution 1:

From the rank-nullity theorem, we know that the rank of A, ie. the linearly independent columns of A + the dimensions of the kernel equals the number of columns of A. Knowing that $A \in \mathrm{R}^{2\times 2}$ and that $\lambda_1 = 0$, we know that the dimension of the kernel must be 1 (there is a subspace $\in \mathrm{R}^1$ of equilibrium points).

One may also view it from a different perspective: if there exists one eigenvalue $=$ 0, then that must mean that determinant of A is 0, which tells us that one of the columns is linearly dependent on the other. Since the second eigenvalue $\neq 0$ we can conclude that the rank of A must be 1.

Question 3

Consider a generic $\mathbb{R}^{3\times 2}$ matrix.

1. How may one interpret it?
2. How may one interpret its range?
3. What is the usefulness of the range from control perspectives when analysing LTI systems?

Aid all your explanations through opportune drawings.

Solution 1:

Interpretation of A: columns are the elements of the domain basis mapped into codomain; represent stretching of unitary hypercube; linear transformation that warps fabric but keeps parallel lines in the domain parallel in the codomain. Interpretation of range: elements of the codomain that can be reached. Usefulness of range: types of derivatives patterns that are naturally occurring.

Question 4

Find, in the simplest way possible, the inverse of

$$A = \begin{bmatrix} 4 & 2 & 1 \\ 3 & 2 & 1 \\ 0 & 5 & 4 \end{bmatrix}.$$

Be creative!

Solution 1:

use Matlab, or copy from a friend

Question 5

Consider a generic $\mathbb{R}^{3\times 3}$ matrix.

1. How may one interpret its kernel?
2. What is the usefulness of the kernel from control perspectives when analysing LTI systems?
3. How may one interpret its determinant?
4. What is the usefulness of the determinant from control perspectives when analysing LTI systems?

Aid all your explanations through opportune drawings.

Solution 1:

Interpretation of kernel: elements of the domain that are mapped in zero. Usefulness of kernel: defines the equilibria. Interpretation of determinant: how much the fabric of space is transformed. Usefulness of determinant: determines invertibility, determines if range is full, and thus if there are non-trivial equilibria.