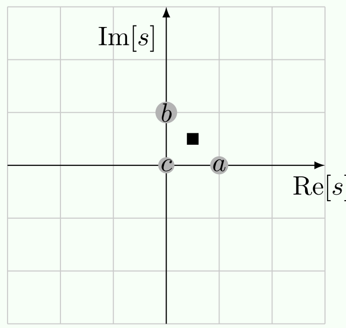
Question 1

Content units indexing this question:

- sequences
- power of complex numbers

Where will the sequence of the powers of the complex number $\blacksquare$, i.e., $\blacksquare^k$ for $k \to +\infty$, converge?



Potential answers:

I: (**wrong**) a

II: (**wrong**) b

III: (**correct**) c

IV: (**wrong**) it will diverge

V: (**wrong**) I do not know

Solution 1:

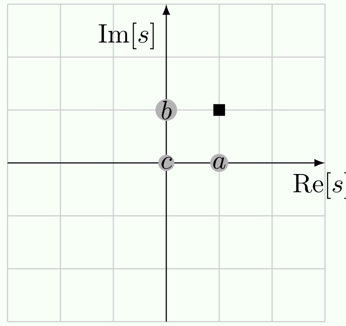
The product of two complex numbers is a complex number whose modulus is the product of the two original moduli, and phase sum of the two original phases. Thus in this case the modulus of $\blacksquare^k$ is going to be the positive number $|\blacksquare|^k$. Given that $|\blacksquare| = \frac{\sqrt{2}}{2}$ and that this number is smaller than 1, $\lim_{k \to +\infty} \left(\frac{\sqrt{2}}{2} \right)^k = 0$. Thus the correct answer is c.

Question 2

Content units indexing this question:

- sequences
- power of complex numbers

Where will the sequence of the powers of the complex number $\blacksquare$, i.e., $\blacksquare^k$ for $k \to +\infty$, converge?



Potential answers:

I: (**wrong**) a

II: (**wrong**) b

III: (**wrong**) c

IV: (**correct**) it will diverge

V: (**wrong**) I do not know

Solution 1:

The product of two complex numbers is a complex number whose modulus is the product of the two original moduli, and phase sum of the two original phases. Thus in this case the modulus of $\blacksquare^k$ is going to be the positive number $|\blacksquare|^k$. Given that $|\blacksquare| = \sqrt{2}$ and that this number is bigger than 1, $\lim_{k \to +\infty} \left(\sqrt{2}\right)^k = +\infty$. Since the phase moreover will increase with k, the correct answer is that this sequence diverges.

Question 3

Content units indexing this question:

- continuous-time LTI systems
- impulse response
- nonlinear systems

One may use the concept of "impulse response" to describe a nonlinear system.

Potential answers:

I: (**wrong**) true

II: (**correct**) false

III: (**wrong**) it depends on the nonlinear system

IV: (**wrong**) I do not know

Solution 1:

Responding wrongly to this question means making one of the most serious mistakes one can do in this course, and should not be taken lightly. If one is convinced that one can talk about the impulse response of a nonlinear system then that person is very off-track.
The impulse response is a concept that can be used only in the context of linear time-invariant (LTI) systems. In linear systems, the principle of superposition holds, meaning that the response to a sum of inputs is equal to the sum of the responses to each individual input. This property allows for a straightforward and useful characterization of system behavior through the impulse response and the convolution operator.
The impulse response of an LTI system is the system's output when the input is an impulse function (or Dirac delta function). It is a fundamental concept because any input to the system can be represented as a sum (or integral) of scaled and shifted impulse functions. The convolution integral between the input signal and the impulse response gives the system's output.
Nonlinear systems, on the other hand, do not obey the principle of superposition. In a nonlinear system, the response to a sum of inputs is generally not equal to the sum of the responses to each individual input. This lack of superposition makes it challenging to define a unique and useful impulse response for a nonlinear system.
The response of a nonlinear system depends on the specific form of the nonlinearity, and the behavior may not be easily characterized by a simple impulse response function. Nonlinear systems can exhibit complex and varied behaviors, including limit cycles, bifurcations, and chaotic dynamics, which are not captured by the linear impulse response framework.
In summary, while the concept of impulse response is powerful and widely used for linear systems, it is not applicable to nonlinear systems due to the lack of superposition in the latter. Instead, nonlinear systems are often analyzed using other tools and techniques, such as bifurcation diagrams, phase portraits, and numerical simulations.

Question 4

Content units indexing this question:

- continuous-time LTI systems
- transfer function
- nonlinear systems

One may use the concept of "transfer function" to describe a nonlinear system.

Potential answers:

I: **(wrong)** true

II: **(correct)** false

III: **(wrong)** it depends on the nonlinear system

IV: **(wrong)** I do not know

Solution 1:

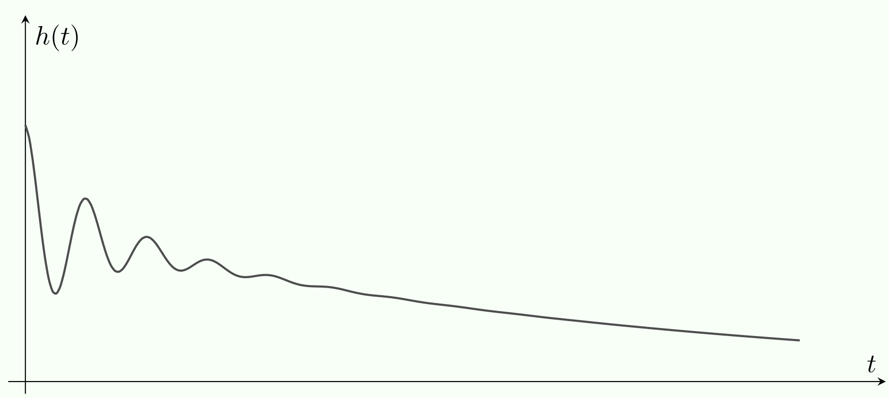
Responding wrongly to this question means making one of the most serious mistakes one can do in this course, and should not be taken lightly. If one is convinced that one can talk about the transfer function of a nonlinear system then that person is very off-track.
The transfer function is a concept that is specifically defined for linear time-invariant (LTI) systems. It describes the relationship between the input and output of a system in the frequency domain, and it's a fundamental tool for analyzing and designing linear systems.
The transfer function is derived from the Laplace transform of the system's impulse response, and it represents the system's response to complex exponential inputs. Importantly, it relies on the principle of superposition, which states that the response to a sum of inputs is equal to the sum of the responses to each individual input. This property is a key characteristic of linear systems.
Nonlinear systems, in contrast, do not satisfy the principle of superposition. In a nonlinear system, the response to a sum of inputs is generally not equal to the sum of the responses to each individual input. This fundamental difference makes it inappropriate and incorrect to talk about the transfer function of a nonlinear system.
In nonlinear systems, the relationship between input and output is typically characterized by nonlinear equations, and the system's behavior can be much more complex compared to linear systems. Nonlinear systems can exhibit phenomena such as bifurcations, limit cycles, chaos, and other nonlinear dynamics that are not captured by the linear transfer function framework.
While linear systems are conveniently analyzed and designed using transfer functions, nonlinear systems often require different tools and methods, such as state-space representation, phase portraits, and numerical simulations, to capture and understand their complex behavior. In summary, the transfer function is a concept applicable only to linear systems, and attempting to apply it to nonlinear systems can lead to incorrect results.

Question 5

Content units indexing this question:

- continuous-time LTI systems
- modal analysis
- third order systems
- impulse response

Which type of LTI system may produce the impulse response $h(t)$ represented in the picture?



Potential answers:

I: (**wrong**) first order

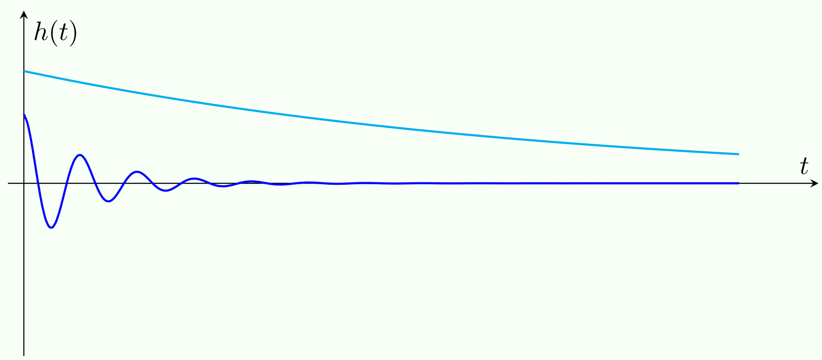
II: (**wrong**) second order

III: (**correct**) at least third order

IV: (**wrong**) I do not know

Solution 1:

Looking at the graph of $h(t)$, we can notice two different behaviours: the first part behaves like $e^{\alpha t}\cos(\omega t)$ and it decays way faster than the second part, which behaves like $e^{\beta t}$. Here both α and β are negative real numbers, with $\alpha < \beta$. Hence this impulse response associates with two modes, the oscillating one relating to a second order subsystem, and the non-oscillating one relating to a first order subsystem. Thus the correct is that the system is at least third order.

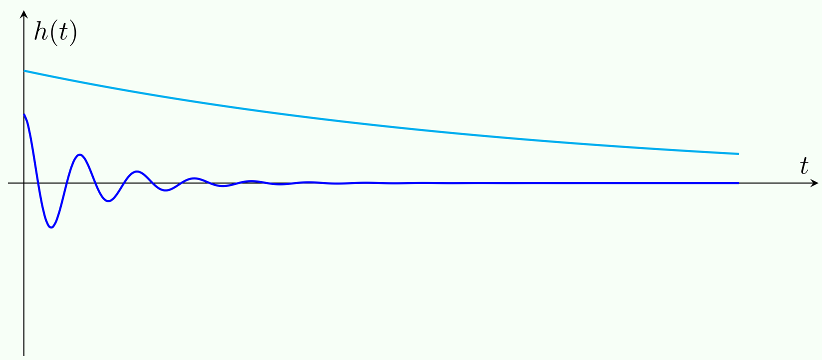


Solution 2:

Content units indexing this solution:

- transfer function

Looking at the graph of $h(t)$, we can notice two different behaviours: the first part behaves like $e^{\alpha t}\cos(\omega t)$ and it decays way faster than the second part, which behaves like $e^{\beta t}$. Hence this impulse response associates with two modes whose Laplace transforms indicate a transfer function with two complex conjugates stable poles on the left of a real stable pole. Given that we can identify at least 3 distinct poles, it means that the system is of order at least three.

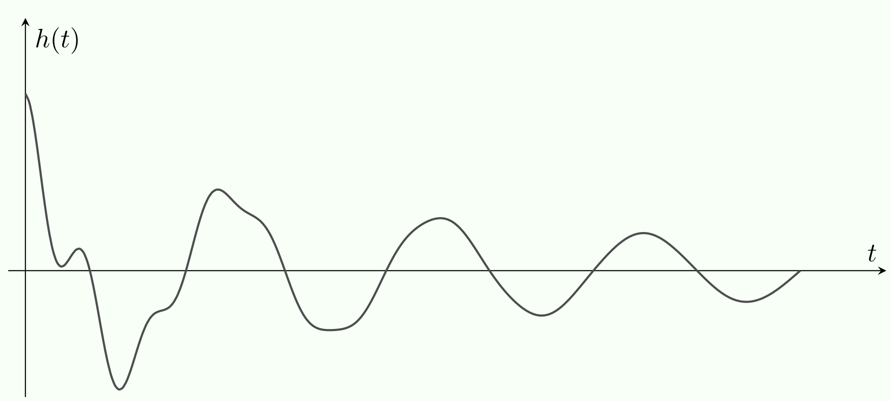


Question 6

Content units indexing this question:

- continuous-time LTI systems
- fourh order systems
- impulse response
- modal analysis

Which type of LTI system may produce the impulse response $h(t)$ represented in the picture?



Potential answers:

I: (**wrong**) first order

II: (**wrong**) second order

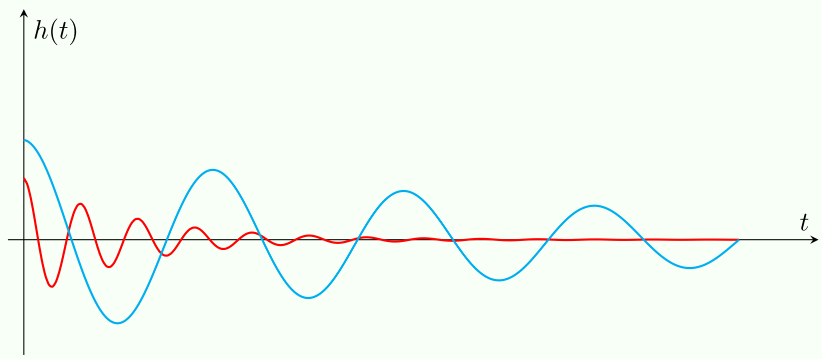
III: (**wrong**) third order

IV: (**correct**) at least fourth order

V: (**wrong**) I do not know

Solution 1:

The impulse response may be decomposed as a sum of two decaying oscillatory behaviors, i.e., as $h(t) = e^{\alpha t} \cos(\omega_1 t) + e^{\beta t} \cos(\omega_2 t)$, as in the figure below. The first part $e^{\alpha t} \cos(\omega_1 t)$ decays faster than the second part $e^{\beta t} \cos(\omega_2 t)$, and is also associated to a cosine oscillating at a higher frequency than the second. Hence this impulse response associates with two modes, both relating to a second order subsystem. Thus the correct answer is a system whose order is at least four.

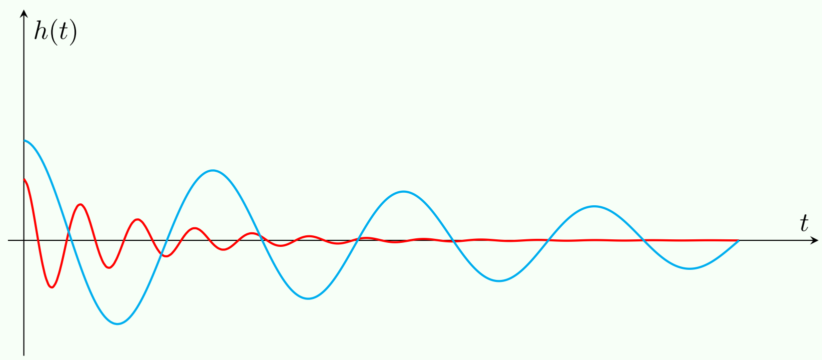


Solution 2:

Content units indexing this solution:

- transfer function

The impulse response may be decomposed as a sum of two decaying oscillatory behaviors, i.e., as $h(t) = e^{\alpha t} \cos(\omega_1 t) + e^{\beta t} \cos(\omega_2 t)$, as in the figure below. The first part $e^{\alpha t} \cos(\omega_1 t)$ decays faster than the second part $e^{\beta t} \cos(\omega_2 t)$, and is also associated to a cosine oscillating at a higher frequency than the second. Hence this impulse response associates with two modes whose Laplace transforms indicate a transfer function with two complex conjugates stable poles on the left and with a bigger imaginary part of an other pair of complex conjugate stable poles. Given that we can identify at least 4 distinct poles, it means that the system is of order at least four.

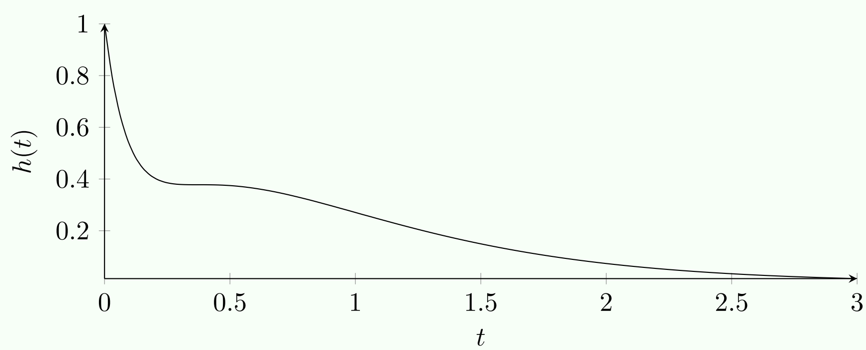


Question 7

Content units indexing this question:

- continuous-time LTI systems
- modal analysis
- third order systems
- impulse response

Which type of LTI system may produce the impulse response $h(t)$ represented in the picture?



Potential answers:

I: **(wrong)** first order

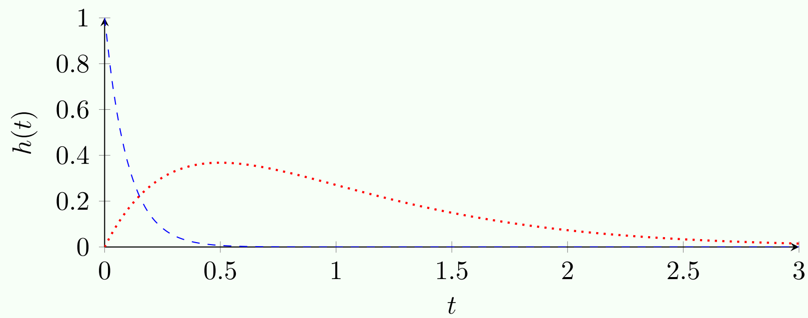
II: **(wrong)** second order

III: **(correct)** at least third order

IV: **(wrong)** I do not know

Solution 1:

Looking at the graph of $h(t)$, we decompose it in the sum of two different modes: the first one behaves like $e^{\alpha t}$, and the second part behaves like $te^{\beta t}$.



Indeed $h(t)$ above is given by the expression

$$\exp(-10t) + 2t\exp(-2t).$$

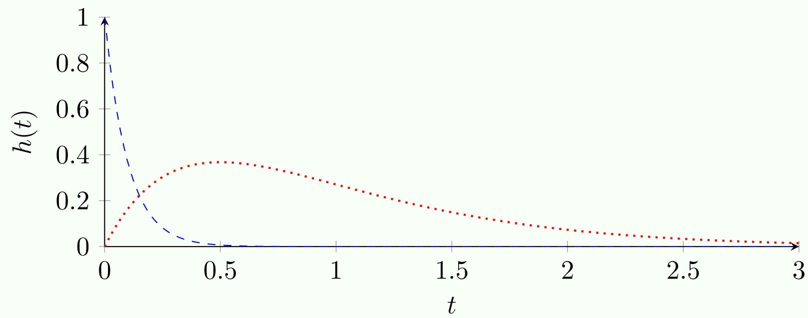
Hence this impulse response associates with two modes, one relating to a second order subsystem and one relating to a first order system. Thus the correct answer is a system whose order is at three.

Solution 2:

Content units indexing this solution:

- transfer function

Looking at the graph of $h(t)$, we decompose it in the sum of two different modes: the first one behaves like $e^{\alpha t}$, and the second part behaves like $te^{\beta t}$.



Indeed $h(t)$ above is given by the expression

$$\exp(-10t) + 2t\exp(-2t).$$

Hence this impulse response associates with two modes whose Laplace transforms indicate a transfer function with a single real stable pole on the left of a double real stable pole. Given that we can identify at least 2 distinct poles, one with multiplicity 2, it means that the system is of order at least three. Indeed, working with Laplace transforms, the transfer function has to be, from a structural point of view,

$$H(s) = \frac{n(s)}{(s-\alpha)(s-\beta)^2}$$

with $n(s)$ a polynomial in s that is not easily identifiable given just a qualitative plot.

Question 8

Content units indexing this question:

- asymptotic stability
- continuous-time LTI systems
- autonomous systems
- asymptotically stable equilibrium
- free evolution

For which value of a are the equilibria of the continuous-time autonomous LTI system $\dot{y} = ay$ asymptotically stable?

Potential answers:

I: (**correct**) $a < 0$

II: (**wrong**) $a \leq 0$

III: (**wrong**) $a = 0$

IV: (**wrong**) $a \geq 0$

V: (**wrong**) $a > 0$

VI: (**wrong**) I do not know

Solution 1:

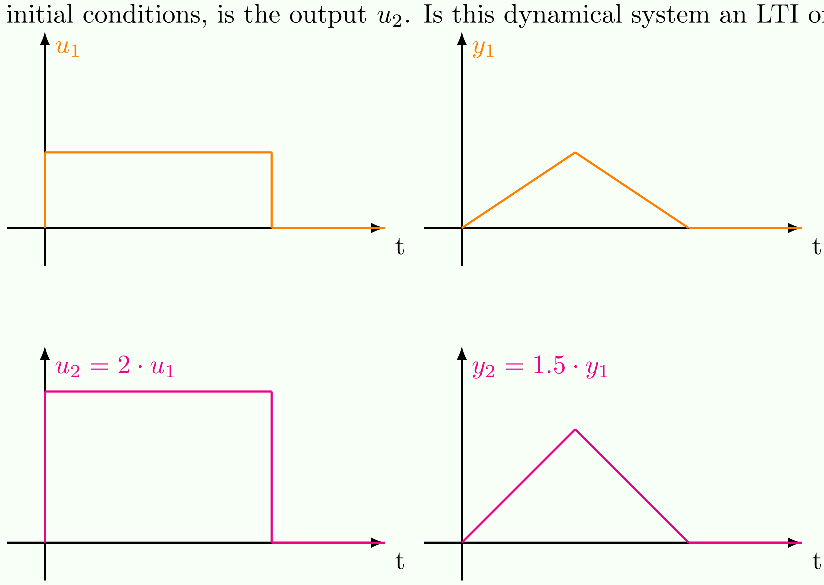
The free evolution associated to this ODE is, given a generic initial condition y_0, given by $y(t) = y_0 e^{at}$. If $a \neq 0$, then the unique equilibrium is $y_{\mathrm{eq}} = 0$. Otherwise, if $a = 0$, then any value in $\mathbb{R}$ is an equilibrium (and more precisely a marginally stable one). So to have asymptotic stability one needs $a < 0$.

Question 9

Content units indexing this question:

- linearity
- superposition principle

Consider a dynamical system whose response to the input u_1 below, starting from null initial conditions, is the output y_1. Consider also that the response of this system to the input u_2 below, again starting from null initial conditions, is the output u_2. Is this dynamical system an LTI one?



Potential answers:

I: (**wrong**) yes

II: (**correct**) no

III: (**wrong**) it depends on the actual values of u_1 and y_1

IV: (**wrong**) I do not know

Solution 1:

The system doesn't satisfy the superposition principle, since $u_2 = 2u_1 = u_1 + u_1$ produces the output $y_2 = 1.5y_1 \neq y_1 + y_1 = 2y_1$. Was the system an LTI one, then the second output should indeed have been $2y_1$.

Question 10

Content units indexing this question:

- impulse response
- first order systems
- continuous-time LTI systems

The impulse response associated to the system $\dot{y} = -0.5y + 3u$ is equal to ...

Potential answers:

I: (**wrong**) $e^{0.5t}$

II: (**wrong**) $e^{-0.5t}$

III: (**wrong**) $0.5e^{0.5t}$

IV: (**wrong**) $-0.5e^{-0.5t}$

V: (**wrong**) $3e^{0.5t}$

VI: (**wrong**) $3e^{-0.5t}$

VII: (**wrong**) $e^{0.5t}$ for $t \geq 0$, 0 otherwise

VIII: (**wrong**) $e^{-0.5t}$ for $t \geq 0$, 0 otherwise

IX: (**wrong**) $0.5e^{0.5t}$ for $t \geq 0$, 0 otherwise

X: (**wrong**) $-0.5e^{-0.5t}$ for $t \geq 0$, 0 otherwise

XI: (**wrong**) $3e^{0.5t}$ for $t \geq 0$, 0 otherwise

XII: (**correct**) $3e^{-0.5t}$ for $t \geq 0$, 0 otherwise

XIII: (**wrong**) I do not know

Solution 1:

In general the impulse response of the ODE $\dot{y} = ay + bu$ is $h(t) = be^{at}$ for $t \geq 0$, 0 otherwise, thus the solution.

Question 11

Content units indexing this question:

- impulse response
- LTI systems

The impulse response of a LTI system contains all the information that is needed to compute the trajectories of that system for every input u and initial condition y_0.

Potential answers:

I: (**correct**) true

II: (**wrong**) false

III: (**wrong**) it depends

IV: (**wrong**) I do not know

Solution 1:

Content units indexing this solution:

- free evolution
- forced response

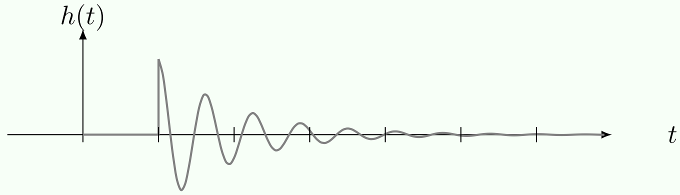
An LTI system benefits from the superposition principle, so that its total output is always expressible as the sum of its forced response and the free evolution. With the impulse response one can compute explicitly both the free evolution for any initial condition and the forced response for any input (the latter through the convolution operator). The answer is thus true, in the sense that knowing $h(t)$ provides a mean to compute the trajectories of the associated system.

Question 12

Content units indexing this question:

- impulse response
- Dirac delta
- delay
- continuous-time LTI systems

Consider the impulse response $h(t)$ given by the plot below, where the distance between consecutive marks in the axes indicate one unit.



Assume that for $t < 0$ the LTI system characterized by this impulse response is in equilibrium, i.e., that $y(t) = 0$ for $t < 0$, and that and also $u(t) = 0$ for $t < 0$. Assume then $u(t)$ to be a Dirac delta centered in $t - 10$, i.e., $u(t) = \delta(t - 10)$. Then the output of the system at time 10.0001 is ...

Potential answers:

I: **(wrong)** $y(10.0001) < 0$

II: **(correct)** $y(10.0001) = 0$

III: **(wrong)** $y(10.0001) > 0$

IV: **(wrong)** I do not know

Solution 1:

The input is an impulse centered at time $t = 10$. Since the system is an LTI, its output will thus be the output that one would obtain with an impulse centered in 0 but delayed of 10. In other words, the total response of the system in this case will be its impulse response delayed of 10. Since the impulse response has a delay too of 1 time unit (since starting being different from zero from $t \geq 1$), this impulse-response-induced delay somehow 'adds up' to the input-signal-delay of 10 mentioned before. Thus the correct answer is $y(10.0001) = 0$.

Solution 2:

Content units indexing this solution:

- convolution

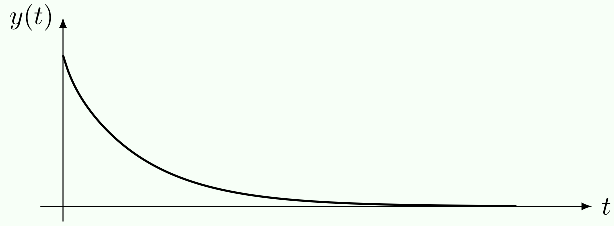
The input is an impulse centered at time $t = 10$. The impulse response has moreover a delay too of 1 time unit (since starting being different from zero from $t \geq 1$). When applying the convolution operator, the product $u(10.0001 - \tau)h(\tau)$ is 0 everywhere. Thus the correct answer is $y(10.0001) = 0$.

Question 13

Content units indexing this question:

- overdamping
- underdamping
- continuous-time system

The following response



corresponds to a . . . response.

Potential answers:

I: (**wrong**) underdamped

II: (**correct**) overdamped

III: (**wrong**) I do not know

Solution 1:

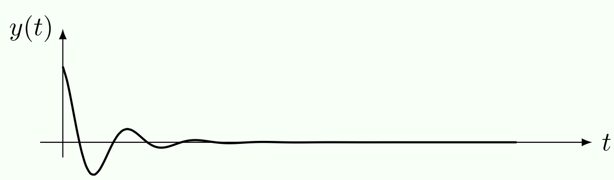
Here the output exhibits no oscillations, so it is an overdamped one.

Question 14

Content units indexing this question:

- overdamping
- underdamping
- continuous-time system

The following response



corresponds to a . . . response

Potential answers:

I: (**correct**) underdamped

II: (**wrong**) overdamped

III: (**wrong**) I do not know

Solution 1:

Here the output exhibits oscillations, so it is an underdamped one.

Question 15

Content units indexing this question:

- convolution
- continuous-time LTI systems

The convolution of a rectangular signal with itself leads to . . .

Potential answers:

I: (**wrong**) another rectangle

II: (**correct**) a triangle

III: (**wrong**) a trapezoid

IV: (**wrong**) it depends on the length of the rectangle

V: (**wrong**) I do not know

Solution 1:

The solution follows immediately considering how to compute the convolution of two continuous time signals.

Question 16

Content units indexing this question:

- convolution
- linearity

Convolution is a nonlinear operator.

Potential answers:

I: (**wrong**) true

II: (**correct**) false

III: (**wrong**) it depends on the actual signals that are convolved

IV: (**wrong**) I do not know

Solution 1:

Consider for example the convolution of two continuous time scalar signals - extensions to discrete or multidimensional convolutions being immediate - and thus the operator defined by

$$u * h(T) = \int_{-\infty}^{+\infty} u(T-\tau)h(\tau)d\tau.$$

Let then $h(t) = \alpha h_1(t) + \beta h_2(t)$, so that

$$u * h(T) = \int_{-\infty}^{+\infty} u(T-\tau)h(\tau)d\tau = \int_{-\infty}^{+\infty} u(T-\tau)\Big(\alpha h_1(\tau) + \beta h_2(\tau)\Big)d\tau.$$

Since integrals are linear operators,

$$\int_{-\infty}^{+\infty} u(T-\tau)\Big(\alpha h_1(\tau) + \beta h_2(\tau)\Big)d\tau = \alpha \int_{-\infty}^{+\infty} u(T-\tau)\alpha h_1(\tau)d\tau + \beta \int_{-\infty}^{+\infty} u(T-\tau)h_2(\tau)d\tau.$$

and this means that

$$u * (\alpha h_1 + \beta h_2)(T) = \alpha u * h_1(T) + \beta u * h_2(T),$$

i.e., convolution is a linear operator too.

Question 17

Content units indexing this question:

- equilibrium
- continuous-time systems

The equilibria of the system

$$\dot{x} = f(x) = x^2 - 2x - 3$$

are ...

Potential answers:

I: (**wrong**) -1

II: (**wrong**) 3

III: (**correct**) both -1 and 3

IV: (**wrong**) I do not know

Solution 1:

We can rewrite the system as

$$\dot{x} = f(x) = x^2 - 2x - 3 = (x-3)(x+1)$$

and thus finding its equilibria corresponds to set $(x-3)(x+1) = 0$. In other words the equilibria are all the zeros of that polynomial, thus

$$\text{equilibria: } \begin{cases} \overline{x}_1 = -1 \\ \overline{x}_2 = 3 \end{cases}$$

Question 18

Content units indexing this question:

- continuous-time systems
- nonlinear systems
- finite escape time
- bounded signal

Consider the dynamics $\dot{x} = x^2$, and the trajectory corresponding to $x_0 = c$, given by

$$x(t) = \frac{c}{1-t}.$$

This trajectory ...

Potential answers:

I: (**wrong**) is bounded

II: (**wrong**) diverges to $+\infty$

III: (**correct**) presents a finite escape time

IV: (**wrong**) I do not know

Solution 1:

As for how to compute the solution to this differential equation, consider that $\dot{x} = x^2$ implies $\frac{dx}{dt} = x^2$, thus $\int \frac{1}{x^2} dx = \int dt$, thus $-\frac{1}{x} = t + c$, implying $x = \frac{-1}{t+c}$. Now setting $x_0 = 1$ means setting $1 = \frac{-1}{0+c}$ and thus $c = -1$. Now the signal $\frac{c}{1-t}$ for $t \to 1$ diverges to either $+\infty$ or $-\infty$, depending on the sign of c (and assuming time to reach 1 'from the left'). Thus the option "presents a finite escape time" is more precise than the "diverges to $+\infty$" one.

Question 19

Content units indexing this question:

- simply stable equilibrium

The following definition of simple stability for an equilibrium is correct:

$$\boldsymbol{y}_{eq} \textit{ is simply stable if } \forall \delta > 0 \; \exists \varepsilon > 0 \textit{ s.t. if } \|\boldsymbol{y}_0 - \boldsymbol{y}_{eq}\| \leq \delta \textit{ then } \|\boldsymbol{y}(t) - \boldsymbol{y}_{eq}\| \leq \varepsilon \quad \forall t \geq 0$$

Potential answers:

I: **(wrong)** true

II: **(correct)** false

III: **(wrong)** it depends

IV: **(wrong)** I do not know

Solution 1:

In words, the definition for simple stability reads as *for every outer ball there must exist an inner ball such that if starting in the inner ball then the trajectory stays within the outer.* For how the mathematical definition in the body of the exercise starts, it seems that the symbol δ should be associated to the outer ball, and the symbol ε to the inner one. But if so then the second part of the definition is clearly wrong, since the roles of the δ and ε symbols are then inverted after the "s.t.".

Question 20

Content units indexing this question:

- simply stable equilibrium

The following definition of simple stability is correct:

$$\boldsymbol{y}_{eq} \textit{ is simply stable if } \exists \varepsilon > 0 \; \forall \delta > 0 \textit{ s.t. if } \|\boldsymbol{y}_0 - \boldsymbol{y}_{eq}\| \leq \delta \textit{ then } \|\boldsymbol{y}(t) - \boldsymbol{y}_{eq}\| \leq \varepsilon \quad \forall t \geq 0$$

Potential answers:

I: **(wrong)** true

II: **(correct)** false

III: **(wrong)** it depends

IV: **(wrong)** I do not know

Solution 1:

In words, the definition for simple stability reads as *for every outer ball there must exist an inner ball such that if starting in the inner ball then the trajectory stays within the outer.* For how the mathematical definition in the body of the exercise starts, it uses the symbol $\exists$ instead of $\forall$. Thus this definition clearly starts in a wrong way, and it makes no sense.

Question 21

Content units indexing this question:

- simply stable equilibrium

The following definition of simple stability is correct:

$$\boldsymbol{y}_{eq} \textit{ is simply stable if } \forall \varepsilon > 0 \ \exists \delta > 0 \textit{ s.t. if } \|\boldsymbol{y}_0 - \boldsymbol{y}_{eq}\| \leq \varepsilon \textit{ then } \|\boldsymbol{y}(t) - \boldsymbol{y}_{eq}\| \leq \delta \quad \forall t \geq 0$$

Potential answers:

I: **(wrong)** true

II: **(correct)** false

III: **(wrong)** it depends

IV: **(wrong)** I do not know

Solution 1:

In words, the definition for simple stability reads as *for every outer ball there must exist an inner ball such that if starting in the inner ball then the trajectory stays within the outer.* For how the mathematical definition in the body of the exercise starts, it seems that the symbol ε should be associated to the outer ball, and the symbol δ to the inner one. But if so then the second part of the definition is clearly wrong, since the roles of the δ and ε symbols are then inverted after the "s.t.".

Question 22

Content units indexing this question:

- simply stable equilibrium

The following definition of simple stability is correct:

$$\boldsymbol{y}_{eq} \textit{ is simply stable if } \forall \varepsilon > 0 \ \exists \delta > 0 \textit{ s.t. if } \|\boldsymbol{y}_0 - \boldsymbol{y}_{eq}\| \leq \delta \textit{ then } \|\boldsymbol{y}(t) - \boldsymbol{y}_{eq}\| \leq \varepsilon \quad \forall t \geq 0$$

Potential answers:

I: (**correct**) true

II: (**wrong**) false

III: (**wrong**) it depends

IV: (**wrong**) I do not know

Solution 1:

In words, the definition for simple stability reads as *for every outer ball there must exist an inner ball such that if starting in the inner ball then the trajectory stays within the outer.* For how the mathematical definition in the body of the exercise starts, it seems that the symbol ε should be associated to the outer ball, and the symbol δ to the inner one. The second part of the definition respects the roles of the δ and ε symbols. This definition is thus correct.

Question 23

Content units indexing this question:

- equilibria
- continuous-time LTI systems
- state space representations
- kernel

The origin $(\boldsymbol{u}, \boldsymbol{y}) = (\boldsymbol{0}, \boldsymbol{0})$ is always an equilibrium for a LTI system of the type $\dot{\boldsymbol{y}} = A\boldsymbol{y} + B\boldsymbol{u}$.

Potential answers:

I: (**correct**) true

II: (**wrong**) false

III: (**wrong**) it depends

IV: (**wrong**) I do not know

Solution 1:

Substituting $(\boldsymbol{u}, \boldsymbol{y}) = (\boldsymbol{0}, \boldsymbol{0})$ in the equation leads naturally to $\dot{\boldsymbol{y}} = \boldsymbol{0}$, thus the answer is true. Actually this is also true for all the pairs $(\boldsymbol{u}, \boldsymbol{y})$ where simultaneously $\boldsymbol{y}$ is in the kernel of A and $\boldsymbol{u}$ is in the kernel of B.

Question 24

Content units indexing this question:

- equilibria
- nonlinear systems

The origin is always an equilibrium for a generic system of the type $\dot{\boldsymbol{y}} = \boldsymbol{f}(\boldsymbol{y}, \boldsymbol{u})$.

Potential answers:

I: **(wrong)** true

II: **(correct)** false

III: **(wrong)** it depends

IV: **(wrong)** I do not know

Solution 1:

This is not true in general, and we can verify this by making an example. E.g., $\dot{y} = (y+1)^2$ is so that $y = 0, u = 0$ is not an equilibrium.

Question 25

Content units indexing this question:

- kernel
- matrix

If one says that the matrix A has a trivial kernel, what does this mean?

Potential answers:

I: **(wrong)** $\mathrm{Ker}(A) = 0$

II: **(correct)** $\mathrm{Ker}(A) = \{0\}$

III: **(wrong)** it depends

IV: **(wrong)** I do not know

Solution 1:

The name indicates the fact that the matrix maps only one element from the domain into the zero of the codomain. The correct answer is $\mathrm{Ker}(A) = \{0\}$, because the kernel is a set since it is a subspace of the domain. Writing $\mathrm{Ker}(A) = 0$ indicates that the kernel is an element of the domain, and this is improper - the kernel is a subspace, not an element.

Question 26

Content units indexing this question:

- LTI systems
- equilibria

Can an autonomous LTI system of the type $\dot{\boldsymbol{x}} = A\boldsymbol{x}$ have equilibria of different types? (i.e., have a $\boldsymbol{x}_1$ that is an asymptotically stable equilibrium for the system, another $\boldsymbol{x}_2$ that is instead a marginally stable equilibrium, etc.)

Potential answers:

I: **(wrong)** yes

II: **(correct)** no

III: **(wrong)** it depends

IV: **(wrong)** I do not know

Solution 1:

The set of equilibria for the system is given by the kernel of A. This kernel may be trivial, i.e., formed by just an element, or a proper subspace of the domain with dimension bigger or equal than 1. For sure $\mathbf{0}$ is in the kernel of A, i.e., it is for sure an equilibrium for the system.

Case 1: assume the kernel of A to be trivial. In this case the system has only one type of equilibria since there is only one equilibrium. Note that this implies the kernel of A to be the set composed by only the element $\mathbf{0}$ (note that $0 \neq \mathbf{0}$, the former is in $\mathbb{C}$, the second is the origin of the domain and thus multidimensional as soon as A is not a scalar).

Case 2: assume the kernel of A to be non-trivial, that implies that there exists at least one eigenvalue of A that is equal to 0.

Then every $\overline{\boldsymbol{x}}$ in the kernel cannot be an asymptotically stable equilibrium, since it should in this case be also convergent. Convergency would indeed require that there exists a $\delta > 0$ so that for any initial condition in the δ-ball centered in $\overline{\boldsymbol{x}}$ the limit of the trajectory for $t \to +\infty$ is $\overline{\boldsymbol{x}}$. But this δ-ball intersects the kernel, that is non trivial and thus contains other elements than $\overline{\boldsymbol{x}}$, and these elements are equilibria. Starting from these equilibria the trajectory won't move and thus won't converge to $\overline{\boldsymbol{x}}$.

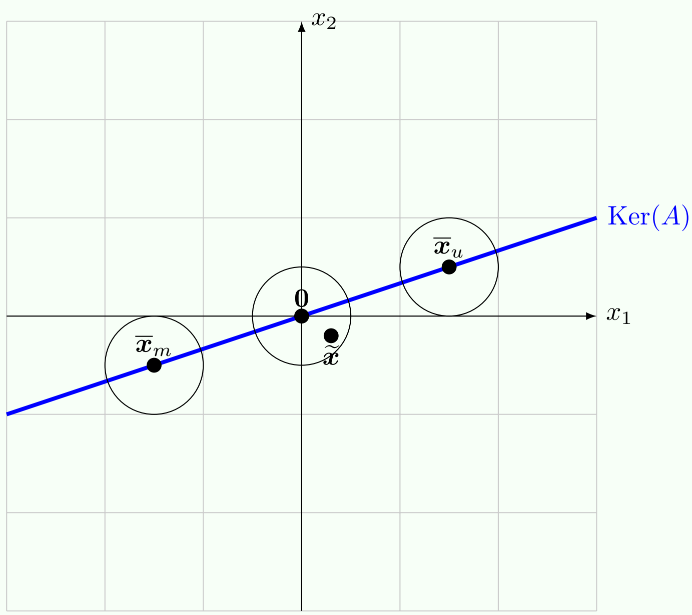
If $\overline{\boldsymbol{x}}$ is in the kernel of A thus as an equilibrium it can be either marginally stable or unstable.

Say then that *ab absurdo* the system is so that $\overline{\boldsymbol{x}}_m$ and $\overline{\boldsymbol{x}}_u$ are two distinct vectors both in the kernel of A, and the former is marginally stable while the latter is unstable.

But then this conflicts with the concept that the trajectories starting around the origin $\mathbf{0}$ form a sort of template for the trajectories starting around the elements in the kernel of A, because of the superposition principle. As in the figure below, consider the $\widetilde{\boldsymbol{x}}$ in the circle drawn around the origin. Assume $\mathbf{0}$ to be marginally stable; starting from $\widetilde{\boldsymbol{x}}$ as initial condition, the trajectory will be confined in another opportune ball (not drawn here). Then consider the initial condition $\widetilde{\boldsymbol{x}} + \overline{\boldsymbol{x}}_u$. Due to the superposition effect the free evolution has to be the sum of the evolution from $\widetilde{\boldsymbol{x}}$ plus the evolution from $\overline{\boldsymbol{x}}_u$. The latter is just $\overline{\boldsymbol{x}}_u$, since the point is an equilibrium, while the former is the trajectory we were seeing around $\mathbf{0}$.

Since this reasoning can be made in the opposite direction, if $\overline{\boldsymbol{x}}_u$ is unstable it means that there is some initial condition $\widetilde{\boldsymbol{x}} + \overline{\boldsymbol{x}}_u$ in some neighborhood of $\overline{\boldsymbol{x}}_u$ that diverges. But then $\mathbf{0}$ could not be marginally stable.

Summarizing, all the equilibria along the kernel of A share the same stability properties of $\mathbf{0}$.



Question 27

Content units indexing this question:

- BIBO stability
- impulse response
- continuous time LTI systems

Is the continuous time LTI system characterized by the impulse response

$$h(t) = \begin{cases} e^{-2t} & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

BIBO stable?

Potential answers:

I: (**correct**) yes

II: (**wrong**) no

III: (**wrong**) it depends

IV: (**wrong**) I don't know

Solution 1:

Content units indexing this solution:

- absolute integrability

This impulse response is absolutely integrable, i.e., so that $\int_{-\infty}^{+\infty} |h(t)| dt < +\infty$. In this specific case we indeed have

$$\int_{0}^{+\infty} |e^{-2t}| dt = \left[-\frac{1}{2} e^{-2t} \right]_{0}^{+\infty} = 0.5,$$

thus finite and thus BIBO.

Question 28

Content units indexing this question:

- BIBO stability
- impulse response

Is the continuous time LTI system characterized by the impulse response

$$h(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

BIBO stable?

Potential answers:

I: (**wrong**) yes

II: (**correct**) no

III: (**wrong**) it depends

IV: (**wrong**) I don't know

Solution 1:

Content units indexing this solution:

- absolute integrability

This impulse response is not absolutely integrable, i.e., not so that $\int_{-\infty}^{+\infty} |h(t)| dt < +\infty$. In this specific case we indeed have

$$\int_{0}^{+\infty} |1| dt = +\infty,$$

thus infinite and thus not BIBO.

Question 29

Content units indexing this question:

- BIBO stability
- impulse response
- continuous time LTI systems

Is the continuous time LTI system characterized by the impulse response

$$h(t) = \begin{cases} \frac{1}{t+1} & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

BIBO stable?

Potential answers:

I: (**wrong**) yes

II: (**correct**) no

III: (**wrong**) it depends

IV: (**wrong**) I don't know

Solution 1:

Content units indexing this solution:

- absolute integrability

This impulse response is not absolutely integrable, i.e., not so that $\int_{-\infty}^{+\infty} |h(t)| dt < +\infty$. In this specific case we indeed have

$$\int_0^{+\infty} \frac{1}{t+1} = [\log(t+1)]_0^{+\infty} = +\infty,$$

thus infinite and thus not BIBO.

Question 30

Content units indexing this question:

- impulse response
- continuous time LTI systems
- rational transfer function

Is the transfer function corresponding to the impulse response

$$h(t) = \begin{cases} \dfrac{1}{t+1} & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

a rational transfer function?

Potential answers:

I: **(wrong)** yes

II: **(correct)** no

III: **(wrong)** it depends

IV: **(wrong)** I don't know

Solution 1:

Content units indexing this solution:

- modal analysis
- partial fraction decomposition

A transfer function is rational if and only if it can be expressed as

$$H(s) = \frac{N(s)}{D(s)}$$

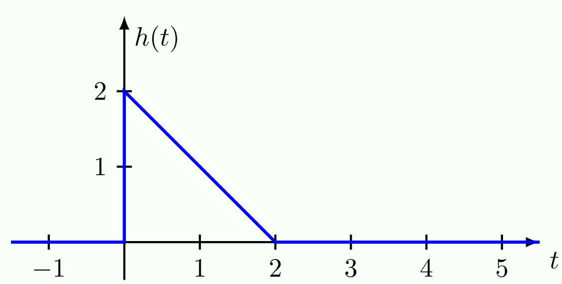
with both the numerator and the denominator finite order polynomials in s. If a transfer function is rational, then doing a partial fraction decomposition in the Laplace's domain will translate into a finite number of terms like $\frac{c}{(s-\lambda_i)^{\mu_i}}$ for opportune values of λ and μ. Taking the inverse-Laplace transform of these terms means eventually that the associated impulse response, in the time domain, is a finite sum of terms like $\alpha t^{\mu} \exp^{\gamma t} \cos(\beta t)$. However, the considered $h(t) = \frac{1}{t+1}$ can be represented only with an infinite number of terms $\alpha t^{\mu} \exp^{\gamma t} \cos(\beta t)$. In other words, to obtain this impulse response there is the need for an infinite number of elementary modes, and this means an infinitely long partial fraction decomposition in Laplace. So the associated transfer function is not rational, since its denominator will be a polynomial with infinite order.

Question 31

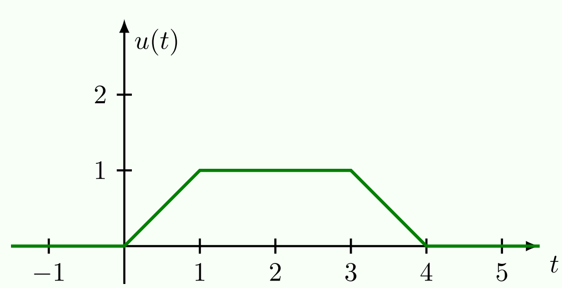
Content units indexing this question:

- continuous time LTI systems
- forced response

Consider a continuous time LTI system with impulse response $h(t)$ is equal to



and the input signal $u(t)$ equal to



The forced response of the system at $t = 5$ is then equal to ...

Potential answers:

I: **(wrong)** 1

II: **(correct)** 1/6

III: **(wrong)** 6

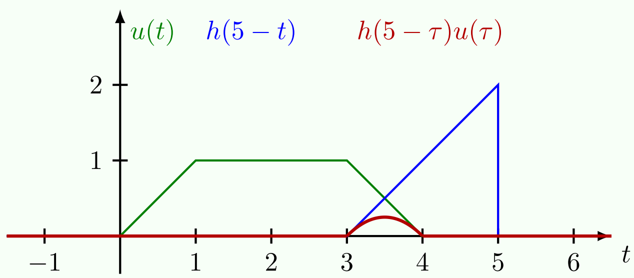
IV: **(wrong)** I don't know

Solution 1:

Content units indexing this solution:

- convolution

We need to calculate $y_f(5) = u * h(t = 5)$. From a graphical perspective, to compute this we shall "flip" $h(t)$ then shift this flipped version in $t = 5$, and then multiply the two signals point by point as in the figure below:



The next step is calculating the integral of the product $u(\tau)h(5-\tau)$. Noting that it is zero everywhere but in $(3, 4)$, it follows that

$$\begin{aligned}
\int_{-\infty}^{+\infty} u(\tau)h(5-\tau)d\tau &= \int_3^4 (\tau-3)(4-\tau)d\tau = \\
&= \int_0^1 \tau(1-\tau)d\tau \\
&= \int_0^1 \left(\tau-\tau^2\right) d\tau \\
&= \left[\frac{\tau^2}{2} - \frac{\tau^3}{3}\right]_0^1 \\
&= \frac{1}{6}
\end{aligned}$$

In conclusion $y_f(5) = \dfrac{1}{6}$.

Question 32

Content units indexing this question:

- BIBO stability
- impulse response
- delayed systems

Can a delayed LTI system (i.e., a LTI system whose impulse response contains a delay) be BIBO stable?

Potential answers:

I: (**correct**) yes

II: (**wrong**) no

III: (**wrong**) it depends

IV: (**wrong**) I do not know

Solution 1:

Content units indexing this solution:

- absolute integrability

As soon as the impulse response of the system is absolutely integrable then the associated LTI system is BIBO stable. So adding delays does not modify the BIBO nature of a system.

Question 33

Content units indexing this question:

- BIBO stability
- impulse response
- non-causal systems

Can a non-causal LTI system be BIBO stable?

Potential answers:

I: (**correct**) yes

II: (**wrong**) no

III: (**wrong**) it depends

IV: (**wrong**) I do not know

Solution 1:

Content units indexing this solution:

- absolute integrability

As soon as the impulse response of the system is absolutely integrable then the associated LTI system is BIBO stable. So the fact that there is some non-causal phenomenon does not necessarily imply a specific BIBO nature of the system.

Question 34

Content units indexing this question:

- BIBO stability
- LTI systems

Can a LTI system whose transfer function have some poles on the imaginary axis be BIBO stable?

Potential answers:

I: **(wrong)** yes

II: **(correct)** no

III: **(wrong)** it depends

IV: **(wrong)** I do not know

Solution 1:

If the transfer function has some poles on the imaginary axis, then the corresponding impulse response includes at least a mode of the type $\theta \cos(\omega t + \phi)$, for opportune parameters θ, ω, ϕ. This means the impulse reponse contains a non-decaying sinusoidal component that makes the impulse response not absolutely integrable. Choosing an input $u(t) = \sin(\omega t)$ will thus surely induce an unbounded output.

Question 35

Content units indexing this question:

- kernel
- matrices

How may one interpret the kernel of a $\mathbb{R}^{n \times m}$ matrix A?

Potential answers:

I: **(wrong)** as the set of equilibria of the system $\dot{\boldsymbol{x}} = A\boldsymbol{x}$

II: **(wrong)** as the set of equilibria of the system $\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u}$

III: **(correct)** as the set of zeros of the linear map induced by A

IV: **(wrong)** as the domain of the linear map induced by A

V: **(wrong)** it depends

VI: **(wrong)** I do not know

Solution 1:

Formally the kernel of a transformation is the set of elements of the domain that are mapped in the null element of the codomain. The first answer is thus imprecise, as n may be different from m and thus in this case A cannot be defining an ODE. The second answer is totally wrong since the equilibria of that system do not coincide in general with the kernel of A even if that matrix is square (indeed if $\boldsymbol{u} \neq \boldsymbol{0}$ the equilibria do not coincide in general with the kernel of A). The fourth answer does not make sense. Thus the correct is the third one.

Question 36

Content units indexing this question:

- determinant
- matrices

How may one interpret the determinant of a $\mathbb{R}^{n \times n}$ matrix A?

Potential answers:

I: **(wrong)** as a measure of the size of the matrix

II: **(correct)** as a measure of the stretching induced by the matrix when transforming the domain into the codomain

III: **(wrong)** as a measure of how big the eigenvalues of A are

IV: **(wrong)** as a measure of the degree of diagonalizability of the matrix A

V: **(wrong)** as a measure of the degree of invertibility of the matrix A

VI: **(wrong)** I do not know

Solution 1:

Geometrically speaking, it can be interpreted as a signed expansion/compression factor of the fabric of space, being indeed equal to the ratio of the volume of the parallelepiped formed by the transformed elements of the standard basis of the domain (with coordinates given though by the initial basis system) over the volume of the unitary parallelepiped defined by the elements of the standard basis of the domain.

Question 37

Content units indexing this question:

- linear transformations
- basis

Which is more correct to say among these two options?

1. a matrix defines a specific linear transformation
2. a matrix defines a specific linear transformation from a specific basis into another

Potential answers:

I: (**wrong**) the first

II: (**correct**) the second

III: (**wrong**) they are equivalent

IV: (**wrong**) I don't know

Solution 1:

Any matrix can be interpreted 'by columns' in the sense that each of its columns corresponds to how the corresponding element of the basis in the domain transforms into something in the codomain (but written in terms of a specific basis in the codomain). More precisely, if $\boldsymbol{a}_i$ is the i-th column of A, then $\boldsymbol{a}_i = A\boldsymbol{e}_i$ with $\boldsymbol{e}_i$ the i-th element of the standard basis of the domain. Thus a given linear transformation $\mathcal{A}$ can be expressed by a specific A if we choose which basis we are using to describe both the domain and the codomain. Without specifying that bases, we are unable to "give a name" to the elements in the domain, and to where these map in the codomain.

Question 38

Content units indexing this question:

- diagonalizability
- eigenvalues

If a $n \times n$ square matrix has n different eigenvalues then it is diagonalizable

Potential answers:

I: (**correct**) true

II: (**wrong**) false

III: (**wrong**) it depends

IV: (**wrong**) I don't know

Solution 1:

Content units indexing this solution:

- geometric multiplicity
- algebraic multiplicity

In this case every eigenvalue will have an algebraic multiplicity of at most 1. This implicitly means that – since to each eigenvalue corresponds to an eigenspace of dimension at least 1 – that all the algebraic multiplicities will be equal to the corresponding geometric multiplicities. In other words, there are enough independent eigenspaces that we can find a basis formed by eigenvectors, thus the matrix can be diagonalized by that basis.

Question 39

Content units indexing this question:

- diagonalizability
- eigenvalues

A $n \times n$ square matrix needs to have n different eigenvalues to be diagonalizable

Potential answers:

I: (**wrong**) true

II: (**correct**) false

III: (**wrong**) it depends

IV: (**wrong**) I don't know

Solution 1:

This is clearly not true, and we can find an example for it. The simples example is the identity matrix, that is clearly diagonalizable (being already diagonal) and has all its eigenvalues equal to 1.

Question 40

Content units indexing this question:

- diagonalizability
- determinant

A $n \times n$ square matrix needs to have its determinant different from zero to be diagonalizable

Potential answers:

I: (**wrong**) true

II: (**correct**) false

III: (**wrong**) it depends

IV: (**wrong**) I don't know

Solution 1:

This is clearly not true, and we can find an example for it. The simples example is the zero matrix, that is clearly diagonalizable (being already diagonal) and has its determinant equal to zero.

Question 41

Content units indexing this question:

- Cayley-Hamilton theorem
- characteristic polynomial

Consider a generic matrix $A \in \mathbb{R}^{n \times n}$, and its characteristic polynomial, i.e., the scalars $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$ such that

$$A^n = -\alpha_{n-1} A^{n-1} - \ldots - \alpha_1 A - \alpha_0 I$$

and forming the polynomial

$$s^n + \alpha_{n-1} s^{n-1} + \ldots + \alpha_1 s + \alpha_0.$$

Then the characteristic polynomial is the lowest order polynomial that is nullified by A. I.e., there are no other scalars $\beta_0, \beta_1, \ldots, \beta_{m-1}$ with $m < n$ such that

$$A^m = -\beta_{m-1} A^{m-1} - \ldots - \beta_1 A - \beta_0 I$$

Potential answers:

I: (**wrong**) true

II: (**correct**) false

III: (**wrong**) it depends

IV: (**wrong**) I don't know

Solution 1:

Content units indexing this solution:

- minimal polynomial

This is not true in general, since in general the minimal polynomial (the one the question refers to, basically) may be different from the characteristic polynomial. E.g., $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ has characteristic polynomial $-s^3$, but also $A^2 = 0$. The multiplicities that appear in the minimal polynomial are thus connected with the sizes of the Jordan miniblocks in A.

Question 42

Content units indexing this question:

- continuous time LTI systems
- state space representations
- autonomous system
- free evolution

Assume that an autonomous continuous time LTI system $\dot{\boldsymbol{x}} = A\boldsymbol{x}$ is s.t. its state update matrix A is s.t. $A^m = \mathbf{0}$ for some $m \geq 0$. How does this help computing the free evolution of the system?

Potential answers:

I: **(<u>correct</u>)** it helps computing the matrix exponential e^{At} associated to the system

II: **(<u>wrong</u>)** it helps simplifying the convolution operation one should solve to find the response

III: **(<u>wrong</u>)** it helps identifying the stability properties of the system (i.e., it show that the system is marginally stable)

IV: **(<u>wrong</u>)** it helps computing the determinant of the system

V: **(<u>wrong</u>)** I don't know

Solution 1:

Content units indexing this solution:

- matrix exponential

The free evolution of the system is $\boldsymbol{x}(t) = e^{At}\boldsymbol{x}_0$. The matrix exponential can then in general be expressed via the Taylor expansion of e^{At}. Since in this case we have $A^m = \mathbf{0}$ for some $m \geq 0$, this Taylor expansion stops at most at the $m-1$-th term, and this means that one may compute it algebrically directly via this Taylor expansion.

Question 43

Content units indexing this question:

- poles
- modes
- continuous time LTI systems
- transfer function

Consider a continuous time input output LTI system of order 4 for which all the poles of its transfer function are distinct. Must the associated impulse response comprise at least one mode of the type $e^{\lambda t}$ with $\lambda \in \mathbb{R}$?

Potential answers:

I: (**wrong**) yes

II: (**correct**) no

III: (**wrong**) it depends

IV: (**wrong**) I don't know

Solution 1:

Since the poles may come in complex conjugate pairs, and the system is of order four, there may be two pairs of complex conjugate poles. Thus there may be no pole that is purely real, and thus no - the system may not exhibit a purely exponential mode.

Question 44

Content units indexing this question:

- poles
- modes
- continuous time LTI systems
- transfer function

Consider a continuous time input output LTI system of order 3 for which all the poles of its transfer function are distinct. Must the associated impulse response comprise at least one mode of the type $e^{\lambda t}$ with $\lambda \in \mathbb{R}$?

Potential answers:

I: (**correct**) yes

II: (**wrong**) no

III: (**wrong**) it depends

IV: (**wrong**) I don't know

Solution 1:

The poles may come in complex conjugate pairs, and the system is of order three. There may be thus be a pair of complex conjugate poles, but for sure there will be a pole that is purely real, and thus yes - the system shall exhibit a purely exponential mode.

Question 45

Content units indexing this question:

- Laplace transforms
- ODE

How would one Laplace-transform the ODE $\dddot{y} = \dot{y} + u$, assuming that all the initial conditions are 0?

Potential answers:

I: (**wrong**) $s^{-3}Y = s^{-1}Y + U$

II: (**correct**) $s^3 Y = sY + U$

III: (**wrong**) I do not know

Solution 1:

If the initial conditions are null, then in simplified terms every derivative contributes with an s.

Question 46

Content units indexing this question:

- Laplace transforms
- integrators

To what does $\frac{1}{s}$ correspond, from an intuitive perspective, if we consider Laplace transforms of continuous time signals?

Potential answers:

I: (**wrong**) a derivative

II: (**correct**) an integrator

III: (**wrong**) a multiplication in frequency

IV: (**wrong**) I do not know

Solution 1:

s corresponds to derivation (i.e., $y \mapsto Y$ means somehow $\dot{y} \mapsto sY$); thus the inverse operation is an integration.

Question 47

Content units indexing this question:

- Laplace transforms
- single poles
- region of convergence

What is the region of convergence of the unilateral Laplace transform of the signal e^{at}?

Potential answers:

I: (**wrong**) $\mathrm{Re}\,[s] < 0$

II: (**wrong**) $\mathrm{Re}\,[s] < a$

III: (**wrong**) $\mathrm{Re}\,[s] > 0$

IV: (**correct**) $\mathrm{Re}\,[s] > a$

V: (**wrong**) I do not know

Solution 1:

This follows directly from the computation of the unilateral Laplace transform of e^{at} as

$$\mathcal{L}\left[e^{at}\right](s) = \int_{0}^{+\infty} e^{at} e^{-st} dt.$$

The correct answer can be also be found using the intuition if thinking at the meaning of the term e^{-st} in the definition of the Laplace transform – when s is a real and fixed value, then that exponential corresponds to a compression or expansion factor of the original signal. If $s = a$ then one obtain $e^{(a-st)} = e^{0t}$, whose integral diverges. But then as soon as $s > a$, $e^{(a-st)} = e^{\alpha t}$ with α negative, thus an exponential whose integral converges.

Question 48

Content units indexing this question:

- transfer function
- first order systems
- time constant

What is the time constant associated to the continuous time LTI system whose transfer function is $\frac{1}{s+3}$?

Potential answers:

I: **(wrong)** 0.3

II: **(wrong)** 3

III: **(correct)** 1/3

IV: **(wrong)** undefined

V: **(wrong)** I do not know

Solution 1:

The impulse response of the system is proportional to e^{-3t}, that means that we have a BIBO stable system, and its time constant is 1/3.

Question 49

Content units indexing this question:

- properties of Laplace transforms

$\mathcal{L}(\ddot{x}) =?$

Potential answers:

I: **(wrong)** $s^2 X(s) + s x(0) + \dot{x}(0)$

II: **(correct)** $s^2 X(s) - s x(0) - \dot{x}(0)$

III: **(wrong)** $s^2 X(s) + s \dot{x}(0) + x(0)$

IV: **(wrong)** $s^2 X(s) - s \dot{x}(0) - x(0)$

V: **(wrong)** I do not know

Solution 1:

This is a standard transform whose computations can be carried out as follows:

$$
\begin{aligned}
&\mathcal{L}(\ddot{y}(t); s) = \int_0^{+\infty} \ddot{y}(t) e^{-st} dt \\
&= \left[\dot{y}(t) e^{-st}\right]_0^{+\infty} - \int_0^{+\infty} \dot{y}(t)(-s) e^{-st} dt \qquad \text{(integrating by parts)} \\
&= \left[\dot{y}(t) e^{-st}\right]_0^{+\infty} + s \int_0^{+\infty} \dot{y}(t) e^{-st} dt \\
&= \left[\dot{y}(t) e^{-st}\right]_0^{+\infty} + s \left[y(t) e^{-st}\right]_0^{+\infty} - s \int_0^{+\infty} y(t)(-s) e^{-st} dt \quad \text{(again by parts)} \\
&= \left[\dot{y}(t) e^{-st}\right]_0^{+\infty} + s \left[y(t) e^{-st}\right]_0^{+\infty} + s^2 \int_0^{+\infty} y(t) e^{-st} dt \\
&= -\dot{y}(0) - s y(0) + s^2 \mathcal{L}(y(t); s)
\end{aligned}
$$

Question 50

Content units indexing this question:

- examples of Laplace transforms

$\mathcal{L}\left(t^n e^{at}\right) = ?$

Potential answers:

I: (**wrong**) $\frac{n!}{(s-a)^n}$

II: (**correct**) $\frac{n!}{(s-a)^{n+1}}$

III: (**wrong**) $\frac{n!}{(s+a)^n}$

IV: (**wrong**) $\frac{n!}{(s+a)^{n+1}}$

V: (**wrong**) I do not know

Solution 1:

This is a standard transform, whose derivation can be found in any textbook or wikipedia. Being the most important transform you see in this course, you shall remember it by heart.

Question 51

Content units indexing this question:

- damping
- second order systems
- poles
- continuous time LTI systems

In which situation is a second order continuous time LTI system said to be critically damped? When the poles of its transfer function are ...

Potential answers:

I: (**wrong**) both real and distinct

II: (**correct**) coinciding and real, i.e., the transfer function has a double real pole

III: (**wrong**) a complex conjugate pair

IV: (**wrong**) I do not know

Solution 1:

In brief, this happens when the system has a double real pole. In more details, a second-order continuous-time linear time-invariant (LTI) system is said to be critically damped when its damping ratio (denoted by ζ, pronounced zeta) is equal to 1. The damping ratio is a dimensionless parameter that describes the level of damping in a system. If we write the general transfer function for a second-order LTI system, this is given by:

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

where:

- ω_n is the natural frequency of the system,
- ζ is the damping ratio, and
- s is the complex frequency variable proper of the Laplace domain (i.e., where the transfer function is defined).

When $\zeta = 1$, the system is critically damped. In this case, the system responds to inputs quickly without oscillations or overshoot. More precisely, critically damped systems have the fastest possible response without overshooting the final steady-state value. This is often desirable in applications where a rapid response to disturbances is required, and overshooting is to be avoided. Note that critical damping means that the impulse response of the system exhibit a mode of the type te^{at}.

Question 52

Content units indexing this question:

- damping
- second order systems
- poles
- continuous time LTI systems

In which situation is a second order continuous time LTI system said to be overdamped? When the poles of its transfer function are ...

Potential answers:

I: (**correct**) both real and distinct

II: (**wrong**) coinciding and real, i.e., the transfer function has a double real pole

III: (**wrong**) a complex conjugate pair

IV: (**wrong**) I do not know

Solution 1:

In brief, this happens when the transfer function has two distinct real roots. In more details, a second-order continuous-time linear time-invariant (LTI) system is considered overdamped when the damping ratio (ζ, pronounced zeta) is greater than 1. In the general transfer function for a second-order system is given by

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

where:

- ω_n is the natural frequency of the system,
- ζ is the damping ratio, and
- s is the complex frequency variable proper of the Laplace domain (i.e., where the transfer function is defined).

If $\zeta > 1$, the system is overdamped. In overdamped systems, the damping is high enough to prevent oscillations, but the trade-off is a slower response compared to critically damped or underdamped systems. Overdamped systems do not exhibit overshooting, and they take a longer time to reach the final steady-state value compared to critically damped or underdamped systems. The overdamped response is characterized by a set of distinct real roots in the transfer function, and the transient response decays without oscillations since the impulse response is composed by two distinct modes of the type $e^{\alpha_1 t}$ and $e^{\alpha_2 t}$ with $\alpha_1 \neq \alpha_2$. Overdamping means thus the absence of modes of the type te^{at} and of modes of the type $e^{at}\cos(\omega t)$. Note also that the higher the value of ζ, the more overdamped the system becomes, leading to a slower but stable response.

Question 53

Content units indexing this question:

- damping
- second order systems
- poles
- continuous time LTI systems

In which situation is a second order continuous time LTI system said to be underdamped? When the poles of its transfer function are ...

Potential answers:

I: **(wrong)** both real and distinct

II: **(wrong)** coinciding and real, i.e., the transfer function has a double real pole

III: **(correct)** a complex conjugate pair

IV: **(wrong)** I do not know

Solution 1:

In brief, this happens when the transfer function has two distinct complex conjugate roots. In more details, a second-order continuous-time linear time-invariant (LTI) system is considered underdamped when the damping ratio (ζ, pronounced zeta) is smaller than 1. In the general transfer function for a second-order system is given by

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

where:

- ω_n is the natural frequency of the system,
- ζ is the damping ratio, and
- s is the complex frequency variable proper of the Laplace domain (i.e., where the transfer function is defined).

If $\zeta < 1$, the system is overdamped. The underdamped response is characterized by a pair of complex conjugate roots in the transfer function. The damping ratio (ζ) influences the decay of the oscillatory behavior. A smaller ζ leads to more pronounced oscillations and a slower settling time, while a larger ζ reduces oscillations but speeds up the settling time. Underdamped systems are often encountered in applications where a rapid response is needed, and a moderate amount of overshooting is acceptable. The underdamped response is characterized thus a mode of the type $e^{at}\cos(\omega t)$.

Question 54

Content units indexing this question:

- impulse response
- free evolution
- initial conditions

Consider writing the free evolution of a continuous time LTI system as a sum of modes, i.e.,

$$y_{\mathrm{fe}}(t) = \sum_i c_i t^{m_i} \exp(\alpha_i t) \cos(\omega_i t + \phi_i).$$

Which of the various parameters above may change with the initial conditions (i.e., $y(0)$, $\dot{y}(0)$, $\ddot{y}(0)$, $\dots$) of the system?

Potential answers:

I: **(correct)** only the residuals c_i and the phase shifts ϕ_i

II: **(wrong)** only the orders of the modes m_i

III: **(wrong)** only the time constants $\left|\frac{1}{\alpha_i}\right|$

IV: **(wrong)** only the frequencies ω_i

V: **(wrong)** I do not know

Solution 1:

The solution can be found immediately considering writing the free evolution in Laplace form, i.e., as

$$Y_{\text{fe}}(s) = H(s)M(s)$$

where $M(s)$ is a polynomial that is due to the initial conditions. The c_i's and ϕ_i's are then obtained when performing a partial fraction decomposition of the product HM, whose denominator is the characteristic polynomial associated to the LTI system independently of M. Since that polynomial does not change with the initial conditions, the parameters that structurally define the modes $(m_i, \alpha_i, \omega_i)$ they are in this case fixed.

Question 55

Content units indexing this question:

- Laplace transforms
- measurement unit

Which measurement unit is associated to s in a Laplace transform of a signal $y(t)$?

Potential answers:

I: (**wrong**) seconds

II: (**wrong**) seconds^{-1}

III: (**wrong**) hours

IV: (**wrong**) hours^{-1}

V: (**correct**) none of the above

VI: (**wrong**) I do not know

Solution 1:

In general it is "time^{-1}", with the same units of t. Thus if one did not specify the measurement unit associated to t in the time domain (e.g., millseconds, hours), then one cannot know what is the measurement unit for s.

Question 56

Content units indexing this question:

- transfer functions
- modal analysis

The number of potentially different modes that compose the impulse response of a continuous time LTI system is ...

Potential answers:

I: (**wrong**) equal to the number of zeros of its transfer function, counted with their multiplicity

II: (**wrong**) at most equal to the number of zeros of its transfer function, counted with their multiplicity

III: (**wrong**) equal to the number of poles of its transfer function, counted with their multiplicity

IV: (**correct**) at most equal to the number of poles of its transfer function, counted with their multiplicity

V: (**wrong**) I do not know

Solution 1:

Content units indexing this solution:

- partial fraction decomposition

The number of modes constituting the impulse response $h(t)$ is equal to the number of factors that one obtain when doing a partial fraction decomposition of the corresponding transfer function $H(s)$. The number of such factors is in general at most equal to the number of poles of its transfer function, counted with their multiplicity, and may be potentially smaller than this number since there may be some zeros-poles cancellation (that in practice make some potential modes disappear, in the sense that their residual is equal to zero).

Question 57

Content units indexing this question:

- rational transfer functions

Every continuous time LTI system admits a rational transfer function.

Potential answers:

I: (**wrong**) true

II: (**correct**) false

III: (**wrong**) it depends on the system

IV: (**wrong**) I do not know

Solution 1:

Content units indexing this solution:

- time delay
- Taylor expansion of exponentials

To be rational, a transfer function must have polynomials of finite order both at the numerator and the denominator. If the system is though delayed, the time delay will translate, in the Laplace domain, into a term of the type e^{-st} in the transfer function. This, though, if one considers the Taylor expansion of this exponential ($e^{\star} = \star^0 + \star^1 + \star^2/2! + \star^3/3! + \ldots$), corresponds to having a polynomial at the denominator of the transfer function that is of infinite order. In this case thus the transfer function is not rational anymore.

Question 58

Content units indexing this question:

- transfer functions
- zeros
- stability

The BIBO stability properties of a continuous time LTI system depend on the position of the zeros of the transfer function of the system, assuming there are no zero poles cancellations.

Potential answers:

I: (**wrong**) true

II: (**correct**) false

III: (**wrong**) it depends on the system

IV: (**wrong**) I do not know

Solution 1:

Content units indexing this solution:

- absolute integrability
- modal analysis
- partial fraction decomposition

The zeros of a transfer function affect the residuals that follow from performing a partial fraction decomposition of that transfer function, and thus the amplitude of the modes that one obtains inverse-Laplacing that partial fractions. Changing the amplitude of a generic mode $t^m e^{\alpha t} \cos(\omega t)$ does though not change its absolute integrability properties. Thus the zeros of a transfer function do not affect the absolute integrability properties of the associated impulse response, and thus the BIBO stability properties of the corresponding LTI system.

Question 59

Content units indexing this question:

- transfer functions
- zeros
- transient response

Changing the zeros of a transfer function of an LTI system means changing the transient associated to the step response of that system.

Potential answers:

I: (**correct**) true

II: (**wrong**) false

III: (**wrong**) it depends on the system

IV: (**wrong**) I do not know

Solution 1:

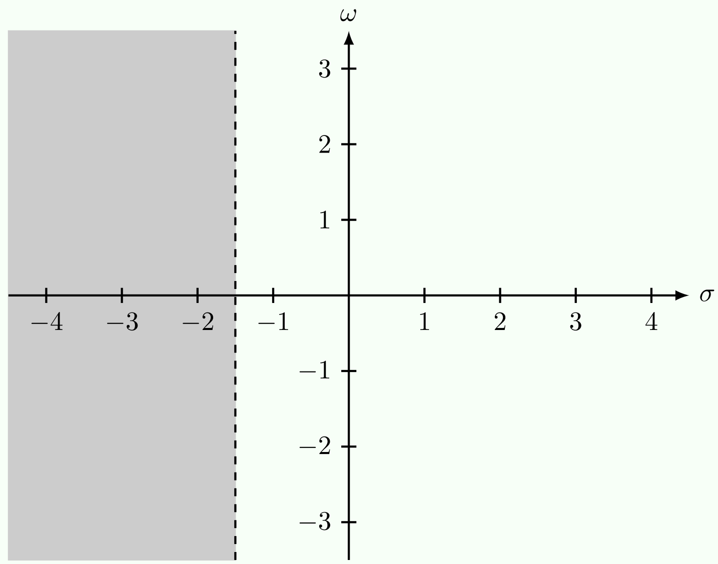
The zeros of a transfer function affect the residuals that follow from performing a partial fraction decomposition of that transfer function, and thus the amplitude of the modes that one obtains inverse-Laplacing that partial fractions. Changing the amplitude of a generic mode $t^m e^{\alpha t} \cos(\omega t)$ does thus change where it starts from (i.e., its amplitude at time $t = 0$), even if it does not change its limit for $t \mapsto +\infty$. Since the transient of the forced response depends on the amplitude of these modes, the transients depends thus on the zeros of the transfer function. Changing its zeros will change its transient.

Question 60

Content units indexing this question:

- Laplace transform
- region of convergence

Assume to know that the region of convergence of the Laplace transform of a time signal $f(t)$ is as in the figure below (i.e., the shaded area is where the Laplace transform does **not** converge). Then ...



Potential answers:

I: (**wrong**) $\lim_{t\to 0} f(t) = 0$

II: (**wrong**) $\lim_{t\to 0} |f(t)| = +\infty$

III: (**correct**) $\lim_{t\to +\infty} f(t) = 0$

IV: (**wrong**) $\lim_{t\to +\infty} |f(t)| = +\infty$

V: (**wrong**) it depends on $f(t)$

VI: (**wrong**) I do not know

Solution 1:

The figure indicates that (note that here we are using the bilateral transform, but the same concepts would apply for the monolateral one)

$$\int_{-\infty}^{+\infty} f(t) e^{t} dt$$

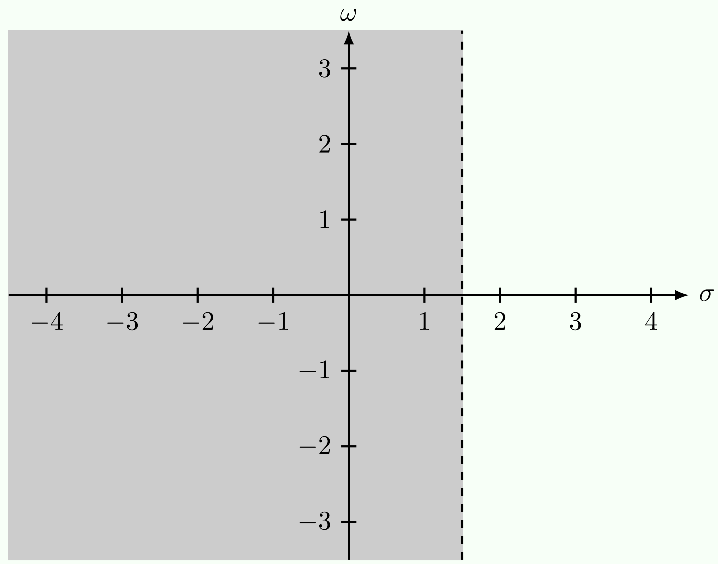
converges (note how this corresponds to setting $\sigma = -1$ in the integral defining the Laplace transform). But for this to happen it has to be that $f(t)$ decays to zero for growing t at least as fast as e^{-t}.

Question 61

Content units indexing this question:

- Laplace transform
- region of convergence

Assume to know that the region of convergence of the Laplace transform of a time signal $f(t)$ is as in the figure below (i.e., the shaded area is where the Laplace transform does **not** converge). Then ...



Potential answers:

I: (**wrong**) $\lim_{t\to 0} f(t) = 0$

II: (**wrong**) $\lim_{t\to 0} |f(t)| = +\infty$

III: (**wrong**) $\lim_{t\to +\infty} f(t) = 0$

IV: (**correct**) $\lim_{t\to +\infty} |f(t)| = +\infty$

V: (**wrong**) it depends on $f(t)$

VI: (**wrong**) I do not know

Solution 1:

The figure indicates that (note that here we are using the bilateral transform, but the same concepts would apply for the monolateral one)

$$\int_{-\infty}^{+\infty} f(t) e^{-t} dt$$

diverges (note how this corresponds to setting $\sigma = +1$ in the integral defining the Laplace transform). But for this to happen it has to be that $f(t)$ diverges for growing t at least as fast as e^t.

Index of the Content Units