TTK4225 System theory, Autumn 2023

Assignment 10

The expected output is a .pdf written in LaTeX or a Python notebook exported to .pdf, even if photos of your handwritten notes or drawings will work. Every person shall hand in her/his assignment, independently of whether it has been done together with others. When dealing with mathematical derivations, unless otherwise stated, explain how you got your answer (tips: use programming aids like Python, Matlab, Maple, or compendia like Rottmann's to check if you have obtained the right answer).

Question 1

Content units indexing this question:

- Laplace transforms
- multiple poles
- LTI systems

Consider the time signal

$$x_1(t) = e^{3t} - 2te^{3t}.$$

Describe which dynamical system may admit $x_1(t)$ as a solution, and from which initial conditions.

Solution 1:

Content units indexing this solution:

- Laplace transforms
- ARMA model

The Laplace transform of $x_1(t)$ is

$$X(s) = \frac{1}{s-3} - \frac{2}{(s-3)^2} = \frac{s-5}{(s-3)^2}$$

that basically can be rewritten as

$$s^2 X(s) - 6sX(s) + 9X(s) - s + 5 = 0.$$

This implies that the autoregressive part of an ARMA like ODE leading to this Laplace transform looks like

$$\ddot{x} = 6\dot{x} - 9x.$$

To find the initial conditions that lead to that Laplace transform for $x(t)$, assuming that $x(t)$ is actually a free evolution then we may consider that

$$\mathcal{L}(\dot{x}) = sX - x_0 \qquad \mathcal{L}(\ddot{x}) = s^2 X - sx_0 - \dot{x}_0.$$

This implies that applying these equations to the previous ARMA model we get

$$s^2 X - sx_0 - \dot{x}_0 - 6\left(sX - x_0\right) + 9X = 0$$

This must be equal to $s^2 X(s) - 6sX(s) + 9X(s) - s + 5 = 0$, implying $x_0 = 1$, $\dot{x}_0 = -6$.

Question 2

Content units indexing this question:

- Laplace transforms
- multiple poles
- LTI systems

Consider the formula

$$\mathcal{L}\left[t^n e^{\lambda t}\right] = \frac{n!}{(s-\lambda)^{n+1}}.$$

Describe which concepts from the course may be somehow connected with this formula, and how you would use them to create a mnemonic for this formula.

Solution 1:

Content units indexing this solution:

- Laplace transforms
- multiple poles
- LTI systems

The concepts that can be seen and connected are:

1. λ is an eigenvalue
2. $e^{\lambda t}$ is the mode associated to an eigenvalue with multiplicity one
3. multiplicity one means a Jordan miniblock of dimension 1
4. the corresponding Laplace transform is $\frac{1}{s-\lambda}$
5. the Laplace transform of $te^{\lambda t}$ is $\frac{1}{(s-\lambda)^2}$
6. for this, the Jordan miniblock needs to have dimension 2
7. $t^n e^{\lambda t}$ is the highest order mode of the matrix exponential of a Jordan miniblock of dimension $n+1$
8. so t^n kind of implies a Jordan miniblock of dimension $n+1$, and thus also a Laplace transform of "order" $n+1$
9. as for the $n!$ at the numerator, it connects to Taylor expansions, that also have factorials $n!$ in them

Question 3

Content units indexing this question:

- modal analysis
- LTI systems
- zeros

1. Design a LTI system in terms of an opportune ODE (thus, decide both the order of the system, and its actual parameters) for which the associate impulse response looks like a fastly vanishing oscillatory behavior on top of a slowly vanishing exponential decay;
2. create a poles-zeros map of the system you designed (hint, if you want to do it in a computer-aided fashion: use the available python notebooks code in GitHub);
3. describe how you may make the amplitude of the oscillatory behavior (note: only the amplitude; not the frequency of the oscillations) bigger or smaller by changing the parameters of the ODE describing the system.

Solution 1:

Content units indexing this solution:

- ARMA model
- poles
- modal analysis
- LTI systems
- zeros

Point 1: anything for which there are three (stable) poles, a real dominant one (say with real part $\overline{\sigma} < 0$) and two non-dominant complex conjugate poles with real part $\widetilde{\sigma} \ll \overline{\sigma}$ will suffice. E.g., an ARMA model whose transfer function that looks like

$$H(s) = \frac{1}{(s - \overline{\sigma})\left((s - \widetilde{\sigma})^2 + \omega^2\right)}$$

for an opportune ω. This will lead to the modes of the system be of the type $e^{\overline{\sigma}t}$ and $e^{\widetilde{\sigma}t} \cos(\omega t)$, with the second thus vanishing faster w.r.t. the first one

Point 2: read and see the examples in `https://se.mathworks.com/help/control/ref/lti.pzmap.html`

Point 3: one needs to change the zeros of the transfer function, that determine the amplitude of the various modes (something that can be seen by inspecting the effects of such thing on the partial fraction decomposition of the transfer function). To make the oscillations of the fast vanishing mode arbitrarily small one may consider adding the complex conjugate zeros $\left((s - \widetilde{sigma} + \epsilon_1)^2 + (\omega + \epsilon_2)^2\right)$ to the numerator of the transfer function. Moreover if one wants the total gain to be the same as before, one shall also put in front of the new transfer function an additioanl constant κ whose value guarantees $|H(0)|$ and $|H'(0)|$ to be the same (where H' is the so-modified transfer function)