Assignment 10 Answer - v2

January 3, 2024

1 Assignment 10 Answer - v2

- There are some lecture references in this document. They are notes to myself and can be ignored.

[ ]:
```
from IPython.display import display
import sympy as sp
import matplotlib.pyplot as plt
```

1.1 Q1

- Lecture 237 S is a noncausal system
- Lecture 212 E^{At} = transition matrix

The system:

$x_1(t) = e^{3t} - 2te^{3t}$

May be generated by the system:

$A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$

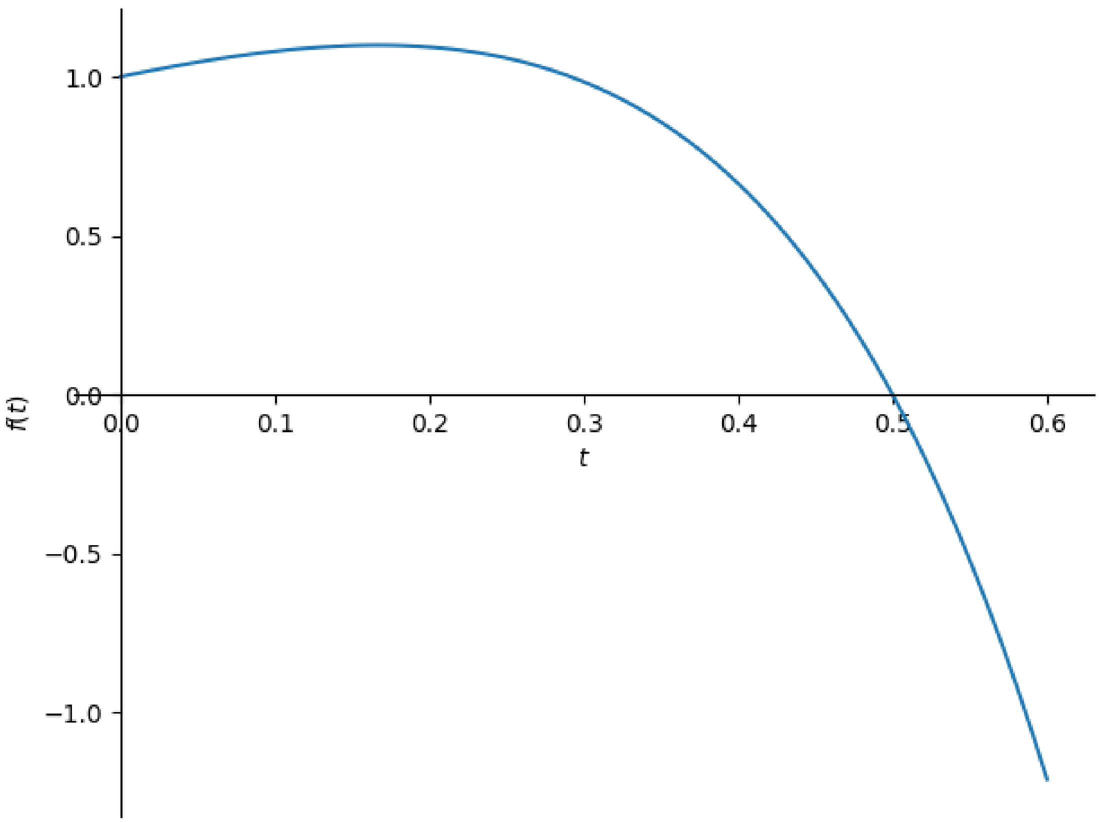
with the initial conditions $(1, -2)$.

[ ]:
```
A = sp.Matrix([
    [3, 1],
    [0, 3]
])
```

[ ]:
```
t = sp.symbols('t')

y0 = sp.Matrix([1, -2])
y, dy = sp.exp(A*t)@y0
y
```

[ ]: $-2te^{3t} + e^{3t}$

[ ]:
```
sp.plot(y, (t, 0, 0.6))
plt.show()
```



1.2 Q2

Lecture 235 Partial fraction decompositions and free evolutions

- For $n > 1$ the system is not diagonisable
- That means the system does not have a full set of independent eigenvectors.

Honestly i find the Jordan representation of this quite helpful in remembering:

$$e^{At} = \begin{bmatrix} e^{\lambda} & te^{\lambda} & t^2e^{\lambda} \\ 0 & e^{\lambda} & te^{\lambda} \\ 0 & 0 & e^{\lambda} \end{bmatrix}$$

1.3 Q3

1.3.1 1

I will create the two separate systems, then combine them.

Fastly decaying oscillatory behaviour: This system should be a damped harmonic oscillator:

$\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2 x = 0$

$\ddot{x} = -2\zeta\omega_0\dot{x} - \omega_0^2 x$

$\vec{x} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$

$A = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -2\zeta\omega_0 \end{bmatrix}$

[ ]:
```
def create_oscilator(damping, frequency):
    zeta = damping
    omega = frequency*2*sp.pi
    return sp.Matrix([
        [0, 1],
        [-omega**2, -2*zeta*omega]
    ]).applyfunc(sp.simplify)
```

[ ]:
```
A = create_oscilator(0.07, 1)
A
```

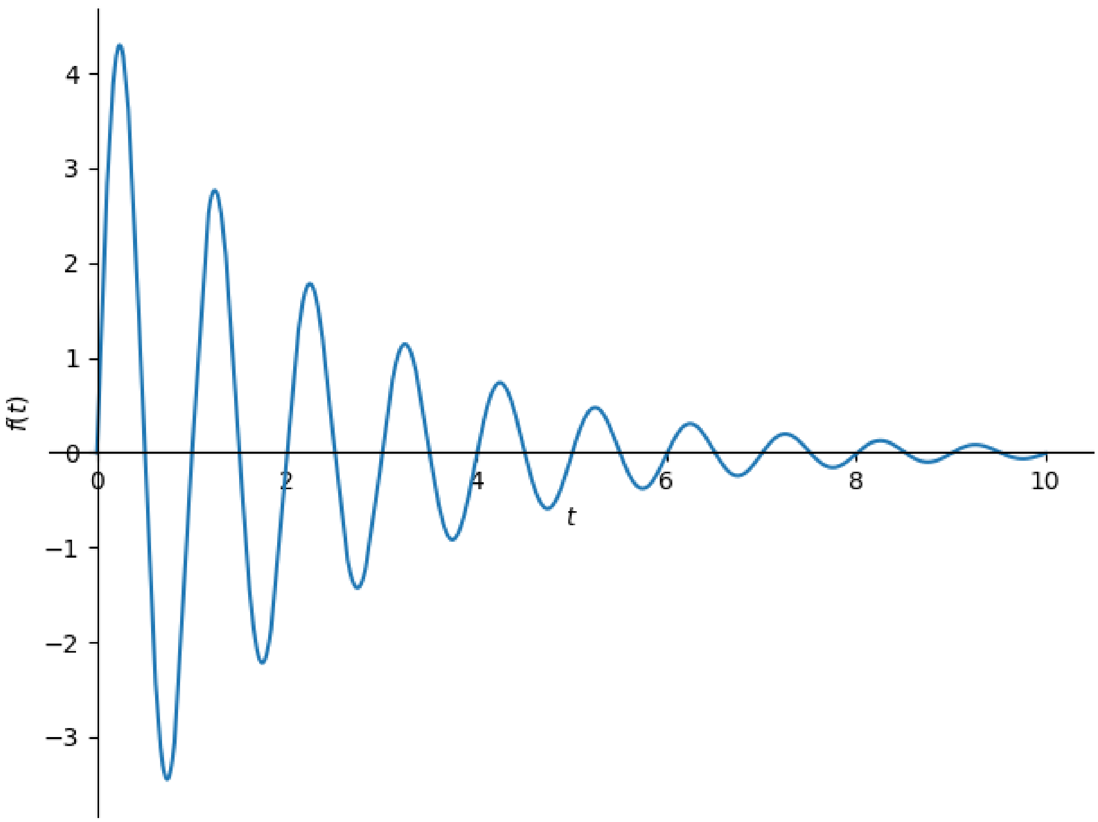
[ ]: $\begin{bmatrix} 0 & 1 \\ -4\pi^2 & -0.28\pi \end{bmatrix}$

The free evolution of the system is given by:

$\vec{x}(t) = e^{At}\vec{x}(0)$

where e^{At} is the transition matrix given by the taylor series.

[ ]:
```
t = sp.symbols('t')

# Adjusts the amplitude which is affected by the damping
gain = 30
x0 = sp.Matrix([0, gain])
x = sp.exp(A*t)@x0

x1 = x[0]

sp.plot(x1, (t, 0, 10))
plt.show()
```



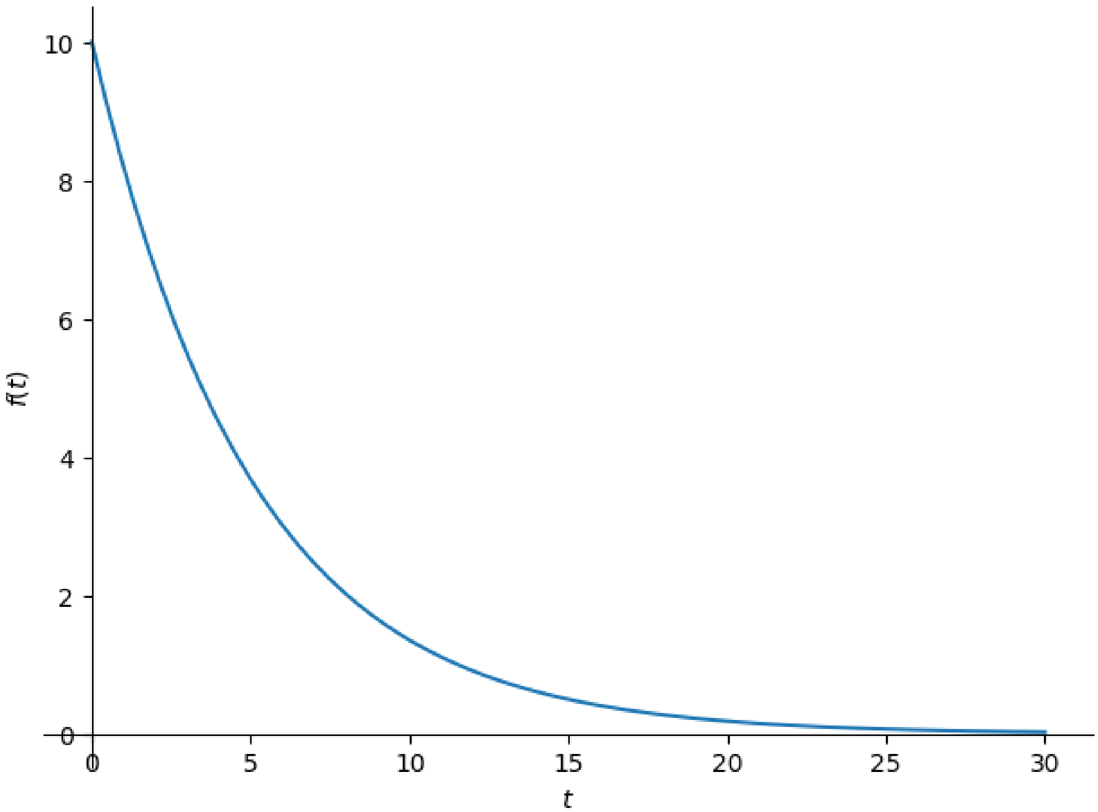
Slowly decaying exponential: This is a simple first order exponential decay:

$\dot{x} = -\lambda x$

The free evolution is given by:

$x(t) = e^{-\lambda t} x(0)$

[ ]:
```
x0 = 10
lambda1 = 1/5
sp.plot(x0*sp.exp(-lambda1*t), (t, 0, 30))
plt.show()
```



Combine them:

These are two independent systems, so their free evolutions must remain separate. A separate state must represent the linear combination.

Creating a new state x_1 that is driven by the derivatives of the two systems:

x_1: The combined state
x_2: The state of the first order system
x_3: The state of the second order system
x_4: $\dot{x_3}$

$\dot{x_1} = \dot{x_2} + \dot{x_3}$

$\dot{x_2} = -\lambda x_2$

$\dot{x_3} = x_4$

$\dot{x_4} = -2\zeta\omega_0 x_4 - \omega_0^2 x_3$

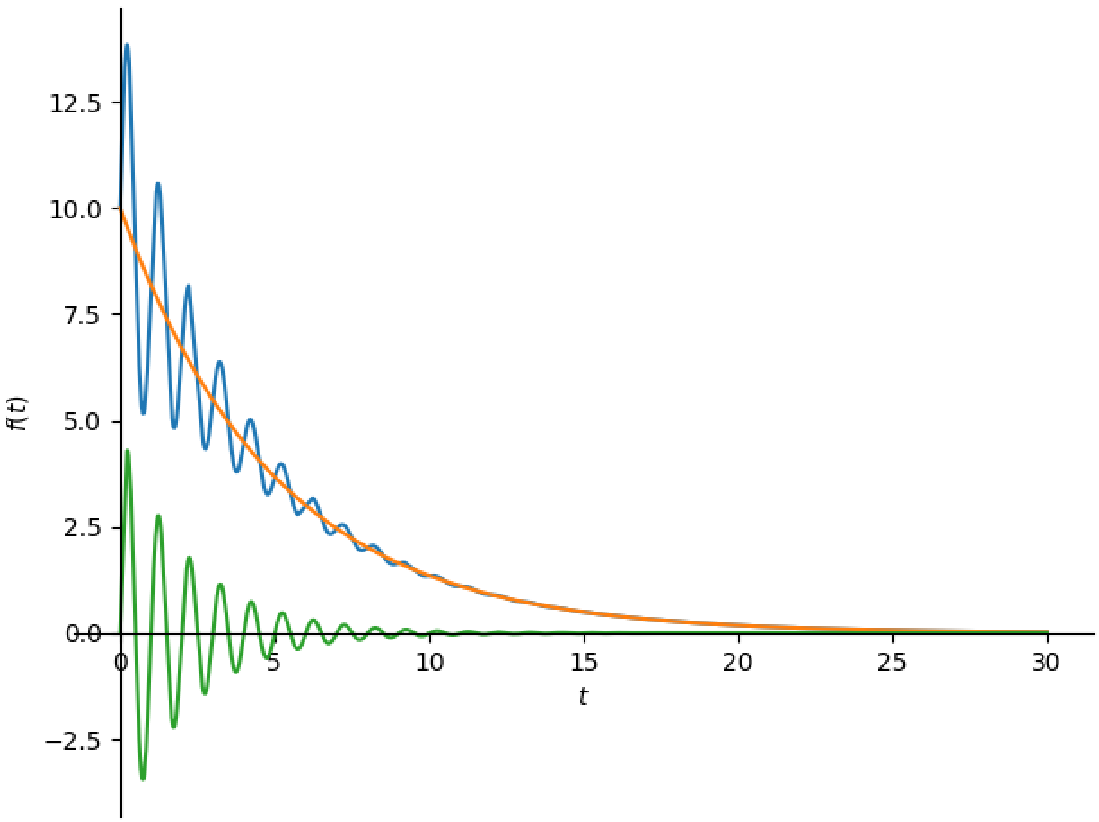
$$A = \begin{bmatrix} 0 & -\lambda & 0 & 1 \\ 0 & -\lambda & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\omega_0^2 & -2\zeta\omega_0 \end{bmatrix}$$

[ ]:
```python
def create_model(tau, damping, frequency):

    A = sp.Matrix([
        [0, -tau, 0, 1],
        [0, -tau, 0, 0],
        [0, 0, 0, 0],
        [0, 0, 0, 0],

    ])
    A[2,2] = create_oscilator(damping, frequency)

    return A.applyfunc(sp.simplify)
```

[ ]:
```python
A = create_model(1/5, 0.07, 1)
A
```

[ ]:
$$\begin{bmatrix} 0 & -0.2 & 0 & 1 \\ 0 & -0.2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -4\pi^2 & -0.28\pi \end{bmatrix}$$

[ ]:
```python
# Adjusts the amplitude which is affected by the damping
y0 = 10
gain = 30
x0 = sp.Matrix([y0, y0, 0, gain])
x1, x2, x3, x4 = sp.exp(A*t)@x0

sp.plot(x1, x2, x3, (t, 0, 30))
plt.show()
```



The plot above shows a system with a fastly vanishing oscillatory behavior on top of a slowly vanishing exponential decay.

1.3.2 2

The poles are the solutions for s in the polynomial of the denominator.

[ ]:
```python
def characteristic(matrix):
    I = sp.eye(matrix.shape[0])
    M = matrix - sp.symbols('s')*I
    return M.det()
```

[ ]:
```python
characteristic(A)
```

[ ]:

$$-s\left(-s-0.2\right)\left(s^{2}+0.28\pi s+4\pi^{2}\right)$$

S has the solutions:

[ ]:
```python
solutions = sp.solve(characteristic(A))

for solution in solutions:
    display(sp.Eq(sp.symbols('s'), solution))
```

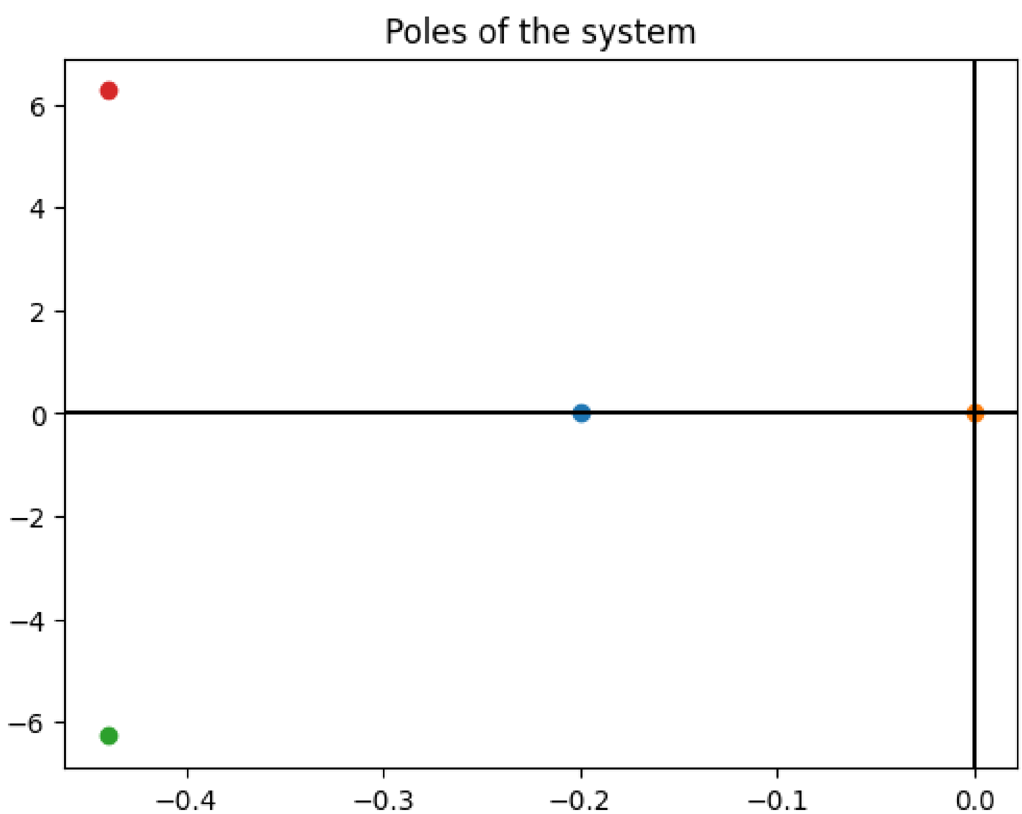
$s = -0.2$

$s = 0.0$

$s = -0.439822971502571 - 6.26777259942446i$

$s = -0.439822971502571 + 6.26777259942446i$

[ ]:
```
for pole in solutions:
    if type(pole) == 'sympy.core.numbers.Float':
        p = (pole, 0)
    else:
        p = pole.as_real_imag()
    plt.scatter(*p)

plt.axhline(0, color='k')
plt.axvline(0, color='k')
plt.title("Poles of the system")
plt.show()
```



1.3.3 3

The amplitude can be changed by increasing or decreasing the gain of the system, which is $x_4(0)$. Below is an example of different gains when the system is undamped:

[ ]:
```
def simulate_undamped(gain):
    A = create_model(0, 0, 1/4)
    x0 = sp.Matrix([0, 0, 0, gain])
    x = sp.exp(A*t)@x0
    return x[0]
```

[ ]:
```
sp.plot(
    simulate_undamped(gain=1),
    simulate_undamped(gain=2),
    simulate_undamped(gain=3),
    (t, 0, 30),
)

plt.show()
```