TTK4225 System theory, Autumn 2023

Assignment 8

The expected output is a .pdf written in LaTeX or a Python notebook exported to .pdf, even if photos of your handwritten notes or drawings will work. Every person shall hand in her/his assignment, independently of whether it has been done together with others. When dealing with mathematical derivations, unless otherwise stated, explain how you got your answer (tips: use programming aids like Python, Matlab, Maple, or compendia like Rottmann's to check if you have obtained the right answer).

Question 1

Content units indexing this question:

- matrices
- linear transformations
- range
- kernel
- determinant

Consider the single Jordan miniblock

$$A = \begin{bmatrix} 5 & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{bmatrix}$$

1. show that the chain of subspaces $\ker (A - \lambda_i I)^h$, $h = 1, 2, \ldots$, is strictly increasing up the multiplicity of that eigenvalue λ_i in the minimal polynomial of A (in this case $\lambda_i = 5$), and that this chain of subspaces becomes stationary (i.e., it does not grow in dimension) for higher powers $h > m_i$. In other words, show that for matrices that are single Jordan miniblocks then

$$\ker (A - \lambda_i I) \subset \ker (A - \lambda_i I)^2 \subset \ldots \subset \ker (A - \lambda_i I)^{m(\lambda_i)} = \ker (A - \lambda_i I)^{m(\lambda_i)+1} = \ldots$$

2. describe which subspaces are invariant for the system $\dot{\boldsymbol{x}} = A\boldsymbol{x}$, i.e., which subspaces $\mathcal{X}$ are such that if one choose an initial condition on $\mathcal{X}$, then the whole free evolution $x(t)$ is contained in $\mathcal{X}$. How are these subspaces nested into each other (i.e., which subspace is part of a bigger subspace?). Hints: the smallest invariant subspace is constituted by the x_1 axis. The second…

Solution 1:

Content units indexing this solution:

- TODO

Point 1:

$$A = \begin{bmatrix} 5 & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\ker(A - \lambda I) = \left\langle \begin{matrix} 1 \\ 0 \\ 0 \end{matrix} \right\rangle$$

$$(A - \lambda I)^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\ker(A - \lambda I)^2 = \left\langle \begin{matrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{matrix} \right\rangle$$

$$(A - \lambda I)^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\ker(A - \lambda I)^3 = \left\langle \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \right\rangle$$

$$(A - \lambda I)^4 = (A - \lambda I)^3$$

$$\ker(1 + \lambda I)^4 = \ker(A - \lambda I)^3$$

$$m\left(\lambda_2\right) = 3$$

Point 2: the invariant subspaces are then $\ker(A - \lambda I)$, $\ker(A - \lambda I)^2$, and $\ker(A - \lambda I)^3$. They are progressively nested in each other.

Question 2

Content units indexing this question:

- matrices
- linear transformations
- range
- kernel
- determinant

Consider

$$A = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

1. show that for every eigenvalue λ_i of A the chain of subspaces $\ker (A - \lambda_i I)^h$, $h = 1, 2, \ldots$, is strictly increasing up to m_i, i.e., the multiplicity of that eigenvalue in the minimal polynomial of A, and it is stationary for higher powers. In other words, show that

$$\ker (A - \lambda_i I) \subset \ker (A - \lambda_i I)^2 \subset \ldots \subset \ker (A - \lambda_i I)^{m(\lambda_i)} = \ker (A - \lambda_i I)^{m(\lambda_i)+1} = \ldots$$

2. describe which subspaces are invariant for the system $\dot{\boldsymbol{x}} = A\boldsymbol{x}$, and how these subspaces are nested into each other (i.e., which subspace is part of a bigger subspace?)

Solution 1:

Content units indexing this solution:

- TODO

Point 1:

$$A - 5I = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\ker(A - 5I) = \left\langle \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \right\rangle$$

$$
(A-5I)^2 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \star & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \star & \star \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \star & \star
\end{bmatrix}
$$

where $\star$ indicates elements that will generically be different from zero and that somehow do not matter.

$$\ker(A - 5I)^2 = \left\langle \begin{matrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{matrix} \right\rangle$$

$$(A-5I)^3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \star & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \star & \star \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \star & \star \end{bmatrix}$$

$$\ker(A - 5I)^3 = \left\langle \begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{matrix} \right\rangle$$

Similar considerations for the eigenvalue 4 hold. Thus the first point is proven.
Point 2: the subspaces are defined first of all by the various Jordan blocks, then by the Jordan

miniblocks, and then by the chain of generalized eigenspaces for each miniblock. Thus, focusing on $\lambda = 5$, and per step as just said:

- subspaces defined by the various Jordan blocks:

$$S_{\lambda=5} = \left\langle \begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{matrix} \right\rangle \qquad S_{\lambda=4} = \left\langle \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \right\rangle$$

- subspaces defined by the Jordan miniblocks:

$$S^{(1)}_{\lambda=5} = \left\langle \begin{matrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix} \right\rangle \qquad S^{(2)}_{\lambda=5} = \left\langle \begin{matrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} \right\rangle \qquad S^{(3)}_{\lambda=5} = \left\langle \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \right\rangle$$

$$S^{(1)}_{\lambda=4} = \left\langle \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \right\rangle \qquad S^{(2)}_{\lambda=4} = \left\langle \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{matrix} \right\rangle$$

- for each Jordan miniblock, then, there is a behavior of the type

$$\ker\left(A - \lambda_i I\right) \subset \ker\left(A - \lambda_i I\right)^2 \subset \ldots \subset \ker\left(A - \lambda_i I\right)^d = \ker\left(A - \lambda_i I\right)^{d+1} = \ldots$$

where d is the dimension of that specific miniblock.

Question 3

Content units indexing this question:

- matrices
- linear transformations
- range
- kernel
- determinant

Consider

$$A = \begin{bmatrix} 5 & 1 & -2 & 4 \\ 0 & 5 & 2 & 2 \\ 0 & 0 & 5 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix},$$

and show that for every eigenvalue λ_i of A the chain of subspaces $\ker{(A - \lambda_i I)}^h$, $h = 1, 2, \ldots$, is strictly increasing up to the m_i, i.e., the multiplicity of that eigenvalue in the minimal polynomial of A, and it is stationary for higher powers. In this case it is better to do not do the computations by hand, but rather use wolfram alpha or some other programming tool to compute the various powers of the various matrices.

Solution 1:

Content units indexing this solution:

- TODO

Considering $\lambda_1 = 5$,

$$A - \lambda_1 I = \begin{bmatrix} 0 & 1 & -2 & 4 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\ker\left(A - \lambda_1 I\right) = \left\langle \begin{matrix} 1 \\ 0 \\ 0 \\ 0 \end{matrix} \right\rangle$$

$$\left(A - \lambda_1 I\right)^2 = \begin{bmatrix} 0 & 0 & 2 & -8 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\ker\left(A - \lambda_1 I\right)^2 = \left\langle \begin{matrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{matrix} \right\rangle$$

$$\left(A - \lambda_1 I\right)^3 = \begin{bmatrix} 0 & 0 & 0 & 14 \\ 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\ker\left(A - \lambda_1 I\right)^3 = \left\langle \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{matrix} \right\rangle$$

$$\ker\left(A - \lambda_1 I\right)^4 = \ker\left(A - \lambda_1 I\right)^3$$

As for the second eigenvalue, i.e., $\lambda_2 = 4$,

$$\left(A - \lambda_2 I\right) = \begin{bmatrix} 1 & 1 & -2 & 4 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

thus

$$\ker\left(A - \lambda_2 I\right) = \left\langle \begin{matrix} 0 \\ 0 \\ 0 \\ 1 \end{matrix} \right\rangle$$

and, moreover,

$$\ker\left(A - \lambda_2 I\right)^2 = \ker\left(A - \lambda_2 I\right)$$

Question 4

Content units indexing this question:

- matrices
- linear transformations
- range
- kernel
- determinant
- generalized eigenvectors
- Jordan chains

(Optional, for who wants to see formally how one computes the change of basis that brings a generic square matrix into its Jordan form) A non-null vector $\boldsymbol{v}$ is said to be a ***generalized eigenvector of order*** k corresponding to the eigenvalue λ if

$$\boldsymbol{v} \in \ker(A - \lambda I)^k \qquad \text{but} \qquad \boldsymbol{v} \notin \ker(A - \lambda I)^{k-1}.$$

Moreover, for each generalized eigenvector $\boldsymbol{v}^{(k)}$ of order k corresponding to the eigenvalue λ, one can find the relative ***Jordan chain***, i.e., a chain of generalized eigenvectors of decreasing order, by choosing

$$\begin{aligned}
&\boldsymbol{v}^{(k)} && && (1)\\
&\boldsymbol{v}^{(k-1)} &&:= (A - \lambda I)\boldsymbol{v}^{(k)} && (2)\\
&\boldsymbol{v}^{(k-2)} &&:= (A - \lambda I)\boldsymbol{v}^{(k-1)} = (A - \lambda I)^2\boldsymbol{v}^{(k)} && (3)\\
&\vdots && \quad\vdots && (4)\\
&\boldsymbol{v}^{(1)} &&:= (A - \lambda I)\boldsymbol{v}^{(2)} = (A - \lambda I)^{k-1}\boldsymbol{v}^{(k)}. && (5)
\end{aligned}$$

Considering

$$A = \begin{bmatrix}
5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 5 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 5 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 5 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4
\end{bmatrix},$$

find enough Jordan chains to form a basis for $\mathbb{R}^9$.

Solution 1:

Being the matrix in Jordan form it is block diagonal. Being block diagonal, each block is independent of the others and we can thus reason by block. Moreover, each block somehow behaves in the same generic way. So we can safely focus on the miniblock associated to $\lambda = 5$ of dimension three, i.e.,

$$\begin{bmatrix} 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{5} & \mathbf{1} & \mathbf{0} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{0} & \mathbf{5} & \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{0} & \mathbf{0} & \mathbf{5} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix},$$

and pad with zeros the three dimensions "before" and the three "after".
Thus, reasoning on

$$\overline{A} = \begin{bmatrix} 5 & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{bmatrix},$$

the Jordan chain can be found considering that

$$\overline{A} - \lambda I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\ker(\overline{A} - \lambda I) = \left\langle \begin{matrix} 1 \\ 0 \\ 0 \end{matrix} \right\rangle$$

$$(\overline{A} - \lambda I)^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\ker(\overline{A} - \lambda I)^2 = \left\langle \begin{matrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{matrix} \right\rangle$$

$$(\overline{A} - \lambda I)^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\ker(\overline{A} - \lambda I)^3 = \left\langle \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \right\rangle$$

This means that we can choose as generalized eigenvector of order 3 the vector $\boldsymbol{v}^{(3)} = \begin{matrix} 0 \\ 0 \\ 1 \end{matrix}$. Following the procedures indicated in the assignment, we then get that

$$\boldsymbol{v}^{(2)} = \begin{matrix} 0 \\ 1 \\ 0 \end{matrix}$$

and

$$\boldsymbol{v}^{(1)} = \begin{matrix} 1 \\ 0 \\ 0 \end{matrix}.$$

Considering similar derivations for the other miniblocks, the sought basis is thus a set of vectors that correspond to the canonical basis for $\mathbb{R}^9$. This means that in this specific case $T = I$, as it should be since A is already in Jordan canonical form.

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