TTK4225 System theory, Autumn 2023

Assignment 8

The expected output is a .pdf written in LaTeX or a Python notebook exported to .pdf, even if photos of your handwritten notes or drawings will work. Every person shall hand in her/his assignment, independently of whether it has been done together with others. When dealing with mathematical derivations, unless otherwise stated, explain how you got your answer (tips: use programming aids like Python, Matlab, Maple, or compendia like Rottmann's to check if you have obtained the right answer).

Question 1

Consider the single Jordan miniblock

$$A = \begin{bmatrix} 5 & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{bmatrix}$$

1. show that the chain of subspaces $\ker (A - \lambda_i I)^h$, $h = 1, 2, \ldots$, is strictly increasing up the multiplicity of that eigenvalue λ_i in the minimal polynomial of A (in this case $\lambda_i = 5$), and that this chain of subspaces becomes stationary (i.e., it does not grow in dimension) for higher powers $h > m_i$. In other words, show that for matrices that are single Jordan miniblocks then

$$\ker (A - \lambda_i I) \subset \ker (A - \lambda_i I)^2 \subset \ldots \subset \ker (A - \lambda_i I)^{m(\lambda_i)} = \ker (A - \lambda_i I)^{m(\lambda_i)+1} = \ldots$$

2. describe which subspaces are invariant for the system $\dot{\boldsymbol{x}} = A\boldsymbol{x}$, i.e., which subspaces $\mathcal{X}$ are such that if one choose an initial condition on $\mathcal{X}$, then the whole free evolution $x(t)$ is contained in $\mathcal{X}$. How are these subspaces nested into each other (i.e., which subspace is part of a bigger subspace?). Hints: the smallest invariant subspace is constituted by the x_1 axis. The second…

Question 2

Consider

$$A = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

1. show that for every eigenvalue λ_i of A the chain of subspaces $\ker (A - \lambda_i I)^h$, $h = 1, 2, \ldots$, is strictly

increasing up to m_i, i.e., the multiplicity of that eigenvalue in the minimal polynomial of A, and it is stationary for higher powers. In other words, show that

$$\ker\left(A-\lambda_i I\right) \subset \ker\left(A-\lambda_i I\right)^2 \subset \ldots \subset \ker\left(A-\lambda_i I\right)^{m(\lambda_i)} = \ker\left(A-\lambda_i I\right)^{m(\lambda_i)+1} = \ldots$$

2. describe which subspaces are invariant for the system $\dot{\boldsymbol{x}} = A\boldsymbol{x}$, and how these subspaces are nested into each other (i.e., which subspace is part of a bigger subspace?)

Question 3

Consider

$$A = \begin{bmatrix} 5 & 1 & -2 & 4 \\ 0 & 5 & 2 & 2 \\ 0 & 0 & 5 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix},$$

and show that for every eigenvalue λ_i of A the chain of subspaces $\ker\left(A-\lambda_i I\right)^h$, $h = 1, 2, \ldots$, is strictly increasing up to the m_i, i.e., the multiplicity of that eigenvalue in the minimal polynomial of A, and it is stationary for higher powers. In this case it is better to do not do the computations by hand, but rather use wolfram alpha or some other programming tool to compute the various powers of the various matrices.

Question 4

(Optional, for who wants to see formally how one computes the change of basis that brings a generic square matrix into its Jordan form) A non-null vector $\boldsymbol{v}$ is said to be a ***generalized eigenvector of order*** k corresponding to the eigenvalue λ if

$$\boldsymbol{v} \in \ker(A-\lambda I)^k \qquad \text{but} \qquad \boldsymbol{v} \notin \ker(A-\lambda I)^{k-1}.$$

Moreover, for each generalized eigenvector $\boldsymbol{v}^{(k)}$ of order k corresponding to the eigenvalue λ, one can find the relative ***Jordan chain***, i.e., a chain of generalized eigenvectors of decreasing order, by choosing

$$\begin{aligned} &\boldsymbol{v}^{(k)} && && (1) \\ &\boldsymbol{v}^{(k-1)} && := (A-\lambda I)\boldsymbol{v}^{(k)} && (2) \\ &\boldsymbol{v}^{(k-2)} && := (A-\lambda I)\boldsymbol{v}^{(k-1)} = (A-\lambda I)^2\boldsymbol{v}^{(k)} && (3) \\ &\quad\vdots && \quad\vdots && (4) \\ &\boldsymbol{v}^{(1)} && := (A-\lambda I)\boldsymbol{v}^{(2)} = (A-\lambda I)^{k-1}\boldsymbol{v}^{(k)}. && (5) \end{aligned}$$

Considering

$$A = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix},$$

find enough Jordan chains to form a basis for $\mathbb{R}^9$.