TTK4225 System theory, Autumn 2023

Assignment 5

The expected output is a .pdf written in LaTeX or a Python notebook exported to .pdf, even if photos of your handwritten notes or drawings will work. Every person shall hand in her/his assignment, independently of whether it has been done together with others. When dealing with mathematical derivations, unless otherwise stated, explain how you got your answer (tips: use programming aids like Python, Matlab, Maple, or compendia like Rottmann's to check if you have obtained the right answer).

Question 1

Content units indexing this question:

- BIBO stability
- LTI systems

Explain from **both** graphical and mathematical perspectives why an harmonic oscillator (`https://en.wikipedia.org/wiki/Harmonic_oscillator`) is not a BIBO stable system.

Solution 1:

Content units indexing this solution:

- BIBO stability
- LTI systems

From a graphical point of view: an harmonic oscillator has an impulse response that is made of non-vanishing oscillations. One knows that the to get the forced response at time T one takes the impulse response of the system, flips it, puts it on top of the input up to time T, point-wise multiply, and then integrates (if this is not clear then one should definitely re-study how to use impulse responses for computing forced responses of LTIs). Choosing an input that is the same persisting oscillation as the impulse response and choosing T opportunely so that the two signals constructively interfere then leads to a point-wise multiplication between input and impulse response that is the oscillation squared (and thus always positive). The more time passes, the more this oscillation squared lasts, thus as $t \to +\infty$ we have the crests of the output oscillation that grows unbounded.
From a mathematical point of view: since the impulse response is non-vanishing, it is not absolutely integrable, that is the condition for BIBO stability for LTI systems.

Question 2

Content units indexing this question:

- linearization
- marginally stable equilibrium

Find two first-order autonomous nonlinear dynamical systems (say, system S_1 and system S_2) that have both 0 as an equilibrium, and that for S_1 the origin is unstable, while for S_2 it is asymptotically stable, and for which when linearizing that two systems around 0 the origin becomes marginally stable for both the linearized versions of these systems.

Solution 1:

An example is $\pm x^3$.

Question 3

Content units indexing this question:

- linearization
- marginally stable equilibrium

Assume that certain linear system has actually been obtained by linearizing an unknown nonlinear system around a given equilibrium. Assume the equilibrium for the linearized system to be marginally stable. Explain from **both** graphical and mathematical perspectives why given this information it is not possible to know which type of stability properties that equilibrium point has for the original nonlinear system.

Solution 1:

From a graphical point of view: think at x^3 and $-x^3$, for which both they have zero derivative at $x_{eq} = 0$. The fact is that the derivative does not capture whether the $f(x)$ is "positive before the equilibrium and negative after" or viceversa, that eventually is what drives the equilibrium to be asymptotically stable or unstable.
From a mathematical point of view: the derivative is not capturing higher order details of $f(x)$. In a sense, considering a Taylor approximation means losing information that is still important for characterizing the stability properties of the points.

Question 4

Content units indexing this question:

- linear systems of equations

1. Consider the system
$$\begin{bmatrix} 2 & -1 \\ -3 & 3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} .$$
Has the system a solution? If so, which one? If not, why?
2. Consider the system
$$\begin{bmatrix} 2 & -1 \\ -3 & 3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$
Has the system a solution? If so, which one? If not, why?

Solution 1:

Content units indexing this solution:

- systems of linear equations

The systems have solutions as soon as $\boldsymbol{b}$ is in range(A). If this happens, then the solution $\boldsymbol{x}$ is that linear combination of the columns of A that gives $\boldsymbol{b}$

Question 5

Content units indexing this question:

- equilibria
- autonomous system

Compute the equilibria of the system $\dot{\boldsymbol{x}} = A\boldsymbol{x}$ with

$$A = \begin{bmatrix} J_{\lambda_1}^{(n_{1,1})} & & & & \\ & J_{\lambda_1}^{(n_{1,2})} & & & \\ & & \ddots & & \\ & & & J_{\lambda_2}^{(n_{2,1})} & \\ & & & & \ddots \end{bmatrix}$$

i.e., A block-diagonal with the various blocks being of the type

$$J_{\lambda_\star}^{(n_\star)} = \begin{bmatrix} \lambda_\star & 1 & & & \\ & \lambda_\star & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda_\star & 1 \\ & & & & \lambda_\star \end{bmatrix} \in \mathbb{C}^{n_\star \times n_\star}$$

Solution 1:

Content units indexing this solution:

- equilibria

The problem of computing the equilibria is the problem of computing the kernel. To make some examples of the miniblocks above,

$$J_1^{(3)} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

while

$$J_2^{(5)} = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

Now each of these mini-blocks have a trivial kernel as soon as $\lambda_\star \neq 0$. If though $\lambda_\star = 0$ then the block looks like

$$J_0^{(5)} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and its kernel is spanned by $[0\ 0\ 0\ 0\ 1]^T$. I.e., the kernel of each of such miniblocks is given by a vector that has the same side-size of the miniblock, and all zeros but 1 on the bottom position. Then the kernel of A is the union of all such kernels. E.g.,

$$\ker \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \operatorname{span}\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

Index of the Content Units