TTK4225 System theory, Autumn 2023

Assignment 9

The expected output is a .pdf written in LaTeX or a Python notebook exported to .pdf, even if photos of your handwritten notes or drawings will work. Every person shall hand in her/his assignment, independently of whether it has been done together with others. When dealing with mathematical derivations, unless otherwise stated, explain how you got your answer (tips: use programming aids like Python, Matlab, Maple, or compendia like Rottmann's to check if you have obtained the right answer).

Question 1

Content units indexing this question:

- matrix exponential
- convolution

Consider the autonomous system

$$\begin{cases} \dot{\boldsymbol{x}} &= \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \boldsymbol{x} \\ y &= \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \boldsymbol{x} \end{cases}$$

Compute $y(T)$ for $T = 2$ assuming the initial condition for the system to be

$$\boldsymbol{x}_0 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

Solution 1:

Content units indexing this solution:

- transition matrix
- Jordan form

Once we have the transition matrix, that in this case is

$$e^{At} = \begin{bmatrix} e^{-t} & te^{-t} & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

we get immediately $y(t)$ as

$$y(t) = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} e^{At} \boldsymbol{x}_0.$$

Thus

$$y(2) = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} e^{-2} & 2e^{-2} & 0 \\ 0 & e^{-2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

Question 2

Content units indexing this question:

- matrix exponential
- free evolution

Consider the system

$$\begin{cases} \dot{\boldsymbol{x}} &= \begin{bmatrix} 1 & 3 & 5 \\ 2 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \boldsymbol{x} \end{cases}$$

How may one do to compute its free evolution?

Solution 1:

one should do a change of basis first, so to get A in either a diagonal or Jordan form, and then do as before. Note though that one should also perform the change of basis back to go to the original coordinates

Question 3

Content units indexing this question:

- matrix exponential

- transition matrix
- characteristic polynomial
- minimal polynomial
- modal analysis
- Jordan form

Assume $A \in \mathbb{R}^{n \times n}$ to be so that its characteristic polynomial is

$$(s-5)^3 (s-4)^2$$

and its minimal polynomial, instead,

$$(s-5)^2 (s-4) .$$

Each element in the transition matrix e^{At} will be then a combination of exponentials and exponentials multiplied by t to some power, i.e.,

$$\left[e^{At}\right]_{ij} = \sum_k \alpha_k t^{(\beta_k)} e^{\lambda_k t}$$

where i, j indicate the row and column of the element of the transition matrix. Which types of $t^(\beta_k) e^{\lambda_k t}$ do we expect to see in e^{At}? And which Jordan structure does A have? And why?

Solution 1:

The characteristic polynomial indicates that there is a Jordan block of dimension 3 associated to the eigenvalue 5, and a Jordan block of dimension 2 associated to the eigenvalue 4.
The minimal polynomial indicates that there is a Jordan miniblock of dimension 2 associated to the eigenvalue 5, and a Jordan miniblock of dimension 1 associated to the eigenvalue 4.

Considering that n has to be 5, the Jordan structure of A needs then to be:

- two Jordan miniblocks, one of dimension 2 and one of dimension 1 associated to the eigenvalue 5,
- two Jordan miniblocks, each of dimension 1, associated to the eigenvalue 4.

This implies that there will be a mode te^{5t}, in addition of the two modes e^{5t} and e^{4t}.