Calculus III Final Review

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Final Review Questions

Convergence: 10.3–10.5

Convergence Notes

- Let $\sum_{n=1}^{\infty} a_n$ be given and note for which series convergence is known, i.e.:

  Geometric: let $c \neq 0$, if $|r| < 1$, then

  $$\sum_{n=0}^{\infty} cr^n = \frac{c}{1-r}$$

  $|r| > 1 \implies$ diverges

  ***p*-Series**: converges if $p > 1$.

  $$\sum_{n=0}^{\infty} \frac{1}{n^p}$$

  $p < 1 \implies$ diverges

- **The n^{th} Term Divergence Test**: a relatively easy test that can be used to quickly determine if a test diverges if the $\lim_{n\to\infty} a_n \neq 0$. If $\lim_{n\to\infty} a_n = 0$, then the test is inconclusive and other tests must be applied.

Tests for Positive Series

- **Direct Comparison Test**: use if dropping terms from the denominator or numerator gives a series b_n wherein convergence is easily found, then compare to the original series a_n as follows:

  $$\sum_{n=1}^{\infty} b_n \text{ converges} \implies \sum_{n=1}^{\infty} a_n \text{ converges} \quad \leftarrow 0 \leq a_n \leq b_n$$

  $$\sum_{n=1}^{\infty} b_n \text{ diverges} \implies \sum_{n=1}^{\infty} a_n \text{ diverges} \quad \leftarrow 0 \leq b_n \leq a_n$$

- **Limit Comparison Test**: use when the direct comparison test isn't convenient or when comparing two series. One can to take the dominant term in the numerator and denominator from a_n to form a new positive sequence b_n if needed.

  Assuming the following limit $L = \lim_{n\to\infty} \frac{a_n}{b_n}$ exists, then:

  $$L > 0 \implies \sum_{n=1}^{\infty} a_n \text{ converges} \iff \sum_{n=1}^{\infty} b_n \text{ converges}$$

  $$L = 0 \text{ and } \sum_{n=1}^{\infty} b_n \text{ converges} \implies \sum_{n=1}^{\infty} a_n \text{ converges}$$

  $$L = \infty \text{ and } \sum_{n=1}^{\infty} a_n \text{ converges} \implies \sum_{n=1}^{\infty} b_n \text{ converges}$$

- **Ratio Test**: often used in the presence of a factorial $(n!)$ or when the are constants raised to the power of n (c^n).

  Assuming the following limit $\rho = \lim_{n\to\infty} \left|\frac{a_n+1}{a_n}\right|$ exists, then

$$\begin{aligned}
\rho < 1 &\implies \sum a_n \text{ converges absolutely} \\
\rho > 1 &\implies \sum a_n \text{ diverges} \\
\rho = 1 &\implies \text{test is inconclusive}
\end{aligned}$$

- **Root Test**: used when there is a term in the form of $f(n)^{g(n)}$.

  Assuming the following limit $C = \lim_{n\to\infty} |a_n|^{\frac{1}{n}}$ exists, then

$$\begin{aligned}
C < 1 &\implies \sum a_n \text{ converges absolutely} \\
C > 1 &\implies \sum a_n \text{ diverges} \\
C = 1 &\implies \text{test is inconclusive}
\end{aligned}$$

- **Integral Test**: if the other tests fail and $a_n = f(n)$ is a decreasing function, then one can use the improper integral $\int_1^\infty f(x)dx$ to test for convergence.

  Let $a_n = f(n)$ be a positive, decreasing, and continuous function $\forall x \geq 1$, then:

$$\begin{aligned}
\int_1^\infty f(x)dx \text{ converges} &\implies \sum_{n=1}^\infty a_n \text{ converges} \\
\int_1^\infty f(x)dx \text{ diverges} &\implies \sum_{n=1}^\infty a_n \text{ diverges}
\end{aligned}$$

Tests for Non-Positive Series

- **Alternating Series Test**: used for series in the form $\sum_{n=0}^\infty (-1)^n a_n$

  Converges if $|a_n|$ decreases monotonically $(|a_n+1| \leq |a_n|)$ and if $\lim_{n\to\infty} a_n = 0$

- **Absolute Convergence**: used if the series $\sum a_n$ is not alternating (if it is alternating, use the alternating test in conjunction); simply test if $\sum |a_n|$ converges using the test for positive series.

  $\sum a_n$ converges **conditionally** if $\sum a_n$ converges, but $\sum |a_n|$ diverges.

  $\sum a_n$ converges **absolutely** if $\sum |a_n|$ converges.

Convergence Problems

10.5 Exercises

Determine convergence or divergence using any method.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n + 4^n}{7^n}$

$$\implies \sum_{n=1}^{\infty} \frac{2^n}{7^n} + \sum_{n=1}^{\infty} \frac{4^n}{7^n} \qquad \text{Separate into two geometric series}^{\uparrow}$$
$$\implies r = \frac{2}{7} < 1, \quad r = \frac{4}{7} < 1$$

Both geometric series converge, thus the original series converges.

2. $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{n!}$

$$\begin{aligned}
\implies \rho &= \lim_{n\to\infty} \left| \frac{(n+1)^3}{(n+1)!} \cdot \frac{n!}{n^3} \right| \qquad \text{Apply the ratio test}^{\uparrow} \\
&= \lim_{n\to\infty} \frac{n^3 + 3n^2 + 3n + 1}{(n+1)n!} \cdot \frac{n!}{n^3} \\
&= \lim_{n\to\infty} \frac{n^3 + 3n^2 + 3n + 1}{n^4 + n^3} \\
&= \lim_{n\to\infty} \frac{n^3 + 3n^2 + 3n + 1}{n^4 + n^3} \cdot \frac{n^{-4}}{n^{-4}} \\
&= \lim_{n\to\infty} \frac{n^{-1} + 3n^{-2} + 3n^{-3} + n^{-4}}{1 + n^{-1}} = 0
\end{aligned}$$

$\rho = 0 < 1$, thus the series converges.

3. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2n+1}$

$$\implies \lim_{n\to\infty} \frac{n}{2n+1} \qquad \text{Apply the } n^{th} \text{ term test}^{\uparrow}$$

$$\implies \lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2} \qquad \text{By L'Hôpital's Rule}$$

$\lim\limits_{n\to\infty} a_n \neq 0$, thus the series diverges.

4. $\displaystyle\sum_{n=1}^{\infty} 2^{\frac{1}{n}}$

$$\implies \lim_{n\to\infty} 2^{\frac{1}{n}} = 2^0 = 1 \qquad \text{Apply the } n^{th} \text{ term test}^{\uparrow}$$

$\lim\limits_{n\to\infty} a_n \neq 0$, thus the series diverges.

5. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$

$$0 \leq \sin n \leq 1 \qquad \leftarrow \forall n \geq 1$$

$$0 \leq \frac{\sin n}{n^2} \leq \frac{1}{n^2} \qquad \text{Apply the direct comparison test}^{\uparrow}$$

$$b_n = \frac{1}{n^2} \rightarrow \text{converges} \qquad \text{by } p\text{-series}^{\uparrow}$$

The larger (b_n) series converges, thus the smaller (a_n) converges.

6. $\sum_{n=1}^{\infty} \frac{n!}{(2n)!}$

$$\begin{aligned}
\implies \rho &= \lim_{n\to\infty} \left| \frac{(n+1)!}{(2n+2)!} \cdot \frac{(2n)!}{n!} \right| && \text{Apply the ratio test}^{\uparrow} \\
&= \lim_{n\to\infty} \frac{(n+1)n!}{(2n+2)(2n+1)2n!} \cdot \frac{(2n)!}{n!} \\
&= \lim_{n\to\infty} \frac{n+1}{(2n+2)(2n+1)} = \frac{n+1}{4n^2+6n+2} \\
&= \lim_{n\to\infty} \frac{1}{8n+6} = 0 && \text{By L'Hôpital's Rule}
\end{aligned}$$

$\rho = 0 < 1$, thus the series converges.

7. $\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$

$$\begin{aligned}
0 \le n &\le n+\sqrt{n} && \leftarrow \forall n \ge 1 \\
0 \le \frac{1}{n+\sqrt{n}} &\le \frac{1}{n} && \text{Apply the direct comparison test}^{\uparrow} \\
b_n = \frac{1}{n} &\to \text{ diverges}
\end{aligned}$$

The smaller (b_n) series diverges, thus the larger a_n original series diverges.

8. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$

f is positive, decreasing, and continuous for $x \geq 2$ Apply the integral test↑

$$\implies \int_2^{\infty} f(x)dx = \lim_{R\to\infty} \int_2^R \frac{1}{x(\ln x)^3}dx \qquad \ln x = u, \quad xdu = dx$$

$$\begin{aligned}\implies \lim_{R\to\infty} \int_2^R \frac{1}{x(u)^3} xdu &= \int_2^R \frac{1}{u}^3 du \\ &= -\frac{1}{2(u)^2} \\ &= -\frac{1}{2\ln^2(x)} + C \Bigg|_2^{\infty}\end{aligned}$$

$$\implies 0 - \left(-\frac{1}{2\ln^2(2)}\right) = \frac{1}{2\ln^2(2)}$$

The improper integral converges, thus the original series converges.

9. $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{5^n}$

$$\begin{aligned}\implies \rho &= \lim_{n\to\infty} \left| \frac{(n+1)^3}{5^{n+1}} \cdot \frac{5^n}{n^3} \right| \\ &= \lim_{n\to\infty} \frac{n^3+1}{5^n+5^1} \cdot \frac{5^n}{n^3} = \frac{1}{5}\end{aligned}$$

Apply the ratio test↑

$\rho = \dfrac{1}{5} < 1$, thus the series converges.

10. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^3 - n^2}}$

$$\begin{aligned} L &= \lim_{n\to\infty} \frac{a_n}{b_n}, \quad b_n = \frac{1}{\sqrt{n^3}} \\ \Longrightarrow L &= \lim_{n\to\infty} \frac{1}{\sqrt{n^3 - n^2}} \cdot \frac{\sqrt{n^3}}{1} \\ &= \lim_{n\to\infty} \sqrt{\frac{n^3}{n^3(1 - n^{-1})}} \\ &= \sqrt{\frac{1}{1(1-0)}} = 1 \end{aligned}$$

Apply the limit comparison test↑

$L > 0$, thus a_n converges if b_n converges.

b_n converges by the p-series test, as $\frac{3}{2} > 1$, thus a_n converges.

11. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 + 4n}{3n^4 + 9}$

$$\begin{aligned} L &= \lim_{n\to\infty} \frac{a_n}{b_n}, \quad b_n = \frac{1}{n^2} \\ &= \lim_{n\to\infty} \frac{n^2 + 4n}{3n^4 + 9} \cdot n^2 \\ &= \lim_{n\to\infty} \frac{n^4 + 4n^3}{3n^4 + 9} \cdot \frac{n^{-4}}{n^{-4}} \\ &= \lim_{n\to\infty} \frac{1 + 4n^{-1}}{3 + 9n^{-4}} = \frac{1}{3} \end{aligned}$$

Apply the limit comparison test↑

$L > 0$, thus a_n converges if b_n converges.

b_n converges by the p-series test, as $2 > 1$, thus a_n converges.

12. $\displaystyle\sum_{n=1}^{\infty}(0.8)^{-n}n^{-0.8}$

$$\begin{aligned}
\rho &= \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \\
&= \lim_{n\to\infty}\left|\frac{(0.8)^{-(n+1)}(n+1)^{-0.8}}{(0.8)^{-n}n^{-0.8}}\right| \\
&= \lim_{n\to\infty}\frac{(0.8)^{n}n^{0.8}}{(0.8)^{n+1}(n+1)^{0.8}} \\
&= \lim_{n\to\infty}\frac{1}{0.8} = 1.25
\end{aligned}$$

Apply the ratio test[↑]

$\rho = 1.25 > 1$, thus a_n diverges.

13. $\displaystyle\sum_{n=1}^{\infty}4^{-2n+1}$

$$\begin{aligned}
\rho &= \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \\
&= \lim_{n\to\infty}\frac{4^{-2(n+1)+1}}{4^{-2n+1}} \\
&= \lim_{n\to\infty}\frac{4^{-2n-1}}{4^{-2n+1}} \\
&= \lim_{n\to\infty}\frac{4^{-2n}4^{-1}}{4^{-2n}4} = \frac{1}{16}
\end{aligned}$$

Apply the ratio test[↑]

$\rho = \frac{1}{16} < 1$, thus a_n converges.

14. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$

$$\sum_{n=1}^{\infty} |a_n| \qquad \text{Apply the Absolute convergence test}^{\uparrow}$$

$$\implies \sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$$

$|a_n|$ diverges by the p-series, as $\frac{1}{2} < 1$, meaning a_n converges conditionally since $|a_n|$ decreases monotonically and $\lim_{n\to\infty} a_n = 0$

15. $\displaystyle\sum_{n=1}^{\infty} \sin \frac{1}{n^2}$

$$L = \lim_{n\to\infty} \frac{a_n}{b_n}, \quad b_n = \frac{1}{n^2} \qquad \text{Apply the limit comparison test}^{\uparrow}$$

$$\implies \lim_{n\to\infty} \frac{\sin(n^{-2})}{n^{-2}} = \frac{0}{0}$$

$$= \lim_{n\to\infty} \frac{\cos(n^{-2})(-2n^{-3})}{-2n^{-3}} \qquad \text{by L'Hôpital's Rule}$$

$$= \lim_{n\to\infty} \cos(n^{-2}) = 1$$

$L > 0$, thus a_n converges if b_n converges.

b_n converges by the p-series test, as $2 > 1$, thus a_n converges.

16. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \cos n^{-1}$

$$\sum_{n=1}^{\infty} (-1)^n a_n, \quad a_n = \cos(n^{-1}) \qquad \text{Apply the alternating series test}^{\uparrow}$$

$$\implies L = \lim_{n\to\infty} \cos(n^{-1}) = 1$$

$L \neq 0$, thus the series diverges

17. $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{\sqrt{n}}$

$$\sum_{n=1}^{\infty} (-1)^n a_n, \quad a_n = \frac{2^n}{\sqrt{n}} \qquad \text{Apply the alternating series test}^{\uparrow}$$

$$\begin{aligned}
\implies L = \lim_{n\to\infty} \frac{2^n}{\sqrt{n}} &= \frac{\infty}{\infty} \\
&= \frac{2^n \ln 2}{\frac{1}{2} n^{-\frac{1}{2}}} = 2^n \ln 2 \cdot 2\sqrt{n} \qquad \text{By L'Hôpital's Rule} \\
&= 2 \lim_{n\to\infty} 2^n \ln(2)\sqrt{n} = \infty
\end{aligned}$$

$L \neq 0$, thus the series diverges

18. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{n+12}\right)^n$

$$\begin{aligned}
L &= \lim_{n\to\infty} a_n \neq 0 \to \text{ diverges} && \text{Apply the } n^{th} \text{ term test}^{\uparrow} \\
\implies L &= \lim_{n\to\infty} \left(\frac{n}{n+12}\right)^n \\
&= \lim_{n\to\infty} e^{-12} && \text{By common limit } \left(\frac{x}{x+k}\right)^x = e^{-k}
\end{aligned}$$

$L \neq 0$, thus the series diverges.

Power/Taylor Series: 10.6–10.8

Power/Taylor Series Notes

Power Series

- **Power series**: a infinite series in the form:

$$F(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

  Where the constant c is the *center* of the power series F(x).

- **Radius of convergence** R: the range of values of the variable x whereby the power series $F(x)$ converges.
  - Every power series converges at $x = c$, as $(x-c)^0 = 1$, though the series may diverge for other values of x.
  - $F(x)$ converges for $|x-c| < R$ and diverges for $|x-c| > R$
  - $F(x)$ may converge of diverge at endpoints $c - R$ and $c + R$
  - **Interval of convergence**: the open interval $(c-R, c+R)$ and possibly one of both of the endpoints, each must be tested.
    - In most cases, the ratio test↑ can be used to find R.
    - If $R > 0$, then F is differentiable over the interval of convergence; the derivative and antiderivative can be obtained using the following:

$$F'(x) = \sum_{n=1}^{\infty} n a_n (x-c)^{n-1} \qquad F(x)dx = C + \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-c)^{n+1}$$

  - **Useful Power Series**: the following power series (more examples: Taylor series↓) can be used to drive expansions of other related functions via substitution, integration, or differentiation:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \leftarrow |x| < 1 \qquad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Taylor Series

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Power/Taylor Series Problems

10.6 Exercises

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10.8 Exercises

1.

Parametric Equations: 11.1

Parametric Problems

11.1 Exercises

1.

Arc Length, Polar Coordinates: 11.2–11.4

11.2–11.4 Notes

Arc Length and Speed

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Polar Coordinates

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Area and Arc Length in Polar Coordinates

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Polar Coordinate Problems

11.2 Exercises

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11.3 Exercises

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11.4 Exercises

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Conic Section Problems

11.5 Exercises

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Quiz 3

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Quiz 4

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Final Review Questions