

Applied Linear Algebra



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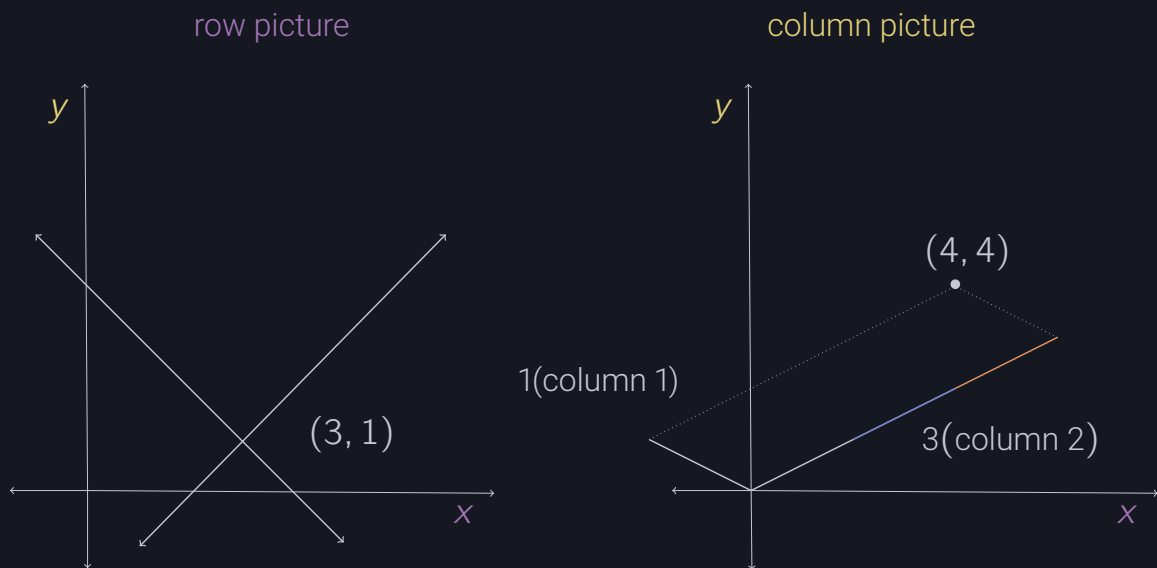
1 Matrices and Gaussian Elimination



1.2 The Geometry of Linear Equations

Problems 1–12

- For the equations $x + y = 4$, $2x - 2y = 4$, draw the row picture (two intersecting lines) and the column picture (combination of two columns equal to the column vector $(4, 4)$ on the right side).



1.2.1

- Solve to find a combination of the columns that equals b :

$$u - v - w = b_1$$

$$v + w = b_2$$

$$w = b_3$$

$$\implies w = b_3$$

$$\implies v = b_2 - b_3$$

$$\implies u = b_1 + v + w = b_1 + b_2$$

- Describe the intersection of the three planes $u + v + w + z = 6$ and $u + w + z = 4$ and $u + w = 2$ (all in four-dimensional space). Is it a line or a point or an empty set? What is the intersection if the fourth plane $u = -1$ is included? Find a fourth equation that leaves us with no solution.

- A line; as $u + w = 2$ is only a line[?]. A fourth plane with $u = -1$ would produce a normally intersecting point. Any addition equation when $u + w \neq 2$ would produce an inconsistent equation.

4. Sketch these three lines and decide if the equations are solvable:

$$x + 2y = 2$$

$$x - y = 2$$

$$y = 1$$



1.2.4

Inconsistent; multiple points of intersect

What happens if all right-hand sides are zero? Is there any nonzero choice of right-hand sides that allows the three lines to intersect at the same point?

- If all the solutions were zero, then it would be a trivial solution.
 - Yes, e.g., $x - y = -1$ would produce a single point of intersection.
5. Find two points on the line of intersection of the three planes $t = 0$ and $z = 0$ and $x + y + z + t = 1$ in four-dimensional space.

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

6. When $b = (2, 5, 7)$, find a solution (u, v, w) to equation (4) different from the solution $(1, 0, 1)$ mentioned in the text.
- Since there are infinite solutions, and if \mathbf{s} vector describing one solution and λ is any scalar, then $\mathbf{s}\lambda$ is also a solution. E.g., $(1, 0, 1)42 = (42, 0, 42)$

8. Explain why the system

$$\begin{aligned}u + v + w &= 2 \\u + 2v + 3w &= 1 \\v + 2w &= 0\end{aligned}$$

is singular by finding a combination of the three equations that adds up to $0 = 1$. What value should replace the last zero on the right side to allow the equations to have solutions—and what is one of the solutions?

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 0 \end{array} \right] \xrightarrow{R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 2 & 0 \end{array} \right] \xrightarrow{R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

- Replacing the last zero with -1 would yield infinite solutions. One solution would be $[3, -1, 0]^T$
9. The column picture for the previous exercise (singular system) is

$$u \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + v \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + w \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = b$$

Show that the three columns on the left lie in the same plane by expressing the third as a combination of the first two. What are all the solutions (u, v, w) if b is the zero vector $(0, 0, 0)$?

$$-1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

- If b is equal to the zero vector $\mathbf{0}$ then the solutions are equal to the kernel² i.e., $-1x_1, 2x_2, 0x_3 = \mathbf{0}$
10. Under what condition on y_1, y_2, y_3 do the points $(0, y_1), (1, y_2), (2, y_3)$ lie on a straight line?
- Question 9 describes the state at which they are collinear, i.e., $y_3 = 2y_2 - y_1$
11. These equations are certain to have the solution $x = y = 0$. For which values of a is there a whole line of solutions?

$$\begin{aligned}ax + 2y &= 0 \\2x + ay &= 0\end{aligned}$$

- Only the scalars that make the lines linearly dependent, i.e., $a = 2, -2$

Problems 17–23

17. The first of these equations plus the second equals the third:

$$\begin{aligned}x + y + z &= 2 \\x + 2y + z &= 3 \\2x + 3y + 2z &= 5\end{aligned}$$

The first two planes meet along a line. The third plane contains that line, because if x, y, z satisfy the first two equations then they also **span all of \mathbb{R}^3** . The equations have infinitely many solutions (the whole line L). Find three solutions.

◦ $\mathbf{v} = (4, 4, 0)$, $\mathbf{w} = (6, 3, 2)$, $\mathbf{u} = 2\mathbf{v} + -1\mathbf{w}$

18. Move the third plane in Problem 17 to a parallel plane $2x + 3y + 2z = 9$. Now the three equations have no solution—*why not*? The first two planes meet along the line L , but the third plane doesn't that **cross** that line.
19. In Problem 17 the columns are $(1, 1, 2)$ and $(1, 2, 3)$ and $(1, 1, 2)$. This is a “singular case” because the third column is **linearly dependent**. Find two combinations of the columns that give $\mathbf{b} = (2, 3, 5)$. This is only possible for $\mathbf{b} = (4, 6, c)$ if $c = 10$
20. Normally 4 “planes” in four-dimensional space meet at a **tensor**. Normally 4 column vectors in four-dimensional space can combine to produce \mathbf{b} . What combination of $(1, 0, 0, 0)$, $(1, 1, 0, 0)$, $(1, 1, 1, 0)$, $(1, 1, 1, 1)$ produces $\mathbf{b} = (3, 3, 3, 2)$? $(1, 0, 0, -2)$? What 4 equations for x, y, z, t are you solving? A **lower triangular matrix**, i.e.,

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3 \\ 1 & 1 & 0 & 0 & 3 \\ 1 & 1 & 1 & 0 & 3 \\ 1 & 1 & 1 & 1 & 1 \end{array} \right]$$

21. When equation 1 is added to equation 2, which of these are changed: the planes in the row picture, the column picture, the coefficient matrix, the solution?
- Row operations do not change the solution. Row 2 is changed, thus the second plane is changed. **All columns are changed.**?

1.3 Gaussian Elimination

Problems 6, 7

6. Choose a coefficient b that makes this system singular. Then choose a right-hand side g that makes it solvable. Find two solutions in that singular case.

$$2x + by = 16$$

$$4x + 8y = g$$

$$2x + 4y = 16$$

$$4x + 8y = 32$$

- Since R_2 is just a multiple of R_1 , then solving for x, y , with one variable = 0, in the first equation will yield two solutions, i.e., $(8, 0), (0, 4)$
7. For which numbers a does elimination break down (a) permanently, and (b) temporarily?

$$ax + 3y = -3$$

$$4x + 6y = 6$$

Solve for x and y after fixing the second breakdown by a row exchange.

- Permanently: $a = 2$ (linearly dependent, no solution)
- Temporarily: $a = 0$;

$$\left[\begin{array}{cc|c} 4 & 6 & 6 \\ 0 & 3 & -3 \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -1 \end{array} \right]$$
$$y = -1, \quad x = 3$$

Problems 17, 18, 19

17. Which number q makes this system singular and which right-hand side t gives it infinitely many solutions? Find the solution that has $z = 1$.

$$x + 4y - 2z = 1$$

$$x + 7y - 6z = 6$$

$$3y + qz = t$$

$$x + 4y - 2z = 1$$

$$x + 7y - 6z = 6$$

$$3y + -4z = 5$$

- If $q = -4$, then R_3 would have no pivot
- If $t = 5$, then there would be finite solutions, R_3 would be linearly dependent with R_2

18. It is impossible for a system of linear equations to have exactly two solutions. Explain why.

- If (x, y, z) and (X, Y, Z) are two solutions, what is the other one?
 - There is no other *one*, there would be infinitely many.
- If 25 planes meet at two points, where else do they meet?
 - Every other single point, they would span all of \mathbb{R}^3

19. Three planes can fail to have an intersection point, when no two planes are parallel. The system is singular if row 3 of \mathbf{A} is a **linearly dependent; a combination** of the first two rows. Find a third equation that can't be solved if $x + y + z = 0$ and $x - 2y - z = 1$.

$$x + y + z = 0$$

$$x - 2y - z = 1$$

$R_1 + R_2 \neq 1 \rightarrow$ parallel; no solution, e.g.,

$$2x - y = 42$$

Problems 30, 31

30. Use elimination to solve

$$u + v + w = 6$$

$$u + 2v + 2w = 11$$

$$2u + 3v - 4w = 3$$

$$u + v + w = 7$$

$$u + 2v + 2w = 10$$

$$2u + 3v - 4w = 3$$

$$\text{rref} \left(\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 2 & 11 \\ 2 & 3 & -4 & 3 \end{array} \right] \right) \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

$$\text{rref} \left(\left[\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 1 & 2 & 2 & 10 \\ 2 & 3 & -4 & 3 \end{array} \right] \right) \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

31. For which three numbers a will elimination fail to give three pivots?

$$ax + 2y + 3z = b_1$$

$$ax + ay + 4z = b_2$$

$$ax + ay + az = b_3$$

- For $a = 0$, multiple failures.
- For $a = 2$, columns 0, 1 would be equal.
- For $a = 4$, rows 1, 2 would be equal.

1.4 Matrix Notation and Matrix Multiplication

Problems 4, 10, 17, 19

4. If an $m \times n$ matrix \mathbf{A} multiplies an n -dimensional vector \mathbf{x} , how many separate multiplications are involved? What if \mathbf{A} multiplies an $n \times p$ matrix \mathbf{B} ?

- $m \cdot n$ multiplications; number of rows times the length of \mathbf{x} .
- $m \cdot n \cdot p$; same as above, except accounting for each additional column p .

10. True or false? Give a specific counterexample when false.

- If rows 1 and 3 of \mathbf{B} are the same, so are rows 1 and 3 of \mathbf{AB} .
- **✗ false**; matrix multiplication is done by the rows of the left matrix and the columns of the right, the rows may be the same, but if a column between the two are different, then there would be different multiplications occurring, e.g.,

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 12 & 8 & 4 \\ 30 & 20 & 10 \\ 38 & 32 & 16 \end{bmatrix}$$

- If columns 1 and 3 of \mathbf{B} are the same, so are columns 1 and 3 of \mathbf{AB} .
- **✓ true**;
- If rows 1 and 3 of \mathbf{A} are the same, so are rows 1 and 3 of \mathbf{AB} .
- **✓ true**
- $(\mathbf{AB})^2 = \mathbf{A}^2 \mathbf{B}^2$.
- **✗ false** (most of the time), e.g.,

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\mathbf{AB}^2 = \begin{bmatrix} 144 & 64 & 16 \\ 900 & 400 & 100 \\ 2304 & 1024 & 256 \end{bmatrix} \neq \begin{bmatrix} 74 & 26 & 10 \\ 452 & 152 & 52 \\ 1154 & 386 & 130 \end{bmatrix} = \mathbf{A}^2 \mathbf{B}^2$$

17. Which of the following matrices are guaranteed to equal $(\mathbf{A} + \mathbf{B})^2$?

- $\mathbf{A}^2 + 2\mathbf{AB} + \mathbf{B}^2$,
- ✓** $\mathbf{A}(\mathbf{A} + \mathbf{B}) + \mathbf{B}(\mathbf{A} + \mathbf{B})$
- ✓** $(\mathbf{A} + \mathbf{B})(\mathbf{B} + \mathbf{A})$,
- ✓** $\mathbf{A}^2 + \mathbf{AB} + \mathbf{BA} + \mathbf{B}^2$

19. A fourth way to multiply matrices is columns of **A** times rows of **B**:

$$\mathbf{AB} = (\text{column 1})(\text{row 1}) + \cdots + (\text{column } n)(\text{row } n) = \text{sum of simple matrices.}$$

Give a 2×2 example of this important rule for matrix multiplication.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \left(a \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right) b \begin{bmatrix} 1 \\ 3 \end{bmatrix} + d \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

Useful, as the right matrix can be thought of as the **weights that scale** the elements of the columns of the left matrix.

Problems 30–31

30. Multiply these matrices:

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \\ 1 & 4 & 0 \end{bmatrix}.$$

$$\begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 0 & 2 & -3 \end{bmatrix} \quad \text{respectively}$$

- The former multiplication performs two operations (left: swaps top and bottom columns, right: swaps left and right columns), while the latter subtracts row 1 from both row 2 and row 3.

31. This 4×4 matrix needs which elimination matrices **E**₂₁ and **E**₃₂ and **E**₄₃?

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

- $e_{21} = -\frac{1}{2}$, $e_{32} = -\frac{2}{3}$, $e_{43} = -\frac{3}{4}$
- I suspect the fractions will tend towards -1 if the matrix was expanded upon in a similar fusion?

Problems 34, 35, 38, 42

34. Multiply these matrices in the orders FE and EF and E^2 :

$$E = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & 0 & 1 \end{bmatrix} \quad F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{bmatrix}$$

$$FE = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ ac + b & c & 1 \end{bmatrix} \quad EF = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix} \quad E^2 = \begin{bmatrix} 1 & 0 & 0 \\ 2a & 1 & 0 \\ 2b & 0 & 1 \end{bmatrix}$$

35. ↓

- (a) Suppose all columns of B are the same. Then all columns of EB are the same, because each one is E times B_{1n} .
- (b) Suppose all rows of B are $[1 \ 2 \ 4]$. Show by example that all rows of EB are not $[1 \ 2 \ 4]$. It is true that those rows are multiples of $[1 \ 2 \ 4]$
 - E.g., if $e_{12} = 2$, then m_2 of EB would be $[3 \ 6 \ 12]$

38. If $AB = I$ and $BC = I$, use the associative law to prove $A = C$.

$$A = A(BC)$$

$$A = (AB)C$$

$$A = C$$

42. True or false?


- (a) If A^2 is defined then A is necessarily square.
 - ✓ true; inner dimensions must match, i.e., dimensions of $n_1 = m_2$. Thus, A must be square.
- (b) If AB and BA are defined, then A and B are square.
 - ✗ false; if $A = 6 \times 9$ and $B = 9 \times 6$ allows for valid pre- and post-multiplication of B .
- (c) If AB and BA are defined, then AB and BA are square.
 - ✓ true; see above example, each case will still yield square matrices. Not a proof, but I can't see another way to falsify (b).
- (d) If $AB = B$ then $A = I$
 - ✗ false; e.g., $B = 0$

1.5 Triangular Factors and Row Exchanges

Problems 1, 6, 7, 14, 18

- When is an upper triangular matrix nonsingular (a full set of pivots)?
 - Every pivot **must be nonzero**. If there is a zero on one of the pivots, then it indicates that one of the columns is a linear combination of one or more of the other columns.
- Find E^2 and E^8 and E^{-1} if


$$E = \begin{bmatrix} 1 & 0 \\ 6 & 1 \end{bmatrix}$$

matrices 

$$E^2 = \begin{bmatrix} 1 & 0 \\ 36 & 1 \end{bmatrix} \quad E^8 = \begin{bmatrix} 1 & 0 \\ 1679616 & 1 \end{bmatrix} \quad E^{-1} = \begin{bmatrix} 1 & 0 \\ -6 & 1 \end{bmatrix}$$

- Find the products FGH and HGF if (with upper triangular zeros omitted)

$$F = \begin{bmatrix} 1 & & & \\ 2 & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad G = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 2 & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad H = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

matrices 


$$FGH = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix} \quad HGF = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 4 & 2 & 1 & 0 \\ 8 & 4 & 2 & 1 \end{bmatrix}$$

14. 14. Write down all six of the 3×3 permutation matrices, including $P = I$. Identify their inverses, which are also permutation matrices. The inverses satisfy $PP^{-1} = I$ and are on the same list.

$$\begin{array}{ll}
 P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
 P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} & P^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \\
 P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} & P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \\
 P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} & P^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \\
 P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} & P^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
 P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} & P^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}
 \end{array}$$

18. 18. Decide whether the following systems are singular or nonsingular, and whether they have no solution, one solution, or infinitely many solutions:

$$\begin{bmatrix} 0 & 1 & -1 & | & 2 \\ 1 & -1 & 0 & | & 2 \\ 1 & 0 & -1 & | & 2 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & -1 & | & 0 \\ 1 & -1 & 0 & | & 0 \\ 1 & 0 & -1 & | & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 1 & | & 1 \\ 1 & 1 & 0 & | & 1 \\ 1 & 0 & 1 & | & 1 \end{bmatrix}$$

matrices 

- Performing rref on above matrices yields:
 - Singular — no solution
 - Singular — ∞ solutions.
 - Nonsingular — one solution $[0.5 \ 0.5 \ 0.5]$

Problems 26, 28

26. Which number c leads to zero in the second pivot position? A row exchange is needed and $\mathbf{A} = \mathbf{L}\mathbf{U}$ is not possible. Which c produces zero in the third pivot position? Then a row exchange can't help and elimination fails.

$$\mathbf{A} = \begin{bmatrix} 1 & c & 0 \\ 2 & 4 & 1 \\ 3 & 5 & 1 \end{bmatrix}$$

- If $c = 2$ then row 2 would have a 0 in the pivot, yielding:

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 3 & 5 & 1 \end{bmatrix}$$

- If $c = 1$, then you could take the matrix down to the following form,

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

which would yield a singular matrix with infinite solutions.

28. \mathbf{A} and \mathbf{B} are symmetric across the diagonal (because $4 = 4$). Find their triple factorizations $\mathbf{L}\mathbf{U}$ and say how \mathbf{U} is related to \mathbf{L} for these symmetric matrices:

$$\mathbf{A} = \begin{bmatrix} 2 & 4 \\ 4 & 11 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 & 4 & 0 \\ 4 & 12 & 4 \\ 0 & 4 & 0 \end{bmatrix}$$

$$\mathbf{U}_A = \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix}, \quad \mathbf{U}_B = \begin{bmatrix} 1 & 4 & 0 \\ 0 & -4 & 4 \\ 0 & 0 & -4 \end{bmatrix}$$

$$\mathbf{V}_A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{V}_B = \begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$


$$\mathbf{L}_A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \quad \mathbf{L}_B = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

- $\mathbf{A}, \mathbf{B} = \mathbf{L}\mathbf{D}\mathbf{V} \implies \mathbf{L} = \mathbf{V}^T$; the diagonal of the upper matrix, if reduced to 1's in the pivot positions, yields the transpose of the lower triangular matrix.

Problems 33, 43

33. Solve $Lc = b$ to find c . Then solve $Ux = c$ to find x . What was A ?

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

matrices 

$$c = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} \quad x = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \implies A = LU = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

43. (Try this question) Which permutation makes PA upper triangular? Which permutations make P_1AP_2 lower triangular? Multiplying A on the right by P_2 exchanges the columns of A .

$$A = \begin{bmatrix} 0 & 0 & 6 \\ 1 & 2 & 3 \\ 0 & 4 & 5 \end{bmatrix}$$

$$U = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 6 \\ 1 & 2 & 3 \\ 0 & 4 & 5 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 6 \\ 1 & 2 & 3 \\ 0 & 4 & 5 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

1.6 Inverses and Transposes

Problems 3, 12, 18

3. From $\mathbf{AB} = \mathbf{C}$ find a formula for \mathbf{A}^{-1} . Also find \mathbf{A}^{-1} from $\mathbf{PA} = \mathbf{LU}$.

$$\mathbf{AB} = \mathbf{C}$$

$$\mathbf{A} = \mathbf{CB}^{-1}$$

$$\mathbf{A}^{-1} = \mathbf{BC}^{-1}$$

$$\mathbf{PA} = \mathbf{LU}$$

$$\mathbf{A} = \mathbf{P}^{-1}\mathbf{LU}$$

$$\mathbf{A}^{-1} = \mathbf{U}^{-1}\mathbf{L}^{-1}\mathbf{P}$$

12. If \mathbf{A} is invertible, which properties of \mathbf{A} remain true for \mathbf{A}^{-1} ?

(a) \mathbf{A} is triangular. ✓ true

(b) \mathbf{A} is symmetric. ✓ true

(c) \mathbf{A} is tridiagonal. ✗ false

(d) All entries are whole ✗ false

(e) All entries are fractions (including numbers like $\frac{3}{1}$). ✓ true;

18. Under what conditions on their entries are \mathbf{A} and \mathbf{B} invertible?

$$\mathbf{A} = \begin{bmatrix} a & b & c \\ d & e & 0 \\ f & 0 & 0 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & e \end{bmatrix}$$

matrices 🧩

$$\begin{aligned} \text{rref} \left(\left[\begin{array}{ccc|ccc} a & b & c & 1 & 0 & 0 \\ d & e & 0 & 0 & 1 & 0 \\ f & 0 & 0 & 0 & 0 & 1 \end{array} \right] \right) &\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & f^{-1} \\ 0 & 1 & 0 & 0 & e^{-1} & -\frac{d}{ef} \\ 0 & 0 & 1 & c^{-1} & -\frac{b}{ce} & \frac{-ae+bd}{cef} \end{array} \right] \\ \text{rref} \left(\left[\begin{array}{ccc|ccc} a & b & 0 & 1 & 0 & 0 \\ c & d & 0 & 0 & 1 & 0 \\ 0 & 0 & e & 0 & 0 & 1 \end{array} \right] \right) &\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{d}{ad-bc} & -\frac{b}{ad-bc} & 0 \\ 0 & 1 & 0 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} & 0 \\ 0 & 0 & 1 & 0 & 0 & e^{-1} \end{array} \right] \end{aligned}$$

$$\bullet \mathbf{A} \rightarrow \mathbf{A}^{-1} \iff c, e, f \neq 0$$

$$\bullet \mathbf{B} \rightarrow \mathbf{B}^{-1} \iff e \neq 0 \wedge ad \neq bc$$

Problems 21, 28, 41, 56, 58

21. (Remarkable) If \mathbf{A} and \mathbf{B} are square matrices, show that $\mathbf{I} - \mathbf{BA}$ is invertible if $\mathbf{I} - \mathbf{AB}$ is invertible. Start from $\mathbf{B}(\mathbf{I} - \mathbf{AB}) = (\mathbf{I} - \mathbf{BA})\mathbf{B}$

$$\begin{aligned}\mathbf{B}(\mathbf{I} - \mathbf{AB}) &= (\mathbf{I} - \mathbf{BA})\mathbf{B} \\ (\mathbf{I} - \mathbf{AB}) &= \mathbf{B}^{-1}(\mathbf{I} - \mathbf{BA})\mathbf{B} \\ (\mathbf{I} - \mathbf{AB})^{-1} &= \mathbf{B}(\mathbf{I} - \mathbf{BA})^{-1}\mathbf{B}^{-1}\end{aligned}$$

- Thus, as long as $\mathbf{I} - \mathbf{BA}$ is invertible, then the inverse is defined.
28. If the product $\mathbf{M} = \mathbf{ABC}$ of three square matrices is invertible, then \mathbf{A} , \mathbf{B} , \mathbf{C} are invertible. Find a formula for \mathbf{B}^{-1} that involves \mathbf{M}^{-1} and \mathbf{A} and \mathbf{C} .

$$\begin{aligned}\mathbf{M} &= \mathbf{ABC} \\ \mathbf{M}^{-1} &= \mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1} \\ \mathbf{CM}^{-1} &= \mathbf{B}^{-1}\mathbf{A}^{-1} \\ \mathbf{CM}^{-1}\mathbf{A} &= \mathbf{B}^{-1}\end{aligned}$$

41. For which three numbers c is this matrix not invertible, and why not?

$$\mathbf{A} = \begin{bmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{bmatrix}$$

- If $c = 0$, then there would be multiple unavoidable zeros in the pivots.
 - If $c = 2$, then row 2 would just be duplicate of row 1.
 - If $c = 7$, then column 2 and 3 would be equal.
57. If $\mathbf{A} = \mathbf{A}^T$ needs a row exchange, then it also needs a column exchange to stay symmetric. In matrix language, \mathbf{PA} loses the symmetry of \mathbf{A} but \mathbf{PAP}^T recovers the symmetry.

58. ↓

- (a) How many entries of \mathbf{A} can be chosen independently, if $\mathbf{A} = \mathbf{A}^T$ is 5×5 ?
- 25 total choices — 10 under the diagonal = 15
- (b) How do \mathbf{L} and \mathbf{D} (5×5) give the same number of choice in \mathbf{LDL}^T ?
- Oh, I kind of used this to find (a). Well, the diagonal doesn't matter ($\neq 0$), since a transpose can simply be thought of a rotation around the diagonal elements. But, every element must match \mathbf{U} , thus only the 10 choices below matter, yielding 15 total choices.

2 Vector Spaces



2.1 Vector Spaces and Subspaces

Problems 25, 26, 30, 31

25. If we add an extra column \mathbf{b} to a matrix \mathbf{A} , then the column space gets larger unless they are linearly dependent..

- Give an example in which the column space gets larger and an example in which it doesn't.

$$\text{Larger: } \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 2 & 1 \end{array} \right] \quad \text{No change: } \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 2 & 4 & 6 \\ 3 & 2 & 9 \end{array} \right]$$

- Why is $\mathbf{Ax} = \mathbf{b}$ solvable exactly when the column space doesn't get larger by including \mathbf{b} ?
 - Because the solution would be in the image, leading to infinite solutions since it could be written as a linear combination of the vectors already in \mathbf{A} .
26. The columns of \mathbf{AB} are combinations of the columns of \mathbf{A} . This means: the column space of \mathbf{AB} contained in (possibly equal to) the column space of \mathbf{A} . Give an example where the column spaces of \mathbf{A} and \mathbf{AB} are not equal.

vectorSpaces 🧩

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 4 & 6 \\ 3 & 2 & 9 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- The column space of \mathbf{B} is clearly contained in \mathbf{A} , but since the dimension of the null space of \mathbf{B} is 2, then \mathbf{AB} will not have the same column space of \mathbf{A} .
30. If the 9×12 system $\mathbf{Ax} = \mathbf{b}$ is solvable for every \mathbf{b} , then $C(\mathbf{A}) = 9$.
- This follows from the Rank-nullity theorem, where the dimensionality of the column space and cokernel is equal to the rows of the original matrix.
 - This also implies the dimensionality of the cokernel is 3.
31. Why isn't \mathbb{R}^2 a subspace of \mathbb{R}^3 ?
- \mathbb{R}^2 could be a *subset*, but not a subspace; there are infinite 2-dimensional planes in \mathbb{R}^3 .
 - If you included a third point of 0 in \mathbb{R}^2 , then it would indicate that includes the origin, which could make it a subspace or \mathbb{R}^3 . However, that extra coordinate would be make it be a vector in \mathbb{R}^3 .

2.2 Solving $Ax = 0$ and $Ax = b$

Problems 12, 24, 25, 70

12. Which of these rules give a correct definition of the rank of A ?

- (a) The number of nonzero rows in R . ✓ true

$$\max(r) = r \in \mathbb{N} : 0 \leq r \leq \min(m, n)$$

- (b) The number of columns minus the total number of rows.

◦ This would yield dimensionality of the null space.

- (c) The number of columns minus the number of free columns.

◦ This would yield the dimensionality of the left null space.

- (d) The number of 1s in R .

◦ This wouldn't tell you much of anything.

24. Every column of AB is a combination of the columns of A . Then the dimensions of the column spaces give $\text{rank}(AB) \leq \text{rank}(A)$.

Problem: Prove that $\text{rank}(AB) \leq \text{rank}(B)$.

$$\text{rank}(A) = C(A) = C(A^T)$$

$$\implies \text{rank}(A) = \text{rank}(A^T)$$

$$\therefore \text{rank}(AB) = \text{rank}((AB)^T) = \text{rank}(B^T A^T) \leq \text{rank}(B^T) = \text{rank}(B)$$

◦ The rank of AB can only be decreased, if B is not full rank itself.

◦ If, both A and B are full rank, then $\text{rank}(AB) = \text{rank}(B)$

25. (Important) Suppose A and B are $n \times n$ matrices, and $AB = I$. Prove from $\text{rank}(AB) \leq \text{rank}(A)$ that the rank of A is n . So A is invertible and B must be its two-sided inverse. Therefore $BA = I$

◦ A must have same size of B , given they are both $n \times n$.

◦ If A was rank deficient, but B was full rank, then $\text{rank}(AB) \leq \text{rank}(A)$ would be invalid, forcing $\text{rank}(A) = n$.

70. Explain why A and $-A$ always have the same reduced echelon form R .

◦ Signed solutions are arbitrary; $-A$ would have permutations that flip the sign and yield the same, reversible, solution.

2.3 Linear Independence, Basis, and Dimension

Problems 9, 13, 28, 36

9. Suppose $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ are vectors in \mathbb{R}^3 .

- (a) There four vectors are dependent because \mathbb{R}^3 can only have 3 linearly independent vectors.
- (b) The two vectors \mathbf{v}_1 and \mathbf{v}_2 will be dependent if they are multiples of each other.
- (c) The vectors \mathbf{v}_1 and $(0, 0, 0)$ are dependent because $\mathbf{v}_1(0, 0, 0) = \mathbf{0}$

13. Find the dimensions of:

- (a) the column space of \mathbf{A} :
- (b) the column space of \mathbf{U} :
- (c) the row space of \mathbf{A} :
- (d) the row space of \mathbf{U} .

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 3 & 1 & -1 \end{bmatrix} \quad \text{and} \quad \mathbf{U} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

vectorSpaces 

◦ Which two of the spaces are the same?

- The row space of \mathbf{A} and \mathbf{U} are the same. Why? Taking \mathbf{A} to rref shows it's also rank 2, with a linearly dependent row $(0, 0, 0)$, just like \mathbf{U} .

28. True or false (give a good reason)?

- (a) If the columns of a matrix are dependent, so are the rows.
 - **✗ false**; as shown in [problem 2.2.12↑](#), the max rank is the minimum of either the rows or columns. You could have linearly dependent columns, if $n > m$, but no linearly dependent rows.
- (b) The column space of a 2×2 matrix is the same as its row space.
 - **✗ false**; e.g., $\begin{bmatrix} 4 & 2 \\ 0 & 0 \end{bmatrix}$
- (c) The column space of a 2×2 matrix has the same dimension as its row space.
 - **✓ true**; $\text{rank}(\mathbf{A}) = C(\mathbf{A}) = C(\mathbf{A}^T)$
- (d) The columns of a matrix are a basis for the column space.
 - **✗ false**; one of the columns could be linearly dependent with another; only linearly independent columns forms a basis for the column space.

36. If \mathbf{A} is a 64×17 matrix of rank 11, how many independent vectors satisfy $\mathbf{A}\mathbf{x} = \mathbf{0}$? How many independent vectors satisfy $\mathbf{A}^T\mathbf{y} = \mathbf{0}$?

- $17 - 11 = 6$, $64 - 11 = 53$, respectively.

2.4 The Four Fundamental Subspaces

Problems 6, 14, 15, 27

6. Suppose \mathbf{A} is an $m \times n$ matrix of rank r . Under what conditions on those numbers does

(a) \mathbf{A} have a two-sided inverse: $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$?

◦ Square, nonsingular (full rank).

(b) $\mathbf{A}\mathbf{x} = \mathbf{b}$ have infinitely many solutions for every \mathbf{b} ?

◦ If $\text{rank}(\mathbf{A}) < n$ (n = number of columns)

14. Find a left-inverse and/or a right-inverse (when they exist) for

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{M} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{T} = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$$

vectorSpaces 

◦ $\mathbf{A}\mathbf{A}^T$ (right inverse) $\mathbf{M}^T\mathbf{M}$ (left inverse) yield the same inverse,

$$\begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

which makes sense, since $\mathbf{M} = \mathbf{A}^T$

$$\mathbf{T}^{-1} = \begin{bmatrix} a^{-1} & -ba^{-2} \\ 0 & a^{-1} \end{bmatrix}$$

• \mathbf{T} has an inverse, left and right, as long as $a \neq 0$.

• \mathbf{T} does not have a normal inverse if $a = b$ (singular), but the left and right inverse both yield the above matrix.

15. If the columns of \mathbf{A} are linearly independent, then the rank is the number of columns, n , the nullspace is empty, the row space is \mathbb{R}^n , and there exists a left-inverse.

27. (Important) \mathbf{A} is an $m \times n$ matrix of rank r . Suppose there are right-hand sides \mathbf{b} for which $\mathbf{A}\mathbf{x} = \mathbf{b}$ has no solution.

(a) What inequalities must be true between m , n , and r ?

◦ No solution means a row of zeros, thus $r < m$

◦ $r \leq n$, nothing changed.

(b) How do you know that $\mathbf{A}^T\mathbf{y} = \mathbf{0}$ has a nonzero solution?

◦ $\mathbf{A}^T\mathbf{y} = \mathbf{0}$ raises investigation of the left null space. Since the column space does not span all of \mathbb{R}^m ($r < m$), then it means it contains a nonzero solution.

2.6 Linear Transformations

Problems 6, 34, 39

6. What 3 by 3 matrices represent the transformation that

- (a) project every vector onto the x-y plane?
- (b) reflect every vector through the x-y plane?
- (c) rotate the x-y plane through 90° , leaving the z-axis alone?
- (d) rotate the x-y plane, then x-z, then y-z, through 90° ?
- (e) carry out the same three rotations, but each one through 180° ?

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{(a)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}_{(b)} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{(c)} \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}_{(d)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{(e)}$$

34. The transformation T that transposes every matrix is definitely linear. Which of these extra properties are true?

(a) $T^2 = \text{identity transformation}$.

◦ ✓ true; given $T(A) = A^T$

$$\begin{aligned} T^2(A) &= T(T(A)) = T(A^T) = (A^T)^T = A \\ &\implies T^2 = I \end{aligned}$$

(b) The kernel of T is the zero matrix.

◦ ✓ true; only matrix that can take the identity to 0 is the trivial case, i.e., when the null space is empty.

(c) Every matrix is in the range of T .

🌐 Definition of column space, Wikipedia 10/16 🐞

◦ ✓ true; the image (range) consists of all possible products $Ax \forall x \in \mathbb{C}^n$.

(d) $T(M) = -M$ is impossible.

◦ ✗ false; e.g.,

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

39. If you keep the same basis vectors but put them in a different order, the change-of-basis matrix M is a **permutation matrix** P . If you keep the basis vectors in order but change their lengths, M is a **diagonal matrix** D .

3 Orthogonality



3.1 Orthogonal Vectors and Subspaces

Problems 6, 46, 47

6. Find all vectors in \mathbb{R}^3 that are orthogonal to $(1, 1, 1)$ and $(1, -1, 0)$. Produce an orthonormal basis from these vectors (mutually orthogonal unit vectors).

orthogonality 🧩

- Simply taking the cross product between two vectors in \mathbb{R}^3 yields a new vector that is normal to the plane containing them.
- $(1, 1, 1) \times (1, -1, 0) = (1, 1, -2)$ yields us orthogonal columns.
- Dividing each vector by their norms yields orthonormal columns, i.e.,

$$\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{3}} \end{bmatrix}$$

46. Find $\mathbf{A}^T \mathbf{A}$ if the columns of \mathbf{A} are unit vectors, all mutually perpendicular.

$$\begin{bmatrix} \|\mathbf{a}_1\|^2 & a_{\perp} & \cdots & a_{\perp} \\ a_{\perp} & \|\mathbf{a}_2\|^2 & a_{\perp} & \vdots \\ \vdots & a_{\perp} & \ddots & a_{\perp} \\ a_{\perp} & \cdots & a_{\perp} & \|\mathbf{a}_n\|^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix} = \mathbf{I}$$

47. Construct a 3×3 matrix \mathbf{A} with no zero entries whose columns are mutually perpendicular. Compute $\mathbf{A}^T \mathbf{A}$. Why is it a diagonal matrix?

orthogonality 🧩

- Note: I tried to set up with random matrices each time, sometimes it fails and gives the zero matrix.
- The way I set up only yields a diagonal matrix with $\mathbf{A}\mathbf{A}^T$, $\mathbf{A}^T \mathbf{A}$ yields a symmetric matrix. Why?

3.2 Cosines and Projections onto Lines

Projection Proof (class problem)

- If I recall the problem correctly, we were requested to **prove what $\text{proj}_{\mathbf{v}} \mathbf{w}$ is equal to**.
- In my notes I have that a orthogonal projection occurs when the dot product between \mathbf{v} and distance \mathbf{w} from \mathbf{v} is equal to zero. This follows from the definition of the inner product, i.e.,

$$\lambda = \mathbf{v}^T \mathbf{w} = \|\mathbf{v}\| \|\mathbf{w}\| \cos \theta$$

- If $\theta = 90^\circ$, then the vectors are perpendicular, i.e., orthogonal. What we are missing is **the distance** from \mathbf{w} to \mathbf{v} ; the distance that yields an inner product of zero with a normalized \mathbf{v} **is the projection**.
- This means we need a scaled version of \mathbf{v} , let's call it $\mathbf{v}\beta$, at which such inner product is equal to zero. At this point, the difference between \mathbf{w} and $\mathbf{v}\beta$ is exactly what we need in order to solve for a β that maintains an inner product of zero with the original vector \mathbf{v} , i.e.,

$$\mathbf{v}^T (\mathbf{w} - \mathbf{v}\beta) = 0$$

$$\mathbf{v}^T \mathbf{w} - \mathbf{v}^T \mathbf{v}\beta = 0$$

$$\mathbf{v}^T \mathbf{v}\beta = \mathbf{v}^T \mathbf{w}$$

$$\beta = \frac{\mathbf{v}^T \mathbf{w}}{\mathbf{v}^T \mathbf{v}}$$

$$\implies \text{proj}_{\mathbf{v}} \mathbf{w} = \mathbf{v}\beta = \mathbf{v} \frac{\mathbf{v}^T \mathbf{w}}{\mathbf{v}^T \mathbf{v}}$$

- I've internalized this as a mapping of \mathbf{w} onto \mathbf{v} over a magnitude (the norm) of \mathbf{v} .
 - The mapping is important because it tells us the shortest distance from \mathbf{w} onto \mathbf{v} , i.e., when they are orthogonal.
 - The magnitude is important, because it is the basis at which \mathbf{w} is parallel to \mathbf{v} , which when added mapping distance, yields \mathbf{w} .

Problems 10, 13, 15

10. Is the projection matrix P invertible? Why or why not?

$$P = \frac{aa^T}{a^T a}$$

$$C(P) = a = \text{rank}(P) = n = 1$$

- The rank is equal to the number of columns, thus it is invertible.

13. Prove that the trace of $P = \frac{aa^T}{a^T a}$ always equals 1.

$$\text{tr}(A) = \sum_{i=1}^n e_{ij} = e_{11} + e_{22} + \cdots + e_{nn}$$

$$\Rightarrow \text{tr}(P) = \frac{a_1 a_1}{a^T a} + \cdots + \frac{a_n a_n}{a^T a} = \frac{a^T a}{a^T a} = 1$$

15. Show that the length of Ax equals the length of $A^T x$ if $AA^T = A^T A$.

$$\|Ax\|^2 = (Ax)^T (Ax) = xA^T Ax$$

$$\|A^T x\|^2 = (A^T x)^T (A^T x) = xAA^T x$$

$$xA^T Ax = xAA^T x \iff AA^T = A^T A$$

3.3 Projections and Least Squares

Problems 8, 19, 20


8. If \mathbf{P} is the projection matrix onto a k -dimensional subspace \mathbf{S} of the whole space \mathbb{R}^n , what is the column space of \mathbf{P} and what is its rank?
- Given $\mathbf{P} = \mathbf{A}(\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T$, then \mathbf{P} will project any vector onto the image of \mathbf{A} , if \mathbf{A} has independent columns.
 - This implies that \mathbf{S} is the image of \mathbf{A} , i.e., $C(\mathbf{A}) = \mathbf{S}$.
 - Since the k -dimensional subspace \mathbf{S} is the whole of \mathbb{R}^n , and $\text{rank}(\mathbf{A}) = n$, then $C(\mathbf{P}) = \mathbf{S} \wedge \text{rank}(\mathbf{P}) = k$
19. If $\mathbf{P}_C = \mathbf{A}(\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T$ is the projection onto the column space of \mathbf{A} , what is the projection \mathbf{P}_R onto the row space?
- $\mathbf{P}_C = C(\mathbf{A})$, $\mathbf{P}_R = C(\mathbf{A}^T)$, thus, taking the transpose of every instance of \mathbf{A} will yield the row space, i.e.,

$$\mathbf{P}_R = \mathbf{A}^T (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A} = \mathbf{A}^T (\mathbf{A} \mathbf{A}^T)^{-1} \mathbf{A}$$

20. If \mathbf{P} is the projection onto the column space of \mathbf{A} , what is the projection onto the left nullspace?
- The column space and left null space are orthogonal to each other, thus the projection onto the left null space is simply $\mathbf{I} - \mathbf{P}$.

3.4 Orthogonal Bases and Gram-Schmidt

Implementation of the Gram-Schmidt Process

Implementing Gram-Schmidt 

Problems 10, 13, 18

10. 10. If \mathbf{q}_1 and \mathbf{q}_2 are the outputs from Gram-Schmidt, what were the possible input vectors \mathbf{a} and \mathbf{b} ?

◦ Any two linearly independent vectors.

13. Apply the Gram-Schmidt process to

$$\mathbf{a} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

and write the result in the form $\mathbf{A} = \mathbf{QR}$.

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$


◦ All that is needed is a permutation to achieve orthonormal columns; switching rows 1 and 3 will achieve this, yielding \mathbf{Q} :

$$\mathbf{Q} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

◦ Thus,

$$\mathbf{A} = \mathbf{QR} \rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

18. If $\mathbf{A} = \mathbf{QR}$, find a simple formula for the projection matrix \mathbf{P} onto the column space of \mathbf{A} .

◦ I was lost on this one, but I wanted to know how to do it. Unfortunately, while looking for help I came across one of the [solutions documents](#)  again.

◦ Normally I would just do a different problem, but I wanted to write it out, for personal reference.

$$\mathbf{A} = \mathbf{QR} \implies \mathbf{A}^T \mathbf{A} = (\mathbf{QR})^T (\mathbf{QR}) = \mathbf{R}^T \mathbf{Q}^T \mathbf{Q} \mathbf{R} = \mathbf{R}^T \mathbf{R}$$

$$\mathbf{P} = \mathbf{A}(\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T$$

$$\implies \mathbf{QR}(\mathbf{R}^T \mathbf{R})^{-1}(\mathbf{QR})^T = \mathbf{Q} \mathbf{R} \mathbf{R}^{-1} (\mathbf{R}^T)^{-1} \mathbf{R}^T \mathbf{Q}^T = \mathbf{Q} \mathbf{Q}^T = \mathbf{I}$$

$$\implies \mathbf{P}^2 = \mathbf{P} \checkmark$$

3.5 The Fast Fourier Transform

Problems 1, 2

1. What are F^2 and F^4 for the 4×4 Fourier matrix F ?

$$\begin{aligned}
 F^2 &= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix} \\
 &= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} \\
 &= \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 4 & 0 \\ 0 & 4 & 0 & 0 \end{bmatrix}
 \end{aligned}$$

$$F^4 = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 4 & 0 \\ 0 & 4 & 0 & 0 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 4 & 0 \\ 0 & 4 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 16 & 0 & 0 & 0 \\ 0 & 16 & 0 & 0 \\ 0 & 0 & 16 & 0 \\ 0 & 0 & 0 & 16 \end{bmatrix}$$

2. Find a permutation P of the columns of F that produces $FP = \overline{F}$ ($n \times n$).
Combine with $F\overline{F} = nI$ to find F^2 and F^4 for the $n \times n$ Fourier matrix.

$$P = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & \vdots \\ 0 & 1 & 0 & \cdots & 0 \end{bmatrix}$$

$$F\overline{F} = n \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix} \quad (n \times n)$$

$$F^2 = \begin{bmatrix} n & 0 & \cdots & 0 \\ 0 & n & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & n \end{bmatrix} \quad F^4 = \begin{bmatrix} n^2 & 0 & \cdots & 0 \\ 0 & n^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & n^2 \end{bmatrix}$$

Problems 5, 6

5. Find all solutions to the equation $e^{ix} = -1$, and all solutions to $e^{i\theta} = i$.

$$e^{ix} = -1$$

$$\implies \cos x + i \sin x = -1 + 0i$$

$$\implies \cos x = -1 \wedge \sin x = 0$$

$$\implies x = \pi + 2k\pi, \quad k \in \mathbb{Z}$$

$$e^{i\theta} = i$$

$$\implies \cos \theta + i \sin \theta = 0 + 1i$$

$$\implies \cos \theta = 0 \wedge \sin \theta = 1$$

$$\implies \theta = \frac{\pi}{2} + 2k\pi, \quad k \in \mathbb{Z}$$

6. What are the square and square roots of w_{128} , the primitive 128^{th} root of 1?

$$| w_n^2 = w_m \iff m = \frac{1}{2}n$$

$$\implies w_{128}^2 = w_{64} \wedge \sqrt{w_{128}} = w_{256}$$

$$\begin{aligned} w_{128} &= e^{2\pi i/128} = \cos \frac{2\pi}{128} + i \sin \frac{2\pi}{128} \\ &= \cos 0 + i \sin 0 = 1 + i0 = 1 \end{aligned}$$