Calculus III Midterm Review

This review contains questions from quiz 1 & 2, in class midterm review questions, and preliminary questions from sections 9.1–9.5 (excluding 9.3) & 10.1–10.3.

Questions are listed in no particular.

1. True or false? (with justification)

(a) If a_n is bounded, then it converges.

(b) If a_n converges, then is must be bounded.

(c) If a_n is not bounded, then it diverges.

(d) If a_n diverges, then it is not bounded.

2. A hot anvil with cooling constant $k = 0.02\,\mathrm{s}^{-1}$ is submerged in a large pool of water whose temperature is $10\,^{\circ}\mathrm{C}$. Let $y(t)$ be the anvil's temperature t seconds later.

   (a) What is the differential equation satisfied by $y(t)$?

   (b) Find a formula for $y(t)$, assuming the object initial temperature is $100\,^{\circ}\mathrm{C}$.

3. As an object cools, its rate of cooing slows. Explain how this follows from Newton's Law of Cooling.

4. In Yellowstone park there were approximately 500 bison in 1970 and 3,000 bison in 1990.
   (a) Using the model that the rate of change of the population is proportional to the population itself, set up (do not solve) an initial value problem to model this situation.

   (b) Now find the particular solution satisfying your initial value problem.

5. Which of the following are first-order linear equations?
   (a) $y' + x^2 y = 1$
   (b) $y' + xy^2 = 1$
   (c) $x^5 y' + y = e^x$
   (d) $x^5 y' + y = e^y$

6. For what function P is the integrating factor $\alpha(x)$ equal to x? What about e^x?

7. Find the limit of the sequence $a_n = \dfrac{n+1}{3n+2}$. Be sure to justify your answer.

8. What is a_4 for the sequence $a_n = n^2 - n$?

9. Which of the following sequences converge to zero?

$$\frac{n^2}{n^2+1} \qquad\qquad 2^n \qquad\qquad \left(-\frac{1}{2}\right)^n$$

10. Which of the following sequences is defined recursively?

$$a_n = \sqrt{4+n} \qquad\qquad b_n = \sqrt{4+b_{n-1}}$$

11. Let a_n be the n^{th} decimal approximation to $\sqrt{2}$. I.e., $a_1 = 1$, $a_2 = 1.4$, $a_3 = 1.41$ and so on. What is $\lim\limits_{n\to\infty} a_n$?

12. Biologists stocked a lake with 400 fish and estimated the carrying capacity to be 10,000. The number of fish tripled in the first year. Assuming the size of the fish population satisfies the logistic equation, find an expression for the size of the population after t years.

13. Find the general term, a_n, for the sequence given below. Assume that we start our sequence at $n = 1$.

$$1, \frac{4}{2}, \frac{9}{4}, \frac{16}{8}, \frac{25}{16}, \frac{36}{32} \cdots$$

14. Determine whether each of the following series converge or diverge by using either the Comparison Test or the Limit Comparison Test to justify your answer.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{a}{e^n + n^2}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{a}{e^n - n^2}$

15. True or false? If $k > 0$, then all solutions of $y' = -k(y - b)$ approach the same limit as $t \to \infty$.

16. Write a solution to $y' = 4(y - 5)$ that tends to $-\infty$ as $t \to \infty$.

17. Does $y' = -4(y - 5)$ have a solution that tends to ∞ and $t \to \infty$?

18. Find the general solution $y' = xe^{-\sin x} - y \cos x$.

19. True or false?

(a) $t\frac{dy}{dt} = 3\sqrt{1+y}$ is a separable differential equation.

(b) $yy' + x + y = 0$ is a first-order linear differential equation.

20. Determine the order of the following differential equations:

(a) $x^5 y' = 1$

(b) $(y')^3 + x = 1$

(c) $y''' + x^4 y' = 2$

(d) $\sin(y'') + x = y$

21. Which of the following differential equations are directly integrable?

(a) $y' = x + y$

(b) $x\frac{dy}{dx} = 3$

(c) $\frac{dP}{dt} = 4P + 1$

(d) $\frac{dw}{dt} = \frac{2t}{1+4t}$

(e) $\frac{dx}{dt} = t^2 e^{-3t}$

(f) $t^2\frac{dx}{dt} = x - 1$

22. Which of the following differential equations are separable?

(a) $\frac{dy}{dx} = x - 2y$

(b) $xy' + 8ye^x = 0$

(c) $y' = x^2y^2$

(d) $y' = 1 - y^2$

(e) $t\frac{dy}{dt} = 3\sqrt{1+y}$

(f) $\frac{dP}{dt} = \frac{P+t}{t}$

23. Which of the following equations are first-order?

(a) $y' = x^2$

(b) $y'' = y^2$

(c) $(y')^3 + yy' = \sin x$

(d) $x^2y' - e^xy = \sin y$

(e) $y'' + 3y' = \frac{y}{x}$

(f) $yy' + x + y = 0$

24. Water is draining from a cylindrical tank with cross sectional area $4m^2$ and height $5m$. Torricelli's Law says that the rate of change of the height of the water in such a cylindrical tank is proportional to the square root of the height of the water in the tank. Suppose that the tank starts full of water and after 30 minutes the height of the water has decreased to $4m$. Set up an initial value problem to model this situation.

25. Find the limit of each of the following sequences. Justify your answer using limit laws, the squeeze theorem, and/or L'Hospital's Rule.

(a) $a_n = \dfrac{2(-1)^{n+1}}{2n-1}$

(b) $a_n = \dfrac{n^2}{2^{n-1}}$

26. Does the series $\sum_{n=1}^{\infty} \cos\left(\frac{1}{n}\right)$ converge or diverge? How do you know?

27. Find the sum of each of the following geometric series or state that it diverges. Be sure to explain how you arrived at your solution.

(a) $\frac{1}{6} + \frac{1}{12} + \frac{1}{24} + \frac{1}{48} + \frac{1}{96} + \cdots$

(b) $-2 + \frac{2}{5} - \frac{2}{25} + \frac{2}{125} - \frac{2}{625} + \cdots$

28. What role do partial sums play in defining the sum of an infinite series?

29. Indicate whether of not the reasoning in the following statements are correct:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = 0$ because $\frac{1}{n^2}$ tends to zero.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ because $\displaystyle\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

30. Find an N such that $S_N > 25$ for the series $\displaystyle\sum_{n=1}^{\infty} 2$.

31. Does there exist an N such that $S_N > 25$ for the series $\sum_{n=1}^{\infty} 2^{-n}$? Explain.

32. Give an example of a divergent infinite series whose general term tends to zero.

33. For the series $\sum_{n=1}^{\infty} a_n$, if the partial sums S_N are increasing, then (choose the correct conclusion):

(a) a_n is an increasing sequence.

(b) a_n is a positive sequence.

34. What are the hypotheses of the Integral Test?

35. Which test would you use to determine whether the following series converge?

(a) $\sum_{n=1}^{\infty} n^{-3.2}$

(b) $\sum_{n=1}^{\infty} \frac{1}{2^n + \sqrt{n}}$