

Applied Linear Algebra



1 Matrices and Gaussian Elimination

1.2 The Geometry of Linear Equations	2
Problems 1–12.....	2
Problems 17–23	5
1.3 Gaussian Elimination	7
Problems 1–9.....	7
Problems 10–19	7
Problems 20–22.....	7
Problems 23–31	7
1.4 Matrix Notation and Matrix Multiplication	8
1.5 Triangular Factors and Row Exchanges	9
1.6 Inverses and Transposes	10
1.7 Special Matrices and Applications	11

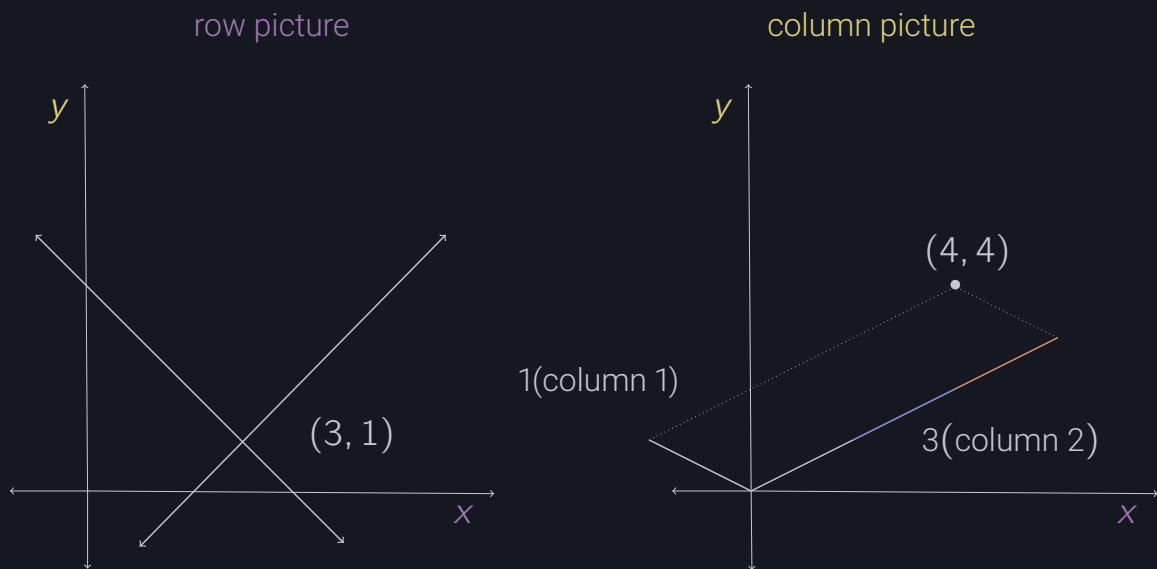
1 Matrices and Gaussian Elimination



1.2 The Geometry of Linear Equations

Problems 1–12

- For the equations $x + y = 4$, $2x - 2y = 4$, draw the row picture (two intersecting lines) and the column picture (combination of two columns equal to the column vector $(4, 4)$ on the right side).



1.2.1

- Solve to find a combination of the columns that equals b :

$$u - v - w = b_1$$

$$v + w = b_2$$

$$w = b_3$$

$$\implies w = b_3$$

$$\implies v = b_2 - b_3$$

$$\implies u = b_1 + v + w = b_1 + b_2$$

- Describe the intersection of the three planes $u + v + w + z = 6$ and $u + w + z = 4$ and $u + w = 2$ (all in four-dimensional space). Is it a line or a point or an empty set? What is the intersection if the fourth plane $u = -1$ is included? Find a fourth equation that leaves us with no solution.

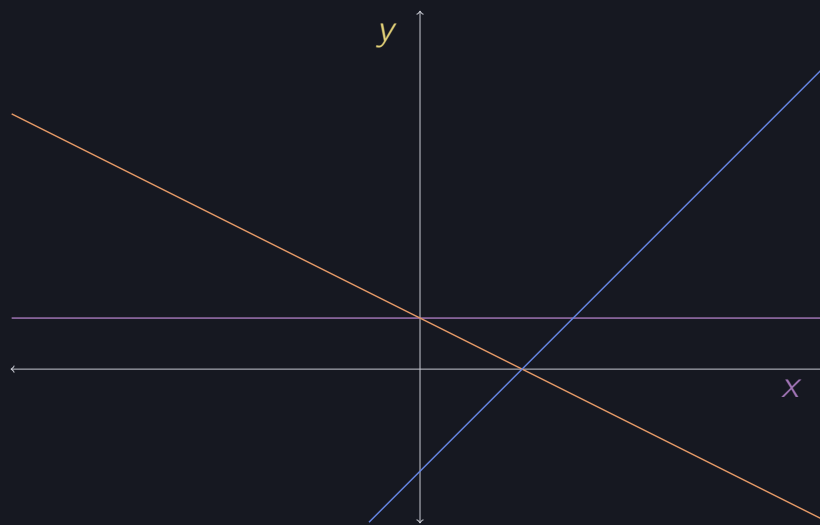
- A line; as $u + w = 2$ is only a line[?]. A fourth plane with $u = -1$ would produce a normally intersecting point. Any addition equation when $u + w \neq 2$ would produce an inconsistent equation.

4. Sketch these three lines and decide if the equations are solvable:

$$x + 2y = 2$$

$$x - y = 2$$

$$y = 1$$



1.2.4

Inconsistent; multiple points of intersect

What happens if all right-hand sides are zero? Is there any nonzero choice of right-hand sides that allows the three lines to intersect at the same point?

- If all the solutions were zero, then it would be a trivial solution.
- Yes, e.g., $x - y = -1$ would produce a single point of intersection.

5. Find two points on the line of intersection of the three planes $t = 0$ and $z = 0$ and $x + y + z + t = 1$ in four-dimensional space.

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

6. When $b = (2, 5, 7)$, find a solution (u, v, w) to equation (4) different from the solution $(1, 0, 1)$ mentioned in the text.

- Since there are infinite solutions, and if \mathbf{s} vector describing one solution and λ is any scalar, then $\mathbf{s}\lambda$ is also a solution. E.g., $(1, 0, 1) 42 = (42, 0, 42)$

8. Explain why the system

$$\begin{aligned}u + v + w &= 2 \\u + 2v + 3w &= 1 \\v + 2w &= 0\end{aligned}$$

is singular by finding a combination of the three equations that adds up to $0 = 1$. What value should replace the last zero on the right side to allow the equations to have solutions—and what is one of the solutions?

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 0 \end{array} \right] \xrightarrow{R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 2 & 0 \end{array} \right] \xrightarrow{R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

- Replacing the last zero with -1 would yield infinite solutions. One solution would be $[3, -1, 0]^T$

9. The column picture for the previous exercise (singular system) is

$$u \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + v \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + w \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = b$$

Show that the three columns on the left lie in the same plane by expressing the third as a combination of the first two. What are all the solutions (u, v, w) if b is the zero vector $(0, 0, 0)$?

$$-1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

- If b is equal to the zero vector $\mathbf{0}$ then the solutions are equal to the kernel² i.e., $-1x_1, 2x_2, 0x_3 = \mathbf{0}$
10. Under what condition on y_1, y_2, y_3 do the points $(0, y_1), (1, y_2), (2, y_3)$ lie on a straight line?
- Question 9 describes the state at which they are collinear, i.e., $y_3 = 2y_2 - y_1$
11. These equations are certain to have the solution $x = y = 0$. For which values of a is there a whole line of solutions?

$$\begin{aligned}ax + 2y &= 0 \\2x + ay &= 0\end{aligned}$$

- Only the scalars that make the lines linearly dependent, i.e., $a = 2, -2$

Problems 17–23

17. The first of these equations plus the second equals the third:

$$x + y + z = 2$$

$$x + 2y + z = 3$$

$$2x + 3y + 2z = 5$$

The first two planes meet along a line. The third plane contains that line, because if x, y, z satisfy the first two equations then they also **span all of \mathbb{R}^3** . The equations have infinitely many solutions (the whole line **L**). Find three solutions.

◦ $\mathbf{v} = (4, 4, 0)$, $\mathbf{w} = (6, 3, 2)$, $\mathbf{u} = 2\mathbf{v} + -1\mathbf{w}$

18. Move the third plane in Problem 17 to a parallel plane $2x + 3y + 2z = 9$. Now the three equations have no solution—*why not*? The first two planes meet along the line **L** , but the third plane doesn't that **cross** that line.

19. In Problem 17 the columns are $(1, 1, 2)$ and $(1, 2, 3)$ and $(1, 1, 2)$. This is a “singular case” because the third column is **linearly dependent**. Find two combinations of the columns that give $\mathbf{b} = (2, 3, 5)$. This is only possible for $\mathbf{b} = (4, 6, c)$ if $c = 10$

20. Normally 4 “planes” in four-dimensional space meet at a **tensor**. Normally 4 column vectors in four-dimensional space can combine to produce \mathbf{b} . What combination of $(1, 0, 0, 0)$, $(1, 1, 0, 0)$, $(1, 1, 1, 0)$, $(1, 1, 1, 1)$ produces $\mathbf{b} = (3, 3, 3, 2)$? **$(1, 0, 0, -2)$** ? What 4 equations for x, y, z, t are you solving? A **lower triangular matrix**, i.e.,

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3 \\ 1 & 1 & 0 & 0 & 3 \\ 1 & 1 & 1 & 0 & 3 \\ 1 & 1 & 1 & 1 & 1 \end{array} \right]$$

21. When equation 1 is added to equation 2, which of these are changed: the planes in the row picture, the column picture, the coefficient matrix, the solution?

◦ Row operations do not change the solution. Row 2 is changed, thus the second plane is changed. **All columns are changed.**?

1.3 Gaussian Elimination

Problems 1–9

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Problems 10–19

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Problems 20–22

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Problems 23–31

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1.4 Matrix Notation and Matrix Multiplication

1.5 Triangular Factors and Row Exchanges

1.6 Inverses and Transposes

1.7 Special Matrices and Applications