1. Find two orthogonal vectors that are both orthogonal $\boldsymbol{u} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

- A vector is orthogonal when the dot product is equal to zero. Thus, if $\boldsymbol{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, then any vector that satisfies the following equation is valid:

$$-2a + b + c = 0$$

- For any vector $\boldsymbol{v}$ that satisfies the above equation, then one can compute the cross product ($\times$) with the given vector $\boldsymbol{u}$ and $\boldsymbol{v}$ in order to find a new vector that is orthogonal to both $\boldsymbol{u}$ and $\boldsymbol{v}$, i.e.,

$$\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \times \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -b + c \\ a + 2c \\ -a - 2b \end{bmatrix}$$

- For example, $a = 0,\ b = 1,\ c = -1$ satisfies the original equation, therefor:

$$\boldsymbol{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad \langle \boldsymbol{u}, \boldsymbol{v} \rangle = 0, \quad \boldsymbol{u} \times \boldsymbol{v} = \begin{bmatrix} -2 \\ -2 \\ -2 \end{bmatrix}, \quad \text{and} \quad \langle \boldsymbol{v}, \boldsymbol{u} \times \boldsymbol{v} \rangle = 0$$

2. Find a parametrization of the plane (write the solutions as linear combinations of vectors with some parameters) given by the scalar equation

$$x - 2y - 3z = 5$$

- A parametrization for a plane can be written as:

$$\boldsymbol{x} = \boldsymbol{c} + r\boldsymbol{a} + s\boldsymbol{b}$$

where $\boldsymbol{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel to the plane, and $\boldsymbol{c}$ is a point on the plane.

- Letting $y = 0$ and $z = 0$ yields a point on the plane $\begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}$
- Finding two vectors on the plane can be done by subtracting a point on the plane from another, which yields a new vector on the plane. Taking an approach to find a point

similar to the first yields:

$$\boldsymbol{a} = \begin{bmatrix} 0 \\ -5/2 \\ 0 \end{bmatrix} - \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -5 \\ -5/2 \\ 0 \end{bmatrix}$$

$$\boldsymbol{b} = \begin{bmatrix} 0 \\ 0 \\ -5/3 \end{bmatrix} - \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -5 \\ 0 \\ -5/3 \end{bmatrix}$$

- Putting it all together yields:

$$\boldsymbol{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix} + r \begin{bmatrix} -5 \\ -5/2 \\ 0 \end{bmatrix} + s \begin{bmatrix} -5 \\ 0 \\ -3/2 \end{bmatrix}$$

3. Show that the following parametrization produces solutions to the scalar equation given in number 2.

$$\boldsymbol{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix} + r \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + s \begin{bmatrix} 7 \\ 2 \\ 1 \end{bmatrix}$$

$$\begin{aligned}
x - 2y - 3z &= 5 && \text{original equation} \\
\begin{bmatrix} 1 & -2 & -3 \end{bmatrix} &= \boldsymbol{n} && \text{vector orthogonal to plane} \\
\begin{bmatrix} 5 & -3 & 2 \end{bmatrix} &= \boldsymbol{c} && \text{given point on plane} \\
\begin{bmatrix} 1 & 2 & -1 \end{bmatrix} &= \boldsymbol{a} && \text{vector describing } r \\
\begin{bmatrix} 7 & 2 & 1 \end{bmatrix} &= \boldsymbol{b} && \text{vector describing } s \\
&\downarrow \\
\langle \boldsymbol{n}, \boldsymbol{c} \rangle &= 5 && \text{yields solution to equation} \checkmark \\
\langle \boldsymbol{n}, \boldsymbol{a} \rangle &= 0 && \boldsymbol{a} \text{ is orthogonal to } \boldsymbol{n} \checkmark \\
\langle \boldsymbol{n}, \boldsymbol{b} \rangle &= 0 && \boldsymbol{b} \text{ is orthogonal to } \boldsymbol{n} \checkmark
\end{aligned}$$

Additionally, one can show dot product of $\boldsymbol{c} - \boldsymbol{a}$ and $\boldsymbol{n}$ yields solution to scalar equation:

$$\begin{aligned}
\langle \boldsymbol{n}, \boldsymbol{c} - \boldsymbol{a} \rangle &= 5 && \text{yields solution to equation} \checkmark \\
\langle \boldsymbol{n}, \boldsymbol{c} - \boldsymbol{b} \rangle &= 5 && \text{yields solution to equation} \checkmark
\end{aligned}$$

Replacing $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ with examples provided in question two yields the same results.