

Applied Linear Algebra



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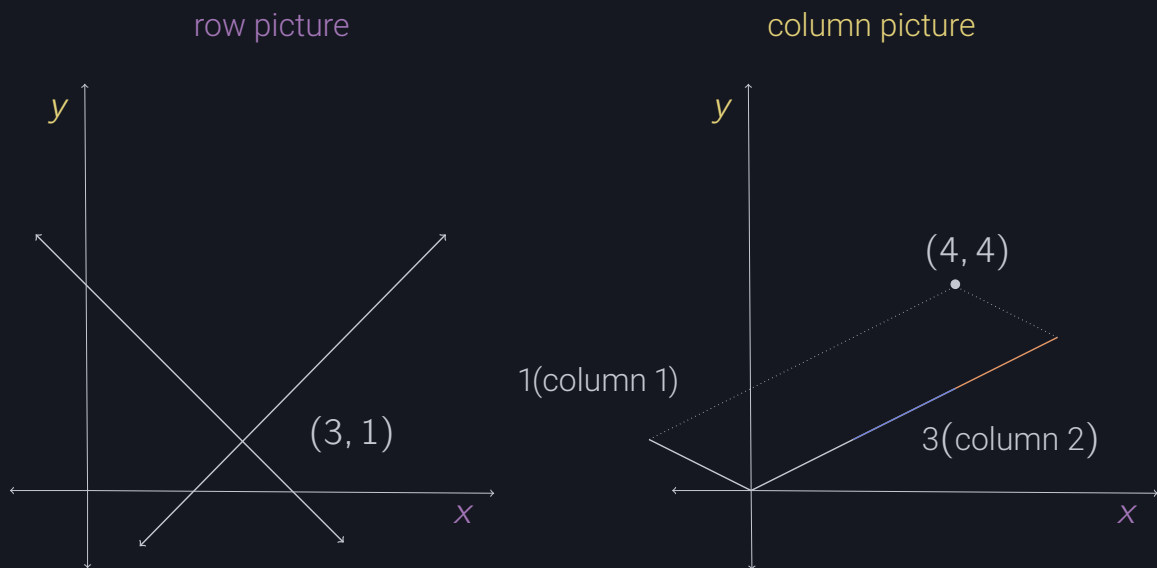
1 Matrices and Gaussian Elimination



1.2 The Geometry of Linear Equations

Problems 1–12

- For the equations $x + y = 4$, $2x - 2y = 4$, draw the row picture (two intersecting lines) and the column picture (combination of two columns equal to the column vector $(4, 4)$ on the right side).



1.2.1

- Solve to find a combination of the columns that equals b :

$$u - v - w = b_1$$

$$v + w = b_2$$

$$w = b_3$$

$$\implies w = b_3$$

$$\implies v = b_2 - b_3$$

$$\implies u = b_1 + v + w = b_1 + b_2$$

- Describe the intersection of the three planes $u + v + w + z = 6$ and $u + w + z = 4$ and $u + w = 2$ (all in four-dimensional space). Is it a line or a point or an empty set? What is the intersection if the fourth plane $u = -1$ is included? Find a fourth equation that leaves us with no solution.

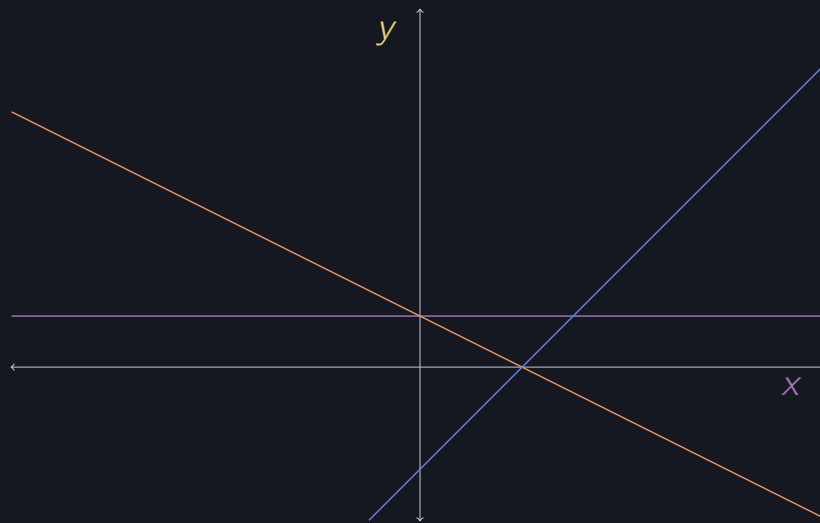
- A line; as $u + w = 2$ is only a line[?]. A fourth plane with $u = -1$ would produce a normally intersecting point. Any addition equation when $u + w \neq 2$ would produce an inconsistent equation.

4. Sketch these three lines and decide if the equations are solvable:

$$x + 2y = 2$$

$$x - y = 2$$

$$y = 1$$



1.2.4

Inconsistent; multiple points of intersect

What happens if all right-hand sides are zero? Is there any nonzero choice of right-hand sides that allows the three lines to intersect at the same point?

- If all the solutions were zero, then it would be a trivial solution.
 - Yes, e.g., $x - y = -1$ would produce a single point of intersection.
5. Find two points on the line of intersection of the three planes $t = 0$ and $z = 0$ and $x + y + z + t = 1$ in four-dimensional space.

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

6. When $b = (2, 5, 7)$, find a solution (u, v, w) to equation (4) different from the solution $(1, 0, 1)$ mentioned in the text.
- Since there are infinite solutions, and if \mathbf{s} vector describing one solution and λ is any scalar, then $\mathbf{s}\lambda$ is also a solution. E.g., $(1, 0, 1)42 = (42, 0, 42)$

8. Explain why the system

$$\begin{aligned}u + v + w &= 2 \\u + 2v + 3w &= 1 \\v + 2w &= 0\end{aligned}$$

is singular by finding a combination of the three equations that adds up to $0 = 1$. What value should replace the last zero on the right side to allow the equations to have solutions—and what is one of the solutions?

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 0 \end{array} \right] \xrightarrow{R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 2 & 0 \end{array} \right] \xrightarrow{R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

- Replacing the last zero with -1 would yield infinite solutions. One solution would be $[3, -1, 0]^T$
9. The column picture for the previous exercise (singular system) is

$$u \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + v \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + w \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = b$$

Show that the three columns on the left lie in the same plane by expressing the third as a combination of the first two. What are all the solutions (u, v, w) if b is the zero vector $(0, 0, 0)$?

$$-1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

- If b is equal to the zero vector $\mathbf{0}$ then the solutions are equal to the kernel² i.e., $-1x_1, 2x_2, 0x_3 = \mathbf{0}$
10. Under what condition on y_1, y_2, y_3 do the points $(0, y_1), (1, y_2), (2, y_3)$ lie on a straight line?
- Question 9 describes the state at which they are collinear, i.e., $y_3 = 2y_2 - y_1$
11. These equations are certain to have the solution $x = y = 0$. For which values of a is there a whole line of solutions?

$$\begin{aligned}ax + 2y &= 0 \\2x + ay &= 0\end{aligned}$$

- Only the scalars that make the lines linearly dependent, i.e., $a = 2, -2$

Problems 17–23

17. The first of these equations plus the second equals the third:

$$\begin{aligned}x + y + z &= 2 \\x + 2y + z &= 3 \\2x + 3y + 2z &= 5\end{aligned}$$

The first two planes meet along a line. The third plane contains that line, because if x, y, z satisfy the first two equations then they also **span all of \mathbb{R}^3** . The equations have infinitely many solutions (the whole line L). Find three solutions.

◦ $\mathbf{v} = (4, 4, 0)$, $\mathbf{w} = (6, 3, 2)$, $\mathbf{u} = 2\mathbf{v} + -1\mathbf{w}$

18. Move the third plane in Problem 17 to a parallel plane $2x + 3y + 2z = 9$. Now the three equations have no solution—*why not*? The first two planes meet along the line L , but the third plane doesn't that **cross** that line.

19. In Problem 17 the columns are $(1, 1, 2)$ and $(1, 2, 3)$ and $(1, 1, 2)$. This is a “singular case” because the third column is **linearly dependent**. Find two combinations of the columns that give $\mathbf{b} = (2, 3, 5)$. This is only possible for $\mathbf{b} = (4, 6, c)$ if $c = 10$

20. Normally 4 “planes” in four-dimensional space meet at a **tensor**. Normally 4 column vectors in four-dimensional space can combine to produce \mathbf{b} . What combination of $(1, 0, 0, 0)$, $(1, 1, 0, 0)$, $(1, 1, 1, 0)$, $(1, 1, 1, 1)$ produces $\mathbf{b} = (3, 3, 3, 2)$? $(1, 0, 0, -2)$? What 4 equations for x, y, z, t are you solving? A **lower triangular matrix**, i.e.,

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3 \\ 1 & 1 & 0 & 0 & 3 \\ 1 & 1 & 1 & 0 & 3 \\ 1 & 1 & 1 & 1 & 1 \end{array} \right]$$

21. When equation 1 is added to equation 2, which of these are changed: the planes in the row picture, the column picture, the coefficient matrix, the solution?

◦ Row operations do not change the solution. Row 2 is changed, thus the second plane is changed. **All columns are changed.**?

1.3 Gaussian Elimination

Problems 6, 7

6. Choose a coefficient b that makes this system singular. Then choose a right-hand side g that makes it solvable. Find two solutions in that singular case.

$$2x + by = 16$$

$$4x + 8y = g$$

$$2x + 4y = 16$$

$$4x + 8y = 32$$

- Since R_2 is just a multiple of R_1 , then solving for x, y , with one variable = 0, in the first equation will yield two solutions, i.e., $(8, 0), (0, 4)$
7. For which numbers a does elimination break down (a) permanently, and (b) temporarily?

$$ax + 3y = -3$$

$$4x + 6y = 6$$

Solve for x and y after fixing the second breakdown by a row exchange.

- Permanently: $a = 2$ (linearly dependent, no solution)
- Temporarily: $a = 0$;

$$\left[\begin{array}{cc|c} 4 & 6 & 6 \\ 0 & 3 & -3 \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -1 \end{array} \right]$$
$$y = -1, \quad x = 3$$

Problems 17, 18, 19

17. Which number q makes this system singular and which right-hand side t gives it infinitely many solutions? Find the solution that has $z = 1$.

$$x + 4y - 2z = 1$$

$$x + 7y - 6z = 6$$

$$3y + qz = t$$

$$x + 4y - 2z = 1$$

$$x + 7y - 6z = 6$$

$$3y + -4z = 5$$

- If $q = -4$, then R_3 would have no pivot
- If $t = 5$, then there would be finite solutions, R_3 would be linearly dependent with R_2

18. It is impossible for a system of linear equations to have exactly two solutions. Explain why.

- If (x, y, z) and (X, Y, Z) are two solutions, what is the other one?
 - There is no other *one*, there would be infinitely many.
- If 25 planes meet at two points, where else do they meet?
 - Every other single point, they would span all of \mathbb{R}^3

19. Three planes can fail to have an intersection point, when no two planes are parallel. The system is singular if row 3 of \mathbf{A} is a **linearly dependent; a combination** of the first two rows. Find a third equation that can't be solved if $x + y + z = 0$ and $x - 2y - z = 1$.

$$x + y + z = 0$$

$$x - 2y - z = 1$$

$$R_1 + R_2 \neq 1 \rightarrow \text{parallel; no solution, e.g.,}$$

$$2x - y = 42$$

Problems 30, 31

30. Use elimination to solve

$$u + v + w = 6$$

$$u + 2v + 2w = 11$$

$$2u + 3v - 4w = 3$$

$$u + v + w = 7$$

$$u + 2v + 2w = 10$$

$$2u + 3v - 4w = 3$$

$$\begin{aligned} \text{rref} \left(\begin{bmatrix} 1 & 1 & 1 & | & 6 \\ 1 & 2 & 2 & | & 11 \\ 2 & 3 & -4 & | & 3 \end{bmatrix} \right) &\rightarrow \begin{bmatrix} 1 & 0 & 0 & | & 1 \\ 0 & 1 & 0 & | & 3 \\ 0 & 0 & 1 & | & 2 \end{bmatrix} \\ \text{rref} \left(\begin{bmatrix} 1 & 1 & 1 & | & 7 \\ 1 & 2 & 2 & | & 10 \\ 2 & 3 & -4 & | & 3 \end{bmatrix} \right) &\rightarrow \begin{bmatrix} 1 & 0 & 0 & | & 4 \\ 0 & 1 & 0 & | & 1 \\ 0 & 0 & 1 & | & 2 \end{bmatrix} \end{aligned}$$

31. For which three numbers a will elimination fail to give three pivots?

$$ax + 2y + 3z = b_1$$

$$ax + ay + 4z = b_2$$

$$ax + ay + az = b_3$$

- For $a = 0$, multiple failures.
- For $a = 2$, columns 0, 1 would be equal.
- For $a = 4$, rows 1, 2 would be equal.

1.4 Matrix Notation and Matrix Multiplication

Problems 4, 10, 17, 19

4. If an $m \times n$ matrix \mathbf{A} multiplies an n -dimensional vector \mathbf{x} , how many separate multiplications are involved? What if \mathbf{A} multiplies an $n \times p$ matrix \mathbf{B} ?

- $m \cdot n$ multiplications; number of rows times the length of \mathbf{x} .
- $m \cdot n \cdot p$; same as above, except accounting for each additional column p .

10. True or false? Give a specific counterexample when false.

- If rows 1 and 3 of \mathbf{B} are the same, so are rows 1 and 3 of \mathbf{AB} .
- **✗ false**; matrix multiplication is done by the rows of the left matrix and the columns of the right, the rows may be the same, but if a column between the two are different, then there would be different multiplications occurring, e.g.,

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 12 & 8 & 4 \\ 30 & 20 & 10 \\ 38 & 32 & 16 \end{bmatrix}$$

- If columns 1 and 3 of \mathbf{B} are the same, so are columns 1 and 3 of \mathbf{AB} .
- **✓ true**;
- If rows 1 and 3 of \mathbf{A} are the same, so are rows 1 and 3 of \mathbf{AB} .
- **✓ true**
- $(\mathbf{AB})^2 = \mathbf{A}^2 \mathbf{B}^2$.
- **✗ false** (most of the time), e.g.,

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\mathbf{AB}^2 = \begin{bmatrix} 144 & 64 & 16 \\ 900 & 400 & 100 \\ 2304 & 1024 & 256 \end{bmatrix} \neq \begin{bmatrix} 74 & 26 & 10 \\ 452 & 152 & 52 \\ 1154 & 386 & 130 \end{bmatrix} = \mathbf{A}^2 \mathbf{B}^2$$

17. Which of the following matrices are guaranteed to equal $(\mathbf{A} + \mathbf{B})^2$?

- $\mathbf{A}^2 + 2\mathbf{AB} + \mathbf{B}^2$,
- ✓** $\mathbf{A}(\mathbf{A} + \mathbf{B}) + \mathbf{B}(\mathbf{A} + \mathbf{B})$
- ✓** $(\mathbf{A} + \mathbf{B})(\mathbf{B} + \mathbf{A})$,
- ✓** $\mathbf{A}^2 + \mathbf{AB} + \mathbf{BA} + \mathbf{B}^2$

19. A fourth way to multiply matrices is columns of **A** times rows of **B**:

$$\mathbf{AB} = (\text{column 1})(\text{row 1}) + \cdots + (\text{column } n)(\text{row } n) = \text{sum of simple matrices.}$$

Give a 2×2 example of this important rule for matrix multiplication.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \left(a \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right) b \begin{bmatrix} 1 \\ 3 \end{bmatrix} + d \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

Useful, as the right matrix can be thought of as the **weights that scale** the elements of the columns of the left matrix.

Problems 30–31

30. Multiply these matrices:

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \\ 1 & 4 & 0 \end{bmatrix}.$$

$$\begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 0 & 2 & -3 \end{bmatrix} \quad \text{respectively}$$

- The former multiplication performs two operations (left: swaps top and bottom columns, right: swaps left and right columns), while the latter subtracts row 1 from both row 2 and row 3.

31. This 4×4 matrix needs which elimination matrices **E**₂₁ and **E**₃₂ and **E**₄₃?

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

- $e_{21} = -\frac{1}{2}$, $e_{32} = -\frac{2}{3}$, $e_{43} = -\frac{3}{4}$
- I suspect the fractions will tend towards -1 if the matrix was expanded upon in a similar fusion?

Problems 34, 35, 38, 42

34. Multiply these matrices in the orders FE and EF and E^2 :

$$E = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & 0 & 1 \end{bmatrix} \quad F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{bmatrix}$$

$$FE = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ ac + b & c & 1 \end{bmatrix} \quad EF = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix} \quad E^2 = \begin{bmatrix} 1 & 0 & 0 \\ 2a & 1 & 0 \\ 2b & 0 & 1 \end{bmatrix}$$

35. ↓

- (a) Suppose all columns of B are the same. Then all columns of EB are the same, because each one is E times B_{1n} .
- (b) Suppose all rows of B are $[1 \ 2 \ 4]$. Show by example that all rows of EB are not $[1 \ 2 \ 4]$. It is true that those rows are multiples of $[1 \ 2 \ 4]$
 - E.g., if $e_{12} = 2$, then m_2 of EB would be $[3 \ 6 \ 12]$

38. If $AB = I$ and $BC = I$, use the associative law to prove $A = C$.

$$A = A(BC)$$

$$A = (AB)C$$

$$A = C$$

42. True or false?


- (a) If A^2 is defined then A is necessarily square.
 - ✓ true; inner dimensions must match, i.e., dimensions of $n_1 = m_2$. Thus, A must be square.
- (b) If AB and BA are defined, then A and B are square.
 - ✗ false; if $A = 6 \times 9$ and $B = 9 \times 6$ allows for valid pre- and post-multiplication of B .
- (c) If AB and BA are defined, then AB and BA are square.
 - ✓ true; see above example, each case will still yield square matrices. Not a proof, but I can't see another way to falsify (b).
- (d) If $AB = B$ then $A = I$
 - ✗ false; e.g., $B = 0$

1.5 Triangular Factors and Row Exchanges

Problems 1, 6, 7, 14, 18

- When is an upper triangular matrix nonsingular (a full set of pivots)?
 - Every pivot **must be nonzero**. If there is a zero on one of the pivots, then it indicates that one of the columns is a linear combination of one or more of the other columns.
- Find E^2 and E^8 and E^{-1} if


$$E = \begin{bmatrix} 1 & 0 \\ 6 & 1 \end{bmatrix}$$

matrices 

$$E^2 = \begin{bmatrix} 1 & 0 \\ 36 & 1 \end{bmatrix} \quad E^8 = \begin{bmatrix} 1 & 0 \\ 1679616 & 1 \end{bmatrix} \quad E^{-1} = \begin{bmatrix} 1 & 0 \\ -6 & 1 \end{bmatrix}$$

- Find the products FGH and HGF if (with upper triangular zeros omitted)

$$F = \begin{bmatrix} 1 & & & \\ 2 & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad G = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 2 & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad H = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

matrices 

$$FGH = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix} \quad HGF = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 4 & 2 & 1 & 0 \\ 8 & 4 & 2 & 1 \end{bmatrix}$$

14. 14. Write down all six of the 3×3 permutation matrices, including $P = I$. Identify their inverses, which are also permutation matrices. The inverses satisfy $PP^{-1} = I$ and are on the same list.

$$\begin{array}{ll}
 P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
 P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} & P^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \\
 P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} & P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \\
 P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} & P^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \\
 P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} & P^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
 P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} & P^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}
 \end{array}$$

18. 18. Decide whether the following systems are singular or nonsingular, and whether they have no solution, one solution, or infinitely many solutions:

$$\begin{bmatrix} 0 & 1 & -1 & | & 2 \\ 1 & -1 & 0 & | & 2 \\ 1 & 0 & -1 & | & 2 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & -1 & | & 0 \\ 1 & -1 & 0 & | & 0 \\ 1 & 0 & -1 & | & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 1 & | & 1 \\ 1 & 1 & 0 & | & 1 \\ 1 & 0 & 1 & | & 1 \end{bmatrix}$$

matrices 🧠

- Performing rref on above matrices yields:
 - Singular — no solution
 - Singular — ∞ solutions.
 - Nonsingular — one solution $[0.5 \ 0.5 \ 0.5]$

Problems 26, 28

26. Which number c leads to zero in the second pivot position? A row exchange is needed and $\mathbf{A} = \mathbf{L}\mathbf{U}$ is not possible. Which c produces zero in the third pivot position? Then a row exchange can't help and elimination fails.

$$\mathbf{A} = \begin{bmatrix} 1 & c & 0 \\ 2 & 4 & 1 \\ 3 & 5 & 1 \end{bmatrix}$$

- If $c = 2$ then row 2 would have a 0 in the pivot, yielding:

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 3 & 5 & 1 \end{bmatrix}$$

- If $c = 1$, then you could take the matrix down to the following form,

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

which would yield a singular matrix with infinite solutions.

28. \mathbf{A} and \mathbf{B} are symmetric across the diagonal (because $4 = 4$). Find their triple factorizations $\mathbf{L}\mathbf{U}$ and say how \mathbf{U} is related to \mathbf{L} for these symmetric matrices:

$$\mathbf{A} = \begin{bmatrix} 2 & 4 \\ 4 & 11 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 & 4 & 0 \\ 4 & 12 & 4 \\ 0 & 4 & 0 \end{bmatrix}$$

$$\mathbf{U}_A = \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix}, \quad \mathbf{U}_B = \begin{bmatrix} 1 & 4 & 0 \\ 0 & -4 & 4 \\ 0 & 0 & -4 \end{bmatrix}$$

$$\mathbf{V}_A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{V}_B = \begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{L}_A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \quad \mathbf{L}_B = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

- $\mathbf{A}, \mathbf{B} = \mathbf{L}\mathbf{D}\mathbf{V} \implies \mathbf{L} = \mathbf{V}^T$; the diagonal of the upper matrix, if reduced to 1's in the pivot positions, yields the transpose of the lower triangular matrix.

Problems 33, 35, 42, 43

33. Solve $Lc = b$ to find c . Then solve $Ux = c$ to find x . What was A ?

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

35. (Important) If A has pivots 2, 7, 6 with no row exchanges, what are the pivots for the upper left 2×2 submatrix B (without row 3 and column 3)? Explain why.
42. If P_1 and P_2 are permutation matrices, so is P_1P_2 . This still has the rows of I in some order. Give examples with $P_1P_2 \neq P_2P_1$ and $P_3P_4 = P_4P_3$.
43. (Try this question) Which permutation makes PA upper triangular? Which permutations make P_1AP_2 lower triangular? Multiplying A on the right by P_2 exchanges the $\langle ? \rangle$ of A .

1.6 Inverses and Transposes

Problems 3, 10, 12, 13, 18, 20, 21

3. From $\mathbf{AB} = \mathbf{C}$ find a formula for \mathbf{A}^{-1} . Also find \mathbf{A}^{-1} from $\mathbf{PA} = \mathbf{LU}$.

10. Find the inverses (in any legal way) of

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{bmatrix},$$

12. If \mathbf{A} is invertible, which properties of \mathbf{A} remain true for \mathbf{A}^{-1} ?

(a) \mathbf{A} is triangular.

(b) \mathbf{A} is symmetric.

(c) \mathbf{A} is tridiagonal.

(d) All entries are whole

(e) All entire are fractions (including numbers like $\frac{3}{1}$)

13. If $\mathbf{A} = [3 \ 1]$ and $\mathbf{B} = [2 \ 2]$, compute $\mathbf{A}^T \mathbf{B}$, $\mathbf{B}^T \mathbf{A}$, \mathbf{AB}^T , and \mathbf{BA}^T .

18. Under what conditions on their entries are \mathbf{A} and \mathbf{B} invertible?

$$\mathbf{A} = \begin{bmatrix} a & b & c \\ d & e & 0 \\ f & 0 & 0 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & e \end{bmatrix}$$

20. Find the inverse of

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{4} & 1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}$$

21. (Remarkable) If \mathbf{A} and \mathbf{B} are square matrices, show that $\mathbf{I} - \mathbf{BA}$ is invertible if $\mathbf{I} - \mathbf{AB}$ is invertible. Start from $\mathbf{B}(\mathbf{I} - \mathbf{AB}) = (\mathbf{I} - \mathbf{BA})\mathbf{B}$

Problems 28–30

28. If the product $\mathbf{M} = \mathbf{ABC}$ of three square matrices is invertible, then \mathbf{A} , \mathbf{B} , \mathbf{C} are invertible. Find a formula for \mathbf{B}^{-1} that involves \mathbf{M}^{-1} and \mathbf{A} and \mathbf{C} .

29. Prove that a matrix with a column of zeros cannot have an inverse.

30. Multiply $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ times $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. What is the inverse of each matrix if $ad \neq bc$?

Problems 40–43

40. True or false (with a counterexample if false and a reason if true):

- (a) A 4×4 matrix with a row of zeros is not invertible.
- (b) A matrix with 1s down the main diagonal is invertible.
- (c) If \mathbf{A} is invertible then \mathbf{A}^{-1} is invertible.
- (d) If \mathbf{A}^T is invertible then \mathbf{A} is invertible.

41. For which three numbers c is this matrix not invertible, and why not?

$$\mathbf{A} = \begin{bmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{bmatrix}$$

42. Prove that \mathbf{A} is invertible if $a \neq 0$ and $a \neq b$ (find the pivots and \mathbf{A}^{-1}):

$$\mathbf{A} = \begin{bmatrix} a & b & b \\ a & a & b \\ a & a & a \end{bmatrix}$$

Problems 56–58

56. If $\mathbf{A} = \mathbf{A}^T$ and $\mathbf{B} = \mathbf{B}^T$, which of these matrices are certainly symmetric?

- (a) $\mathbf{A}^2 - \mathbf{B}^2$
- (b) $(\mathbf{A} + \mathbf{B})(\mathbf{A} - \mathbf{B})$
- (c) \mathbf{ABA}
- (d) \mathbf{ABAB}

57. If $\mathbf{A} = \mathbf{A}^T$ needs a row exchange, then it also needs a column exchange to stay symmetric. In matrix language, \mathbf{PA} loses the symmetry of \mathbf{A} but \mathbf{PAP}^T recovers the symmetry.

58. ↓

- (a) How many entries of \mathbf{A} can be chosen independently, if $\mathbf{A} = \mathbf{A}^T$ is 5×5 ?
- (b) How do \mathbf{L} and \mathbf{D} (5×5) give the same number of choice in \mathbf{LCL}^T ?