Calculus III Final Review

Convergence Notes

- Let $\sum_{n=1}^{\infty} a_n$ be given and note for which series convergence is known, i.e.:

  Geometric: let $c \neq 0$, if $|r| < 1$, then

  $$\sum_{n=0}^{\infty} cr^n = \frac{c}{1-r}$$

  $|r| > 1 \implies$ diverges

  ***p*-Series**: converges if $p > 1$.

  $$\sum_{n=0}^{\infty} \frac{1}{n^p}$$

  $p < 1 \implies$ diverges

- **The n^{th} Term Divergence Test**: a relatively easy test that can be used to quickly determine if a test diverges if the $\lim_{n\to\infty} a_n \neq 0$. If $\lim_{n\to\infty} a_n = 0$, then the test is inconclusive and other tests must be applied.

Tests for Positive Series

- **Direct Comparison Test**: use if dropping terms from the denominator or numerator gives a series b_n wherein convergence is easily found, then compare to the original series a_n as follows:

  $$\sum_{n=1}^{\infty} b_n \text{ converges} \implies \sum_{n=1}^{\infty} a_n \text{ converges} \quad \leftarrow 0 \leq a_n \leq b_n$$

  $$\sum_{n=1}^{\infty} b_n \text{ diverges} \implies \sum_{n=1}^{\infty} a_n \text{ diverges} \quad \leftarrow 0 \leq b_n \leq a_n$$

- **Limit Comparison Test**: use when the direct comparison test isn't convenient or when comparing two series. One can to take the dominant term in the numerator and denominator from a_n to form a new positive sequence b_n if needed.

  Assuming the following limit $L = \lim_{n\to\infty} \frac{a_n}{b_n}$ exists, then:

  $$L > 0 \implies \sum_{n=1}^{\infty} a_n \text{ converges} \iff \sum_{n=1}^{\infty} b_n \text{ converges}$$

  $$L = 0 \text{ and } \sum_{n=1}^{\infty} b_n \text{ converges} \implies \sum_{n=1}^{\infty} a_n \text{ converges}$$

  $$L = \infty \text{ and } \sum_{n=1}^{\infty} a_n \text{ converges} \implies \sum_{n=1}^{\infty} b_n \text{ converges}$$

- **Ratio Test**: often used in the presence of a factorial $(\boldsymbol{n!})$ or when the are constants raised to the power of n $(\boldsymbol{c^n})$.

  Assuming the following limit $\rho = \lim_{n\to\infty} \left|\frac{a_n+1}{a_n}\right|$ exists, then

$$\begin{aligned} \rho < 1 &\implies \sum a_n \text{ converges absolutely} \\ \rho > 1 &\implies \sum a_n \text{ diverges} \\ \rho = 1 &\implies \text{test is inconclusive} \end{aligned}$$

- **Root Test**: used when there is a term in the form of $\boldsymbol{f(n)^{g(n)}}$.

  Assuming the following limit $C = \lim_{n\to\infty} |a_n|^{\frac{1}{n}}$ exists, then

$$\begin{aligned} C < 1 &\implies \sum a_n \text{ converges absolutely} \\ C > 1 &\implies \sum a_n \text{ diverges} \\ C = 1 &\implies \text{test is inconclusive} \end{aligned}$$

- **Integral Test**: if the other tests fail and $a_n = f(n)$ is a decreasing function, then one can use the improper integral $\int_1^\infty f(x)dx$ to test for convergence.

  Let $a_n = f(n)$ be a positive, decreasing, and continuous function $\forall x \geq 1$, then:

$$\begin{aligned} \int_1^\infty f(x)dx \text{ converges} &\implies \sum_{n=1}^\infty a_n \text{ converges} \\ \int_1^\infty f(x)dx \text{ diverges} &\implies \sum_{n=1}^\infty a_n \text{ diverges} \end{aligned}$$

Tests for Non-Positive Series

- **Alternating Series Test**: used for series in the form $\sum_{n=0}^\infty (-1)^n a_n$

  Converges if $|a_n|$ decreases monotonically $(|a_n+1| \leq |a_n|)$ and if $\lim_{n\to\infty} a_n = 0$

- **Absolute Convergence**: used if the series $\sum a_n$ is not alternating; simply test if $\sum |a_n|$ converges using the test for positive series.

Convergence Problems

10.4 Preliminary Questions

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10.5 Preliminary Questions

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10.5 Exercises

Determine convergence or divergence using any method.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n + 4^n}{7^n}$

$$\implies \sum_{n=1}^{\infty} \frac{2^n}{7^n} + \sum_{n=1}^{\infty} \frac{4^n}{7^n}$$

Separate into two geometric series↑

$$\implies r = \frac{2}{7} < 1, \quad r = \frac{4}{7} < 1$$

Both geometric series converge, thus the original series converges.

2. $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{n!}$

$$\implies \rho = \lim_{n\to\infty} \left| \frac{(n+1)^3}{(n+1)!} \cdot \frac{n!}{n^3} \right|$$

Apply the ratio test↑

$$\implies \rho = \lim_{n\to\infty} \frac{n^3 + 3n^2 + 3n + 1}{(n+1)n!} \cdot \frac{n!}{n^3}$$

Expand; all positive

$$\implies \rho = \lim_{n\to\infty} \frac{3n^2 + 3n + 1}{(n+1)}$$

$$\implies \rho = \lim_{n\to\infty} \frac{3n^2 + 3n}{(n+1)} \cdot \lim_{n\to\infty} \frac{1}{(n+1)} = 0$$

$\rho = 0 < 1$, thus the series converges.

3. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2n+1}$

$$\implies \lim_{n\to\infty} \frac{n}{2n+1} \qquad \text{Apply the } n^{th} \text{ term test}^{\uparrow}$$

$$\implies \lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2} \qquad \text{by L'Hôpital's Rule}$$

$\lim_{n\to\infty} a_n \neq 0$, thus the series diverges.

4. $\displaystyle\sum_{n=1}^{\infty} 2^{\frac{1}{n}}$

$$\implies \lim_{n\to\infty} 2^{\frac{1}{n}} = 2^0 = 1 \qquad \text{Apply the } n^{th} \text{ term test}^{\uparrow}$$

$\lim_{n\to\infty} a_n \neq 0$, thus the series diverges.

5. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$

$$0 \leq \sin n \leq 1 \qquad \leftarrow \forall n \geq 1$$

$$0 \leq \frac{\sin n}{n^2} \leq \frac{1}{n^2} \qquad \text{Apply the direct comparison test}^{\uparrow}$$

$$b_n = \frac{1}{n^2} \to \text{converges} \qquad \text{by } p\text{-series}^{\uparrow}$$

The larger (b_n) series converges, thus the smaller (a_n) converges.

6. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{(2n)!}$

$$\implies \rho = \lim_{n\to\infty} \left| \frac{(n+1)!}{(2n+2)!} \cdot \frac{(2n)!}{n!} \right| \qquad \text{Apply the ratio test}^{\uparrow}$$

$$\implies \rho = \lim_{n\to\infty} \frac{(n+1)n!}{(2n+2)(2n+1)2n!} \cdot \frac{(2n)!}{n!}$$

$$\implies \rho = \lim_{n\to\infty} \frac{n+1}{(2n+2)(2n+1)} = \frac{n+1}{4n^2+6n+2}$$

$$\implies \rho = \lim_{n\to\infty} \frac{1}{8n+6} = 0 \qquad \text{By L'Hôpital's Rule}$$

$\rho = 0 < 1$, thus the series converges.

7. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$

$$0 \le n \le n+\sqrt{n} \qquad \leftarrow \forall n \ge 1$$

$$0 \le \frac{1}{n+\sqrt{n}} \le \frac{1}{n}$$

Apply the direct comparison test ↑

$$b_n = \frac{1}{n} \to \text{ diverges}$$

The smaller (b_n) series diverges, thus the larger a_n original series diverges.

8. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$

f is poisitved, decreasing, and continuous for $x \ge 2$ Apply the integral test ↑

$$\implies \int_2^{\infty} f(x)dx = \lim_{R\to\infty} \int_2^R \frac{1}{x(\ln x)^3}dx \qquad \ln x = u, \quad xdu = dx$$

$$\implies \lim_{R\to\infty} \int_2^R \frac{1}{x(u)^3}xdu = \int_{i2}^R \frac{1}{u}^3 du$$

$$\implies -\frac{1}{2(u)^2} = -\frac{1}{2\ln^2(x)} + C\Big|_2^{\infty}$$

$$\implies 0 - \left(-\frac{1}{2\ln^2(2)}\right) = \frac{1}{2\ln^2(2)}$$

The improper integral converges, thus the original series converges.

9. $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{5^n}$

10. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2(\ln n)^3}$

11. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3-n^2}}$

12. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+4n}{3n^4+9}$

13. $\sum_{n=1}^{\infty} n^{-0.8}$

14. $\sum_{n=1}^{\infty} (0.8)^{-n} n^{-0.8}$

15. $\sum_{n=1}^{\infty} 4^{-2n+1}$

16. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$

17. $\sum_{n=1}^{\infty} \sin \frac{1}{n^2}$

18. $\sum_{n=1}^{\infty} (-1)^n \cos n^{-1}$

19. $\sum_{n=1}^{\infty} \frac{(-2)^n}{\sqrt{n}}$

20. $\sum_{n=1}^{\infty} \left(\frac{n}{n+12}\right)^n$

21. $\sum_{n=1}^{\infty} (-1)^n \cos n^{-1}$

Power Series: 10.6

Power Series Notes

- **Power series**: a infinite series in the form:

$$F(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

  Where the constant c is the *center* of the power series F(x).

- **Radius of convergence** R: the range of values of the variable x whereby the power series $F(x)$ converges.
  - Every power series converges at $x = c$, as $(x-c)^0 = 1$, though the series may diverge for other values of x.
  - $F(x)$ converges for $|x-c| < R$ and diverges for $|x-c| > R$
  - $F(x)$ may converge of diverge at endpoints $c - R$ and $c + R$
  - **Interval of convergence**: the open interval $(c-R, c+R)$ and possibly one of both of the endpoints, each must be tested.
    - In most cases, the ratio test↑ can be used to find R.
    - If $R > 0$, then F is differentiable over the interval of convergence; the derivative and antiderivative can be obtained using the following:

$$F'(x) = \sum_{n=1}^{\infty} n a_n (x-c)^{n-1} \qquad F(x)dx = C + \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-c)^{n+1}$$

  - **Useful Power Series**: the following power series (more examples: Taylor series↓) can be used to drive expansions of other related functions via substitution, integration, or differentiation:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \leftarrow |x| < 1 \qquad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Power Series Problems

10.6 Preliminary Questions

1. Suppose that $\sum a_n x^n$ converges for $x = 5$. Must it also converge for $x = 4$? What about $x = -3$?

$$\begin{aligned} R = 5, \quad c = 0 &\implies \text{series converges for } |x| < 5 \\ &\implies -5 < x < 5 \end{aligned}$$

Both -3 and 4 are inside the interval, thus it must converge for both.

2. Suppose that $\sum a_n (x-6)^n$ converges for $x = 10$. At which of the following points must it also converge?

$$\begin{aligned} R = 10, c = 6 &\implies |x - 6| < 4 \\ &\implies 2 < x < 10 \end{aligned}$$

(a) $x = 8$ converges

(b) $x = 11$ uncertain

(c) $x = 3$ converges

(d) $x = 0$ uncertain

3. What is the radius of converges of $F(3x)$ if $F(x)$ is a power series with $R = 12$?

$$R = \frac{12}{3} = 4$$

4. The power series $F(x) = \sum_{n=1}^{\infty} nx^n$ has a radius of converge $R = 1$.

What is the power series expansion of $F'(x)$ and what is its radius of convergence?

$$F'(x) = \sum_{n=1}^{\infty} n^2 x^{n-1} \qquad \leftarrow F'(x) = \sum_{n=1}^{\infty} na_n (x-c)^{n-1}$$

$$R = 1$$

10.6 Exercises

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Taylor Series Notes

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Taylor Series Problems

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Parametric Notes

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Parametric Problems

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Polar Coordinates Notes

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Polar Coordinate Problems

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Conic Sections Notes

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Conic Section Problems

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Quiz 3

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Quiz 4

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Final Review Questions