

# Applied Linear Algebra



## 1 Matrices and Gaussian Elimination

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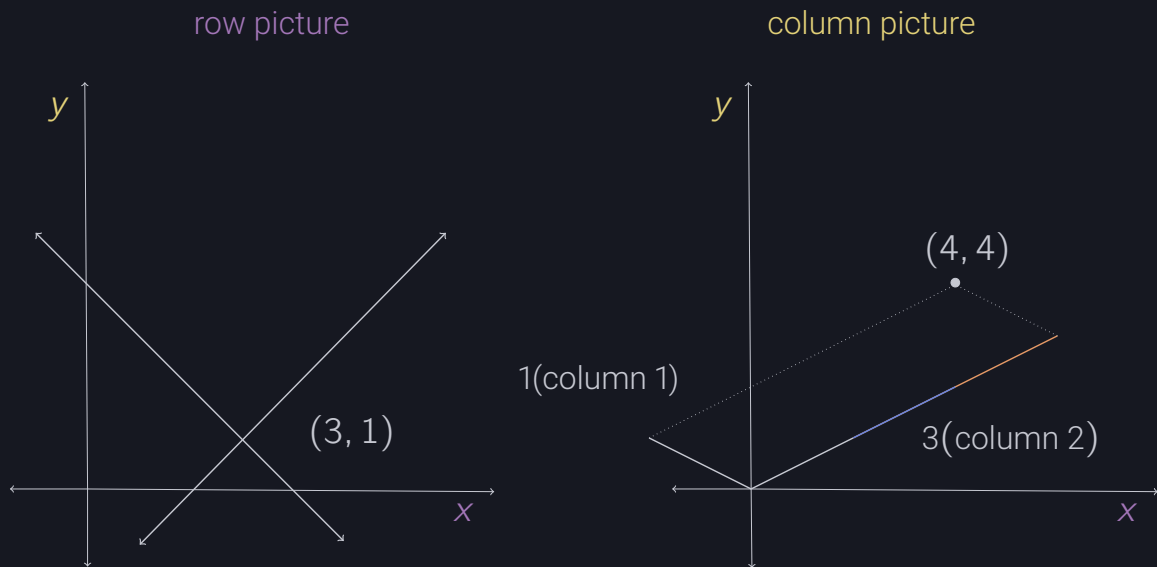
# 1 Matrices and Gaussian Elimination



## 1.2 The Geometry of Linear Equations

### Problems 1–12

1. For the equations  $x + y = 4$ ,  $2x - 2y = 4$ , draw the row picture (two intersecting lines) and the column picture (combination of two columns equal to the column vector  $(4, 4)$  on the right side).



### 1.2.1

2. Solve to find a combination of the columns that equals  $b$ :

$$u - v - w = b_1$$

$$v + w = b_2$$

$$w = b_3$$

$$\implies w = b_3$$

$$\implies v = b_2 - b_3$$

$$\implies u = b_1 + v + w = b_1 + b_2$$

3. Describe the intersection of the three planes  $u + v + w + z = 6$  and  $u + w + z = 4$  and  $u + w = 2$  (all in four-dimensional space). Is it a line or a point or an empty set? What is the intersection if the fourth plane  $u = -1$  is included? Find a fourth equation that leaves us with no solution.

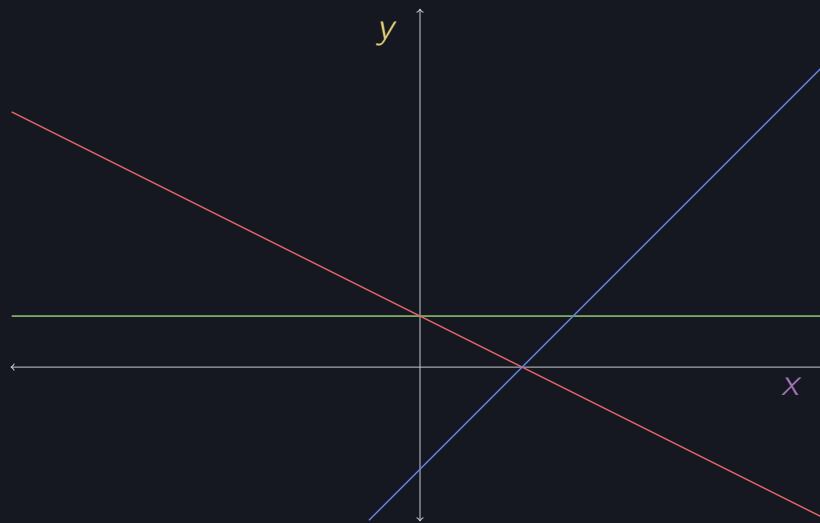
- **A line**; as  $u + w = 2$  is only a line (\*?). A fourth plane with  $u = -1$  would produce a normally intersecting point. Any addition equation when  $u + w \neq 2$  would produce an inconsistent equation.

4. Sketch these three lines and decide if the equations are solvable:

$$x + 2y = 2$$

$$x - y = 2$$

$$y = 1$$



#### 1.2.4

What happens if all right-hand sides are zero? Is there any nonzero choice of right-hand sides that allows the three lines to intersect at the same point?

- If all the solutions were zero, then it would be a trivial solution.
  - Yes, e.g.,  $x - y = -1$  would produce a single point of intersection.
5. Find two points on the line of intersection of the three planes  $t = 0$  and  $z = 0$  and  $x + y + z + t = 1$  in four-dimensional space.

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

6. When  $b = (2, 5, 7)$ , find a solution  $(u, v, w)$  to equation (4) different from the solution  $(1, 0, 1)$  mentioned in the text.
- Since there are infinite solutions, and if  $\mathbf{s}$  vector describing one solution and  $\lambda$  is any scalar, then  $\mathbf{s}\lambda$  is also a solution. E.g.,  $(1, 0, 1)42 = (42, 0, 42)$
7. Give two more right-hand sides in addition to  $b = (2, 5, 7)$  for which equation (4) can be solved. Give two more right-hand sides in addition to  $b = (2, 5, 6)$  for which it cannot be solved.

8. Explain why the system

$$\begin{aligned}u + v + w &= 2 \\u + 2v + 3w &= 1 \\v + 2w &= 0\end{aligned}$$

is singular by finding a combination of the three equations that adds up to  $0 = 1$ . What value should replace the last zero on the right side to allow the equations to have solutions—and what is one of the solutions?

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 0 \end{array} \right] \xrightarrow{R_2 - R_1} \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 2 & 0 \end{array} \right] \xrightarrow{R_3 - R_2} \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

- Replacing the last zero with  $-1$  would yield infinite solutions. One solution would be  $[3, -1, 0]^T$

9. The column picture for the previous exercise (singular system) is

$$u \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + v \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + w \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = b$$

Show that the three columns on the left lie in the same plane by expressing the third as a combination of the first two. What are all the solutions  $(u, v, w)$  if  $b$  is the zero vector  $(0, 0, 0)$ ?

$$-1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

- If  $b$  is equal to the zero vector  $\mathbf{0}$  then the solutions are equal to the kernel (\*)  
i.e.,  $-1x_1, 2x_2, 0x_3 = \mathbf{0}$
10. Under what condition on  $y_1, y_2, y_3$  do the points  $(0, y_1), (1, y_2), (2, y_3)$  lie on a straight line?
- Question 9 describes the state at which they are co linear, i.e.,  $y_3 = 2y_2 - y_1$
11. These equations are certain to have the solution  $x = y = 0$ . For which values of  $a$  is there a whole line of solutions?

$$\begin{aligned}ax + 2y &= 0 \\2x + ay &= 0\end{aligned}$$

12. Starting with  $x + 4y = 7$ , find the equation for the parallel line through  $x = 0, y = 0$ .

## Problems 13–15

13. Draw the two pictures in two planes for the equations  $x - 2y = 0$ ,  $x + y = 6$ .
14. For two linear equations in three unknowns  $x$ ,  $y$ ,  $z$ , the row picture will show (2 or 3) (lines or planes) in (two or three)-dimensional space. The column picture is in (two or three)-dimensional space. The solutions normally lie on a  $\langle \rangle$ .
15. For four linear equations in two unknowns  $x$  and  $y$ , the row picture shows four  $\langle \rangle$ . The column picture is in  $\langle \rangle$ -dimensional space. The equations have no solution unless the vector on the right-hand side is a combination of  $\langle \rangle$ .

## Problems 16–23

16. Find a point with  $z = 2$  on the intersection line of the planes  $x + y + 3z = 6$  and  $x - y + z = 4$ . Find the point with  $z = 0$  and a third point halfway between.
17. The first of these equations plus the second equals the third:

$$x + y + z = 2$$

$$x + 2y + z = 3$$

$$2x + 3y + 2z = 5$$

The first two planes meet along a line. The third plane contains that line, because if  $x$ ,  $y$ ,  $z$  satisfy the first two equations then they also  $\langle \rangle$ . The equations have infinitely many solutions (the whole line  $\mathbf{L}$ ). Find three solutions.

18. Move the third plane in Problem 17 to a parallel plane  $2x + 3y + 2z = 9$ . Now the three equations have no solution—*why not*? The first two planes meet along the line  $\mathbf{L}$ , but the third plane doesn't that line.
19. In Problem 17 the columns are  $(1, 1, 2)$  and  $(1, 2, 3)$  and  $(1, 1, 2)$ . This is a "singular case" because the third column is  $\langle \rangle$ . Find two combinations of the columns that give  $b = (2, 3, 5)$ . This is only possible for  $b = (4, 6, c)$  if  $c =$
20. Normally 4 "planes" in four-dimensional space meet at a  $\langle \rangle$ . Normally 4 column vectors in four-dimensional space can combine to produce  $b$ . What combination of  $(1, 0, 0, 0)$ ,  $(1, 1, 0, 0)$ ,  $(1, 1, 1, 0)$ ,  $(1, 1, 1, 1)$  produces  $b = (3, 3, 3, 2)$ ? What 4 equations for  $x$ ,  $y$ ,  $z$ ,  $t$  are you solving?
21. When equation 1 is added to equation 2, which of these are changed: the planes in the row picture, the column picture, the coefficient matrix, the solution?
22. If  $(a, b)$  is a multiple of  $(c, d)$  with  $abcd \neq 0$ , show that  $(a, c)$  is a multiple of  $(b, d)$ . This is surprisingly important: call it a challenge question. You could use numbers first to see how  $a$ ,  $b$ ,  $c$ , and  $d$  are related. The question will lead to:  $\langle \rangle$   
If  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  has dependent rows then it has dependent columns.



## 1.3 Gaussian Elimination

### Problems 1–9

1.

### Problems 10–19

1.

### Problems 20–22

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### Problems 23–31

1.

## 1.4 Matrix Notation and Matrix Multiplication



## 1.5 Triangular Factors and Row Exchanges

## 1.6 Inverses and Transposes

## 1.7 Special Matrices and Applications