1. Determine if the following are 1 to 1, onto, or total. Justify your answer.

- **Total (total)**: $\forall x \in A \implies f(x)$ is defined.
- **1 to 1 (injective)**: $\forall x, y \in X, \quad f(x) = f(y) \implies x = y.$
- **Onto (surjective)**: $f : X \to Y, \quad \forall y \in Y, \quad \exists x \in X \implies f(x) = y$

(a) $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \sin x$

- Total: ✓ true: $\forall x \in \mathbb{R}, \quad f(x) \in \mathbb{R}$
- Injective: ✘ false: $\forall x, y \in \mathbb{R} : x \neq y,$ some $f(x) = f(y)$
- Surjective: ✘ false: e.g., $2 \in \mathbb{R}$ but $\notin [-1, 1]$

(b) $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \sqrt{x}$

- Total: ✘ false: e.g., $f(-1) = i, i \notin \mathbb{R}$
- Injective: ✓ true (partial): $\forall x, y \in \mathbb{R}, : x = y \iff f(x) = f(y)$
- Surjective: ✘ false: $\forall y \in \mathbb{R}^-, \quad \nexists x \in \mathbb{R} \implies f(x) \neq y$

(c) $f : \mathbb{N} \to \mathbb{R}^+, \quad f(x) = \sqrt{x}$

- Total: ✓ true: $\forall x \in \mathbb{N}, \quad f(x) \in \mathbb{R}^+$
- Injective: ✓ true: $\forall x, y \in \mathbb{N} : x = y \implies f(x) = f(y)$
- Surjective: ✓ true: $\forall y \in \mathbb{R}^+, \exists x \in \mathbb{N} : f(x) = y$

(d) $f : \mathbb{R}^+ \to \mathbb{N}, \quad f(x) = \sqrt{x}$

- Total: ✘ false: e.g., $\sqrt{42} \notin \mathbb{N}$
- Injective: ✓ true (partial):
  $\forall x, y \in \mathbb{R}^+ : x = y \implies f(x) = f(y) \iff f(x) \in \mathbb{N}$
- Surjective: ✓ true: $\forall y \in \mathbb{N}, \quad \exists x \in \mathbb{R}^+ : f(x) = y$

(e) $f : \mathbb{R} \to \mathbb{R}^+, \quad f(x) = x^2$

- Total: ✓ true: $\forall x \in \mathbb{R}, \quad f(x) \in \mathbb{R}^+$
- Injective: ✘ false: e.g., $f(-2) = 4, f(2) = 4$, i.e, $x \neq y, f(x) = f(y)$
- Surjective: ✓ true: $\forall \in \mathbb{R}^+, \quad \exists x \in \mathbb{R} : f(x) = y$

(f) $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = x^3$

- Total: ✓ true: $\forall x \in \mathbb{R}, \quad f(x) \in \mathbb{R}$
- Injective: ✓ true: $\forall x, y \in \mathbb{R}, : x = y \implies f(x) = f(y)$
- Surjective: ✓ true: $\forall y \in \mathbb{R}, \quad \exists x \in \mathbb{R} : f(x) = y$

2. Determine if the following relations are reflexive, symmetric, antisymmetric, or transitive. For this question

$$a, b \in \mathbb{N}, \quad \frac{p}{q}, \frac{m}{n} \in \mathbb{Q}, \quad s, t \in \Sigma^*$$

Justify your answer.

- **Equivalence**: $\sim, \equiv \iff$ a relation is reflexive, symmetric, and transitive.
- **Reflexive**: $\forall a \in X, \quad a \sim a$
- **Symmetric**: $\forall a, b \in X, \quad a \sim b \iff b \sim a$
- **Antisymmetric**:
  $\forall a, b \in X, \quad a \sim b, a \neq b \implies b \nsim a$ ...equiv... $a \sim b, b \sim a \implies a = b$
- **Transitive**: $\forall a, b, c \in X, \quad : a \sim b, b \sim c \implies a \sim c$

(a) $a \sim b$ if $a + b = 10$

- Reflexive: ✘ false: $6 \nsim 6$
- Symmetric: ✓ true: $a = 10 - b, b = 10 - a$
- Antisymmetric: ✘ false: $6 \sim 4, 4 \sim 6, \quad 6 \neq 4$
- Transitive: ✘ false: $4 \sim 6, 6 \sim 4, 4 \nsim 4$

(b) $a \sim b$ if a and b are both even (E) or both odd (O)

- Reflexive: ✓ true: $a \sim a, b \sim b \iff a \wedge b \in (E \oplus O)$
- Symmetric: ✓ true: $\forall a, b \in (O \oplus E), \quad a \sim b \iff b \sim a$
- Antisymmetric: ✘ false: $2 \sim 4, 4 \sim 2, \quad 2 \neq 4$
- Transitive: ✓ true: $\forall a, b \in (O \oplus E), \quad a \sim b, b \sim c \implies a \sim c$

(c) $\frac{p}{q} \sim \frac{r}{s}$ if $q \leq s$

- let $a = \frac{p}{q}, b = \frac{r}{s}, X = \{a \sim b\} \impliedby q \leq s$
- Reflexive: ✓ true: $\forall x \in X, \quad x \sim x$
- Symmetric: ✓ true: $\forall x, y \in X, \quad x \sim y \iff y \sim x$
- Antisymmetric: ✘ false: $\frac{1}{2} \sim \frac{2}{2}, \frac{2}{2} \sim \frac{1}{2}, \quad \frac{1}{2} \neq \frac{2}{2}$
- Transitive: ✓ true: $\forall x, y, z \in X, \quad x \sim z \impliedby x \sim y, y \sim z$

(d) $s \sim t$ if $s = reverse(t)$

- Reflexive: ✘ false: hannah $\nsim$ bob
- Symmetric: ✓ true: $\forall s, t \in \Sigma^*, \quad s \sim t \iff t \sim s$
- Antisymmetric: ✓ true: $\forall s, t \in \Sigma^*, \quad s \sim t, t \sim s \iff s = t$
- Transitive: ✓ true: $\forall s, t, w \in \Sigma^*, \quad s \sim w \iff s = t = w$

(e) $a \sim b$ if $b = ca$ for some c

- Reflexive: ✓ true: $\forall x \in \mathbb{N}, x \sim x \impliedby c = 1$
- Symmetric: ✘ false: $2 \sim 4 \impliedby c = 2, \quad 4 \nsim 2$
  - ✓ true: if c is allowed to vary between relations.
- Antisymmetric: ✓ true: $\forall a, b \in \mathbb{N}, a \sim b, b \sim a \iff c = 1 \land a = b$
  - ✘ false: if c is allowed to vary between relations.
- Transitive: ✘ false: $2 \sim 4, \quad 4 \sim 8, \quad 2 \nsim 8$
  - ✓ true: if c is allowed to vary between relations.

(f) $a \sim b$ if $a^b = b^a$

- let $X = \{a \sim b\} \impliedby a^b = b^a$
- Reflexive: ✓ true: $\forall x \in X, \quad x^x = x^x$
- Symmetric: ✓ true: $\forall a, b \in \mathbb{N}, : a^b = b^a \implies b^a = a^b$
- Antisymmetric: ✘ false: $2 \sim 4, 4 \sim 2, \quad 2 \neq 4$
- Transitive: ✓ true: $\forall a, b, c \in \mathbb{N}, a \sim b$

3. Prove that if $f : B \to C$ is 1 to 1, and $g : A \to B$ is 1 to 1, then $f \circ g$ is also 1 to 1.

$$f \circ g = f(g(x)) = f : (g : A \to B) \to C$$

$$\begin{aligned}
&: \forall x, y \in A, \quad g(x) = g(y) \implies x = y \\
&: \forall x, y \in B, \quad f(x) = f(y) \implies x = y \\
&\therefore \forall x, y \in C, \quad f(g(x)) = f(g(y)) \implies x = y
\end{aligned}$$

4. Prove or disprove (where P is the power set):

(a) for any sets A and B, $P(A \cap B) = P(A) \cap P(B)$

✓ true:

$$\begin{aligned}
P(A) = \{S : S \subseteq A\}, \quad & P(B) = \{S : S \subseteq B\} \\
: A \cap B \subseteq A, \quad & : B \cap A \subseteq B \\
\implies & P(A \cap B) = \{S : S \subseteq A \cap B\}
\end{aligned}$$

(b) for any sets A and B, $P(A \cup B) = P(A) \cup P(B)$

✘ false:

$$\begin{aligned}
A = \{42\}, \quad B = \{69\}, \quad & A \cup B = \{42, 69\} \\
P(A) \cup P(B) &= \{\emptyset, \{42\}, \{69\}\} \\
P(A \cap B) &= \{\emptyset, \{42\}, \{69\}, \{42, 69\}\} \\
\implies & P(A) \cup P(B) \neq P(A \cup B)
\end{aligned}$$

5. De Morgan's rule is a logical equivalence $(\neg a) \vee (\neg b) = \neg(a \wedge b)$
   You can verify this equivalence with a truth table.

   Set theory also has a version of De Morgan's rule. Let A and B be sets in universe U. Prove that $A' \cup B' = (A \cap B)'$

$$
\begin{aligned}
A \cap B &= \{x : x \in A \wedge x \in B\} \\
(A \cap B)' &= \{x : x \in U \wedge x \notin (A \cap B)\} \\
A' \cup B' &= \{x : x \notin A \vee x \notin B\} \\
&= \{x : x \in U \wedge x \notin (A \cap B)\} \\
&= (A \cap B)'
\end{aligned}
$$

6. Mathologer is a YouTube channel that does videos illustrating math concepts. He did a video on the fair division problem

   NWT: Spanner's lemma defeats the rental harmony problem

   It's an interesting proof about coloring triangles. Unfortunately, He doesn't complete the proof. At about the 5 minute mark he says that "there will always be an odd number of doors at the bottom of the triangle", but he never proves this. Show that there will always be an odd number of doors on the bottom of the triangle.

   - Vertices of the main triangle must be of different colors, thus every time the triangle is increased in size, then at least one connection is split, increasing number of connections on the bottom side by two.
   - Since the red cannot go on the bottom side, then that leaves only BG/BB/GG connections, which yields $\{BBB, BBG, BGB, BGG, GGG, GGB\}$
   - Starting with BG, then only options are BBG or GBB, thus number of odd doors is at least 1.
   - Each new increase will do one of two options:
     (a) Split a door: yielding a miniature starting case (BBG, GBB) (doors += 0)
     (b) Split a non-door:
        i. into two doors (BGB, GBG) (doors += 2)
        ii. into two non-doors (GGG, BBB) (doors += 0)
   - Thus, max number of doors $= 1 + 2n$ (always odd)
   - note: this proof inspired by Fiefo's YouTube comment, but I think I made it more clear.

7. In class I said that two functions were equal if they give the same output for every input. So, if

$$f : A \to B \text{ and } g : A \to B, \text{ and } \forall x \in A, f(x) = g(x) \implies f = g$$

Show that this is actually an equivalence relation.

- Reflexive: $\forall x \in X, f(x) = g(x) \implies x \sim x$
- Symmetric:
$\forall x, y \in X, \quad f(x) = f(y) \implies x \sim y \iff g(x) = g(y) \implies y \sim x$
- Transitive:
$\forall x, y, z \in X, \quad x \sim y, y \sim z, : f(x) = f(y) \wedge f(y) = f(z) \implies x \sim z$
- All three relations hold, thus the function can be said to be equivalent by it's equivalence relations.

8. In a dictionary all of the words are arranged in a specific order. "a", "aardvark", and so on. This is called the dictionary order for words.

Our ordering here is very simple. Look at the first character of two words w_1 and w_2. If the first character of w_1 comes before w_2 ($w_1[0] < w_2[0]$) then $w_1 < w_2$.

If the characters are equal, then we move on to the second character. We continue this until we find a different character, or one of the words ends. Show that this dictionary order is a partial order (reflexive, antisymmetric, and transitive).

- Reflexive: $\forall s \in \Sigma^*, s \sim s$
- Antisymmetric: $\forall s, t \in \Sigma^*, s \sim t, t \sim s \iff s = t$ (not symmetric)
- Transitive: $\forall s, t, w \in \Sigma^*, s \sim t, t \sim w \iff s = t = w \implies s \sim w$