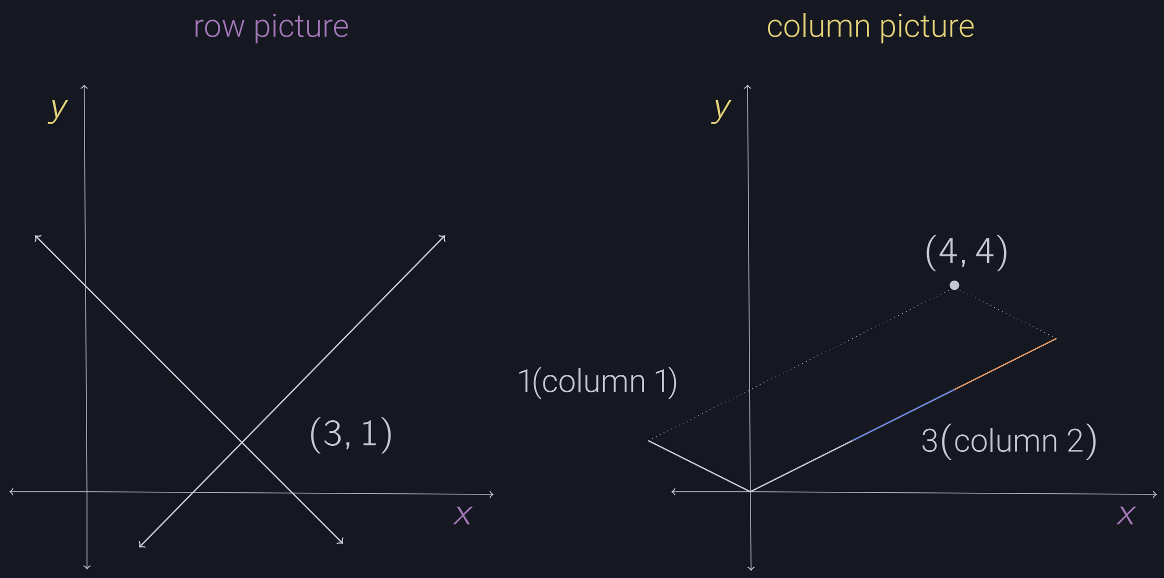
Applied Linear Algebra

1 Matrices and Gaussian Elimination

1.2 The Geometry of Linear Equations

Problems 1–12

1. For the equations $x + y = 4,\ 2x - 2y = 4$, draw the row picture (two intersecting lines) and the column picture (combination of two columns equal to the column vector (4,4) on the right side).



1.2.1

2. Solve to find a combination of the columns that equals b:

$$
\begin{aligned}
u - v - w &= b_1 \\
v + w &= b_2 \\
w &= b_3
\end{aligned}
$$

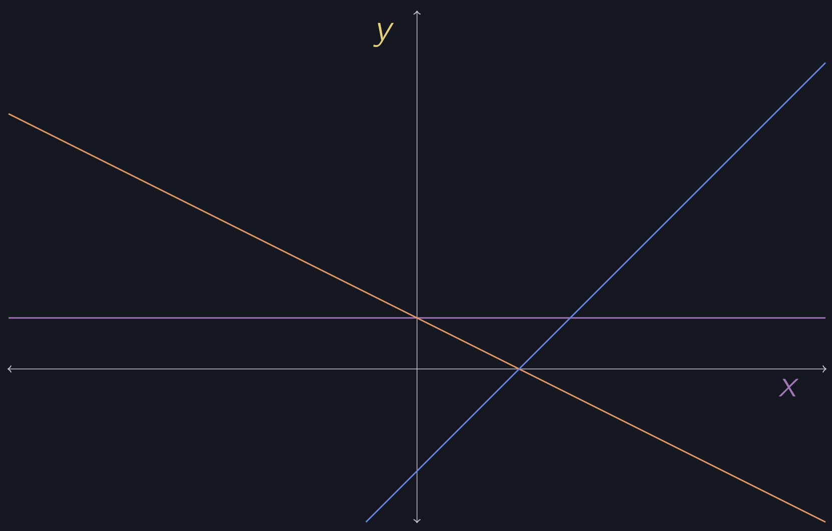
$$
\begin{aligned}
&\implies w = b_3 \\
&\implies v = b_2 - b_3 \\
&\implies u = b_1 + v + w = b_1 + b_2
\end{aligned}
$$

3. Describe the intersection of the three planes $u + v + w + z = 6$ and $u + w + z = 4$ and $u + w = 2$ (all in four-dimensional space). Is it a line or a point or an empty set? What is the intersection if the fourth plane $u = -1$ is included? Find a fourth equation that leaves us with no solution.

    - A line; as $u + w = 2$ is only a line[?]. A fourth plane with $u = -1$ would produce a normally intersecting point. Any addition equation when $u + w \neq 2$ would produce an inconsistent equation.

4. Sketch these three lines and decide if the equations are solvable:

$$x + 2y = 2$$
$$x - y = 2$$
$$y = 1$$



1.2.4

Inconsistent; multiple points of intersect

What happens if all right-hand sides are zero? Is there any nonzero choice of right-hand sides that allows the three lines to intersect at the same point?

- If all the solutions were zero, then it would be a trivial solution.
- Yes, e.g., $x - y = -1$ would produce a single point of intersection.

5. Find two points on the line of intersection of the three planes $t = 0$ and $z = 0$ and $x + y + z + t = 1$ in four-dimensional space.

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

6. When $b = (2, 5, 7)$, find a solution (u, v, w) to equation (4) different from the solution $(1, 0, 1)$ mentioned in the text.
   - Since there are infinite solutions, and if $\boldsymbol{s}$ vector describing one solution and λ is any scalar, then $\boldsymbol{s}\lambda$ is also a solution. E.g., $(1, 0, 1)\,42 = (42, 0, 42)$

8. Explain why the system

$$\begin{aligned} u+v+w&=2\\ u+2v+3w&=1\\ v+2w&=0 \end{aligned}$$

is singular by finding a combination of the three equations that adds up to $0=1$. What value should replace the last zero on the right side to allow the equations to have solutions—and what is one of the solutions?

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2\\ 1 & 2 & 3 & 1\\ 0 & 1 & 2 & 0 \end{array}\right] \xrightarrow{R_2-R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2\\ 0 & 1 & 2 & -1\\ 0 & 1 & 2 & 0 \end{array}\right] \xrightarrow{R_3-R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2\\ 0 & 1 & 2 & -1\\ 0 & 0 & 0 & 1 \end{array}\right]$$

- Replacing the last zero with -1 would yield infinite solutions. One solution would be $[3,-1,0]^T$

9. The column picture for the previous exercise (singular system) is

$$u\begin{bmatrix}1\\1\\0\end{bmatrix}+v\begin{bmatrix}1\\2\\1\end{bmatrix}+w\begin{bmatrix}1\\3\\2\end{bmatrix}=b$$

Show that the three columns on the left lie in the same plane by expressing the third as a combination of the first two. What are all the solutions (u,v,w) if b is the zero vector $(0,0,0)$?

$$-1\begin{bmatrix}1\\1\\0\end{bmatrix}+2\begin{bmatrix}1\\2\\1\end{bmatrix}=\begin{bmatrix}1\\3\\2\end{bmatrix}$$

- If is $\boldsymbol{b}$ equal to the zero vector $\mathbf{0}$ then the solutions are equal to the kernel? i.e., $-1x_1, 2x_2, 0x_3=\mathbf{0}$

10. Under what condition on y_1, y_2, y_3 do the points $(0,y_1)$, $(1,y_2)$, $(2,y_3)$ lie on a straight line?

- Question 9 describes the state at which they are collinear, i.e., $y_3=2y_2-y_1$

11. These equations are certain to have the solution $x=y=0$. For which values of a is there a whole line of solutions?

$$\begin{aligned} ax+2y&=0\\ 2x+ay&=0 \end{aligned}$$

- Only the scalars that make the lines linearly dependent, i.e., $a=2,-2$

Problems 17–23

17. The first of these equations plus the second equals the third:

$$\begin{aligned} x + y + z &= 2 \\ x + 2y + z &= 3 \\ 2x + 3y + 2z &= 5 \end{aligned}$$

The first two planes meet along a line. The third plane contains that line, because if x, y, z satisfy the first two equations then they also span all of $\mathbb{R}^3$. The equations have infinitely many solutions (the whole line $\boldsymbol{L}$). Find three solutions.

   - $\boldsymbol{v} = (4, 4, 0)$, $\boldsymbol{w} = (6, 3, 2)$, $\boldsymbol{u} = 2v + -1w$

18. Move the third plane in Problem 17 to a parallel plane $2x + 3y + 2z = 9$. Now the three equations have no solution—*why not*? The first two planes meet along the line $\boldsymbol{L}$, but the third plane doesn't that cross that line.

19. In Problem 17 the columns are $(1, 1, 2)$ and $(1, 2, 3)$ and $(1, 1, 2)$. This is a "singular case" because the third column is linearly dependent Find two combinations of the columns that give $b = (2, 3, 5)$. This is only possible for $b = (4, 6, c)$ if $c = 10$

20. Normally 4 "planes" in four-dimensional space meet at a tensor. Normally 4 column vectors in four-dimensional space can combine to produce b. What combination of $(1, 0, 0, 0)$, $(1, 1, 0, 0)$, $(1, 1, 1, 0)$, $(1, 1, 1, 1)$ produces $b = (3, 3, 3, 2)$? $(1, 0, 0, -2)^?$ What 4 equations for x, y, z, t are you solving? A lower triangular matrix, i.e.,

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3 \\ 1 & 1 & 0 & 0 & 3 \\ 1 & 1 & 1 & 0 & 3 \\ 1 & 1 & 1 & 1 & 1 \end{array}\right]$$

21. When equation 1 is added to equation 2, which of these are changed: the planes in the row picture, the column picture, the coefficient matrix, the solution?

   - Row operations do not change the solution. Row 2 is changed, thus the second plane is changed. All columns are changed.$^?$

22. If (a, b) is a multiple of (c, d) with $abcd \neq 0$, show that (a, c) is a multiple of (b, d). This is surprisingly important: call it a challenge question. You could use numbers first to see how a, b, c, and d are related. The question will lead to:

    If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has dependent rows then it has dependent columns.

    Received help from this document (pg. 3), accessed 10/01/2021

$$\begin{aligned} \lambda \in \mathbb{R}, \quad (a, b) &= \lambda(c, d) = (\lambda c, \lambda d) \\ \implies a &= \lambda c = \lambda c d^{-1} d = d^{-1} c \lambda d = d^{-1} c b \\ \implies (a, c) &= \left(d^{-1} c b, d^{-1} d c\right) = c d^{-1}(b, d) \end{aligned}$$

    Thus, (b, d) is a multiple of (a, c)

23. In these equations, the third column (multiplying w) is the same as the right side b. The column form of the equations immediately gives what solution for (u, v, w)?

$$\begin{aligned} 6u + 7v + 8w &= 8 \\ 4u + 5v + 9w &= 9 \\ 2u - 2v + 7w &= 7 \end{aligned}$$

    - First two columns are irrelevant, $u = 0$, $v = 0$, only need w

1.3 Gaussian Elimination

Problems 6, 7

6. Choose a coefficient b that makes this system singular. Then choose a right-hand side g that makes it solvable. Find two solutions in that singular case.

$$2x + by = 16$$
$$4x + 8y = g$$

$$2x + 4y = 16$$
$$4x + 8y = 32$$

- Since R_2 is just a multiple of R_1, then solving for x, y, with one variable = 0, in the first equation will yield two solutions, i.e., $(8, 0)$, $(0, 4)$

7. For which numbers a does elimination break down (a) permanently, and (b) temporarily?

$$ax + 3y = -3$$
$$4x + 6y = 6$$

Solve for x and y after fixing the second breakdown by a row exchange.

- Permanently: $a = 2$ (linearly dependent, no solution)
- Temporarily: $a = 0$;

$$\left[\begin{array}{cc|c} 4 & 6 & 6 \\ 0 & 3 & -3 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -1 \end{array}\right]$$
$$y = -1, \quad x = 3$$

Problems 17, 18, 19

17. Which number q makes this system singular and which right-hand side t gives it infinitely many solutions? Find the solution that has $z = 1$.

$$x + 4y - 2z = 1$$
$$x + 7y - 6z = 6$$
$$3y + qz = t$$

$$x + 4y - 2z = 1$$
$$x + 7y - 6z = 6$$
$$3y + -4z = 5$$

- If $q = -4$, then R_3 would have no pivot
- If $t = 5$, then there would be finite solutions, R_3 would be linearly dependent with R_2

18. It is impossible for a system of linear equations to have exactly two solutions. Explain why.
    - If (x, y, z) and (X, Y, Z) are two solutions, what is the other one?
        - There is no other *one*, there would be infinitely many.
    - If 25 planes meet at two points, where else do they meet?
        - Every other single point, they would span all of $\mathbb{R}^3$
19. Three planes can fail to have an intersection point, when no two planes are parallel. The system is singular if row 3 of $\boldsymbol{A}$ is a linearly dependent; a combination of the first two rows. Find a third equation that can't be solved if $x + y + z = 0$ and $x - 2y - z = 1$.

$$\begin{aligned} x + y + z &= 0 \\ x - 2y - z &= 1 \\ R_1 + R_2 \neq 1 \rightarrow \text{ parallel; no solution, e.g.,} \\ 2x - y &= 42 \end{aligned}$$

Problems 30, 31

30. Use elimination to solve

$$\begin{aligned} u + v + w &= 6 \\ u + 2v + 2w &= 11 \\ 2u + 3v - 4w &= 3 \end{aligned} \qquad \text{and} \qquad \begin{aligned} u + v + w &= 7 \\ u + 2v + 2w &= 10 \\ 2u + 3v - 4w &= 3 \end{aligned}$$

$$\text{rref}\left(\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 2 & 11 \\ 2 & 3 & -4 & 3 \end{array}\right]\right) \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$
$$\text{rref}\left(\left[\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 1 & 2 & 2 & 10 \\ 2 & 3 & -4 & 3 \end{array}\right]\right) \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

31. For which three numbers a will elimination fail to give three pivots?

$$\begin{aligned} ax + 2y + 3z &= b_1 \\ ax + ay + 4z &= b_2 \\ ax + ay + az &= b_3 \end{aligned}$$

- For $a = 0$, multiple failures.
- For $a = 2$, columns $0, 1$ would be equal.
- For $a = 4$, rows $1, 2$ would be equal.

1.4 Matrix Notation and Matrix Multiplication

Problems 4, 10, 17, 19

4. If an $m \times n$ matrix $\boldsymbol{A}$ multiplies an n-dimensional vector $\boldsymbol{x}$, how many separate multiplications are involved? What if A multiplies an $n \times p$ matrix $\boldsymbol{B}$?
   - $m \cdot n$ multiplications; number of rows times the length of $\boldsymbol{x}$.
   - $m \cdot n \cdot p$; same as above, except accounting for each additional column p.
10. True or false? Give a specific counterexample when false.
    - If rows 1 and 3 of $\boldsymbol{B}$ are the same, so are rows 1 and 3 of $\boldsymbol{AB}$.
      - ✘ false; matrix multiplication is done by the rows of the left matrix and the columns of the right, the rows may be the same, but if a column between the two are different, then there would be different multiplications occurring, e.g.,

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 12 & 8 & 4 \\ 30 & 20 & 10 \\ 38 & 32 & 16 \end{bmatrix}$$

    - If columns 1 and 3 of $\boldsymbol{B}$ are the same, so are columns 1 and 3 of $\boldsymbol{AB}$.
      - ✓ true;
    - If rows 1 and 3 of $\boldsymbol{A}$ are the same, so are rows 1 and 3 of $\boldsymbol{AB}$.
      - ✓ true
    - $(AB)^2 = A^2B^2$.
      - ✘ false (most of the time), e.g.,

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \boldsymbol{B} = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\boldsymbol{AB}^2 = \begin{bmatrix} 144 & 64 & 16 \\ 900 & 400 & 100 \\ 2304 & 1024 & 256 \end{bmatrix} \neq \begin{bmatrix} 74 & 26 & 10 \\ 452 & 152 & 52 \\ 1154 & 386 & 130 \end{bmatrix} = \boldsymbol{A}^2\boldsymbol{B}^2$$

17. Which of the following matrices are guaranteed to equal $(A+B)^2$?

$$\begin{aligned} A^2 + 2AB + B^2, \\ ✓\, A(A+B) + B(A+B) \\ ✓\, (A+B)(B+A), \\ ✓\, A^2 + AB + BA + B^2 \end{aligned}$$

19. A fourth way to multiply matrices is columns of $\boldsymbol{A}$ times rows of $\boldsymbol{B}$:

$\boldsymbol{AB} =$ (column 1)(row 1) $+ \cdots +$ (column n)(row n) $=$ sum of simple matrices.

Give a 2×2 example of this important rule for matrix multiplication.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \left(a \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c \begin{bmatrix} 2 \\ 4 \end{bmatrix} b \begin{bmatrix} 1 \\ 3 \end{bmatrix} + d \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right)$$

Useful, as the right matrix can be thought of as the weights that scale the elements of the columns of the left matrix.

Problems 30–31

30. Multiply these matrices:

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \\ 1 & 4 & 0 \end{bmatrix}.$$

$$\begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 0 & 2 & -3 \end{bmatrix} \quad \text{respectively}$$

- The former multiplication performs two operations (left: swaps top and bottom columns, right: swaps left and right columns), while the latter subtracts row 1 from both row 2 and row 3.

31. This 4×4 matrix needs which elimination matrices $\boldsymbol{E}_{21}$ and $\boldsymbol{E}_{32}$ and $\boldsymbol{E}_{43}$?

$$\boldsymbol{A} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

- $e_{21} = -\frac{1}{2}, \quad e_{32} = -\frac{2}{3}, \quad e_{43} = -\frac{3}{4}$
- I suspect the factions will tend towards -1 if the matrix was expanded upon in a similar fusion?

Problems 34, 35, 38, 42

34. Multiply these matrices in the orders FE and FE and E^2:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & 0 & 1 \end{bmatrix} \qquad F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{bmatrix}$$

$$FE = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ ac+b & c & 1 \end{bmatrix} \quad EF = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix} \quad E^2 = \begin{bmatrix} 1 & 0 & 0 \\ 2a & 1 & 0 \\ 2b & 0 & 1 \end{bmatrix}$$

35. $\downarrow$

   (a) Suppose all columns of B are the same. Then all columns of EB are the same, because each one is E times B_{1n}.

   (b) Suppose all rows of B are $[1\ 2\ 4]$. Show by example that all rows of EB are not $[1\ 2\ 4]$. It is true that those rows are multiples of $[1\ 2\ 4]$

   - E.g., if $e_{12} = 2$, then m_2 of EB would be $[3\ 6\ 12]$

38. If $AB = I$ and $BC = I$, use the associative law to prove $A = C$.

$$A = A(BC)$$
$$A = (AB)C$$
$$A = C$$

42. True of false?

   (a) If A^2 is defined then A is necessarily square.

   - ✓ true; inner dimensions much match, i.e., dimensions of $n_1 = m_2$. Thus, A must be square.

   (b) If AB and BA are defined, then A and B are square.

   - ✘ false; if $A = 6 \times 9$ and $B = 9 \times 6$ allows for valid pre- and post-multiplication of B.

   (c) If AB and BA are defined, then AB and BA are square.

   - ✓ true; see above example, each case will still yield square matrices. Not a proof, but I can't see another way to falsify (b).

   (d) If $AB = B$ then $A = I$

   - ✘ false; e.g.,$B = \mathtt{0}$

1.5 Triangular Factors and Row Exchanges

Problems 1, 6, 7, 14, 18

1. When is an upper triangular matrix nonsingular (a full set of pivots)?
    - Every pivot must be nonzero. If there is a zero on one of the pivots, then it indicates that one of the columns is a linear combination of one or more of the other columns.
6. Find E^2 and E^8 and E^{-1} if

$$E = \begin{bmatrix} 1 & 0 \\ 6 & 1 \end{bmatrix}$$

matrices

$$E^2 = \begin{bmatrix} 1 & 0 \\ 36 & 1 \end{bmatrix} \quad E^8 = \begin{bmatrix} 1 & 0 \\ 1679616 & 1 \end{bmatrix} \quad E^{-1} = \begin{bmatrix} 1 & 0 \\ -6 & 1 \end{bmatrix}$$

7. Find the products FGH and HGF if (with upper triangular zeros omitted)

$$F = \begin{bmatrix} 1 & & & \\ 2 & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix} G = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 2 & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix} H = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

matrices

$$FGH = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix} \quad HGF = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 4 & 2 & 1 & 0 \\ 8 & 4 & 2 & 1 \end{bmatrix}$$

14. 14. Write down all six of the 3×3 permutation matrices, including $P = I$. Identify their inverses, which are also permutation matrices. The inverses satisfy $PP^{-1} = I$ and are on the same list.

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad P^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad P^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad P^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad P^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

18. Decide whether the following systems are singular or nonsingular, and whether they have no solution, one solution, or infinitely many solutions:

$$\left[\begin{array}{ccc|c} 0 & 1 & -1 & 2 \\ 1 & -1 & 0 & 2 \\ 1 & 0 & -1 & 2 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{array}\right]$$

matrices

- Performing rref on above martices yields:
  - Singular — no solution
  - Singular — ∞ solutions.
  - Nonsingular — one solution [0.5 0.5 0.5]

Problems 26, 28

26. Which number c leads to zero in the second pivot position? A row exchange is needed and $\boldsymbol{A} = \boldsymbol{LU}$ is not possible. Which c produces zero in the third pivot position? Then a row exchange can't help and elimination fails.

$$\boldsymbol{A} = \begin{bmatrix} 1 & c & 0 \\ 2 & 4 & 1 \\ 3 & 5 & 1 \end{bmatrix}$$

- If $c = 2$ then row 2 would have a 0 in the pivot, yielding:

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 3 & 5 & 1 \end{bmatrix}$$

- If $c = 1$, then you could take the matrix down to the following form,

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

  which would yield a singular matrix with infinite solutions.

28. $\boldsymbol{A}$ and $\boldsymbol{B}$ are symmetric across the diagonal (because 4 = 4). Find their triple factorizations $\boldsymbol{LU}$ and say how $\boldsymbol{U}$ is related to $\boldsymbol{L}$ for these symmetric matrices:

$$\boldsymbol{A} = \begin{bmatrix} 2 & 4 \\ 4 & 11 \end{bmatrix} \quad \textit{and} \quad \boldsymbol{B} = \begin{bmatrix} 1 & 4 & 0 \\ 4 & 12 & 4 \\ 0 & 4 & 0 \end{bmatrix}$$

$$\boldsymbol{U}_A = \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix}, \qquad \boldsymbol{U}_B = \begin{bmatrix} 1 & 4 & 0 \\ 0 & -4 & 4 \\ 0 & 0 & -4 \end{bmatrix}$$

$$\boldsymbol{V}_A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \qquad \boldsymbol{V}_B = \begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{L}_A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \qquad \boldsymbol{L}_B = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

- $\boldsymbol{A}, \boldsymbol{B} = \boldsymbol{LDV} \implies \boldsymbol{L} = \boldsymbol{V}^T$; the diagonal of the upper matrix, if reduced to 1's in the pivot positions, yields the transpose of the lower triangular matrix.

Problems 33, 43

33. Solve $\boldsymbol{Lc} = \boldsymbol{b}$ to find $\boldsymbol{c}$. Then solve $\boldsymbol{Ux} = \boldsymbol{c}$ to find $\boldsymbol{x}$. What was $\boldsymbol{A}$?

$$\boldsymbol{L} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad \textit{and} \quad \boldsymbol{U} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \textit{and} \quad \boldsymbol{b} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

matrices

$$\boldsymbol{c} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} \quad \boldsymbol{x} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \implies \boldsymbol{A} = \boldsymbol{LU} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

43. (Try this question) Which permutation makes $\boldsymbol{PA}$ upper triangular? Which permutations make $\boldsymbol{P}_1\boldsymbol{A}\boldsymbol{P}_2$ lower triangular? Multiplying $\boldsymbol{A}$ on the right by $\boldsymbol{P}_2$ exchanges the columns of $\boldsymbol{A}$.

$$\boldsymbol{A} = \begin{bmatrix} 0 & 0 & 6 \\ 1 & 2 & 3 \\ 0 & 4 & 5 \end{bmatrix}$$

$$\boldsymbol{U} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 6 \\ 1 & 2 & 3 \\ 0 & 4 & 5 \end{bmatrix}$$

$$\boldsymbol{L} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 6 \\ 1 & 2 & 3 \\ 0 & 4 & 5 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

1.6 Inverses and Transposes

Problems 3, 12, 18

3. From $AB = C$ find a formula for A^{-1}. Also find A^{-1} from $PA = LU$.

$$\begin{aligned} AB &= C \\ A &= CB^{-1} \\ A^{-1} &= BC^{-1} \\ PA &= LU \\ A &= P^{-1}LU \\ A^{-1} &= U^{-1}L^{-1}P \end{aligned}$$

12. If A is invertible, which properties of A remain true for A^{-1}?
    (a) A is triangular. ✓ true
    (b) A is symmetric. ✓ true
    (c) A is tridiagonal. ✘ false
    (d) All entries are whole ✘ false
    (e) All entire are fractions (including numbers like $\frac{3}{1}$). ✓ true;

18. Under what conditions on their entries are A and B invertible?

$$A = \begin{bmatrix} a & b & c \\ d & e & 0 \\ f & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & e \end{bmatrix}$$

matrices

$$\operatorname{rref}\left(\left[\begin{array}{ccc|ccc} a & b & c & 1 & 0 & 0 \\ d & e & 0 & 0 & 1 & 0 \\ f & 0 & 0 & 0 & 0 & 1 \end{array}\right]\right) \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & f^{-1} \\ 0 & 1 & 0 & 0 & e^{-1} & -\frac{d}{ef} \\ 0 & 0 & 1 & c^{-1} & -\frac{b}{ce} & \frac{-ae+bd}{cef} \end{array}\right]$$

$$\operatorname{rref}\left(\left[\begin{array}{ccc|ccc} a & b & 0 & 1 & 0 & 0 \\ c & d & 0 & 0 & 1 & 0 \\ 0 & 0 & e & 0 & 0 & 1 \end{array}\right]\right) \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{d}{ad-bc} & -\frac{b}{ad-bc} & 0 \\ 0 & 1 & 0 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} & 0 \\ 0 & 0 & 1 & 0 & 0 & e^{-1} \end{array}\right]$$

- $A \to A^{-1} \iff c, e, f \neq 0$
- $B \to B^{-1} \iff e \neq 0 \wedge ad \neq bc$

Problems 21, 28, 41, 56, 58

21. (Remarkable) If A and B are square matrices, show that $I - BA$ is invertible if $I - AB$ is invertible. Sart from $B(I - AB) = (I - BA)B$

$$\begin{aligned} B(I - AB) &= (I - BA)B \\ (I - AB) &= B^{-1}(I - BA)B \\ (I - AB)^{-1} &= B(I - BA)^{-1}B^{-1} \end{aligned}$$

   - Thus, as long as $I - BA$ is invertible, then the inverse is defined.

28. If the product $M = ABC$ of three square matrices is invertible, then A, B, C are invertible. Find a formula for B^{-1} that involves M^{-1} and A and C.

$$\begin{aligned} M &= ABC \\ M^{-1} &= C^{-1}B^{-1}A^{-1} \\ CM^{-1} &= B^{-1}A^{-1} \\ CM^{-1}A &= B^{-1} \end{aligned}$$

41. For which three numbers c is this matrix not invertible, and why not?

$$A = \begin{bmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{bmatrix}$$

   - If $c = 0$, then there would be multile unavoidable zeros in the pivots.
   - If $c = 2$, then row 2 would just be duplicate of row 1.
   - If $c = 7$, then column 2 and 3 would be equal.

57. If $A = A^T$ needs a row exchange, then it also needs a column exchange to stay symmetric. In matrix language, PA loses the symmetry of A but PAP^T recovers the symmetry.

58. ↓
   (a) How many entries of A can be chosen independently, if $A = A^T$ is 5×5?
      - 25 total choices — 10 under the diagonal = 15
   (b) How do L and D (5×5) give the same number of choice in LDL^T?
      - Oh, I kind of used this to find (a). Well, the diagonal doesn't matter ($\neq 0$), since a transponse can simply be thought of a rotation around the diagonal elements. But, every element must match U, thus only the 10 choices below matter, yieldng 15 total choices.

2 Vector Spaces

2.1 Vector Spaces and Subspaces

Problems 25, 26, 30, 31

25. If we add an extra column $\boldsymbol{b}$ to a matrix $\boldsymbol{A}$, then the column space gets larger unless <?>.
    - Give an example in which the column space gets larger and an example in which it doesn't.
    - Why is Ax = b solvable exactly when the column space doesn't get larger by including b?
26. The columns of $\boldsymbol{AB}$ are combinations of the columns of $\boldsymbol{A}$. This means: the column space of $\boldsymbol{AB}$ contained in (possibly equal to) the column space of $\boldsymbol{A}$. Give an example where the column spaces of $\boldsymbol{A}$ and $\boldsymbol{AB}$ are not equal.
30. If the 9×12 system $\boldsymbol{Ax} = \boldsymbol{b}$ is solvable for every $\boldsymbol{b}$, then $C(A) =$.
31. Why isn't $\mathbb{R}^2$ a subspace of $\mathbb{R}^3$?

2.2 Solving Ax = 0 and Ax = b

Problems 12, 24, 25, 70

12. Which of these rules give a correct definition of the rank of $\boldsymbol{A}$?
    (a) The number of nonzero rows in $\boldsymbol{R}$.
    (b) The number of columns minus the total number of rows.
    (c) The number of columns minus the number of free columns.
    (d) The number of 1s in $\boldsymbol{R}$.
24. Every column of $\boldsymbol{AB}$ is a combination of the columns of $\boldsymbol{A}$. Then the dimensions of the column spaces give $\operatorname{rank}(\boldsymbol{AB}) \leq \operatorname{rank}(\boldsymbol{A})$.

    Problem: Prove also that $\operatorname{rank}(\boldsymbol{AB}) \leq \operatorname{rank}(\boldsymbol{B})$.
25. (Important) Suppose $\boldsymbol{A}$ and $\boldsymbol{B}$ are $n \times n$ matrices, and $\boldsymbol{AB} = \boldsymbol{I}$. Prove from $\operatorname{rank}(AB) \leq \operatorname{rank}(A)$ that the rank of $\boldsymbol{A}$ is n. So A is invertible and $\boldsymbol{B}$ must be its two-sided inverse. Therefore $\boldsymbol{BA} = \boldsymbol{I}$
70. Explain why $\boldsymbol{A}$ and $-\boldsymbol{A}$ always have the same reduced echelon form $\boldsymbol{R}$.

2.3 Linear Independence, Basis, and Dimension

Problems 9, 13, 28, 36

9. Suppose $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4$ are vectors in $\mathbb{R}^3$.
   (a) There four vectors are dependent because <?>
   (b) The two vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ will be dependent if <?>
   (c) The vectors $\boldsymbol{v}_1$ and $(0, 0, 0)$ are dependent because <?>
13. Find the dimensions of:
   (a) the column space of $\boldsymbol{A}$:
   (b) the column space of $\boldsymbol{U}$:
   (c) the row space of $\boldsymbol{A}$:
   (d) the row space of $\boldsymbol{U}$.

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 3 & 1 & -1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{U} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

   - Which two of the spaces are the same?
28. True or false (give a good reason)?
   (a) If the columns of a matrix are dependent, so are the rows.
   (b) The column space of a 2×2 matrix is the same as its row space.
   (c) The column space of a 2×2 matrix has the same dimension as its row space.
   (d) The columns of a matrix are a basis for the column space.
36. If $\boldsymbol{A}$ is a 64×17 matrix of rank 11, how many independent vectors satisfy $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$?

2.4 The Four Fundamental Subspaces

Problems 6, 14, 15, 27

6. Suppose $\boldsymbol{A}$ is an $m \times n$ matrix of rank r. Under what conditions on those numbers does

   (a) A have a two-sided inverse: $\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}$?

   (b) $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ have infinitely many solutions for every $\boldsymbol{b}$?

14. 14. Find a left-inverse and/or a right-inverse (when they exist) for

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{M} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{T} = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$$

15. 15. If the columns of A are linearly independent, then the rank is <?>, the nullspace is <?>, the row space is <?>, and there exists a <?>-inverse.

27. (Important) $\boldsymbol{A}$ is an $m \times n$ matrix of rank r. Suppose there are right-hand sides $\boldsymbol{b}$ for which $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ has no solution.

   (a) What inequalities must be true between m, n, and r?

   (b) How do you know that $\boldsymbol{A}^T\boldsymbol{y} = \boldsymbol{0}$ has a nonzero solution?

2.5 Graphs and Networks

2.6 Linear Transformations

3 Orthogonality

3.1 Orthogonal Vectors and Subspaces

3.2 Cosines and Projections onto Lines

Projection Proof (class problem)

- If I recall the problem correctly, we were requested to prove what $\text{proj}_{\boldsymbol{v}}\, \boldsymbol{w}$ is equal to.
- In my notes I have that a orthogonal projection occurs when the dot product between $\boldsymbol{v}$ and distance $\boldsymbol{w}$ from $\boldsymbol{v}$ is equal to zero. This follows from the definition of the inner product, i.e.,

$$\lambda = \boldsymbol{v}^T \boldsymbol{w} = ||v||||w|| \cos\theta$$

- If $\theta = 90^{\circ}$, then the vectors are perpendicular, i.e., orthogonal. What we are missing is the distance from $\boldsymbol{w}$ to $\boldsymbol{v}$; the distance that yields an inner product of zero with a normalized $\boldsymbol{v}$ *is the projection*.
- This means we need a scaled version of $\boldsymbol{v}$, let's call it $\boldsymbol{v}\beta$, at which such inner product is equal to zero. At this point, the difference between $\boldsymbol{w}$ and $\boldsymbol{v}\beta$ is exactly what we need in order to solve for a β that maintains an inner product of zero with the original vector $\boldsymbol{v}$, i.e.,

$$\begin{aligned} \boldsymbol{v}^T(\boldsymbol{w} - \boldsymbol{v}\beta) &= 0 \\ \boldsymbol{v}^T\boldsymbol{w} - \boldsymbol{v}^T\boldsymbol{v}\beta &= 0 \\ \boldsymbol{v}^T\boldsymbol{v}\beta &= \boldsymbol{v}^T\boldsymbol{w} \\ \beta &= \frac{\boldsymbol{v}^T\boldsymbol{w}}{\boldsymbol{v}^T\boldsymbol{v}} \end{aligned}$$

$$\implies \text{proj}_{\boldsymbol{v}}\, \boldsymbol{w} = \boldsymbol{v}\beta = \boldsymbol{v}\frac{\boldsymbol{v}^T\boldsymbol{w}}{\boldsymbol{v}^T\boldsymbol{v}}$$

- I've internalized this as a mapping of $\boldsymbol{w}$ onto $\boldsymbol{v}$ over a magnitude (the norm) of $\boldsymbol{v}$.
  - The mapping is important because it tells us the shortest distance from $\boldsymbol{w}$ onto $\boldsymbol{v}$, i.e., when they are orthogonal.
  - The magnitude is important, because it is the basis at which $\boldsymbol{w}$ is parallel to $\boldsymbol{v}$, which when added mapping distance, yields $\boldsymbol{w}$.

3.3 Projections and Least Squares

3.4 Orthogonal Bases and Gram-Schmidt

3.5 The Fast Fourier Transform