

# Calculus III Exercises



## Introduction to Differential Equations

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# Introduction to Differential Equations



## 9.1 Edfinity: Solving Differential Equations

### 9.1.5

- Solve  $y' = x^5 y^2$ , using separation of variables, given the initial condition  $y(0) = 9$

$$\begin{aligned}\frac{\frac{dy}{dx}}{y^2} &= x^5 \\ \int \frac{\frac{dy}{dx}}{y(x)^2} &= \int x^5 dx \\ -\frac{1}{y(x)} &= \frac{x^6}{6} + c_1 \\ y(x) &= -\frac{6}{x^6 + c_1} \\ 9 &= -\frac{6}{c}, \quad c = -\frac{6}{9} \\ \boxed{y(x) &= -\frac{18}{2x^6 - 2}}\end{aligned}$$

### 9.1.6

- Solve the initial value problem  $\frac{dy}{dx} + 3y = 0$ ,  $y(\ln 4) = 3$ .

$$\begin{aligned}\frac{dy}{dx} &= -3y \\ \int \frac{dy}{dx} y^{-1} dx &= \int -3 dx \\ \ln |y| &= -3x + \lambda \\ y &= e^{-3x} \lambda \\ 3 &= e^{3(\ln 4)} \lambda \implies \lambda = 192 \\ y &= 192e^{-3x}\end{aligned}$$

### 9.1.7

- Solve  $(t^2 + 36)\frac{dx}{dt} = (x^2 + 9)$ , using separation of variables, given the initial condition  $x(0) = 3$ .

$$\begin{aligned}\frac{dx}{dt} &= (t^2 + 36)^{-1} \\ \frac{dx}{dt} (x^2 + 9)^{-1} &= (t^2 + 36)^{-1} \\ \int \frac{dx}{dt} (x^2 + 9)^{-1} &= \int (t^2 + 36)^{-1} dt \\ \frac{1}{9} \int \left( \frac{x^2}{9} + 1 \right)^{-1} dx &= \frac{1}{36} \int \left( \frac{t^2}{36} + 1 \right)^{-1} dt \\ \frac{1}{3} \tan^{-1} \left( \frac{x}{3} \right) &= \frac{1}{6} \tan^{-1} \left( \frac{t}{6} \right) + \lambda \\ \tan^{-1} \left( \frac{x}{3} \right) &= \frac{1}{2} \tan^{-1} \left( \frac{t}{6} \right) + \lambda \\ \tan^{-1} \left( \frac{3}{3} \right) &= \frac{1}{2} \tan^{-1} \left( \frac{0}{6} \right) + \lambda \\ \frac{\pi}{4} &= \lambda \\ x &= 3 \tan \left( \frac{1}{2} \tan^{-1} \left( \frac{t}{6} \right) + \frac{\pi}{4} \right)\end{aligned}$$

### 9.1.8

- Solve the initial value problem  $\frac{dy}{dx} = (x - 7)(y - 8)$ ,  $y(0) = 4$

$$\begin{aligned}\frac{dy}{dx} (y - 8)^{-1} &= (x - 7) \\ \int dy (y - 8)^{-1} &= \int (x - 7) dx \\ \ln y - 8 &= x^{-2} - 7x + \lambda \\ y &= e^{\frac{x^2}{2} - 7x} \lambda + 8 \\ -4 &= \lambda \\ y &= -4e^{\frac{x^2}{2} - 7x} + 8\end{aligned}$$

### 9.1.9

- Solve the initial value problem  $t^2 \frac{dy}{dt} - t = 1 + y + ty$ ,  $y(1) = 7$

$$\int (y + 1) dy = \int \frac{1 + t}{t^2} dt$$

$$\ln |1 + y| = -t^{-1} + \ln t + \lambda$$

$$y = \lambda e^{-t^{-1} + \ln t} - 1 = \lambda \frac{t}{e^{t^{-1}}} - 1$$

$$7 = \lambda e^{-1} - 1$$

$$\lambda = 8e$$

$$y = 8te^{1-t^{-1}} - 1$$

### 9.1.10

- Solve the initial value problem  $y' = 2y^2 \sin x$ ,  $y(0) = 6$

$$\frac{dy}{dx}(2y^2)^{-1} = \sin x$$

$$\int \frac{dy}{dx} y^{-2} dx = \int 2 \sin x dx$$

$$-y^{-1} = -2 \cos x + \lambda$$

$$y = (2 \cos x + \lambda)^{-1}$$

$$6 = (2 \cos 0 + \lambda)^{-1}$$

$$-\frac{11}{6} = \lambda$$

$$y = \left( 2 \cos x - \frac{11}{6} \right)^{-1}$$

## 9.2 Edfinity: Models Involving $y' = k(y - b)$

### 9.2.2

- Find the general solution of  $y' = 5(y - 16)$ .

$$y(t) = b + Ce^{kt}$$

$$y' = k(y - b)$$

$$y(t) = 16 + Ce^{5t}$$

$$30 = 16 + C$$

$$C = 14$$

$$y(t) = 16 + 14e^{5t}$$

$$1 = 16 + C$$

$$C = -15$$

$$y(t) = 16 + -15e^{5t}$$

### 9.2.3

- A 62 kg skydiver jumps out of an airplane. What is her terminal velocity in miles per hour, assuming that  $k = 10 \frac{kg}{s}$  for free fall?

$$-\frac{gm}{k} = -\frac{9.8(62)}{10} = -60.76 \frac{m}{s} = 199.343 \frac{ft}{s} = -134.916 \text{ mph}$$

### 9.2.4

- A continuous annuity with withdrawal rate  $N = \$600$  y and interest rate  $r = 5\%$  is funded by an initial deposit  $P_0$
- When will the annuity run out of funds if  $P_0 = \$10,000$ ?

$$P(t) = Nr^{-1} + Ce^{rt} = 600(0.05)^{-1} + Ce^{0.05t} = 12,000 + Ce^{0.05t}$$

$$10,000 = 12,000 + C$$

$$C = -2,000$$

$$t = 0.05^{-1} \ln \frac{12,000}{2,000} = 35.83 \approx 38 \text{ years}$$

- Which initial deposit  $P_0$  yields a constant balance?

$$P(t) = 12,000 + Ce^{0.05t}, \quad C = 0$$

$$P_0 = 12,000$$

### 9.2.5

- A cup of coffee, cooling off in a room temperature  $20^\circ\text{C}$ , has cooling constant  $k = 0.085 \text{ min}^{-1}$ .
- How fast is the coffee cooling when its temperature is  $T = 70^\circ\text{C}$ ?

$$k(T - T_0)$$
$$0.085(70 - 20) = 4.25^\circ\text{C min}^{-1}$$

- Use the Linear Approximation to estimate the change in temperature over the next 4 seconds when  $T = 70^\circ\text{C}$

$$4.25^\circ\text{C min}^{-1}(4\text{s})60\text{s min}^{-1} = 0.283^\circ\text{C}$$

- The coffee is served at a temperature of  $86^\circ\text{C}$ . How long should you wait before drinking it if the optimal temperature is  $65^\circ\text{C}$ ?

$$65 = 20 + 66e^{-0.085t}$$
$$t = -(0.085)^{-1} \ln \left( \frac{45}{66} \right)$$
$$t \approx 4.5 \text{ min}$$

## 9.3 Edfinity: Graphical and Numerical Methods

### 4. 9.3.4

- User Euler's method with step size 0.2 to estimate  $y(1)$ , where  $y(x)$  is the solution of the initial-value problem.

$$y' = 4x + y^2, \quad y(0) = 1$$

$$y_k = y_{k-1} + hF(t_{k-1}, y_{k-1}), \quad F = 4x + y^2, \quad h = 0.2$$

$$y(0.2) \approx 1 + 0.2(4(0) + 1^2) = 1.2$$

$$y(0.4) \approx 1.2 + 0.2(4(0.2) + 1.2^2) \approx 1.648$$

$$y(0.6) \approx 1.648 + 0.2(4(0.4) + 1.648^2) \approx 2.511$$

$$y(0.8) \approx 2.511 + 0.2(4(0.6) + 2.511^2) \approx 4.092$$

$$y(1) \approx 4.092 + 0.2(4(0.8) + 4.092^2) \approx 8.578$$

### 9.3.5

- User Euler's method with  $\Delta x = 0.1$  to estimate  $y(1.4)$ .

$$y' = -x - y, \quad y(1) = 1$$

$$y(1) \approx 1 + 0.1(-1 - 1) = 0.8$$

$$y(1.1) \approx 0.8 + 0.1(-1.1 - 0.8) = 0.61$$

$$y(1.2) \approx 0.61 + 0.1(-1.2 - 0.61) = 0.429$$

$$y(1.3) \approx 0.429 + 0.1(-1.3 - 0.429) = 0.2561$$

$$y(1.4) \approx 0.2561$$

## 9.4 Edfinity: The Logistic Equation

- The logistic equation and general non-equilibrium solution ( $k > 0$  and  $A > 0$ )

$$\frac{dy}{dt} = ky \left(1 - \frac{y}{A}\right), \quad y = \frac{A}{1 - e^{-kt}/B} \Leftrightarrow \frac{y}{y - A} = Be^{kt}$$

- $y = 0 \rightarrow$  unstable equilibrium
- $y = A \rightarrow$  stable equilibrium
- If the initial value  $y_0 = y(0)$  satisfies  $y_0 > 0$ , then  $y(t)$  approach the stable equilibrium  $y = A$ , i.e.,  $\lim_{t \rightarrow \infty} y(t) = A$

### 9.4.1

- A population  $P$  obeys the logistic model. It satisfies the equation

$$\frac{dP}{dt} = \frac{7}{1100}P(11 - P) \text{ for } P > 0$$

$$\frac{dP}{dt} = \frac{7}{1100}P\left(1 - \frac{P}{11}\right) \text{ for } P > 0$$

- c. Assume that  $P(0) = 4$ . Find  $P(87)$

$$\frac{y}{y - A} = Be^{kt}$$

$$\frac{4}{4 - 11} = Be^{\frac{7}{1100}0}$$

$$-0.571 \approx B$$

$$\Rightarrow P(87) = \frac{11}{1 - e^{-0.06 \cdot 87} / -0.571} \approx -10.9$$

### 9.4.2

- Assuming  $P \geq 0$ , suppose that a population develops according to the logistic equation

$$\frac{dP}{dt} = 0.01P - 5 \times 10^{-5}P^2$$

where  $t$  is measured in weeks.

$$\begin{aligned} \frac{dP}{dt} &= 0.01P - 5 \times 10^{-5}P^2 \\ &= 0.01P \left(1 - \frac{5 \times 10^{-5}}{0.01}P\right) \\ &= 0.01P \left(1 - \frac{P}{200}\right) \end{aligned}$$

Carrying capacity =  $A = 200$

$k = 0.01$

increasing =  $(0, 200)$

decreasing =  $(200, \infty)$



### 9.4.3

- A population of squirrels lives in a forest with a carrying capacity of 1600. Assume logistic growth with growth constant  $k = 1 \text{ yr}^{-1}$
- Find a formula for the squirrel population  $P(t)$ , assuming an initial population of 400 squirrels.

$$\begin{aligned}\frac{dP}{dt} &= 1P \left( 1 - \frac{P}{1600} \right) \\ B &= \frac{400}{400 - 1600} = -\frac{1}{3} \\ P(t) &= 1600 / 1 - \frac{e^{-t}}{-\frac{1}{3}} = \frac{1600}{1 + 3e^{-t}} \\ 800 &= \frac{1600}{1 + 3e^{-t}} \\ 1 + 3e^{-t} &= 2 \\ e^{-t} &= \frac{1}{3} \\ t &= -\ln \frac{1}{3} = 1.098 \text{ yr}\end{aligned}$$

### 9.4.4

- Sunset Lake is stocked with 2700 rainbow trout and after 1 year the population has grown to 7050. Assuming logistic growth with a carrying capacity of 27,000, find the growth constant  $k$ , and determine when the population will increase to 13600.

$$\frac{dP}{dt} = kP \left( 1 - \frac{P}{27000} \right) \quad P(0) = 2700, \quad P(t) = 13600$$

$$\begin{aligned}B &= \frac{2700}{2700 - 27000} = -\frac{1}{9} \\ 7050 &= \frac{27000}{1 + 9e^{-k \cdot 1}} \\ 1 + 9e^{-k} &= \frac{27000}{7050} \\ k &= -\ln \frac{\frac{27000}{7050} - 1}{9} = 1.157 \\ 13600 &= \frac{27000}{1 + 9e^{1.157t}} \\ e^{1.157t} &= \frac{\frac{27000}{13600} - 1}{9} \\ t &= \ln \left( \frac{\frac{27000}{13600} - 1}{9} \right) (1.157)^{-1} \\ t &\approx 1.911 \text{ yr}\end{aligned}$$

### 9.4.5

- Suppose that a population develops according to the logistic equation

$$\frac{dP}{dt} = 0.05P - 6.667 \times 10^{-5}P^2$$

where  $t$  is measure in weeks.

$$\frac{dP}{dt} = 0.05P \left( 1 - \frac{6.667 \times 10^{-5}}{0.05}P \right) = 0.05P \left( 1 - \frac{P}{750} \right)$$

## 9.5 Edfinity: First-Order Linear Equations

- Hammers:

$$y' + P(x)y = Q(x)$$

$$\alpha(x) = e^{\int P(x)dx}$$

$$y = \alpha(x)^{-1} \left( \int \alpha(x)Q(x)dx + C \right)$$

### 9.5.1

- Solve  $y' + 3x^{-1}y = x^2$ ,  $y(1) = -9$

- Identify the integrating factor,  $\alpha(x)$

$$\alpha(x) = e^{\int 3x^{-1}dx}$$

$$\alpha(x) = e^{3 \ln x}$$

$$\alpha(x) = x^3$$

- Find the general solution,  $y(x)$

$$y = x^{-3} \left( \int x^3 x^2 dx + C \right)$$

$$y = x^{-3} \left( \frac{x^6}{6} + C \right)$$

$$y = \frac{x^3}{6} + Cx^{-3}$$

- Solve the initial value problem,  $y(1) = -9$

$$-9 = \frac{1^3}{6} + C^{-3}$$

$$-9.5 = C^{-3}$$

$$C = - \left( \frac{55}{6} \right)^{3^{-1}}$$

$$\Rightarrow y = \frac{x^3}{6} - \frac{55^{\frac{1}{3}}}{6} x^{-3}$$

$$y = \frac{x^3}{6} - 9.167x^{-3}$$

### 9.5.2

- Solve  $4xy' - 8y = x^{-1}$ ,  $y(1) = 6$

$$\implies y' - 2x^{-1}y = \frac{1}{4}x^{-2}$$

- Identify the integrating factor,  $\alpha(x)$

$$\alpha(x) = e^{\int -2x^{-1}dx}$$

$$\alpha(x) = e^{-2\ln x}$$

$$\alpha(x) = x^{-2}$$

- Find the general solution,  $y(x)$

$$y = x^2 \left( \int x^{-2} \frac{1}{4} x^{-2} dx + C \right)$$

$$y = x^2 \left( -\frac{1}{12} x^{-3} + C \right)$$

$$y = -\frac{1}{12} x^{-1} + Cx^2$$

- Solve the initial value problem,  $y(1) = 6$

$$6 = -\frac{1}{12} 1^{-1} + C$$

$$C = 6 + \frac{1}{12} = 6.083$$

$$\implies y = \frac{1}{12} x^{-1} + 6.083x^2$$

### 9.5.3

- Solve  $xy' = 2y - 9x$ ,  $y(1) = -2$

$$\implies y' - 2x^{-1}y = -9$$

- Identify the integrating factor,  $\alpha(x)$

$$\alpha(x) = e^{\int -2x^{-1}dx}$$

$$\alpha(x) = e^{-2\ln x}$$

$$\alpha(x) = x^{-2}$$

- Find the general solution,  $y(x)$

$$y = x^2 \left( \int -9x^{-2} dx + C \right)$$

$$y = x^2(9x^{-1} + C)$$

$$y = 9x + Cx^2$$

- Solve the initial value problem,  $y(1) = -2$

$$-2 = 9 + C$$

$$C = -11$$

$$\implies y = 9x - 11x^2$$

### 9.5.4

- Find the general solution of the first-order linear differential equation

$$y' - (\ln x)y = 2x^x$$

$$\alpha(x) = e^{\int -\ln x dx}$$

$$\alpha(x) = e^{x-x \ln x}$$

$$\alpha(x) = e^x x^{-x}$$

$$y = e^{-x} x^x \left( \int e^x x^{-x} 2x^x dx + C \right)$$

$$y = e^{-x} x^x (2e^x + C)$$

$$y = 2x^x + C e^{-x} x^x$$

### 9.5.5

- Solve the initial value problem  $y' + 4y = e^{8x}$ ,  $y(0) = -7$

$$\alpha(x) = e^{4x}$$

$$y = e^{-4x} \left( \int e^{4x} e^{8x} dx + C \right)$$

$$y = e^{-4x} \left( \frac{e^{12x}}{12} + C \right)$$

$$y = \frac{e^{8x}}{12} + C e^{-4x}$$

$$-7 = \frac{1}{12} + C$$

$$C = -\frac{85}{12}$$

$$y = \frac{e^{8x}}{12} + -\frac{85}{12} e^{-4x}$$