1. A function $f : A \to B$ is *linear* if, $\forall a, b \in \mathbb{R}$, $f(ax+b) = af(x) + b$.
   Apply the definition of linear to:

   (a) $f(x) = 2x$

   $$\implies \forall a, b \in \mathbb{R}, \quad 2(ax+b) = a2x + b$$

   (b) $f(x) = x^2$

   $$\implies \forall a, b \in \mathbb{R}, \quad (ax+b)^2 = ax^2 + b$$

   (c) $f(x) = \sum_{i=0}^{\infty} a_i x^i$

   $$\implies \forall a, b \in \mathbb{R}, \quad \sum_{i=0}^{\infty} a_i(ax+b)^i = a\left(\sum_{i=0}^{\infty} a_i x^i\right) + b$$

2. A function $f : \mathbb{R} \to \mathbb{R}$ is *continuous* if, $\forall \epsilon > 0, \exists \delta > 0 : f(x+\delta) - f(x) < \epsilon$.
   Apply the definition of continuous to:

   (a) $f(x) = |2x - 1|$

   $$\forall \epsilon > 0, \exists \delta > 0 : |2(x+\delta) - 1| - |2x - 1| < \epsilon$$

   (b) $f(x) = x^{-1}$

   $$\forall \epsilon > 0, \exists \delta > 0 : (x+\delta)^{-1} - x^{-1} < \epsilon$$

   (c) $f(x) = \sum_{n=0}^{\infty} \cos(b^n \pi x)$

   $$\forall \epsilon > 0, \exists \delta > 0 : \sum_{n=0}^{\infty} \cos(b^n \pi(x+\delta)) - \sum_{n=0}^{\infty} \cos(b^n \pi x) < \epsilon$$

3. A relation $\sim: A \times A$ is a *chain* if, $\forall x, y \in A, x \sim y \vee y \sim x$
   Apply the definition of chain to:

   (a) $x \sim y, : x, y \in \mathbb{R} \wedge |x| \leq |y|$

   $$\forall x, y \in \mathbb{R}, \quad |x| \leq |y| \vee |y| \leq |x|$$

   (b) $S \sim T \impliedby S \in P(T)$, where S, T are sets and $P()$ denotes power set.

   $$\begin{gathered} \forall S \in P(T) \implies S \sim T \\ \vee \\ \forall T \in P(S) \implies T \sim S \end{gathered}$$

   (c) $\sigma_1 \sim \sigma_2 \impliedby \sigma_1, \sigma_2, : A \rightarrow A$ are functions and $\exists \tau : \sigma_1 = \tau \circ \sigma_2$

   $$\forall \sigma_1, \sigma_2 \in A, \quad \exists \tau : \sigma_1 = \tau \circ \sigma_2 \vee \exists \tau : \sigma_2 = \tau \circ \sigma_1$$

4. (a) Prove that there is no smallest positive rational number greater than 0.

   (b) Prove that for every positive real number greater than 0 there is a smaller positive rational number.

   (c) Prove that there is no smallest positive real number greater than 0.

5. Fermat's Last theorem is a famous theorem in Math that was unproven for 200 years. The theorem says $\forall n > 2,\ a, b, c \in \mathbb{N} \implies a^n + b^n \neq c^n$. Another way to state this is $a^n + b^n = c^n$ has no integer solutions for n larger than 2. Use this theorem to prove that $\sqrt[n]{2}$ is irrational for n larger than 2.