Calculus III Exercises

Introduction to Differential Equations

Introduction to Differential Equations

9.1 Edfinity: Solving Differential Equations

9.1.5

- Solve $y' = x^5 y^2$, using separation of variables, given the initial condition $y\left(0\right) = 9$

$$\frac{\frac{dy}{dx}}{y^2} = x^5$$
$$\int \frac{\frac{dy}{dx}}{y(x)^2} = \int x^5 dx$$
$$-\frac{1}{y(x)} = \frac{x^6}{6} + c_1$$
$$y(x) = -\frac{6}{x^6 + c_1}$$
$$9 = -\frac{6}{c}, \quad c = -\frac{6}{9}$$
$$\boxed{y(x) = -\frac{18}{2x^6 - 2}}$$

9.1.6

- Solve the initial value problem $\frac{dy}{dx} + 3y = 0, \quad y(\ln 4) = 3.$

$$\frac{dy}{dx} = -3y$$
$$\int \frac{dy}{dx} y^{-1} dx = \int -3 dx$$
$$\ln|y| = -3x + \lambda$$
$$y = e^{-3x}\lambda$$
$$3 = e^{3(\ln 4)}\lambda \implies \lambda = 192$$
$$y = 192e^{-3x}$$

9.1.7

- Solve $(t^2+36)\frac{dx}{dt}=(x^2+9)$, using separation of variables, given the initial condition $x(0)=3$.

$$\begin{aligned}
\frac{dx}{dt} &= \left(t^2+36\right)^{-1} \\
\frac{dx}{dt}\left(x^2+9\right)^{-1} &= (t^2+36)^{-1} \\
\int \frac{dx}{dt}\left(x^2+9\right)^{-1} &= \int (t^2+36)^{-1} dt \\
\frac{1}{9}\int\left(\frac{x^2}{9}+1\right)^{-1} dx &= \frac{1}{36}\int\left(\frac{t^2}{36}+1\right)^{-1} dt \\
\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) &= \frac{1}{6}\tan^{-1}\left(\frac{t}{6}\right)+\lambda \\
\tan^{-1}\left(\frac{x}{3}\right) &= \frac{1}{2}\tan^{-1}\left(\frac{t}{6}\right)+\lambda \\
\tan^{-1}\left(\frac{3}{3}\right) &= \frac{1}{2}\tan^{-1}\left(\frac{0}{6}\right)+\lambda \\
\frac{\pi}{4} &= \lambda \\
x &= 3\tan\left(\frac{1}{2}\tan^{-1}\left(\frac{t}{6}\right)+\frac{\pi}{4}\right)
\end{aligned}$$

9.1.8

- Solve the initial value problem $\frac{dy}{dx}=(x-7)(y-8), \quad y(0)=4$

$$\begin{aligned}
\frac{dy}{dx}(y-8)^{-1} &= (x-7) \\
\int dy\,(y-8)^{-1} &= \int (x-7)\,dx \\
\ln y-8 &= x^{-2}-7x+\lambda \\
y &= e^{\frac{x^2}{2}-7x}\lambda+8 \\
-4 &= \lambda \\
y &= -4e^{\frac{x^2}{2}-7x}+8
\end{aligned}$$

9.1.9

- Solve the initial value problem $t^2\frac{dy}{dt} - t = 1 + y + ty, \quad y(1) = 7$

$$\begin{aligned}\int (y+1)\,dy &= \int \frac{1+t}{t^2} dt \\ \ln|1+y| &= -t^{-1} + \ln t + \lambda \\ y &= \lambda e^{-t^{-1}+\ln t} - 1 = \lambda \frac{t}{e^{t^{-1}}} - 1 \\ 7 &= \lambda e^{-1} - 1 \\ \lambda &= 8e \\ y &= 8te^{1-t^{-1}} - 1\end{aligned}$$

9.1.10

- Solve the initial value problem $y' = 2y^2 \sin x, \quad y(0) = 6$

$$\begin{aligned}\frac{dy}{dx}(2y^2)^{-1} &= \sin x \\ \int \frac{dy}{dx} y^{-2} dx &= \int 2\sin x dx \\ -y^{-1} &= -2\cos x + \lambda \\ y &= (2\cos x + \lambda)^{-1} \\ 6 &= (2\cos 0 + \lambda)^{-1} \\ -\frac{11}{6} &= \lambda \\ y &= \left(2\cos x - \frac{11}{6}\right)^{-1}\end{aligned}$$

9.2 Edfinity: Models Involving y'=k(y-b)

9.2.2

- Find the general solution of $y' = 5\,(y - 16)$.

$$y(t) = b + Ce^{kt} \qquad\qquad y' = k(y-b)$$

$$\begin{aligned} y(t) &= 16 + Ce^{5t} \\ 30 &= 16 + C \\ C &= 14 \\ y(t) &= 16 + 14e^{5t} \\ 1 &= 16 + C \\ C &= -15 \\ y(t) &= 16 + -15e^{5t} \end{aligned}$$

9.2.3

- A 62 kg skydiver jumps out of an airplane. What is her terminal velocity in miles per hour, assuming that $k = 10\frac{kg}{s}$ for free fall?

$$-\frac{gm}{k} = -\frac{9.8(62)}{10} = -60.76\frac{m}{s} = 199.343\frac{ft}{s} = -134.916 \text{ mph}$$

9.2.4

- A continuous annuity with withdrawal rate $N = \$600$ y and interest rate $r = 5\%$ is funded by an initial deposit P_0
- When will the annuity run out of funds if $P_0 = \$10,000$?

$$\begin{aligned} P(t) = Nr^{-1} + Ce^{rt} = 600(0.05)^{-1} + Ce^{0.05t} &= 12,000 + Ce^{0.05t} \\ 10,000 &= 12,000 + C \\ C &= -2,000 \\ t = 0.05^{-1} \ln \frac{12,000}{2,000} &= 35.83 \approx 38 \text{ years} \end{aligned}$$

- Which initial deposit P_0 yields a constant balance?

$$\begin{aligned} P(t) &= 12,000 + Ce^{0.05^t}, \quad C = 0 \\ P_0 &= 12,000 \end{aligned}$$

9.2.5

- A cup of coffee, cooling off in a room temperature $20\,°\mathrm{C}$, has cooling constant $k = 0.085\ \mathrm{min}^{-1}$.
- How fast is the coffee cooling when its temperature is $T = 70\,°\mathrm{C}$?

$$k(T - T_0)$$
$$0.085(70 - 20) = 4.25\,°\mathrm{C{\cdot}min}^{-1}$$

- Use the Linear Approximation to estimate the change in temperature over the next 4 seconds when $T = 70\,°\mathrm{C}$

$$4.25\,°\mathrm{C{\cdot}min}^{-1}(4\mathrm{s})60\,\mathrm{s{\cdot}min}^{-1} = 0.283\,°\mathrm{C}$$

- The coffee is served at a temperature of $86\,°\mathrm{C}$. How long should you wait before drinking it if the optimal temperature is $65\,°\mathrm{C}$?

$$65 = 20 + 66e^{-0.085t}$$
$$t = -(0.085)^{-1}\ln\left(\frac{45}{66}\right)$$
$$t \approx 4.5\ \mathrm{min}$$