

CALCULUS III FINAL REVIEW

CONVERGENCE: 10.3–10.5

Convergence Notes	2
<i>Tests for Positive Series</i>	2
<i>Tests for Non-Positive Series</i>	3
Convergence Problems	4
10.5 Exercises	4

POWER/TAYLOR SERIES: 10.6–10.8

Power/Taylor Series Notes	12
<i>Power Series</i>	12
<i>Taylor Series</i>	13
Power/Taylor Series Problems	14
10.6 Exercises	14
10.8 Exercises	18

PARAMETRIC EQUATIONS: 11.1

Parametric Equations Notes	20
Parametric Problems	21
11.1 Exercises	21

ARC LENGTH, POLAR COORDINATES: 11.2–11.4

11.2–11.4 Notes	22
<i>Arc Length and Speed</i>	22
<i>Polar Coordinates</i>	23
<i>Area and Arc Length in Polar Coordinates</i>	24
Polar Coordinate Problems	25
11.2 Exercises	25
11.3 Exercises	26
11.4 Exercises	27

QUIZ QUESTIONS

Quiz 3	28
Quiz 4	31

FINAL REVIEW QUESTIONS

CONVERGENCE: 10.3–10.5

Convergence Notes

- Let $\sum_{n=1}^{\infty} a_n$ be given and note for which series convergence is known, i.e.:

Geometric: let $c \neq 0$, if $|r| < 1$, then

p-Series: converges if $p > 1$.

$$\sum_{n=0}^{\infty} cr^n = \frac{c}{1-r}$$

$$\sum_{n=0}^{\infty} \frac{1}{n^p}$$

$|r| \geq 1 \implies$ diverges

$p \leq 1 \implies$ diverges

- The n^{th} Term Divergence Test:** a relatively easy test that can be used to quickly determine if a test diverges if the $\lim_{n \rightarrow \infty} a_n \neq 0$. If $\lim_{n \rightarrow \infty} a_n = 0$, then the test is inconclusive and other tests must be applied.

Tests for Positive Series

- Direct Comparison Test:** use if dropping terms from the denominator or numerator gives a series b_n wherein convergence is easily found, then compare to the original series a_n as follows:

$$\sum_{n=1}^{\infty} b_n \text{ converges} \implies \sum_{n=1}^{\infty} a_n \text{ converges} \quad \leftarrow 0 \leq a_n \leq b_n$$

$$\sum_{n=1}^{\infty} b_n \text{ diverges} \implies \sum_{n=1}^{\infty} a_n \text{ diverges} \quad \leftarrow 0 \leq b_n \leq a_n$$

- Limit Comparison Test:** use when the direct comparison test isn't convenient or when comparing two series. One can take the dominant term in the numerator and denominator from a_n to form a new positive sequence b_n if needed.

Assuming the following limit $L = \lim_{n \rightarrow \infty} \frac{a_n}{b_n}$ exists, then:

$$L > 0 \implies \sum_{n=1}^{\infty} a_n \text{ converges} \iff \sum_{n=1}^{\infty} b_n \text{ converges}$$

$$L = 0 \text{ and } \sum_{n=1}^{\infty} b_n \text{ converges} \implies \sum_{n=1}^{\infty} a_n \text{ converges}$$

$$L = \infty \text{ and } \sum_{n=1}^{\infty} a_n \text{ converges} \implies \sum_{n=1}^{\infty} b_n \text{ converges}$$

- **Ratio Test:** often used in the presence of a factorial ($n!$) or when the are constants raised to the power of n (c^n).

Assuming the following limit $\rho = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$ exists, then

$$\rho < 1 \implies \sum a_n \text{ converges absolutely}$$

$$\rho > 1 \implies \sum a_n \text{ diverges}$$

$$\rho = 1 \implies \text{test is inconclusive}$$

- **Root Test:** used when there is a term in the form of $f(n)^{g(n)}$.

Assuming the following limit $C = \lim_{n \rightarrow \infty} |a_n|^{\frac{1}{n}}$ exists, then

$$C < 1 \implies \sum a_n \text{ converges absolutely}$$

$$C > 1 \implies \sum a_n \text{ diverges}$$

$$C = 1 \implies \text{test is inconclusive}$$

- **Integral Test:** if the other tests fail and $a_n = f(n)$ is a decreasing function, then one can use the improper integral $\int_1^\infty f(x)dx$ to test for convergence.

Let $a_n = f(n)$ be a positive, decreasing, and continuous function $\forall x \geq 1$, then:

$$\int_1^\infty f(x)dx \text{ converges} \implies \sum_{n=1}^\infty a_n \text{ converges}$$

$$\int_1^\infty f(x)dx \text{ diverges} \implies \sum_{n=1}^\infty a_n \text{ diverges}$$

Tests for Non-Positive Series

- **Alternating Series Test:** used for series in the form $\sum_{n=0}^\infty (-1)^n a_n$

Converges if $|a_n|$ decreases monotonically ($|a_{n+1}| \leq |a_n|$) and if $\lim_{n \rightarrow \infty} a_n = 0$

- **Absolute Convergence:** used if the series $\sum a_n$ is not alternating (if it is alternating, use the alternating test in conjunction); simply test if $\sum |a_n|$ converges using the test for positive series.

$\sum a_n$ converges **conditionally** if $\sum a_n$ converges, but $\sum |a_n|$ diverges.

$\sum a_n$ converges **absolutely** if $\sum |a_n|$ converges.

Convergence Problems

10.5 Exercises

Determine convergence or divergence using any method.

$$1. \sum_{n=1}^{\infty} \frac{2^n + 4^n}{7^n}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{2^n}{7^n} + \sum_{n=1}^{\infty} \frac{4^n}{7^n}$$

Separate into two geometric series[†]

$$\Rightarrow r = \frac{2}{7} < 1, \quad r = \frac{4}{7} < 1$$

Both geometric series converge, thus the original series **converges**.

$$2. \sum_{n=1}^{\infty} \frac{n^3}{n!}$$

$$\Rightarrow \rho = \lim_{n \rightarrow \infty} \left| \frac{(n+1)^3}{(n+1)!} \cdot \frac{n!}{n^3} \right|$$

Apply the ratio test[†]

$$= \lim_{n \rightarrow \infty} \frac{n^3 + 3n^2 + 3n + 1}{(n+1)n!} \cdot \frac{n!}{n^3}$$

$$= \lim_{n \rightarrow \infty} \frac{n^3 + 3n^2 + 3n + 1}{n^4 + n^3}$$

$$= \lim_{n \rightarrow \infty} \frac{n^3 + 3n^2 + 3n + 1}{n^4 + n^3} \cdot \frac{n^{-4}}{n^{-4}}$$

$$= \lim_{n \rightarrow \infty} \frac{n^{-1} + 3n^{-2} + 3n^{-3} + n^{-4}}{1 + n^{-1}} = 0$$

$\rho = 0 < 1$, thus the series **converges**.

$$3. \sum_{n=1}^{\infty} \frac{n}{2n+1}$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{n}{2n+1}$$

Apply the n^{th} term test[†]

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{n}{2n+1} = \frac{1}{2}$$

By L'Hôpital's Rule

$\lim_{n \rightarrow \infty} a_n \neq 0$, thus the series **diverges**.

$$4. \sum_{n=1}^{\infty} 2^{\frac{1}{n}}$$

$$\Rightarrow \lim_{n \rightarrow \infty} 2^{\frac{1}{n}} = 2^0 = 1$$

Apply the n^{th} term test[†]

$\lim_{n \rightarrow \infty} a_n \neq 0$, thus the series **diverges**.

$$5. \sum_{n=1}^{\infty} \frac{\sin n}{n^2}$$

$$a_n \leq b_n, \quad b_n = 1$$

b_n converges $\rightarrow a_n$ converges

Apply the **direct comparison test**[†]

$$\sin n \leq 1$$

$$\leftarrow \forall n \geq 1$$

$$\frac{\sin n}{n^2} \leq \frac{1}{n^2}$$

$$\frac{1}{n^2} \rightarrow \text{converges}$$

by **p-series**[†]

The larger (b_n) series converges, thus the smaller (a_n) **converges**.

$$6. \sum_{n=1}^{\infty} \frac{n!}{(2n)!}$$

$$\Rightarrow \rho = \lim_{n \rightarrow \infty} \left| \frac{(n+1)!}{(2n+2)!} \cdot \frac{(2n)!}{n!} \right|$$

Apply the **ratio test** [†]

$$= \lim_{n \rightarrow \infty} \frac{(n+1)n!}{(2n+2)(2n+1)2n!} \cdot \frac{(2n)!}{n!}$$

$$= \lim_{n \rightarrow \infty} \frac{n+1}{(2n+2)(2n+1)} = \frac{n+1}{4n^2 + 6n + 2}$$

$$= \lim_{n \rightarrow \infty} \frac{1}{8n+6} = 0$$

By L'Hôpital's Rule

$\rho = 0 < 1$, thus the series **converges**.

$$7. \sum_{n=1}^{\infty} \frac{1}{n + \sqrt{n}}$$

$$b_n \leq a_n, \quad b_n = n$$

$$a_n \text{ diverges} \iff b_n \text{ diverges}$$

Apply the **direct comparison test** [†]

$$n \leq n + \sqrt{n}$$

$$\leftarrow \forall n \geq 1$$

$$\frac{1}{n + \sqrt{n}} \leq \frac{1}{n}$$

$$\frac{1}{n} \rightarrow \text{diverges}$$

by **p-series** [†]

The smaller (b_n) series diverges, thus the larger (a_n) **diverges**.

$$8. \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$$

f is positive, decreasing, and continuous for $x \geq 2$ Apply the [integral test](#)†

$$\Rightarrow \int_2^{\infty} f(x) dx = \lim_{R \rightarrow \infty} \int_2^R \frac{1}{x(\ln x)^3} dx \quad \ln x = u, \quad x du = dx$$

$$\begin{aligned} \Rightarrow \lim_{R \rightarrow \infty} \int_2^R \frac{1}{x(u)^3} x du &= \int_2^R \frac{1}{u^3} du \\ &= -\frac{1}{2(u)^2} \Big|_2^R = \frac{1}{2R^2} - \frac{1}{8} \\ &= -\frac{1}{2\ln^2(x)} + C \Big|_2^R = \frac{1}{8} - \frac{1}{2R^2} \\ &= \lim_{R \rightarrow \infty} \left(\frac{1}{8} - \frac{1}{2R^2} \right) = \frac{1}{8} \end{aligned}$$

The improper integral converges, thus the original series [converges](#).

$$9. \sum_{n=1}^{\infty} \frac{n^3}{5^n}$$

$$\begin{aligned} \Rightarrow \rho &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)^3}{5^{n+1}} \cdot \frac{5^n}{n^3} \right| && \text{Apply the [ratio test](#)†} \\ &= \lim_{n \rightarrow \infty} \frac{(n+1)^3}{5^n \cdot 5} \cdot \frac{5^n}{n^3} \\ &= \frac{1}{5} \lim_{n \rightarrow \infty} \frac{(n+1)^3}{n^3} \\ &= \frac{1}{5} \lim_{n \rightarrow \infty} \left(\frac{n+1}{n} \right)^3 = \frac{1}{5} \end{aligned}$$

$\rho = \frac{1}{5} < 1$, thus the series [converges](#).

$$10. \sum_{n=2}^{\infty} \frac{1}{\sqrt{n^3 - n^2}}$$

$$L = \lim_{n \rightarrow \infty} \frac{a_n}{b_n}, \quad b_n = \frac{1}{\sqrt{n^3}}$$

Apply the [limit comparison test](#) [†]

$$\Rightarrow L = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n^3 - n^2}} \cdot \frac{\sqrt{n^3}}{1}$$

$$= \lim_{n \rightarrow \infty} \sqrt{\frac{n^3}{n^3(1 - n^{-1})}}$$

$$= \sqrt{\frac{1}{1(1 - 0)}} = 1$$

$L > 0$, thus a_n converges if b_n converges.

b_n converges by the p -series test, as $\frac{3}{2} > 1$, thus a_n [converges](#).

$$11. \sum_{n=1}^{\infty} \frac{n^2 + 4n}{3n^4 + 9}$$

$$L = \lim_{n \rightarrow \infty} \frac{a_n}{b_n}, \quad b_n = \frac{1}{n^2}$$

Apply the [limit comparison test](#) [†]

$$= \lim_{n \rightarrow \infty} \frac{n^2 + 4n}{3n^4 + 9} \cdot n^2$$

$$= \lim_{n \rightarrow \infty} \frac{n^4 + 4n^3}{3n^4 + 9} \cdot \frac{n^{-4}}{n^{-4}}$$

$$= \lim_{n \rightarrow \infty} \frac{1 + 4n^{-1}}{3 + 9n^{-4}} = \frac{1}{3}$$

$L > 0$, thus a_n converges if b_n converges.

b_n converges by the p -series test, as $2 > 1$, thus a_n [converges](#).

$$12. \sum_{n=1}^{\infty} (0.8)^{-n} n^{-0.8}$$

$$\begin{aligned} \rho &= \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{(0.8)^{-(n+1)} (n+1)^{-0.8}}{(0.8)^{-n} n^{-0.8}} \right| \\ &= \lim_{n \rightarrow \infty} \frac{(0.8)^{-n} \cdot 0.8^{-1} \cdot (n+1)^{-0.8}}{(0.8)^{-n} n^{-0.8}} \\ &= 1.25 \lim_{n \rightarrow \infty} \frac{(n+1)^{-0.8}}{n^{-0.8}} \\ &= 1.25 \lim_{n \rightarrow \infty} \left(\frac{n+1}{n} \right)^{-0.8} = 1.25 \end{aligned}$$

Apply the ratio test \uparrow

$\rho = 1.25 > 1$, thus a_n diverges.

$$13. \sum_{n=1}^{\infty} 4^{-2n+1}$$

$$\sum_{n=1}^{\infty} cr^n \implies \sum_{n=1}^{\infty} 4 \cdot (4^{-2})^n$$

Convert into geometric series \uparrow

$|r| = 2^{-2} = \frac{1}{4} < 1$ and $c \neq 0$, thus a_n converges.

$$14. \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$$

$$\sum_{n=1}^{\infty} (-1)^n a_n, \quad a_n = \frac{1}{\sqrt{n}}$$

Apply the [alternating series test](#) [†]

$$\frac{1}{\sqrt{n}} \rightarrow \text{diverges}$$

by [p-series](#) [†]

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} = 0$$

$|a_n|$ decreases monotonically and $\lim_{n \rightarrow \infty} a_n = 0$, but a_n diverges, thus the series [converges conditionally](#)

$$15. \sum_{n=1}^{\infty} \sin \frac{1}{n^2}$$

$$L = \lim_{n \rightarrow \infty} \frac{a_n}{b_n}, \quad b_n = \frac{1}{n^2}$$

Apply the [limit comparison test](#) [†]

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{\sin(n^{-2})}{n^{-2}} = \frac{0}{0}$$

$$= \lim_{n \rightarrow \infty} \frac{\cos(n^{-2})(-2n^{-3})}{-2n^{-3}}$$

by L'Hôpital's Rule

$$= \lim_{n \rightarrow \infty} \cos(n^{-2}) = 1$$

$L > 0$, thus a_n converges if b_n converges.

b_n converges by the [p-series test](#), thus a_n [converges](#).

$$16. \sum_{n=1}^{\infty} (-1)^n \cos n^{-1}$$

$$\sum_{n=1}^{\infty} (-1)^n a_n, \quad a_n = \cos(n^{-1})$$

Apply the [nth term test](#) [†]

$$\Rightarrow L = \lim_{n \rightarrow \infty} \cos(n^{-1}) = 1$$

$L \neq 0$, thus the series [diverges](#)

$$17. \sum_{n=1}^{\infty} \frac{(-2)^n}{\sqrt{n}}$$

$$\sum_{n=1}^{\infty} (-1)^n a_n, \quad a_n = \frac{2^n}{\sqrt{n}}$$

Apply the n^{th} term test \uparrow

$$\Rightarrow L = \lim_{n \rightarrow \infty} \frac{2^n}{\sqrt{n}} = \frac{\infty}{\infty}$$

$$= \frac{2^n \ln 2}{\frac{1}{2} n^{-\frac{1}{2}}} = 2^n \ln 2 \cdot 2\sqrt{n} \quad \text{By L'Hôpital's Rule}$$

$$= 2 \lim_{n \rightarrow \infty} 2^n \ln(2) \sqrt{n} = \infty$$

$L \neq 0$, thus the series **diverges**

$$18. \sum_{n=1}^{\infty} \left(\frac{n}{n+12} \right)^n$$

$$L = \lim_{n \rightarrow \infty} \left(\frac{n}{n+12} \right)^n$$

Apply the n^{th} term test \uparrow

$$= \lim_{n \rightarrow \infty} e^{-12}$$

By common limit $\left(\frac{x}{x+k} \right)^x = e^{-k}$

$L \neq 0$, thus the series **diverges**.

POWER/TAYLOR SERIES: 10.6–10.8

Power/Taylor Series Notes

Power Series

- **Power series:** a infinite series in the form:

$$F(x) = \sum_{n=0}^{\infty} a_n(x - c)^n$$

Where the constant c is the *center* of the power series $F(x)$.

- **Radius of convergence R :** the range of values of the variable x whereby the power series $F(x)$ converges.
 - Every power series converges at $x = c$, as $(x - c)^0 = 1$, though the series may diverge for other values of x .
 - $F(x)$ converges for $|x - c| < R$ and diverges for $|x - c| > R$
 - $F(x)$ may converge or diverge at endpoints $c - R$ and $c + R$
- **Interval of convergence:** the open interval $(c - R, c + R)$ and possibly one of both of the endpoints, each must be tested.
 - In most cases, the **ratio test**† can be used to find R .
 - If $R > 0$, then F is differentiable over the interval of convergence; the derivative and antiderivative can be obtained using the following:

$$F'(x) = \sum_{n=1}^{\infty} n a_n (x - c)^{n-1} \qquad F(x) dx = C + \sum_{n=0}^{\infty} \frac{a_n}{n+1} (x - c)^{n+1}$$

Taylor Series

- **Taylor series:** the power series of a infinitely differentiable function $f(x)$ centered at c ,

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x - c)^n$$

- **n^{th} Taylor polynomial:** a polynomial of degree n that is formed partial sum formed by the first $n + 1$ terms of a Taylor series, i.e.,

$$f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \frac{f'''(c)}{3!}(x - c)^3 + \cdots + \frac{f^{(n)}(c)}{n!}(x - c)^n$$

- **Maclaurin series:** when $c = 0$, i.e.,

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} (x)$$

- **Useful Maclaurin Series:** useful Taylor series centered at 0 that can be used to derive other series via differentiation, integration, multiplication, or substitution.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \leftarrow \forall x$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad \leftarrow \forall x$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \quad \leftarrow \forall x$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \leftarrow |x| < 1$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n \quad \leftarrow |x| < 1$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} \quad \leftarrow |x| < 1 \wedge x \neq -1$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \quad \leftarrow |x| \leq 1$$

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n \quad \leftarrow |x| < 1$$

$$\text{where } \binom{\alpha}{n} = \sum_{k=0}^n \prod_{k=1}^n \frac{\alpha - k + 1}{k}$$

Power/Taylor Series Problems

10.6 Exercises

Find the interval of convergence.

1. $\sum_{n=0}^{\infty} (-1)^n \frac{n}{4^n} x^{2n}$

$$\rho = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \quad \text{Apply the ratio test}^\uparrow$$

$$= \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} (n+1) (x^{2(n+1)})}{4^{n+1}} \cdot \frac{4^n}{(-1)^n n (x^{2n})} \right|$$

$$= \lim_{n \rightarrow \infty} \frac{(n+1) (|x|^{2n} \cdot |x|^2)}{4^n \cdot 4} \cdot \frac{4^n}{n \cdot |x|^{2n}}$$

$$= \lim_{n \rightarrow \infty} \frac{(n+1) |x|^2}{4n} \cdot \frac{n^{-1}}{n^{-1}}$$

$$= \lim_{n \rightarrow \infty} \frac{(1 + n^{-1}) |x|^2}{4} = \frac{|x|^2}{4}$$

$$\Rightarrow \frac{|x|^2}{4} < 1 \Rightarrow |x| < 2 \quad \text{converges for } \rho < 1$$

Both endpoints tend toward ∞ (diverge), thus the interval of convergence is $(-2, 2)$.

2. $\sum_{n=8}^{\infty} n^7 x^n$

$$\rho = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \quad \text{Apply the ratio test}^\uparrow$$

$$\Rightarrow \lim_{n \rightarrow \infty} \left| \frac{(n+1)^7 x^{n+1}}{n^7 x^n} \right|$$

$$= \lim_{n \rightarrow \infty} \frac{(n+1)^7 \cdot |x|^n \cdot |x|}{n^7 |x|^n}$$

$$= \lim_{n \rightarrow \infty} \frac{(n+1)^7 |x|}{n^7}$$

$$= |x| \lim_{n \rightarrow \infty} \left(\frac{n+1}{n} \right)^7 = |x|$$

$$\Rightarrow |x| < 1 \quad \text{converges for } \rho < 1$$

Both endpoints tend toward ∞ (diverge), thus the interval of convergence is $(-1, 1)$.

$$3. \sum_{n=2}^{\infty} \frac{x^n}{\ln n}$$

$$\rho = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

Apply the **ratio test** \uparrow

$$\Rightarrow \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{\ln(n+1)} \cdot \frac{\ln n}{x^n} \right|$$

$$= \lim_{n \rightarrow \infty} \frac{|x| \ln n}{\ln(n+1)}$$

$$= |x| \lim_{n \rightarrow \infty} \frac{\ln n}{\ln(n+1)} = \frac{\infty}{\infty}$$

$$= |x| \lim_{n \rightarrow \infty} \frac{n^{-1}}{(n+1)^{-1}}$$

By L'Hôpital's Rule

$$= |x| \lim_{n \rightarrow \infty} \frac{n+1}{n}$$

$$= |x| \lim_{n \rightarrow \infty} 1 + n^{-1} = |x|$$

$$\Rightarrow |x| < 1$$

converges for $\rho < 1$

$$f(1) = \sum_{n=1}^{\infty} \frac{1}{\ln n}$$

$$\Rightarrow 0 \leq b_n \leq a_n$$

Apply the **direct comparison test** \uparrow

$$\Rightarrow 0 \leq \frac{1}{n} \leq \frac{1}{\ln n}$$

$$\Rightarrow a_n \text{ diverges}$$

$\frac{1}{n} \rightarrow$ diverges as $p \leq 1$

$$f(-1) = \sum_{n=1}^{\infty} \frac{(-1)^n}{\ln n}$$

$$\sum_{n=1}^{\infty} (-1)^n a_n, \quad a_n = \frac{1}{\ln n}$$

Apply the **alternating series test** \uparrow

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{1}{\ln n} = 0$$

Note: $|a_n|$ decreases monotonically

$f(-1)$ converges and $f(1)$ diverges, thus the interval of convergence is $[-1, 1)$

$$4. \sum_{n=1}^{\infty} \frac{(-5)^n (x-3)^n}{n^2}$$

$$\rho = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

Apply the **ratio test** [†]

$$\Rightarrow \lim_{n \rightarrow \infty} \left| \frac{(-5)^{n+1} (x-3)^{n+1}}{(n+1)^2} \cdot \frac{n^2}{(-5)^n (x-3)^n} \right|$$

$$= \lim_{n \rightarrow \infty} \frac{5^n \cdot 5 \cdot (|x-3|)^n \cdot |x-3|}{n^2} \cdot \frac{n^2}{5^n (|x-3|)^n}$$

$$= \lim_{n \rightarrow \infty} 5|x-3|$$

$$\Rightarrow |x-3| < \frac{1}{5}$$

$$\Rightarrow -\frac{14}{5} < x < \frac{16}{5}$$

converges for $\rho < 1$

$$f\left(\frac{-14}{5}\right) = \frac{(-5)^n \left(-\frac{1}{5}\right)^n}{n^2} = \frac{1}{n^2}$$

$$f\left(\frac{16}{5}\right) = \frac{(-5)^n \left(\frac{1}{5}\right)^n}{n^2} = \frac{(-1)^n}{n^2}$$

$\lim_{n \rightarrow \infty} a_n$ (of both points) = 0 and the $|a_n|$ of both endpoints decrease monotonically;

$R = \frac{1}{5}$, $c = 3$, thus the interval of convergence is $\left[-\frac{14}{5}, \frac{16}{5}\right]$

Use the following equation to expand the function in a power series with $c = 0$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \leftarrow |x| < 1$$

and determine the interval of convergence.

5. $f(x) = \frac{1}{4+3x}$

$$\frac{1}{4+3x} = \frac{\frac{1}{4}}{1 - (-\frac{3x}{4})}$$

$$= \frac{1}{4} \sum_{n=0}^{\infty} \left(-\frac{3x}{4}\right)^n = \sum_{n=0}^{\infty} (-1)^n \frac{3^n x^n}{4^{n+1}} \quad \text{Expansion}$$

$$\Rightarrow \left| \frac{3x}{4} \right| < 1 \quad \sum_{n=1}^{\infty} \leftarrow |x| < 1$$

$$\Rightarrow -\frac{4}{3} < x < \frac{4}{3} \quad \text{Interval of convergence}$$

Thus, $\frac{1}{4+3x} = \sum_{n=0}^{\infty} (-1)^n \frac{3^n x^n}{4^{n+1}}$ with an interval of convergence of $\left(-\frac{4}{3}, \frac{4}{3}\right)$

6. $f(x) = \frac{1}{1-x^4}$

$$\frac{1}{1-x^4} = \sum_{n=0}^{\infty} (x^4)^n = \sum_{n=0}^{\infty} x^{4n} \quad \text{Expansion}$$

$$\Rightarrow |x^4| < 1 \quad \sum_{n=1}^{\infty} \leftarrow |x| < 1$$

$$\Rightarrow -1 < x < 1 \quad \text{Interval of convergence}$$

Thus, $\frac{1}{1-x^4} = \sum_{n=0}^{\infty} x^{4n}$ with an interval of convergence of $(-1, 1)$

10.8 Exercises

Find the Maclaurin series and find the interval on which the expression is valid.

1. $f(x) = \sin(2x)$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

Use relevant [Maclaurin series](#) [†]

$$\Rightarrow \sin 2x = (-1)^n \frac{(2x)^{2n+1}}{(2n+1)!}$$

$\sin(x)$ converges $\forall x$, thus $\sin(2x)$ also converges $\forall x$. Therefore:

$$f(x) = (-1)^n \frac{(2x)^{2n+1}}{(2n+1)!} \quad \leftarrow \forall x \in \mathbb{R}$$

2. $f(x) = x^2 e^{x^2}$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Use relevant [Maclaurin series](#) [†]

$$\Rightarrow e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!}$$

$$\Rightarrow x^2 e^{x^2} = \sum_{n=0}^{\infty} \frac{x^2 \cdot (x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n+2}}{n!}$$

e^x converges $\forall x$, thus e^{x^2} also converges $\forall x$. Therefore:

$$f(x) = \sum_{n=0}^{\infty} \frac{x^{2n+2}}{n!} \quad \leftarrow \forall x \in \mathbb{R}$$

Find the Taylor series centered at c and the interval on which the expansion is valid.

3. $f(x) = e^{3x}$, $c = -1$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Use relevant [Maclaurin series](#) [†]

$$\Rightarrow f(x) = e^{3(x-1)} = e^{-3} \cdot e^{3(x+1)}$$

Center series $(x - c)$

$$= e^{-3} \sum_{n=0}^{\infty} \frac{(3(x+1))^n}{n!}$$

$$= e^{-3} \sum_{n=0}^{\infty} \frac{3^n (x+1)^n}{n!}$$

e^x converges $\forall x$, thus e^{3x} also converges $\forall x$. Therefore:

$$f(x) = e^{-3} \sum_{n=0}^{\infty} \frac{3^n (x+1)^n}{n!} \quad \leftarrow \text{convergence interval: } (-\infty, \infty)$$

4. $f(x) = \sin(x)$, $c = \frac{\pi}{2}$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

Use relevant [Maclaurin series](#) [†]

$$\Rightarrow f(x) = \sin\left(x - \frac{\pi}{2}\right)$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{\left(x - \frac{\pi}{2}\right)^{2n+1}}{(2n+1)!}$$

Center series $(x - c)$

$\sin(x)$ converges $\forall x$, therefore:

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{\left(x - \frac{\pi}{2}\right)^{2n+1}}{(2n+1)!} \quad \leftarrow \text{convergence interval: } (-\infty, \infty)$$

PARAMETRIC EQUATIONS: 11.1

Parametric Equations Notes

- **Parametric equation:** defines a group of quantities as functions of one or more independent variables called parameters, commonly expressed as coordinates of points that make up a geometric object.

- **Parametrization:** the representation of a geometrical curve \mathcal{C} with parameter t , i.e.,

$$c(t) = (x(t), y(t))$$

- Note: parametrizations are not unique; the path $c(t)$ may traverse all or part of \mathcal{C} more than once.

- **Parametrization of a line:** a line through point $P = (a, b)$ with slope m :

$$x = a + t, \quad y = b + mt \quad \leftarrow -\infty < t < \infty$$

- **Parametrization of a circle** with radius R and center (a, b) :

$$c(t) = (a + R \cos \theta, b + R \sin \theta)$$

- **Parametrization of an ellipse:**

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \quad \rightarrow \quad c(\theta) = (a \cos \theta, b \sin \theta)$$

- **Parametrization of a cycloid:** generated by a circle of radius R ,

$$c(\theta) = (R(t - \sin \theta), R(1 - \cos \theta))$$

- **Graph of $y = f(x)$:**

$$c(t) = (t, f(t))$$

- **Slope of tangent line at $c(t)$:**

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{y'(t)}{x'(t)} \quad \leftarrow x'(t) \neq 0$$

- **Area under a parametric curve:** valid when the curve $y = h(x)$ is traced **once** by the parametric curve $c(t) = (x(t), y(t))$.

$$\begin{aligned} y = h(x) &\rightarrow y(t), & dx &\rightarrow x'(t)dt \\ \Rightarrow A &= \int_{t_0}^{t_1} y(t)x'(t)dt \end{aligned}$$

Parametric Problems

11.1 Exercises

1.

ARC LENGTH, POLAR COORDINATES: 11.2–11.4

11.2–11.4 Notes

Arc Length and Speed

- **Arc Length of \mathcal{C}** : valid if $c(t) = (x(t), y(t))$ directly traverses \mathcal{C} for $a \leq t \leq b$, then

$$s = \int_a^b \sqrt{x'(t)^2 + y'(t)^2} dt$$

- Can be interpreted as the **distance traveled** along the path from $t = a \rightarrow b$
- **Displacement**: less than or equal to the distance traveled; simply the distance from starting point $c(a)$ to endpoint $c(b)$.
- Distance traveled as as **function of t** , starting at t_0 :

$$s(t) = \int_{t_0}^t \sqrt{x'(u)^2 + y'(u)^2} du$$

- **Speed** at time t :

$$\frac{ds}{dt} = \sqrt{x'(t)^2 + y'(t)^2}$$

- **Surface area**: obtained via rotation of the parametric equation about the x-axis for $a \leq t \leq b$, given $y(t) \geq 0$, $x(t)$ is increasing, and $x'(t) \wedge y'(t)$ are continuous:

$$S = 2\pi \int_a^b y(t) \sqrt{x'(t)^2 + y'(t)^2} dt$$

Polar Coordinates

- **Polar coordinate system:** a two-dimensional coordinate system wherein each point is determined by the distance and angle from a reference point and direction.
 - **Radial coordinate, r :** the distance from reference point.
 - **Angular coordinate, θ :** the angle from reference direction.
 - A point P has polar coordinates (r, θ) with the angle measured in the counterclockwise direction by convention.
- **Conversion between polar and rectangular coordinates:**

$$x = r \cos \theta \quad y = r \sin \theta \quad r = \sqrt{x^2 + y^2} \quad \tan \theta = \frac{y}{x} \quad \leftarrow x \neq 0$$

- **If $r > 0$ then:** (r, θ) must lie in quadrant I or IV;

$$\theta = \begin{cases} \tan^{-1} \frac{y}{x} & \leftarrow x > 0 \\ \tan^{-1} \frac{y}{x} + \pi & \leftarrow x < 0 \\ \pm \frac{\pi}{2} & \leftarrow x = 0 \end{cases}$$

- **Non-uniqueness:** Multiple representations can represent the same point, i.e.,

$$(r, \theta) \equiv (r, \theta + 2n\pi) \equiv (-r, \theta + (2n + 1)\pi) \quad \leftarrow n \in \mathbb{Z}$$

- **Polar Equations:**

Curve	Polar Equation
Circle of radius R , center at origin	$r = R$
Line through origin slope $m = \tan \theta_0$	$\theta = \theta_0$
Line, where $P_0 = (d, \alpha)$ is closest to the origin	$r = d \sec(\theta - \alpha)$
Circle radius a , center at $(a, 0)$ $(x - a)^2 + y^2 = a^2$	$r = 2a \cos \theta$
Circle radius a , center at $(0, a)$ $x^2 + (y - a)^2 = a^2$	$r = 2a \sin \theta$

Area and Arc Length in Polar Coordinates

- **Area in Polar Coordinates:** given that f is continuous, then the sector is bounded by:
 - **Polar curve, r :** $r = f(\theta)$
 - **Two rays, α, β :** where each ray is an angle θ with $\alpha < \beta$, $\beta = \theta - \alpha$
 - Thus, the area is equal to the integral between α and β , i.e.

$$A = \frac{1}{2} \int_{\alpha}^{\beta} f(\theta)^2 d\theta$$

- **Arc length of polar curve:** given $\alpha \leq \theta \leq \beta$:

$$s = \int_{\alpha}^{\beta} \sqrt{f(\theta)^2 + f'(\theta)^2} d\theta$$

Polar Coordinate Problems

11.2 Exercises

1.

11.3 Exercises

1.

11.4 Exercises

1.

QUIZ QUESTIONS

Quiz 3

1. Indicate whether the following statements are **True** or **False**, with justification.

(a) The series $\sum_{n=1}^{\infty} (-1)^n \cos\left(\frac{1}{n}\right)$ converges.

✗ False:

$$\lim_{n \rightarrow \infty} a_n \stackrel{?}{=} 0, \quad a_n = \cos\left(\frac{1}{n}\right) \quad \text{Apply the } n^{\text{th}} \text{ term test}^{\uparrow}$$
$$\Rightarrow \lim_{n \rightarrow \infty} \cos\left(\frac{1}{n}\right) = 1$$

$\lim_{n \rightarrow \infty} a_n \neq 0$, thus the series **diverges**.

(b) If the radius of converges of the power series $\sum_{n=0}^{\infty} a_n x^n$ is $R = 5$, then the series must converge for $x = -3$ and $x = -4$.

✓ True:

$$c = 0, \quad R = 5 \Rightarrow \text{converges } \forall x \in (-5, 5)$$

By the **Interval of convergence**[↑]

$x = -3 \wedge x = 4 \in (-5, 5)$, thus the series **must converge** at these values.

2. Determine whether the following series converge absolutely/conditionally, or diverge.

$$(a) \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{2n+5}$$

$$\sum_{n=1}^{\infty} (-1)^n a_n, \quad a_n = \frac{n}{2n+5} \quad \text{Apply the alternating series test}^\uparrow$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{n}{2n+5} = \frac{\infty}{\infty}$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{n}{2n+5} = \frac{1}{2} \quad \text{By L'Hôpital's Rule}$$

$\lim_{n \rightarrow \infty} a_n \neq 0$, thus the series **diverges**

$$(b) \sum_{n=1}^{\infty} \frac{(-1)^n}{2\sqrt{n}-1}$$

$$\sum_{n=1}^{\infty} (-1)^n a_n, \quad a_n = \frac{1}{2\sqrt{n}-1} \quad \text{Apply the alternating series test}^\uparrow$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{1}{2\sqrt{n}-1} = 0$$

$$\Rightarrow a_n \text{ converges} \quad \text{Note: } |a_n| \text{ decreases monotonically}$$

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{2\sqrt{n}-1} \right| \stackrel{?}{=} \text{converges} \quad \text{Apply the absolute convergence test}^\uparrow$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{1}{2\sqrt{n}-1}$$

$$\lim_{n \rightarrow \infty} a_n = 0 \rightarrow n^{\text{th}} \text{ term inconclusive...}$$

$$L = \lim_{n \rightarrow \infty} \frac{a_n}{b_n}, \quad b_n = \frac{1}{\sqrt{n}}$$

$$\text{Apply the limit comparison test}^\uparrow$$

$$= \lim_{n \rightarrow \infty} \frac{1}{2\sqrt{n}-1} \cdot \sqrt{n}$$

$$= \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{2\sqrt{n}-1} \cdot \frac{n^{-\frac{1}{2}}}{n^{-\frac{1}{2}}}$$

$$= \lim_{n \rightarrow \infty} \frac{1}{2 - n^{-\frac{1}{2}}} = \frac{1}{2}$$

$L > 0$, and b_n diverges by the p -series, implying the $|a_n|$ diverges. Thus, the original series **converges conditionally**.

3. Find a power series expansion with the center $c = 0$ for

$$f(x) = \frac{1}{1+x^3}$$

and find the interval of convergence. Hint: use $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \leftarrow |x| < 1$

$$\begin{aligned} \Rightarrow \frac{1}{1+x^3} &= \frac{1}{1-(-x^3)} \\ &= \sum_{n=1}^{\infty} (-x^3)^n = \sum_{n=1}^{\infty} (-1)^n x^{3n} && \text{Apply hint} \\ \Rightarrow \frac{1}{1+x^3} &= \sum_{n=1}^{\infty} (-1)^n x^{3n} && \leftarrow |x| < 1 \end{aligned}$$

Thus, the interval of convergence is all values in the interval $(-1, 1)$.

4. Find the radius of convergence of the power series given by

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+1}}{2^n n}$$

$$\begin{aligned} \rho &= \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| && \text{Apply the ratio test}^\uparrow \\ \Rightarrow \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} x^{2(n+1)+1}}{2^{n+1}(n+1)} \cdot \frac{2^n n}{(-1)^n x^{2n+1}} \right| \\ &= \lim_{n \rightarrow \infty} \frac{|x|^{2n+3}}{2^{2n}(n+1)} \cdot \frac{2^n n}{|x|^{2n+1}} \\ &= \lim_{n \rightarrow \infty} \frac{|x|^{2n+1} \cdot |x|^2}{2^n \cdot 2(n+1)} \cdot \frac{2^n n}{|x|^{2n+1}} \\ &= |x|^2 \lim_{n \rightarrow \infty} \frac{n}{2n+2} = \frac{\infty}{\infty} \\ &= |x|^2 \lim_{n \rightarrow \infty} \frac{1}{2} && \text{By L'Hôpital's Rule} \\ \Rightarrow \frac{|x|^2}{2} < 1 && \text{converges when } \rho < 1 \\ &= |x| < \sqrt{2} \end{aligned}$$

Thus, the interval of convergence is $(-\sqrt{2}, \sqrt{2})$ with $R = \sqrt{2}$, and $c = 0$ (endpoints not required to be tested for this problem).

Quiz 4

1. Indicate whether the following statements are **True** or **False**, with justification.

- (a) The curve with parametric representations $c(t) = (4 + 3 \cos t, 5 + 3 \sin t)$ is a circle with radius $R = 3$ centered at the origin.

✗ False:

$$c(t) = (a + R \cos \theta, b + R \sin \theta) \quad \text{Parametrization of a circle}^\uparrow$$

$$c(t) = (3 \cos \theta, 3 \sin \theta) \quad \leftarrow R = 3, (0, 0)$$

Note: $a = 4$, $b = 5$, thus it's a circle with radius 3, but not centered at the origin.

- (b) The parametric representation given by $c(t) = (\sin t, t)$ can be represented by function of the form $y = f(x)$.

✗ False:

Note that the y component of the parametric representation is given by $y = t$.

Substituting y for t in the x component yields $x = \sin y$, which is a function of x in terms of y , but NOT a function of y in terms of x .

2. Determine whether the following series converge or diverge, with justification.

$$(a) \sum_{n=1}^{\infty} \frac{n^3}{n!}$$

$$\begin{aligned} \rho &= \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)^3}{(n+1)!} \cdot \frac{n!}{n^3} \right| \\ &= \lim_{n \rightarrow \infty} \frac{(n+1)^3}{(n+1)n!} \cdot \frac{n!}{n^3} \\ &= \lim_{n \rightarrow \infty} \frac{(n+1)^2}{n^3} \\ &= \lim_{n \rightarrow \infty} \frac{n^2 + 2n + 1}{n^3} \cdot \frac{n^{-3}}{n^{-3}} \\ &= \lim_{n \rightarrow \infty} n^{-1} + 2n^{-2} + n^{-3} = 0 \end{aligned}$$

Apply the **ratio test** [†]

$\rho < 1$, thus the series **converges absolutely**.

$$(b) \sum_{n=0}^{\infty} \left(\frac{n}{3n+1} \right)^n$$

$$\begin{aligned} C &= \lim_{n \rightarrow \infty} |a_n|^{\frac{1}{n}} \\ &= \lim_{n \rightarrow \infty} \left| \left(\frac{n}{3n+1} \right)^n \right|^{\frac{1}{n}} \\ &= \lim_{n \rightarrow \infty} \frac{n}{3n+1} = \frac{1}{3} \end{aligned}$$

Apply the **root test** [†]

By L'Hôpital's Rule

$C = \frac{1}{3} < 1$, thus the series **converges absolutely**.

3. Find the Maclaurin series of (using substitution and/or multiplication)

$$f(x) = x \cos(x^2)$$

$$\begin{aligned} \cos x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} && \text{Use relevant Maclaurin series}^\dagger \\ \Rightarrow \cos(x^2) &= \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n}}{(2n)!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!} && \text{Substitution of } x^2 \\ x \cdot \cos(x^2) &= x \cdot \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{(2n)!} && \text{Multiply by } x \end{aligned}$$

$\cos x$ converges $\forall x$, thus $x \cos x^2$ also converges $\forall x$. Therefore:

$$f(x) = x \cos(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{(2n)!} \quad \leftarrow \forall x \in \mathbb{R}$$

4. Express the following integral as a infinite series, first by finding the Maclaurin series of the integrand, then integrating this series.

$$\int_0^1 e^{-x^2} dx$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Use relevant [Maclaurin series](#) [†]

$$\Rightarrow f(x) = e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}$$

← $\forall x \in \mathbb{R}$

$$\Rightarrow \int_0^1 e^{-x^2} dx = \int_0^1 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!} dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^1 x^{2n} dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot \frac{x^{2n+1}}{2n+1} \Big|_0^1$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)n!} - 0$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)n!}}$$

5. Consider the curve with parametric representation

$$c(t) = (\sin 2t + \cos t, \cos 2t - \sin t)$$

Find an equation of the tangent line at $t = \pi$

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} \quad \leftarrow x'(t) \neq 0$$

Note: slope of a tangent line \uparrow

$$\Rightarrow \frac{dy}{dx} = \frac{-2 \sin(2t) - \cos t}{2 \cos(2t) - \sin t}$$

$$\Rightarrow m = \left. \frac{dy}{dx} \right|_{t=\pi} = \frac{0 - (-1)}{2 - 0} = \frac{1}{2}$$

$$\Rightarrow \boxed{y - 1 = \frac{1}{2}(x + 1)}$$

$$c(\pi) = (-1, 1)$$

FINAL REVIEW QUESTIONS

Note: these questions were taken from a provided review sheet; they focus on sections 10.6–11.4. Some questions already exist on the quizzes, but will be duplicated here.

1. Find the interval of convergence of the following power series.

(a) $\sum_{n=1}^{\infty} \frac{5^n}{n} x^n$

$$\begin{aligned} C &= \lim_{n \rightarrow \infty} |a_n|^{\frac{1}{n}} \\ &= \lim_{n \rightarrow \infty} \left| \frac{5^n x^n}{n} \right|^{\frac{1}{n}} \\ &= 5|x| \lim_{n \rightarrow \infty} \frac{1}{\sqrt[n]{n}} = 5|x| \end{aligned}$$

Apply the **root test**[†]

$$\Rightarrow |x| < \frac{1}{5}$$

converges for $C < 1$

$$\begin{aligned} f\left(-\frac{1}{5}\right) &= \sum_{n=1}^{\infty} \frac{5^n}{n} \left(-\frac{1}{5}\right)^n \\ &= \sum_{n=1}^{\infty} (-1)^n \frac{1}{n}, \quad a_n = \frac{1}{n} \end{aligned}$$

Apply the **alternating series test**[†]

$$\lim_{n \rightarrow \infty} a_n = 0 \wedge |a_n| \text{ decreases monotonically}$$

\Rightarrow converges

$$f\left(\frac{1}{5}\right) = \sum_{n=1}^{\infty} \frac{5^n}{n} \left(\frac{1}{5}\right)^n = \sum_{n=1}^{\infty} \frac{1}{n}$$

\Rightarrow diverges by **p-series**[†]

$$\Rightarrow \text{Interval of convergence: } \left[-\frac{1}{5}, \frac{1}{5}\right)$$

$$(b) \sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2+1}$$

$$\begin{aligned} \rho &= \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{(x-2)^{n+1}}{(n+1)^2+1} \cdot \frac{n^2+1}{(x-2)^n} \right| \\ &= \lim_{n \rightarrow \infty} \frac{(|x-2|)^n \cdot |x-2|}{n^2+2n+2} \cdot \frac{n^2+1}{(|x-2|)^n} \\ &= |x-2| \lim_{n \rightarrow \infty} \frac{n^2+1}{n^2+2n+2} \\ &= |x-2| \lim_{n \rightarrow \infty} \frac{2}{2} \end{aligned}$$

Apply the **ratio test** [†]

By L'Hôpital's Rule

$$\Rightarrow |x-2| < 1$$

converges for $\rho < 1$

$$\Rightarrow 1 < x < 3$$

$$f(1) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1} \quad a_n = \frac{1}{n^2+1}$$

Apply the **alternating series test** [†]

$$\lim_{n \rightarrow \infty} a_n = 0 \wedge |a_n| \text{ decreases monotonically} \Rightarrow \text{converges}$$

$$f(3) = \sum_{n=1}^{\infty} \frac{1}{n^2+1}$$

$$b_n \text{ converges} \rightarrow a_n \text{ converges}$$

Apply the **direct comparison test** [†]

$$b_n = \frac{1}{n^2} \Rightarrow b_n \text{ converges}$$

By **p-series** [†]

$$\Rightarrow \text{Interval of convergence: } [1, 3]$$

2. Find the Taylor series of the following functions $f(x)$ centered at the given value of c using the definition.

(a) $f(x) = e^x, \quad c = 2$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \leftarrow \forall n \geq 0$$

Use relevant [Maclaurin series](#) [†]

$$f^{(n)}(2) = e^2 \leftarrow f^{(n)}(x) = e^x$$

$$\Rightarrow \boxed{\sum_{n=0}^{\infty} \frac{e^2(x-2)^n}{n!}}$$

Center series, $c = 2$

(b) $f(x) = \sqrt{x}, \quad c = 1$

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n \quad \leftarrow |x| < 1 \quad \text{Use relevant [Maclaurin series](#) [†]}$$

3. Find the Maclaurin series of the following functions using substitution and/or multiplication.

(a) $f(x) = x \cos(2x)$

(b) $f(x) = \frac{x^3}{1+x}$

4. Express the following integral as a power series, first by finding the Maclaurin series of the integrand, then integrating this series term-by-term:

$$\int_0^1 e^{-x^2} dx$$

5. Find the parametric equations for the following curves.

(a) The line through $(3, 6)$ and $(-2, 0)$.

(b) The circle of radius 5 centered at $(1, 7)$.

(c) The ellipse

$$\left(\frac{x-1}{2}\right)^2 + \left(\frac{y+1}{3}\right)^2 = 1$$

6. Find the equation of the tangent line to the curve

$$x = \sin(2t) + \cos(t), \quad y = \cos(2t) - \sin(t), \quad \leftarrow t = \pi$$

7. Find the arc length of the curve

$$x = \frac{2}{3}t^2, \quad y = t^2 - 2, \quad \leftarrow 0 \leq t \leq 2$$

8. Find the surface area obtained by rotating the following around the x-axis;

$$x = e^t - t, \quad y = 4e^{\frac{t}{2}}, \quad \leftarrow 0 \leq t \leq 1$$

9. Match each equation in rectangular coordinates with its equation in polar coordinates.

(a) $x^2 + y^2 = 4$

(i) $r^2(1 - 2\sin^2 \theta) = 4$

(b) $x^2 + (y - 1)^2 = 1$

(ii) $r(\cos \theta + \sin \theta) = 4$

(c) $x^2 - y^2 = 4$

(iii) $r = \sin \theta$

(d) $x + y = 4$

(iv) $r = 2$

10. Find the area enclosed by one loop of the curve

$$r^2 \cos 2\theta$$