

# Calculus



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# Limits and Continuity



## Limits

🌐 Limit 📖 | Thomas (2.2–2.4) 📖

- **Limit**  $\lim_{x \rightarrow c}$ : the value of a function (or sequence) as the input (or index) approaches some value (note: an informal definition).
  - Limits are used to define **continuity** ↓, **derivatives** ↓, and **integrals** ↓.

## Limits of a Functions and Sequences

🌐 Limit of a function 📖 | Limit of a sequence 📖 | Essence of Calculus, Ea 📖

- **Limit of a function**: a fundamental concept in calculus and analysis concerning the behavior of a function near a particular input  $c$ , i.e.,

$$\lim_{x \rightarrow c} f(x) = L$$

- Reads as “ $f$  of  $x$  tends to  $L$  as  $x$  tends to  $c$ ”
- $\epsilon, \delta$  **Limit of a function**: a formalized definition, wherein  $f(x)$  is defined on an open interval  $\mathcal{I}$ , except possibly at  $c$  itself, leading to the informal definition, if and only if

$$f : \mathbb{R} \rightarrow \mathbb{R}, c, L \in \mathbb{R} \Rightarrow \lim_{x \rightarrow c} f(x) = L$$



$$\forall \epsilon > 0 (\exists \delta > 0 : \forall x \in \mathcal{I} (0 < |x - c| < \delta \Rightarrow |f(x) - L| < \epsilon))$$

- Functions **do not have a limit** when the function:
  - has a **unit step**, i.e., it “jumps” at a point;
  - is **not bounded**, i.e., it tends towards infinity;
  - or it **oscillate**, i.e., it does not stay close to any single number.
- **Limit of a sequence**: the value that the terms of a sequence  $(x_n)_{n \in \mathbb{N}}$  “tends to” (and not to any other) as  $n$  approaches infinity (or some other point), i.e.,

$$\lim_{n \rightarrow \infty} x_n = x$$

- $\epsilon$  **Limit of a sequence**: for every measure of closeness  $\epsilon$ , the sequence’s  $x_n$  term eventually converges to the limit, i.e.,

$$\forall \epsilon > 0 (\exists N \in \mathbb{N} (\forall n \in \mathbb{N} (n \geq N \Rightarrow |x_n - x| < \epsilon)))$$

- **Convergent**: when a limit of a sequence **exists**.
- **Divergent**: a sequence that **does not** converge.

## Properties of Limits

📌 List of limits 📌 | Squeeze theorem 📌

- **Operations on a single known limit:** if  $\lim_{x \rightarrow c} f(x) = L$ , then:
  - $\lim_{x \rightarrow c} [f(x) \pm \alpha] = L \pm \alpha$
  - $\lim_{x \rightarrow c} \alpha f(x) = \alpha L$
  - $\lim_{x \rightarrow c} f(x)^{-1} = L^{-1}, L \neq 0$
  - $\lim_{x \rightarrow c} f(x)^n = L^n, n \in \mathbb{N}$
  - $\lim_{x \rightarrow c} f(x)^{n^{-1}} = L^{n^{-1}}, \text{ if } n \in \mathbb{N}_e \Rightarrow L > 0$
- **Operations on two known limits:** if  $\lim_{x \rightarrow c} f(x) = L_1$  and  $\lim_{x \rightarrow c} g(x) = L_2$ , then:
  - $\lim_{x \rightarrow c} [f(x) \pm g(x)] = L_1 \pm L_2$
  - $\lim_{x \rightarrow c} [f(x)g(x)] = L_1 L_2$
  - $\lim_{x \rightarrow c} f(x)g(x)^{-1} = L_1 L_2^{-1}$
- **Squeeze theorem:** used to confirm the limit of a difficult to compute function via comparison with two other functions whose limits are easily known or computed.
  - Let  $\mathcal{I}$  be an interval having the point  $c$  as a limit point.
  - Let  $g, f$ , and  $h$  be functions defined on  $\mathcal{I}$ , except possibly at  $c$  itself.
  - Suppose that  $\forall x \in \mathcal{I} \wedge x \neq c \Rightarrow g(x) \leq f(x) \leq h(x)$
  - and  $\lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} h(x) = L$
  - then,  $\lim_{x \rightarrow c} f(x) = L$
  - Essentially, the hard to compute limit of the “middle function” can be found by finding the limit of two other “easier” functions that “squeeze” the middle function at a point of interest.

## One-Sided Limit

📌 One-Sided Limit 📌

- **One-sided limit:** one of two limits of  $f(x)$  as  $x$  approaches a specified point from either the left or from the right.
- From the left:  $\lim_{x \rightarrow c^-} f(x) = L$
- From the right:  $\lim_{x \rightarrow c^+} f(x) = L$
- If the left and right limits exist and are equal, then

$$\lim_{x \rightarrow c} f(x) = L \Leftrightarrow \lim_{x \rightarrow c^-} f(x) = L \wedge \lim_{x \rightarrow c^+} f(x) = L$$

- Limits can still exist, even if the function is defined at a different point, as long as both one-sided limits approach the same value near the given input.

# Continuity

🌐 Thomas (2.5) 📖

- Continuity of functions is one of the core concepts of topology, however, there are definitions in terms of limits that prove useful; the following is only a primer.

## Continuous Functions

🌐 Continuous function 🔗 | Discontinuities 🔗

- **Continuous function:** a function that does not have any abrupt changes in value.
  - I.e., a function is continuous if and only if arbitrarily small changes in its output can be assured by restricting to sufficiently small changes in its input.
- **Discontinuous:** when a function is not continuous at a point in its domain, leading to a discontinuity; there are three classifications:
  - **Removable:** when both **one-sided limits** <sup>↑</sup> exist, are finite, and are equal, but the actual value of  $f(x)$  is not equal to the limit and instead equal to some other value.
    - The discontinuity can be removed to regain continuity.
    - Sometimes the term *removable discontinuity* is mistaken for a *removable singularity*, or a “whole” in the function (the point is not defined elsewhere).
  - **Jump:** when a single limit does not exist because the one-sided limits exist and are finite, but not equal.
    - Points can be defined at the discontinuity, but the function can not be made continuous.
  - **Essential:** when at least one of the two one-sided limits do not exist; can be the result of oscillating or unbounded functions.

## Intermediate Value Theorem

🌐 Intermediate value theorem 🔗

- **Intermediate value theorem:** if  $f$  is a continuous function whose domain contains the interval  $[a, b]$ , then it **takes on any given value between  $f(a)$  and  $f(b)$**  at some point within the intervals.
- Relevant deductions, i.e., important corollaries:
  - **Bolzano's theorem:** if a continuous function has values of opposite sign inside an interval, then it **has a root** in that interval.
  - The image of a continuous function over an interval is itself an interval.
- Thus, the image set  $f(\mathcal{I})$  (which has no gaps) is also an interval, and it contains:

$$[\min(f(a), f(b)), \max(f(a), f(b))]$$

# Limits Involving Infinity

🌐 Thomas (2.6) 📖

- Let  $S \subset \mathbb{R}$ ,  $x \in S$  and  $f : S \mapsto \mathbb{R}$ , then limits of these functions can approach arbitrarily large ( $\pm$ ) values, providing a connection to asymptotes, and thus, analysis.

## Limits at Infinity and Infinite Limits

🌐 Limits involving infinity 🌀

- **Limits at infinity:** limits defined as  $f(x) \pm$  infinity are defined much like normal limits:

$$\lim_{x \rightarrow -\infty} f(x) = L \quad \lim_{x \rightarrow \infty} f(x) = L$$

- Formally, for all measures of closeness  $\epsilon$  there exists a point  $c$  such that  $|f(x) - L| < \epsilon$  whenever  $x < c \vee x > c$  (respectively), i.e.,

$$\forall \epsilon > 0 (\exists c (\forall x \{< \vee >\} c : |f(x) - L| < \epsilon))$$

- Basic rules for rational functions  $f(x) = p(x)q(x)^{-1}$ , where  $p$  and  $q$  are polynomials, where the degree of each is denoted as  $\{p \vee q\}^\circ$ , and where the leading coefficients are denoted as  $P, Q$ , then:

- $p^\circ > q^\circ \Rightarrow \pm L$ , depending on the sign of the leading coefficient.

- $p^\circ = q^\circ \Rightarrow L = PQ^{-1}$

- $p^\circ < q^\circ \Rightarrow L = 0$

- **Infinite limits:** the usual limit does not exist for a limit that grows out of bounds, however, limits with infinite values can be introduced:

$$\lim_{x \rightarrow c} f(x) = \infty, \quad \text{i.e.,} \quad \forall n > 0 (\exists \delta > 0 : f(x) > n \Leftrightarrow 0 < |x - c| < \delta)$$

## Asymptotes of functions

🌐 Asymptotes 🌀

- **Asymptote:** a tangent line of a curve at a point at infinity; the distance between the curve and the line approaches zero as a coordinate tends to infinity.
- There are three kinds of asymptotes: *horizontal*, *vertical* and *oblique*; the nature of the asymptote is dependent on a function's relation to infinity.

- **Horizontal asymptote:** a result of **limits at infinity**, i.e., when  $x \rightarrow \pm\infty$ .

- **Vertical asymptote:** a result of **infinite limits**, i.e., when  $x \rightarrow \pm c = \pm\infty$

- **Oblique asymptote:** when a linear asymptote is not parallel to either axis;  $f(x)$  is asymptotic to the straight line  $y = mx + n$  ( $m \neq 0$ ) if:

$$\lim_{x \rightarrow \pm\infty} [f(x) - (mx + n)] = 0$$



# Derivatives



## Derivative Fundamentals

🌐 Derivative 📖 | Thomas (3.2, 3.4) 📖

- **Derivative:** the measure of **sensitivity to change** of the function **value** with respect to some change in its **in argument**.
  - Often described as the **instantaneous rate of change** of a single variable function, since it is the slope of a tangent line at a particular point, when it exists.
  - **Tangent line:** the line through a pair of points on a curve (secant line), except the points are **infinitely close**, hence, it is the rate of change at that “instant”.

## Derivative Notation

- Formally, a derivative of the function  $f(x)$ , with respect to the variable  $x$ , is the function  $f'$  whose value at  $x$  is (provided the limit exists):

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

- Let  $z = x + h$ , then  $h = z - x \wedge h \rightarrow 0 \Leftrightarrow z \rightarrow x$ ; this leads to an equivalent definition of the derivative (sometimes more convenient):

$$f'(x) = \lim_{z \rightarrow x} \frac{f(z) - f(x)}{z - x}$$

- **Notation:** there are many ways to denote the derivative; different notation can be useful in various contexts, some common notations (for  $y = f(x)$ ):

$$f'(x) = y' = \dot{y} = \frac{dy}{dx} = \frac{d}{dx}f(x) = D(f)(x) = D_x f(x)$$

- **Differentiation:** the process of finding a derivative; if  $f'$  exists at a particular point, then  $f$  is said to be differentiable at that point.
  - If  $f'$  exists at every point on an interval, then  $f$  is differentiable on that interval.
  - $f'$  is differentiable on a closed interval  $[a, b]$  if both **one-sided limits**  $\uparrow$  of the function ( $h \rightarrow \{0^+ : a, 0^- : b\}$ ) exist at the end points, and it is differentiable on the interior.
  - Not all continuous functions have a derivative, but **functions with a derivative are continuous**; functions with any of the following **do not have derivatives**:
    - **corners** (one-sided derivatives differ at a point),
    - **cusps** (slope approaches alternating  $\pm\infty$  on both sides of a point),
    - **discontinuities**, or **vertical tangent lines**.

# Differentiation Rules

🌐 Differentiation rules 📖 | Thomas (3.3, 3.5, 3.6) 📖

- Derivatives can be found by computing the limit, but there are several methods that use combinations of simpler functions to make computation easier.

## Linear, Product, Chain, Inverse

🌐 Product 📖 | Chain 📖 | Inverse 📖

- Linear:** differentiation of linear functions consists of the constant and sum rules, given the following:

$$\forall (f \wedge g) \wedge \forall (a \wedge b \in \mathbb{R}) \Rightarrow \frac{d(af + bg)}{dx} = a \frac{df}{dx} + b \frac{dg}{dx}$$

**Constant**

$$\frac{d}{dx}(c) = 0$$

**Constant factor**

$$(af)' = af'$$

**Sum / Difference**

$$(f + g)' = f' + g'$$

- Product rule:** used for the product of two functions; can be generalized<sup>↓</sup>

$$\frac{d(fg)}{dx} = g \frac{df}{dx} + f \frac{dg}{dx}$$

- Chain rule:** used for the composition of two functions  $f(g(x))$ ; if  $z$  depends on  $y$ , which is dependent on  $x$ , then  $z$  depends on  $x$  as well, i.e.,

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$$

- The following is used to indicate points of evaluation:

$$\left. \frac{dz}{dx} \right|_x = \left. \frac{dz}{dy} \right|_{y(x)} \cdot \left. \frac{dy}{dx} \right|_x$$

- Outside-Inside rule:** take the derivative of the “outside” function, leave the “inside” alone, and multiply it by the derivative of the “inside.”
- This method must be recursively “chained” when there are further compositions in the inside function, hence the name.
- Inverse function rule:** can be applied if the function  $f$  has an inverse function  $g$ , i.e., “undoes” the effect of  $f$ .

$$\{g(f(x)) = x \wedge f(g(y)) = y\} \Rightarrow \frac{dx}{dy} = \left( \frac{dy}{dx} \right)^{-1}$$

- Application of the chain rule on  $f^{-1}(y) = x$  in terms of  $x$  clearly shows the result, if the derivatives exist and are reciprocal,

$$\frac{dx}{dy} \cdot \frac{dy}{dx} = \frac{dx}{dx} = 1$$

## Power, Polynomial, Reciprocal, Quotient

🌐 Power 🌐 | Reciprocal 🌐 | Quotient 🌐

- **Power rule:** used to differentiate functions in the form of  $f(x) = x^r$ ; can be applied to polynomials since differentiation is linear.
  - Let  $f : \mathbb{R} \mapsto \mathbb{R}$  be a function satisfying

$$f(x) = x^r, \quad \forall x (r \in \mathbb{R}) \Rightarrow \frac{d}{dx} = rx^{r-1}$$

- **Reciprocal rule:** yields the derivative of the reciprocal (multiplicative inverse) of a function  $f$  in terms of the derivative of  $f$ .
  - Can be used to show that the power rule holds for negative exponents.
  - The product and reciprocal rules can be used to deduce the quotient rule.
  - Let  $f$  be differentiable at  $x$  and  $f(x) \neq 0$ , then  $g(x) = f(x)^{-1}$  is also differentiable and

$$\frac{d(f^{-1})}{dx} = -f^{-2} \frac{df}{dx} \quad \text{i.e.,} \quad g' = -\frac{f'}{f^2}$$

- **Quotient rule:** used to find the derivative of a function that is a ratio of two differentiable functions.
  - Let  $f$  and  $g$  be differentiable and  $g(x) \neq 0$ , then

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

## Trigonometric Differentiation

🌐 Trigonometric functions 🌐

- All derivatives of circular trigonometric functions can be found from those of  $\sin(x)$  and  $\cos(x)$  by means of the quotient rule.

$\sin(x) \rightarrow \cos(x)$	$\arcsin(x) \rightarrow \left(\sqrt{1-x^2}\right)^{-1}$
$\cos(x) \rightarrow -\sin(x)$	$\arccos(x) \rightarrow -\left(\sqrt{1-x^2}\right)^{-1}$
$\tan(x) \rightarrow \sec^2(x)$	$\arctan(x) \rightarrow \left(x^2+1\right)^{-1}$
$\cot(x) \rightarrow -\csc^2(x)$	$\operatorname{arccot}(x) \rightarrow -\left(x^2+1\right)^{-1}$
$\sec(x) \rightarrow \sec(x)\tan(x)$	$\operatorname{arcsec}(x) \rightarrow \left( x \sqrt{x^2-1}\right)^{-1}$
$\csc(x) \rightarrow -\csc(x)\cot(x)$	$\operatorname{arccsc}(x) \rightarrow -\left( x \sqrt{x^2-1}\right)^{-1}$

- Inverse trigonometric functions are found using **implicit differentiation** ↓.

# Differentials and Related Concepts

- Differentials

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## Linearization

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## Implicit Differentiation

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## Related Rates

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# Applications of Derivatives



## Stationary Point

### Maxima and Minima

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### Extreme Value Theorem

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### Interior Extremum Theorem

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# Mean Value Theorem

## Rolle's Theorem

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## Corollaries of the Mean Value Theorem

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## Monotonic Functions

# Derivative Tests

## First-Derivative Test

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## Second-Derivative Test

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## Concavity

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## Higher-Order Derivative Test

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# Differential Methods

## Newton's Method

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## Taylor's Theorem

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## General Leibniz Rule

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# Integrals



## Integral Fundamentals

### Terminology and Notation

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### Primer: Formal Definitions

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# Definite Integrals

## Riemann Integral

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## Integrability

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## Properties of Definite Integrals

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# The Fundamental Theorem of Calculus

## Fundamental Theorem, Part 1

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## Fundamental Theorem, Part 2

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## The Integral of a Rate

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## Total Area

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# Integration By Substitution

## Indefinite Integrals

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## Definite Integrals

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## Symmetric Functions

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## Area Between Curves

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# Applications of Definite Integrals



## Solid of Revolution

### Disc Integration

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### Shell Integration

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# Arc Length

## Dealing with Discontinuities

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## Differential Arc Length

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# Surface of Revolution

## Revolution about the y-Axis

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# Transcendental Functions



## Inverse Functions

### One-to-One Functions

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### Derivative Rule for Inverses

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# Logarithmic Functions

## Natural Logarithm

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## Properties of Logarithms

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## Trigonometric Integrals

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## Logarithmic Differentiation

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# Exponential Functions

## Euler's Number

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## Natural Exponential Function

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## Laws of Exponents

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## General Exponential Function

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# Exponential Change

- Separable Differential Equations

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## Examples of Exponential Change

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# Indeterminate Forms

## Indeterminate Form 0/0

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## L'Hôpital's Rule

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## Infinite Indeterminate Forms

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## Indeterminate Powers

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# Inverse Trigonometric Functions

## Principal Trigonometric Values

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## Inverse Trigonometric Tables

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# Hyperbolic Functions

## Hyperbolic Function Tables

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# Techniques of Integration



## Integration by Parts

### Definite Integrals by Parts

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# Trigonometric Integral Methods

## Trigonometric Products and Powers

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## Trigonometric Square Roots

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## Trigonometric Substitutions

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# Partial Fraction Decomposition

## Partial Fraction Principles

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## General Statement

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# Numerical Integration

## Trapezoidal Rule

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## Simpson's Rule

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# Improper Integrals

## Indirect Evaluation

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# Infinite Sequences and Series



# First-Order Differential Equations



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# Parametric Equations and Polar Coordinates



# Vectors and Vector-Valued Functions



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# Partial Derivatives





# Multiple Integrals



# Vector Calculus



# Second-Order Differential Equations

