1. Show that if $\boldsymbol{B}$ has a column of zeros, so too does $\boldsymbol{AB}$.

- Visualizing matrix multiplication as building of the product matrix via scaling the columns of the left matrix by the columns of the right matrix allows for a good example of why the above is true, e.g.,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} \alpha & 0 \\ \lambda & 0 \end{bmatrix} = \left(\alpha\begin{bmatrix} a \\ c \end{bmatrix} + \lambda\begin{bmatrix} b \\ d \end{bmatrix} \quad 0\begin{bmatrix} a \\ c \end{bmatrix} + 0\begin{bmatrix} b \\ d \end{bmatrix}\right) = \begin{bmatrix} \alpha a + \lambda b & 0+0 \\ \alpha c + \lambda d & 0+0 \end{bmatrix}$$

- Under standard matrix multiplication, the product matrix dimensions are equal to the rows of the left matrix $\times$ the columns of the right matrix—thus, the product matrix must have a column of zeros if the right matrix contains a column of zeros.

2. Create an example where $\boldsymbol{AB}$ has a column of zeros, but $\boldsymbol{B}$ does not.

- Again, the column perspective is very useful here—if the scaled columns of the left matrix summed together equal zero, then the entire column will be zero, e.g.,

$$\begin{bmatrix} 4 & 2 \\ 6 & 3 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 1 & -2 \end{bmatrix} = \left(3\begin{bmatrix} 4 \\ 6 \end{bmatrix} + 1\begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad 1\begin{bmatrix} 4 \\ 6 \end{bmatrix} + -2\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} (12+2) & (4-4) \\ (18+3) & (6-6) \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 0 \\ 21 & 0 \end{bmatrix}$$

3. For two numbers a and b, note it's always true that

$$(a+b)^2 = a^2 + 2ab + b^2$$

Find two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ so that

$$(\boldsymbol{A}+\boldsymbol{B})^2 \neq \boldsymbol{A}^2 + 2\boldsymbol{AB} + \boldsymbol{B}^2$$

- The above is true if and only if $\boldsymbol{AB} = \boldsymbol{BA}$, i.e.,

$$(\boldsymbol{A}+\boldsymbol{B})^2 = \boldsymbol{A}(\boldsymbol{A}+\boldsymbol{B}) + \boldsymbol{B}(\boldsymbol{A}+\boldsymbol{B}) = \boldsymbol{A}^2 + \boldsymbol{AB} + \boldsymbol{BA} + \boldsymbol{B}^2$$

  - This is true if $\boldsymbol{B}$ is the identity matrix, in which case $\boldsymbol{AI} = \boldsymbol{IA} = \boldsymbol{A}$.
  - Or if $\boldsymbol{A}$ is invertible and $\boldsymbol{B}$ is equal to the inverse, in which case $\boldsymbol{AA}^{-1} = \boldsymbol{I} = \boldsymbol{A}^{-1}\boldsymbol{A}$

- Thus, any two square matrices of equal size such that $\boldsymbol{AB} \neq \boldsymbol{BA}$ will yield a counter example, e.g,

$$\boldsymbol{A} = \begin{bmatrix} 4 & 2 & 0 \\ 0 & 0 & 6 \\ 0 & 0 & 9 \end{bmatrix} \quad \begin{bmatrix} 3 & 1 & 4 \\ 0 & 0 & 0 \\ 8 & 2 & 6 \end{bmatrix} = \boldsymbol{B}$$

$$\boldsymbol{AB} = \begin{bmatrix} 12 & 4 & 16 \\ 48 & 12 & 36 \\ 72 & 18 & 54 \end{bmatrix} \neq \begin{bmatrix} 12 & 6 & 42 \\ 0 & 0 & 0 \\ 32 & 16 & 66 \end{bmatrix} = \boldsymbol{BA}$$

$$2\boldsymbol{AB} = \begin{bmatrix} 24 & 8 & 32 \\ 96 & 24 & 72 \\ 144 & 36 & 108 \end{bmatrix} \neq \begin{bmatrix} 24 & 10 & 58 \\ 48 & 12 & 36 \\ 104 & 34 & 120 \end{bmatrix} = \boldsymbol{AB} + \boldsymbol{BA}$$

4. Let A and B denote invertible $n \times n$ matrices. Show that if $A^{-1} = B^{-1}$, then $A = B$.

$$A^{-1} = B^{-1}$$

$$\downarrow$$

$$A^{-1}A = AA^{-1} = AB^{-1} = I = B^{-1}A = BB^{-1} = B^{-1}B$$

$$\downarrow$$

$$A = AI = A(B^{-1}B) = (AB^{-1})B = IB = B$$