Calculus

Limits and Continuity

Derivatives

Applications of Derivatives

Limits and Continuity

Limits

Limit | Thomas (2.2–2.4)

- **Limit** $\lim_{x \to c}$: the value of a function (or sequence) as the input (or index) approaches some value (note: an informal definition).
  - Limits are used to define continuity↓, derivatives↓, and integrals↓.

Limits of a Functions and Sequences

Limit of a function | Limit of a sequence | Essence of Calculus, Ea

- **Limit of a function**: a fundamental concept in calculus and analysis concerning the behavior of a function near a particular input c, i.e.,

$$\lim_{x \to c} f(x) = L$$

  - Reads as "f of x tends to L as x tends to c"

- ϵ, δ **Limit of a function**: a formalized definition, wherein $f(x)$ is defined on an open interval $\mathcal{I}$, except possibly at c itself, leading to the informal definition, if and only if

$$f : \mathbb{R} \to \mathbb{R}, c, L \in \mathbb{R} \Rightarrow \lim_{x \to c} f(x) = L$$

$$\Updownarrow$$

$$\forall \varepsilon > 0 \, (\exists \delta > 0 : \forall x \in \mathcal{I} \, (0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon))$$

- Functions do not have a limit when the function:
  - has a unit step, i.e., it "jumps" at a point;
  - is not bounded, i.e., it tends towards infinity;
  - or it oscillate, i.e., it does not stay close to any single number.

- **Limit of a sequence**: the value that the terms of a sequence $(x_n)_{n \in \mathbb{N}}$ "tends to" (and not to any other) as n approaches infinity (or some other point), i.e.,

$$\lim_{n \to \infty} x_n = x$$

- $\mathcal{E}$ **Limit of a sequence**: for every measure of closeness $\mathcal{E}$, the sequence's x_n term eventually converges to the limit, i.e.,

$$\forall \varepsilon > 0 \, (\exists N \in \mathbb{N} \, (\forall n \in \mathbb{N} \, (n \geq N \Rightarrow |x_n - x| < \varepsilon)))$$

  - **Convergent**: when a limit of a sequence exists.
  - **Divergent**: a sequence that does not converge.

Properties of Limits

List of limits | Squeeze theorem

- **Operations on a single known limit**: if $\lim_{x \to c} f(x) = L$, then:
  - $\lim_{x \to c} \left[f(x) \pm \alpha \right] = L \pm \alpha$
  - $\lim_{x \to c} \alpha f(x) = \alpha L$
  - $\lim_{x \to c} f(x)^{-1} = L^{-1}, L \neq 0$
  - $\lim_{x \to c} f(x)^{n} = L^{n}, n \in \mathbb{N}$
  - $\lim_{x \to c} f(x)^{n^{-1}} = L^{n^{-1}}$, if $n \in \mathbb{N}_e \Rightarrow L > 0$
- **Operations on two known limits**: if $\lim_{x \to c}$ and $\lim_{x \to c} g(x) = L_2$, then:
  - $\lim_{x \to c} \left[f(x) \pm g(x) \right] = L_1 \pm L_2$
  - $\lim_{x \to c} [f(x)g(x)] = L_1 L_2$
  - $\lim_{x \to c} f(x)g(x)^{-1} = L_1 L_2^{-1}$
- **Squeeze theorem**: used to confirm the limit of a difficult to compute function via comparison with two other functions whose limits are easily known or computed.
  - Let $\mathcal{I}$ be an interval having the point c as a limit point.
  - Let g, f, and h be functions defined on $\mathcal{I}$, except possibly at c itself.
  - Suppose that $\forall x \in \mathcal{I} \land x \neq \ \Rightarrow g(x) \leq f(x) \leq h(x)$
  - and $\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L$
  - then, $\lim_{x \to c} f(x) = L$
  - Essentially, the hard to compute limit of the "middle function" can be found by finding the limit of two other "easier" functions that that "squeeze" the middle function at a point of interest.

One-Sided Limit

One-Sided Limit

- **One-sided limit**: one of two limits of $f(x)$ as x approaches a specified point from either the left or from the right right.
  - From the left: $\lim_{x \to c^-} = L$
  - From the right: $\lim_{x \to c^+} = L$
- If the left and right limits exist and are equal, then

$$\lim_{x \to c} f(x) = L \Leftrightarrow \lim_{x \to c^-} f(x) = L \land \lim_{x \to c^+} f(x) = L$$

- Limits can still exist, even if the function is defined at a different point, as long as both one-sided limits approach the same value near the given input.

Continuity

Continuous Functions

-

Intermediate Value Theorem

-

Limits Involving Infinity

Limits at Infinity and Infinite Limits

-

Asymptotes of functions

-

Derivatives

Derivative Fundamentals

Derivative Notation

- ...

Differentiation Rules

Linear, Product, Chain, Inverse

-

Powers, Polynomials, Quotients, Reciprocals

-

Exponential, Logarithmic

-

Trigonometric, Hyperbolic

-

Differentials and Related Concepts

Differentials

-

Linearization

-

Implicit Differentiation

-

Related Rates

-

Applications of Derivatives

Stationary Point

Maxima and Minima

-

Extreme Value Theorem

-

Interior Extremum Theorem

-

Mean Value Theorem

Rolle's Theorem

- ◦

Corollaries of the Mean Value Theorem

- ◦

Monotonic Functions

Derivative Tests

First-Derivative Test

-

Second-Derivative Test

-

Concavity

-

Higher-Order Derivative Test

-

Differential Methods

Newton’s Method

-

Taylor’s Theorem

-

General Leibniz Rule

-

Integrals

Integral Fundamentals

Terminology and Notation

-

Primer: Formal Definitions

-

Definite Integrals

Riemann Integral

-

Integrability

-

Properties of Definite Integrals

-

The Fundamental Theorem of Calculus

Fundamental Theorem, Part 1

-

Fundamental Theorem, Part 2

-

The Integral of a Rate

-

Total Area

-

Integration By Substitution

Indefinite Integrals

-

Definite Integrals

-

Symmetric Functions

-

Area Between Curves

-

Applications of Definite Integrals

Solid of Revolution

Disc Integration

-

Shell Integration

-

Arc Length

Dealing with Discontinuities

-

Differential Arc Length

-

Surface of Revolution

Revolution about the y-Axis

- ◦

Transcendental Functions

Inverse Functions

One-to-One Functions

-

Derivative Rule for Inverses

-

Logarithmic Functions

Natural Logarithm

-

Properties of Logarithms

-

Trigonometric Integrals

-

Logarithmic Differentiation

-

Exponential Functions

Euler's Number

-

Natural Exponential Function

-

Laws of Exponents

-

General Exponential Function

-

Exponential Change

- **Separable Differential Equations**
    -

  Examples of Exponential Change
    -

Indeterminate Forms

Indeterminate Form 0/0

-

L'Hôpital's Rule

-

Infinite Indeterminate Forms

-

Indeterminate Powers

-

Inverse Trigonometric Functions

Principal Trigonometric Values

-

Inverse Trigonometric Tables

-

Hyperbolic Functions

Hyperbolic Function Tables

-

Integration by Parts

Definite Integrals by Parts

-

Trigonometric Integral Methods

Trigonometric Products and Powers

-

Trigonometric Square Roots

-

Trigonometric Substitutions

-

Partial Fraction Decomposition

Partial Fraction Principles

-

General Statement

-

Numerical Integration

Trapezoidal Rule

-

Simpson's Rule

-

Improper Integrals

Indirect Evaluation

-