

# CALCULUS III FINAL REVIEW

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## FINAL REVIEW QUESTIONS

# CONVERGENCE: 10.3–10.5

## Convergence Notes

- Let  $\sum_{n=1}^{\infty} a_n$  be given and note for which series convergence is known, i.e.:

**Geometric:** let  $c \neq 0$ , if  $|r| < 1$ , then

**p-Series:** converges if  $p > 1$ .

$$\sum_{n=0}^{\infty} cr^n = \frac{c}{1-r}$$

$$\sum_{n=0}^{\infty} \frac{1}{n^p}$$

$|r| \geq 1 \implies$  diverges

$p \leq 1 \implies$  diverges

- The  $n^{\text{th}}$  Term Divergence Test:** a relatively easy test that can be used to quickly determine if a test diverges if the  $\lim_{n \rightarrow \infty} a_n \neq 0$ . If  $\lim_{n \rightarrow \infty} a_n = 0$ , then the test is inconclusive and other tests must be applied.

## Tests for Positive Series

- Direct Comparison Test:** use if dropping terms from the denominator or numerator gives a series  $b_n$  wherein convergence is easily found, then compare to the original series  $a_n$  as follows:

$$\sum_{n=1}^{\infty} b_n \text{ converges} \implies \sum_{n=1}^{\infty} a_n \text{ converges} \quad \leftarrow 0 \leq a_n \leq b_n$$

$$\sum_{n=1}^{\infty} b_n \text{ diverges} \implies \sum_{n=1}^{\infty} a_n \text{ diverges} \quad \leftarrow 0 \leq b_n \leq a_n$$

- Limit Comparison Test:** use when the direct comparison test isn't convenient or when comparing two series. One can take the dominant term in the numerator and denominator from  $a_n$  to form a new positive sequence  $b_n$  if needed.

Assuming the following limit  $L = \lim_{n \rightarrow \infty} \frac{a_n}{b_n}$  exists, then:

$$L > 0 \implies \sum_{n=1}^{\infty} a_n \text{ converges} \iff \sum_{n=1}^{\infty} b_n \text{ converges}$$

$$L = 0 \text{ and } \sum_{n=1}^{\infty} b_n \text{ converges} \implies \sum_{n=1}^{\infty} a_n \text{ converges}$$

$$L = \infty \text{ and } \sum_{n=1}^{\infty} a_n \text{ converges} \implies \sum_{n=1}^{\infty} b_n \text{ converges}$$

- **Ratio Test:** often used in the presence of a factorial ( $n!$ ) or when the are constants raised to the power of  $n$  ( $c^n$ ).

Assuming the following limit  $\rho = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$  exists, then

$$\rho < 1 \implies \sum a_n \text{ converges absolutely}$$

$$\rho > 1 \implies \sum a_n \text{ diverges}$$

$$\rho = 1 \implies \text{test is inconclusive}$$

- **Root Test:** used when there is a term in the form of  $f(n)^{g(n)}$ .

Assuming the following limit  $C = \lim_{n \rightarrow \infty} |a_n|^{\frac{1}{n}}$  exists, then

$$C < 1 \implies \sum a_n \text{ converges absolutely}$$

$$C > 1 \implies \sum a_n \text{ diverges}$$

$$C = 1 \implies \text{test is inconclusive}$$

- **Integral Test:** if the other tests fail and  $a_n = f(n)$  is a decreasing function, then one can use the improper integral  $\int_1^\infty f(x)dx$  to test for convergence.

Let  $a_n = f(n)$  be a positive, decreasing, and continuous function  $\forall x \geq 1$ , then:

$$\int_1^\infty f(x)dx \text{ converges} \implies \sum_{n=1}^\infty a_n \text{ converges}$$

$$\int_1^\infty f(x)dx \text{ diverges} \implies \sum_{n=1}^\infty a_n \text{ diverges}$$

## Tests for Non-Positive Series

- **Alternating Series Test:** used for series in the form  $\sum_{n=0}^\infty (-1)^n a_n$

Converges if  $|a_n|$  decreases monotonically ( $|a_{n+1}| \leq |a_n|$ ) and if  $\lim_{n \rightarrow \infty} a_n = 0$

- **Absolute Convergence:** used if the series  $\sum a_n$  is not alternating (if it is alternating, use the alternating test in conjunction); simply test if  $\sum |a_n|$  converges using the test for positive series.

$\sum a_n$  converges **conditionally** if  $\sum a_n$  converges, but  $\sum |a_n|$  diverges.

$\sum a_n$  converges **absolutely** if  $\sum |a_n|$  converges.

# Convergence Problems

## 10.5 Exercises

Determine convergence or divergence using any method.

$$1. \sum_{n=1}^{\infty} \frac{2^n + 4^n}{7^n}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{2^n}{7^n} + \sum_{n=1}^{\infty} \frac{4^n}{7^n}$$

Separate into two geometric series<sup>†</sup>

$$\Rightarrow r = \frac{2}{7} < 1, \quad r = \frac{4}{7} < 1$$

Both geometric series converge, thus the original series **converges**.

$$2. \sum_{n=1}^{\infty} \frac{n^3}{n!}$$

$$\Rightarrow \rho = \lim_{n \rightarrow \infty} \left| \frac{(n+1)^3}{(n+1)!} \cdot \frac{n!}{n^3} \right|$$

Apply the ratio test<sup>†</sup>

$$= \lim_{n \rightarrow \infty} \frac{n^3 + 3n^2 + 3n + 1}{(n+1)n!} \cdot \frac{n!}{n^3}$$

$$= \lim_{n \rightarrow \infty} \frac{n^3 + 3n^2 + 3n + 1}{n^4 + n^3}$$

$$= \lim_{n \rightarrow \infty} \frac{n^3 + 3n^2 + 3n + 1}{n^4 + n^3} \cdot \frac{n^{-4}}{n^{-4}}$$

$$= \lim_{n \rightarrow \infty} \frac{n^{-1} + 3n^{-2} + 3n^{-3} + n^{-4}}{1 + n^{-1}} = 0$$

$\rho = 0 < 1$ , thus the series **converges**.

$$3. \sum_{n=1}^{\infty} \frac{n}{2n+1}$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{n}{2n+1}$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{n}{2n+1} = \frac{1}{2}$$

Apply the  $n^{th}$  term test<sup>†</sup>

By L'Hôpital's Rule

$\lim_{n \rightarrow \infty} a_n \neq 0$ , thus the series **diverges**.

$$4. \sum_{n=1}^{\infty} 2^{\frac{1}{n}}$$

$$\Rightarrow \lim_{n \rightarrow \infty} 2^{\frac{1}{n}} = 2^0 = 1$$

Apply the  $n^{th}$  term test<sup>†</sup>

$\lim_{n \rightarrow \infty} a_n \neq 0$ , thus the series **diverges**.

$$5. \sum_{n=1}^{\infty} \frac{\sin n}{n^2}$$

$$0 \leq \sin n \leq 1$$

$$\leftarrow \forall n \geq 1$$

$$0 \leq \frac{\sin n}{n^2} \leq \frac{1}{n^2}$$

Apply the **direct comparison test**<sup>†</sup>

$$b_n = \frac{1}{n^2} \rightarrow \text{converges}$$

by  **$p$ -series**<sup>†</sup>

The larger  $(b_n)$  series converges, thus the smaller  $(a_n)$  **converges**.

$$6. \sum_{n=1}^{\infty} \frac{n!}{(2n)!}$$

$$\Rightarrow \rho = \lim_{n \rightarrow \infty} \left| \frac{(n+1)!}{(2n+2)!} \cdot \frac{(2n)!}{n!} \right|$$

Apply the **ratio test**  $\uparrow$

$$= \lim_{n \rightarrow \infty} \frac{(n+1)n!}{(2n+2)(2n+1)2n!} \cdot \frac{(2n)!}{n!}$$

$$= \lim_{n \rightarrow \infty} \frac{n+1}{(2n+2)(2n+1)} = \frac{n+1}{4n^2 + 6n + 2}$$

$$= \lim_{n \rightarrow \infty} \frac{1}{8n+6} = 0$$

By L'Hôpital's Rule

$\rho = 0 < 1$ , thus the series **converges**.

$$7. \sum_{n=1}^{\infty} \frac{1}{n + \sqrt{n}}$$

$$0 \leq n \leq n + \sqrt{n} \quad \leftarrow \forall n \geq 1$$

$$0 \leq \frac{1}{n + \sqrt{n}} \leq \frac{1}{n}$$

Apply the **direct comparison test**  $\uparrow$

$$b_n = \frac{1}{n} \rightarrow \text{diverges}$$

The smaller ( $b_n$ ) series diverges, thus the larger  $a_n$  original series **diverges**.

$$8. \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$$

$f$  is positive, decreasing, and continuous for  $x \geq 2$  Apply the **integral test** <sup>†</sup>

$$\Rightarrow \int_2^{\infty} f(x) dx = \lim_{R \rightarrow \infty} \int_2^R \frac{1}{x(\ln x)^3} dx \quad \ln x = u, \quad x du = dx$$

$$\begin{aligned} \Rightarrow \lim_{R \rightarrow \infty} \int_2^R \frac{1}{x(u)^3} x du &= \int_2^R \frac{1^3}{u} du \\ &= -\frac{1}{2(u)^2} \\ &= -\frac{1}{2 \ln^2(x)} + C \Big|_2^{\infty} \end{aligned}$$

$$\Rightarrow 0 - \left( -\frac{1}{2 \ln^2(2)} \right) = \frac{1}{2 \ln^2(2)}$$

The improper integral converges, thus the original series **converges**.

$$9. \sum_{n=1}^{\infty} \frac{n^3}{5^n}$$

$$\begin{aligned} \Rightarrow \rho &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)^3}{5^{n+1}} \cdot \frac{5^n}{n^3} \right| \\ &= \lim_{n \rightarrow \infty} \frac{n^3 \cdot 1}{5^n \cdot 5^1} \cdot \frac{5^n}{n^3} = \frac{1}{5} \end{aligned}$$

Apply the **ratio test** <sup>†</sup>

$\rho = \frac{1}{5} < 1$ , thus the series **converges**.

$$10. \sum_{n=2}^{\infty} \frac{1}{\sqrt{n^3 - n^2}}$$

$$L = \lim_{n \rightarrow \infty} \frac{a_n}{b_n}, \quad b_n = \frac{1}{\sqrt{n^3}}$$

Apply the [limit comparison test](#) <sup>†</sup>

$$\Rightarrow L = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n^3 - n^2}} \cdot \frac{\sqrt{n^3}}{1}$$

$$= \lim_{n \rightarrow \infty} \sqrt{\frac{n^3}{n^3(1 - n^{-1})}}$$

$$= \sqrt{\frac{1}{1(1 - 0)}} = 1$$

$L > 0$ , thus  $a_n$  converges if  $b_n$  converges.

$b_n$  converges by the  $p$ -series test, as  $\frac{3}{2} > 1$ , thus  $a_n$  [converges](#).

$$11. \sum_{n=1}^{\infty} \frac{n^2 + 4n}{3n^4 + 9}$$

$$L = \lim_{n \rightarrow \infty} \frac{a_n}{b_n}, \quad b_n = \frac{1}{n^2}$$

Apply the [limit comparison test](#) <sup>†</sup>

$$= \lim_{n \rightarrow \infty} \frac{n^2 + 4n}{3n^4 + 9} \cdot n^2$$

$$= \lim_{n \rightarrow \infty} \frac{n^4 + 4n^3}{3n^4 + 9} \cdot \frac{n^{-4}}{n^{-4}}$$

$$= \lim_{n \rightarrow \infty} \frac{1 + 4n^{-1}}{3 + 9n^{-4}} = \frac{1}{3}$$

$L > 0$ , thus  $a_n$  converges if  $b_n$  converges.

$b_n$  converges by the  $p$ -series test, as  $2 > 1$ , thus  $a_n$  [converges](#).



$$12. \sum_{n=1}^{\infty} (0.8)^{-n} n^{-0.8}$$

$$\begin{aligned} \rho &= \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{(0.8)^{-(n+1)} (n+1)^{-0.8}}{(0.8)^{-n} n^{-0.8}} \right| \\ &= \lim_{n \rightarrow \infty} \frac{(0.8)^{-n} \cdot 0.8^{-1} \cdot n^{-0.8}}{(0.8)^{-n} n^{-0.8}} \\ &= \lim_{n \rightarrow \infty} 0.8^{-1} = 1.25 \end{aligned}$$

Apply the ratio test  $\uparrow$

$\rho = 1.25 > 1$ , thus  $a_n$  diverges.

$$13. \sum_{n=1}^{\infty} 4^{-2n+1}$$

$$\begin{aligned} \rho &= \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \\ &= \lim_{n \rightarrow \infty} \frac{4^{-2(n+1)+1}}{4^{-2n+1}} \\ &= \lim_{n \rightarrow \infty} \frac{4^{-2n-1}}{4^{-2n+1}} \\ &= \lim_{n \rightarrow \infty} \frac{4^{-2n} 4^{-1}}{4^{-2n} 4} = \frac{1}{16} \end{aligned}$$

Apply the ratio test  $\uparrow$

$\rho = \frac{1}{16} < 1$ , thus  $a_n$  converges.

$$14. \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$$

$$\sum_{n=1}^{\infty} |a_n|$$

Apply the **Absolute convergence test** <sup>†</sup>

$$\Rightarrow \sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$$

$|a_n|$  diverges by the  $p$ -series, as  $\frac{1}{2} < 1$ , meaning  $a_n$  **converges conditionally** since  $|a_n|$  decreases monotonically and  $\lim_{n \rightarrow \infty} a_n = 0$

$$15. \sum_{n=1}^{\infty} \sin \frac{1}{n^2}$$

$$L = \lim_{n \rightarrow \infty} \frac{a_n}{b_n}, \quad b_n = \frac{1}{n^2}$$

Apply the **limit comparison test** <sup>†</sup>

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{\sin(n^{-2})}{n^{-2}} = \frac{0}{0}$$

$$= \lim_{n \rightarrow \infty} \frac{\cos(n^{-2})(-2n^{-3})}{-2n^{-3}}$$

by L'Hôpital's Rule

$$= \lim_{n \rightarrow \infty} \cos(n^{-2}) = 1$$

$L > 0$ , thus  $a_n$  converges if  $b_n$  converges.

$b_n$  converges by the  $p$ -series test, as  $2 > 1$ , thus  $a_n$  **converges**.

$$16. \sum_{n=1}^{\infty} (-1)^n \cos n^{-1}$$

$$\sum_{n=1}^{\infty} (-1)^n a_n, \quad a_n = \cos(n^{-1})$$

Apply the **alternating series test** <sup>†</sup>

$$\Rightarrow L = \lim_{n \rightarrow \infty} \cos(n^{-1}) = 1$$

$L \neq 0$ , thus the series **diverges**

$$17. \sum_{n=1}^{\infty} \frac{(-2)^n}{\sqrt{n}}$$

$$\sum_{n=1}^{\infty} (-1)^n a_n, \quad a_n = \frac{2^n}{\sqrt{n}}$$

Apply the **alternating series test** <sup>†</sup>

$$\begin{aligned} \Rightarrow L &= \lim_{n \rightarrow \infty} \frac{2^n}{\sqrt{n}} = \frac{\infty}{\infty} \\ &= \frac{2^n \ln 2}{\frac{1}{2} n^{-\frac{1}{2}}} = 2^n \ln 2 \cdot 2\sqrt{n} \quad \text{By L'Hôpital's Rule} \\ &= 2 \lim_{n \rightarrow \infty} 2^n \ln(2) \sqrt{n} = \infty \end{aligned}$$

$L \neq 0$ , thus the series **diverges**

$$18. \sum_{n=1}^{\infty} \left( \frac{n}{n+12} \right)^n$$

$$L = \lim_{n \rightarrow \infty} a_n \neq 0 \rightarrow \text{diverges} \quad \text{Apply the } n^{\text{th}} \text{ term test}^{\dagger}$$

$$\Rightarrow L = \lim_{n \rightarrow \infty} \left( \frac{n}{n+12} \right)^n$$

$$= \lim_{n \rightarrow \infty} e^{-12}$$

$$\text{By common limit } \left( \frac{x}{x+k} \right)^x = e^{-k}$$

$L \neq 0$ , thus the series **diverges**.

# POWER/TAYLOR SERIES: 10.6–10.8

## Power/Taylor Series Notes

### Power Series

- **Power series:** a infinite series in the form:

$$F(x) = \sum_{n=0}^{\infty} a_n(x - c)^n$$

Where the constant  $c$  is the *center* of the power series  $F(x)$ .

- **Radius of convergence  $R$ :** the range of values of the variable  $x$  whereby the power series  $F(x)$  converges.
  - Every power series converges at  $x = c$ , as  $(x - c)^0 = 1$ , though the series may diverge for other values of  $x$ .
  - $F(x)$  converges for  $|x - c| < R$  and diverges for  $|x - c| > R$
  - $F(x)$  may converge or diverge at endpoints  $c - R$  and  $c + R$
- **Interval of convergence:** the open interval  $(c - R, c + R)$  and possibly one of both of the endpoints, each must be tested.
  - In most cases, the **ratio test**† can be used to find  $R$ .
  - If  $R > 0$ , then  $F$  is differentiable over the interval of convergence; the derivative and antiderivative can be obtained using the following:

$$F'(x) = \sum_{n=1}^{\infty} n a_n (x - c)^{n-1} \qquad F(x) dx = C + \sum_{n=0}^{\infty} \frac{a_n}{n+1} (x - c)^{n+1}$$

## Taylor Series

- **Taylor series:** the power series of a infinitely differentiable function  $f(x)$  centered at  $c$ ,

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x - c)^n$$

- **$n^{th}$  Taylor polynomial:** a polynomial of degree  $n$  that is formed partial sum formed by the first  $n + 1$  terms of a Taylor series, i.e.,

$$f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \frac{f'''(c)}{3!}(x - c)^3 + \cdots + \frac{f^{(n)}(c)}{n!}(x - c)^n$$

- **Maclaurin series:** when  $c = 0$ , i.e.,

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} (x)$$

- **Useful Maclaurin Series:** useful Taylor series centered at 0 that can be used to derive other series via differentiation, integration, multiplication, or substitution.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \leftarrow \forall x$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad \leftarrow \forall x$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \quad \leftarrow \forall x$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \leftarrow |x| < 1$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n \quad \leftarrow |x| < 1$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} \quad \leftarrow |x| < 1 \wedge x \neq -1$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \quad \leftarrow |x| \leq 1$$

$$(1+x)^{\alpha} = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n \quad \leftarrow |x| < 1$$

$$\text{where } \binom{\alpha}{n} = \sum_{k=0}^{\infty} \prod_{k=1}^n \frac{\alpha - k + 1}{k}$$

# Power/Taylor Series Problems

## 10.6 Exercises

Find the interval of convergence.

1.  $\sum_{n=0}^{\infty} (-1)^n \frac{n}{4^n} x^{2n}$

$$\begin{aligned}\rho &= \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| && \text{Apply the ratio test}^\uparrow \\&= \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} (n+1) (x^{2(n+1)})}{4^{n+1}} \cdot \frac{4^n}{(-1)^n n (x^{2n})} \right| \\&= \lim_{n \rightarrow \infty} \frac{(n+1) (|x|^{2n} \cdot |x|^2)}{4^n \cdot 4} \cdot \frac{4^n}{n \cdot |x|^{2n}} \\&= \lim_{n \rightarrow \infty} \frac{(n+1) |x|^2}{4n} \cdot \frac{n^{-1}}{n^{-1}} \\&= \lim_{n \rightarrow \infty} \frac{(1 + n^{-1}) |x|^2}{4} = \frac{|x|^2}{4} \\&\Rightarrow \frac{|x|^2}{4} < 1 \Rightarrow |x| < 2 && \text{converges for } \rho < 1\end{aligned}$$

Both endpoints are alternating and tend toward  $\infty$  (diverge),  $R = 2$ ,  $c = 0$ , thus the interval of convergence is  $(-2, 2)$ .

2.  $\sum_{n=8}^{\infty} n^7 x^n$

$$\begin{aligned}\rho &= \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| && \text{Apply the ratio test}^\uparrow \\&\Rightarrow \lim_{n \rightarrow \infty} \left| \frac{(n+1)^7 x^{n+1}}{n^7 x^n} \right| \\&= \lim_{n \rightarrow \infty} \frac{(n+1)^n \cdot |x|^n \cdot |x|}{n^7 |x|^n} \\&= \lim_{n \rightarrow \infty} \frac{(n+1)^n |x|}{n^7} \cdot \frac{n^{-7}}{n^{-7}} \\&= \lim_{n \rightarrow \infty} (1 + n^{-1})^{-7} |x| = |x| \\&\Rightarrow |x| < 1 && \text{converges for } \rho < 1\end{aligned}$$

Both endpoints tend toward  $\infty$  (diverge),  $R = 1$ ,  $c = 0$ , thus the interval of convergence is  $(-1, 1)$ .

$$3. \sum_{n=2}^{\infty} \frac{x^n}{\ln n}$$

$$\rho = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

Apply the **ratio test**  $\uparrow$

$$\Rightarrow \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{\ln(n+1)} \cdot \frac{\ln n}{x^n} \right|$$

$$= \lim_{n \rightarrow \infty} \frac{|x| \ln n}{\ln(n+1)}$$

$$= |x| \lim_{n \rightarrow \infty} \frac{\ln n}{\ln(n+1)} = \frac{\infty}{\infty}$$

$$= |x| \lim_{n \rightarrow \infty} \frac{n^{-1}}{(n+1)^{-1}}$$

By L'Hôpital's Rule

$$= |x| \lim_{n \rightarrow \infty} \frac{n+1}{n}$$

$$= |x| \lim_{n \rightarrow \infty} 1 + n^{-1} = |x|$$

$$\Rightarrow |x| < 1$$

converges for  $\rho < 1$

$$f(1) = \sum_{n=1}^{\infty} \frac{1}{\ln n}$$

$$\Rightarrow 0 \leq b_n \leq a_n$$

Apply the **Direct comparison test**  $\uparrow$

$$\Rightarrow 0 \leq \frac{1}{n} \leq \frac{1}{\ln n}$$

$$\Rightarrow a_n \text{ diverges}$$

$\frac{1}{n} \rightarrow$  diverges as  $p \leq 1$

$$f(-1) = \sum_{n=1}^{\infty} \frac{(-1)^n}{\ln n}$$

$$\sum_{n=1}^{\infty} (-1)^n a_n, \quad a_n = \frac{1}{\ln n}$$

Apply the **alternating series test**  $\uparrow$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{1}{\ln n} = 0$$

Note:  $|a_n|$  decreases monotonically

$$\lim_{n \rightarrow \infty} a_n = 0 \rightarrow \text{converges}$$

By  $n^{\text{th}}$  test

$R = 1, c = 0$ ;  $f(1)$  diverges and  $f(-1)$  converges, thus the interval of convergence is  $[-1, 1)$

$$4. \sum_{n=1}^{\infty} \frac{(-5)^n (x-3)^n}{n^2}$$

$$\rho = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

Apply the **ratio test** <sup>†</sup>

$$\Rightarrow \lim_{n \rightarrow \infty} \left| \frac{(-5)^{n+1} (x-3)^{n+1}}{(n+1)^2} \cdot \frac{n^2}{(-5)^n (x-3)^n} \right|$$

$$= \lim_{n \rightarrow \infty} \frac{5^n \cdot 5 \cdot (|x-3|)^n \cdot |x-3|}{n^2} \cdot \frac{n^2}{5^n (|x-3|)^n}$$

$$= \lim_{n \rightarrow \infty} 5|x-3|$$

$$\Rightarrow |x| < \frac{1}{5} + 3$$

converges for  $\rho < 1$

$$f\left(\frac{14}{5}\right) = \left(\frac{(-5)^n}{(-5)^n n^2}\right) = \frac{1}{n^2}$$

$$f\left(\frac{16}{5}\right) = \left(\frac{(-5)^n}{5^n n^2}\right) = \frac{(-1)^n}{n^2}$$

$\lim_{n \rightarrow \infty} a_n$  (of both points) = 0 and the  $|a_n|$  of both endpoints decreases monotonically;

$R = \frac{1}{5}$ ,  $c = 3$ , thus the interval of convergence is  $\left[\frac{14}{5}, \frac{16}{5}\right]$



Use  $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \leftarrow |x| < 1$  to expand the function in a power series with  $c = 0$ , and determine interval of convergence.

5.  $f(x) = \frac{1}{4+3x}$

6.  $f(x) = \frac{1}{1-x^4}$

## 10.8 Exercises

Find the Maclaurin series and find the interval on which the expression is valid.

1.  $f(x) = \sin(2x)$

2.  $f(x) = x^2 e^{x^2}$

Find the Taylor series centered at  $c$  and the interval on which the expansion is valid.

3.  $f(x) = e^{3x}, \quad c = -1$

4.  $f(x) = \sin(x), \quad c = \frac{\pi}{2}$

5.  $f(x) = \frac{1}{1-x^2}, \quad c = 3$

# PARAMETRIC EQUATIONS: 11.1

## Parametric Equations Notes

- **Parametric equation:** defines a group of quantities as functions of one or more independent variables called parameters, commonly expressed as coordinates of points that make up a geometric object.

- **Parametrization:** the representation of a geometrical curve  $\mathcal{C}$  with parameter  $t$ , i.e.,

$$c(t) = (x(t), y(t))$$

- Note: parametrizations are not unique; the path  $c(t)$  may traverse all or part of  $\mathcal{C}$  more than once.

- **Parametrization of a line:** a line through point  $P = (a, b)$  with slope  $m$ :

$$x = a + t, \quad y = b + mt \quad \leftarrow -\infty < t < \infty$$

- **Parametrization of a circle** with radius  $R$  and center  $(a, b)$ :

$$c(t) = (a + R \cos \theta, b + R \sin \theta)$$

- **Parametrization of an ellipse:**

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \quad \rightarrow \quad c(\theta) = (a \cos \theta, b \sin \theta)$$

- **Parametrization of a cycloid:** generated by a circle of radius  $R$ ,

$$c(\theta) = (R(t - \sin \theta), R(1 - \cos \theta))$$

- **Graph of  $y = f(x)$ :**

$$c(t) = (t, f(t))$$

- **Slope of tangent line at  $c(t)$ :**

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{y'(t)}{x'(t)} \quad \leftarrow x'(t) \neq 0$$

- **Areas under a parametric curve:** valid when the curve  $y = h(x)$  is traced **once** by the parametric curve  $c(t) = (x(t), y(t))$ .

$$y = h(x) \rightarrow y(t), \quad dx \rightarrow x'(t)dt$$

$$\Rightarrow A = \int_{t_0}^{t_1} y(t)x'(t)dt$$

# Parametric Problems

## 11.1 Exercises

1.

# ARC LENGTH, POLAR COORDINATES: 11.2–11.4

## 11.2–11.4 Notes

### Arc Length and Speed

- **Arc Length of  $\mathcal{C}$** : valid if  $c(t) = (x(t), y(t))$  directly traverses  $\mathcal{C}$  for  $a \leq t \leq b$ , then

$$s = \int_a^b \sqrt{x'(t)^2 + y'(t)^2} dt$$

- Can be interpreted as the **distance traveled** along the path from  $t = a \rightarrow b$
- **Displacement**: less than or equal to the distance traveled; simply the distance from starting point  $c(a)$  to endpoint  $c(b)$ .
- Distance traveled as as **function of  $t$** , starting at  $t_0$ :

$$s(t) = \int_{t_0}^t \sqrt{x'(u)^2 + y'(u)^2} du$$

- **Speed** at time  $t$ :

$$\frac{ds}{dt} = \sqrt{x'(t)^2 + y'(t)^2}$$

- **Surface area**: obtained via rotation of the parametric equation about the x-axis for  $a \leq t \leq b$ , given  $y(t) \geq 0$ ,  $x(t)$  is increasing, and  $x'(t) \wedge y'(t)$  are continuous:

$$S = 2\pi \int_a^b y(t) \sqrt{x'(t)^2 + y'(t)^2} dt$$

## Polar Coordinates

- **Polar coordinate system:** a two-dimensional coordinate system wherein each point is determined by the distance and angle from a reference point and direction.
  - **Radial coordinate,  $r$ :** the distance from reference point.
  - **Angular coordinate,  $\theta$ :** the angle from reference direction.
  - A point  $P$  has polar coordinates  $(r, \theta)$  with the angle measured in the counterclockwise direction by convention.
- **Conversion between polar and rectangular coordinates:**

$$x = r \cos \theta \quad y = r \sin \theta \quad r = \sqrt{x^2 + y^2} \quad \tan \theta = \frac{y}{x} \quad \leftarrow x \neq 0$$

- **If  $r > 0$  then:**  $(r, \theta)$  must lie in quadrant I or IV;

$$\theta = \begin{cases} \tan^{-1} \frac{y}{x} & \leftarrow x > 0 \\ \tan^{-1} \frac{y}{x} + \pi & \leftarrow x < 0 \\ \pm \frac{\pi}{2} & \leftarrow x = 0 \end{cases}$$

- **Non-uniqueness:** Multiple representations can represent the same point, i.e.,

$$(r, \theta) \equiv (r, \theta + 2n\pi) \equiv (-r, \theta + (2n + 1)\pi) \quad \leftarrow n \in \mathbb{Z}$$

- **Polar Equations:**

Curve	Polar Equation
Circle of radius $R$ , center at origin	$r = R$
Line through origin slope $m = \tan \theta_0$	$\theta = \theta_0$
Line, where $P_0 = (d, \alpha)$ is closest to the origin	$r = d \sec(\theta - \alpha)$
Circle radius $a$ , center at $(a, 0)$ $(x - a)^2 + y^2 = a^2$	$r = 2a \cos \theta$
Circle radius $a$ , center at $(0, a)$ $x^2 + (y - a)^2 = a^2$	$r = 2a \sin \theta$

## Area and Arc Length in Polar Coordinates

- **Area in Polar Coordinates:** given that  $f$  is continuous, then the sector is bounded by:
  - **Polar curve,  $r$ :**  $r = f(\theta)$
  - **Two rays,  $\alpha, \beta$ :** where each ray is an angle  $\theta$  with  $\alpha < \beta$ ,  $\beta = \theta - \alpha$
  - Thus, the area is equal to the integral between  $\alpha$  and  $\beta$ , i.e.

$$A = \frac{1}{2} \int_{\alpha}^{\beta} f(\theta)^2 d\theta$$

- **Arc length of polar curve:** given  $\alpha \leq \theta \leq \beta$ :

$$s = \int_{\alpha}^{\beta} \sqrt{f(\theta)^2 + f'(\theta)^2} d\theta$$



# Polar Coordinate Problems

## 11.2 Exercises

1.

## 11.3 Exercises

1.

## 11.4 Exercises

1.

# QUIZ QUESTIONS

## Quiz 3

1. For each statement below, indicate whether it is **True** or **False** and provide a brief justification.

(a) The series  $\sum_{n=1}^{\infty} (-1)^n \cos\left(\frac{1}{n}\right)$  converges.

**✗ False:**

$$\lim_{n \rightarrow \infty} a_n \stackrel{?}{=} 0, \quad a_n = \cos\left(\frac{1}{n}\right) \quad \text{Apply the } n^{\text{th}} \text{ term test}^\uparrow$$
$$\Rightarrow \lim_{n \rightarrow \infty} \cos\left(\frac{1}{n}\right) = 1$$

$\lim_{n \rightarrow \infty} a_n \neq 0$ , thus the series **diverges**.

(b) If the radius of convergence of the power series  $\sum_{n=0}^{\infty} a_n x^n$  is  $R = 5$ , then the series must converge for  $x = -3$  and  $x = -4$ .

**✓ True:**

$$c = 0, \quad R = 5 \Rightarrow \text{converges } \forall x \in (-5, 5)$$

By the **Interval of convergence**  $^\uparrow$

$x = -3 \wedge x = -4 \in (-5, 5)$ , thus the series **must converge** at these values.

2. Determine whether the following series converge absolutely/conditionally, or diverge.

$$(a) \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{2n+5}$$

$$\sum_{n=1}^{\infty} (-1)^n a_n, \quad a_n = \frac{n}{2n+5} \quad \text{Apply the alternating series test}^\uparrow$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{n}{2n+5} = \frac{\infty}{\infty}$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{n}{2n+5} = \frac{1}{2}$$

By L'Hôpital's Rule

$\lim_{n \rightarrow \infty} a_n \neq 0$ , thus the series **diverges**

$$(b) \sum_{n=1}^{\infty} \frac{(-1)^n}{2\sqrt{n}-1}$$

$$\sum_{n=1}^{\infty} (-1)^n a_n, \quad a_n = \frac{1}{2\sqrt{n}-1} \quad \text{Apply the alternating series test}^\uparrow$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{1}{2\sqrt{n}-1} = 0$$

$$\Rightarrow a_n \text{ converges}$$

Note:  $|a_n|$  decreases monotonically

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{2\sqrt{n}-1} \right| \stackrel{?}{=} \text{converges} \quad \text{Apply the absolute convergence test}^\uparrow$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{1}{2\sqrt{n}-1}$$

$\lim_{n \rightarrow \infty} a_n = 0 \rightarrow n^{\text{th}}$  term inconclusive...

$$L = \lim_{n \rightarrow \infty} \frac{a_n}{b_n}, \quad b_n = \frac{1}{\sqrt{n}}$$

Apply the limit comparison test<sup>†</sup>

$$= \lim_{n \rightarrow \infty} \frac{1}{2\sqrt{n}-1} \cdot \sqrt{n}$$

$$= \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{2\sqrt{n}-1} \cdot \frac{n^{-\frac{1}{2}}}{n^{-\frac{1}{2}}}$$

$$= \lim_{n \rightarrow \infty} \frac{1}{2 - n^{-\frac{1}{2}}} = \frac{1}{2}$$

$L > 0$ , and  $b_n$  diverges by the  $p$ -series, implying the  $|a_n|$  diverges. Thus, the original series **converges conditionally**.

3. Find a power series expansion with the center  $c = 0$  for

$$f(x) = \frac{1}{1+x^3}$$

and find the interval of convergence. Hint: use  $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \leftarrow |x| < 1$

$$\begin{aligned} \Rightarrow \frac{1}{1+x^3} &= \frac{1}{1-(-x^3)} \\ &= \sum_{n=1}^{\infty} (-x^3)^n = \sum_{n=1}^{\infty} (-1)^n x^{3n} && \text{Apply hint} \\ \Rightarrow \frac{1}{1+x^3} &= \sum_{n=1}^{\infty} (-1)^n x^{3n} && \leftarrow |x| < 1 \end{aligned}$$

Thus, the interval of convergence is all values in the interval  $(-1, 1)$ .

4. Find the radius of convergence of the power series given by

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+1}}{2^n n}$$

$$\begin{aligned} \rho &= \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| && \text{Apply the ratio test}^\uparrow \\ \Rightarrow \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} x^{2(n+1)+1}}{2^{n+1}(n+1)} \cdot \frac{2^n n}{(-1)^n x^{2n+1}} \right| \\ &= \lim_{n \rightarrow \infty} \frac{|x|^{2n+3}}{2^{2n}(n+1)} \cdot \frac{2^n n}{|x|^{2n+1}} \\ &= \lim_{n \rightarrow \infty} \frac{|x|^{2n+1} \cdot |x|^2}{2^n \cdot 2(n+1)} \cdot \frac{2^n n}{|x|^{2n+1}} \\ &= |x|^2 \lim_{n \rightarrow \infty} \frac{n}{2n+2} = \frac{\infty}{\infty} \\ &= |x|^2 \lim_{n \rightarrow \infty} \frac{1}{2} && \text{By L'Hôpital's Rule} \\ \Rightarrow \frac{|x|^2}{2} &< 1 && \text{converges when } \rho < 1 \\ &= |x| < \sqrt{2} \end{aligned}$$

Thus, the interval of convergence is  $(-\sqrt{2}, \sqrt{2})$  with  $R = \sqrt{2}$ , and  $c = 0$  (endpoints not required to be tested for this problem).

## Quiz 4

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# FINAL REVIEW QUESTIONS

Note: these questions were taken from a provided review sheet; they focus on sections 10.6–11.4. Some questions already exist on the quizzes, but will be duplicated here.

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1. Find the interval of convergence of the following power series.

(a)  $\sum_{n=1}^{\infty} \frac{5^n}{n} x^n$

(b)  $\frac{(x-2)^n}{n^2+1}$



2. Find the Taylor series of the following functions  $f(x)$  centered at the given value of  $c$  using the definition.

(a)  $f(x) = e^x, \quad c = 2$

(b)  $f(x) = \sqrt{x}, \quad c = 1$

3. Find the Maclaurin series of the following functions using substitution and/or multiplication.

(a)  $f(x) = x \cos(2x)$

(b)  $f(x) = \frac{x^3}{1+x}$

4. Express the following integral as a power series, first by finding the Maclaurin series of the integrand, then integrating this series term-by-term:

$$\int_0^1 e^{-x^2} dx$$

5. Find the parametric equations for the following curves.

(a) The line through  $(3, 6)$  and  $(-2, 0)$ .

(b) The circle of radius 5 centered at  $(1, 7)$ .

(c) The ellipse

$$\left(\frac{x-1}{2}\right)^2 + \frac{y+1}{3} = 1$$

6. Find the equation of the tangent line to the curve

$$x = \sin(2t) + \cos(t), \quad y = \cos(2t) - \sin(t), \quad \leftarrow t = \pi$$

7. Find the arc length of the curve

$$x = \frac{2}{3}t^2, \quad y = t^2 - 2, \quad \leftarrow 0 \leq t \leq 2$$

8. Find the surface area obtained by rotating the following around the x-axis;

$$x = e^t - t, \quad y = 4e^{\frac{t}{2}}, \quad \leftarrow 0 \leq t \leq 1$$

9. Match each equation in rectangular coordinates with its equation in polar coordinates.

(a)  $x^2 + y^2 = 4$

(i)  $r^2(1 - 2\sin^2 \theta) = 4$

(b)  $x^2 + (y - 1)^2 = 1$

(ii)  $r(\cos \theta + \sin \theta) = 4$

(c)  $x^2 - y^2 = 4$

(iii)  $r = \sin \theta$

(d)  $x + y = 4$

(iv)  $r = 2$

10. Find the area enclosed by one loop of the curve

$$r^2 \cos 2\theta$$