

1. Compute the following summations

$$(a) \sum_{i=1}^n 6 = \boxed{6n}$$

$$(b) \sum_{i=1}^n 5i = 5 \left( \frac{n(n+1)}{2} \right)$$

$$(c) \sum_{i=1}^n 3i + 2 + 2^i = 3 \left( \frac{n(n+1)}{2} \right) + 2n + 2^{n+1} - 1$$

$$(d) \sum_{i=1}^n 3^i = \sum_{i=0}^{n-1} 3^i = \frac{3^n - 1}{2}$$

$$(e) \sum_{i=1}^n \frac{1}{3^i} - \sum_{i=1}^n \frac{1}{3^i + 1} = \sum_{i=1}^n \frac{1}{3^i} - \frac{1}{3^i + 1} = -\frac{1}{3^i + 1} - \frac{3^{-n}}{2} + \frac{1}{2}$$

$$(f) \sum_{i=1}^n \sum_{j=1}^n j + ij = \sum_{j=1}^n j + \sum_{j=1}^n ij = \sum_{i=1}^n \frac{n(n+1) + in(n+1)}{2} = \frac{n^4 + 4n^3 + 3n^2}{4}$$

$$(g) \sum_{k=0}^n \binom{n}{k} 2^k = (1+2)^n = 3^n \quad \text{by binomial theorem } (a+b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k}$$

2. Complete the following:

(a) You have 10 people, how many ways are there to split into two teams of 5?

$$\binom{10-1}{5-1} = 126$$

(b) If you have a red die and green die, how many different ways are there to roll a 7?

$$S_7 = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\} \implies |S_7| = 6$$

(c) How many ways are there to draw 5 cards from a deck of 52?

$$\binom{52}{5} = 2,598,960$$

(d) How many possible full house hands are there in poker?

$$13 \binom{4}{3} 12 \binom{4}{2} = 3,744$$

(e) Out of a bag of red, blue and green marbles, how many ways are there to draw three marbles where the order doesn't matter?

$$\binom{3+2}{3} = 10$$

3. There are 38 people in this class. Prove that at least 4 people were born in the same month

*Proof.*

There are 12 possible months.

In the most distributed case, then there are 12 months with 3 people.

Therefore, by the pigeon hole principle, at least 1 of the months must have 4 or more people born in that month. ■

4. A round robin tournament is one in which every team plays every other team. Assuming that there are no ties, and every team wins at least one game, prove that two teams won the same number of games.

*Proof.*

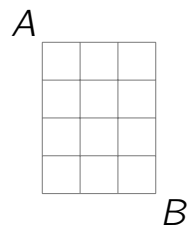
5. We saw a lot of summation properties in class. Fortunately we don't need to prove all of them separately. We can prove several of them at the same time.

Use induction to prove that summations are linear:

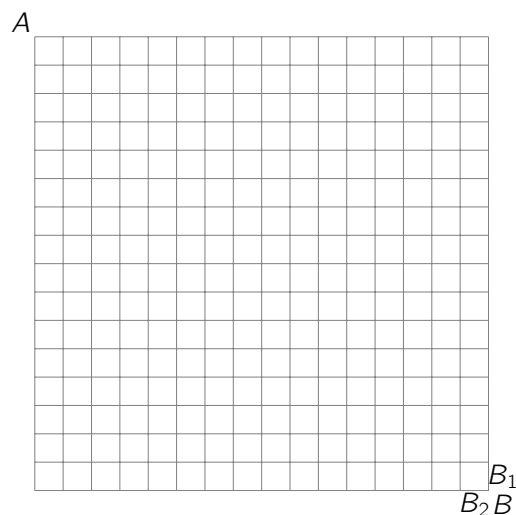
$$\sum_{i=m}^n c \cdot a_i + d \cdot b_i = c \cdot \sum_{i=m}^n a_i + d \cdot \sum_{i=m}^n b_i$$

6. Complete the following:

- (a) How many paths are there from  $A$  to  $B$  in the following grid if we're only allowed to move right and down.



- (b) Now we have a much larger grid. This would be very difficult to count the paths by hand. Instead, let's try to come up with a recursive solution. Let's have the number of paths from  $A$  to  $B$  be  $P$ , the number of paths from  $A$  to  $B_1$  be  $P_1$ , and the number of paths from  $A$  to  $B_2$  be  $P_2$ . Now give a formula for  $P$  in terms of  $P_1$  and  $P_2$



$$P =$$

- (c) Now given a grid of  $r$  rows and  $c$  columns, give a recursive formula for the number of paths from  $A$  to  $B$

## 7. (Extra Credit)

I want to answer a simple question from class. If I roll  $n$  dice, how many possible rolls are there when order doesn't matter? We saw in class that if I have 2 dice, the answer is 21. Let's generalize this.

First Let's start by calculating the number of ways of rolling 3 dice.

- (a) How many ways are to roll 3 dice where all numbers are distinct?
- (b) How many ways are to roll 3 dice two dice have the same value? While the order of the dice doesn't matter, the number that rolled twice does.
- (c) How many ways are to roll 3 dice where all dice are the same?

This way of counting dice seems like a good start, but it seems like it could get complicated pretty quickly. We'll need to break it down.

Notice that in the last part we had 3 cases: all 3 numbers are distinct, 2 numbers are distinct, and only 1 number is distinct. So, we should really ask the smaller question:

"how many ways are there to roll  $n$  dice, where  $k$  dice are distinct?"

We can again break this into 2 questions.

- (d) How many ways are there to choose the  $k$  values for the  $n$  dice?

For the second question we need to how we group the dice of the same value.

This doesn't seem like a big deal with 3 dice, but let's look at what happens when we have 5. If I roll 5 dice and get 2 distinct values (4 and 1 for example), how many ways can I break that up?

I could have (1 1 1 1 4), (1 1 1 4 4), (1 1 4 4 4), or (1 4 4 4 4). So there are 4 possible ways.

This gets more complicated with 3 distinct values, so we need a better way to solve the problem. I could also write this as I have  $m_1$  1s and  $m_2$  4s. I'll call this the multiplicity of a value. So the multiplicity of 1 is  $m_1$ .

The problem is really asking how many different ways can  $m_1 + m_2 = 5$ . This is good, because it generalizes. If I have 3 distinct values, then I'm asking how many ways can  $m_1 + m_2 + m_3 = 5$ .

- (e) How many ways are there to assign multiplicities to a roll of  $n$  dice with  $k$  distinct values?

Or to rephrase that, how many solutions are there to  $m_1 + m_2 + \dots + m_k = n$ ?

Now we can put the last two parts together.

- (f) How many ways are there to roll  $n$  dice with  $k$  distinct values?

- (g) Give a summation for the number of ways to roll  $n$  dice where order doesn't matter.

Ok, that was ... something. I'm kind of concerned about that answer though. What we really need is a way to check the problem. We can do that by solving the problem again, but in a different way. This time we're going to take our roll and sort it.

So if I roll 7 dice (4 3 4 1 6 2 2), I'll get (1 2 2 3 4 4 6). If two rolls sort to the same thing, then they're the same roll.

Now that I have the numbers grouped together, this looks like a good spot for stars and bars. In This case a I'll put a star every time there's a number, and a bar every time a number goes up by 1. So the roll (1 2 2 3 4 4 6) will be (\* | \*\* | \* | \*\* | | \*). Since I didn't use the number 5, I have 2 bars in a row. I'll always have  $n$  stars, and 5 bars.

- (h) use this stars and bars set up to get a new formula for the number of ways to roll  $n$  dice where order doesn't matter.
- (i) Finally, use the recursive formula  $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$  to show that part (g) and part (h) are really the same.