Calculus III Final Review

CONVERGENCE: 10.3–10.5

Convergence Notes

- Let $\sum_{n=1}^{\infty} a_n$ be given and note for which series convergence is known, i.e.:

  Geometric: let $c \neq 0$, if $|r| < 1$, then

  $$\sum_{n=0}^{\infty} cr^n = \frac{c}{1-r}$$

  $|r| \geq 1 \implies$ diverges

  ***p*-Series**: converges if $p > 1$.

  $$\sum_{n=0}^{\infty} \frac{1}{n^p}$$

  $p \leq 1 \implies$ diverges

- **The n^{th} Term Divergence Test**: a relatively easy test that can be used to quickly determine if a test diverges if the $\lim_{n\to\infty} a_n \neq 0$. If $\lim_{n\to\infty} a_n = 0$, then the test is inconclusive and other tests must be applied.

Tests for Positive Series

- **Direct Comparison Test**: use if dropping terms from the denominator or numerator gives a series b_n wherein convergence is easily found, then compare to the original series a_n as follows:

  $$\sum_{n=1}^{\infty} b_n \text{ converges} \implies \sum_{n=1}^{\infty} a_n \text{ converges} \quad \leftarrow 0 \leq a_n \leq b_n$$

  $$\sum_{n=1}^{\infty} b_n \text{ diverges} \implies \sum_{n=1}^{\infty} a_n \text{ diverges} \quad \leftarrow 0 \leq b_n \leq a_n$$

- **Limit Comparison Test**: use when the direct comparison test isn't convenient or when comparing two series. One can to take the dominant term in the numerator and denominator from a_n to form a new positive sequence b_n if needed.

  Assuming the following limit $L = \lim_{n\to\infty} \frac{a_n}{b_n}$ exists, then:

  $$L > 0 \implies \sum_{n=1}^{\infty} a_n \text{ converges} \iff \sum_{n=1}^{\infty} b_n \text{ converges}$$

  $$L = 0 \text{ and } \sum_{n=1}^{\infty} b_n \text{ converges} \implies \sum_{n=1}^{\infty} a_n \text{ converges}$$

  $$L = \infty \text{ and } \sum_{n=1}^{\infty} a_n \text{ converges} \implies \sum_{n=1}^{\infty} b_n \text{ converges}$$

- **Ratio Test**: often used in the presence of a factorial $(n!)$ or when the are constants raised to the power of n (c^n).

  Assuming the following limit $\rho = \lim_{n\to\infty} \left|\frac{a_n+1}{a_n}\right|$ exists, then

$$\rho < 1 \implies \sum a_n \text{ converges absolutely}$$
$$\rho > 1 \implies \sum a_n \text{ diverges}$$
$$\rho = 1 \implies \text{test is inconclusive}$$

- **Root Test**: used when there is a term in the form of $f(n)^{g(n)}$.

  Assuming the following limit $C = \lim_{n\to\infty} |a_n|^{\frac{1}{n}}$ exists, then

$$C < 1 \implies \sum a_n \text{ converges absolutely}$$
$$C > 1 \implies \sum a_n \text{ diverges}$$
$$C = 1 \implies \text{test is inconclusive}$$

- **Integral Test**: if the other tests fail and $a_n = f(n)$ is a decreasing function, then one can use the improper integral $\int_1^\infty f(x)dx$ to test for convergence.

  Let $a_n = f(n)$ be a positive, decreasing, and continuous function $\forall x \geq 1$, then:

$$\int_1^\infty f(x)dx \text{ converges} \implies \sum_{n=1}^{\infty} a_n \text{ converges}$$
$$\int_1^\infty f(x)dx \text{ diverges} \implies \sum_{n=1}^{\infty} a_n \text{ diverges}$$

Tests for Non-Positive Series

- **Alternating Series Test**: used for series in the form $\sum_{n=0}^{\infty}(-1)^n a_n$

  Converges if $|a_n|$ decreases monotonically $(|a_n + 1| \leq |a_n|)$ and if $\lim_{n\to\infty} a_n = 0$

- **Absolute Convergence**: used if the series $\sum a_n$ is not alternating (if it is alternating, use the alternating test in conjunction); simply test if $\sum |a_n|$ converges using the test for positive series.

  $\sum a_n$ converges **conditionally** if $\sum a_n$ converges, but $\sum |a_n|$ diverges.

  $\sum a_n$ converges **absolutely** if $\sum |a_n|$ converges.

Convergence Problems

10.5 Exercises

Determine convergence or divergence using any method.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n + 4^n}{7^n}$

$$\implies \sum_{n=1}^{\infty} \frac{2^n}{7^n} + \sum_{n=1}^{\infty} \frac{4^n}{7^n}$$

Separate into two geometric series↑

$$\implies r = \frac{2}{7} < 1, \quad r = \frac{4}{7} < 1$$

Both geometric series converge, thus the original series converges.

2. $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{n!}$

$$\implies \rho = \lim_{n\to\infty} \left| \frac{(n+1)^3}{(n+1)!} \cdot \frac{n!}{n^3} \right|$$

Apply the ratio test↑

$$= \lim_{n\to\infty} \frac{n^3 + 3n^2 + 3n + 1}{(n+1)n!} \cdot \frac{n!}{n^3}$$
$$= \lim_{n\to\infty} \frac{n^3 + 3n^2 + 3n + 1}{n^4 + n^3}$$
$$= \lim_{n\to\infty} \frac{n^3 + 3n^2 + 3n + 1}{n^4 + n^3} \cdot \frac{n^{-4}}{n^{-4}}$$
$$= \lim_{n\to\infty} \frac{n^{-1} + 3n^{-2} + 3n^{-3} + n^{-4}}{1 + n^{-1}} = 0$$

$\rho = 0 < 1$, thus the series converges.

3. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2n+1}$

$$\Longrightarrow \lim_{n\to\infty} \frac{n}{2n+1} \qquad \text{Apply the } n^{th} \text{ term test}^{\uparrow}$$

$$\Longrightarrow \lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2} \qquad \text{By L'Hôpital's Rule}$$

$\lim_{n\to\infty} a_n \neq 0$, thus the series diverges.

4. $\displaystyle\sum_{n=1}^{\infty} 2^{\frac{1}{n}}$

$$\Longrightarrow \lim_{n\to\infty} 2^{\frac{1}{n}} = 2^0 = 1 \qquad \text{Apply the } n^{th} \text{ term test}^{\uparrow}$$

$\lim_{n\to\infty} a_n \neq 0$, thus the series diverges.

5. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$

$$a_n \leq b_n, \quad b_n = 1$$

$$b_n \text{ converges} \rightarrow a_n \text{ converges} \qquad \text{Apply the direct comparison test}^{\uparrow}$$

$$\sin n \leq 1 \qquad \leftarrow \forall n \geq 1$$

$$\frac{\sin n}{n^2} \leq \frac{1}{n^2}$$

$$\frac{1}{n^2} \rightarrow \text{ converges} \qquad \text{by } p\text{-series}^{\uparrow}$$

The larger (b_n) series converges, thus the smaller (a_n) converges.

6. $\sum_{n=1}^{\infty} \frac{n!}{(2n)!}$

$$\begin{aligned}
\Longrightarrow \rho &= \lim_{n\to\infty} \left| \frac{(n+1)!}{(2n+2)!} \cdot \frac{(2n)!}{n!} \right| && \text{Apply the ratio test}^{\uparrow} \\
&= \lim_{n\to\infty} \frac{(n+1)n!}{(2n+2)(2n+1)2n!} \cdot \frac{(2n)!}{n!} \\
&= \lim_{n\to\infty} \frac{n+1}{(2n+2)(2n+1)} = \frac{n+1}{4n^2+6n+2} \\
&= \lim_{n\to\infty} \frac{1}{8n+6} = 0 && \text{By L'Hôpital's Rule}
\end{aligned}$$

$\rho = 0 < 1$, thus the series converges.

7. $\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$

$$b_n \leq a_n, \quad b_n = n$$

$$a_n \text{ diverges} \iff b_n \text{ diverges} \qquad \text{Apply the direct comparison test}^{\uparrow}$$

$$\begin{aligned}
n &\leq n + \sqrt{n} && \leftarrow \forall n \geq 1 \\
\frac{1}{n+\sqrt{n}} &\leq \frac{1}{n} \\
\frac{1}{n} &\rightarrow \text{ diverges} && \text{by } p\text{-series}^{\uparrow}
\end{aligned}$$

The smaller (b_n) series diverges, thus the larger (a_n) diverges.

8. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$

f is positive, decreasing, and continuous for $x \geq 2$ Apply the integral test↑

$$\implies \int_2^{\infty} f(x)dx = \lim_{R\to\infty} \int_2^R \frac{1}{x(\ln x)^3} dx \qquad \ln x = u, \quad xdu = dx$$

$$\implies \lim_{R\to\infty} \int_2^R \frac{1}{x(u)^3} xdu = \int_2^R \frac{1}{u}^3 du$$

$$= -\frac{1}{2(u)^2}\bigg|_2^R = \frac{1}{2R^2} - \frac{1}{8}$$

$$= -\frac{1}{2\ln^2(x)} + C\bigg|_2^R = \frac{1}{8} - \frac{1}{2R^2}$$

$$= \lim_{R\to\infty} \left(\frac{1}{8} - \frac{1}{2R^2}\right) = \frac{1}{8}$$

The improper integral converges, thus the original series converges.

9. $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{5^n}$

$$\implies \rho = \lim_{n\to\infty} \left| \frac{(n+1)^3}{5^{n+1}} \cdot \frac{5^n}{n^3} \right| \qquad \text{Apply the ratio test}^{\uparrow}$$

$$= \lim_{n\to\infty} \frac{(n+1)^3}{5^n \cdot 5} \cdot \frac{5^n}{n^3}$$

$$= \frac{1}{5} \lim_{n\to\infty} \frac{(n+1)^3}{n^3}$$

$$= \frac{1}{5} \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^3 = \frac{1}{5}$$

$\rho = \dfrac{1}{5} < 1$, thus the series converges.

10. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^3 - n^2}}$

$$\begin{aligned} L &= \lim_{n\to\infty} \frac{a_n}{b_n}, \quad b_n = \frac{1}{\sqrt{n^3}} \\ \Longrightarrow L &= \lim_{n\to\infty} \frac{1}{\sqrt{n^3 - n^2}} \cdot \frac{\sqrt{n^3}}{1} \\ &= \lim_{n\to\infty} \sqrt{\frac{n^3}{n^3(1 - n^{-1})}} \\ &= \sqrt{\frac{1}{1(1-0)}} = 1 \end{aligned}$$

Apply the limit comparison test↑

$L > 0$, thus a_n converges if b_n converges.

b_n converges by the p-series test, as $\frac{3}{2} > 1$, thus a_n converges.

11. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 + 4n}{3n^4 + 9}$

$$\begin{aligned} L &= \lim_{n\to\infty} \frac{a_n}{b_n}, \quad b_n = \frac{1}{n^2} \\ &= \lim_{n\to\infty} \frac{n^2 + 4n}{3n^4 + 9} \cdot n^2 \\ &= \lim_{n\to\infty} \frac{n^4 + 4n^3}{3n^4 + 9} \cdot \frac{n^{-4}}{n^{-4}} \\ &= \lim_{n\to\infty} \frac{1 + 4n^{-1}}{3 + 9n^{-4}} = \frac{1}{3} \end{aligned}$$

Apply the limit comparison test↑

$L > 0$, thus a_n converges if b_n converges.

b_n converges by the p-series test, as $2 > 1$, thus a_n converges.

12. $\sum_{n=1}^{\infty}(0.8)^{-n}n^{-0.8}$

$$\begin{aligned}
\rho &= \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \\
&= \lim_{n\to\infty}\left|\frac{(0.8)^{-(n+1)}(n+1)^{-0.8}}{(0.8)^{-n}n^{-0.8}}\right| \\
&= \lim_{n\to\infty}\frac{(0.8)^{-n}\cdot 0.8^{-1}\cdot(n+1)^{-0.8}}{(0.8)^{-n}n^{-0.8}} \\
&= 1.25\lim_{n\to\infty}\frac{(n+1)^{-0.8}}{n^{-0.8}} \\
&= 1.25\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^{-0.8} = 1.25
\end{aligned}$$

Apply the ratio test↑

$\rho = 1.25 > 1$, thus a_n diverges.

13. $\sum_{n=1}^{\infty}4^{-2n+1}$

$$\sum_{n=1}^{\infty}cr^n \implies \sum_{n=1}^{\infty}4\cdot(4^{-2})^n$$

Convert into geometric series↑

$|r| = 2^{-2} = \dfrac{1}{16} < 1$ and $c \neq 0$, thus a_n converges.

14. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$

$$\sum_{n=1}^{\infty}(-1)^n a_n, \quad a_n = \frac{1}{\sqrt{n}}$$

Apply the alternating series test ↑

$$\frac{1}{\sqrt{n}} \to \text{diverges}$$

by p-series ↑

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$$

$|a_n|$ decreases monotonically and $\lim_{n\to\infty} a_n = 0$, but a_n diverges, thus the series.
converges conditionally

15. $\displaystyle\sum_{n=1}^{\infty} \sin\frac{1}{n^2}$

$$L = \lim_{n\to\infty} \frac{a_n}{b_n}, \quad b_n = \frac{1}{n^2}$$

Apply the limit comparison test ↑

$$\implies \lim_{n\to\infty} \frac{\sin(n^{-2})}{n^{-2}} = \frac{0}{0}$$

$$= \lim_{n\to\infty} \frac{\cos(n^{-2})(-2n^{-3})}{-2n^{-3}}$$

by L'Hôpital's Rule

$$= \lim_{n\to\infty} \cos(n^{-2}) = 1$$

$L > 0$, thus a_n converges if b_n converges.

b_n converges by the p-series test, thus a_n converges.

16. $\displaystyle\sum_{n=1}^{\infty}(-1)^n \cos n^{-1}$

$$\sum_{n=1}^{\infty}(-1)^n a_n, \quad a_n = \cos(n^{-1})$$

Apply the n^{th} term test ↑

$$\implies L = \lim_{n\to\infty} \cos(n^{-1}) = 1$$

$L \neq 0$, thus the series diverges

17. $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{\sqrt{n}}$

$$\sum_{n=1}^{\infty}(-1)^n a_n, \quad a_n = \frac{2^n}{\sqrt{n}}$$ Apply the n^{th} term test↑

$$\implies L = \lim_{n\to\infty} \frac{2^n}{\sqrt{n}} = \frac{\infty}{\infty}$$

$$= \frac{2^n \ln 2}{\frac{1}{2}n^{-\frac{1}{2}}} = 2^n \ln 2 \cdot 2\sqrt{n}$$ By L'Hôpital's Rule

$$= 2 \lim_{n\to\infty} 2^n \ln(2)\sqrt{n} = \infty$$

$L \neq 0$, thus the series diverges

18. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{n+12}\right)^n$

$$L = \lim_{n\to\infty} \left(\frac{n}{n+12}\right)^n$$ Apply the n^{th} term test↑

$$= \lim_{n\to\infty} e^{-12}$$ By common limit $\left(\frac{x}{x+k}\right)^x = e^{-k}$

$L \neq 0$, thus the series diverges.

Power/Taylor Series: 10.6–10.8

Power/Taylor Series Notes

Power Series

- **Power series**: a infinite series in the form:

  $$F(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

  Where the constant c is the *center* of the power series F(x).

- **Radius of convergence** R: the range of values of the variable x whereby the power series $F(x)$ converges.
  - Every power series converges at $x = c$, as $(x-c)^0 = 1$, though the series may diverge for other values of x.
  - $F(x)$ converges for $|x-c| < R$ and diverges for $|x-c| > R$
  - $F(x)$ may converge of diverge at endpoints $c - R$ and $c + R$
- **Interval of convergence**: the open interval $(c-R, c+R)$ and possibly one of both of the endpoints, each must be tested.
  - In most cases, the ratio test↑ can be used to find R.
  - If $R > 0$, then F is differentiable over the interval of convergence; the derivative and antiderivative can be obtained using the following:

$$F'(x) = \sum_{n=1}^{\infty} n a_n (x-c)^{n-1} \qquad F(x)dx = C + \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-c)^{n+1}$$

Taylor Series

- **Taylor series**: the power series of a infinitely differentiable function $f(x)$ centered at c,

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

  - n^{th} **Taylor polynomial**: a polynomial of degree n that is formed partial sum formed by the first $n+1$ terms of a Taylor series, i.e.,

$$f(c)+f'(c)(x-c)+\frac{f''(c)}{2!}(x-c)^2+\frac{f'''(c)}{3!}(x-c)^3+\cdots+\frac{f^n(c)}{n!}(x-c)^n$$

  - **Maclaurin series**: when $c=0$, i.e.,

$$\sum_{n=0}^{\infty} \frac{f^{(n)}0}{n!}(x)$$

- **Useful Maclaurin Series**: useful Taylor series centered at 0 that can be used to derive other series via differentiation, integration, multiplication, or substitution.

$$\begin{aligned}
e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} && \leftarrow \forall x \\
\sin x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} && \leftarrow \forall x \\
\cos x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} && \leftarrow \forall x \\
\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n && \leftarrow |x|<1 \\
\frac{1}{1+x} &= \sum_{n=0}^{\infty} (-1)^n x^n && \leftarrow |x|<1 \\
\ln(1+x) &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} && \leftarrow |x|<1 \land x=1 \\
\tan^{-1} x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} && \leftarrow |x|\leq 1 \\
(1+x)^\alpha &= \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n && \leftarrow |x|<1 \\
\text{where } \binom{\alpha}{n} &= \sum_{n=0}^{\infty} \prod_{k=1}^{n} \frac{\alpha-k+1}{k}
\end{aligned}$$

Power/Taylor Series Problems

10.6 Exercises

Find the interval of convergence.

1. $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{n}{4^n}x^{2n}$

$$\begin{aligned}
\rho &= \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| && \text{Apply the ratio test}^{\uparrow}\\
&= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+1)(x^{2(n+1)})}{4^{n+1}}\cdot\frac{4^n}{(-1)^n n(x^{2n})}\right|\\
&= \lim_{n\to\infty}\frac{(n+1)(|x|^{2n}\cdot|x|^2)}{4^n\cdot 4}\cdot\frac{4^n}{n\cdot|x|^{2n}}\\
&= \lim_{n\to\infty}\frac{(n+1)|x|^2}{4n}\cdot\frac{n^{-1}}{n^{-1}}\\
&= \lim_{n\to\infty}\frac{(1+n^{-1})|x|^2}{4}=\frac{|x|^2}{4}
\end{aligned}$$

$$\implies \frac{|x|^2}{4}<1 \implies |x|<2 \qquad \text{converges for } \rho<1$$

Both endpoints tend toward ∞ (diverge), thus the interval of convergence is $(-2,2)$.

2. $\displaystyle\sum_{n=8}^{\infty}n^7x^n$

$$\begin{aligned}
\rho &= \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| && \text{Apply the ratio test}^{\uparrow}\\
&\implies \lim_{n\to\infty}\left|\frac{(n+1)^7x^{n+1}}{n^7x^n}\right|\\
&= \lim_{n\to\infty}\frac{(n+1)^7\cdot|x|^n\cdot|x|}{n^7|x|^n}\\
&= \lim_{n\to\infty}\frac{(n+1)^7|x|}{n^7}\\
&= |x|\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^7=|x|
\end{aligned}$$

$$\implies |x|<1 \qquad \text{converges for } \rho<1$$

Both endpoints tend toward ∞ (diverge), thus the interval of convergence is $(-1,1)$.

3. $\displaystyle\sum_{n=2}^{\infty} \frac{x^n}{\ln n}$

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$ Apply the ratio test ↑

$$\implies \lim_{n\to\infty} \left|\frac{x^{n+1}}{\ln(n+1)} \cdot \frac{\ln n}{x^n}\right|$$

$$= \lim_{n\to\infty} \frac{|x| \ln n}{\ln(n+1)}$$

$$= |x| \lim_{n\to\infty} \frac{\ln n}{\ln(n+1)} = \frac{\infty}{\infty}$$

$$= |x| \lim_{n\to\infty} \frac{n^{-1}}{(n+1)^{-1}}$$ By L'Hôpital's Rule

$$= |x| \lim_{n\to\infty} \frac{n+1}{n}$$

$$= |x| \lim_{n\to\infty} 1 + n^{-1} = |x|$$

$$\implies |x| < 1$$ converges for $\rho < 1$

$$f(1) = \sum_{n=1}^{\infty} \frac{1}{\ln n}$$

$$\implies 0 \leq b_n \leq a_n$$ Apply the direct comparison test ↑

$$\implies 0 \leq \frac{1}{n} \leq \frac{1}{\ln n}$$

$\implies a_n$ diverges $\frac{1}{n} \to$ diverges as $p \leq 1$

$$f(-1) = \sum_{n=1}^{\infty} \frac{(-1)^n}{\ln n}$$

$$\sum_{n=1}^{\infty} (-1)^n a_n, \quad a_n = \frac{1}{\ln n}$$ Apply the alternating series test ↑

$$\implies \lim_{n\to\infty} \frac{1}{\ln n} = 0$$ Note: $|a_n|$ decreases monotonically

$f(-1)$ converges and $f(1)$ diverges, thus the interval of convergence is $[-1, 1)$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{(-5)^n(x-3)^n}{n^2}$

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

Apply the ratio test[↑]

$$\implies \lim_{n\to\infty} \left|\frac{(-5)^{n+1}(x-3)^{n+1}}{(n+1)^2} \cdot \frac{n^2}{(-5)^n(x-3)^n}\right|$$

$$= \lim_{n\to\infty} \frac{5^n \cdot 5 \cdot (|x-3|)^n \cdot |x-3|}{n^2} \cdot \frac{n^2}{5^n(|x-3|)^n}$$

$$= \lim_{n\to\infty} 5|x-3|$$

$$\implies |x-3| < \frac{1}{5}$$

$$\implies -\frac{14}{5} < x < \frac{16}{5}$$

converges for $\rho < 1$

$$f\left(\frac{-14}{5}\right) = \frac{(-5)^n\left(-\frac{1}{5}\right)^n}{n^2} = \frac{1}{n^2}$$

$$f\left(\frac{16}{5}\right) = \frac{(-5)^n\left(\frac{1}{5}\right)^n}{n^2} = \frac{(-1)^n}{n^2}$$

$\lim\limits_{n\to\infty} a_n$ (of both points) $= 0$ and the $|a_n|$ of both endpoints decrease monotonically;
$R = \frac{1}{5}$, $c = 3$, thus the interval of convergence is $\left[-\frac{14}{5}, \frac{16}{5}\right]$

Use the following equation to expand the function in a power series with $c = 0$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \leftarrow |x| < 1$$

and determine the interval of convergence.

5. $f(x) = \dfrac{1}{4+3x}$

$$\begin{aligned} \frac{1}{4+3x} &= \frac{\frac{1}{4}}{1-(-\frac{3x}{4})} \\ &= \frac{1}{4}\sum_{n=0}^{\infty}\left(-\frac{3x}{4}\right)^n = \sum_{n=0}^{\infty}(-1)^n\frac{3^n x^n}{4^{n+1}} && \text{Expansion} \\ &\implies \left|\frac{3x}{4}\right| < 1 && \sum_{n=1}^{\infty} \leftarrow |x| < 1 \\ &\implies -\frac{4}{3} < x < \frac{4}{3} && \text{Interval of convergence} \end{aligned}$$

Thus, $\dfrac{1}{4+3x} = \displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{3^n x^n}{4^{n+1}}$ with an interval of convergence of $\left(-\dfrac{4}{3}, \dfrac{4}{3}\right)$

6. $f(x) = \dfrac{1}{1-x^4}$

$$\begin{aligned} \frac{1}{1-x^4} &= \sum_{n=0}^{\infty}\left(x^4\right)^n = \sum_{n=0}^{\infty} x^{4n} && \text{Expansion} \\ &\implies |x^4| < 1 && \sum_{n=1}^{\infty} \leftarrow |x| < 1 \\ &\implies -1 < x < 1 && \text{Interval of convergence} \end{aligned}$$

Thus, $\dfrac{1}{1-x^4} = \displaystyle\sum_{n=0}^{\infty} x^{4n}$ with an interval of convergence of $(-1, 1)$

10.8 Exercises

Find the Maclaurin series and find the interval on which the expression is valid.

1. $f(x) = \sin(2x)$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \qquad \text{Use relevant Maclaurin series}^{\uparrow}$$

$$\implies \sin 2x = (-1)^n \frac{(2x)^{2n+1}}{(2n+1)!}$$

$\sin(x)$ converges $\forall x$, thus $\sin(2x)$ also converges $\forall x$. Therefore:

$$f(x) = (-1)^n \frac{(2x)^{2n+1}}{(2n+1)!} \qquad \leftarrow \forall x \in \mathbb{R}$$

2. $f(x) = x^2 e^{x^2}$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \text{Use relevant Maclaurin series}^{\uparrow}$$

$$\implies e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!}$$

$$\implies x^2 e^{x^2} = \sum_{n=0}^{\infty} \frac{x^2 \cdot (x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n+2}}{n!}$$

e^x converges $\forall x$, thus e^{x^2} also converges $\forall x$. Therefore:

$$f(x) = \sum_{n=0}^{\infty} \frac{x^{2n+2}}{n!} \qquad \leftarrow \forall x \in \mathbb{R}$$

Find the Taylor series centered at c and the interval on which the expansion is valid.

3. $f(x) = e^{3x}, \quad c = -1$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Use relevant Maclaurin series↑

$$\implies f(x) = e^{3(x-1)} = e^{-3} \cdot e^{3(x+1)}$$

Center series $(x - c)$

$$= e^{-3} \sum_{n=0}^{\infty} \frac{(3(x+1))^n}{n!}$$

$$= e^{-3} \sum_{n=0}^{\infty} \frac{3^n(x+1)^n}{n!}$$

e^x converges $\forall x$, thus e^{3x} also converges $\forall x$. Therefore:

$$f(x) = e^{-3} \sum_{n=0}^{\infty} \frac{3^n(x+1)^n}{n!}$$

$\leftarrow$ convergence interval: $(-\infty, \infty)$

4. $f(x) = \sin(x), \quad c = \dfrac{\pi}{2}$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

Use relevant Maclaurin series↑

$$\implies f(x) = \sin\left(x - \frac{\pi}{2}\right)$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{\left(x - \frac{\pi}{2}\right)^{2n+1}}{(2n+1)!}$$

Center series$(x - c)$

$\sin(x)$ converges $\forall x$, therefore:

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{\left(x - \frac{\pi}{2}\right)^{2n+1}}{(2n+1)!}$$

$\leftarrow$ convergence interval: $(-\infty, \infty)$

PARAMETRIC EQUATIONS: 11.1

Parametric Equations Notes

- **Parametric equation**: defines a group of quantities as functions of one or more independent variables called parameters, commonly expressed as coordinates of points that make up a geometric object.
  - **Parametrization**: the representation of a geometrical curve $\mathcal{C}$ with parameter t, i.e.,

    $$c(t) = (x(t), y(t))$$

    - Note: parametrizations are not unique; the path $c(t)$ may traverse all or part of $\mathcal{C}$ more than once.
  - **Parametrization of a line**: a line through point $P = (a, b)$ with slope m:

    $$x = a + t, \quad y = b + mt \qquad \leftarrow -\infty < t < \infty$$

  - **Parametrization of a circle** with radius R and center (a, b):

    $$c(t) = (a + R\cos\theta, b + R\sin\theta)$$

  - **Parametrization of an ellipse**:

    $$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \qquad \rightarrow \qquad c(\theta) = (a\cos\theta, b\sin\theta)$$

  - **Parametrization of a cycloid**: generated by a circle of radius R,

    $$c(\theta) = (R(t - \sin\theta), R(1 - \cos\theta))$$

  - **Graph of $y = f(x)$**:

    $$c(t) = (t, f(t))$$

- **Slope of tangent lie at $c(t)$**:

  $$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{y'(t)}{x'(t)} \qquad \leftarrow x'(t) \neq 0$$

- **Area under a parametric curve**: valid when the curve $y = h(x)$ is traced once by the parametric curve $c(t) = (x(t), y(t))$.

  $$y = h(x) \rightarrow y(t), \qquad dx \rightarrow x'(t)dt$$

  $$\Longrightarrow A = \int_{t_0}^{t_1} y(t)x'(t)dt$$

Parametric Problems

11.1 Exercises

Find parametric equations for the given curve.

1. Line through $(3, 1)$ and $(-5, 4)$.

$$P(3,1), \quad Q(-5,4)$$
$$\implies m = \frac{y_Q - y_P}{x_Q - x_P} = \frac{4-1}{-5-3} = -\frac{3}{8}$$
$$\implies y = m(x - x_P) + y_P = -\frac{3}{8}(x-3) + 1 = -\frac{3}{8}x + \frac{17}{8}$$
$$\begin{cases} x = t \\ y = -\frac{3}{8}t + \frac{17}{8} \end{cases}$$
$$\implies \boxed{c(t) = \left(t, -\frac{3}{8}t + \frac{17}{8}\right)}$$

2. Circle of radius 4 with center $(3, 9)$.

$$c(t) = (a + R\cos\theta, b + R\sin\theta) \quad \leftarrow \text{ Parametrization of a circle}^{\uparrow}$$

$$\implies c(t) = (3 + 4\cos\theta, 9 + 4\sin\theta) \in [0, 2\pi)$$

3. The following ellipse its center translated center to $(7, 4)$

$$\left(\frac{x}{5}\right)^2 + \left(\frac{y}{12}\right)^2 = 1$$

$$c(\theta) = (a\cos\theta, b\sin\theta) \qquad \text{Parametrization of a ellipse}^{\uparrow}$$

$$c(\theta) = (5\cos\theta, 12\sin\theta) \qquad \leftarrow \left(\frac{x}{5}\right)^2 + \left(\frac{y}{12}\right)^2 = 1$$

$$c(\theta) = \boxed{(7 + 5\cos\theta, 4 + 12\sin\theta)} \qquad \leftarrow \left(\frac{x-7}{5}\right)^2 + \left(\frac{y-4}{12}\right)^2 = 1$$

Arc Length, Polar Coordinates: 11.2–11.4

11.2–11.4 Notes

Arc Length and Speed

- **Arc Length of $\mathcal{C}$**: valid if $c(t) = (x(t), y(t))$ directly traverses $\mathcal{C}$ for $a \leq t \leq b$, then

$$s = \int_a^b \sqrt{x'(t)^2 + y'(t)^2} dt$$

  - Can be interpreted as the **distance traveled** along the path from $t = a \to b$
  - **Displacement**: less than or equal to the distance traveled; simply the distance from starting point $c(a)$ to endpoint $c(b)$.
  - Distance traveled as as **function of** t, starting at t_0:

$$s(t) = \int_{t_0}^{t_1} \sqrt{x'(u)^2 + y'(u)^2} du$$

- **Speed** at time t:

$$\frac{ds}{dt} = \sqrt{x'(t)^2 + y'(t)^2}$$

- **Surface area**: obtained via rotation of the parametric equation about the x-axis for $a \leq t \leq b$, given $y(t) \geq 0$, $x(t)$ is increasing, and $x'(t) \wedge y'(t)$ are continuous:

$$S = 2\pi \int_a^b y(t)\sqrt{x'(t)^2 + y'(t)^2} dt$$

Polar Coordinates

- **Polar coordinate system**: a two-dimensional coordinate system wherein each point is determined by the distance and angle from a reference point and direction.
  - **Radial coordinate, r**: the distance from reference point.
  - **Angular coordinate, θ**: the angle from reference direction.
  - A point P has polar coordinates (r, θ) with the angle measured in the counterclockwise direction by convention.
- **Conversion between polar and rectangular coordinates**:

$$x = r\cos\theta \qquad y = r\sin\theta \qquad r = \sqrt{x^2 + y^2} \qquad \tan\theta = \frac{y}{x} \leftarrow x \neq 0$$

- **If $r > 0$ then**: (r, θ) must lie in quadrant I or IV;

$$\theta = \begin{cases} \tan^{-1}\frac{y}{x} & \leftarrow x > 0 \\ \tan^{-1}\frac{y}{x} + \pi & \leftarrow x < 0 \\ \pm\frac{\pi}{2} & \leftarrow x = 0 \end{cases}$$

- **Non-uniqueness**: Multiple representations can represent the same point, i.e.,

$$(r, \theta) \equiv (r, \theta + 2n\pi) \equiv (-r, \theta + (2n+1)\pi) \qquad \leftarrow n \in \mathbb{Z}$$

- **Polar Equations**:

Curve	**Polar Equation**
Circle of radius R, center at origin	$r = R$
Line through origin slope $m = \tan\theta_0$	$\theta = \theta_0$
Line, where $P_0 = (d, \alpha)$ is closest to the origin	$r = d\sec(\theta - \alpha)$
Circle radius a, center at $(a, 0)$ $(x-a)^2 + y^2 = a^2$	$r = 2a\cos\theta$
Circle radius a, center at $(0, a)$ $x^2 + (y-a)^2 = a^2$	$r = 2a\sin\theta$

Area and Arc Length in Polar Coordinates

- **Area in Polar Coordinates**: given that f is continuous, then the sector is bounded by:
  - **Polar curve, r**: $r = f(\theta)$
  - **Two rays, α, β**: where each ray is an angle θ with $\alpha < \beta, \quad \beta = \theta - \alpha$
  - Thus, the area is equal to the integral between α and β, i.e.

$$A = \frac{1}{2} \int_{\alpha}^{\beta} f(\theta)^2 d\theta$$

- **Arc length of polar curve**: given $\alpha \leq \theta \leq \beta$:

$$s = \int_{\alpha}^{\beta} \sqrt{f(\theta)^2 + f'(\theta)^2} d\theta$$

Quiz 3

1. Indicate whether the following statements are **True** or **False**, with justification.

   (a) The series $\sum_{n=1}^{\infty}(-1)^n \cos\left(\frac{1}{n}\right)$ converges.

   ✘ **False:**

$$\lim_{n\to\infty} a_n \stackrel{?}{=} 0, \quad a_n = \cos\left(\frac{1}{n}\right) \qquad \text{Apply the } n^{th} \text{ term test}^{\uparrow}$$

$$\implies \lim_{n\to\infty} \cos\left(\frac{1}{n}\right) = 1$$

   $\lim_{n\to\infty} a_n \neq 0$, thus the series diverges.

   (b) If the radius of converges of the power series $\sum_{n=0}^{\infty} a_n x^n$ is $R = 5$, then the series must converge for $x = -3$ and $x = -4$.

   ✓ **True:**

$$c = 0, \quad R = 5 \implies \text{converges } \forall x \in (-5, 5)$$

   By the Interval of convergence↑

   $x = -3 \wedge x = 4 \in (-5, 5)$, thus the series must converge at these values.

2. Determine whether the following series converge absolutely/conditionally, or diverge.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{2n+5}$

$$\sum_{n=1}^{\infty} (-1)^n a_n, \quad a_n = \frac{n}{2n+5} \qquad \text{Apply the alternating series test}^{\uparrow}$$

$$\implies \lim_{n\to\infty} \frac{n}{2n+5} = \frac{\infty}{\infty}$$

$$\implies \lim_{n\to\infty} \frac{n}{2n+5} = \frac{1}{2} \qquad \text{By L'Hôpital's Rule}$$

$\lim\limits_{n\to\infty} a_n \neq 0$, thus the series diverges

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{2\sqrt{n}-1}$

$$\sum_{n=1}^{\infty} (-1)^n a_n, \quad a_n = \frac{1}{2\sqrt{n}-1} \qquad \text{Apply the alternating series test}^{\uparrow}$$

$$\implies \lim_{n\to\infty} \frac{1}{2\sqrt{n}-1} = 0$$

$$\implies a_n \text{ converges} \qquad \text{Note: } |a_n| \text{ decreases monotonically}$$

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{2\sqrt{n}-1} \right| \stackrel{?}{=} \text{converges} \qquad \text{Apply the absolute convergence test}^{\uparrow}$$

$$\implies \lim_{n\to\infty} \frac{1}{2\sqrt{n}-1} \qquad \lim_{n\to\infty} a_n = 0 \to n^{th} \text{ term inconclusive...}$$

$$L = \lim_{n\to\infty} \frac{a_n}{b_n}, \quad b_n = \frac{1}{\sqrt{n}} \qquad \text{Apply the limit comparison test}^{\uparrow}$$

$$= \lim_{n\to\infty} \frac{1}{2\sqrt{n}-1} \cdot \sqrt{n}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n}}{2\sqrt{n}-1} \cdot \frac{n^{-\frac{1}{2}}}{n^{-\frac{1}{2}}}$$

$$= \lim_{n\to\infty} \frac{1}{2-n^{-\frac{1}{2}}} = \frac{1}{2}$$

$L > 0$, and b_n diverges by the p-series, implying the $|a_n|$ diverges. Thus, the original series converges conditionally.

3. Find a power series expansion with the center $c = 0$ for

$$f(x) = \frac{1}{1+x^3}$$

and find the interval of convergence. Hint: use $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \leftarrow |x| < 1$

$$\Longrightarrow \frac{1}{1+x^3} = \frac{1}{1-(-x^3)}$$

$$= \sum_{n=1}^{\infty} (-x^3)^n = \sum_{n=1}^{\infty} (-1)^n x^{3n} \qquad \text{Apply hint}$$

$$\Longrightarrow \frac{1}{1+x^3} = \sum_{n=1}^{\infty} (-1)^n x^{3n} \qquad \leftarrow |x| < 1$$

Thus, the interval of convergence is all values in the interval $(-1, 1)$.

4. Find the radius of convergence of the power series given by

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+1}}{2^n n}$$

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \qquad \text{Apply the ratio test}^{\uparrow}$$

$$\Longrightarrow \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2(n+1)+1}}{2^{n+1}(n+1)} \cdot \frac{2^n n}{(-1)^n x^{2n+1}} \right|$$

$$= \lim_{n\to\infty} \frac{|x|^{2n+3}}{2^{2n}(n+1)} \cdot \frac{2^n n}{|x|^{2n+1}}$$

$$= \lim_{n\to\infty} \frac{|x|^{2n+1} \cdot |x|^2}{2^n \cdot 2(n+1)} \cdot \frac{2^n n}{|x|^{2n+1}}$$

$$= |x|^2 \lim_{n\to\infty} \frac{n}{2n+2} = \frac{\infty}{\infty}$$

$$= |x|^2 \lim_{n\to\infty} \frac{1}{2} \qquad \text{By L'Hôpital's Rule}$$

$$\Longrightarrow \frac{|x|^2}{2} < 1 \qquad \text{converges when } \rho < 1$$

$$= |x| < \sqrt{2}$$

Thus, the interval of convergence is $(-\sqrt{2}, \sqrt{2})$ with $R = \sqrt{2},$ and $c = 0$ (endpoints not required to be tested for this problem).

Quiz 4

1. Indicate whether the following statements are **True** or **False**, with justification.

(a) The curve with parametric representations $c(t) = (4 + 3\cos t, 5 + 3\sin t)$ is a circle with radius $R = 3$ centered art the origin.

✘ False:

$$c(t) = (a + R\cos\theta, b + R\sin\theta) \qquad \text{Parametrization of a circle}^{\uparrow}$$
$$c(t) = (3\cos\theta, 3\sin\theta) \qquad \leftarrow R = 3, (0,0)$$

Note: $a = 4$, $b = 5$, thus it's a circle with radius 3, but not centered at the origin.

(b) The parametric representation given by $c(t) = (\sin t, t)$ can be represented by function of the form $y = f(x)$.

✘ False:

Note that the y component of the parametric representation is given by $y = t$. Substituting y for t in the x component yields $x = \sin y$, which is a function of x in terms of y, but NOT a function of y in terms of x.

2. Determine whether the following series converge of diverge, with justification.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{n!}$

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| && \text{Apply the ratio test}^{\uparrow} \\
&= \lim_{n\to\infty} \left|\frac{(n+1)^3}{(n+1)!}\cdot\frac{n!}{n^3}\right| \\
&= \lim_{n\to\infty} \frac{(n+1)^3}{(n+1)n!}\cdot\frac{n!}{n^3} \\
&= \lim_{n\to\infty} \frac{(n+1)^2}{n^3} \\
&= \lim_{n\to\infty} \frac{n^2+2n+1}{n^3}\cdot\frac{n^{-3}}{n^{-3}} \\
&= \lim_{n\to\infty} n^{-1}+2n^{-2}+n^{-3} = 0
\end{aligned}$$

$\rho < 1$, thus the series converges absolutely.

(b) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{n}{3n+1}\right)^n$

$$\begin{aligned}
C &= \lim_{n\to\infty} |a_n|^{\frac{1}{n}} && \text{Apply the root test}^{\uparrow} \\
&= \lim_{n\to\infty} \left|\left(\frac{n}{3n+1}\right)^n\right|^{\frac{1}{n}} \\
&= \lim_{n\to\infty} \frac{n}{3n+1} = \frac{1}{3} && \text{By L'Hôpital's Rule}
\end{aligned}$$

$C = \dfrac{1}{3} < 1$, thus the series converges absolutely.

3. Find the Maclaurin series of (using substitution and/or multiplication)

$$f(x) = x\cos(x^2)$$

$$\begin{aligned}
\cos x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} && \text{Use relevant Maclaurin series}^{\uparrow} \\
\implies \cos(x^2) &= \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n}}{(2n)!} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!} && \text{Substitution of } x^2 \\
x \cdot \cos(x^2) &= x \cdot \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{(2n)!} && \text{Multiply by } x
\end{aligned}$$

$\cos x$ converges $\forall x$, thus $x\cos x^2$ also converges $\forall x$. Therefore:

$$f(x) = x\cos(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{(2n)!} \qquad \leftarrow \forall x \in \mathbb{R}$$

4. Express the following integral as a infinite series, first by finding the Maclaurin series of the integrand, then integrating this series.

$$\int_0^1 e^{-x^2}\, dx$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Use relevant Maclaurin series ↑

$$\implies f(x) = e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!} \qquad \leftarrow \forall x \in \mathbb{R}$$

$$\implies \int_0^1 e^{-x^2}\, dx = \int_0^1 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}\, dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^1 x^{2n}\, dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot \left. \frac{x^{2n+1}}{2n+1} \right|_0^1$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)n!} - 0$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)n!}}$$

5. Consider the curve with parametric representation

$$c(t) = (\sin 2t + \cos t, \cos 2t - \sin t)$$

Find an equation of the tangent line at $t = \pi$

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} \quad \leftarrow x'(t) \neq 0$$

Note: slope of a tangent line↑

$$\Longrightarrow \frac{dy}{dx} = \frac{-2\sin(2t) - \cos t}{2\cos(2t) - \sin t}$$

$$\Longrightarrow m = \left.\frac{dy}{dx}\right|_{t=\pi} = \frac{0-(-1)}{2-0} = \frac{1}{2}$$

$$\Longrightarrow \boxed{y - 1 = \frac{1}{2}(x+1)}$$

$$c(\pi) = (-1, 1)$$

Final Review Questions

Note: these questions were taken form a provided review sheet; they focus on sections 10.6–11.4. Some questions already exist on the quizzes, but will be duplicated here.

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1. Find the interval of convergence of the following power series.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{n} x^n$

$$\begin{aligned} C &= \lim_{n\to\infty} |a_n|^{\frac{1}{n}} && \text{Apply the root test}^{\uparrow} \\ &= \lim_{n\to\infty} \left| \frac{5^n x^n}{n} \right|^{\frac{1}{n}} \\ &= 5|x| \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 5|x| \end{aligned}$$

$$\implies |x| < \frac{1}{5} \qquad \text{converges for } C < 1$$

$$\begin{aligned} f\left(-\frac{1}{5}\right) &= \sum_{n=1}^{\infty} \frac{5^n}{n} \left(-\frac{1}{5}\right)^n \\ &= \sum_{n=1}^{\infty} (-1)^n \frac{1}{n}, \quad a_n = \frac{1}{n} && \text{Apply the alternating series test}^{\uparrow} \end{aligned}$$

$$\lim_{n\to\infty} a_n = 0 \land |a_n| \text{ decreases monotonically} \qquad \implies \text{converges}$$

$$f\left(\frac{1}{5}\right) = \sum_{n=1}^{\infty} \frac{5^n}{n} \left(\frac{1}{5}\right)^n = \sum_{n=1}^{\infty} \frac{1}{n} \qquad \implies \text{diverges by } p\text{-series}^{\uparrow}$$

$$\implies \text{Interval of convergence: } \left[-\frac{1}{5}, \frac{1}{5}\right)$$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2+1}$

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| && \text{Apply the ratio test}^{\uparrow} \\
&= \lim_{n\to\infty} \left|\frac{(x-2)^{n+1}}{(n+1)^2+1} \cdot \frac{n^2+1}{(x-2)^n}\right| \\
&= \lim_{n\to\infty} \frac{(|x-2|)^n \cdot |x-2|}{n^2+2n+2} \cdot \frac{n^2+1}{(|x-2|)^n} \\
&= |x-2| \lim_{n\to\infty} \frac{n^2+1}{n^2+2n+2} \\
&= |x-2| \lim_{n\to\infty} \frac{2}{2} && \text{By L'Hôpital's Rule}
\end{aligned}$$

$$\begin{aligned}
&\implies |x-2| < 1 && \text{converges for } \rho < 1 \\
&\implies 1 < x < 3
\end{aligned}$$

$$f(1) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1} \quad a_n = \frac{1}{n^2+1} \qquad \text{Apply the alternating series test}^{\uparrow}$$

$$\lim_{n\to\infty} a_n = 0 \implies \text{converges} \qquad |a_n| \text{ decreases monotonically}$$

$$f(3) = \sum_{n=1}^{\infty} \frac{1}{n^2+1}$$

b_n converges $\rightarrow$ a_n converges — Apply the direct comparison test↑

$$b_n = \frac{1}{n^2} \implies b_n \text{ converges} \qquad \text{By p-series}^{\uparrow}$$

$\implies$ Interval of convergence: $[1, 3]$

2. Find the Taylor series of the following functions $f(x)$ centered at the given value of c using the definition.

(a) $f(x) = e^x, \quad c = 2$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \leftarrow \forall n \geq 0$$

Use relevant Maclaurin series↑

$$f^{(n)}(2) = e^2 \leftarrow f^{(n)}(x) = e^x$$

$$\implies \boxed{\sum_{n=0}^{\infty} \frac{e^2(x-2)^n}{n!}}$$

Center series, $c = 2$

(b) $f(x) = \sqrt{x}, \quad c = 1$

$$(1+x)^{\alpha} = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n \quad \leftarrow |x| < 1$$

Use relevant Maclaurin series↑

$$\implies \boxed{\sum_{n\geq 0} \binom{\frac{1}{2}}{n} (x-1)^n}$$

note: $(1+(x-1))^{\frac{1}{2}} = \sqrt{n}$

3. Find the Maclaurin series of the following functions using substitution and/or multiplication.

(a) $f(x) = x\cos(2x)$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \quad \leftarrow \forall x$$

Use relevant Maclaurin series↑

$$\implies \cos(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 4^n x^{2n}}{(2n)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-4)^n x^{2n}}{(2n)!}$$

Substitution of x^2

$$x \cdot \cos(2x) = x \cdot \sum_{n=0}^{\infty} \frac{(-4)^n x^{2n}}{(2n)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-4)^n x^{2n+1}}{(2n)!}$$

Multiply by x

$$\implies f(x) = \boxed{\sum_{n=0}^{\infty} \frac{(-4)^n x^{2n+1}}{(2n)!}} \quad \leftarrow \forall x$$

(b) $f(x) = \dfrac{x^3}{1+x}$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n \quad \leftarrow |x| < 1$$

Use relevant Maclaurin series↑

$$\implies \frac{x^3}{1+x} = \sum_{n=1}^{\infty} (-1)^n (x^3)^n$$

$$= \boxed{\sum_{n=1}^{\infty} (-1)^n x^{n+3}} \quad \leftarrow |x| < 1$$

4. Express the following integral as a power series, first by finding the Maclaurin series of the integrand, then integrating this series term-by-term:

$$\int_0^1 e^{-x^2}\,dx$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \leftarrow \forall x \qquad \text{Use relevant Maclaurin series}^{\uparrow}$$

$$\begin{aligned}
\implies f(x) = e^{-x^2} &= \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!} \qquad \leftarrow \forall x
\end{aligned}$$

$$\begin{aligned}
\implies \int_0^1 e^{-x^2}\,dx &= \int_0^1 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}\,dx \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^1 x^{2n}\,dx \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot \left.\frac{x^{2n+1}}{2n+1}\right|_0^1 \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)n!} - 0 \\
&= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)n!}}
\end{aligned}$$

5. Find the parametric equations for the following curves.

(a) The line through $(3, 6)$ and $(-2, 0)$.

$$P(-2,0), \quad Q(3,6)$$
$$\implies m = \frac{y_Q - y_P}{x_Q - x_P} = \frac{6-0}{3+2} = \frac{6}{5}$$
$$\implies y = m(x - x_P) + y_P = \frac{6}{5}(x+2) + 0 = \frac{6}{5}x + \frac{12}{5}$$
$$\begin{cases} x = t \\ y = \frac{6}{5}x + \frac{12}{5} \end{cases}$$
$$\implies \boxed{c(t) = \left(t, \frac{6}{5}t + \frac{12}{5}\right)}$$

(b) The circle of radius 5 centered at $(1, 7)$.

$$c(t) = (a + R\cos\theta, b + R\sin\theta) \quad \leftarrow \text{Parametrization of a circle}^{\uparrow}$$

$$\implies c(t) = (1 + 5\cos\theta, 7 + 5\sin\theta) \in [0, 2\pi)$$

(c) The ellipse
$$\left(\frac{x-1}{2}\right)^2 + \left(\frac{y+1}{3}\right)^2 = 1$$

$$c(\theta) = (a\cos\theta, b\sin\theta) \qquad \text{Parametrization of a ellipse}^{\uparrow}$$

$$c(\theta) = (2\cos\theta, 3\sin\theta) \qquad \leftarrow \left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$$

$$c(\theta) = \boxed{(1 + 2\cos\theta, -1 + 3\sin\theta)} \qquad \leftarrow \left(\frac{x-1}{2}\right)^2 + \left(\frac{y+1}{3}\right)^2 = 1$$

6. Find the equation of the tangent line to the curve

$$x = \sin(2t) + \cos(t), \quad y = \cos(2t) - \sin(t), \qquad \leftarrow t = \pi$$

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} \quad \leftarrow x'(t) \neq 0$$

Slope of a tangent line↑

$$\Longrightarrow \frac{dy}{dx} = \frac{-2\sin(2t) - \cos t}{2\cos(2t) - \sin t}$$

$$\Longrightarrow m = \left.\frac{dy}{dx}\right|_{t=\pi} = \frac{0-(-1)}{2-0} = \frac{1}{2}$$

$$\Longrightarrow \boxed{y - 1 = \frac{1}{2}(x+1)} \qquad c(\pi) = (-1, 1)$$

7. Find the arc length of the curve

$$x = \frac{2}{3}t^2, \quad y = t^2 - 2, \qquad \leftarrow 0 \leq t \leq 2$$

$$s = \int_a^b \sqrt{x'(t)^2 + y'(t)^2} dt \qquad \leftarrow \text{Arch length}\uparrow$$

$$= \int_0^2 \sqrt{\left(\frac{4}{3}t\right)^2 + (2t)^2} dt$$

$$= \int_0^2 \sqrt{\frac{16}{9}t^2 + 4t^2}\, dt$$

$$= \int_0^2 \sqrt{\frac{52t^2}{9}}\, dt = \int_0^2 \frac{2\sqrt{13}t}{3}\, dt$$

$$= \left.\frac{2\sqrt{13}t^2}{6}\right|_0^2 = \left.\frac{t^2\sqrt{13}}{3}\right|_0^2 = \frac{4\sqrt{13}}{3} - 0$$

$$= \boxed{\frac{4\sqrt{13}}{3}}$$

8. Find the surface area obtained by rotating the following around the x-axis;

$$x = e^t - t, \quad y = 4e^{\frac{t}{2}}, \qquad \leftarrow 0 \le t \le 1$$

$$\begin{aligned}
S &= 2\pi \int_a^b y(t)\sqrt{x'(t)^2 + y'(t)^2}\,dt \qquad \leftarrow \text{Surface area}^{\uparrow} \\
&= 2\pi \int_0^1 4e^{\frac{t}{2}}\sqrt{(e^t - 1)^2 + (2e^{\frac{t}{2}})^2}\,dt \\
&= 8\pi \int_0^1 e^{\frac{t}{2}}\sqrt{e^{2t} - 2e^t + 1 + 4e^t}\,dt \\
&= 8\pi \int_0^1 e^{\frac{t}{2}}\sqrt{e^{2t} + 2e^t + 1}\,dt \\
&= 8\pi \int_0^1 e^{\frac{t}{2}}\sqrt{(e^t + 1)^2}\,dt \\
&= 8\pi \int_0^1 e^{\frac{t}{2}}(e^t + 1)dt \\
&= 8\pi \int_0^1 \left(e^{\frac{3t}{2}} + e^{\frac{t}{2}}\right)dt \\
&= 8\pi \left(\frac{2}{3}e^{\frac{3t}{2}} + 2e^{\frac{t}{2}}\right)\Bigg|_0^1 = 8\pi\left(\left(\frac{2}{3}e^{\frac{3}{2}} + 2e^{\frac{1}{2}}\right) - \left(\frac{3}{2} + 2\right)\right) \\
&= \boxed{8\pi\left(\frac{2}{3}e^{\frac{3}{2}} + 2e^{\frac{1}{2}} - \frac{7}{2}\right)}
\end{aligned}$$

9. Match each equation in rectangular coordinates with its equation in polar coordinates.

(a) $x^2 + y^2 = 4$

(b) $x^2 + (y-1)^2 = 1$

(c) $x^2 - y^2 = 4$

(d) $x + y = 4$

(i) $r^2(1 - 2\sin^2\theta) = 4$

(ii) $r(\cos\theta + \sin\theta) = 4$

(iii) $r = 2\sin\theta$

(iv) $r = 2$

Polar Equations ↑

(a) $\leftarrow r = \sqrt{x^2+y^2} \implies r = \sqrt{4} \implies$ (iv)

(b) $\leftarrow a^2 = \sqrt{x^2+(y-a)^2} \implies r = 2a\sin\theta \implies$ (iii)

(c) $\implies$ (i) ? Not sure how.

(d) $\leftarrow x = r\cos\theta, \quad y = r\sin\theta \implies r(\cos\theta + \sin\theta) \implies$ (ii)

10. Find the area enclosed by one loop of the curve

$$r^2 = \cos 2\theta$$

$$A = \frac{1}{2}\int_\alpha^\beta f(\theta)^2 d\theta, \quad \alpha = -\frac{\pi}{4}, \beta = \frac{\pi}{4}$$

Area in polar coordinates ↑

$$\implies A = \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{4}} (\cos 2\theta)^2\, d\theta$$

$$= \frac{1}{2}\cdot\frac{-\sin 2\theta}{2}\Bigg|_{\frac{-\pi}{4}}^{\frac{\pi}{4}} = \frac{-\sin 2\theta}{4}\Bigg|_{\frac{\pi}{4}}^{\frac{-\pi}{4}}$$

$$= \frac{\sin\frac{\pi}{2}}{4} - \frac{-\sin\frac{\pi}{2}}{4} = \frac{1}{4} - \frac{-1}{4}$$

$$= \boxed{\frac{1}{2}}$$