

1. Use induction to prove the following summation formulas. Remember rules of logs. Specifically $\log(ab) = \log(a) + \log(b)$.

$$(a) \sum_{i=1}^n a_i + b_i = \sum_{i=1}^n a_i + \sum_{i=1}^n b_i$$

Proof.

$$\text{Base case: } n = 1 \implies (a_1 + b_1) = (a_1) + (b_1)$$

$$\text{Inductive hypothesis: } n = k \implies \sum_{i=1}^k a_i + b_i = \sum_{i=1}^k a_i + \sum_{i=1}^k b_i$$

Inductive case:

$$\begin{aligned} \sum_{i=1}^{k+1} a_i + b_i &= \left(\sum_{i=1}^k a_i + b_i \right) + (a_{k+1} + b_{k+1}) \\ &= \left(\sum_{i=1}^k a_i + \sum_{i=1}^k b_i \right) + (a_{k+1} + b_{k+1}) \quad \text{by I.H.} \\ &= \left(\sum_{i=1}^k a_i \right) + a_{k+1} + \left(\sum_{i=1}^k b_i \right) + b_{k+1} \\ &= \sum_{i=1}^{k+1} a_i + \sum_{i=1}^{k+1} b_i \end{aligned}$$

■

$$(b) \sum_{i=1}^n ca_i = c \sum_{i=1}^n a_i$$

Proof.

$$\text{Base case: } n = 1 \implies ca_1 = ca_1$$

$$\text{Inductive hypothesis: } n = k \implies \sum_{i=1}^k ca_i = c \sum_{i=1}^k a_i$$

Inductive case:

$$\begin{aligned} \sum_{i=1}^{k+1} ca_i &= \left(\sum_{i=1}^k ca_i \right) + ca_{k+1} \\ &= \left(c \sum_{i=1}^k a_i \right) + ca_{k+1} \quad \text{by I.H.} \\ &= c \sum_{i=1}^{k+1} a_i \end{aligned}$$

■

$$(c) \sum_{i=1}^n \log(a_i) = \log \left(\prod_{i=1}^n a_i \right)$$

Proof.

$$\text{Base case: } n = 1 \implies \log(a_1) = \log(a_1)$$

$$\text{Inductive hypothesis: } n = k \implies \sum_{i=1}^k \log(a_i) = \log \left(\prod_{i=1}^k a_i \right)$$

Inductive case:

$$\begin{aligned} \sum_{i=1}^{k+1} \log(a_i) &= \sum_{i=1}^k \log(a_i) + \log(a_{k+1}) \\ &= \log \left(\prod_{i=1}^k a_i \right) + \log(a_{k+1}) && \text{by I.H.} \\ &= \log \left(\prod_{i=1}^{k+1} a_i \right) && \blacksquare \end{aligned}$$

$$(d) \sum_{i=1}^n 2i - 1 = n^2$$

Proof.

$$\text{Base case: } n = 1 \implies 2 - 1 = 1 = 1^2$$

$$\text{Inductive hypothesis: } n = k \implies \sum_{i=1}^k 2i - 1 = k^2$$

Inductive case:

$$\begin{aligned} \sum_{i=1}^{k+1} 2i - 1 &= \left(\sum_{i=1}^k 2i - 1 \right) + 2(k+1) - 1 \\ &= k^2 + 2(k+1) - 1 && \text{by I.H.} \\ &= k^2 + 2k + 1 \\ &= (k+1)^2 && \blacksquare \end{aligned}$$

$$(e) \sum_{i=1}^n i^3 = \left(\frac{n(n+1)}{2} \right)^2$$

Proof.

$$\text{Base case: } n = 1 \implies \frac{1(1+1)}{2} = 1 = 1^3$$

$$\text{Inductive hypothesis: } n = k \implies \sum_{i=1}^k i^3 = \left(\frac{k(k+1)}{2} \right)^2$$

Inductive case:

$$\begin{aligned} \sum_{i=1}^{k+1} i^3 &= \sum_{i=1}^k i^3 + (k+1)^3 \\ &= \left(\frac{k(k+1)}{2} \right)^2 + (k+1)^3 && \text{by I.H.} \\ &= \left(\frac{k^2(k+1)^2}{4} \right) + (k+1)^3 \\ &= \frac{k^4 + 2k^3 + k^2}{4} + \frac{4k^3 + 12k^2 + 12k + 4}{4} \\ &= \frac{k^4 + 6k^3 + 13k^2 + 4}{4} \\ &= \frac{(k+1)^2(k+2)^2}{4} \\ &= \left(\frac{(k+1)(k+2)}{2} \right)^2 \quad \blacksquare \end{aligned}$$

2. Prove that $n! > 2^n \quad \forall n > 4$

Proof.

Base case: $n = 5 \implies 5! > 2^5 \equiv 120 > 32$

Inductive hypothesis: $k \geq n \implies k! > 2^k$

Inductive case:

$$(k+1)! > 2^{k+1}$$

$$\implies k!(k+1) > 2^k(k+1) > 2^k(2) = 2^{k+1} \quad \text{by I.H.}$$

$$\therefore k!(k+1) > 2^{k+1}$$



3. A prime number p is a natural number greater than 1 where $1|p$ and $p|p$ and nothing else divides p . In class we showed that if n^2 is even, then n is even. Extend this to any prime number.

Prove that if p is prime and $p|n^2 \implies p|n$.

Equivalent Theorem: $p \nmid n \implies p \nmid n^2$.

Proof.

$$p \nmid n \implies n = pq + r, \quad \exists r > 0$$

$$\implies n^2 = nn = (pq + r)(pq + r) = p(pq^2 + 2qr) + r^2$$

$$\implies p \nmid p(pq^2 + 2qr) + r^2$$



4. Prove:

$$\sum_{i=1}^n ia^i = \frac{a - (n+1)a^{n+1} + na^{n+2}}{(a-1)^2}$$

Proof.

Base case: $n = 1 \implies$

$$a = \frac{a - 2a^2 + a^3}{(a-1)^2} = \frac{a(a^2 - 2a + 1)}{a^2 - 2a + 1} = a$$

Inductive hypothesis: $n = k \implies \sum_{i=1}^k ia^i = \frac{a - (k+1)a^{k+1} + ka^{k+2}}{(a-1)^2}$

Inductive case:

$$\begin{aligned} p \sum_{i=1}^{k+1} ia^i &= \frac{a - (k+1)a^{k+1} + ka^{k+2}}{(a-1)^2} + (k+1)a^{k+1} \\ &= \frac{a - (k+1+1)a^{k+1+1} + (k+1)a^{k+1+2}}{(a-1)^2} = \frac{a - (k+1)a^{k+1} + ka^{k+2}}{(a-1)^2} + (k+1)a^{k+1} \text{ by I.H.} \\ &= \frac{a - (k+1+1)a^{k+1+1} + (k+1)a^{k+1+2}}{(a-1)^2} - \frac{a - (k+1)a^{k+1} + ka^{k+2}}{(a-1)^2} - (k+1)a^{k+1} = 0 \\ &= (k+1)a^{k+1} - (k+1)a^{k+1} = 0 \quad \text{by lots of algebra (used wolfram)} \\ 0 &= 0 \end{aligned}$$

5. Let's prove some theorems about cardinality. For each of the following equations, give a bijection between the two sets.

Hint: you can (should) reuse a bijection you've already defined.

(a) $|E| = |\mathbb{N}|, f : E \rightarrow \mathbb{N}, \quad f(e) = \frac{e}{2}$

(b) $|\mathbb{N}| = |\mathbb{Z}|, f : \mathbb{N} \rightarrow \mathbb{Z}, \quad f(n) = n + 1$

(c) $|\mathbb{N}| = |\mathbb{Q}^+|, f : \mathbb{N} \rightarrow \mathbb{Q}^+, \quad f(n) = \frac{1}{n+1}$

(d) $|\mathbb{Z}| = |\mathbb{Q}|, f : \mathbb{Z} \rightarrow \mathbb{Q}, \quad f(z) = \frac{1}{z}$

(e) $|E| = |\mathbb{Q}|, f : E \rightarrow \mathbb{Q}, \quad f(e) = \frac{1}{e}$

6. Prove that $\forall n, |\mathbb{N}| = |\mathbb{N}^n|$.

You can assume that $d : \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}$ is the diagonal bijection from class.

Proof.

Base case: $|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$ (proved in class)

Inductive hypothesis: $n = k \implies |\mathbb{N}| = |\mathbb{N}^k|$

Inductive case:

$$\begin{aligned} |\mathbb{N}^{k+1}| &= |(\mathbb{N}_1 \times \dots \times \mathbb{N}_k) \times \mathbb{N}_{k+1}| \\ &= |\mathbb{N}^k \times \mathbb{N}_{k+1}| && \text{by I.H.} \\ &= |\mathbb{N}^{k+1}| \quad \blacksquare \end{aligned}$$

7. Let's try to find a use for this infinity nonsense.

In computer science it can be useful to look at problems as a language. A language is just a set of finite strings, so we can make a language describing π as:

$$L_\pi = \{ "3", "3.1", "3.14", "3.141", \dots \}$$

So why do we care about languages? Well, we can phrase all of our problems in CS as different languages. For example $L_{factor} = \{ "6 = 2 \cdot 3", "12 = 2 \cdot 2 \cdot 3", \dots \}$ is the language of numbers and their factors. We can make the language of graphs and their shortest paths, the languages of lists and their sorting. Really we can make a language for any problem.

A language is decidable if there is a program that can (eventually) produce any string in that language. We want to prove that there is at least 1 undecidable language. That is, there is a problem that can't be solved by a program.

(a) Show that there are countably many programs we can write.

Need to show: there exists an injective function from $S \rightarrow \mathbb{N}$

Let f be a function that takes program and returns a binary sequence that represents the instructions of that program and vice versa.

If two binary sequences are equal, then they must produce the same program, implying the function is injective ■

(b) Show that there are uncountably many languages.

Since π can be described above as L_π , then two unique strings could easily map the same output. Thus, it cannot be an injective function and $|L_\pi| > |\mathbb{N}|$ ■

8. Show why the following two lemmas are actually invalid.

Note: the theorem is actually true despite this (haha)

Hint: part 1 is a problem with induction and part 2 is a problem with infinity.

Lemma 0.1. *There are infinitely many green vegetables.*

Proof.

Base case: There exists a head of lettuce.

Inductive hypothesis: there is a set of k green vegetables.

↑ ERROR: the inductive hypothesis must say something about the set of vegetables, not just that the set exists. This is **not a valid hypothesis**.

Inductive case: Suppose, by the inductive hypothesis, there are k green vegetables.

Remember that head of lettuce from the base case? Let's add that into our set of k vegetables. Now we have a set of $k + 1$ green vegetables.

Therefore we must have a set of n vegetables for any n , which means we have infinitely many green vegetables. ■

Lemma 0.2. *Every green edible thing is a vegetable.*

Proof.

Let's consider the set of all possible green edible things G . By the previous lemma G is clearly an infinite set. I'll sort the set putting all of the green vegetables at the front.

Now we want to show that there are ONLY vegetables in this set.

↑ ERROR: (1) G is clearly not an infinite set. (2) You can't sort an infinite set, therefore, the following hypothesis is an impossible claim.

Base case: the first element is that dang head of lettuce from the last proof.

Inductive hypothesis: The first k elements of G are vegetables.

Inductive case: Assume, by the inductive hypothesis, that the first k elements are vegetables. Then the next element must also be a vegetable. If it wasn't then there would only be a finite number of vegetables, but by the last theorem this is impossible.

Therefore, the first $k + 1$ elements of G must be vegetables. Thus, for all n , the first n elements of G are vegetables. Now, since G is a countably infinite set, and we've shown that each position in the set is a vegetable, it must be the case that the set contains only vegetables.

Theorem 0.2.1: *Mountain Dew is a vegetable.*

Proof.

It's green and edible. From the previous two lemmas, it's clearly a vegetable. ■