

# Applied Linear Algebra



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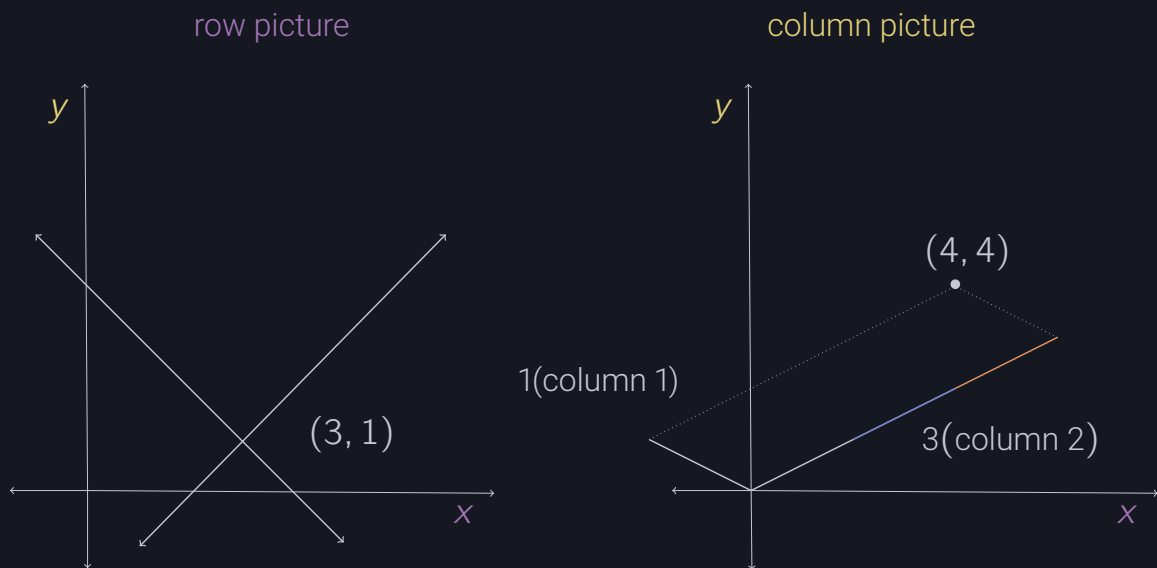
# 1 Matrices and Gaussian Elimination



## 1.2 The Geometry of Linear Equations

### Problems 1–12

- For the equations  $x + y = 4$ ,  $2x - 2y = 4$ , draw the row picture (two intersecting lines) and the column picture (combination of two columns equal to the column vector  $(4, 4)$  on the right side).



### 1.2.1

- Solve to find a combination of the columns that equals  $b$ :

$$u - v - w = b_1$$

$$v + w = b_2$$

$$w = b_3$$

$$\implies w = b_3$$

$$\implies v = b_2 - b_3$$

$$\implies u = b_1 + v + w = b_1 + b_2$$

- Describe the intersection of the three planes  $u + v + w + z = 6$  and  $u + w + z = 4$  and  $u + w = 2$  (all in four-dimensional space). Is it a line or a point or an empty set? What is the intersection if the fourth plane  $u = -1$  is included? Find a fourth equation that leaves us with no solution.

- A line; as  $u + w = 2$  is only a line? A fourth plane with  $u = -1$  would produce a normally intersecting point. Any addition equation when  $u + w \neq 2$  would produce an inconsistent equation.

4. Sketch these three lines and decide if the equations are solvable:

$$x + 2y = 2$$

$$x - y = 2$$

$$y = 1$$



#### 1.2.4

##### Inconsistent; multiple points of intersect

What happens if all right-hand sides are zero? Is there any nonzero choice of right-hand sides that allows the three lines to intersect at the same point?

- If all the solutions were zero, then it would be a trivial solution.
  - Yes, e.g.,  $x - y = -1$  would produce a single point of intersection.
5. Find two points on the line of intersection of the three planes  $t = 0$  and  $z = 0$  and  $x + y + z + t = 1$  in four-dimensional space.

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

6. When  $b = (2, 5, 7)$ , find a solution  $(u, v, w)$  to equation (4) different from the solution  $(1, 0, 1)$  mentioned in the text.
- Since there are infinite solutions, and if  $\mathbf{s}$  vector describing one solution and  $\lambda$  is any scalar, then  $\mathbf{s}\lambda$  is also a solution. E.g.,  $(1, 0, 1)42 = (42, 0, 42)$

8. Explain why the system

$$\begin{aligned}u + v + w &= 2 \\ u + 2v + 3w &= 1 \\ v + 2w &= 0\end{aligned}$$

is singular by finding a combination of the three equations that adds up to  $0 = 1$ . What value should replace the last zero on the right side to allow the equations to have solutions—and what is one of the solutions?

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 0 \end{array} \right] \xrightarrow{R_2 - R_1} \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 2 & 0 \end{array} \right] \xrightarrow{R_3 - R_2} \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

- Replacing the last zero with  $-1$  would yield infinite solutions. One solution would be  $[3, -1, 0]^T$

9. The column picture for the previous exercise (singular system) is

$$u \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + v \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + w \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = b$$

Show that the three columns on the left lie in the same plane by expressing the third as a combination of the first two. What are all the solutions  $(u, v, w)$  if  $b$  is the zero vector  $(0, 0, 0)$ ?

$$-1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

- If  $b$  is equal to the zero vector  $\mathbf{0}$  then the solutions are equal to the kernel<sup>2</sup> i.e.,  $-1x_1, 2x_2, 0x_3 = \mathbf{0}$
10. Under what condition on  $y_1, y_2, y_3$  do the points  $(0, y_1), (1, y_2), (2, y_3)$  lie on a straight line?
- Question 9 describes the state at which they are collinear, i.e.,  $y_3 = 2y_2 - y_1$
11. These equations are certain to have the solution  $x = y = 0$ . For which values of  $a$  is there a whole line of solutions?

$$\begin{aligned}ax + 2y &= 0 \\ 2x + ay &= 0\end{aligned}$$

- Only the scalars that make the lines linearly dependent, i.e.,  $a = 2, -2$

## Problems 17–23

17. The first of these equations plus the second equals the third:

$$\begin{aligned}x + y + z &= 2 \\x + 2y + z &= 3 \\2x + 3y + 2z &= 5\end{aligned}$$

The first two planes meet along a line. The third plane contains that line, because if  $x, y, z$  satisfy the first two equations then they also **span all of  $\mathbb{R}^3$** . The equations have infinitely many solutions (the whole line  $L$ ). Find three solutions.

◦  $\mathbf{v} = (4, 4, 0)$ ,  $\mathbf{w} = (6, 3, 2)$ ,  $\mathbf{u} = 2\mathbf{v} + -1\mathbf{w}$

18. Move the third plane in Problem 17 to a parallel plane  $2x + 3y + 2z = 9$ . Now the three equations have no solution—*why not*? The first two planes meet along the line  $L$ , but the third plane doesn't that **cross** that line.
19. In Problem 17 the columns are  $(1, 1, 2)$  and  $(1, 2, 3)$  and  $(1, 1, 2)$ . This is a “singular case” because the third column is **linearly dependent**. Find two combinations of the columns that give  $\mathbf{b} = (2, 3, 5)$ . This is only possible for  $\mathbf{b} = (4, 6, c)$  if  $c = 10$
20. Normally 4 “planes” in four-dimensional space meet at a **tensor**. Normally 4 column vectors in four-dimensional space can combine to produce  $\mathbf{b}$ . What combination of  $(1, 0, 0, 0)$ ,  $(1, 1, 0, 0)$ ,  $(1, 1, 1, 0)$ ,  $(1, 1, 1, 1)$  produces  $\mathbf{b} = (3, 3, 3, 2)$ ?  $(1, 0, 0, -2)$ ? What 4 equations for  $x, y, z, t$  are you solving? A **lower triangular matrix**, i.e.,

$$\left[ \begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3 \\ 1 & 1 & 0 & 0 & 3 \\ 1 & 1 & 1 & 0 & 3 \\ 1 & 1 & 1 & 1 & 1 \end{array} \right]$$

21. When equation 1 is added to equation 2, which of these are changed: the planes in the row picture, the column picture, the coefficient matrix, the solution?
- Row operations do not change the solution. Row 2 is changed, thus the second plane is changed. **All columns are changed.**?



## 1.3 Gaussian Elimination

### Problems 6, 7

6. Choose a coefficient  $b$  that makes this system singular. Then choose a right-hand side  $g$  that makes it solvable. Find two solutions in that singular case.

$$2x + by = 16$$

$$4x + 8y = g$$

$$2x + 4y = 16$$

$$4x + 8y = 32$$

- Since  $R_2$  is just a multiple of  $R_1$ , then solving for  $x, y$ , with one variable = 0, in the first equation will yield two solutions, i.e.,  $(8, 0), (0, 4)$
7. For which numbers  $a$  does elimination break down (a) permanently, and (b) temporarily?

$$ax + 3y = -3$$

$$4x + 6y = 6$$

Solve for  $x$  and  $y$  after fixing the second breakdown by a row exchange.

- Permanently:  $a = 2$  (linearly dependent, no solution)
- Temporarily:  $a = 0$ ;

$$\left[ \begin{array}{cc|c} 4 & 6 & 6 \\ 0 & 3 & -3 \end{array} \right] \rightarrow \left[ \begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -1 \end{array} \right]$$
$$y = -1, \quad x = 3$$

### Problems 17, 18, 19

17. Which number  $q$  makes this system singular and which right-hand side  $t$  gives it infinitely many solutions? Find the solution that has  $z = 1$ .

$$x + 4y - 2z = 1$$

$$x + 7y - 6z = 6$$

$$3y + qz = t$$

$$x + 4y - 2z = 1$$

$$x + 7y - 6z = 6$$

$$3y + -4z = 5$$

- If  $q = -4$ , then  $R_3$  would have no pivot
- If  $t = 5$ , then there would be finite solutions,  $R_3$  would be linearly dependent with  $R_2$



18. It is impossible for a system of linear equations to have exactly two solutions. Explain why.

- If  $(x, y, z)$  and  $(X, Y, Z)$  are two solutions, what is the other one?
  - There is no other *one*, there would be infinitely many.
- If 25 planes meet at two points, where else do they meet?
  - Every other single point, they would span all of  $\mathbb{R}^3$

19. Three planes can fail to have an intersection point, when no two planes are parallel. The system is singular if row 3 of  $\mathbf{A}$  is a **linearly dependent; a combination** of the first two rows. Find a third equation that can't be solved if  $x + y + z = 0$  and  $x - 2y - z = 1$ .

$$x + y + z = 0$$

$$x - 2y - z = 1$$

$R_1 + R_2 \neq 1 \rightarrow$  parallel; no solution, e.g.,

$$2x - y = 42$$

## Problems 30, 31

30. Use elimination to solve

$$u + v + w = 6$$

$$u + 2v + 2w = 11$$

$$2u + 3v - 4w = 3$$

$$u + v + w = 7$$

$$u + 2v + 2w = 10$$

$$2u + 3v - 4w = 3$$

$$\text{rref} \left( \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 2 & 11 \\ 2 & 3 & -4 & 3 \end{array} \right] \right) \rightarrow \left[ \begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

$$\text{rref} \left( \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 1 & 2 & 2 & 10 \\ 2 & 3 & -4 & 3 \end{array} \right] \right) \rightarrow \left[ \begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

31. For which three numbers  $a$  will elimination fail to give three pivots?

$$ax + 2y + 3z = b_1$$

$$ax + ay + 4z = b_2$$

$$ax + ay + az = b_3$$

- For  $a = 0$ , multiple failures.
- For  $a = 2$ , columns 0, 1 would be equal.
- For  $a = 4$ , rows 1, 2 would be equal.

## 1.4 Matrix Notation and Matrix Multiplication

### Problems 4, 10, 17, 19

4. If an  $m \times n$  matrix  $\mathbf{A}$  multiplies an  $n$ -dimensional vector  $\mathbf{x}$ , how many separate multiplications are involved? What if  $\mathbf{A}$  multiplies an  $n \times p$  matrix  $\mathbf{B}$ ?

- $m \cdot n$  multiplications; number of rows times the length of  $\mathbf{x}$ .
- $m \cdot n \cdot p$ ; same as above, except accounting for each additional column  $p$ .

10. True or false? Give a specific counterexample when false.

- If rows 1 and 3 of  $\mathbf{B}$  are the same, so are rows 1 and 3 of  $\mathbf{AB}$ .
- **✗ false**; matrix multiplication is done by the rows of the left matrix and the columns of the right, the rows may be the same, but if a column between the two are different, then there would be different multiplications occurring, e.g.,

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 12 & 8 & 4 \\ 30 & 20 & 10 \\ 38 & 32 & 16 \end{bmatrix}$$

- If columns 1 and 3 of  $\mathbf{B}$  are the same, so are columns 1 and 3 of  $\mathbf{AB}$ .
- **✓ true**;
- If rows 1 and 3 of  $\mathbf{A}$  are the same, so are rows 1 and 3 of  $\mathbf{AB}$ .
- **✓ true**
- $(\mathbf{AB})^2 = \mathbf{A}^2 \mathbf{B}^2$ .
- **✗ false** (most of the time), e.g.,

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\mathbf{AB}^2 = \begin{bmatrix} 144 & 64 & 16 \\ 900 & 400 & 100 \\ 2304 & 1024 & 256 \end{bmatrix} \neq \begin{bmatrix} 74 & 26 & 10 \\ 452 & 152 & 52 \\ 1154 & 386 & 130 \end{bmatrix} = \mathbf{A}^2 \mathbf{B}^2$$

17. Which of the following matrices are guaranteed to equal  $(\mathbf{A} + \mathbf{B})^2$ ?

- $\mathbf{A}^2 + 2\mathbf{AB} + \mathbf{B}^2$ ,
- ✓**  $\mathbf{A}(\mathbf{A} + \mathbf{B}) + \mathbf{B}(\mathbf{A} + \mathbf{B})$
- ✓**  $(\mathbf{A} + \mathbf{B})(\mathbf{B} + \mathbf{A})$ ,
- ✓**  $\mathbf{A}^2 + \mathbf{AB} + \mathbf{BA} + \mathbf{B}^2$

19. A fourth way to multiply matrices is columns of **A** times rows of **B**:

$$\mathbf{AB} = (\text{column 1})(\text{row 1}) + \cdots + (\text{column } n)(\text{row } n) = \text{sum of simple matrices.}$$

Give a  $2 \times 2$  example of this important rule for matrix multiplication.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \left( a \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right) b \begin{bmatrix} 1 \\ 3 \end{bmatrix} + d \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

Useful, as the right matrix can be thought of as the **weights that scale** the elements of the columns of the left matrix.

### Problems 30–31

30. Multiply these matrices:

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \\ 1 & 4 & 0 \end{bmatrix}.$$

$$\begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 0 & 2 & -3 \end{bmatrix} \quad \text{respectively}$$

- The former multiplication performs two operations (left: swaps top and bottom columns, right: swaps left and right columns), while the latter subtracts row 1 from both row 2 and row 3.

31. This  $4 \times 4$  matrix needs which elimination matrices **E**<sub>21</sub> and **E**<sub>32</sub> and **E**<sub>43</sub>?

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

- $e_{21} = -\frac{1}{2}$ ,  $e_{32} = -\frac{2}{3}$ ,  $e_{43} = -\frac{3}{4}$
- I suspect the fractions will tend towards  $-1$  if the matrix was expanded upon in a similar fusion?

## Problems 34, 35, 38, 42

34. Multiply these matrices in the orders  $FE$  and  $EF$  and  $E^2$ :

$$E = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & 0 & 1 \end{bmatrix} \quad F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{bmatrix}$$

$$FE = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ ac + b & c & 1 \end{bmatrix} \quad EF = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix} \quad E^2 = \begin{bmatrix} 1 & 0 & 0 \\ 2a & 1 & 0 \\ 2b & 0 & 1 \end{bmatrix}$$

35. ↓

- (a) Suppose all columns of  $B$  are the same. Then all columns of  $EB$  are the same, because each one is  $E$  times  $B_{1n}$ .
- (b) Suppose all rows of  $B$  are  $[1 \ 2 \ 4]$ . Show by example that all rows of  $EB$  are not  $[1 \ 2 \ 4]$ . It is true that those rows are multiples of  $[1 \ 2 \ 4]$ 
  - E.g., if  $e_{12} = 2$ , then  $m_2$  of  $EB$  would be  $[3 \ 6 \ 12]$

38. If  $AB = I$  and  $BC = I$ , use the associative law to prove  $A = C$ .

$$A = A(BC)$$

$$A = (AB)C$$

$$A = C$$

42. True or false?


- (a) If  $A^2$  is defined then  $A$  is necessarily square.
  - ✓ true; inner dimensions must match, i.e., dimensions of  $n_1 = m_2$ . Thus,  $A$  must be square.
- (b) If  $AB$  and  $BA$  are defined, then  $A$  and  $B$  are square.
  - ✗ false; if  $A = 6 \times 9$  and  $B = 9 \times 6$  allows for valid pre- and post-multiplication of  $B$ .
- (c) If  $AB$  and  $BA$  are defined, then  $AB$  and  $BA$  are square.
  - ✓ true; see above example, each case will still yield square matrices. Not a proof, but I can't see another way to falsify (b).
- (d) If  $AB = B$  then  $A = I$ 
  - ✗ false; e.g.,  $B = 0$

## 1.5 Triangular Factors and Row Exchanges

### Problems 1, 6, 7, 14, 18

- When is an upper triangular matrix nonsingular (a full set of pivots)?
  - Every pivot **must be nonzero**. If there is a zero on one of the pivots, then it indicates that one of the columns is a linear combination of one or more of the other columns.
- Find  $E^2$  and  $E^8$  and  $E^{-1}$  if


$$E = \begin{bmatrix} 1 & 0 \\ 6 & 1 \end{bmatrix}$$

matrices 

$$E^2 = \begin{bmatrix} 1 & 0 \\ 36 & 1 \end{bmatrix} \quad E^8 = \begin{bmatrix} 1 & 0 \\ 1679616 & 1 \end{bmatrix} \quad E^{-1} = \begin{bmatrix} 1 & 0 \\ -6 & 1 \end{bmatrix}$$

- Find the products  $FGH$  and  $HGF$  if (with upper triangular zeros omitted)

$$F = \begin{bmatrix} 1 & & & \\ 2 & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad G = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 2 & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad H = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

matrices 

$$FGH = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix} \quad HGF = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 4 & 2 & 1 & 0 \\ 8 & 4 & 2 & 1 \end{bmatrix}$$

14. 14. Write down all six of the  $3 \times 3$  permutation matrices, including  $P = I$ . Identify their inverses, which are also permutation matrices. The inverses satisfy  $PP^{-1} = I$  and are on the same list.

$$\begin{array}{ll}
 P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
 P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} & P^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \\
 P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} & P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \\
 P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} & P^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \\
 P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} & P^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
 P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} & P^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}
 \end{array}$$

18. 18. Decide whether the following systems are singular or nonsingular, and whether they have no solution, one solution, or infinitely many solutions:

$$\begin{bmatrix} 0 & 1 & -1 & | & 2 \\ 1 & -1 & 0 & | & 2 \\ 1 & 0 & -1 & | & 2 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & -1 & | & 0 \\ 1 & -1 & 0 & | & 0 \\ 1 & 0 & -1 & | & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 1 & | & 1 \\ 1 & 1 & 0 & | & 1 \\ 1 & 0 & 1 & | & 1 \end{bmatrix}$$

matrices 🧠

- Performing rref on above matrices yields:
  - Singular — no solution
  - Singular —  $\infty$  solutions.
  - Nonsingular — one solution  $[0.5 \ 0.5 \ 0.5]$

## Problems 26, 28

26. Which number  $c$  leads to zero in the second pivot position? A row exchange is needed and  $\mathbf{A} = \mathbf{L}\mathbf{U}$  is not possible. Which  $c$  produces zero in the third pivot position? Then a row exchange can't help and elimination fails.

$$\mathbf{A} = \begin{bmatrix} 1 & c & 0 \\ 2 & 4 & 1 \\ 3 & 5 & 1 \end{bmatrix}$$

- If  $c = 2$  then row 2 would have a 0 in the pivot, yielding:

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 3 & 5 & 1 \end{bmatrix}$$

- If  $c = 1$ , then you could take the matrix down to the following form,

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

which would yield a singular matrix with infinite solutions.

28.  $\mathbf{A}$  and  $\mathbf{B}$  are symmetric across the diagonal (because  $4 = 4$ ). Find their triple factorizations  $\mathbf{L}\mathbf{U}$  and say how  $\mathbf{U}$  is related to  $\mathbf{L}$  for these symmetric matrices:

$$\mathbf{A} = \begin{bmatrix} 2 & 4 \\ 4 & 11 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 & 4 & 0 \\ 4 & 12 & 4 \\ 0 & 4 & 0 \end{bmatrix}$$

$$\mathbf{U}_A = \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix}, \quad \mathbf{U}_B = \begin{bmatrix} 1 & 4 & 0 \\ 0 & -4 & 4 \\ 0 & 0 & -4 \end{bmatrix}$$

$$\mathbf{V}_A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{V}_B = \begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$


$$\mathbf{L}_A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \quad \mathbf{L}_B = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

- $\mathbf{A}, \mathbf{B} = \mathbf{L}\mathbf{D}\mathbf{V} \implies \mathbf{L} = \mathbf{V}^T$ ; the diagonal of the upper matrix, if reduced to 1's in the pivot positions, yields the transpose of the lower triangular matrix.

## Problems 33, 43

33. Solve  $Lc = b$  to find  $c$ . Then solve  $Ux = c$  to find  $x$ . What was  $A$ ?

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

matrices 

$$c = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} \quad x = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \implies A = LU = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

43. (Try this question) Which permutation makes  $PA$  upper triangular? Which permutations make  $P_1AP_2$  lower triangular? Multiplying  $A$  on the right by  $P_2$  exchanges the columns of  $A$ .

$$A = \begin{bmatrix} 0 & 0 & 6 \\ 1 & 2 & 3 \\ 0 & 4 & 5 \end{bmatrix}$$

$$U = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 6 \\ 1 & 2 & 3 \\ 0 & 4 & 5 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 6 \\ 1 & 2 & 3 \\ 0 & 4 & 5 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$



## 1.6 Inverses and Transposes

### Problems 3, 12, 18

3. From  $\mathbf{AB} = \mathbf{C}$  find a formula for  $\mathbf{A}^{-1}$ . Also find  $\mathbf{A}^{-1}$  from  $\mathbf{PA} = \mathbf{LU}$ .

$$\mathbf{AB} = \mathbf{C}$$

$$\mathbf{A} = \mathbf{CB}^{-1}$$

$$\mathbf{A}^{-1} = \mathbf{BC}^{-1}$$

$$\mathbf{PA} = \mathbf{LU}$$

$$\mathbf{A} = \mathbf{P}^{-1}\mathbf{LU}$$

$$\mathbf{A}^{-1} = \mathbf{U}^{-1}\mathbf{L}^{-1}\mathbf{P}$$

12. If  $\mathbf{A}$  is invertible, which properties of  $\mathbf{A}$  remain true for  $\mathbf{A}^{-1}$ ?

(a)  $\mathbf{A}$  is triangular. ✓ true

(b)  $\mathbf{A}$  is symmetric. ✓ true

(c)  $\mathbf{A}$  is tridiagonal. ✗ false

(d) All entries are whole ✗ false

(e) All entries are fractions (including numbers like  $\frac{3}{1}$ ). ✓ true;

18. Under what conditions on their entries are  $\mathbf{A}$  and  $\mathbf{B}$  invertible?

$$\mathbf{A} = \begin{bmatrix} a & b & c \\ d & e & 0 \\ f & 0 & 0 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & e \end{bmatrix}$$

matrices 🧩

$$\begin{aligned} \text{rref} \left( \left[ \begin{array}{ccc|ccc} a & b & c & 1 & 0 & 0 \\ d & e & 0 & 0 & 1 & 0 \\ f & 0 & 0 & 0 & 0 & 1 \end{array} \right] \right) &\rightarrow \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & f^{-1} \\ 0 & 1 & 0 & 0 & e^{-1} & -\frac{d}{ef} \\ 0 & 0 & 1 & c^{-1} & -\frac{b}{ce} & \frac{-ae+bd}{cef} \end{array} \right] \\ \text{rref} \left( \left[ \begin{array}{ccc|ccc} a & b & 0 & 1 & 0 & 0 \\ c & d & 0 & 0 & 1 & 0 \\ 0 & 0 & e & 0 & 0 & 1 \end{array} \right] \right) &\rightarrow \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{d}{ad-bc} & -\frac{b}{ad-bc} & 0 \\ 0 & 1 & 0 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} & 0 \\ 0 & 0 & 1 & 0 & 0 & e^{-1} \end{array} \right] \end{aligned}$$

$$\bullet \mathbf{A} \rightarrow \mathbf{A}^{-1} \iff c, e, f \neq 0$$

$$\bullet \mathbf{B} \rightarrow \mathbf{B}^{-1} \iff e \neq 0 \wedge ad \neq bc$$

## Problems 21, 28, 41, 56, 58

21. (Remarkable) If  $\mathbf{A}$  and  $\mathbf{B}$  are square matrices, show that  $\mathbf{I} - \mathbf{BA}$  is invertible if  $\mathbf{I} - \mathbf{AB}$  is invertible. Start from  $\mathbf{B}(\mathbf{I} - \mathbf{AB}) = (\mathbf{I} - \mathbf{BA})\mathbf{B}$

$$\begin{aligned}\mathbf{B}(\mathbf{I} - \mathbf{AB}) &= (\mathbf{I} - \mathbf{BA})\mathbf{B} \\ (\mathbf{I} - \mathbf{AB}) &= \mathbf{B}^{-1}(\mathbf{I} - \mathbf{BA})\mathbf{B} \\ (\mathbf{I} - \mathbf{AB})^{-1} &= \mathbf{B}(\mathbf{I} - \mathbf{BA})^{-1}\mathbf{B}^{-1}\end{aligned}$$

- Thus, as long as  $\mathbf{I} - \mathbf{BA}$  is invertible, then the inverse is defined.
28. If the product  $\mathbf{M} = \mathbf{ABC}$  of three square matrices is invertible, then  $\mathbf{A}$ ,  $\mathbf{B}$ ,  $\mathbf{C}$  are invertible. Find a formula for  $\mathbf{B}^{-1}$  that involves  $\mathbf{M}^{-1}$  and  $\mathbf{A}$  and  $\mathbf{C}$ .

$$\begin{aligned}\mathbf{M} &= \mathbf{ABC} \\ \mathbf{M}^{-1} &= \mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1} \\ \mathbf{CM}^{-1} &= \mathbf{B}^{-1}\mathbf{A}^{-1} \\ \mathbf{CM}^{-1}\mathbf{A} &= \mathbf{B}^{-1}\end{aligned}$$

41. For which three numbers  $c$  is this matrix not invertible, and why not?

$$\mathbf{A} = \begin{bmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{bmatrix}$$

- If  $c = 0$ , then there would be multiple unavoidable zeros in the pivots.
  - If  $c = 2$ , then row 2 would just be duplicate of row 1.
  - If  $c = 7$ , then column 2 and 3 would be equal.
57. If  $\mathbf{A} = \mathbf{A}^T$  needs a row exchange, then it also needs a column exchange to stay symmetric. In matrix language,  $\mathbf{PA}$  loses the symmetry of  $\mathbf{A}$  but  $\mathbf{PAP}^T$  recovers the symmetry.

58. ↓

- (a) How many entries of  $\mathbf{A}$  can be chosen independently, if  $\mathbf{A} = \mathbf{A}^T$  is  $5 \times 5$ ?
- 25 total choices — 10 under the diagonal = 15
- (b) How do  $\mathbf{L}$  and  $\mathbf{D}$  ( $5 \times 5$ ) give the same number of choice in  $\mathbf{LDL}^T$ ?
- Oh, I kind of used this to find (a). Well, the diagonal doesn't matter ( $\neq 0$ ), since a transpose can simply be thought of a rotation around the diagonal elements. But, every element must match  $\mathbf{U}$ , thus only the 10 choices below matter, yielding 15 total choices.

## 2 Vector Spaces



### 2.1 Vector Spaces and Subspaces

#### Problems 25, 26, 30, 31

25. If we add an extra column  $\mathbf{b}$  to a matrix  $\mathbf{A}$ , then the column space gets larger unless they are linearly dependent..

- Give an example in which the column space gets larger and an example in which it doesn't.

$$\text{Larger: } \left[ \begin{array}{cc|c} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 2 & 1 \end{array} \right] \quad \text{No change: } \left[ \begin{array}{cc|c} 1 & 0 & 3 \\ 2 & 4 & 6 \\ 3 & 2 & 9 \end{array} \right]$$

- Why is  $\mathbf{Ax} = \mathbf{b}$  solvable exactly when the column space doesn't get larger by including  $\mathbf{b}$ ?
    - Because the solution would be in the image, leading to infinite solutions since it could be written as a linear combination of the vectors already in  $\mathbf{A}$ .
26. The columns of  $\mathbf{AB}$  are combinations of the columns of  $\mathbf{A}$ . This means: the column space of  $\mathbf{AB}$  contained in (possibly equal to) the column space of  $\mathbf{A}$ . Give an example where the column spaces of  $\mathbf{A}$  and  $\mathbf{AB}$  are not equal.

vectorSpaces 🧩

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 4 & 6 \\ 3 & 2 & 9 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- The column space of  $\mathbf{B}$  is clearly contained in  $\mathbf{A}$ , but since the dimension of the null space of  $\mathbf{B}$  is 2, then  $\mathbf{AB}$  will not have the same column space of  $\mathbf{A}$ .
30. If the  $9 \times 12$  system  $\mathbf{Ax} = \mathbf{b}$  is solvable for every  $\mathbf{b}$ , then  $C(\mathbf{A}) = 9$ .
- This follows from the Rank-nullity theorem, where the dimensionality of the column space and cokernel is equal to the rows of the original matrix.
  - This also implies the dimensionality of the cokernel is 3.
31. Why isn't  $\mathbb{R}^2$  a subspace of  $\mathbb{R}^3$  ?
- $\mathbb{R}^2$  could be a *subset*, but not a subspace; there are infinite 2-dimensional planes in  $\mathbb{R}^3$ .
  - If you included a third point of 0 in  $\mathbb{R}^2$ , then it would indicate that includes the origin, which could make it a subspace or  $\mathbb{R}^3$ . However, that extra coordinate would be make it be a vector in  $\mathbb{R}^3$ .

## 2.2 Solving $Ax = 0$ and $Ax = b$

### Problems 12, 24, 25, 70

12. Which of these rules give a correct definition of the rank of  $A$ ?

- (a) The number of nonzero rows in  $R$ . ✓ true

$$\max(r) = r \in \mathbb{N} : 0 \leq r \leq \min(m, n)$$

- (b) The number of columns minus the total number of rows.

◦ This would yield dimensionality of the null space.

- (c) The number of columns minus the number of free columns.

◦ This would yield the dimensionality of the left null space.

- (d) The number of 1s in  $R$ .

◦ This wouldn't tell you much of anything.

24. Every column of  $AB$  is a combination of the columns of  $A$ . Then the dimensions of the column spaces give  $\text{rank}(AB) \leq \text{rank}(A)$ .

Problem: Prove that  $\text{rank}(AB) \leq \text{rank}(B)$ .

$$\text{rank}(A) = C(A) = C(A^T)$$

$$\implies \text{rank}(A) = \text{rank}(A^T)$$

$$\therefore \text{rank}(AB) = \text{rank}((AB)^T) = \text{rank}(B^T A^T) \leq \text{rank}(B^T) = \text{rank}(B)$$

◦ The rank of  $AB$  can only be decreased, if  $B$  is not full rank itself.

◦ If, both  $A$  and  $B$  are full rank, then  $\text{rank}(AB) = \text{rank}(B)$

25. (Important) Suppose  $A$  and  $B$  are  $n \times n$  matrices, and  $AB = I$ . Prove from  $\text{rank}(AB) \leq \text{rank}(A)$  that the rank of  $A$  is  $n$ . So  $A$  is invertible and  $B$  must be its two-sided inverse. Therefore  $BA = I$

◦  $A$  must have same size of  $B$ , given they are both  $n \times n$ .

◦ If  $A$  was rank deficient, but  $B$  was full rank, then  $\text{rank}(AB) \leq \text{rank}(A)$  would be invalid, forcing  $\text{rank}(A) = n$ .

70. Explain why  $A$  and  $-A$  always have the same reduced echelon form  $R$ .

◦ Signed solutions are arbitrary;  $-A$  would have permutations that flip the sign and yield the same, reversible, solution.

## 2.3 Linear Independence, Basis, and Dimension

### Problems 9, 13, 28, 36

9. Suppose  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$  are vectors in  $\mathbb{R}^3$ .

- (a) There four vectors are dependent because  $\mathbb{R}^3$  can only have 3 linearly independent vectors.
- (b) The two vectors  $\mathbf{v}_1$  and  $\mathbf{v}_2$  will be dependent if they are multiples of each other.
- (c) The vectors  $\mathbf{v}_1$  and  $(0, 0, 0)$  are dependent because  $\mathbf{v}_1(0, 0, 0) = \mathbf{0}$

13. Find the dimensions of:

- (a) the column space of  $\mathbf{A}$ :
- (b) the column space of  $\mathbf{U}$ :
- (c) the row space of  $\mathbf{A}$ :
- (d) the row space of  $\mathbf{U}$ .

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 3 & 1 & -1 \end{bmatrix} \quad \text{and} \quad \mathbf{U} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

vectorSpaces 

◦ Which two of the spaces are the same?

- The row space of  $\mathbf{A}$  and  $\mathbf{U}$  are the same. Why? Taking  $\mathbf{A}$  to rref shows it's also rank 2, with a linearly dependent row  $(0, 0, 0)$ , just like  $\mathbf{U}$ .

28. True or false (give a good reason)?

- (a) If the columns of a matrix are dependent, so are the rows.
  - **✗ false**; as shown in [problem 2.2.12](#)<sup>↑</sup>, the max rank is the minimum of either the rows or columns. You could have linearly dependent columns, if  $n > m$ , but no linearly dependent rows.
- (b) The column space of a  $2 \times 2$  matrix is the same as its row space.
  - **✗ false**; e.g.,  $\begin{bmatrix} 4 & 2 \\ 0 & 0 \end{bmatrix}$
- (c) The column space of a  $2 \times 2$  matrix has the same dimension as its row space.
  - **✓ true**;  $\text{rank}(\mathbf{A}) = C(\mathbf{A}) = C(\mathbf{A}^T)$
- (d) The columns of a matrix are a basis for the column space.
  - **✗ false**; one of the columns could be linearly dependent with another; only linearly independent columns forms a basis for the column space.

36. If  $\mathbf{A}$  is a  $64 \times 17$  matrix of rank 11, how many independent vectors satisfy  $\mathbf{A}\mathbf{x} = \mathbf{0}$ ? How many independent vectors satisfy  $\mathbf{A}^T\mathbf{y} = \mathbf{0}$ ?

- $17 - 11 = 6$ ,  $64 - 11 = 53$ , respectively.

## 2.4 The Four Fundamental Subspaces

### Problems 6, 14, 15, 27

6. Suppose  $\mathbf{A}$  is an  $m \times n$  matrix of rank  $r$ . Under what conditions on those numbers does

(a)  $\mathbf{A}$  have a two-sided inverse:  $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$ ?

◦ Square, nonsingular (full rank).

(b)  $\mathbf{A}\mathbf{x} = \mathbf{b}$  have infinitely many solutions for every  $\mathbf{b}$ ?

◦ If  $\text{rank}(\mathbf{A}) < n$  ( $n$  = number of columns)

14. Find a left-inverse and/or a right-inverse (when they exist) for

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{M} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{T} = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$$

vectorSpaces 

◦  $\mathbf{A}\mathbf{A}^T$  (right inverse)  $\mathbf{M}^T\mathbf{M}$  (left inverse) yield the same inverse,

$$\begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

which makes sense, since  $\mathbf{M} = \mathbf{A}^T$

$$\mathbf{T}^{-1} = \begin{bmatrix} a^{-1} & -ba^{-2} \\ 0 & a^{-1} \end{bmatrix}$$

•  $\mathbf{T}$  has an inverse, left and right, as long as  $a \neq 0$ .

•  $\mathbf{T}$  does not have a normal inverse if  $a = b$  (singular), but the left and right inverse both yield the above matrix.

15. If the columns of  $\mathbf{A}$  are linearly independent, then the rank is the number of columns,  $n$ , the nullspace is empty, the row space is  $\mathbb{R}^n$ , and there exists a left-inverse.

27. (Important)  $\mathbf{A}$  is an  $m \times n$  matrix of rank  $r$ . Suppose there are right-hand sides  $\mathbf{b}$  for which  $\mathbf{A}\mathbf{x} = \mathbf{b}$  has no solution.

(a) What inequalities must be true between  $m$ ,  $n$ , and  $r$ ?

◦ No solution means a row of zeros, thus  $r < m$

◦  $r \leq n$ , nothing changed.

(b) How do you know that  $\mathbf{A}^T\mathbf{y} = \mathbf{0}$  has a nonzero solution?

◦  $\mathbf{A}^T\mathbf{y} = \mathbf{0}$  raises investigation of the left null space. Since the column space does not span all of  $\mathbb{R}^m$  ( $r < m$ ), then it means it contains a nonzero solution.

## 2.6 Linear Transformations

### Problems 6, 34, 39

6. What 3 by 3 matrices represent the transformation that

- (a) project every vector onto the x-y plane?
- (b) reflect every vector through the x-y plane?
- (c) rotate the x-y plane through  $90^\circ$ , leaving the z-axis alone?
- (d) rotate the x-y plane, then x-z, then y-z, through  $90^\circ$ ?
- (e) carry out the same three rotations, but each one through  $180^\circ$ ?

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{(a)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}_{(b)} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{(c)} \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}_{(d)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{(e)}$$

34. The transformation  $T$  that transposes every matrix is definitely linear. Which of these extra properties are true?

(a)  $T^2 = \text{identity transformation}$ .

◦ ✓ true; given  $T(A) = A^T$

$$\begin{aligned} T^2(A) &= T(T(A)) = T(A^T) = (A^T)^T = A \\ &\implies T^2 = I \end{aligned}$$

(b) The kernel of  $T$  is the zero matrix.

◦ ✓ true; only matrix that can take the identity to 0 is the trivial case, i.e., when the null space is empty.

(c) Every matrix is in the range of  $T$ .

🌐 Definition of column space, Wikipedia 10/16

◦ ✓ true; the image (range) consists of all possible products  $Ax \forall x \in \mathbb{C}^n$ .

(d)  $T(M) = -M$  is impossible.

◦ ✗ false; e.g.,

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

39. If you keep the same basis vectors but put them in a different order, the change-of-basis matrix  $M$  is a **permutation matrix**  $P$ . If you keep the basis vectors in order but change their lengths,  $M$  is a **diagonal matrix**  $D$ .

# 3 Orthogonality



## 3.1 Orthogonal Vectors and Subspaces

### Problems 6, 46, 47

6. Find all vectors in  $\mathbb{R}^3$  that are orthogonal to  $(1, 1, 1)$  and  $(1, -1, 0)$ . Produce an orthonormal basis from these vectors (mutually orthogonal unit vectors).

orthogonality 🧠

- Simply taking the cross product between two vectors in  $\mathbb{R}^3$  yields a new vector that is normal to the plane containing them.
- $(1, 1, 1) \times (1, -1, 0) = (1, 1, -2)$  yields us orthogonal columns.
- Dividing each vector by their norms yields orthonormal columns, i.e.,

$$\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{3}} \end{bmatrix}$$

46. Find  $\mathbf{A}^T \mathbf{A}$  if the columns of  $\mathbf{A}$  are unit vectors, all mutually perpendicular.

$$\begin{bmatrix} \|\mathbf{a}_1\|^2 & a_{\perp} & \cdots & a_{\perp} \\ a_{\perp} & \|\mathbf{a}_2\|^2 & a_{\perp} & \vdots \\ \vdots & a_{\perp} & \ddots & a_{\perp} \\ a_{\perp} & \cdots & a_{\perp} & \|\mathbf{a}_n\|^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix} = \mathbf{I}$$

47. Construct a  $3 \times 3$  matrix  $\mathbf{A}$  with no zero entries whose columns are mutually perpendicular. Compute  $\mathbf{A}^T \mathbf{A}$ . Why is it a diagonal matrix?

orthogonality 🧠

- Note: I tried to set up with random matrices each time, sometimes it fails and gives the zero matrix.
- The way I set up only yields a diagonal matrix with  $\mathbf{A}\mathbf{A}^T$ ,  $\mathbf{A}^T \mathbf{A}$  yields a symmetric matrix. Why?



## 3.2 Cosines and Projections onto Lines

### Projection Proof (class problem)

- If I recall the problem correctly, we were requested to prove what  $\text{proj}_{\mathbf{v}} \mathbf{w}$  is equal to.
- In my notes I have that a orthogonal projection occurs when the dot product between  $\mathbf{v}$  and distance  $\mathbf{w}$  from  $\mathbf{v}$  is equal to zero. This follows from the definition of the inner product, i.e.,

$$\lambda = \mathbf{v}^T \mathbf{w} = \|\mathbf{v}\| \|\mathbf{w}\| \cos \theta$$

- If  $\theta = 90^\circ$ , then the vectors are perpendicular, i.e., orthogonal. What we are missing is the distance from  $\mathbf{w}$  to  $\mathbf{v}$ ; the distance that yields an inner product of zero with a normalized  $\mathbf{v}$  is the projection.
- This means we need a scaled version of  $\mathbf{v}$ , let's call it  $\mathbf{v}\beta$ , at which such inner product is equal to zero. At this point, the difference between  $\mathbf{w}$  and  $\mathbf{v}\beta$  is exactly what we need in order to solve for a  $\beta$  that maintains an inner product of zero with the original vector  $\mathbf{v}$ , i.e.,

$$\mathbf{v}^T (\mathbf{w} - \mathbf{v}\beta) = 0$$

$$\mathbf{v}^T \mathbf{w} - \mathbf{v}^T \mathbf{v}\beta = 0$$

$$\mathbf{v}^T \mathbf{v}\beta = \mathbf{v}^T \mathbf{w}$$

$$\beta = \frac{\mathbf{v}^T \mathbf{w}}{\mathbf{v}^T \mathbf{v}}$$

$$\implies \text{proj}_{\mathbf{v}} \mathbf{w} = \mathbf{v}\beta = \mathbf{v} \frac{\mathbf{v}^T \mathbf{w}}{\mathbf{v}^T \mathbf{v}}$$

- I've internalized this as a mapping of  $\mathbf{w}$  onto  $\mathbf{v}$  over a magnitude (the norm) of  $\mathbf{v}$ .
  - The mapping is important because it tells us the shortest distance from  $\mathbf{w}$  onto  $\mathbf{v}$ , i.e., when they are orthogonal.
  - The magnitude is important, because it is the basis at which  $\mathbf{w}$  is parallel to  $\mathbf{v}$ , which when added mapping distance, yields  $\mathbf{w}$ .

## Problems 10, 13, 15

10. Is the projection matrix  $P$  invertible? Why or why not?

$$P = \frac{aa^T}{a^T a}$$
$$C(P) = a = \text{rank}(P) = n = 1$$

- The rank is equal to the number of columns, thus it is invertible.

13. Prove that the trace of  $P = \frac{aa^T}{a^T a}$  always equals 1.

$$\text{tr}(A) = \sum_{i=1}^n e_{ij} = e_{11} + e_{22} + \cdots + e_{nn}$$

$$\Rightarrow \text{tr}(P) = \frac{a_1 a_1}{a^T a} + \cdots + \frac{a_n a_n}{a^T a} = \frac{a^T a}{a^T a} = 1$$

15. Show that the length of  $Ax$  equals the length of  $A^T x$  if  $AA^T = A^T A$ .

$$\begin{aligned}\|Ax\|^2 &= (Ax)^T (Ax) = xA^T Ax \\ \|A^T x\|^2 &= (A^T x)^T (A^T x) = xAA^T x \\ xA^T Ax &= xAA^T x \iff AA^T = A^T A\end{aligned}$$

### 3.3 Projections and Least Squares

#### Problems 8, 19, 20


8. If  $\mathbf{P}$  is the projection matrix onto a  $k$ -dimensional subspace  $\mathbf{S}$  of the whole space  $\mathbb{R}^n$ , what is the column space of  $\mathbf{P}$  and what is its rank?
- Given  $\mathbf{P} = \mathbf{A}(\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T$ , then  $\mathbf{P}$  will project any vector onto the image of  $\mathbf{A}$ , if  $\mathbf{A}$  has independent columns.
  - This implies that  $\mathbf{S}$  is the image of  $\mathbf{A}$ , i.e.,  $C(\mathbf{A}) = \mathbf{S}$ .
  - Since the  $k$ -dimensional subspace  $\mathbf{S}$  is the whole of  $\mathbb{R}^n$ , and  $\text{rank}(\mathbf{A}) = n$ , then  $C(\mathbf{P}) = \mathbf{S} \wedge \text{rank}(\mathbf{P}) = k$
19. If  $\mathbf{P}_C = \mathbf{A}(\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T$  is the projection onto the column space of  $\mathbf{A}$ , what is the projection  $\mathbf{P}_R$  onto the row space?
- $\mathbf{P}_C = C(\mathbf{A})$ ,  $\mathbf{P}_R = C(\mathbf{A}^T)$ , thus, taking the transpose of every instance of  $\mathbf{A}$  will yield the row space, i.e.,

$$\mathbf{P}_R = \mathbf{A}^T (\mathbf{A}^{TT} \mathbf{A}^T)^{-1} \mathbf{A}^{TT} = \mathbf{A}^T (\mathbf{A} \mathbf{A}^T)^{-1} \mathbf{A}$$

20. If  $\mathbf{P}$  is the projection onto the column space of  $\mathbf{A}$ , what is the projection onto the left nullspace?
- The column space and left null space are orthogonal to each other, thus the projection onto the left null space is simply  $\mathbf{I} - \mathbf{P}$ .

## 3.4 Orthogonal Bases and Gram-Schmidt

### Implementation of the Gram-Schmidt Process

Implementing Gram-Schmidt 

#### Problems 10, 13, 18

10. 10. If  $\mathbf{q}_1$  and  $\mathbf{q}_2$  are the outputs from Gram-Schmidt, what were the possible input vectors  $\mathbf{a}$  and  $\mathbf{b}$ ?

- Any two linearly independent vectors.

13. Apply the Gram-Schmidt process to

$$\mathbf{a} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

and write the result in the form  $\mathbf{A} = \mathbf{QR}$ .

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$


- All that is needed is a permutation to achieve orthonormal columns; switching rows 1 and 3 will achieve this, yielding  $\mathbf{Q}$ :

$$\mathbf{Q} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

- Thus,

$$\mathbf{A} = \mathbf{QR} \rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

18. If  $\mathbf{A} = \mathbf{QR}$ , find a simple formula for the projection matrix  $\mathbf{P}$  onto the column space of  $\mathbf{A}$ .

- I was lost on this one, but I wanted to know how to do it. Unfortunately, while looking for help I came across one of the [solutions documents](#)  again.
- Normally I would just do a different problem, but I wanted to write it out, for personal reference.

$$\mathbf{A} = \mathbf{QR} \implies \mathbf{A}^T \mathbf{A} = (\mathbf{QR})^T (\mathbf{QR}) = \mathbf{R}^T \mathbf{Q}^T \mathbf{Q} \mathbf{R} = \mathbf{R}^T \mathbf{R}$$

$$\mathbf{P} = \mathbf{A}(\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T$$

$$\implies \mathbf{QR}(\mathbf{R}^T \mathbf{R})^{-1}(\mathbf{QR})^T = \mathbf{QRR}^{-1}(\mathbf{R}^T)^{-1} \mathbf{R}^T \mathbf{Q}^T = \mathbf{QQ}^T = \mathbf{I}$$

$$\implies \mathbf{P}^2 = \mathbf{P} \checkmark$$

## 3.5 The Fast Fourier Transform

Problems 1, 5, 6, 15, 16

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