1. The system of linear equations

$$\begin{aligned} 3x_1 + 2x_2 - \phantom{2}x_3 &= -15 \\ 5x_1 + 3x_2 + 2x_3 &= 0 \\ 3x_1 + \phantom{2}x_2 + 3x_3 &= 11 \\ -6x_1 - 4x_2 + 2x_3 &= 30 \end{aligned}$$

is consistent. The reduced row-echelon form of the augmented matrix associated with the system is

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

(a) Determine a solution to the system:

$$x_1 = -4 \quad x_2 = 2 \quad x_3 = 7$$

(b) Are there any more solutions to the system? How do you know?

- No, the rank of the matrix is three since there are three pivots (leading 1s) and there are only 3 columns in the coefficient matrix. Thus, $r = n$, which means there is exactly one unique solution.

2. Solve the following system of linear equations

$$\begin{aligned} x_1 - \phantom{1}2x_2 + \phantom{11}x_3 - \phantom{1}4x_4 &= 1 \\ x_1 + \phantom{1}3x_2 + \phantom{1}7x_3 + \phantom{1}2x_4 &= 2 \\ x_1 - 12x_2 - 11x_3 - 16x_4 &= 5 \end{aligned}$$

$$\left[\begin{array}{cccc|c} 1 & -2 & 1 & -4 & 1 \\ 1 & 3 & 7 & 2 & 2 \\ 1 & -12 & -11 & -16 & 5 \end{array}\right] R_2, R_3 - R_1 \left[\begin{array}{cccc|c} 1 & -2 & 1 & -4 & 1 \\ 0 & 5 & 6 & 6 & 1 \\ 0 & -10 & -12 & -12 & 4 \end{array}\right]$$

$$R_3 + 2R_2 \left[\begin{array}{cccc|c} 1 & -2 & 1 & -4 & 1 \\ 0 & 5 & 6 & 6 & 1 \\ 0 & 0 & 0 & 0 & 6 \end{array}\right]$$

- Solution is inconsistent—no solution possbile.

3. Consider the homogeneous system of linear equations

$$\begin{cases} 3x_1 + x_2 + x_3 + x_4 = 0 \\ 5x_1 - x_2 + x_3 - x_4 = 0 \end{cases} \quad \text{and let } \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \ \boldsymbol{u} = \begin{bmatrix} -1 \\ -1 \\ 4 \\ 0 \end{bmatrix}, \text{ and } \boldsymbol{v} = \begin{bmatrix} 1 \\ -4 \\ -4 \\ 5 \end{bmatrix}$$

(a) Show that $\boldsymbol{x} = \boldsymbol{u}$ is a solution to the system.

$$\begin{aligned} 3(-1) + -1 + 4 + 0 &= 0 \\ 5(-1) - -1 + 4 - 0 &= 0 \\ &\downarrow \\ -3 - 1 + 4 &= 0 \\ -5 + 1 + 4 &= 0 \\ &\downarrow \\ 0 &= 0 \\ 0 &= 0 \end{aligned}$$

$\boldsymbol{x} = \boldsymbol{u}$ is a valid solution.

(b) Show that $\boldsymbol{x} = \boldsymbol{v}$ is a solution to the system.

$$\begin{aligned} 3(1) + -4 + -4 + 5 &= 0 \\ 5(1) - -4 + -4 - 5 &= 0 \\ &\downarrow \\ 3 - 4 - 4 + 5 &= 0 \\ 5 + 4 - 4 - 5 &= 0 \\ &\downarrow \\ 0 &= 0 \\ 0 &= 0 \end{aligned}$$

$\boldsymbol{x} = \boldsymbol{v}$ is a valid solution.

(c) Show that $\boldsymbol{x} = 3u + 2v$ is a solution to the system.

$$3\boldsymbol{u} + 2\boldsymbol{v} = 3\begin{bmatrix} -1 \\ -1 \\ 4 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 1 \\ -4 \\ -4 \\ 5 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \\ 12 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ -8 \\ -8 \\ 10 \end{bmatrix} = \begin{bmatrix} -1 \\ -11 \\ 4 \\ 10 \end{bmatrix}$$

$$\begin{aligned} &\downarrow \\ 3(-1) + -11 + 4 + 10 &= 0 \\ 5(-1) - -11 + 4 - 10 &= 0 \\ &\downarrow \\ -3 - 11 + 4 + 10 &= 0 \\ -5 + 11 + 4 - 10 &= 0 \\ &\downarrow \\ 0 &= 0 \\ 0 &= 0 \end{aligned}$$

$\boldsymbol{x} = 3\boldsymbol{u} + 2\boldsymbol{v}$ is a valid solution.