Calculus III Exercises

9 Introduction to Differential Equations

9.1 Edfinity: Solving Differential Equations

9.1.5

- Solve $y' = x^5 y^2$, using separation of variables, given the initial condition $y(0) = 9$

$$\frac{\frac{dy}{dx}}{y^2} = x^5$$
$$\int \frac{\frac{dy}{dx}}{y(x)^2} = \int x^5 dx$$
$$-\frac{1}{y(x)} = \frac{x^6}{6} + c_1$$
$$y(x) = -\frac{6}{x^6 + c_1}$$
$$9 = -\frac{6}{c}, \quad c = -\frac{6}{9}$$
$$\boxed{y(x) = -\frac{18}{2x^6 - 2}}$$

9.1.6

- Solve the initial value problem $\frac{dy}{dx} + 3y = 0, \quad y(\ln 4) = 3.$

$$\frac{dy}{dx} = -3y$$
$$\int \frac{dy}{dx} y^{-1} dx = \int -3dx$$
$$\ln|y| = -3x + \lambda$$
$$y = e^{-3x}\lambda$$
$$3 = e^{3(\ln 4)}\lambda \implies \lambda = 192$$
$$y = 192e^{-3x}$$

9.1.7

- Solve $(t^2+36)\frac{dx}{dt} = (x^2+9)$, using separation of variables, given the initial condition $x(0)=3$.

$$
\begin{aligned}
\frac{dx}{dt} &= \left(t^2+36\right)^{-1} \\
\frac{dx}{dt}\left(x^2+9\right)^{-1} &= \left(t^2+36\right)^{-1} \\
\int \frac{dx}{dt}\left(x^2+9\right)^{-1} &= \int \left(t^2+36\right)^{-1} dt \\
\frac{1}{9}\int \left(\frac{x^2}{9}+1\right)^{-1} dx &= \frac{1}{36}\int \left(\frac{t^2}{36}+1\right)^{-1} dt \\
\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) &= \frac{1}{6}\tan^{-1}\left(\frac{t}{6}\right)+\lambda \\
\tan^{-1}\left(\frac{x}{3}\right) &= \frac{1}{2}\tan^{-1}\left(\frac{t}{6}\right)+\lambda \\
\tan^{-1}\left(\frac{3}{3}\right) &= \frac{1}{2}\tan^{-1}\left(\frac{0}{6}\right)+\lambda \\
\frac{\pi}{4} &= \lambda \\
x &= 3\tan\left(\frac{1}{2}\tan^{-1}\left(\frac{t}{6}\right)+\frac{\pi}{4}\right)
\end{aligned}
$$

9.1.8

- Solve the initial value problem $\frac{dy}{dx} = (x-7)(y-8), \quad y(0)=4$

$$
\begin{aligned}
\frac{dy}{dx}(y-8)^{-1} &= (x-7) \\
\int dy\,(y-8)^{-1} &= \int (x-7)\,dx \\
\ln y - 8 &= x^{-2} - 7x + \lambda \\
y &= e^{\frac{x^2}{2}-7x}\lambda + 8 \\
-4 &= \lambda \\
y &= -4e^{\frac{x^2}{2}-7x} + 8
\end{aligned}
$$

9.1.9

- Solve the initial value problem $t^2\frac{dy}{dt} - t = 1 + y + ty, \quad y(1) = 7$

$$\begin{aligned}\int (y+1)\,dy &= \int \frac{1+t}{t^2} dt \\ \ln|1+y| &= -t^{-1} + \ln t + \lambda \\ y &= \lambda e^{-t^{-1}+\ln t} - 1 = \lambda \frac{t}{e^{t^{-1}}} - 1 \\ 7 &= \lambda e^{-1} - 1 \\ \lambda &= 8e \\ y &= 8te^{1-t^{-1}} - 1\end{aligned}$$

9.1.10

- Solve the initial value problem $y' = 2y^2 \sin x, \quad y(0) = 6$

$$\begin{aligned}\frac{dy}{dx}(2y^2)^{-1} &= \sin x \\ \int \frac{dy}{dx} y^{-2} dx &= \int 2 \sin x dx \\ -y^{-1} &= -2\cos x + \lambda \\ y &= (2\cos x + \lambda)^{-1} \\ 6 &= (2\cos 0 + \lambda)^{-1} \\ -\frac{11}{6} &= \lambda \\ y &= \left(2\cos x - \frac{11}{6}\right)^{-1}\end{aligned}$$

9.2 Edfinity: Models Involving y'=k(y-b)

9.2.2

- Find the general solution of $y' = 5\,(y - 16)$.

$$y(t) = b + Ce^{kt} \qquad\qquad y' = k(y-b)$$

$$\begin{aligned} y(t) &= 16 + Ce^{5t} \\ 30 &= 16 + C \\ C &= 14 \\ y(t) &= 16 + 14e^{5t} \\ 1 &= 16 + C \\ C &= -15 \\ y(t) &= 16 + -15e^{5t} \end{aligned}$$

9.2.3

- A 62 kg skydiver jumps out of an airplane. What is her terminal velocity in miles per hour, assuming that $k = 10\frac{kg}{s}$ for free fall?

$$-\frac{gm}{k} = -\frac{9.8(62)}{10} = -60.76\frac{m}{s} = 199.343\frac{ft}{s} = -134.916 \text{ mph}$$

9.2.4

- A continuous annuity with withdrawal rate $N = \$600$ y and interest rate $r = 5\%$ is funded by an initial deposit P_0
- When will the annuity run out of funds if $P_0 = \$10,000$?

$$\begin{aligned} P(t) = Nr^{-1} + Ce^{rt} = 600(0.05)^{-1} + Ce^{0.05t} &= 12,000 + Ce^{0.05t} \\ 10,000 &= 12,000 + C \\ C &= -2,000 \\ t = 0.05^{-1} \ln \frac{12,000}{2,000} &= 35.83 \approx 38 \text{ years} \end{aligned}$$

- Which initial deposit P_0 yields a constant balance?

$$\begin{aligned} P(t) &= 12,000 + Ce^{0.05^t}, \quad C = 0 \\ P_0 &= 12,000 \end{aligned}$$

9.2.5

- A cup of coffee, cooling off in a room temperature $20\,°\mathrm{C}$, has cooling constant $k = 0.085\,\mathrm{min}^{-1}$.
- How fast is the coffee cooling when its temperature is $T = 70\,°\mathrm{C}$?

$$\begin{gathered} k(T - T_0) \\ 0.085(70 - 20) = 4.25\,°\mathrm{C}\,\mathrm{min}^{-1} \end{gathered}$$

- Use the Linear Approximation to estimate the change in temperature over the next 4 seconds when $T = 70\,°\mathrm{C}$

$$4.25\,°\mathrm{C}\,\mathrm{min}^{-1}(4\mathrm{s})60\,\mathrm{s}\,\mathrm{min}^{-1} = 0.283\,°\mathrm{C}$$

- The coffee is served at a temperature of $86\,°\mathrm{C}$. How long should you wait before drinking it if the optimal temperature is $65\,°\mathrm{C}$?

$$\begin{aligned} 65 &= 20 + 66e^{-0.085t} \\ t &= -(0.085)^{-1} \ln\left(\frac{45}{66}\right) \\ t &\approx 4.5\,\mathrm{min} \end{aligned}$$

9.3 Edfinity: Graphical and Numerical Methods

4. **9.3.4**

   - User Euler's method with step size 0.2 to estimate $y(1)$, where $y(x)$ is the solution of the initial-value problem.

$$y' = 4x + y^2, \quad y(0) = 1$$

$$y_k = y_{k-1} + hF(t_{k-1}, y_{k-1}), \quad F = 4x + y^2, \quad h = 0.2$$

$$\begin{aligned} y(0.2) &\approx 1 + 0.2(4(0) + 1^2) = 1.2 \\ y(0.4) &\approx 1.2 + 0.2(4(0.2) + 1.2^2) \approx 1.648 \\ y(0.6) &\approx 1.648 + 0.2(4(0.4) + 1.648^2) \approx 2.511 \\ y(0.8) &\approx 2.511 + 0.2(4(0.6) + 2.511^2) \approx 4.092 \\ y(1) &\approx 4.092 + 0.2(4(0.8) + 4.092^2) \approx 8.578 \end{aligned}$$

9.3.5

- User Euler's method with $\Delta x = 0.1$ to estimate $y(1.4)$.

$$y' = -x - y, \quad y(1) = 1$$

$$\begin{aligned} y(1) &\approx 1 + 0.1(-1 - 1) = 0.8 \\ y(1.1) &\approx 0.8 + 0.1(-1.1 - 0.8) = 0.61 \\ y(1.2) &\approx 0.61 + 0.1(-1.2 - 0.61) = 0.429 \\ y(1.3) &\approx 0.429 + 0.1(-1.3 - 0.429) = 0.2561 \\ y(1.4) &\approx 0.2561 \end{aligned}$$

9.4 Edfinity: The Logistic Equation

- The logistic equation and general non-equilibrium solution ($k > 0$ and $A > 0$)

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{A}\right), \quad y = \frac{A}{1 - e^{-kt}/B} \quad \Leftrightarrow \quad \frac{y}{y - A} = Be^{kt}$$

- $y = 0 \rightarrow$ unstable equilibrium
- $y = A \rightarrow$ stable equilibrium
- If the initial value $y_0 = y(0)$ satisfies $y_0 > 0$, then $y(t)$ approach the stable equilibrium $y = A$, i.e., $\lim_{t\to\infty} y(t) = A$

9.4.1

- A population P obeys the logistic model. It satisfies the equation

$$\frac{dP}{dt} = \frac{7}{1100}P(11 - P) \text{ for } P > 0$$
$$\frac{dP}{dt} = \frac{7}{1100}P(1 - \frac{P}{11}) \text{ for } P > 0$$

c. Assume that $P(0) = 4$. Find $P(87)$

$$\frac{y}{y - A} = Be^{kt}$$
$$\frac{4}{4 - 11} = Be^{\frac{7}{1110}0}$$
$$-0.571 \approx B$$
$$\Longrightarrow \quad P(87) = \frac{11}{1 - e^{-0.06 \cdot 87}/ - 0.571} \approx -10.9$$

9.4.2

- Assuming $P \geq 0$, suppose that a population develops according to the logistic equation

$$\frac{dP}{dt} = 0.01P - 5 \times 10^{-5}P^2$$

where t is measured in weeks.

$$\begin{aligned} \frac{dP}{dt} &= 0.01P - 5 \times 10^{-5}P^2 \\ &= 0.01P\left(1 - \frac{5 \times 10^{-5}}{0.01}P\right) \\ &= 0.01P\left(1 - \frac{P}{200}\right) \end{aligned}$$

$$\begin{aligned} \text{Carrying capacity} = A &= 200 \\ k &= 0.01 \\ \text{increasing} &= (0, 200) \\ \text{decreasing} &= (200, \infty) \end{aligned}$$

9.4.3

- A population of squirrels lives in a forest with a carrying capacity of 1600. Assume logistic growth with growth constant $k = 1\,\text{yr}^{-1}$
- Find a formula for the squirrel population $P(t)$, assuming an initial population of 400 squirrels.

$$\begin{aligned}
\frac{dP}{dt} &= 1P\left(1 - \frac{P}{1600}\right) \\
B &= \frac{400}{400 - 1600} = -\frac{1}{3} \\
P(t) &= 1600/1 - \frac{e^{-t}}{-\frac{1}{3}} = \frac{1600}{1 + 3e^{-t}} \\
800 &= \frac{1600}{1 + 3e^{-t}} \\
1 + 3e^{-t} &= 2 \\
e^{-t} &= \frac{1}{3} \\
t &= -\ln\frac{1}{3} = 1.098\,\text{yr}
\end{aligned}$$

9.4.4

- Sunset Lake is stocked with 2700 rainbow trout and after 1 year the population has grown to 7050. Assuming logistic growth with a carrying capacity of 27,000, find the growth constant k, and determine when the population will increase to 13600.

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{27000}\right) \; P(0) = 2700, \quad P(t) = 13600$$

$$\begin{aligned}
B &= \frac{2700}{2700 - 27000} = -\frac{1}{9} \\
7050 &= \frac{27000}{1 + 9e^{-k\cdot 1}} \\
1 + 9e^{-k} &= \frac{27000}{7050} \\
k &= -\ln\frac{\frac{27000}{7050} - 1}{9} = 1.157 \\
13600 &= \frac{27000}{1 + 9e^{1.157t}} \\
e^{1.157t} &= \frac{\frac{27000}{13600} - 1}{9} \\
t &= \ln\left(\frac{\frac{27000}{13600} - 1}{9}\right)(1.157)^{-1} \\
t &\approx 1.911\,\text{yr}
\end{aligned}$$

9.4.5

- Suppose that a population develops according to the logistic equation

$$\frac{dP}{dt} = 0.05P - 6.667 \times 10^{-5} P^2$$

where t is measure in weeks.

$$\frac{dP}{dt} = 0.05P\left(1 - \frac{6.667 \times 10^{-5}}{0.05} P\right) = 0.05P\left(1 - \frac{P}{750}\right)$$

9.5 Edfinity: First-Order Linear Equations

- Hammers:

$$
\begin{aligned}
y' + P(x)y &= Q(x) \\
\alpha(x) &= e^{\int P(x)dx} \\
y &= \alpha(x)^{-1}\left(\int \alpha(x)Q(x)dx + C\right)
\end{aligned}
$$

9.5.1

- Solve $y' + 3x^{-1}y = x^2, \qquad y(1) = -9$
  - Idetify the integrating factor, $\alpha(x)$

$$
\begin{aligned}
\alpha(x) &= e^{\int 3x^{-1}dx} \\
\alpha(x) &= e^{3\ln x} \\
\alpha(x) &= x^3
\end{aligned}
$$

  - Find the general solution, $y(x)$

$$
\begin{aligned}
y &= x^{-3}\left(\int x^3x^2dx + C\right) \\
y &= x^{-3}\left(\frac{x^6}{6} + C\right) \\
y &= \frac{x^3}{6} + Cx^{-3}
\end{aligned}
$$

  - Solve the innitla value problem, $y(1) = -9$

$$
\begin{aligned}
-9 &= \frac{1^3}{6} + C^{-3} \\
1.5 &= C^{-3} \\
C &= -\left(\frac{55}{6}\right)^{3^{-1}} \\
\implies y &= \frac{x^3}{6} - \frac{55}{6}^{\frac{1}{3}}x^{-3} \\
y &= \frac{x^3}{6} - 9.167x^{-3}
\end{aligned}
$$

9.5.2

- Solve $4xy' - 8y = x^{-1}, \qquad y(1) = 6$

$$\implies y' - 2x^{-1}y = \frac{1}{4}x^{-2}$$

  - Idetify the integrating factor, $\alpha(x)$

$$\alpha(x) = e^{\int -2x^{-1}dx}$$
$$\alpha(x) = e^{-2\ln x}$$
$$\alpha(x) = x^{-2}$$

  - Find the general solution, $y(x)$

$$y = x^2\left(\int x^{-2}\frac{1}{4}x^{-2}dx + C\right)$$
$$y = x^2\left(-\frac{1}{12}x^{-3} + C\right)$$
$$y = -\frac{1}{12}x^{-1} + Cx^2$$

  - Solve the innitla value problem, $y(1) = 6$

$$6 = -\frac{1}{12}1^{-1} + C$$
$$C = 6 + \frac{1}{12} = 6.083$$
$$\implies y = \frac{1}{12}x^{-1} + 6.083x^2$$

9.5.3

- Solve $xy' = 2y - 9x, \qquad y(1) = -2$

$$\implies y' - 2x^{-1}y = -9$$

  - Idetify the integrating factor, $\alpha(x)$

$$\alpha(x) = e^{\int -2x^{-1}dx}$$
$$\alpha(x) = e^{-2\ln x}$$
$$\alpha(x) = x^{-2}$$

  - Find the general solution, $y(x)$

$$y = x^2\left(\int -9x^{-2}dx + C\right)$$
$$y = x^2(9x^{-1} + C)$$
$$y = 9x + Cx^2$$

- Solve the innitla value problem, $y(1) = -2$

$$\begin{aligned}-2 &= 9 + C \\ C &= -11 \\ \implies y &= 9x - 11x^2\end{aligned}$$

9.5.4

◦ Find the general solution of the first-order linear differetial equation

$$y' - (\ln x)y = 2x^x$$

$$\begin{aligned}\alpha(x) &= e^{\int -\ln x dx} \\ \alpha(x) &= e^{x - x\ln x} \\ \alpha(x) &= e^x x^{-x} \\ y &= e^{-x}x^x \left(\int e^x x^{-x} 2x^x dx + C \right) \\ y &= e^{-x}x^x(2e^x + C) \\ y &= 2x^x + Ce^{-x}x^x\end{aligned}$$

9.5.5

◦ Solve the initial value problem $y' + 4y = e^{8x}, \qquad y(0) = -7$

$$\begin{aligned}\alpha(x) &= e^{4x} \\ y &= e^{-4x}\left(\int e^{4x}e^{8x} dx + C \right) \\ y &= e^{-4x}\left(\frac{e^{12x}}{12} + C \right) \\ y &= \frac{e^{8x}}{12} + Ce^{-4x} \\ -7 &= \frac{1}{12} + C \\ C &= -\frac{85}{12} \\ y &= \frac{e^{8x}}{12} + -\frac{85}{12}e^{-4x}\end{aligned}$$

9 Rogawski: Review

Chapter 9 Toolbox

- **Separable first-order**: a differential equation in the form

$$\frac{dy}{dx} = f(x)g(y)$$

  - **General solution**: when $\frac{dy}{dt} = ky$, then $y(t) = De^{kt}$

$$\begin{aligned} y^{-1}dy &= kdt \\ \int y^{-1}dy &= \int kdt \\ \ln|y| &= kt + C \\ |y| &= e^{kt} \\ y &= De^{kt} \end{aligned}$$

  - Exponential decay: $k < 0$; half-life: $(\ln 0.5)k^{-1}$
  - Exponential growth: $k > 0$; doubling: $(\ln 2)k^{-1}$
- **First-order linear constant coefficient**: when a quantity y whose rate of change is proportional to the difference $y - b$, i.e.,

$$\frac{dy}{dt} = k(y - b)$$

  - **General solution**: using separation of variables,

$$y(t) = b + Ce^{kt} \quad \leftrightarrow \quad \frac{d}{dt}(y - b) = k(y - b)$$

  - **Newton's law of Cooling**: where k is the cooling constant (dependent on object) and T_0 is the ambient temperature.

$$\frac{dy}{dt} = -k(y - T_0) \implies y(t) = T_0 + C^{-kt}$$

  - **Newton's Second Law of Motion**: $F = ma = mv' = -mg - kv$, i.e.,

$$\frac{dv}{dt} = -\frac{k}{m}\left(v + \frac{mg}{k}\right) \implies v(t) = -\frac{mg}{k} + Ce^{-\frac{k}{m}t}$$

  - **Annuity/Compound interest**: modeling balance in annuity by the differential equation

$$\frac{dP}{dt} = rP - N = r(P - \frac{N}{r}) \implies P(t) = \frac{N}{r} + C^{rt}$$

- **Slope filed**: when a first-order differential equation $\frac{dy}{dt} = F(t, y)$ is obtained by drawing small segments of slope $F(t, y)$ at points t, y.
  - Test points particular points, often two easy tests are enough to match an equation to graph via elimination of potential options.
- **Euler's Method**: an approximate solution to $\frac{dy}{dt} = F(t, y)$ when given an initial condition $y(t_0) = y_0$ and time step h.
  - Setting $t_k = t_0 + kh$ yields $y_1, y_2, \ldots, y_n$ through recursive application of

$$y_k = y_{k-1} + hF(t_{k-1}, y_{k-1})$$

  - I.e.,

$$\begin{aligned} y_1 &= y_0 + hF(t_0, y_0) \\ &\searrow \\ y_2 &= y_1 + hF(t_{0+1h}, y_1) \\ &\searrow \\ y_3 &= y_2 + hF(t_{0+2h}, y_2) \\ &\searrow \\ &\vdots \end{aligned}$$

    where each y_k is an approximate of $y(t_n)$
- **Logistic differential equation**: where $y(t)$ is the population at time t and A denotes the carrying capacity, yielding a representation of room for growth $A - y(t)$.
  - The assumption is that the $\frac{dy}{dt}$ is proportional to the amount of $y(t)$ present and amount of $A - y(t)$ of room for growth, i.e.,

$$\frac{dy}{dt} = Ky(A - y), \qquad K = \text{ proportionality constant}$$

  - Which an be written as

$$\frac{dy}{dt} = ky(1 - \frac{y}{A}), \qquad k = KA$$

  - **General non-equilibrium solution**: when $k > 0 \wedge A > 0$:

$$y = \frac{A}{1 - \frac{e^{-kt}}{B}} \quad \leftrightarrow \quad \frac{y}{y - A} = Be^{kt}$$

- Two equilibrium constant solutions:
  - $y = 0$; unstable equilibrium.
  - $y = A$; a stable equilibrium.
- If the initial value $y_0 = y(0)$ satisfies $y_0 > 0$, then $\lim_{t \to \infty} y(t) = A$

- **First-Order Linear Equations**: method of solving all first-order linear differential equations, separable or not, as long as the equation can be put in the form:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

  - **Integrating factor**:

$$\alpha(x) = e^{\int P(x)dx}$$

  - **General solution**:

$$y = \alpha(x)^{-1}\left(\int \alpha(x)Q(x)dx + C\right)$$

  - Approach to the problems:
    1. Arrange equation in first-order linear form.
    2. Find the Integrating factor.
    3. Solve general solution.
    4. Solve initial value by finding C in solved general solution, if given $y(t)$.

11 Parametric Equations, Polar Coordinates, and Conic Sections