Applied Linear Algebra

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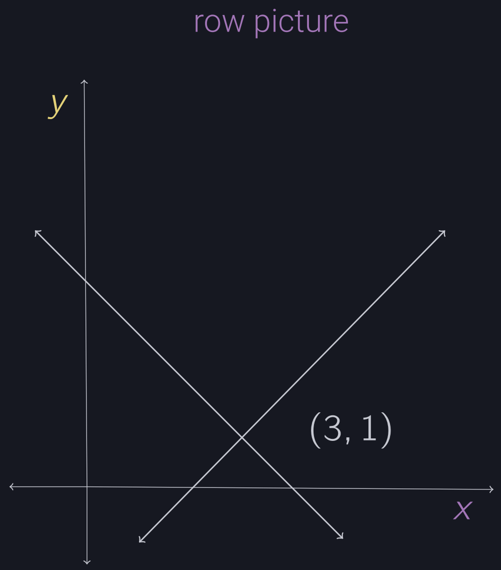
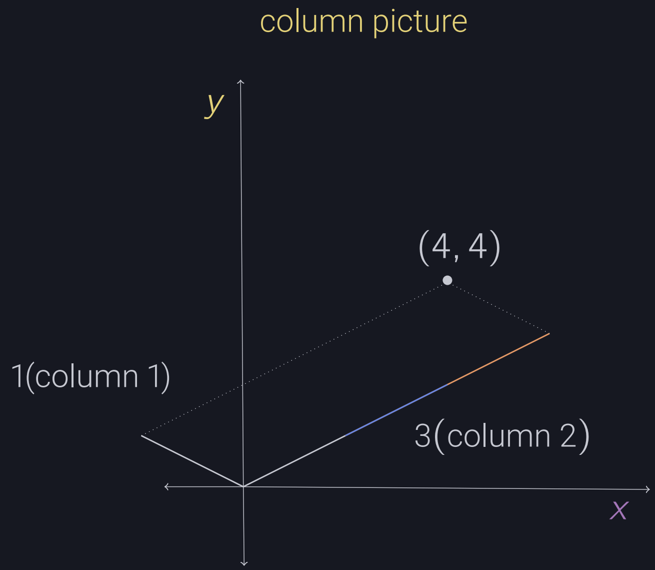
1 Matrices and Gaussian Elimination

1.2 The Geometry of Linear Equations

Problems 1–12

1. For the equations $x + y = 4,\ 2x - 2y = 4$, draw the row picture (two intersecting lines) and the column picture (combination of two columns equal to the column vector (4,4) on the right side).





1.2.1

2. Solve to find a combination of the columns that equals b:

$$\begin{aligned} u - v - w &= b_1 \\ v + w &= b_2 \\ w &= b_3 \end{aligned}$$

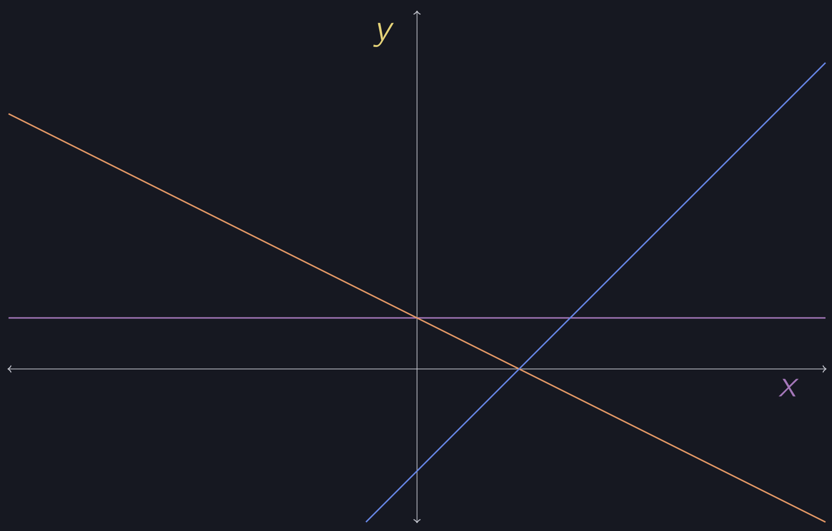
$$\begin{aligned} &\implies w = b_3 \\ &\implies v = b_2 - b_3 \\ &\implies u = b_1 + v + w = b_1 + b_2 \end{aligned}$$

3. Describe the intersection of the three planes $u + v + w + z = 6$ and $u + w + z = 4$ and $u + w = 2$ (all in four-dimensional space). Is it a line or a point or an empty set? What is the intersection if the fourth plane $u = -1$ is included? Find a fourth equation that leaves us with no solution.

   - A line; as $u + w = 2$ is only a line$^?$. A fourth plane with $u = -1$ would produce a normally intersecting point. Any addition equation when $u + w \neq 2$ would produce an inconsistent equation.

4. Sketch these three lines and decide if the equations are solvable:

$$\begin{aligned} x + 2y &= 2 \\ x - y &= 2 \\ y &= 1 \end{aligned}$$



1.2.4

Inconsistent; multiple points of intersect

What happens if all right-hand sides are zero? Is there any nonzero choice of right-hand sides that allows the three lines to intersect at the same point?

- If all the solutions were zero, then it would be a trivial solution.
- Yes, e.g., $x - y = -1$ would produce a single point of intersection.

5. Find two points on the line of intersection of the three planes $t = 0$ and $z = 0$ and $x + y + z + t = 1$ in four-dimensional space.

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

6. When $b = (2, 5, 7)$, find a solution (u, v, w) to equation (4) different from the solution $(1, 0, 1)$ mentioned in the text.

- Since there are infinite solutions, and if $\boldsymbol{s}$ vector describing one solution and λ is any scalar, then $\boldsymbol{s}\lambda$ is also a solution. E.g., $(1, 0, 1)\,42 = (42, 0, 42)$

8. Explain why the system

$$\begin{aligned} u+v+w &= 2 \\ u+2v+3w &= 1 \\ v+2w &= 0 \end{aligned}$$

is singular by finding a combination of the three equations that adds up to $0=1$. What value should replace the last zero on the right side to allow the equations to have solutions—and what is one of the solutions?

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 0 \end{array}\right] \xrightarrow{R_2-R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 2 & 0 \end{array}\right] \xrightarrow{R_3-R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

- Replacing the last zero with -1 would yield infinite solutions. One solution would be $[3,-1,0]^T$

9. The column picture for the previous exercise (singular system) is

$$u\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + v\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + w\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = b$$

Show that the three columns on the left lie in the same plane by expressing the third as a combination of the first two. What are all the solutions (u,v,w) if b is the zero vector $(0,0,0)$?

$$-1\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

- If is $\boldsymbol{b}$ equal to the zero vector $\mathbf{0}$ then the solutions are equal to the kernel? i.e., $-1x_1, 2x_2, 0x_3 = \mathbf{0}$

10. Under what condition on y_1, y_2, y_3 do the points $(0,y_1), (1,y_2), (2,y_3)$ lie on a straight line?

- Question 9 describes the state at which they are collinear, i.e., $y_3 = 2y_2 - y_1$

11. These equations are certain to have the solution $x=y=0$. For which values of a is there a whole line of solutions?

$$\begin{aligned} ax+2y &= 0 \\ 2x+ay &= 0 \end{aligned}$$

- Only the scalars that make the lines linearly dependent, i.e., $a=2,-2$

Problems 17–23

17. The first of these equations plus the second equals the third:

$$\begin{aligned} x + y + z &= 2 \\ x + 2y + z &= 3 \\ 2x + 3y + 2z &= 5 \end{aligned}$$

The first two planes meet along a line. The third plane contains that line, because if x, y, z satisfy the first two equations then they also span all of $\mathbb{R}^3$. The equations have infinitely many solutions (the whole line $\boldsymbol{L}$). Find three solutions.

- $\boldsymbol{v} = (4, 4, 0)$, $\boldsymbol{w} = (6, 3, 2)$, $\boldsymbol{u} = 2v + -1w$

18. Move the third plane in Problem 17 to a parallel plane $2x + 3y + 2z = 9$. Now the three equations have no solution—*why not*? The first two planes meet along the line $\boldsymbol{L}$, but the third plane doesn't that cross that line.

19. In Problem 17 the columns are $(1, 1, 2)$ and $(1, 2, 3)$ and $(1, 1, 2)$. This is a "singular case" because the third column is linearly dependent Find two combinations of the columns that give $b = (2, 3, 5)$. This is only possible for $b = (4, 6, c)$ if $c = 10$

20. Normally 4 "planes" in four-dimensional space meet at a tensor. Normally 4 column vectors in four-dimensional space can combine to produce b. What combination of $(1, 0, 0, 0), (1, 1, 0, 0), (1, 1, 1, 0), (1, 1, 1, 1)$ produces $b = (3, 3, 3, 2)$? $(1, 0, 0, -2)^?$ What 4 equations for x, y, z, t are you solving? A lower triangular matrix, i.e.,

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3 \\ 1 & 1 & 0 & 0 & 3 \\ 1 & 1 & 1 & 0 & 3 \\ 1 & 1 & 1 & 1 & 1 \end{array}\right]$$

21. When equation 1 is added to equation 2, which of these are changed: the planes in the row picture, the column picture, the coefficient matrix, the solution?

- Row operations do not change the solution. Row 2 is changed, thus the second plane is changed. All columns are changed.?

22. If (a, b) is a multiple of (c, d) with $abcd \neq 0$, show that (a, c) is a multiple of (b, d). This is surprisingly important: call it a challenge question. You could use numbers first to see how a, b, c, and d are related. The question will lead to:

    If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has dependent rows then it has dependent columns.

    Received help from this document (pg. 3), accessed 10/01/2021

    $$\begin{aligned} \lambda \in \mathbb{R}, \quad (a, b) &= \lambda(c, d) = (\lambda c, \lambda d) \\ \implies a &= \lambda c = \lambda c d^{-1} d = d^{-1} c \lambda d = d^{-1} c b \\ \implies (a, c) &= \left(d^{-1} cb, d^{-1} dc\right) = cd^{-1}(b, d) \end{aligned}$$

    Thus, (b, d) is a multiple of (a, c)

23. In these equations, the third column (multiplying w) is the same as the right side b. The column form of the equations immediately gives what solution for (u, v, w)?

    $$\begin{aligned} 6u + 7v + 8w &= 8 \\ 4u + 5v + 9w &= 9 \\ 2u - 2v + 7w &= 7 \end{aligned}$$

    - First two columns are irrelevant, $u = 0$, $v = 0$, only need w

1.3 Gaussian Elimination

Problems 6, 7

6. Choose a coefficient b that makes this system singular. Then choose a right-hand side g that makes it solvable. Find two solutions in that singular case.

$$\begin{aligned} 2x + by &= 16 \\ 4x + 8y &= g \end{aligned}$$

$$\begin{aligned} 2x + 4y &= 16 \\ 4x + 8y &= 32 \end{aligned}$$

- Since R_2 is just a multiple of R_1, then solving for x, y, with one variable = 0, in the first equation will yield two solutions, i.e., $(8, 0)$, $(0, 4)$

7. For which numbers a does elimination break down (a) permanently, and (b) temporarily?

$$\begin{aligned} ax + 3y &= -3 \\ 4x + 6y &= 6 \end{aligned}$$

Solve for x and y after fixing the second breakdown by a row exchange.

- Permanently: $a = 2$ (linearly dependent, no solution)
- Temporarily: $a = 0$;

$$\left[\begin{array}{cc|c} 4 & 6 & 6 \\ 0 & 3 & -3 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -1 \end{array}\right]$$
$$y = -1, \quad x = 3$$

Problems 17, 18, 19

17. Which number q makes this system singular and which right-hand side t gives it infinitely many solutions? Find the solution that has $z = 1$.

$$\begin{aligned} x + 4y - 2z &= 1 \\ x + 7y - 6z &= 6 \\ 3y + qz &= t \end{aligned}$$

$$\begin{aligned} x + 4y - 2z &= 1 \\ x + 7y - 6z &= 6 \\ 3y + -4z &= 5 \end{aligned}$$

- If $q = -4$, then R_3 would have no pivot
- If $t = 5$, then there would be finite solutions, R_3 would be linearly dependent with R_2

18. It is impossible for a system of linear equations to have exactly two solutions. Explain why.
    - If (x, y, z) and (X, Y, Z) are two solutions, what is the other one?
        - There is no other *one*, there would be infinitely many.
    - If 25 planes meet at two points, where else do they meet?
        - Every other single point, they would span all of $\mathbb{R}^3$
19. Three planes can fail to have an intersection point, when no two planes are parallel. The system is singular if row 3 of $\boldsymbol{A}$ is a linearly dependent; a combination of the first two rows. Find a third equation that can't be solved if $x + y + z = 0$ and $x - 2y - z = 1$.

$$\begin{aligned} x + y + z &= 0 \\ x - 2y - z &= 1 \\ R_1 + R_2 \neq 1 \to \text{ parallel; no solution, e.g.,} \\ 2x - y &= 42 \end{aligned}$$

Problems 30, 31

30. Use elimination to solve

$$\begin{aligned} u + v + w &= 6 \\ u + 2v + 2w &= 11 \\ 2u + 3v - 4w &= 3 \end{aligned} \qquad \text{and} \qquad \begin{aligned} u + v + w &= 7 \\ u + 2v + 2w &= 10 \\ 2u + 3v - 4w &= 3 \end{aligned}$$

$$\text{rref}\left(\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 2 & 11 \\ 2 & 3 & -4 & 3 \end{array}\right]\right) \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$
$$\text{rref}\left(\left[\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 1 & 2 & 2 & 10 \\ 2 & 3 & -4 & 3 \end{array}\right]\right) \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

31. For which three numbers a will elimination fail to give three pivots?

$$\begin{aligned} ax + 2y + 3z &= b_1 \\ ax + ay + 4z &= b_2 \\ ax + ay + az &= b_3 \end{aligned}$$

- For $a = 0$, multiple failures.
- For $a = 2$, columns $0, 1$ would be equal.
- For $a = 4$, rows $1, 2$ would be equal.

1.4 Matrix Notation and Matrix Multiplication

Problems 4, 10, 17, 19

4. If an $m \times n$ matrix $\boldsymbol{A}$ multiplies an n-dimensional vector $\boldsymbol{x}$, how many separate multiplications are involved? What if A multiplies an $n \times p$ matrix $\boldsymbol{B}$?
   - $m \cdot n$ multiplications; number of rows times the length of $\boldsymbol{x}$.
   - $m \cdot n \cdot p$; same as above, except accounting for each additional column p.
10. True or false? Give a specific counterexample when false.
   - If rows 1 and 3 of $\boldsymbol{B}$ are the same, so are rows 1 and 3 of $\boldsymbol{AB}$.
     - ✘ false; matrix multiplication is done by the rows of the left matrix and the columns of the right, the rows may be the same, but if a column between the two are different, then there would be different multiplications occurring, e.g.,

$$\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\end{bmatrix}\begin{bmatrix}1 & 1 & 1\\4 & 2 & 0\\1 & 1 & 1\end{bmatrix} = \begin{bmatrix}12 & 8 & 4\\30 & 20 & 10\\38 & 32 & 16\end{bmatrix}$$

   - If columns 1 and 3 of $\boldsymbol{B}$ are the same, so are columns 1 and 3 of $\boldsymbol{AB}$.
     - ✓ true;
   - If rows 1 and 3 of $\boldsymbol{A}$ are the same, so are rows 1 and 3 of $\boldsymbol{AB}$.
     - ✓ true
   - $(AB)^2 = A^2B^2$.
     - ✘ false (most of the time), e.g.,

$$\boldsymbol{A} = \begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\end{bmatrix} \boldsymbol{B} = \begin{bmatrix}1 & 1 & 1\\4 & 2 & 0\\1 & 1 & 1\end{bmatrix}$$

$$\boldsymbol{AB}^2 = \begin{bmatrix}144 & 64 & 16\\900 & 400 & 100\\2304 & 1024 & 256\end{bmatrix} \neq \begin{bmatrix}74 & 26 & 10\\452 & 152 & 52\\1154 & 386 & 130\end{bmatrix} = \boldsymbol{A}^2\boldsymbol{B}^2$$

17. Which of the following matrices are guaranteed to equal $(A+B)^2$?

$$A^2 + 2AB + B^2,$$
$$✓\, A(A+B) + B(A+B)$$
$$✓\, (A+B)(B+A),$$
$$✓\, A^2 + AB + BA + B^2$$

19. A fourth way to multiply matrices is columns of $\boldsymbol{A}$ times rows of $\boldsymbol{B}$:

$\boldsymbol{AB}=$ (column 1)(row 1) $+\cdots+$ (column n)(row n) $=$ sum of simple matrices.

Give a 2×2 example of this important rule for matrix multiplication.

$$\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}a & b\\c & d\end{bmatrix}=\left(a\begin{bmatrix}1\\3\end{bmatrix}+c\begin{bmatrix}2\\4\end{bmatrix}b\begin{bmatrix}1\\3\end{bmatrix}+d\begin{bmatrix}2\\4\end{bmatrix}\right)$$

Useful, as the right matrix can be thought of as the weights that scale the elements of the columns of the left matrix.

Problems 30–31

30. Multiply these matrices:

$$\begin{bmatrix}0 & 0 & 1\\0 & 1 & 0\\1 & 0 & 0\end{bmatrix}\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\end{bmatrix}\begin{bmatrix}0 & 0 & 1\\0 & 1 & 0\\1 & 0 & 0\end{bmatrix}\quad\text{and}\quad\begin{bmatrix}1 & 0 & 0\\-1 & 1 & 0\\-1 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & 3\\1 & 3 & 1\\1 & 4 & 0\end{bmatrix}.$$

$$\begin{bmatrix}9 & 8 & 7\\6 & 5 & 4\\3 & 2 & 1\end{bmatrix}\quad\text{and}\quad\begin{bmatrix}1 & 2 & 3\\0 & 1 & -2\\0 & 2 & -3\end{bmatrix}\quad\text{respectively}$$

- The former multiplication performs two operations (left: swaps top and bottom columns, right: swaps left and right columns), while the latter subtracts row 1 from both row 2 and row 3.

31. This 4×4 matrix needs which elimination matrices $\boldsymbol{E}_{21}$ and $\boldsymbol{E}_{32}$ and $\boldsymbol{E}_{43}$?

$$\boldsymbol{A}=\begin{bmatrix}2 & -1 & 0 & 0\\-1 & 2 & -1 & 0\\0 & -1 & 2 & -1\\0 & 0 & -1 & 2\end{bmatrix}$$

- $e_{21}=-\frac{1}{2},\quad e_{32}=-\frac{2}{3},\quad e_{43}=-\frac{3}{4}$
- I suspect the factions will tend towards -1 if the matrix was expanded upon in a similar fusion?

Problems 34, 35, 38, 42

34. Multiply these matrices in the orders FE and FE and E^2:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & 0 & 1 \end{bmatrix} \qquad F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{bmatrix}$$

$$FE = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ ac+b & c & 1 \end{bmatrix} \quad EF = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix} \quad E^2 = \begin{bmatrix} 1 & 0 & 0 \\ 2a & 1 & 0 \\ 2b & 0 & 1 \end{bmatrix}$$

35. ↓
   (a) Suppose all columns of B are the same. Then all columns of EB are the same, because each one is E times B_{1n}.
   (b) Suppose all rows of B are $[1\ 2\ 4]$. Show by example that all rows of EB are not $[1\ 2\ 4]$. It is true that those rows are multiples of $[1\ 2\ 4]$
      - E.g., if $e_{12} = 2$, then m_2 of EB would be $[3\ 6\ 12]$

38. If $AB = I$ and $BC = I$, use the associative law to prove $A = C$.

$$A = A(BC)$$
$$A = (AB)C$$
$$A = C$$

42. True of false?
   (a) If A^2 is defined then A is necessarily square.
      - ✓ true; inner dimensions much match, i.e., dimensions of $n_1 = m_2$. Thus, A must be square.
   (b) If AB and BA are defined, then A and B are square.
      - ✘ false; if $A = 6 \times 9$ and $B = 9 \times 6$ allows for valid pre- and post-multiplication of B.
   (c) If AB and BA are defined, then AB and BA are square.
      - ✓ true; see above example, each case will still yield square matrices. Not a proof, but I can't see another way to falsify (b).
   (d) If $AB = B$ then $A = I$
      - ✘ false; e.g.,$B = 0$

1.5 Triangular Factors and Row Exchanges

Problems 1, 6, 7, 14, 18

1. When is an upper triangular matrix nonsingular (a full set of pivots)?
    - Every pivot must be nonzero. If there is a zero on one of the pivots, then it indicates that one of the columns is a linear combination of one or more of the other columns.
6. Find E^2 and E^8 and E^{-1} if

$$E = \begin{bmatrix} 1 & 0 \\ 6 & 1 \end{bmatrix}$$

matrices

$$E^2 = \begin{bmatrix} 1 & 0 \\ 36 & 1 \end{bmatrix} \quad E^8 = \begin{bmatrix} 1 & 0 \\ 1679616 & 1 \end{bmatrix} \quad E^{-1} = \begin{bmatrix} 1 & 0 \\ -6 & 1 \end{bmatrix}$$

7. Find the products FGH and HGF if (with upper triangular zeros omitted)

$$F = \begin{bmatrix} 1 & & & \\ 2 & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix} G = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 2 & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix} H = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

matrices

$$FGH = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix} \quad HGF = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 4 & 2 & 1 & 0 \\ 8 & 4 & 2 & 1 \end{bmatrix}$$

14. 14. Write down all six of the 3×3 permutation matrices, including $P = I$. Identify their inverses, which are also permutation matrices. The inverses satisfy $PP^{-1} = I$ and are on the same list.

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad P^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad P^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad P^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad P^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

18. Decide whether the following systems are singular or nonsingular, and whether they have no solution, one solution, or infinitely many solutions:

$$\left[\begin{array}{ccc|c} 0 & 1 & -1 & 2 \\ 1 & -1 & 0 & 2 \\ 1 & 0 & -1 & 2 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{array}\right]$$

matrices

- Performing rref on above martices yields:
  - Singular — no solution
  - Singular — ∞ solutions.
  - Nonsingular — one solution [0.5 0.5 0.5]

Problems 26, 28

26. Which number c leads to zero in the second pivot position? A row exchange is needed and $A = LU$ is not possible. Which c produces zero in the third pivot position? Then a row exchange can't help and elimination fails.

$$A = \begin{bmatrix} 1 & c & 0 \\ 2 & 4 & 1 \\ 3 & 5 & 1 \end{bmatrix}$$

- If $c = 2$ then row 2 would have a 0 in the pivot, yielding:

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 3 & 5 & 1 \end{bmatrix}$$

- If $c = 1$, then you could take the matrix down to the following form,

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

  which would yield a singular matrix with infinite solutions.

28. A and B are symmetric across the diagonal (because 4 = 4). Find their triple factorizations LU and say how U is related to L for these symmetric matrices:

$$A = \begin{bmatrix} 2 & 4 \\ 4 & 11 \end{bmatrix} \quad \textit{and} \quad B = \begin{bmatrix} 1 & 4 & 0 \\ 4 & 12 & 4 \\ 0 & 4 & 0 \end{bmatrix}$$

$$U_A = \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix}, \qquad U_B = \begin{bmatrix} 1 & 4 & 0 \\ 0 & -4 & 4 \\ 0 & 0 & -4 \end{bmatrix}$$

$$V_A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \qquad V_B = \begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$L_A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \qquad L_B = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

- $A, B = LDV \implies L = V^T$; the diagonal of the upper matrix, if reduced to 1's in the pivot positions, yields the transpose of the lower triangular matrix.

Problems 33, 43

33. Solve $\boldsymbol{Lc} = \boldsymbol{b}$ to find $\boldsymbol{c}$. Then solve $\boldsymbol{Ux} = \boldsymbol{c}$ to find $\boldsymbol{x}$. What was $\boldsymbol{A}$?

$$\boldsymbol{L} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad \textit{and} \quad \boldsymbol{U} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \textit{and} \quad \boldsymbol{b} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

matrices

$$\boldsymbol{c} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} \quad \boldsymbol{x} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \implies \boldsymbol{A} = \boldsymbol{LU} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

43. (Try this question) Which permutation makes $\boldsymbol{PA}$ upper triangular? Which permutations make $\boldsymbol{P_1 A P_2}$ lower triangular? Multiplying $\boldsymbol{A}$ on the right by $\boldsymbol{P_2}$ exchanges the columns of $\boldsymbol{A}$.

$$\boldsymbol{A} = \begin{bmatrix} 0 & 0 & 6 \\ 1 & 2 & 3 \\ 0 & 4 & 5 \end{bmatrix}$$

$$\boldsymbol{U} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 6 \\ 1 & 2 & 3 \\ 0 & 4 & 5 \end{bmatrix}$$

$$\boldsymbol{L} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 6 \\ 1 & 2 & 3 \\ 0 & 4 & 5 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

1.6 Inverses and Transposes

Problems 3, 12, 18

3. From $AB = C$ find a formula for A^{-1}. Also find A^{-1} from $PA = LU$.

$$\begin{aligned} AB &= C \\ A &= CB^{-1} \\ A^{-1} &= BC^{-1} \\ PA &= LU \\ A &= P^{-1}LU \\ A^{-1} &= U^{-1}L^{-1}P \end{aligned}$$

12. If A is invertible, which properties of A remain true for A^{-1}?
    (a) A is triangular. ✓ true
    (b) A is symmetric. ✓ true
    (c) A is tridiagonal. ✘ false
    (d) All entries are whole ✘ false
    (e) All entire are fractions (including numbers like $\frac{3}{1}$). ✓ true;

18. Under what conditions on their entries are A and B invertible?

$$A = \begin{bmatrix} a & b & c \\ d & e & 0 \\ f & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & e \end{bmatrix}$$

matrices

$$\operatorname{rref}\left(\left[\begin{array}{ccc|ccc} a & b & c & 1 & 0 & 0 \\ d & e & 0 & 0 & 1 & 0 \\ f & 0 & 0 & 0 & 0 & 1 \end{array}\right]\right) \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & f^{-1} \\ 0 & 1 & 0 & 0 & e^{-1} & -\frac{d}{ef} \\ 0 & 0 & 1 & c^{-1} & -\frac{b}{ce} & \frac{-ae+bd}{cef} \end{array}\right]$$

$$\operatorname{rref}\left(\left[\begin{array}{ccc|ccc} a & b & 0 & 1 & 0 & 0 \\ c & d & 0 & 0 & 1 & 0 \\ 0 & 0 & e & 0 & 0 & 1 \end{array}\right]\right) \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{d}{ad-bc} & -\frac{b}{ad-bc} & 0 \\ 0 & 1 & 0 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} & 0 \\ 0 & 0 & 1 & 0 & 0 & e^{-1} \end{array}\right]$$

- $A \rightarrow A^{-1} \iff c, e, f \neq 0$
- $B \rightarrow B^{-1} \iff e \neq 0 \wedge ad \neq bc$

Problems 21, 28, 41, 56, 58

21. (Remarkable) If $\boldsymbol{A}$ and $\boldsymbol{B}$ are square matrices, show that $\boldsymbol{I}-\boldsymbol{B}\boldsymbol{A}$ is invertible if $\boldsymbol{I}-\boldsymbol{A}\boldsymbol{B}$ is invertible. Sart from $\boldsymbol{B}(\boldsymbol{I}-\boldsymbol{A}\boldsymbol{B})=(\boldsymbol{I}-\boldsymbol{B}\boldsymbol{A})\boldsymbol{B}$

$$\begin{aligned} B(I-AB) &= (I-BA)B \\ (I-AB) &= B^{-1}(I-BA)B \\ (I-AB)^{-1} &= B(I-BA)^{-1}B^{-1} \end{aligned}$$

- Thus, as long as $\boldsymbol{I}-\boldsymbol{B}\boldsymbol{A}$ is invertible, then the inverse is defined.

28. If the product $\boldsymbol{M}=\boldsymbol{A}\boldsymbol{B}\boldsymbol{C}$ of three square matrices is invertible, then $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ are invertible. Find a formula for $\boldsymbol{B}^{-1}$ that involves $\boldsymbol{M}^{-1}$ and $\boldsymbol{A}$ and $\boldsymbol{C}$.

$$\begin{aligned} M &= ABC \\ M^{-1} &= C^{-1}B^{-1}A^{-1} \\ CM^{-1} &= B^{-1}A^{-1} \\ CM^{-1}A &= B^{-1} \end{aligned}$$

41. For which three numbers c is this matrix not invertible, and why not?

$$A=\begin{bmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{bmatrix}$$

- If $c=0$, then there would be multile unavoidable zeros in the pivots.
- If $c=2$, then row 2 would just be duplicate of row 1.
- If $c=7$, then column 2 and 3 would be equal.

57. If $\boldsymbol{A}=\boldsymbol{A}^T$ needs a row exchange, then it also needs a column exchange to stay symmetric. In matrix language, $\boldsymbol{P}\boldsymbol{A}$ loses the symmetry of $\boldsymbol{A}$ but $\boldsymbol{P}\boldsymbol{A}\boldsymbol{P}^T$ recovers the symmetry.

58. $\downarrow$
    (a) How many entries of $\boldsymbol{A}$ can be chosen independently, if $\boldsymbol{A}=\boldsymbol{A}^T$ is 5×5?
    - 25 total choices — 10 under the diagonal = 15

    (b) How do $\boldsymbol{L}$ and $\boldsymbol{D}$ (5×5) give the same number of choice in $\boldsymbol{L}\boldsymbol{D}\boldsymbol{L}^T$?
    - Oh, I kind of used this to find (a). Well, the diagonal doesn't matter ($\neq 0$), since a transponse can simply be thought of a rotation around the diagonal elements. But, every element must match $\boldsymbol{U}$, thus only the 10 choices below matter, yieldng 15 total choices.

2 Vector Spaces

2.1 Vector Spaces and Subspaces

Problems 25, 26, 30, 31

25. If we add an extra column $\boldsymbol{b}$ to a matrix $\boldsymbol{A}$, then the column space gets larger unless they are linearly dependent..
    - Give an example in which the column space gets larger and an example in which it doesn't.

$$\text{Larger: } \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 2 & 1 \end{array}\right] \qquad \text{No change: } \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 2 & 4 & 6 \\ 3 & 2 & 9 \end{array}\right]$$

    - Why is $\boldsymbol{Ax} = \boldsymbol{b}$ solvable exactly when the column space doesn't get larger by including $\boldsymbol{b}$?
        - Because the solution would be in the image, leading to infinite solutions since it could be written as a linear combination of the vectors already in $\boldsymbol{A}$.
26. The columns of $\boldsymbol{AB}$ are combinations of the columns of $\boldsymbol{A}$. This means: the column space of $\boldsymbol{AB}$ contained in (possibly equal to) the column space of $\boldsymbol{A}$. Give an example where the column spaces of $\boldsymbol{A}$ and $\boldsymbol{AB}$ are not equal.

    vectorSpaces

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 4 & 6 \\ 3 & 2 & 9 \end{bmatrix} \qquad \boldsymbol{B} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

    - The column space of $\boldsymbol{B}$ is clearly contained in $\boldsymbol{A}$, but since the dimension of the null space of $\boldsymbol{B}$ is 2, then $\boldsymbol{AB}$ will not have the same column space of $\boldsymbol{A}$.
30. If the 9×12 system $\boldsymbol{Ax} = \boldsymbol{b}$ is solvable for every $\boldsymbol{b}$, then $C(A) = 9$.
    - This follows from the Rank-nullity theorem, where the dimensionality of the column space and cokernel is equal to the rows of the original matrix.
    - This also implies the dimensionality of the cokernel is 3.
31. Why isn't $\mathbb{R}^2$ a subspace of $\mathbb{R}^3$?
    - $\mathbb{R}^2$ could be a *subset*, but not a subspace; there are infinite 2-dimensional planes in $\mathbb{R}^3$.
    - If you included a third point of 0 in $\mathbb{R}^2$, then it would indicate that includes the origin, which could make it a subspace or $\mathbb{R}^3$. However, that extra coordinate would be make it be a vector in $\mathbb{R}^3$.

2.2 Solving Ax = 0 and Ax = b

Problems 12, 24, 25, 70

12. Which of these rules give a correct definition of the rank of $\boldsymbol{A}$?
    (a) The number of nonzero rows in $\boldsymbol{R}$. ✓ true

    $$\max(r) = r \in \mathbb{N} : 0 \leq r \leq \min(m, n)$$

    (b) The number of columns minus the total number of rows.
    - This would yield dimensionality of the null space.

    (c) The number of columns minus the number of free columns.
    - This would yield the dimensionality of the left null space.

    (d) The number of 1s in $\boldsymbol{R}$.
    - This wouldn't tell you much of anything.

24. Every column of $\boldsymbol{AB}$ is a combination of the columns of $\boldsymbol{A}$. Then the dimensions of the column spaces give $\text{rank}(\boldsymbol{AB}) \leq \text{rank}(\boldsymbol{A})$.

    Problem: Prove that $\text{rank}(\boldsymbol{AB}) \leq \text{rank}(\boldsymbol{B})$.

    $$\begin{aligned} & \text{rank}(A) = C(A) = C(A^T) \\ \implies & \text{rank}(A) = \text{rank}\left(A^T\right) \\ \therefore\ & \text{rank}(AB) = \text{rank}\left((AB)^T\right) = \text{rank}\left(B^T A^T\right) \leq \text{rank}\left(B^T\right) = \text{rank}(B) \end{aligned}$$

    - The rank of $\boldsymbol{AB}$ can only be decreased, if $\boldsymbol{B}$ is not full rank itself.
    - If, both $\boldsymbol{A}$ and $\boldsymbol{B}$ are full rank, then $\text{rank}(\boldsymbol{AB}) = \text{rank}(\boldsymbol{B})$

25. (Important) Suppose $\boldsymbol{A}$ and $\boldsymbol{B}$ are $n \times n$ matrices, and $\boldsymbol{AB}$ = $\boldsymbol{I}$. Prove from $\text{rank}(AB) \leq \text{rank}(A)$ that the rank of $\boldsymbol{A}$ is n. So $\boldsymbol{A}$ is invertible and $\boldsymbol{B}$ must be its two-sided inverse. Therefore $\boldsymbol{BA} = \boldsymbol{I}$
    - $\boldsymbol{A}$ must have same size of $\boldsymbol{B}$, given they are both $n \times n$.
    - If $\boldsymbol{A}$ was rank deficient, but $\boldsymbol{B}$ was full rank, then $\text{rank}(AB) \leq \text{rank}(\boldsymbol{A})$ would be invalid, forcing $\text{rank}(\boldsymbol{A}) = n$.

70. Explain why $\boldsymbol{A}$ and $-\boldsymbol{A}$ always have the same reduced echelon form $\boldsymbol{R}$.
    - Signed solutions are arbitrary; $-\boldsymbol{A}$ would have permutations that flip the sign and yield the same, reversible, solution.

2.3 Linear Independence, Basis, and Dimension

Problems 9, 13, 28, 36

9. Suppose $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4$ are vectors in $\mathbb{R}^3$.
   (a) There four vectors are dependent because $\mathbb{R}^3$ can only have 3 linearly independent vectors.
   (b) The two vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ will be dependent if they are multiples of each other.
   (c) The vectors $\boldsymbol{v}_1$ and $(0,0,0)$ are dependent because $\boldsymbol{v}_1(0,0,0) = \mathbf{0}$
13. Find the dimensions of:
   (a) the column space of $\boldsymbol{A}$:
   (b) the column space of $\boldsymbol{U}$:
   (c) the row space of $\boldsymbol{A}$:
   (d) the row space of $\boldsymbol{U}$.

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 3 & 1 & -1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{U} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

   vectorSpaces

   - Which two of the spaces are the same?
     - The row space of $\boldsymbol{A}$ and $\boldsymbol{U}$ are the same. Why? Taking $\boldsymbol{A}$ to rref shows it's also rank 2, with a linearly dependent row (0, 0, 0), just like $\boldsymbol{U}$.
28. True or false (give a good reason)?
   (a) If the columns of a matrix are dependent, so are the rows.
   - ✘ false; as shown in problem 2.2.12↑, the max rank is the minimum of either the rows or columns. You could have linearly dependent columns, if $n > m$, but no linearly dependent rows.

   (b) The column space of a 2×2 matrix is the same as its row space.
   - ✘ false; e.g., $\left[\begin{smallmatrix} 4 & 2 \\ 0 & 0 \end{smallmatrix}\right]$

   (c) The column space of a 2×2 matrix has the same dimension as its row space.
   - ✓ true; rank $(\boldsymbol{A}) = C(\boldsymbol{A}) = C(\boldsymbol{A}^T)$

   (d) The columns of a matrix are a basis for the column space.
   - ✘ false; one of the columns could be linearly dependent with another; only linearly independent columns forms a basis for the column space.
36. If $\boldsymbol{A}$ is a 64×17 matrix of rank 11, how many independent vectors satisfy $\boldsymbol{A}\boldsymbol{x} = \mathbf{0}$? How many independent vectors satisfy $\boldsymbol{A}^T\boldsymbol{y} = \mathbf{0}$?
   - $17 - 11 = 6, \quad 64 - 11 = 53$, respectively.

2.4 The Four Fundamental Subspaces

Problems 6, 14, 15, 27

6. Suppose $\boldsymbol{A}$ is an $m \times n$ matrix of rank r. Under what conditions on those numbers does

   (a) $\boldsymbol{A}$ have a two-sided inverse: $\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}$?

   - Square, nonsingular (full rank).

   (b) $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ have infinitely many solutions for every $\boldsymbol{b}$?

   - If rank $(A) < n$ ($n =$ number of columns)

14. 14. Find a left-inverse and/or a right-inverse (when they exist) for

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{M} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{T} = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$$

   vectorSpaces

   - $\boldsymbol{A}\boldsymbol{A}^T$ (right inverse) $\boldsymbol{M}^T\boldsymbol{M}$ (left inverse) yield the same inverse,

$$\begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

     which makes sense, since $\boldsymbol{M} = \boldsymbol{A}^T$

$$\boldsymbol{T}^{-1} = \begin{bmatrix} a^{-1} & -ba^{-2} \\ 0 & a^{-1} \end{bmatrix}$$

     - $\boldsymbol{T}$ has an inverse, left and right, as long as $\boldsymbol{a} \neq 0$.
     - $\boldsymbol{T}$ does not have a normal inverse if $a = b$ (singular), but the left and right inverse both yield the above matrix.

15. 15. If the columns of $\boldsymbol{A}$ are linearly independent, then the rank is the number of columns, n, the nullspace is empty , the row space is $\mathbb{R}^n$, and there exists a left-inverse.

27. (Important) $\boldsymbol{A}$ is an $m \times n$ matrix of rank r. Suppose there are right-hand sides $\boldsymbol{b}$ for which $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ has no solution.

   (a) What inequalities must be true between $m, n,$ and r?

   - No solution means a row of zeros, thus $r < m$
   - $r \leq n$, nothing changed.

   (b) How do you know that $\boldsymbol{A}^T\boldsymbol{y} = \mathbf{0}$ has a nonzero solution?

   - $\boldsymbol{A}^T\boldsymbol{y} = \mathbf{0}$ raises investigation of the left null space. Since the column space does not span all of $\mathbb{R}^m$ ($r < m$), then it means it contains a nonzero solution.

2.6 Linear Transformations

Problems 6, 34, 39

6. What 3 by 3 matrices represent the transformation that
   (a) project every vector onto the x-y plane?
   (b) reflect every vector through the x-y plane?
   (c) rotate the x-y plane through 90°, leaving the z-axis alone?
   (d) rotate the x-y plane, then x-z, then y-z, through 90°?
   (e) carry out the same three rotations, but each one through 180°?

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}_{(a)} \begin{bmatrix}1&0&0\\0&1&0\\0&0&-1\end{bmatrix}_{(b)} \begin{bmatrix}0&-1&0\\1&0&0\\0&0&1\end{bmatrix}_{(c)} \begin{bmatrix}0&0&-1\\0&1&0\\1&0&0\end{bmatrix}_{(d)} \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}_{(e)}$$

34. The transformation T that transposes every matrix is definitely linear. Which of these extra properties are true?
   (a) T^2 = identity transformation.
      - ✓ true; given $T(A) = A^T$

$$T^2(A) = T(T(A)) = T(A^T) = (A^T)^T = A$$
$$\implies T^2 = I$$

   (b) The kernel of T is the zero matrix.
      - ✓ true; only matrix that can take the identity to 0 is the trivial case, i.e., when the null space is empty.
   (c) Every matrix is in the range of T.
      Definition of column space, Wikipedia 10/16
      - ✓ true; the image (range) consists of all possible products $Ax\ \forall x \in C^n$.
   (d) $T(M) = -M$ is impossible.
      - ✘ false; e.g.,

$$\begin{bmatrix}0&1\\-1&0\end{bmatrix}^T = \begin{bmatrix}0&-1\\1&0\end{bmatrix} = -\begin{bmatrix}0&-1\\1&0\end{bmatrix} = \begin{bmatrix}0&1\\-1&0\end{bmatrix}$$

39. If you keep the same basis vectors but put them in a different order, the change-of-basis matrix M is a permutation matrix P. If you keep the basis vectors in order but change their lengths, M is a diagonal matrix D.

3 Orthogonality

3.1 Orthogonal Vectors and Subspaces

Problems 6, 46, 47

6. Find all vectors in R^3 that are orthogonal to $(1, 1, 1)$ and $(1, -1, 0)$. Produce an orthonormal basis from these vectors (mutually orthogonal unit vectors).

   orthogonality

   - Simply taking the cross product between two vectors in $\mathbb{R}^3$ yields a new vector that is normal to the plane containing them.
   - $(1, 1, 1) \times (1, -1, 0) = (1, 1, -2)$ yields us orthogonal columns.
   - Dividing each vector by their norms yields orthonormal columns, i.e.,

$$\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{3}} \end{bmatrix}$$

46. Find $\boldsymbol{A}^T\boldsymbol{A}$ if the columns of $\boldsymbol{A}$ are unit vectors, all mutually perpendicular.

$$\begin{bmatrix} \|\boldsymbol{a_1}\|^2 & a_\perp & \cdots & a_\perp \\ a_\perp & \|\boldsymbol{a}_2\|^2 & a_\perp & \vdots \\ \vdots & a_\perp & \ddots & a_\perp \\ a_\perp & \cdots & a_\perp & \|\boldsymbol{a}_n\|^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix} = \boldsymbol{I}$$

47. Construct a 3×3 matrix $\boldsymbol{A}$ with no zero entries whose columns are mutually perpendicular. Compute $\boldsymbol{A}^T\boldsymbol{A}$. Why is it a diagonal matrix?

    orthogonality

    - Note: I tried to set up with random matrices each time, sometimes it fails and gives the zero matrix.
    - The way I set up only yields a diagonal matrix with $\boldsymbol{A}\boldsymbol{A}^T$, $\boldsymbol{A}^T\boldsymbol{A}$ yields a symmetric matrix. Why?

3.2 Cosines and Projections onto Lines

Projection Proof (class problem)

- If I recall the problem correctly, we were requested to prove what $\text{proj}_{\boldsymbol{v}}\, \boldsymbol{w}$ is equal to.
- In my notes I have that a orthogonal projection occurs when the dot product between $\boldsymbol{v}$ and distance $\boldsymbol{w}$ from $\boldsymbol{v}$ is equal to zero. This follows from the definition of the inner product, i.e.,

$$\lambda = \boldsymbol{v}^T \boldsymbol{w} = ||v||||w|| \cos \theta$$

- If $\theta = 90^\circ$, then the vectors are perpendicular, i.e., orthogonal. What we are missing is the distance from $\boldsymbol{w}$ to $\boldsymbol{v}$; the distance that yields an inner product of zero with a normalized $\boldsymbol{v}$ *is the projection*.
- This means we need a scaled version of $\boldsymbol{v}$, let's call it $\boldsymbol{v}\beta$, at which such inner product is equal to zero. At this point, the difference between $\boldsymbol{w}$ and $\boldsymbol{v}\beta$ is exactly what we need in order to solve for a β that maintains an inner product of zero with the original vector $\boldsymbol{v}$, i.e.,

$$\begin{aligned} \boldsymbol{v}^T(\boldsymbol{w} - \boldsymbol{v}\beta) &= 0 \\ \boldsymbol{v}^T\boldsymbol{w} - \boldsymbol{v}^T\boldsymbol{v}\beta &= 0 \\ \boldsymbol{v}^T\boldsymbol{v}\beta &= \boldsymbol{v}^T\boldsymbol{w} \\ \beta &= \frac{\boldsymbol{v}^T\boldsymbol{w}}{\boldsymbol{v}^T\boldsymbol{v}} \end{aligned}$$

$$\implies \text{proj}_{\boldsymbol{v}}\, \boldsymbol{w} = \boldsymbol{v}\beta = \boldsymbol{v}\frac{\boldsymbol{v}^T\boldsymbol{w}}{\boldsymbol{v}^T\boldsymbol{v}}$$

- I've internalized this as a mapping of $\boldsymbol{w}$ onto $\boldsymbol{v}$ over a magnitude (the norm) of $\boldsymbol{v}$.
  - The mapping is important because it tells us the shortest distance from $\boldsymbol{w}$ onto $\boldsymbol{v}$, i.e., when they are orthogonal.
  - The magnitude is important, because it is the basis at which $\boldsymbol{w}$ is parallel to $\boldsymbol{v}$, which when added mapping distance, yields $\boldsymbol{w}$.

Problems 10, 13, 15

10. Is the projection matrix P invertible? Why or why not?

$$P = \frac{\boldsymbol{a}\boldsymbol{a}^T}{\boldsymbol{a}^T\boldsymbol{a}}$$
$$C(\boldsymbol{P}) = \boldsymbol{a} = \operatorname{rank}(\boldsymbol{P}) = n = 1$$

- The rank in equal to the number of columns, thus it is invertible.

13. Prove that the trace of $\boldsymbol{P} = \frac{\boldsymbol{a}\boldsymbol{a}^T}{\boldsymbol{a}^T\boldsymbol{a}}$ always equals 1.

$$\operatorname{tr}(\boldsymbol{A}) = \sum_{i=1}^{n} e_{ij} = e_{11} + e_{22} + \cdots + e_{nn}$$

$$\implies \operatorname{tr}(\boldsymbol{P}) = \frac{\boldsymbol{a}_1\boldsymbol{a}_1}{\boldsymbol{a}^T\boldsymbol{a}} + \cdots + \frac{\boldsymbol{a}_n\boldsymbol{a}_n}{\boldsymbol{a}^T\boldsymbol{a}} = \frac{\boldsymbol{a}^T\boldsymbol{a}}{\boldsymbol{a}^T\boldsymbol{a}} = 1$$

15. Show that the length of $\boldsymbol{Ax}$ equals the length of $\boldsymbol{A}^T\boldsymbol{x}$ if $\boldsymbol{AA}^T = \boldsymbol{A}^T\boldsymbol{A}$.

$$\|\boldsymbol{Ax}\|^2 = (\boldsymbol{Ax})^T(\boldsymbol{Ax}) = x\boldsymbol{A}^T\boldsymbol{A}x$$
$$\|\boldsymbol{A}^T x\|^2 = (\boldsymbol{A}^T x)^T(\boldsymbol{A}^T x) = x\boldsymbol{AA}^T x$$
$$x\boldsymbol{A}^T\boldsymbol{A}x = x\boldsymbol{AA}^T x \iff \boldsymbol{AA}^T = \boldsymbol{A}^T\boldsymbol{A}$$

3.3 Projections and Least Squares

Problems 8, 19, 20

8. If $\boldsymbol{P}$ is the projection matrix onto a k-dimensional subspace $\boldsymbol{S}$ of the whole space $\mathbb{R}^n$, what is the column space of $\boldsymbol{P}$ and what is its rank?
    - Given $\boldsymbol{P} = \boldsymbol{A}(\boldsymbol{A}^T\boldsymbol{A})^{-1}\boldsymbol{A}^T$, then $\boldsymbol{P}$ will project any vector onto the image of $\boldsymbol{A}$, if $\boldsymbol{A}$ has independent columns.
    - This implies that $\boldsymbol{S}$ is the image of $\boldsymbol{A}$, i.e., $C(\boldsymbol{A}) = \boldsymbol{S}$.
    - Since the k-dimensional subspace $\boldsymbol{S}$ is the whole of $\mathbb{R}^n$, and $\operatorname{rank}(\boldsymbol{A}) = n$, then $C(\boldsymbol{P}) = \boldsymbol{S} \wedge \operatorname{rank}(\boldsymbol{P}) = k$

19. If $\boldsymbol{P}_C = \boldsymbol{A}(\boldsymbol{A}^T\boldsymbol{A})^{-1}\boldsymbol{A}^T$ is the projection onto the column space of $\boldsymbol{A}$, what is the projection $\boldsymbol{P}_R$ onto the row space?
    - $\boldsymbol{P}_C = C(\boldsymbol{A})$, $\boldsymbol{P}_R = C(\boldsymbol{A}^T)$, thus, taking the transpose of every instance of $\boldsymbol{A}$ will yield the row space, i.e.,

$$\boldsymbol{P}_R = \boldsymbol{A}^T\left(\boldsymbol{A}^{TT}\boldsymbol{A}^T\right)^{-1}\boldsymbol{A}^{TT} = \boldsymbol{A}^T(\boldsymbol{A}\boldsymbol{A}^T)^{-1}\boldsymbol{A}$$

20. If $\boldsymbol{P}$ is the projection onto the column space of $\boldsymbol{A}$, what is the projection onto the left nullspace?
    - The column space and left null space are orthogonal to each other, thus the projection onto the left null space is simply $\boldsymbol{I} - \boldsymbol{P}$.

3.4 Orthogonal Bases and Gram-Schmidt

Implementation of the Gram-Schmidt Process

Implementing Gram-Schmidt

Problems 10, 13, 18

10. 10. If $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ are the outputs from Gram-Schmidt, what were the possible input vectors $\boldsymbol{a}$ and $\boldsymbol{b}$?
    - Any two linearly independent vectors.
13. Apply the Gram-Schmidt process to

$$\boldsymbol{a} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \qquad \boldsymbol{c} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

and write the result in the form $\boldsymbol{A} = \boldsymbol{QR}$.

$$\boldsymbol{A} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

- All that is needed is a permutation to achieve orthonormal columns; switching rows 1 and 3 will achieve this, yielding $\boldsymbol{Q}$:

$$\boldsymbol{Q} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

- Thus,

$$\boldsymbol{A} = \boldsymbol{QR} \rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

18. If $\boldsymbol{A} = \boldsymbol{QR}$, find a simple formula for the projection matrix $\boldsymbol{P}$ onto the column space of $\boldsymbol{A}$.
    - I was lost on this one, but I wanted to know how to do it. Unfortunately, while looking for help I came across one of the solutions documents again.
    - Normally I would just do a different problem, but I wanted to write it out, for personal reference.

$$\begin{aligned} \boldsymbol{A} = \boldsymbol{QR} \implies \boldsymbol{A}^T\boldsymbol{A} = (\boldsymbol{QR})^T(\boldsymbol{QR}) = \boldsymbol{R}^T\boldsymbol{Q}^T\boldsymbol{QR} = \boldsymbol{R}^T\boldsymbol{R} \\ \boldsymbol{P} = \boldsymbol{A}(\boldsymbol{A}^T\boldsymbol{A})^{-1}\boldsymbol{A}^T \\ \implies \boldsymbol{QR}(\boldsymbol{R}^T\boldsymbol{R})^{-1}(\boldsymbol{QR})^T = \boldsymbol{QRR}^{-1}(\boldsymbol{R}^T)^{-1}\boldsymbol{R}^T\boldsymbol{Q}^T = \boldsymbol{QQ}^T = \boldsymbol{I} \\ \implies \boldsymbol{P}^2 = \boldsymbol{P}\,\checkmark \end{aligned}$$

3.5 The Fast Fourier Transform

Problems 1, 2

1. What are $\boldsymbol{F}^2$ and $\boldsymbol{F}^4$ for the 4×4 Fourier matrix $\boldsymbol{F}$?

$$\boldsymbol{F}^2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}$$
$$= \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 4 & 0 \\ 0 & 4 & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{F}^4 = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 4 & 0 \\ 0 & 4 & 0 & 0 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 4 & 0 \\ 0 & 4 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 16 & 0 & 0 & 0 \\ 0 & 16 & 0 & 0 \\ 0 & 0 & 16 & 0 \\ 0 & 0 & 0 & 16 \end{bmatrix}$$

2. Find a permutation $\boldsymbol{P}$ of the columns of $\boldsymbol{F}$ that produces $\boldsymbol{F}\boldsymbol{P} = \overline{\boldsymbol{F}}$ $(n \times n)$. Combine with $\boldsymbol{F}\overline{\boldsymbol{F}} = n\boldsymbol{I}$ to find $\boldsymbol{F}^2$ and $\boldsymbol{F}^4$ for the $n \times n$ Fourier matrix.

$$\boldsymbol{P} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ \vdots & \vdots & \iddots & \iddots & 0 \\ 0 & 0 & \iddots & \iddots & \vdots \\ 0 & 1 & 0 & \cdots & 0 \end{bmatrix}$$

$$\boldsymbol{F}\overline{\boldsymbol{F}} = n \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix} \quad (n \times n)$$

$$\boldsymbol{F}^2 = \begin{bmatrix} n & 0 & \cdots & 0 \\ 0 & n & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & n \end{bmatrix} \quad \boldsymbol{F}^4 = \begin{bmatrix} n^2 & 0 & \cdots & 0 \\ 0 & n^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & n^2 \end{bmatrix}$$

Problems 5, 6

5. Find all solutions to the equation $e^{ix} = -1$, and all solutions to $e^{i\theta} = i$.

$$\begin{aligned} e^{ix} &= -1 \\ \implies& \cos x + i \sin x = -1 + 0i \\ \implies& \cos x = -1 \wedge \sin x = 0 \\ \implies& x = \pi + 2k\pi, \quad k \in \mathbb{Z} \end{aligned}$$

$$\begin{aligned} e^{i\theta} &= i \\ \implies& \cos\theta + i\sin\theta = 0 + 1i \\ \implies& \cos\theta = 0 \wedge \sin\theta = 1 \\ \implies& \theta = \frac{\pi}{2} + 2k\pi, \quad k \in \mathbb{Z} \end{aligned}$$

6. What are the square and square roots of w_{128}, the primitive 128^{th} root of 1?

$$\begin{aligned} &\mid w_n^2 = w_m \iff m = \frac{1}{2}n \\ \implies& w_{128}^2 = w_{64} \wedge \sqrt{w_{128}} = w_{256} \end{aligned}$$

$$\begin{aligned} w_{128} = e^{2\pi i/128} &= \cos\frac{2\pi}{128} + i\sin\frac{2\pi}{128} \\ &= \cos 0 + i \sin 0 = 1 + i0 = 1 \end{aligned}$$

4.2 Properties of the Determinant

Problems 8, 14

8. Show how rule 6 (det = 0 if a row is zero) comes directly from rules 2 and 3.
   - Rule 2 states there is nothing special about the first row (only the sign changes). Rule 3 (the determinant depends linearly on the first row) implies that:

     $$\begin{bmatrix} ta & tb \\ c & d \end{bmatrix} = t \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

     Thus, a common factor of row 1 could be zero $(t = 0)$, meaning the determinant itself must be zero.

14. 14. True or false, with reason if true and counterexample if false:
   (a) If $\boldsymbol{A}$ and $\boldsymbol{B}$ are identical except that $b_{11} = 2a_{11}$ then $\det(\boldsymbol{B}) = 2\det(A)$
   - ✘ false; the entire row must be scaled by 2, not just a_{11}, e.g.,

     $$\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \implies \det(\boldsymbol{A}) = -1 = \det(\boldsymbol{B})$$

     Thus, $\det(\boldsymbol{A}) \neq 2\det(\boldsymbol{B})$

   (b) The determinant is the product of the pivots.
   - ✓ true; if it is an upper/lower diagonal matrix. (I am assuming the pivots defined as elements in row-reduced form of a matrix).

   (c) If $\boldsymbol{A}$ is invertible and $\boldsymbol{B}$ is singular, then $\boldsymbol{A} + \boldsymbol{B}$ is invertible.
   - ✘ false; e.g.,

     $$\boldsymbol{A} = \begin{bmatrix} 6 & 9 \\ 1 & 1 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix}, \quad \boldsymbol{A} + \boldsymbol{B} = \begin{bmatrix} 6 & 9 \\ 0 & 0 \end{bmatrix}$$

     Thus, $\det(\boldsymbol{A} + \boldsymbol{B})$ is singular despite $\boldsymbol{A}$ being invertible.

   (d) If $\boldsymbol{A}$ is invertible and $\boldsymbol{B}$ is singular, then $\boldsymbol{AB}$ is singular.
   - ✓ true; given $\det(\boldsymbol{AB}) = \det(\boldsymbol{A})\det(\boldsymbol{B})$, then it must follow that $\det(\boldsymbol{AB} = 0)$ if one of the matrices are singular.

   (e) The determinant of $\boldsymbol{AB} - \boldsymbol{BA}$ is zero.
   - ✘ false; are- and post-multiplication of matrices can yield different results, thus $\det(\boldsymbol{AB} - \boldsymbol{BA})$ is often more than just a sign flip.

4.3 Formulas for the Determinant

Problems 3, 9, 18

3. True or false?

   (a) The determinant of $\boldsymbol{S}^{-1}\boldsymbol{AS}$ equals the determinant of $\boldsymbol{A}$.

   - ✓ true; $\det\left(\boldsymbol{A}^{-1}\right) = \det(\boldsymbol{A})^{-1}$, thus

   $$\begin{aligned}\det\left(\boldsymbol{S}^{-1}\boldsymbol{AS}\right) &= \det\left(\boldsymbol{S}^{-1}\right)\det(\boldsymbol{A})\det(\boldsymbol{S}) \\ \implies \det\left(\boldsymbol{S}^{-1}\right)\det(\boldsymbol{S})\det(\boldsymbol{A}) &= \det(\boldsymbol{A})\end{aligned}$$

   (b) If $\det(\boldsymbol{A}) = 0$ then at least one of those cofactors must be zero.

   - ✘ false; $\det(\boldsymbol{A}) = 0 \iff$ the matrix is singular. All cofactors could be nonzero, but there could still be linearly dependent columns/rows, which would yield a singular matrix.

   (c) A matrix whose entries are 0s and 1s has determinant 1, 0, or -1.

   - ✘ false; a *row-reduced matrix* that contains all 1s or 0s would yield a determinant of 1, 0, or -1. Most matrices will not contain only 1s or 0s after row-reduction.

9. How many multiplications to find an $n \times n$ determinant from

   (a) The big formula (6)?

   - The big formula is the sum of all possible permutations, i.e., $n \cdot n!$

   (b) The cofactor formula (10), building from the count for $n - 1$?

   - The cofactor formula is just an algorithmic way to find all possible permutations by expanding all possible cofactors one element at a time until all are filled. This yields the same complexity as the big formula, i.e., $n \cdot n!$

   (c) The product of the pivots formula (including the elimination steps)?

   - Each row requires $n(n-1)$ steps to reduce, followed by multiplication along the diagonal, i.e., $\sum_{i=2}^{n} i(i-1) + n$

18. Place the smallest number of zeros in a 4×4 matrix that will guarantee $\det(\boldsymbol{A}) = 0$.

   - The smallest number of zeros to *guarantee* $\det(\boldsymbol{A}) = 0$ is n, (a full row). I.e., ; all must be in same row/column.

   Place as many zeros as possible while still allowing $\det(\boldsymbol{A}) \neq 0$

   - The most is $n^2 - n$, i.e., 12; all diagonal elements must be non-zero.

4.4 Applications of Determinants

Problems 6, 10, 19, 22

6. Explain in terms of volumes why $\det(A)\,3 = 3^n \det(A)$ for an $n \times n$ matrix.
   - Scaling a determinant implies stretching/shrinking every side side of the n-dimensional object. The determinant represents the “volume” of this object. Every column represents orthogonal dimension that needs to be scaled.

     Thus, $\det(A)\,3$ scales each column, implying that $3^n \det(A)$ represents the scaling of the volume of the object.

10. If P is an odd permutation, explain why P^2 is even but P^{-1} is odd.
    - P is odd when $\det(P) = -1$, and even if it is 1.
      - Thus, $P^2 = -1^2$, i.e., even.
    - Since $\det\left(P^{-1}\right) = \det(P)^{-1}$, then $P^{-1} = \frac{1}{-1}$, i.e., odd.

19. If all the cofactors are zero, how do you know that A has no inverse? If none of the cofactors are zero, is A sure to be invertible?
    - Yes, if all the cofactors are zero, then it implies there is a row/column of zeros, yielding a singular matrix.
    - Not always, only knowing the cofactors will not tell you if there are linearly dependent dimensions. Only upon summation of the cofactor expansion will you be able to tell for sure. E.g.,

$$A = \begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix}$$

22. From the formula $AC^T = \det(A)\,I$ show that $\det(C) = \det(A)^{n-1}$

$$\begin{aligned}
\implies \det\left(AC^T\right) &= \det\left(\det(A)\,I\right) \\
\det(A)\det\left(C^T\right) &= \det(A)^n \det(I) && \text{Product Rule, Rule 3} \\
\det(C) &= \det(A)^n \left(\det(A)^{-1}\right) && \text{Rearrange, Rule 10} \\
\implies \det(C) &= \det(A)^{n-1} && \text{Simplify}
\end{aligned}$$

5 Eigenvalues and Eigenvectors

5.1 Introduction to Eigenvalues and Eigenvectors

Properties of Eigenvalues and Eigenvectors

Strang provided a list of properties of Eigenvectors and Eigenvalues at the end of chapter 5, but I wanted to explore them first before diving into the practice problems. The following is my attempt at restating such properties and their significance.

Problems 39, 40

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5.2 Diagonalization of a Matrix

Problems 11, 14, 39

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5.5 Complex Matrices

Problems 2, 12, 50

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5.6 Similarity Transformations

Problems 17, 21, 42

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6 Positive Definite Matrices

6.1 Minima, Maxima, and Saddle Points

6.2 Tests for Positive Definiteness

6.3 Singular Value Decomposition

6.4 Minimum Principles

6.5 The Finite Element Method