Calculus III Final Review

Convergence: 10.3–10.5

Power/Taylor Series: 10.6–10.8

Parametric Equations: 11.1

Arc Length, Polar Coordinates: 11.2–11.4

Quiz Questions

Final Review Questions

Convergence Notes

- Let $\displaystyle\sum_{n=1}^{\infty} a_n$ be given and note for which series convergence is known, i.e.:

  Geometric: let $c \neq 0$, if $|r| < 1$, then

  $$\sum_{n=0}^{\infty} cr^n = \frac{c}{1-r}$$

  $|r| \geq 1 \implies$ diverges

  p-Series: converges if $p > 1$.

  $$\sum_{n=0}^{\infty} \frac{1}{n^p}$$

  $p \leq 1 \implies$ diverges

- **The nth Term Divergence Test**: a relatively easy test that can be used to quickly determine if a test diverges if the $\displaystyle\lim_{n\to\infty} a_n \neq 0$. If $\displaystyle\lim_{n\to\infty} a_n = 0$, then the test is inconclusive and other tests must be applied.

Tests for Positive Series

- **Direct Comparison Test**: use if dropping terms from the denominator or numerator gives a series b_n wherein convergence is easily found, then compare to the original series a_n as follows:

  $$\sum_{n=1}^{\infty} b_n \text{ converges} \implies \sum_{n=1}^{\infty} a_n \text{ converges} \quad \leftarrow 0 \leq a_n \leq b_n$$

  $$\sum_{n=1}^{\infty} b_n \text{ diverges} \implies \sum_{n=1}^{\infty} a_n \text{ diverges} \quad \leftarrow 0 \leq b_n \leq a_n$$

- **Limit Comparison Test**: use when the direct comparison test isn't convenient or when comparing two series. One can to take the dominant term in the numerator and denominator from a_n to form a new positive sequence b_n if needed.

  Assuming the following limit $L = \displaystyle\lim_{n\to\infty} \frac{a_n}{b_n}$ exists, then:

  $$L > 0 \implies \sum_{n=1}^{\infty} a_n \text{ converges} \iff \sum_{n=1}^{\infty} b_n \text{ converges}$$

  $$L = 0 \text{ and } \sum_{n=1}^{\infty} b_n \text{ converges} \implies \sum_{n=1}^{\infty} a_n \text{ converges}$$

  $$L = \infty \text{ and } \sum_{n=1}^{\infty} a_n \text{ converges} \implies \sum_{n=1}^{\infty} b_n \text{ converges}$$

- **Ratio Test**: often used in the presence of a factorial $(n!)$ or when the are constants raised to the power of n (c^n).

  Assuming the following limit $\rho = \lim\limits_{n\to\infty} \left| \frac{a_n+1}{a_n} \right|$ exists, then

  $$\rho < 1 \implies \sum a_n \text{ converges absolutely}$$
  $$\rho > 1 \implies \sum a_n \text{ diverges}$$
  $$\rho = 1 \implies \text{ test is inconclusive}$$

- **Root Test**: used when there is a term in the form of $f(n)^{g(n)}$.

  Assuming the following limit $C = \lim\limits_{n\to\infty} |a_n|^{\frac{1}{n}}$ exists, then

  $$C < 1 \implies \sum a_n \text{ converges absolutely}$$
  $$C > 1 \implies \sum a_n \text{ diverges}$$
  $$C = 1 \implies \text{ test is inconclusive}$$

- **Integral Test**: if the other tests fail and $a_n = f(n)$ is a decreasing function, then one can use the improper integral $\int_1^\infty f(x)dx$ to test for convergence.

  Let $a_n = f(n)$ be a positive, decreasing, and continuous function $\forall x \geq 1$, then:

  $$\int_1^\infty f(x)dx \text{ converges} \implies \sum_{n=1}^\infty a_n \text{ converges}$$
  $$\int_1^\infty f(x)dx \text{ diverges} \implies \sum_{n=1}^\infty a_n \text{ diverges}$$

Tests for Non-Positive Series

- **Alternating Series Test**: used for series in the form $\sum\limits_{n=0}^\infty (-1)^n a_n$

  Converges if $|a_n|$ decreases monotonically $(|a_n + 1| \leq |a_n|)$ and if $\lim\limits_{n\to\infty} a_n = 0$

- **Absolute Convergence**: used if the series $\sum a_n$ is not alternating (if it is alternating, use the alternating test in conjunction); simply test if $\sum |a_n|$ converges using the test for positive series.

  $\sum a_n$ converges **conditionally** if $\sum a_n$ converges, but $\sum |a_n|$ diverges.

  $\sum a_n$ converges **absolutely** if $\sum |a_n|$ converges.

Convergence Problems

10.5 Exercises

Determine convergence or divergence using any method.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n + 4^n}{7^n}$

$\implies \displaystyle\sum_{n=1}^{\infty} \frac{2^n}{7^n} + \sum_{n=1}^{\infty} \frac{4^n}{7^n}$ Separate into two geometric series $\uparrow$

$\implies r = \dfrac{2}{7} < 1, \quad r = \dfrac{4}{7} < 1$

Both geometric series converge, thus the original series converges.

2. $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{n!}$

$\implies \rho = \displaystyle\lim_{n\to\infty} \left| \frac{(n+1)^3}{(n+1)!} \cdot \frac{n!}{n^3} \right|$ Apply the ratio test $\uparrow$

$= \displaystyle\lim_{n\to\infty} \frac{n^3 + 3n^2 + 3n + 1}{(n+1)n!} \cdot \frac{n!}{n^3}$

$= \displaystyle\lim_{n\to\infty} \frac{n^3 + 3n^2 + 3n + 1}{n^4 + n^3}$

$= \displaystyle\lim_{n\to\infty} \frac{n^3 + 3n^2 + 3n + 1}{n^4 + n^3} \cdot \frac{n^{-4}}{n^{-4}}$

$= \displaystyle\lim_{n\to\infty} \frac{n^{-1} + 3n^{-2} + 3n^{-3} + n^{-4}}{1 + n^{-1}} = 0$

$\rho = 0 < 1$, thus the series converges.

3. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2n+1}$

$$\implies \lim_{n\to\infty} \frac{n}{2n+1} \qquad\qquad \text{Apply the } n^{th} \text{ term test}^{\uparrow}$$

$$\implies \lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2} \qquad\qquad \text{By L'Hôpital's Rule}$$

$\lim_{n\to\infty} a_n \neq 0$, thus the series diverges.

4. $\displaystyle\sum_{n=1}^{\infty} 2^{\frac{1}{n}}$

$$\implies \lim_{n\to\infty} 2^{\frac{1}{n}} = 2^0 = 1 \qquad\qquad \text{Apply the } n^{th} \text{ term test}^{\uparrow}$$

$\lim_{n\to\infty} a_n \neq 0$, thus the series diverges.

5. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$

$$a_n \leq b_n, \quad b_n = 1$$

$$b_n \text{ converges} \to a_n \text{ converges} \qquad \text{Apply the direct comparison test}^{\uparrow}$$

$$\sin n \leq 1 \qquad\qquad\qquad \leftarrow \forall n \geq 1$$

$$\frac{\sin n}{n^2} \leq \frac{1}{n^2}$$

$$\frac{1}{n^2} \to \text{ converges} \qquad\qquad \text{by } p\text{-series}^{\uparrow}$$

The larger (b_n) series converges, thus the smaller (a_n) converges.

6. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{(2n)!}$

$$\implies \rho = \lim_{n\to\infty} \left| \frac{(n+1)!}{(2n+2)!} \cdot \frac{(2n)!}{n!} \right| \qquad \text{Apply the ratio test}^{\uparrow}$$

$$= \lim_{n\to\infty} \frac{(n+1)n!}{(2n+2)(2n+1)2n!} \cdot \frac{(2n)!}{n!}$$

$$= \lim_{n\to\infty} \frac{n+1}{(2n+2)(2n+1)} = \frac{n+1}{4n^2+6n+2}$$

$$= \lim_{n\to\infty} \frac{1}{8n+6} = 0 \qquad \text{By L'Hôpital's Rule}$$

$\rho = 0 < 1$, thus the series converges.

7. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$

$$b_n \leq a_n, \quad b_n = n$$
$$a_n \text{ diverges} \iff b_n \text{ diverges} \qquad \text{Apply the direct comparison test}^{\uparrow}$$

$$n \leq n + \sqrt{n} \qquad\qquad \leftarrow \forall n \geq 1$$

$$\frac{1}{n+\sqrt{n}} \leq \frac{1}{n}$$

$$\frac{1}{n} \to \text{diverges} \qquad\qquad \text{by } p\text{-series}^{\uparrow}$$

The smaller (b_n) series diverges, thus the larger (a_n) diverges.

8. $\sum_{n=2}^{\infty} \dfrac{1}{n(\ln n)^3}$

f is positive, decreasing, and continuous for $x \geq 2$ Apply the integral test↑

$$\implies \int_2^\infty f(x)\,dx = \lim_{R\to\infty} \int_2^R \frac{1}{x(\ln x)^3}\,dx \qquad \ln x = u, \quad x\,du = dx$$

$$\implies \lim_{R\to\infty} \int_2^R \frac{1}{x(u)^3}x\,du = \int_2^R \frac{1}{u}^3\,du$$

$$= -\frac{1}{2(u)^2}\bigg|_2^R = \frac{1}{2R^2} - \frac{1}{8}$$

$$= -\frac{1}{2\ln^2(x)} + C\bigg|_2^R = \frac{1}{8} - \frac{1}{2R^2}$$

$$= \lim_{R\to\infty} \left(\frac{1}{8} - \frac{1}{2R^2}\right) = \frac{1}{8}$$

The improper integral converges, thus the original series converges.

9. $\sum_{n=1}^{\infty} \dfrac{n^3}{5^n}$

$$\implies \rho = \lim_{n\to\infty} \left| \frac{(n+1)^3}{5^{n+1}} \cdot \frac{5^n}{n^3} \right| \qquad \text{Apply the ratio test↑}$$

$$= \lim_{n\to\infty} \frac{(n+1)^3}{5^n \cdot 5} \cdot \frac{5^n}{n^3}$$

$$= \frac{1}{5} \lim_{n\to\infty} \frac{(n+1)^3}{n^3}$$

$$= \frac{1}{5} \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^3 = \frac{1}{5}$$

$\rho = \dfrac{1}{5} < 1$, thus the series converges.

10. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^3 - n^2}}$

$$L = \lim_{n\to\infty} \frac{a_n}{b_n}, \quad b_n = \frac{1}{\sqrt{n^3}} \qquad \text{Apply the limit comparison test}^{\uparrow}$$

$$\implies L = \lim_{n\to\infty} \frac{1}{\sqrt{n^3 - n^2}} \cdot \frac{\sqrt{n^3}}{1}$$

$$= \lim_{n\to\infty} \sqrt{\frac{n^3}{n^3(1 - n^{-1})}}$$

$$= \sqrt{\frac{1}{1(1 - 0)}} = 1$$

$L > 0$, thus a_n converges if b_n converges.

b_n converges by the p-series test, as $\frac{3}{2} > 1$, thus a_n converges.

11. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 + 4n}{3n^4 + 9}$

$$L = \lim_{n\to\infty} \frac{a_n}{b_n}, \quad b_n = \frac{1}{n^2} \qquad \text{Apply the limit comparison test}^{\uparrow}$$

$$= \lim_{n\to\infty} \frac{n^2 + 4n}{3n^4 + 9} \cdot n^2$$

$$= \lim_{n\to\infty} \frac{n^4 + 4n^3}{3n^4 + 9} \cdot \frac{n^{-4}}{n^{-4}}$$

$$= \lim_{n\to\infty} \frac{1 + 4n^{-1}}{3 + 9n^{-4}} = \frac{1}{3}$$

$L > 0$, thus a_n converges if b_n converges.

b_n converges by the p-series test, as $2 > 1$, thus a_n converges.

12. $\sum_{n=1}^{\infty} (0.8)^{-n} n^{-0.8}$

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \qquad \text{Apply the ratio test}^{\uparrow}$$

$$= \lim_{n\to\infty} \left| \frac{(0.8)^{-(n+1)}(n+1)^{-0.8}}{(0.8)^{-n} n^{-0.8}} \right|$$

$$= \lim_{n\to\infty} \frac{(0.8)^{-n} \cdot 0.8^{-1} \cdot (n+1)^{-0.8}}{(0.8)^{-n} n^{-0.8}}$$

$$= 1.25 \lim_{n\to\infty} \frac{(n+1)^{-0.8}}{n^{-0.8}}$$

$$= 1.25 \lim_{n\to\infty} \left(\frac{n+1}{n} \right)^{-0.8} = 1.25$$

$\rho = 1.25 > 1$, thus a_n diverges.

13. $\sum_{n=1}^{\infty} 4^{-2n+1}$

$$\sum_{n=1}^{\infty} cr^n \implies \sum_{n=1}^{\infty} 4 \cdot (4^{-2})^n \qquad \text{Convert into geometric series}^{\uparrow}$$

$|r| = 2^{-2} = \dfrac{1}{16} < 1$ and $c \neq 0$, thus a_n converges.

14. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$

$$\sum_{n=1}^{\infty}(-1)^n a_n, \quad a_n = \frac{1}{\sqrt{n}} \qquad \text{Apply the alternating series test}^{\uparrow}$$

$$\frac{1}{\sqrt{n}} \rightarrow \text{diverges} \qquad \text{by } p\text{-series}^{\uparrow}$$

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$$

$|a_n|$ decreases monotonically and $\displaystyle\lim_{n\to\infty} a_n = 0$, but a_n diverges, thus the series.

converges conditionally

15. $\displaystyle\sum_{n=1}^{\infty} \sin \frac{1}{n^2}$

$$L = \lim_{n\to\infty} \frac{a_n}{b_n}, \quad b_n = \frac{1}{n^2} \qquad \text{Apply the limit comparison test}^{\uparrow}$$

$$\implies \lim_{n\to\infty} \frac{\sin(n^{-2})}{n^{-2}} = \frac{0}{0}$$

$$= \lim_{n\to\infty} \frac{\cos(n^{-2})(-2n^{-3})}{-2n^{-3}} \qquad \text{by L'Hôpital's Rule}$$

$$= \lim_{n\to\infty} \cos(n^{-2}) = 1$$

$L > 0$, thus a_n converges if b_n converges.

b_n converges by the p-series test, thus a_n converges.

16. $\displaystyle\sum_{n=1}^{\infty}(-1)^n \cos n^{-1}$

$$\sum_{n=1}^{\infty}(-1)^n a_n, \quad a_n = \cos(n^{-1}) \qquad \text{Apply the } n^{th} \text{ term test}^{\uparrow}$$

$$\implies L = \lim_{n\to\infty} \cos(n^{-1}) = 1$$

$L \neq 0$, thus the series diverges

17. $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{\sqrt{n}}$

$$\sum_{n=1}^{\infty}(-1)^n a_n, \quad a_n = \frac{2^n}{\sqrt{n}} \qquad \text{Apply the } n^{th} \text{ term test}^{\uparrow}$$

$$\implies L = \lim_{n\to\infty} \frac{2^n}{\sqrt{n}} = \frac{\infty}{\infty}$$

$$= \frac{2^n \ln 2}{\frac{1}{2}n^{-\frac{1}{2}}} = 2^n \ln 2 \cdot 2\sqrt{n} \qquad \text{By L'Hôpital's Rule}$$

$$= 2 \lim_{n\to\infty} 2^n \ln(2)\sqrt{n} = \infty$$

$L \neq 0$, thus the series diverges

18. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{n+12}\right)^n$

$$L = \lim_{n\to\infty} \left(\frac{n}{n+12}\right)^n \qquad \text{Apply the } n^{th} \text{ term test}^{\uparrow}$$

$$= \lim_{n\to\infty} e^{-12} \qquad \text{By common limit } \left(\frac{x}{x+k}\right)^x = e^{-k}$$

$L \neq 0$, thus the series diverges.

Power/Taylor Series Notes

Power Series

- **Power series**: a infinite series in the form:

$$F(x) = \sum_{n=0}^{\infty} a_n(x - c)^n$$

  Where the constant c is the *center* of the power series F(x).

- **Radius of convergence** R: the range of values of the variable x whereby the power series $F(x)$ converges.

  - Every power series converges at $x = c$, as $(x - c)^0 = 1$, though the series may diverge for other values of x.

  - $F(x)$ converges for $|x - c| < R$ and diverges for $|x - c| > R$

  - $F(x)$ may converge of diverge at endpoints $c - R$ and $c + R$

- **Interval of convergence**: the open interval $(c - R, c + R)$ and possibly one of both of the endpoints, each must be tested.

  - In most cases, the ratio test ↑ can be used to find R.

  - If $R > 0$, then F is differentiable over the interval of convergence; the derivative and antiderivative can be obtained using the following:

$$F'(x) = \sum_{n=1}^{\infty} na_n(x - c)^{n-1} \qquad F(x)dx = C + \sum_{n=0}^{\infty} \frac{a_n}{n + 1}(x - c)^{n+1}$$

Taylor Series

- **Taylor series**: the power series of a infinitely differentiable function $f(x)$ centered at c,

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

  - n^{th} **Taylor polynomial**: a polynomial of degree n that is formed partial sum formed by the first $n+1$ terms of a Taylor series, i.e.,

$$f(c)+f'(c)(x-c)+\frac{f''(c)}{2!}(x-c)^2+\frac{f'''(c)}{3!}(x-c)^3+\cdots+\frac{f^n(c)}{n!}(x-c)^n$$

  - **Maclaurin series**: when $c=0$, i.e.,

$$\sum_{n=0}^{\infty} \frac{f^{(n)}0}{n!}(x)$$

- **Useful Maclaurin Series**: useful Taylor series centered at 0 that can be used to derive other series via differentiation, integration, multiplication, or substitution.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad\qquad \leftarrow \forall x$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \qquad\qquad \leftarrow \forall x$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \qquad\qquad \leftarrow \forall x$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad\qquad \leftarrow |x| < 1$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n \qquad\qquad \leftarrow |x| < 1$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} \qquad\qquad \leftarrow |x| < 1 \wedge x = 1$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \qquad\qquad \leftarrow |x| \leq 1$$

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n \qquad\qquad \leftarrow |x| < 1$$

$$\text{where } \binom{\alpha}{n} = \sum_{n=0}^{\infty} \prod_{k=1}^{n} \frac{\alpha - k + 1}{k}$$

Power/Taylor Series Problems

10.6 Exercises

Find the interval of convergence.

1. $\displaystyle\sum_{n=0}^{\infty}(-1)^{n}\frac{n}{4^{n}}x^{2n}$

$$\begin{aligned}
\rho &= \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_{n}}\right| && \text{Apply the ratio test}^{\uparrow}\\
&= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+1)(x^{2(n+1)})}{4^{n+1}}\cdot\frac{4^{n}}{(-1)^{n}n(x^{2n})}\right|\\
&= \lim_{n\to\infty}\frac{(n+1)(|x|^{2n}\cdot|x|^{2})}{4^{n}\cdot4}\cdot\frac{4^{n}}{n\cdot|x|^{2n}}\\
&= \lim_{n\to\infty}\frac{(n+1)|x|^{2}}{4n}\cdot\frac{n^{-1}}{n^{-1}}\\
&= \lim_{n\to\infty}\frac{(1+n^{-1})|x|^{2}}{4}=\frac{|x|^{2}}{4}
\end{aligned}$$

$$\implies \frac{|x|^{2}}{4}<1 \implies |x|<2 \qquad\qquad \text{converges for } \rho<1$$

Both endpoints tend toward ∞ (diverge), thus the interval of convergence is $(-2,2)$.

2. $\displaystyle\sum_{n=8}^{\infty}n^{7}x^{n}$

$$\begin{aligned}
\rho &= \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_{n}}\right| && \text{Apply the ratio test}^{\uparrow}\\
&\implies \lim_{n\to\infty}\left|\frac{(n+1)^{7}x^{n+1}}{n^{7}x^{n}}\right|\\
&= \lim_{n\to\infty}\frac{(n+1)^{7}\cdot|x|^{n}\cdot|x|}{n^{7}|x|^{n}}\\
&= \lim_{n\to\infty}\frac{(n+1)^{7}|x|}{n^{7}}\\
&= |x|\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^{7}=|x|
\end{aligned}$$

$$\implies |x|<1 \qquad\qquad \text{converges for } \rho<1$$

Both endpoints tend toward ∞ (diverge), thus the interval of convergence is $(-1,1)$.

3. $\displaystyle\sum_{n=2}^{\infty} \frac{x^n}{\ln n}$

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$ 　　　Apply the ratio test $^\uparrow$

$$\implies \lim_{n\to\infty} \left| \frac{x^{n+1}}{\ln(n+1)} \cdot \frac{\ln n}{x^n} \right|$$

$$= \lim_{n\to\infty} \frac{|x|\ln n}{\ln(n+1)}$$

$$= |x| \lim_{n\to\infty} \frac{\ln n}{\ln(n+1)} = \frac{\infty}{\infty}$$

$$= |x| \lim_{n\to\infty} \frac{n^{-1}}{(n+1)^{-1}}$$ 　　　By L'Hôpital's Rule

$$= |x| \lim_{n\to\infty} \frac{n+1}{n}$$

$$= |x| \lim_{n\to\infty} 1 + n^{-1} = |x|$$

$$\implies |x| < 1$$ 　　　converges for $\rho < 1$

$$f(1) = \sum_{n=1}^{\infty} \frac{1}{\ln n}$$

$$\implies 0 \leq b_n \leq a_n$$ 　　　Apply the direct comparison test $^\uparrow$

$$\implies 0 \leq \frac{1}{n} \leq \frac{1}{\ln n}$$

$$\implies a_n \text{ diverges}$$ 　　　$\dfrac{1}{n} \to$ diverges as $p \leq 1$

$$f(-1) = \sum_{n=1}^{\infty} \frac{(-1)^n}{\ln n}$$

$$\sum_{n=1}^{\infty} (-1)^n a_n, \quad a_n = \frac{1}{\ln n}$$ 　　　Apply the alternating series test $^\uparrow$

$$\implies \lim_{n\to\infty} \frac{1}{\ln n} = 0$$ 　　　Note: $|a_n|$ decreases monotonically

$f(-1)$ converges and $f(1)$ diverges, thus the interval of convergence is $[-1, 1)$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{(-5)^n(x-3)^n}{n^2}$

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$
Apply the ratio test$^\uparrow$

$$\implies \lim_{n\to\infty}\left|\frac{(-5)^{n+1}(x-3)^{n+1}}{(n+1)^2}\cdot\frac{n^2}{(-5)^n(x-3)^n}\right|$$

$$= \lim_{n\to\infty}\frac{5^n\cdot 5\cdot(|x-3|)^n\cdot|x-3|}{n^2}\cdot\frac{n^2}{5^n(|x-3|)^n}$$

$$= \lim_{n\to\infty} 5|x-3|$$

$$\implies |x-3| < \frac{1}{5}$$

$$\implies -\frac{14}{5} < x < \frac{16}{5}$$
converges for $\rho < 1$

$$f\left(\frac{-14}{5}\right) = \frac{(-5)^n\left(-\frac{1}{5}\right)^n}{n^2} = \frac{1}{n^2}$$

$$f\left(\frac{16}{5}\right) = \frac{(-5)^n\left(\frac{1}{5}\right)^n}{n^2} = \frac{(-1)^n}{n^2}$$

$\displaystyle\lim_{n\to\infty} a_n$ (of both points) $= 0$ and the $|a_n|$ of both endpoints decrease monotonically;

$R = \dfrac{1}{5}$, $c = 3$, thus the interval of convergence is $\left[-\dfrac{14}{5},\dfrac{16}{5}\right]$

Use the following equation to expand the function in a power series with $c = 0$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \leftarrow |x| < 1$$

and determine the interval of convergence.

5. $f(x) = \dfrac{1}{4+3x}$

$$\frac{1}{4+3x} = \frac{\frac{1}{4}}{1-\left(-\frac{3x}{4}\right)}$$

$$= \frac{1}{4}\sum_{n=0}^{\infty}\left(-\frac{3x}{4}\right)^n = \sum_{n=0}^{\infty}(-1)^n\frac{3^n x^n}{4^{n+1}} \qquad \text{Expansion}$$

$$\implies \left|\frac{3x}{4}\right| < 1 \qquad\qquad\qquad \sum_{n=1}^{\infty} \leftarrow |x| < 1$$

$$\implies -\frac{4}{3} < x < \frac{4}{3} \qquad\qquad \text{Interval of convergence}$$

Thus, $\dfrac{1}{4+3x} = \displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{3^n x^n}{4^{n+1}}$ with an interval of convergence of $\left(-\dfrac{4}{3}, \dfrac{4}{3}\right)$

6. $f(x) = \dfrac{1}{1-x^4}$

$$\frac{1}{1-x^4} = \sum_{n=0}^{\infty}\left(x^4\right)^n = \sum_{n=0}^{\infty} x^{4n} \qquad \text{Expansion}$$

$$\implies |x^4| < 1 \qquad\qquad\qquad \sum_{n=1}^{\infty} \leftarrow |x| < 1$$

$$\implies -1 < x < 1 \qquad\qquad \text{Interval of convergence}$$

Thus, $\dfrac{1}{1-x^4} = \displaystyle\sum_{n=0}^{\infty} x^{4n}$ with an interval of convergence of $(-1, 1)$

10.8 Exercises

Find the Maclaurin series and find the interval on which the expression is valid.

1. $f(x) = \sin(2x)$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$ Use relevant Maclaurin series $^\uparrow$

$$\implies \sin 2x = (-1)^n \frac{(2x)^{2n+1}}{(2n+1)!}$$

$\sin(x)$ converges $\forall x$, thus $\sin(2x)$ also converges $\forall x$. Therefore:

$$f(x) = (-1)^n \frac{(2x)^{2n+1}}{(2n+1)!} \qquad \leftarrow \forall x \in \mathbb{R}$$

2. $f(x) = x^2 e^{x^2}$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$ Use relevant Maclaurin series $^\uparrow$

$$\implies e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!}$$

$$\implies x^2 e^{x^2} = \sum_{n=0}^{\infty} \frac{x^2 \cdot (x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n+2}}{n!}$$

e^x converges $\forall x$, thus e^{x^2} also converges $\forall x$. Therefore:

$$f(x) = \sum_{n=0}^{\infty} \frac{x^{2n+2}}{n!} \qquad \leftarrow \forall x \in \mathbb{R}$$

Find the Taylor series centered at c and the interval on which the expansion is valid.

3. $f(x) = e^{3x}, \quad c = -1$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad\qquad \text{Use relevant Maclaurin series}^{\uparrow}$$

$$\implies f(x) = e^{3(x-1)} = e^{-3} \cdot e^{3(x+1)} \qquad \text{Center series } (x - c)$$

$$= e^{-3} \sum_{n=0}^{\infty} \frac{(3(x+1))^n}{n!}$$

$$= e^{-3} \sum_{n=0}^{\infty} \frac{3^n(x+1)^n}{n!}$$

e^x converges $\forall x$, thus e^{3x} also converges $\forall x$. Therefore:

$$f(x) = e^{-3} \sum_{n=0}^{\infty} \frac{3^n(x+1)^n}{n!} \qquad \leftarrow \text{convergence interval: } (-\infty, \infty)$$

4. $f(x) = \sin(x), \quad c = \dfrac{\pi}{2}$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \qquad\qquad \text{Use relevant Maclaurin series}^{\uparrow}$$

$$\implies f(x) = \sin\left(x - \frac{\pi}{2}\right)$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{\left(x - \frac{\pi}{2}\right)^{2n+1}}{(2n+1)!} \qquad \text{Center series}(x - c)$$

$\sin(x)$ converges $\forall x$, therefore:

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{\left(x - \frac{\pi}{2}\right)^{2n+1}}{(2n+1)!} \qquad \leftarrow \text{convergence interval: } (-\infty, \infty)$$

PARAMETRIC EQUATIONS: 11.1

Parametric Equations Notes

- **Parametric equation**: defines a group of quantities as functions of one or more independent variables called parameters, commonly expressed as coordinates of points that make up a geometric object.

  - **Parametrization**: the representation of a geometrical curve $\mathcal{C}$ with parameter t, i.e.,

  $$c(t) = (x(t), y(t))$$

    - Note: parametrizations are not unique; the path $c(t)$ may traverse all or part of $\mathcal{C}$ more than once.

  - **Parametrization of a line**: a line through point $P = (a, b)$ with slope m:

  $$x = a + t, \quad y = b + mt \qquad \leftarrow -\infty < t < \infty$$

  - **Parametrization of a circle** with radius R and center (a, b):

  $$c(t) = (a + R\cos\theta, b + R\sin\theta)$$

  - **Parametrization of an ellipse**:

  $$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \qquad \rightarrow \qquad c(\theta) = (a\cos\theta, b\sin\theta)$$

  - **Parametrization of a cycloid**: generated by a circle of radius R,

  $$c(\theta) = (R(t - \sin\theta), R(1 - \cos\theta))$$

  - **Graph of $y = f(x)$**:
  $$c(t) = (t, f(t))$$

- **Slope of tangent lie at $c(t)$**:

  $$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{y'(t)}{x'(t)} \qquad \leftarrow x'(t) \neq 0$$

- **Area under a parametric curve**: valid when the curve $y = h(x)$ is traced once by the parametric curve $c(t) = (x(t), y(t))$.

  $$y = h(x) \rightarrow y(t), \qquad dx \rightarrow x'(t)dt$$
  $$\implies A = \int_{t_0}^{t_1} y(t)x'(t)dt$$

Parametric Problems

11.1 Exercises

Find parametric equations for the given curve.

1. Line through $(3, 1)$ and $(-5, 4)$.

$$P(3, 1), \quad Q(-5, 4)$$

$$\implies m = \frac{y_Q - y_P}{x_Q - x_P} = \frac{4 - 1}{-5 - 3} = -\frac{3}{8}$$

$$\implies y = m(x - x_P) + y_P = -\frac{3}{8}(x - 3) + 1 = -\frac{3}{8}x + \frac{17}{8}$$

$$\begin{cases} x = t \\ y = -\frac{3}{8}t + \frac{17}{8} \end{cases}$$

$$\implies \boxed{c(t) = \left(t, -\frac{3}{8}t + \frac{17}{8} \right)}$$

2. Circle of radius 4 with center $(3, 9)$.

$$c(t) = (a + R\cos\theta, b + R\sin\theta) \quad \leftarrow \text{Parametrization of a circle}^{\uparrow}$$

$$\implies c(t) = (3 + 4\cos\theta, 9 + 4\sin\theta) \in [0, 2\pi)$$

3. The following ellipse its center translated center to $(7, 4)$

$$\left(\frac{x}{5}\right)^2 + \left(\frac{y}{12}\right)^2 = 1$$

$$c(\theta) = (a\cos\theta, b\sin\theta) \qquad \text{Parametrization of a ellipse}^{\uparrow}$$

$$c(\theta) = (5\cos\theta, 12\sin\theta) \qquad \leftarrow \left(\frac{x}{5}\right)^2 + \left(\frac{y}{12}\right)^2 = 1$$

$$c(\theta) = \boxed{(7 + 5\cos\theta, 4 + 12\sin\theta)} \qquad \leftarrow \left(\frac{x-7}{5}\right)^2 + \left(\frac{y-4}{12}\right)^2 = 1$$

Arc Length, Polar Coordinates: 11.2–11.4

11.2–11.4 Notes

Arc Length and Speed

- **Arc Length of** $\mathcal{C}$: valid if $c(t) = (x(t), y(t))$ directly traverses $\mathcal{C}$ for $a \leq t \leq b$, then

$$s = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

  - Can be interpreted as the **distance traveled** along the path from $t = a \to b$

  - **Displacement**: less than or equal to the distance traveled; simply the distance from starting point $c(a)$ to endpoint $c(b)$.

  - Distance traveled as as **function of** t, starting at t_0:

$$s(t) = \int_{t_0}^{t_1} \sqrt{x'(u)^2 + y'(u)^2}\, du$$

- **Speed** at time t:

$$\frac{ds}{dt} = \sqrt{x'(t)^2 + y'(t)^2}$$

- **Surface area**: obtained via rotation of the parametric equation about the x-axis for $a \leq t \leq b$, given $y(t) \geq 0$, $x(t)$ is increasing, and $x'(t) \wedge y'(t)$ are continuous:

$$S = 2\pi \int_a^b y(t)\sqrt{x'(t)^2 + y'(t)^2}\, dt$$

Polar Coordinates

- **Polar coordinate system**: a two-dimensional coordinate system wherein each point is determined by the distance and angle from a reference point and direction.

  - **Radial coordinate,** r: the distance from reference point.

  - **Angular coordinate,** θ: the angle from reference direction.

  - A point P has polar coordinates (r, θ) with the angle measured in the counterclockwise direction by convention.

- **Conversion between polar and rectangular coordinates**:

$$x = r\cos\theta \qquad y = r\sin\theta \qquad r = \sqrt{x^2 + y^2} \qquad \tan\theta = \frac{y}{x} \leftarrow x \neq 0$$

- **If $r > 0$ then**: (r, θ) must lie in quadrant I or IV;

$$\theta = \begin{cases} \tan^{-1}\frac{y}{x} & \leftarrow x > 0 \\ \tan^{-1}\frac{y}{x} + \pi & \leftarrow x < 0 \\ \pm\frac{\pi}{2} & \leftarrow x = 0 \end{cases}$$

- **Non-uniqueness**: Multiple representations can represent the same point, i.e.,

$$(r, \theta) \equiv (r, \theta + 2n\pi) \equiv (-r, \theta + (2n+1)\pi) \qquad \leftarrow n \in \mathbb{Z}$$

- **Polar Equations**:

Curve	Polar Equation
Circle of radius R, center at origin	$r = R$
Line through origin slope $m = \tan\theta_0$	$\theta = \theta_0$
Line, where $P_0 = (d, \alpha)$ is closest to the origin	$r = d\sec(\theta - \alpha)$
Circle radius a, center at $(a, 0)$ $(x - a)^2 + y^2 = a^2$	$r = 2a\cos\theta$
Circle radius a, center at $(0, a)$ $x^2 + (y - a)^2 = a^2$	$r = 2a\sin\theta$

Area and Arc Length in Polar Coordinates

- **Area in Polar Coordinates**: given that f is continuous, then the sector is bounded by:

  - **Polar curve, r**: $r = f(\theta)$

  - **Two rays, α, β**: where each ray is an angle θ with $\alpha < \beta, \quad \beta = \theta - \alpha$

  - Thus, the area is equal to the integral between α and β, i.e.

$$A = \frac{1}{2} \int_{\alpha}^{\beta} f(\theta)^2 d\theta$$

- **Arc length of polar curve**: given $\alpha \leq \theta \leq \beta$:

$$s = \int_{\alpha}^{\beta} \sqrt{f(\theta)^2 + f'(\theta)^2} d\theta$$

Quiz 3

1. Indicate whether the following statements are **True** or **False**, with justification.

(a) The series $\sum_{n=1}^{\infty} (-1)^n \cos\left(\frac{1}{n}\right)$ converges.

   ✗ **False:**

$$\lim_{n\to\infty} a_n \overset{?}{=} 0, \quad a_n = \cos\left(\frac{1}{n}\right) \qquad \text{Apply the } n^{th} \text{ term test}^{\uparrow}$$

$$\implies \lim_{n\to\infty} \cos\left(\frac{1}{n}\right) = 1$$

$\lim_{n\to\infty} a_n \neq 0$, thus the series diverges.

(b) If the radius of converges of the power series $\sum_{n=0}^{\infty} a_n x^n$ is $R = 5$, then the series must converge for $x = -3$ and $x = -4$.

   ✓ **True:**

$$c = 0, \quad R = 5 \implies \text{converges } \forall x \in (-5, 5)$$
$$\text{By the Interval of convergence}^{\uparrow}$$

$x = -3 \wedge x = 4 \in (-5, 5)$, thus the series must converge at these values.

2. Determine whether the following series converge absolutely/conditionally, or diverge.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{2n+5}$

$$\sum_{n=1}^{\infty}(-1)^n a_n, \quad a_n = \frac{n}{2n+5} \qquad \text{Apply the \textcolor{green}{alternating series test}}^\uparrow$$

$$\implies \lim_{n\to\infty} \frac{n}{2n+5} = \frac{\infty}{\infty}$$

$$\implies \lim_{n\to\infty} \frac{n}{2n+5} = \frac{1}{2} \qquad \text{By L'Hôpital's Rule}$$

$\displaystyle\lim_{n\to\infty} a_n \neq 0$, thus the series \textcolor{red}{diverges}

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{2\sqrt{n}-1}$

$$\sum_{n=1}^{\infty}(-1)^n a_n, \quad a_n = \frac{1}{2\sqrt{n}-1} \qquad \text{Apply the \textcolor{green}{alternating series test}}^\uparrow$$

$$\implies \lim_{n\to\infty} \frac{1}{2\sqrt{n}-1} = 0$$

$$\implies a_n \text{ converges} \qquad\qquad \text{Note: } |a_n| \text{ decreases monotonically}$$

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^n}{2\sqrt{n}-1}\right| \overset{?}{=} \text{converges} \qquad \text{Apply the \textcolor{green}{absolute convergence test}}^\uparrow$$

$$\implies \lim_{n\to\infty} \frac{1}{2\sqrt{n}-1} \qquad\qquad \lim_{n\to\infty} a_n = 0 \to n^{th} \text{ term inconclusive...}$$

$$L = \lim_{n\to\infty} \frac{a_n}{b_n}, \quad b_n = \frac{1}{\sqrt{n}} \qquad \text{Apply the \textcolor{green}{limit comparison test}}^\uparrow$$

$$= \lim_{n\to\infty} \frac{1}{2\sqrt{n}-1} \cdot \sqrt{n}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n}}{2\sqrt{n}-1} \cdot \frac{n^{-\frac{1}{2}}}{n^{-\frac{1}{2}}}$$

$$= \lim_{n\to\infty} \frac{1}{2 - n^{-\frac{1}{2}}} = \frac{1}{2}$$

$L > 0$, and b_n diverges by the p-series, implying the $|a_n|$ diverges. Thus, the original series \textcolor{blue}{converges conditionally}.

3. Find a power series expansion with the center $c = 0$ for

$$f(x) = \frac{1}{1 + x^3}.$$

and find the interval of convergence. Hint: use $\dfrac{1}{1 - x} = \displaystyle\sum_{n=0}^{\infty} x^n \quad \leftarrow |x| < 1$

$$\implies \frac{1}{1 + x^3} = \frac{1}{1 - (-x^3)}$$

$$= \sum_{n=1}^{\infty} (-x^3)^n = \sum_{n=1}^{\infty} (-1)^n x^{3n} \qquad \text{Apply hint}$$

$$\implies \frac{1}{1 + x^3} = \sum_{n=1}^{\infty} (-1)^n x^{3n} \qquad \leftarrow |x| < 1$$

Thus, the interval of convergence is all values in the interval $(-1, 1)$.

4. Find the radius of convergence of the power series given by

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+1}}{2^n n}$$

$$\rho = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| \qquad\qquad \text{Apply the ratio test}^{\uparrow}$$

$$\implies \lim_{n \to \infty} \left| \frac{(-1)^{n+1} x^{2(n+1)+1}}{2^{n+1}(n+1)} \cdot \frac{2^n n}{(-1)^n x^{2n+1}} \right|$$

$$= \lim_{n \to \infty} \frac{|x|^{2n+3}}{2^{2n}(n+1)} \cdot \frac{2^n n}{|x|^{2n+1}}$$

$$= \lim_{n \to \infty} \frac{|x|^{2n+1} \cdot |x|^2}{2^n \cdot 2(n+1)} \cdot \frac{2^n n}{|x|^{2n+1}}$$

$$= |x|^2 \lim_{n \to \infty} \frac{n}{2n + 2} = \frac{\infty}{\infty}$$

$$= |x|^2 \lim_{n \to \infty} \frac{1}{2} \qquad\qquad \text{By L'Hôpital's Rule}$$

$$\implies \frac{|x|^2}{2} < 1 \qquad\qquad \text{converges when } \rho < 1$$

$$= |x| < \sqrt{2}$$

Thus, the interval of convergence is $(-\sqrt{2}, \sqrt{2})$ with $R = \sqrt{2}$, and $c = 0$
(endpoints not required to be tested for this problem).

Quiz 4

1. Indicate whether the following statements are **True** or **False**, with justification.

(a) The curve with parametric representations $c(t) = (4 + 3 \cos t, 5 + 3 \sin t)$ is a circle with radius $R = 3$ centered art the origin.

✗ **False:**

$$c(t) = (a + R \cos \theta, b + R \sin \theta) \qquad \text{Parametrization of a circle}^\uparrow$$
$$c(t) = (3 \cos \theta, 3 \sin \theta) \qquad\qquad \leftarrow R = 3, (0, 0)$$

Note: $a = 4$, $b = 5$, thus it's a circle with radius 3, but not centered at the origin.

(b) The parametric representation given by $c(t) = (\sin t, t)$ can be represented by function of the form $y = f(x)$.

✗ **False:**

Note that the y component of the parametric representation is given by $y = t$. Substituting y for t in the x component yields $x = \sin y$, which is a function of x in terms of y, but NOT a function of y in terms of x.

2. Determine whether the following series converge of diverge, with justification.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{n!}$

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \qquad\qquad \text{Apply the ratio test}^{\uparrow}$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)^3}{(n+1)!} \cdot \frac{n!}{n^3} \right|$$

$$= \lim_{n\to\infty} \frac{(n+1)^3}{(n+1)n!} \cdot \frac{n!}{n^3}$$

$$= \lim_{n\to\infty} \frac{(n+1)^2}{n^3}$$

$$= \lim_{n\to\infty} \frac{n^2 + 2n + 1}{n^3} \cdot \frac{n^{-3}}{n^{-3}}$$

$$= \lim_{n\to\infty} n^{-1} + 2n^{-2} + n^{-3} = 0$$

$\rho < 1$, thus the series converges absolutely.

(b) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{n}{3n+1} \right)^n$

$$C = \lim_{n\to\infty} |a_n|^{\frac{1}{n}} \qquad\qquad \text{Apply the root test}^{\uparrow}$$

$$= \lim_{n\to\infty} \left| \left(\frac{n}{3n+1} \right)^n \right|^{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{n}{3n+1} = \frac{1}{3} \qquad\qquad \text{By L'Hôpital's Rule}$$

$C = \dfrac{1}{3} < 1$, thus the series converges absolutely.

3. Find the Maclaurin series of (using substitution and/or multiplication)

$$f(x) = x\cos(x^2)$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \qquad \text{Use relevant Maclaurin series}^{\uparrow}$$

$$\implies \cos(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n}}{(2n)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!} \qquad \text{Substitution of } x^2$$

$$x \cdot \cos(x^2) = x \cdot \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{(2n)!} \qquad \text{Multiply by } x$$

$\cos x$ converges $\forall x$, thus $x\cos x^2$ also converges $\forall x$. Therefore:

$$f(x) = x\cos(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{(2n)!} \qquad \leftarrow \forall x \in \mathbb{R}$$

4. Express the following integral as a infinite series, first by finding the Maclaurin series of the integrand, then integrating this series.

$$\int_0^1 e^{-x^2} dx$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$ Use relevant Maclaurin series↑

$$\implies f(x) = e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!} \qquad \leftarrow \forall x \in \mathbb{R}$$

$$\implies \int_0^1 e^{-x^2} dx = \int_0^1 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!} dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^1 x^{2n} dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot \frac{x^{2n+1}}{2n+1} \Big|_0^1$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)n!} - 0$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)n!}}$$

5. Consider the curve with parametric representation

$$c(t) = (\sin 2t + \cos t, \cos 2t - \sin t)$$

Find an equation of the tangent line at $t = \pi$

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} \quad \leftarrow x'(t) \neq 0 \qquad\qquad \text{Note: slope of a tangent line}\uparrow$$

$$\implies \frac{dy}{dx} = \frac{-2\sin(2t) - \cos t}{2\cos(2t) - \sin t}$$

$$\implies m = \left.\frac{dy}{dx}\right|_{t=\pi} = \frac{0 - (-1)}{2 - 0} = \frac{1}{2}$$

$$\implies \boxed{y - 1 = \frac{1}{2}(x + 1)} \qquad\qquad c(\pi) = (-1, 1)$$

Final Review Questions

Note: these questions were taken form a provided review sheet; they focus on sections 10.6−11.4. Some questions already exist on the quizzes, but will be duplicated here.

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1. Find the interval of convergence of the following power series.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{n} x^n$

$$C = \lim_{n\to\infty} |a_n|^{\frac{1}{n}}$$ 　　　　　　　Apply the root test$^\uparrow$

$$= \lim_{n\to\infty} \left| \frac{5^n x^n}{n} \right|^{\frac{1}{n}}$$

$$= 5|x| \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 5|x|$$

$$\implies |x| < \frac{1}{5}$$ 　　　　　　　converges for $C < 1$

$$f\left(-\frac{1}{5}\right) = \sum_{n=1}^{\infty} \frac{5^n}{n} \left(-\frac{1}{5}\right)^n$$

$$= \sum_{n=1}^{\infty} (-1)^n \frac{1}{n}, \quad a_n = \frac{1}{n}$$ 　　Apply the alternating series test$^\uparrow$

$$\lim_{n\to\infty} a_n = 0 \wedge |a_n| \text{ decreases monotonically}$$ 　$\implies$ converges

$$f\left(\frac{1}{5}\right) = \sum_{n=1}^{\infty} \frac{5^n}{n} \left(\frac{1}{5}\right)^n = \sum_{n=1}^{\infty} \frac{1}{n}$$ 　$\implies$ diverges by p-series$^\uparrow$

$$\implies \text{Interval of convergence: } \left[-\frac{1}{5}, \frac{1}{5}\right)$$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2+1}$

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

Apply the ratio test↑

$$= \lim_{n\to\infty} \left| \frac{(x-2)^{n+1}}{(n+1)^2+1} \cdot \frac{n^2+1}{(x-2)^n} \right|$$

$$= \lim_{n\to\infty} \frac{(|x-2|)^n \cdot |x-2|}{n^2+2n+2} \cdot \frac{n^2+1}{(|x-2|)^n}$$

$$= |x-2| \lim_{n\to\infty} \frac{n^2+1}{n^2+2n+2}$$

$$= |x-2| \lim_{n\to\infty} \frac{2}{2}$$

By L'Hôpital's Rule

$$\implies |x-2| < 1$$

converges for $\rho < 1$

$$\implies 1 < x < 3$$

$$f(1) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1} \quad a_n = \frac{1}{n^2+1}$$

Apply the alternating series test↑

$$\lim_{n\to\infty} a_n = 0 \implies \text{converges}$$

$|a_n|$ decreases monotonically

$$f(3) = \sum_{n=1}^{\infty} \frac{1}{n^2+1}$$

b_n converges $\to a_n$ converges

Apply the direct comparison test↑

$$b_n = \frac{1}{n^2} \implies b_n \text{ converges}$$

By p-series↑

$$\implies \text{Interval of convergence: } [1, 3]$$

2. Find the Taylor series of the following functions $f(x)$ centered at the given value of c using the definition.

(a) $f(x) = e^x, \quad c = 2$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \leftarrow \forall n \geq 0 \qquad \text{Use relevant Maclaurin series}^{\uparrow}$$

$$f^{(n)}(2) = e^2 \leftarrow f^{(n)}(x) = e^x$$

$$\implies \boxed{\sum_{n=0}^{\infty} \frac{e^2(x-2)^n}{n!}} \qquad \text{Center series, } c = 2$$

(b) $f(x) = \sqrt{x}, \quad c = 1$

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n \quad \leftarrow |x| < 1 \qquad \text{Use relevant Maclaurin series}^{\uparrow}$$

$$\implies \boxed{\sum_{n \geq 0} \binom{\frac{1}{2}}{n} (x-1)^n} \qquad \text{note: } (1 + (x-1))^{\frac{1}{2}} = \sqrt{n}$$

3. Find the Maclaurin series of the following functions using substitution and/or multiplication.

(a) $f(x) = x \cos(2x)$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \quad \leftarrow \forall x \qquad \text{Use relevant Maclaurin series}^{\uparrow}$$

$$\implies \cos(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 4^n x^{2n}}{(2n)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-4)^n x^{2n}}{(2n)!} \qquad \text{Substitution of } x^2$$

$$x \cdot \cos(2x) = x \cdot \sum_{n=0}^{\infty} \frac{(-4)^n x^{2n}}{(2n)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-4)^n x^{2n+1}}{(2n)!} \qquad \text{Multiply by } x$$

$$\implies f(x) = \boxed{\sum_{n=0}^{\infty} \frac{(-4)^n x^{2n+1}}{(2n)!}} \quad \leftarrow \forall x$$

(b) $f(x) = \dfrac{x^3}{1+x}$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n \quad \leftarrow |x| < 1 \qquad \text{Use relevant Maclaurin series}^{\uparrow}$$

$$\implies \frac{x^3}{1+x} = \sum_{n=1}^{\infty} (-1)^n (x^3)^n$$

$$= \boxed{\sum_{n=1}^{\infty} (-1)^n x^{n+3}} \quad \leftarrow |x| < 1$$

4. Express the following integral as a power series, first by finding the Maclaurin series of the integrand, then integrating this series term-by-term:

$$\int_0^1 e^{-x^2}\,dx$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \leftarrow \forall x \qquad\qquad \text{Use relevant Maclaurin series}^\uparrow$$

$$\implies f(x) = e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!} \qquad\qquad \leftarrow \forall x$$

$$\implies \int_0^1 e^{-x^2}\,dx = \int_0^1 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}\,dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^1 x^{2n}\,dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot \frac{x^{2n+1}}{2n+1}\Bigg|_0^1$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)n!} - 0$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)n!}}$$

5. Find the parametric equations for the following curves.

(a) The line through $(3, 6)$ and $(-2, 0)$.

$$P(-2, 0), \quad Q(3, 6)$$

$$\implies m = \frac{y_Q - y_P}{x_Q - x_P} = \frac{6 - 0}{3 + 2} = \frac{6}{5}$$

$$\implies y = m(x - x_P) + y_P = \frac{6}{5}(x + 2) + 0 = \frac{6}{5}x + \frac{12}{5}$$

$$\begin{cases} x = t \\ y = \frac{6}{5}x + \frac{12}{5} \end{cases}$$

$$\implies \boxed{c(t) = \left(t, \frac{6}{5}t + \frac{12}{5} \right)}$$

(b) The circle of radius 5 centered at $(1, 7)$.

$$c(t) = (a + R\cos\theta, b + R\sin\theta) \quad \leftarrow \text{ Parametrization of a circle}^\uparrow$$

$$\implies c(t) = (1 + 5\cos\theta, 7 + 5\sin\theta) \in [0, 2\pi)$$

(c) The ellipse

$$\left(\frac{x - 1}{2} \right)^2 + \left(\frac{y + 1}{3} \right)^2 = 1$$

$$c(\theta) = (a\cos\theta, b\sin\theta) \qquad \text{Parametrization of a ellipse}^\uparrow$$

$$c(\theta) = (2\cos\theta, 3\sin\theta) \qquad \leftarrow \left(\frac{x}{2} \right)^2 + \left(\frac{y}{3} \right)^2 = 1$$

$$c(\theta) = \boxed{(1 + 2\cos\theta, -1 + 3\sin\theta)} \qquad \leftarrow \left(\frac{x - 1}{2} \right)^2 + \left(\frac{y + 1}{3} \right)^2 = 1$$

6. Find the equation of the tangent line to the curve

$$x = \sin(2t) + \cos(t), \quad y = \cos(2t) - \sin(t), \qquad \leftarrow t = \pi$$

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} \quad \leftarrow x'(t) \neq 0 \qquad\qquad \text{Slope of a tangent line}^\uparrow$$

$$\implies \frac{dy}{dx} = \frac{-2\sin(2t) - \cos t}{2\cos(2t) - \sin t}$$

$$\implies m = \left.\frac{dy}{dx}\right|_{t=\pi} = \frac{0 - (-1)}{2 - 0} = \frac{1}{2}$$

$$\implies \boxed{y - 1 = \frac{1}{2}(x + 1)} \qquad\qquad c(\pi) = (-1, 1)$$

7. Find the arc length of the curve

$$x = \frac{2}{3}t^2, \quad y = t^2 - 2, \qquad \leftarrow 0 \leq t \leq 2$$

8. Find the surface area obtained by rotating the following around the x-axis;

$$x = e^t - t, \quad y = 4e^{\frac{t}{2}}, \qquad \leftarrow 0 \le t \le 1$$

9. Match each equation in rectangular coordinates with its equation in polar coordinates.

(a) $x^2 + y^2 = 4$

(b) $x^2 + (y - 1)^2 = 1$

(c) $x^2 - y^2 = 4$

(d) $x + y = 4$

(i) $r^2(1 - 2\sin^2\theta) = 4$

(ii) $r(\cos\theta + \sin\theta) = 4$

(iii) $r = \sin\theta$

(iv) $r = 2$

10. Find the area enclosed by one loop of the curve

$$r^2 = \cos 2\theta$$