Applied Linear Algebra

1 Matrices and Gaussian Elimination

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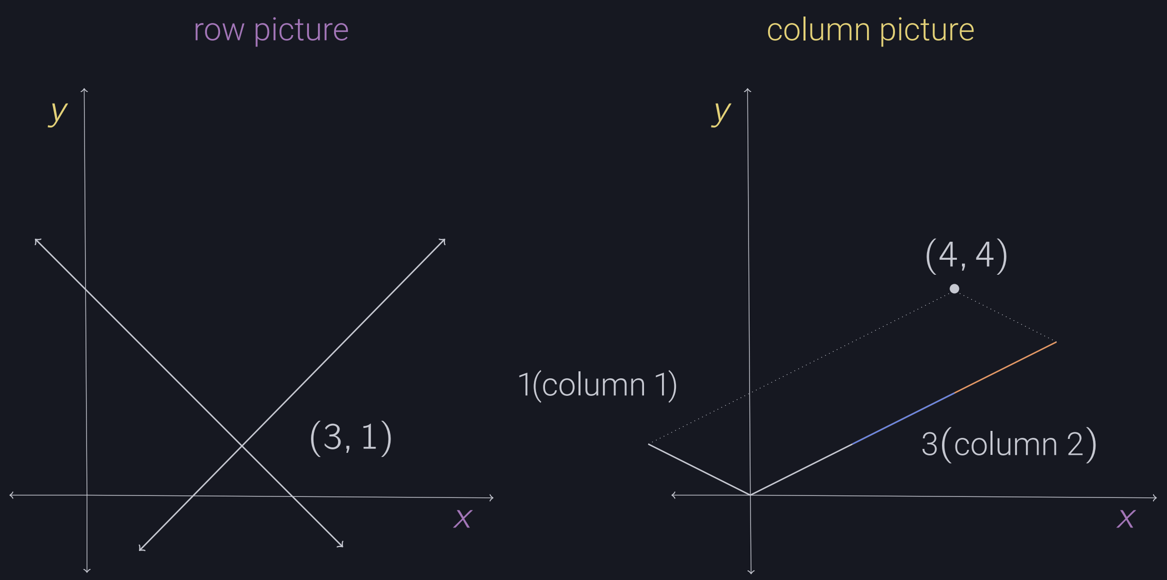
1.2 The Geometry of Linear Equations

Problems 1–12

1. For the equations $x + y = 4,\ 2x - 2y = 4$, draw the row picture (two intersecting lines) and the column picture (combination of two columns equal to the column vector (4,4) on the right side).



1.2.1

2. Solve to find a combination of the columns that equals b:

$$
\begin{aligned}
u - v - w &= b_1 \\
v + w &= b_2 \\
w &= b_3
\end{aligned}
$$

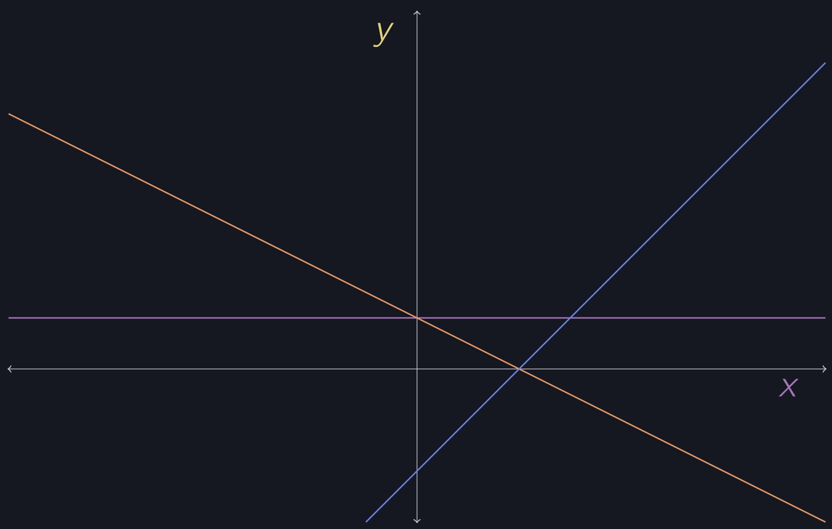
$$
\begin{aligned}
&\implies w = b_3 \\
&\implies v = b_2 - b_3 \\
&\implies u = b_1 + v + w = b_1 + b_2
\end{aligned}
$$

3. Describe the intersection of the three planes $u + v + w + z = 6$ and $u + w + z = 4$ and $u + w = 2$ (all in four-dimensional space). Is it a line or a point or an empty set? What is the intersection if the fourth plane $u = -1$ is included? Find a fourth equation that leaves us with no solution.

   - A line; as $u + w = 2$ is only a line[?]. A fourth plane with $u = -1$ would produce a normally intersecting point. Any addition equation when $u + w \neq 2$ would produce an inconsistent equation.

4. Sketch these three lines and decide if the equations are solvable:

$$x + 2y = 2$$
$$x - y = 2$$
$$y = 1$$



1.2.4

Inconsistent; multiple points of intersect

What happens if all right-hand sides are zero? Is there any nonzero choice of right-hand sides that allows the three lines to intersect at the same point?

- If all the solutions were zero, then it would be a trivial solution.
- Yes, e.g., $x - y = -1$ would produce a single point of intersection.

5. Find two points on the line of intersection of the three planes $t = 0$ and $z = 0$ and $x + y + z + t = 1$ in four-dimensional space.

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

6. When $b = (2, 5, 7)$, find a solution (u, v, w) to equation (4) different from the solution $(1, 0, 1)$ mentioned in the text.

- Since there are infinite solutions, and if $\boldsymbol{s}$ vector describing one solution and λ is any scalar, then $\boldsymbol{s}\lambda$ is also a solution. E.g., $(1, 0, 1)\,42 = (42, 0, 42)$

8. Explain why the system

$$\begin{aligned} u + v + w &= 2 \\ u + 2v + 3w &= 1 \\ v + 2w &= 0 \end{aligned}$$

is singular by finding a combination of the three equations that adds up to $0 = 1$. What value should replace the last zero on the right side to allow the equations to have solutions—and what is one of the solutions?

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 0 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 2 & 0 \end{array}\right] \xrightarrow{R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

- Replacing the last zero with -1 would yield infinite solutions. One solution would be $[3, -1, 0]^T$

9. The column picture for the previous exercise (singular system) is

$$u \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + v \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + w \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = b$$

Show that the three columns on the left lie in the same plane by expressing the third as a combination of the first two. What are all the solutions (u, v, w) if b is the zero vector $(0, 0, 0)$?

$$-1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

- If is $\boldsymbol{b}$ equal to the zero vector $\mathbf{0}$ then the solutions are equal to the kernel? i.e., $-1x_1, 2x_2, 0x_3 = \mathbf{0}$

10. Under what condition on y_1, y_2, y_3 do the points $(0, y_1), (1, y_2), (2, y_3)$ lie on a straight line?

- Question 9 describes the state at which they are collinear, i.e., $y_3 = 2y_2 - y_1$

11. These equations are certain to have the solution $x = y = 0$. For which values of a is there a whole line of solutions?

$$\begin{aligned} ax + 2y &= 0 \\ 2x + ay &= 0 \end{aligned}$$

- Only the scalars that make the lines linearly dependent, i.e., $a = 2, -2$

Problems 17–23

17. The first of these equations plus the second equals the third:

$$\begin{aligned} x+y+z&=2\\ x+2y+z&=3\\ 2x+3y+2z&=5 \end{aligned}$$

The first two planes meet along a line. The third plane contains that line, because if x, y, z satisfy the first two equations then they also span all of $\mathbb{R}^3$. The equations have infinitely many solutions (the whole line $\boldsymbol{L}$). Find three solutions.

- $\boldsymbol{v} = (4, 4, 0)$, $\boldsymbol{w} = (6, 3, 2)$, $\boldsymbol{u} = 2v + -1w$

18. Move the third plane in Problem 17 to a parallel plane $2x + 3y + 2z = 9$. Now the three equations have no solution—*why not*? The first two planes meet along the line $\boldsymbol{L}$, but the third plane doesn't that cross that line.

19. In Problem 17 the columns are $(1, 1, 2)$ and $(1, 2, 3)$ and $(1, 1, 2)$. This is a "singular case" because the third column is linearly dependent Find two combinations of the columns that give $b = (2, 3, 5)$. This is only possible for $b = (4, 6, c)$ if $c = 10$

20. Normally 4 "planes" in four-dimensional space meet at a tensor. Normally 4 column vectors in four-dimensional space can combine to produce b. What combination of $(1, 0, 0, 0), (1, 1, 0, 0), (1, 1, 1, 0), (1, 1, 1, 1)$ produces $b = (3, 3, 3, 2)$? $(1, 0, 0, -2)$? What 4 equations for x, y, z, t are you solving? A lower triangular matrix, i.e.,

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3\\ 1 & 1 & 0 & 0 & 3\\ 1 & 1 & 1 & 0 & 3\\ 1 & 1 & 1 & 1 & 1 \end{array}\right]$$

21. When equation 1 is added to equation 2, which of these are changed: the planes in the row picture, the column picture, the coefficient matrix, the solution?

- Row operations do not change the solution. Row 2 is changed, thus the second plane is changed. All columns are changed.?

22. If (a, b) is a multiple of (c, d) with $abcd \neq 0$, show that (a, c) is a multiple of (b, d). This is surprisingly important: call it a challenge question. You could use numbers first to see how a, b, c, and d are related. The question will lead to:

    If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has dependent rows then it has dependent columns.

    Received help from this document (pg. 3), accessed 10/01/2021

    $$\begin{aligned} \lambda \in \mathbb{R}, \quad (a, b) = \lambda(c, d) &= (\lambda c, \lambda d) \\ \implies a = \lambda c &= \lambda c d^{-1} d = d^{-1} c \lambda d = d^{-1} c b \\ \implies (a, c) &= \left(d^{-1} c b, d^{-1} d c\right) = c d^{-1}(b, d) \end{aligned}$$

    Thus, (b, d) is a multiple of (a, c)

23. In these equations, the third column (multiplying w) is the same as the right side b. The column form of the equations immediately gives what solution for (u, v, w)?

    $$\begin{aligned} 6u + 7v + 8w &= 8 \\ 4u + 5v + 9w &= 9 \\ 2u - 2v + 7w &= 7 \end{aligned}$$

    - First two columns are irrelevant, $u = 0$, $v = 0$, only need w

1.3 Gaussian Elimination

Problems 6, 7

6. Choose a coefficient b that makes this system singular. Then choose a right-hand side g that makes it solvable. Find two solutions in that singular case.

$$2x + by = 16$$
$$4x + 8y = g$$

$$2x + 4y = 16$$
$$4x + 8y = 32$$

   - Since R_2 is just a multiple of R_1, then solving for x, y, with one variable = 0, in the first equation will yield two solutions, i.e., $(8, 0)$, $(0, 4)$

7. For which numbers a does elimination break down (a) permanently, and (b) temporarily?

$$ax + 3y = -3$$
$$4x + 6y = 6$$

   Solve for x and y after fixing the second breakdown by a row exchange.

   - Permanently: $a = 2$ (linearly dependent, no solution)
   - Temporarily: $a = 0$;

$$\left[\begin{array}{cc|c} 4 & 6 & 6 \\ 0 & 3 & -3 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -1 \end{array}\right]$$
$$y = -1, \quad x = 3$$

Problems 17, 18, 19

17. Which number q makes this system singular and which right-hand side t gives it infinitely many solutions? Find the solution that has $z = 1$.

$$x + 4y - 2z = 1$$
$$x + 7y - 6z = 6$$
$$3y + qz = t$$

$$x + 4y - 2z = 1$$
$$x + 7y - 6z = 6$$
$$3y + -4z = 5$$

   - If $q = -4$, then R_3 would have no pivot
   - If $t = 5$, then there would be finite solutions, R_3 would be linearly dependent with R_2

18. It is impossible for a system of linear equations to have exactly two solutions. Explain why.
    - If (x, y, z) and (X, Y, Z) are two solutions, what is the other one?
        - There is no other *one*, there would be infinitely many.
    - If 25 planes meet at two points, where else do they meet?
        - Every other single point, they would span all of $\mathbb{R}^3$
19. Three planes can fail to have an intersection point, when no two planes are parallel. The system is singular if row 3 of $\boldsymbol{A}$ is a linearly dependent; a combination of the first two rows. Find a third equation that can't be solved if $x + y + z = 0$ and $x - 2y - z = 1$.

$$\begin{aligned} x + y + z &= 0 \\ x - 2y - z &= 1 \\ R_1 + R_2 \neq 1 \rightarrow \text{ parallel; no solution, e.g.,} \\ 2x - y &= 42 \end{aligned}$$

Problems 30, 31

30. Use elimination to solve

$$\begin{aligned} u + v + w &= 6 \\ u + 2v + 2w &= 11 \\ 2u + 3v - 4w &= 3 \end{aligned} \qquad \text{and} \qquad \begin{aligned} u + v + w &= 7 \\ u + 2v + 2w &= 10 \\ 2u + 3v - 4w &= 3 \end{aligned}$$

$$\text{rref}\left(\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 2 & 11 \\ 2 & 3 & -4 & 3 \end{array}\right]\right) \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$
$$\text{rref}\left(\left[\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 1 & 2 & 2 & 10 \\ 2 & 3 & -4 & 3 \end{array}\right]\right) \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

31. For which three numbers a will elimination fail to give three pivots?

$$\begin{aligned} ax + 2y + 3z &= b_1 \\ ax + ay + 4z &= b_2 \\ ax + ay + az &= b_3 \end{aligned}$$

- For $a = 0$, multiple failures.
- For $a = 2$, columns $0, 1$ would be equal.
- For $a = 4$, rows $1, 2$ would be equal.

1.4 Matrix Notation and Matrix Multiplication

Problems 4, 10, 17, 19

4. If an $m \times n$ matrix $\boldsymbol{A}$ multiplies an n-dimensional vector $\boldsymbol{x}$, how many separate multiplications are involved? What if A multiplies an $n \times p$ matrix $\boldsymbol{B}$?
   - $m \cdot n$ multiplications; number of rows times the length of $\boldsymbol{x}$.
   - $m \cdot n \cdot p$; same as above, except accounting for each additional column p.

10. True or false? Give a specific counterexample when false.
    - If rows 1 and 3 of $\boldsymbol{B}$ are the same, so are rows 1 and 3 of $\boldsymbol{AB}$.
      - ✘ false; matrix multiplication is done by the rows of the left matrix and the columns of the right, the rows may be the same, but if a column between the two are different, then there would be different multiplications occurring, e.g.,

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 12 & 8 & 4 \\ 30 & 20 & 10 \\ 38 & 32 & 16 \end{bmatrix}$$

    - If columns 1 and 3 of $\boldsymbol{B}$ are the same, so are columns 1 and 3 of $\boldsymbol{AB}$.
      - ✓ true;
    - If rows 1 and 3 of $\boldsymbol{A}$ are the same, so are rows 1 and 3 of $\boldsymbol{AB}$.
      - ✓ true
    - $(AB)^2 = A^2B^2$.
      - ✘ false (most of the time), e.g.,

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \boldsymbol{B} = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\boldsymbol{AB}^2 = \begin{bmatrix} 144 & 64 & 16 \\ 900 & 400 & 100 \\ 2304 & 1024 & 256 \end{bmatrix} \neq \begin{bmatrix} 74 & 26 & 10 \\ 452 & 152 & 52 \\ 1154 & 386 & 130 \end{bmatrix} = \boldsymbol{A}^2\boldsymbol{B}^2$$

17. Which of the following matrices are guaranteed to equal $(A+B)^2$?

$$\begin{aligned} A^2 + 2AB + B^2, \\ ✓\, A(A+B) + B(A+B) \\ ✓\, (A+B)(B+A), \\ ✓\, A^2 + AB + BA + B^2 \end{aligned}$$

19. A fourth way to multiply matrices is columns of $\boldsymbol{A}$ times rows of $\boldsymbol{B}$:

$\boldsymbol{AB}=$ (column 1)(row 1) $+\cdots+$ (column n)(row n) $=$ sum of simple matrices.

Give a 2×2 example of this important rule for matrix multiplication.

$$\begin{bmatrix}1&2\\3&4\end{bmatrix}\begin{bmatrix}a&b\\c&d\end{bmatrix}=\left(a\begin{bmatrix}1\\3\end{bmatrix}+c\begin{bmatrix}2\\4\end{bmatrix}b\begin{bmatrix}1\\3\end{bmatrix}+d\begin{bmatrix}2\\4\end{bmatrix}\right)$$

Useful, as the left matrix can be thought of as the weights that scale the elements of the columns of the right matrix.

Problems 30, 31

30. Multiply these matrices:

$$\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}\quad\text{and}\quad\begin{bmatrix}1&0&0\\-1&1&0\\-1&0&1\end{bmatrix}\begin{bmatrix}1&2&3\\1&3&1\\1&4&0\end{bmatrix}.$$

$$\begin{bmatrix}9&8&7\\6&5&4\\3&2&1\end{bmatrix}\quad\text{and}\quad\begin{bmatrix}1&2&3\\0&1&-2\\0&2&-3\end{bmatrix}\quad\text{respectively}$$

- The former multiplication performs two operations (left: swaps top and bottom columns, right: swaps left and right columns), while the latter subtracts row 1 from both row 2 and row 3.

31. This 4×4 matrix needs which elimination matrices $\boldsymbol{E}_{21}$ and $\boldsymbol{E}_{32}$ and $\boldsymbol{E}_{43}$?

$$\boldsymbol{A}=\begin{bmatrix}2&-1&0&0\\-1&2&-1&0\\0&-1&2&-1\\0&0&-1&2\end{bmatrix}$$

- $e_{21}=-\frac{1}{2},\quad e_{32}=-\frac{2}{3},\quad e_{43}=-\frac{3}{4}$
- I suspect the factions will tend towards -1 if the matrix was expanded upon in a similar fusion?

Problems 34, 35, 38, 42

34. Multiply these matrices in the orders FE and FE and E^2:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & 0 & 1 \end{bmatrix} \qquad F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{bmatrix}$$

$$FE = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ ac+b & c & 1 \end{bmatrix} \quad EF = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix} \quad E^2 = \begin{bmatrix} 1 & 0 & 0 \\ 2a & 1 & 0 \\ 2b & 0 & 1 \end{bmatrix}$$

35. ↓
   (a) Suppose all columns of B are the same. Then all columns of EB are the same, because each one is E times B_{1n}.
   (b) Suppose all rows of B are $[1\ 2\ 4]$. Show by example that all rows of EB are not $[1\ 2\ 4]$. It is true that those rows are multiples of $[1\ 2\ 4]$
      - E.g., if $e_{12} = 2$, then m_2 of EB would be $[3\ 6\ 12]$

38. If $AB = I$ and $BC = I$, use the associative law to prove $A = C$.

$$\begin{aligned} A &= A(BC) \\ A &= (AB)C \\ A &= C \end{aligned}$$

42. True of false?
   (a) If A^2 is defined then A is necessarily square.
      - ✓ true; inner dimensions much match, i.e., dimensions of $n_1 = m_2$. Thus, A must be square.

   (b) If AB and BA are defined, then A and B are square.
      - ✘ false; if $A = 6 \times 9$ and $B = 9 \times 6$ allows for valid pre- and post-multiplication of B.

   (c) If AB and BA are defined, then AB and BA are square.
      - ✓ true; see above example, each case will still yield square matrices. Not a proof, but I can't see another way to falsify (b).

   (d) If $AB = B$ then $A = I$
      - ✘ false; e.g.,$B = \mathtt{0}$

1.5 Triangular Factors and Row Exchanges

Problems 1, 6, 7, 12, 14, 18, 19

1. When is an upper triangular matrix nonsingular (a full set of pivots)?
6. Find $\boldsymbol{E}^2$ and $\boldsymbol{E}^8$ and $\boldsymbol{E}^{-1}$ if

$$\boldsymbol{E} = \begin{bmatrix} 1 & 0 \\ 6 & 1 \end{bmatrix}$$

7. 7. Find the products FGH and HGF if (with upper triangular zeros omitted)

$$\boldsymbol{F} = \begin{bmatrix} 1 & & & \\ 2 & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix} \boldsymbol{G} = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 2 & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix} \boldsymbol{H} = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

12. How could you factor $\boldsymbol{A}$ into a product $\boldsymbol{UL}$, upper triangular times lower triangular? Would they be the same factors as in $\boldsymbol{A} = \boldsymbol{LU}$?
14. 14. Write down all six of the 3×3 permutation matrices, including $\boldsymbol{P} = \boldsymbol{I}$. Identify their inverses, which are also permutation matrices. The inverses satisfy $\boldsymbol{PP}^{-1} = \boldsymbol{I}$ and are on the same list.
18. 18. Decide whether the following systems are singular or nonsingular, and whether they have no solution, one solution, or infinitely many solutions:

$$\left[\begin{array}{ccc|c} 0 & 1 & -1 & 2 \\ 1 & -1 & 0 & 2 \\ 1 & 0 & -1 & 2 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{array}\right]$$

19. Which numbers a, b, c lead to row exchanges? Which make the matrix singular?

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 0 \\ a & 8 & 3 \\ 0 & b & 5 \end{bmatrix} \quad \text{and} \quad \boldsymbol{A} = \begin{bmatrix} c & 2 \\ 6 & 4 \end{bmatrix}$$

Problems 25–30

25. When zero appears in a pivot position, $\boldsymbol{A} = \boldsymbol{LU}$ is not possible (we need nonzero pivots d, f, i in $\boldsymbol{U}$)! Show directly why these are both impossible:

$$\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \ell & 1 \end{bmatrix} \begin{bmatrix} d & e \\ 0 & f \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ \ell & 1 & \\ m & n & 1 \end{bmatrix} \begin{bmatrix} d & e & g \\ & f & h \\ & & i \end{bmatrix}$$

26. Which number c leads to zero in the second pivot position? A row exchange is needed and $A = LU$ is not possible. Which c produces zero in the third pivot position? Then a row exchange can't help and elimination fails.

$$A = \begin{bmatrix} 1 & c & 0 \\ 2 & 4 & 1 \\ 3 & 5 & 1 \end{bmatrix}$$

27. What are L and D for this matrix A? What is U in $A = LU$ and what is the new U in $A = LDU$

$$A = \begin{bmatrix} 2 & 4 & 8 \\ 0 & 3 & 9 \\ 0 & 0 & 0 \end{bmatrix}$$

28. A and B are symmetric across the diagonal (because 4 = 4). Find their triple factorizations LDU and say how U is related to L for these symmetric matrices:

$$A = \begin{bmatrix} 2 & 4 \\ 4 & 11 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 4 & 0 \\ 4 & 12 & 4 \\ 0 & 4 & 0 \end{bmatrix}$$

29. (Recommended) Compute L and U for the symmetric matrix

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}$$

30.

Problems 32, 33, 35, 42, 43

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1.6 Inverses and Transposes

Problems 3, 10, 12, 13, 18, 20, 21

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Problems 28–30

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Problems 40–43

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Problems 49–59

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1.7 Special Matrices and Applications

Problems 1, 2, 5, 6

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Problems 7, 8, 10

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