# Applied Linear Algebra

## 1 Matrices and Gaussian Elimination

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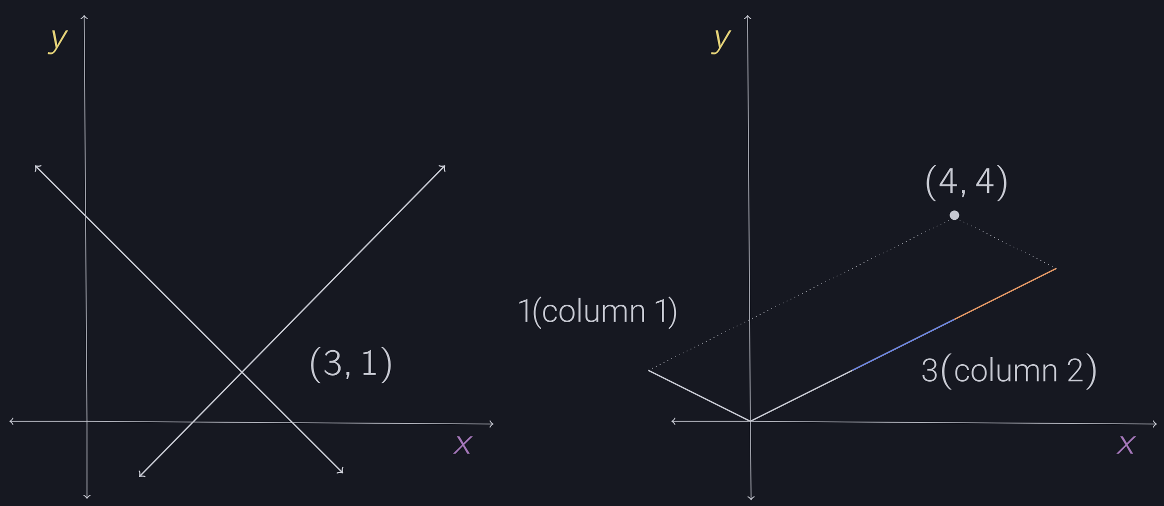
## 1.2 The Geometry of Linear Equations

### Problems 1–12

1. For the equations $x + y = 4,\ 2x - 2y = 4$, draw the row picture (two intersecting lines) and the column picture (combination of two columns equal to the column vector (4,4) on the right side).

row picture column picture




**1.2.1**

2. Solve to find a combination of the columns that equals $b$:

$$\begin{aligned} u - v - w &= b_1 \\ v + w &= b_2 \\ w &= b_3 \end{aligned}$$

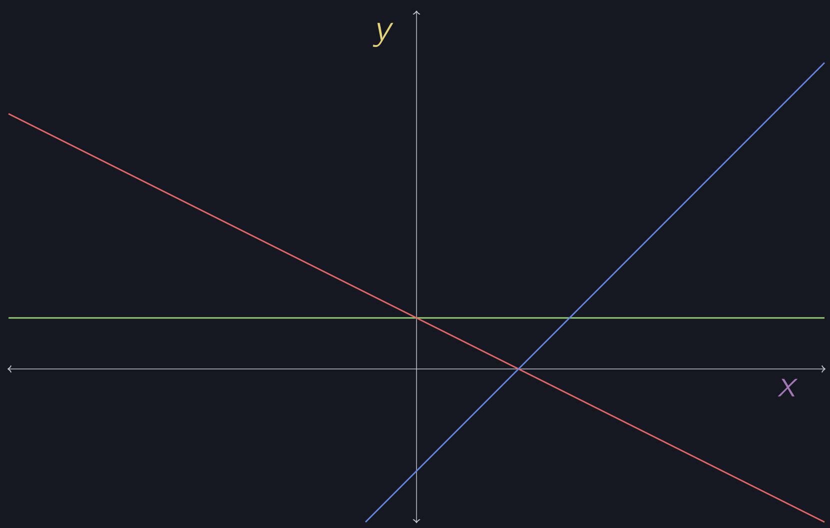
$$\begin{aligned} &\implies w = b_3 \\ &\implies v = b_2 - b_3 \\ &\implies u = b_1 + v + w = b_1 + b_2 \end{aligned}$$

3. Describe the intersection of the three planes $u + v + w + z = 6$ and $u + w + z = 4$ and $u + w = 2$ (all in four-dimensional space). Is it a line or a point or an empty set? What is the intersection if the fourth plane $u = -1$ is included? Find a fourth equation that leaves us with no solution.

   - A line; as $u + w = 2$ is only a line (*?). A fourth plane with $u = -1$ would produce a normally intersecting point. Any addition equation when $u + w \neq 2$ would produce an inconsistent equation.

4. Sketch these three lines and decide if the equations are solvable:

$$\begin{aligned} x + 2y &= 2 \\ x - y &= 2 \\ y &= 1 \end{aligned}$$




**1.2.4**

What happens if all right-hand sides are zero? Is there any nonzero choice of right-hand sides that allows the three lines to intersect at the same point?

- If all the solutions were zero, then it would be a trivial solution.
- Yes, e.g., $x - y = -1$ would produce a single point of intersection.

5. Find two points on the line of intersection of the three planes $t = 0$ and $z = 0$ and $x + y + z + t = 1$ in four-dimensional space.

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

6. When $b = (2, 5, 7)$, find a solution $(u, v, w)$ to equation (4) different from the solution $(1, 0, 1)$ mentioned in the text.
   - Since there are infinite solutions, and if $\boldsymbol{s}$ vector describing one solution and $\lambda$ is any scalar, then $\boldsymbol{s}\lambda$ is also a solution. E.g., $(1, 0, 1)\,42 = (42, 0, 42)$

7. Give two more right-hand sides in addition to $b = (2, 5, 7)$ for which equation (4) can be solved. Give two more right-hand sides in addition to $b = (2, 5, 6)$ for which it cannot be solved.

8. Explain why the system

$$u + v + w = 2$$
$$u + 2v + 3w = 1$$
$$v + 2w = 0$$

is singular by finding a combination of the three equations that adds up to $0 = 1$. What value should replace the last zero on the right side to allow the equations to have solutions—and what is one of the solutions?

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 0 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 2 & 0 \end{array}\right] \xrightarrow{R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

   - Replacing the last zero with $-1$ would yield infinite solutions. One solution would be $[3, -1, 0]^T$

9. The column picture for the previous exercise (singular system) is

$$u \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + v \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + w \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = b$$

Show that the three columns on the left lie in the same plane by expressing the third as a combination of the first two. What are all the solutions $(u, v, w)$ if $b$ is the zero vector $(0, 0, 0)$?

$$-1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

   - If is $\boldsymbol{b}$ equal to the zero vector $\mathbf{0}$ then the solutions are equal to the kernel (*?) i.e., $-1x_1, 2x_2, 0x_3 = \mathbf{0}$

10. Under what condition on $y_1, y_2, y_3$ do the points $(0, y_1), (1, y_2), (2, y_3)$ lie on a straight line?

   - Question 9 describes the state at which they are collinear, i.e., $y_3 = 2y_2 - y_1$

11. These equations are certain to have the solution $x = y = 0$. For which values of $a$ is there a whole line of solutions?

$$ax + 2y = 0$$
$$2x + ay = 0$$

12. Starting with $x + 4y = 7$, find the equation for the parallel line through $x = 0, y = 0$.

## Problems 13–15

13. Draw the two pictures in two planes for the equations $x - 2y = 0$, $x + y = 6$.

14. For two linear equations in three unknowns $x, y, z$, the row picture will show (2 or 3) (lines or planes) in (two or three)-dimensional space. The column picture is in (two or three)-dimensional space. The column picture is in (two or three)-dimensional space. The solutions normally lie on a <?>.

15. For four linear equations in two unknowns $x$ and $y$, the row picture shows four <?>. The column picture is in <?>-dimensional space. The equations have no solution unless the vector on the right-hand side is a combination of <?>.

## Problems 16–23

16. Find a point with $z = 2$ on the intersection line of the planes $x + y + 3z = 6$ and $x - y + z = 4$. Find the point with $z = 0$ and a third point halfway between.

17. The first of these equations plus the second equals the third:

$$\begin{aligned} x + y + z &= 2 \\ x + 2y + z &= 3 \\ 2x + 3y + 2z &= 5 \end{aligned}$$

The first two planes meet along a line. The third plane contains that line, because if $x, y, z$ satisfy the first two equations then they also <?>. The equations have infinitely many solutions (the whole line $\boldsymbol{L}$). Find three solutions.

18. Move the third plane in Problem 17 to a parallel plane $2x + 3y + 2z = 9$. Now the three equations have no solution—*why not*? The first two planes meet along the line $\boldsymbol{L}$, but the third plane doesn't that line.

19. In Problem 17 the columns are $(1, 1, 2)$ and $(1, 2, 3)$ and $(1, 1, 2)$. This is a "singular case" because the third column is <?>. Find two combinations of the columns that give $b = (2, 3, 5)$. This is only possible for $b = (4, 6, c)$ if $c =$

20. Normally 4 "planes" in four-dimensional space meet at a <?>. Normally 4 column vectors in four-dimensional space can combine to produce $b$. What combination of $(1, 0, 0, 0)$, $(1, 1, 0, 0)$, $(1, 1, 1, 0)$, $(1, 1, 1, 1)$ produces $b = (3, 3, 3, 2)$? What 4 equations for $x, y, z, t$ are you solving?

21. When equation 1 is added to equation 2, which of these are changed: the planes in the row picture, the column picture, the coefficient matrix, the solution?

22. If $(a, b)$ is a multiple of $(c, d)$ with $abcd \neq 0$, show that $(a, c)$ is a multiple of $(b, d)$. This is surprisingly important: call it a challenge question. You could use numbers first to see how $a$, $b$, $c$, and $d$ are related. The question will lead to: <?>

    If $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ has dependent rows then it has dependent columns.

23. In these equations, the third column (multiplying w) is the same as the right side $b$. The column form of the equations immediately gives what solution for $(u, v, w)$?

$$6u + 7v + 8w = 8$$
$$4u + 5v + 9w = 9$$
$$2u - 2v + 7w = 7$$

# 1.3 Gaussian Elimination

## Problems 1–9

1.

## Problems 10–19

1.

## Problems 20–22

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## Problems 23–31

1.

## 1.4 Matrix Notation and Matrix Multiplication

## 1.5 Triangular Factors and Row Exchanges

## 1.6 Inverses and Transposes

## 1.7 Special Matrices and Applications