# Calculus III Final Review

# Convergence: 10.3–10.5

## Convergence Notes

- Let $\sum_{n=1}^{\infty} a_n$ be given and note for which series convergence is known, i.e.:

  **Geometric**: let $c \neq 0$, if $|r| < 1$, then

  $$\sum_{n=0}^{\infty} cr^n = \frac{c}{1-r}$$

  $|r| > 1 \implies$ diverges

  ***p*-Series**: converges if $p > 1$.

  $$\sum_{n=0}^{\infty} \frac{1}{n^p}$$

  $p < 1 \implies$ diverges

- **The $n^{th}$ Term Divergence Test**: a relatively easy test that can be used to quickly determine if a test diverges if the $\lim_{n\to\infty} a_n \neq 0$. If $\lim_{n\to\infty} a_n = 0$, then the test is inconclusive and other tests must be applied.

### *Tests for Positive Series*

- **Direct Comparison Test**: use if dropping terms from the denominator or numerator gives a series $b_n$ wherein convergence is easily found, then compare to the original series $a_n$ as follows:

  $$\sum_{n=1}^{\infty} b_n \text{ converges} \implies \sum_{n=1}^{\infty} a_n \text{ converges} \quad \leftarrow 0 \leq a_n \leq b_n$$

  $$\sum_{n=1}^{\infty} b_n \text{ diverges} \implies \sum_{n=1}^{\infty} a_n \text{ diverges} \quad \leftarrow 0 \leq b_n \leq a_n$$

- **Limit Comparison Test**: use when the direct comparison test isn't convenient or when comparing two series. One can to take the dominant term in the numerator and denominator from $a_n$ to form a new positive sequence $b_n$ if needed.

  Assuming the following limit $L = \lim_{n\to\infty} \frac{a_n}{b_n}$ exists, then:

  $$L > 0 \implies \sum_{n=1}^{\infty} a_n \text{ converges} \iff \sum_{n=1}^{\infty} b_n \text{ converges}$$

  $$L = 0 \text{ and } \sum_{n=1}^{\infty} b_n \text{ converges} \implies \sum_{n=1}^{\infty} a_n \text{ converges}$$

  $$L = \infty \text{ and } \sum_{n=1}^{\infty} a_n \text{ converges} \implies \sum_{n=1}^{\infty} b_n \text{ converges}$$

- **Ratio Test**: often used in the presence of a factorial $(\boldsymbol{n!})$ or when the are constants raised to the power of $n$ $(\boldsymbol{c^n})$.

  Assuming the following limit $\rho = \lim_{n\to\infty} \left|\frac{a_n+1}{a_n}\right|$ exists, then

$$\rho < 1 \implies \sum a_n \text{ converges absolutely}$$
$$\rho > 1 \implies \sum a_n \text{ diverges}$$
$$\rho = 1 \implies \text{test is inconclusive}$$

- **Root Test**: used when there is a term in the form of $\boldsymbol{f(n)^{g(n)}}$.

  Assuming the following limit $C = \lim_{n\to\infty} |a_n|^{\frac{1}{n}}$ exists, then

$$C < 1 \implies \sum a_n \text{ converges absolutely}$$
$$C > 1 \implies \sum a_n \text{ diverges}$$
$$C = 1 \implies \text{test is inconclusive}$$

- **Integral Test**: if the other tests fail and $a_n = f(n)$ is a decreasing function, then one can use the improper integral $\int_1^\infty f(x)dx$ to test for convergence.

  Let $a_n = f(n)$ be a positive, decreasing, and continuous function $\forall x \geq 1$, then:

$$\int_1^\infty f(x)dx \text{ converges} \implies \sum_{n=1}^{\infty} a_n \text{ converges}$$
$$\int_1^\infty f(x)dx \text{ diverges} \implies \sum_{n=1}^{\infty} a_n \text{ diverges}$$

## *Tests for Non-Positive Series*

- **Alternating Series Test**: used for series in the form $\sum_{n=0}^{\infty}(-1)^n a_n$

  Converges if $|a_n|$ decreases monotonically $(|a_n + 1| \leq |a_n|)$ and if $\lim_{n\to\infty} a_n = 0$

- **Absolute Convergence**: used if the series $\sum a_n$ is not alternating; simply test if $\sum |a_n|$ converges using the test for positive series.

# Convergence Problems

## *10.5 Exercises*

Determine convergence or divergence using any method.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n + 4^n}{7^n}$

$$\implies \sum_{n=1}^{\infty} \frac{2^n}{7^n} + \sum_{n=1}^{\infty} \frac{4^n}{7^n}$$

Separate into two geometric series↑

$$\implies r = \frac{2}{7} < 1, \quad r = \frac{4}{7} < 1$$

Both geometric series converge, thus the original series converges.

2. $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{n!}$

$$\begin{aligned} \implies \rho &= \lim_{n\to\infty} \left| \frac{(n+1)^3}{(n+1)!} \cdot \frac{n!}{n^3} \right| \\ &= \lim_{n\to\infty} \frac{n^3 + 3n^2 + 3n + 1}{(n+1)n!} \cdot \frac{n!}{n^3} \\ &= \lim_{n\to\infty} \frac{n^3 + 3n^2 + 3n + 1}{n^4 + n^3} \\ &= \lim_{n\to\infty} \frac{n^3 + 3n^2 + 3n + 1}{n^4 + n^3} \cdot \frac{n^{-4}}{n^{-4}} \\ &= \lim_{n\to\infty} \frac{n^{-1} + 3n^{-2} + 3n^{-3} + n^{-4}}{1 + n^{-1}} = 0 \end{aligned}$$

Apply the ratio test↑

$\rho = 0 < 1$, thus the series converges.

3. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2n+1}$

$$\implies \lim_{n\to\infty} \frac{n}{2n+1}$$

Apply the $n^{th}$ term test↑

$$\implies \lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2}$$

by L'Hôpital's Rule

$\displaystyle\lim_{n\to\infty} a_n \neq 0$, thus the series diverges.

4. $\sum_{n=1}^{\infty} 2^{\frac{1}{n}}$

$$\implies \lim_{n\to\infty} 2^{\frac{1}{n}} = 2^0 = 1 \qquad \text{Apply the } n^{th} \text{ term test}^{\uparrow}$$

$\lim_{n\to\infty} a_n \neq 0$, thus the series diverges.

5. $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$

$$\begin{aligned} 0 \leq \sin n &\leq 1 && \leftarrow \forall n \geq 1 \\ 0 \leq \frac{\sin n}{n^2} &\leq \frac{1}{n^2} && \text{Apply the direct comparison test}^{\uparrow} \\ b_n = \frac{1}{n^2} &\to \text{ converges} && \text{by } p\text{-series}^{\uparrow} \end{aligned}$$

The larger ($b_n$) series converges, thus the smaller ($a_n$) converges.

6. $\sum_{n=1}^{\infty} \frac{n!}{(2n)!}$

$$\begin{aligned} \implies \rho &= \lim_{n\to\infty} \left| \frac{(n+1)!}{(2n+2)!} \cdot \frac{(2n)!}{n!} \right| && \text{Apply the ratio test}^{\uparrow} \\ &= \lim_{n\to\infty} \frac{(n+1)n!}{(2n+2)(2n+1)2n!} \cdot \frac{(2n)!}{n!} \\ &= \lim_{n\to\infty} \frac{n+1}{(2n+2)(2n+1)} = \frac{n+1}{4n^2+6n+2} \\ &= \lim_{n\to\infty} \frac{1}{8n+6} = 0 && \text{By L'Hôpital's Rule} \end{aligned}$$

$\rho = 0 < 1$, thus the series converges.

7. $\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$

$$\begin{aligned} 0 \leq n &\leq n+\sqrt{n} && \leftarrow \forall n \geq 1 \\ 0 \leq \frac{1}{n+\sqrt{n}} &\leq \frac{1}{n} && \text{Apply the direct comparison test}^{\uparrow} \\ b_n = \frac{1}{n} &\to \text{ diverges} \end{aligned}$$

The smaller ($b_n$) series diverges, thus the larger $a_n$ original series diverges.

8. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$

$f$ is positive, decreasing, and continuous for $x \geq 2$ Apply the integral test↑

$$\implies \int_2^{\infty} f(x)dx = \lim_{R\to\infty} \int_2^R \frac{1}{x(\ln x)^3}dx \qquad \ln x = u, \quad xdu = dx$$

$$\begin{aligned}\implies \lim_{R\to\infty} \int_2^R \frac{1}{x(u)^3}xdu &= \int_2^R \frac{1}{u}^3 du \\ &= -\frac{1}{2(u)^2} \\ &= -\frac{1}{2\ln^2(x)} + C\Big|_2^{\infty}\end{aligned}$$

$$\implies 0 - \left(-\frac{1}{2\ln^2(2)}\right) = \frac{1}{2\ln^2(2)}$$

The improper integral converges, thus the original series converges.

9. $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{5^n}$

$$\begin{aligned}\implies \rho &= \lim_{n\to\infty} \left|\frac{(n+1)^3}{5^{n+1}} \cdot \frac{5^n}{n^3}\right| \\ &= \lim_{n\to\infty} \frac{n^3+1}{5^n+5^1} \cdot \frac{5^n}{n^3} = \frac{1}{5}\end{aligned}$$

Apply the ratio test↑

$\rho = \dfrac{1}{5} < 1$, thus the series converges.

10. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3 - n^2}}$

11. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 + 4n}{3n^4 + 9}$

12. $\displaystyle\sum_{n=1}^{\infty} n^{-0.8}$

13. $\displaystyle\sum_{n=1}^{\infty} (0.8)^{-n} n^{-0.8}$

14. $\displaystyle\sum_{n=1}^{\infty} 4^{-2n+1}$

15. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$

16. $\displaystyle\sum_{n=1}^{\infty} \sin \frac{1}{n^2}$

17. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \cos n^{-1}$

18. $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{\sqrt{n}}$

19. $\displaystyle\sum_{n=1}^{\infty} \left( \frac{n}{n+12} \right)^{n}$

20. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \cos n^{-1}$

# POWER/TAYLOR SERIES: 10.6–10.8

## Power/Taylor Series Notes

### *Power Series*

- **Power series**: a infinite series in the form:

$$F(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$$

  Where the constant $c$ is the *center* of the power series F(x).

- **Radius of convergence** $R$: the range of values of the variable $x$ whereby the power series $F(x)$ converges.
  - Every power series converges at $x = c$, as $(x-c)^0 = 1$, though the series may diverge for other values of $x$.
  - $F(x)$ converges for $|x-c| < R$ and diverges for $|x-c| > R$
  - $F(x)$ may converge of diverge at endpoints $c - R$ and $c + R$
  - **Interval of convergence**: the open interval $(c - R, c + R)$ and possibly one of both of the endpoints, each must be tested.
    - In most cases, the ratio test↑ can be used to find R.
    - If $R > 0$, then $F$ is differentiable over the interval of convergence; the derivative and antiderivative can be obtained using the following:

$$F'(x) = \sum_{n=1}^{\infty} na_n(x-c)^{n-1} \qquad F(x)dx = C + \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-c)^{n+1}$$

  - **Useful Power Series**: the following power series (more examples: Taylor series↓) can be used to drive expansions of other related functions via substitution, integration, or differentiation:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \leftarrow |x| < 1 \qquad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

### *Taylor Series*

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# Power/Taylor Series Problems

## *10.6 Exercises*

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## *10.8 Exercises*

1.

## Parametric Problems

### *11.1 Exercises*

1.

## 11.2–11.4 Notes

### *Arc Length and Speed*

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### *Polar Coordinates*

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### *Area and Arc Length in Polar Coordinates*

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# Polar Coordinate Problems

## 11.2 Exercises

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## 11.3 Exercises

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## 11.4 Exercises

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# Conic Section Problems

## *11.5 Exercises*

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## Quiz 3

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## Quiz 4

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# Final Review Questions