1. A function $f : A \to B$ is *linear* if, $\forall a, b \in \mathbb{R}$, $f(ax + b) = af(x) + b$.
   Apply the definition of linear to:

   (a) $f(x) = 2x$

   $$\implies \forall a, b \in \mathbb{R}, \quad 2(ax + b) = a2x + b$$

   (b) $f(x) = x^2$

   $$\implies \forall a, b \in \mathbb{R}, \quad (ax + b)^2 = ax^2 + b$$

   (c) $f(x) = \sum_{i=0}^{\infty} a_i x^i$

   $$\implies \forall a, b \in \mathbb{R}, \quad \sum_{i=0}^{\infty} a_i(ax + b)^i = a\left(\sum_{i=0}^{\infty} a_i x^i\right) + b$$

2. A function $f : \mathbb{R} \to \mathbb{R}$ is *continuous* if, $\forall \epsilon > 0, \exists \delta > 0 : f(x + \delta) - f(x) < \epsilon$.
   Apply the definition of continuous to:

   (a) $f(x) = |2x - 1|$

   $$\forall \epsilon > 0, \exists \delta > 0 : |2(x + \delta) - 1| - |2x - 1| < \epsilon$$

   (b) $f(x) = x^{-1}$

   $$\forall \epsilon > 0, \exists \delta > 0 : (x + \delta)^{-1} - x^{-1} < \epsilon$$

   (c) $f(x) = \sum_{n=0}^{\infty} \cos(b^n \pi x)$

   $$\forall \epsilon > 0, \exists \delta > 0 : \sum_{n=0}^{\infty} \cos(b^n \pi(x + \delta)) - \sum_{n=0}^{\infty} \cos(b^n \pi x) < \epsilon$$

3. A relation $\sim: A \times A$ is a *chain* if, $\forall x, y \in A, x \sim y \vee y \sim x$
   Apply the definition of chain to:
   (a) $x \sim y, : x, y \in \mathbb{R} \wedge |x| \leq |y|$

$$\forall x, y \in \mathbb{R} \times \mathbb{R}, \quad |x| \leq |y| \vee |y| \leq |x|$$

   (b) $S \sim T \iff S \in P(T)$, where S, T are sets and $P()$ denotes power set.

$$\begin{gathered} \forall S \in P(T) \implies S \sim T \\ \vee \\ \forall T \in P(S) \implies T \sim S \end{gathered}$$

   (c) $\sigma_1 \sim \sigma_2 \iff \sigma_1, \sigma_2, : A \to A$ are functions and $\exists \tau : \sigma_1 = \tau \circ \sigma_2$

$$\forall \sigma_1, \sigma_2 \in A, \quad \exists \tau : \sigma_1 = \tau \circ \sigma_2 \vee \exists \tau : \sigma_2 = \tau \circ \sigma_1$$

4. (a) Prove that there is no smallest positive rational number greater than 0.

$$\forall p, q \in \mathbb{N}, \exists q : 0 < \frac{p}{q+1} < \frac{p}{q}$$

   (b) Prove that for every positive real number greater than 0 there is a smaller positive rational number.

   There is no smallest positive rational number by theorem (a), thus for any given positive real number there is always some rational number that could be smaller.

   (c) Prove that there is no smallest positive real number greater than 0.

$$\mathbb{Q} \subset \mathbb{R}$$

   Thus, for any given real number there is always a smaller positive real number by theorem (b).

5. Fermat's Last theorem is a famous theorem in Math that was unproven for 200 years. The theorem says $\forall n > 2,\ a, b, c \in \mathbb{N} \implies a^n + b^n \neq c^n$. Another way to state this is $a^n + b^n = c^n$ has no integer solutions for n larger than 2. Use this theorem to prove that $\sqrt[n]{2}$ is irrational for n larger than 2.

$$\begin{aligned}\sqrt[n]{2} \in \mathbb{Q} &\implies \exists a, b \in \mathbb{Z} : \gcd(a, b) = 1 \\ &\implies a^n = 2b^n \implies a^n = b^n + b^n\end{aligned}$$

Thus, this contradicts Fermat's Last theorem implying $\sqrt[n]{2}$ is irrational for $n > 2$.

- *Note: this is essentially zscoder's proof. No credit here, couldn't figure it out myself at first; it's pretty simple, so I couldn't formulate something else that was better without adding unnecessary steps.*