

Calculus



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Parametric Equations and Polar Coordinates

Vectors and Vector-Valued Functions

Partial Derivatives

Multiple Integrals

Vector Calculus

Second-Order Differential Equations

Limits and Continuity



Limits

🌐 Limit 📖 | Thomas (2.2–2.4) 📖

- **Limit** $\lim_{x \rightarrow c}$: the value of a function (or sequence) as the input (or index) approaches some value (note: an informal definition).
 - Limits are used to define **continuity**↓, **derivatives**↓, and **integrals**↓.

Limits of a Functions and Sequences

🌐 Limit of a function 📖 | Limit of a sequence 📖 | Essence of Calculus, Ea 📖

- **Limit of a function**: a fundamental concept in calculus and analysis concerning the behavior of a function near a particular input c , i.e.,

$$\lim_{x \rightarrow c} f(x) = L$$

- Reads as “ f of x tends to L as x tends to c ”
- ϵ, δ **Limit of a function**: a formalized definition, wherein $f(x)$ is defined on an open interval \mathcal{I} , except possibly at c itself, leading to the informal definition, if and only if

$$f : \mathbb{R} \rightarrow \mathbb{R}, c, L \in \mathbb{R} \Rightarrow \lim_{x \rightarrow c} f(x) = L$$



$$\forall \epsilon > 0 (\exists \delta > 0 : \forall x \in \mathcal{I} (0 < |x - c| < \delta \Rightarrow |f(x) - L| < \epsilon))$$

- Functions **do not have a limit** when the function:
 - has a **unit step**, i.e., it “jumps” at a point;
 - is **not bounded**, i.e., it tends towards infinity;
 - or it **oscillate**, i.e., it does not stay close to any single number.
- **Limit of a sequence**: the value that the terms of a sequence $(x_n)_{n \in \mathbb{N}}$ “tends to” (and not to any other) as n approaches infinity (or some other point), i.e.,

$$\lim_{n \rightarrow \infty} x_n = x$$

- ϵ **Limit of a sequence**: for every measure of closeness ϵ , the sequence’s x_n term eventually converges to the limit, i.e.,

$$\forall \epsilon > 0 (\exists N \in \mathbb{N} (\forall n \in \mathbb{N} (n \geq N \Rightarrow |x_n - x| < \epsilon)))$$

- **Convergent**: when a limit of a sequence **exists**.
- **Divergent**: a sequence that **does not** converge.

Properties of Limits

📌 List of limits 📌 | Squeeze theorem 📌

- **Operations on a single known limit:** if $\lim_{x \rightarrow c} f(x) = L$, then:
 - $\lim_{x \rightarrow c} [f(x) \pm \alpha] = L \pm \alpha$
 - $\lim_{x \rightarrow c} \alpha f(x) = \alpha L$
 - $\lim_{x \rightarrow c} f(x)^{-1} = L^{-1}, L \neq 0$
 - $\lim_{x \rightarrow c} f(x)^n = L^n, n \in \mathbb{N}$
 - $\lim_{x \rightarrow c} f(x)^{n^{-1}} = L^{n^{-1}}, \text{ if } n \in \mathbb{N}_e \Rightarrow L > 0$
- **Operations on two known limits:** if $\lim_{x \rightarrow c} f(x) = L_1$ and $\lim_{x \rightarrow c} g(x) = L_2$, then:
 - $\lim_{x \rightarrow c} [f(x) \pm g(x)] = L_1 \pm L_2$
 - $\lim_{x \rightarrow c} [f(x)g(x)] = L_1 L_2$
 - $\lim_{x \rightarrow c} f(x)g(x)^{-1} = L_1 L_2^{-1}$
- **Squeeze theorem:** used to confirm the limit of a difficult to compute function via comparison with two other functions whose limits are easily known or computed.
 - Let \mathcal{I} be an interval having the point c as a limit point.
 - Let g, f , and h be functions defined on \mathcal{I} , except possibly at c itself.
 - Suppose that $\forall x \in \mathcal{I} \wedge x \neq c \Rightarrow g(x) \leq f(x) \leq h(x)$
 - and $\lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} h(x) = L$
 - then, $\lim_{x \rightarrow c} f(x) = L$
 - Essentially, the hard to compute limit of the “middle function” can be found by finding the limit of two other “easier” functions that “squeeze” the middle function at a point of interest.

One-Sided Limit

📌 One-Sided Limit 📌

- **One-sided limit:** one of two limits of $f(x)$ as x approaches a specified point from either the left or from the right.
- From the left: $\lim_{x \rightarrow c^-} f(x) = L$
- From the right: $\lim_{x \rightarrow c^+} f(x) = L$
- If the left and right limits exist and are equal, then

$$\lim_{x \rightarrow c} f(x) = L \Leftrightarrow \lim_{x \rightarrow c^-} f(x) = L \wedge \lim_{x \rightarrow c^+} f(x) = L$$

- Limits can still exist, even if the function is defined at a different point, as long as both one-sided limits approach the same value near the given input.

Continuity

Continuous Functions

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Intermediate Value Theorem

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Limits Involving Infinity

Limits at Infinity and Infinite Limits

-

Asymptotes of functions

-

Derivatives



Derivative Fundamentals

Derivative Notation

- ...

Differentiation Rules

Linear, Product, Chain, Inverse

-

Powers, Polynomials, Quotients, Reciprocals

-

Exponential, Logarithmic

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Trigonometric, Hyperbolic

-

Differentials and Related Concepts

Differentials

-

Linearization

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Implicit Differentiation

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Related Rates

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Applications of Derivatives



Stationary Point

Maxima and Minima

-

Extreme Value Theorem

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Interior Extremum Theorem

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Mean Value Theorem

Rolle's Theorem

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Corollaries of the Mean Value Theorem

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Monotonic Functions

Derivative Tests

First-Derivative Test

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Second-Derivative Test

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Concavity

-

Higher-Order Derivative Test

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Differential Methods

Newton's Method

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Taylor's Theorem

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General Leibniz Rule

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Integrals



Integral Fundamentals

Terminology and Notation

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Primer: Formal Definitions

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Definite Integrals

Riemann Integral

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Integrability

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Properties of Definite Integrals

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The Fundamental Theorem of Calculus

Fundamental Theorem, Part 1

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Fundamental Theorem, Part 2

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The Integral of a Rate

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Total Area

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Integration By Substitution

Indefinite Integrals

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Definite Integrals

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Symmetric Functions

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Area Between Curves

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Applications of Definite Integrals



Solid of Revolution

Disc Integration

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Shell Integration

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Arc Length

Dealing with Discontinuities

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Differential Arc Length

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Surface of Revolution

Revolution about the y-Axis

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Transcendental Functions



Inverse Functions

One-to-One Functions

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Derivative Rule for Inverses

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Logarithmic Functions

Natural Logarithm

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Properties of Logarithms

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Trigonometric Integrals

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Logarithmic Differentiation

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Exponential Functions

Euler's Number

-

Natural Exponential Function

-

Laws of Exponents

-

General Exponential Function

-

Exponential Change

- Separable Differential Equations

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Examples of Exponential Change

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Indeterminate Forms

Indeterminate Form 0/0

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L'Hôpital's Rule

-

Infinite Indeterminate Forms

-

Indeterminate Powers

-

Inverse Trigonometric Functions

Principal Trigonometric Values

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Inverse Trigonometric Tables

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Hyperbolic Functions

Hyperbolic Function Tables

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Techniques of Integration



Integration by Parts

Definite Integrals by Parts

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Trigonometric Integral Methods

Trigonometric Products and Powers

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Trigonometric Square Roots

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Trigonometric Substitutions

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Partial Fraction Decomposition

Partial Fraction Principles

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General Statement

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Numerical Integration

Trapezoidal Rule

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Simpson's Rule

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Improper Integrals

Indirect Evaluation

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Infinite Sequences and Series



First-Order Differential Equations



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Parametric Equations and Polar Coordinates



Vectors and Vector-Valued Functions



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Partial Derivatives



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