

Calculus



Limits and Continuity

Limits	5
Limits of a Functions and Sequences.....	5
Properties of Limits.....	6
One-Sided Limit.....	6
Continuity	7
Continuous Functions	7
Intermediate Value Theorem.....	7
Limits Involving Infinity	8
Limits at Infinity and Infinite Limits	8
Asymptotes of functions.....	8

Derivatives

Derivative Fundamentals	9
Derivative Notation	9
Differentiation Rules	10
Linear, Product, Chain, Inverse	10
Power, Polynomial, Reciprocal, Quotient	11
Trigonometric Differentiation	11
Differentiation Concepts	12
Implicit Differentiation	12
Logarithmic Differentiation	12
Higher Order Derivatives	12
Related Rates.....	12

Applications of Derivatives

Stationary Point	13
Maxima and Minima	13
Extreme Value Theorem	13
Interior Extremum Theorem	13
Mean Value Theorem	14
Rolle's Theorem.....	14
Corollaries of the Mean Value Theorem.....	14
Monotonic Functions	14
Derivative Tests	15
First-Derivative Test.....	15
Second-Derivative Test	15

Concavity	15
Higher-Order Derivative Test	15
Differential Methods	16
Newton's Method	16
Taylor's Theorem	16
General Leibniz Rule	16
Integrals	
Integral Fundamentals	17
Terminology and Notation	17
Primer: Formal Definitions	17
Definite Integrals	18
Riemann Integral	18
Integrability	18
Properties of Definite Integrals.....	18
The Fundamental Theorem of Calculus	19
Fundamental Theorem, Part 1	19
Fundamental Theorem, Part 2	19
The Integral of a Rate.....	19
Total Area.....	19
Integration By Substitution	20
Indefinite Integrals.....	20
Definite Integrals.....	20
Symmetric Functions.....	20
Area Between Curves	20
Applications of Definite Integrals	
Solid of Revolution	21
Disc Integration.....	21
Shell Integration	21
Arc Length	22
Dealing with Discontinuities	22
Differential Arc Length	22
Surface of Revolution	23
Revolution about the y-Axis	23
Transcendental Functions	
Inverse Functions	24
One-to-One Functions	24
Derivative Rule for Inverses	24
Logarithmic Functions	25
Natural Logarithm	25

Properties of Logarithms	25
Trigonometric Integrals	25
Exponential Functions	26
Euler's Number	26
Natural Exponential Function	26
Laws of Exponents	26
General Exponential Function.....	26
Exponential Change	27
Separable Differential Equations	27
Examples of Exponential Change	27
Indeterminate Forms	28
Indeterminate Form 0/0	28
L'Hôpital's Rule.....	28
Infinite Indeterminate Forms.....	28
Indeterminate Powers	28
Inverse Trigonometric Functions	29
Principal Trigonometric Values	29
Inverse Trigonometric Tables	29
Hyperbolic Functions	30
Hyperbolic Function Tables	30
Techniques of Integration	
Integration by Parts	31
Definite Integrals by Parts	31
Trigonometric Integral Methods	32
Trigonometric Products and Powers.....	32
Trigonometric Square Roots.....	32
Trigonometric Substitutions	32
Partial Fraction Decomposition	33
Partial Fraction Principles.....	33
General Statement.....	33
Numerical Integration	34
Trapezoidal Rule	34
Simpson's Rule	34
Improper Integrals	35
Indirect Evaluation	35
First-Order Differential Equations	
Ordinary Differential Equations	36
Solving ODEs	36
Models Involving $y' = k(y-b)$	36

First-Order Methods	37
Euler’s Method.....	37
Slope Fields.....	37
The Logistic Equation.....	37
First-Order Linear Differential Equations	38
Solving LDEs.....	38

Infinite Sequences and Series

Parametric Equations and Polar Coordinates

Vectors and Vector-Valued Functions

Partial Derivatives

Multiple Integrals

Vector Calculus

Second-Order Differential Equations

Limits and Continuity



Limits

🌐 Limit 📖 | Thomas (2.2–2.4) 📖

- **Limit** $\lim_{x \rightarrow c}$: the value of a function (or sequence) as the input (or index) approaches some value (note: an informal definition).
 - Limits are used to define **continuity** ↓, **derivatives** ↓, and **integrals** ↓.

Limits of a Functions and Sequences

🌐 Limit of a function 📖 | Limit of a sequence 📖 | Essence of Calculus, Ea 📖

- **Limit of a function**: a fundamental concept in calculus and analysis concerning the behavior of a function near a particular input c , i.e.,

$$\lim_{x \rightarrow c} f(x) = L$$

- Reads as “ f of x tends to L as x tends to c ”
- ϵ, δ **Limit of a function**: a formalized definition, wherein $f(x)$ is defined on an open interval \mathcal{I} , except possibly at c itself, leading to the informal definition, if and only if

$$f : \mathbb{R} \rightarrow \mathbb{R}, c, L \in \mathbb{R} \Rightarrow \lim_{x \rightarrow c} f(x) = L$$



$$\forall \epsilon > 0 (\exists \delta > 0 : \forall x \in \mathcal{I} (0 < |x - c| < \delta \Rightarrow |f(x) - L| < \epsilon))$$

- Functions **do not have a limit** when the function:
 - has a **unit step**, i.e., it “jumps” at a point;
 - is **not bounded**, i.e., it tends towards infinity;
 - or it **oscillate**, i.e., it does not stay close to any single number.
- **Limit of a sequence**: the value that the terms of a sequence $(x_n)_{n \in \mathbb{N}}$ “tends to” (and not to any other) as n approaches infinity (or some other point), i.e.,

$$\lim_{n \rightarrow \infty} x_n = x$$

- ϵ **Limit of a sequence**: for every measure of closeness ϵ , the sequence’s x_n term eventually converges to the limit, i.e.,

$$\forall \epsilon > 0 (\exists N \in \mathbb{N} (\forall n \in \mathbb{N} (n \geq N \Rightarrow |x_n - x| < \epsilon)))$$

- **Convergent**: when a limit of a sequence **exists**.
- **Divergent**: a sequence that **does not** converge.

Properties of Limits

📌 List of limits 📌 | Squeeze theorem 📌

- **Operations on a single known limit:** if $\lim_{x \rightarrow c} f(x) = L$, then:
 - $\lim_{x \rightarrow c} [f(x) \pm \alpha] = L \pm \alpha$
 - $\lim_{x \rightarrow c} \alpha f(x) = \alpha L$
 - $\lim_{x \rightarrow c} f(x)^{-1} = L^{-1}, L \neq 0$
 - $\lim_{x \rightarrow c} f(x)^n = L^n, n \in \mathbb{N}$
 - $\lim_{x \rightarrow c} f(x)^{n-1} = L^{n-1}, \text{ if } n \in \mathbb{N}_e \implies L > 0$
- **Operations on two known limits:** if $\lim_{x \rightarrow c} f(x) = L_1$ and $\lim_{x \rightarrow c} g(x) = L_2$, then:
 - $\lim_{x \rightarrow c} [f(x) \pm g(x)] = L_1 \pm L_2$
 - $\lim_{x \rightarrow c} [f(x)g(x)] = L_1 L_2$
 - $\lim_{x \rightarrow c} f(x)g(x)^{-1} = L_1 L_2^{-1}$
- **Squeeze theorem:** used to confirm the limit of a difficult to compute function via comparison with two other functions whose limits are easily known or computed.
 - Let \mathcal{I} be an interval having the point c as a limit point.
 - Let g, f , and h be functions defined on \mathcal{I} , except possibly at c itself.
 - Suppose that $\forall x \in \mathcal{I} \wedge x \neq c \implies g(x) \leq f(x) \leq h(x)$
 - and $\lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} h(x) = L$
 - then, $\lim_{x \rightarrow c} f(x) = L$
 - Essentially, the hard to compute limit of the “middle function” can be found by finding the limit of two other “easier” functions that “squeeze” the middle function at a point of interest.

One-Sided Limit

📌 One-Sided Limit 📌

- **One-sided limit:** one of two limits of $f(x)$ as x approaches a specified point from either the left or from the right.
- From the left: $\lim_{x \rightarrow c^-} f(x) = L$
- From the right: $\lim_{x \rightarrow c^+} f(x) = L$
- If the left and right limits exist and are equal, then

$$\lim_{x \rightarrow c} f(x) = L \iff \lim_{x \rightarrow c^-} f(x) = L \wedge \lim_{x \rightarrow c^+} f(x) = L$$

- Limits can still exist, even if the function is defined at a different point, as long as both one-sided limits approach the same value near the given input.

Continuity

🌐 Thomas (2.5) 📖

- Continuity of functions is one of the core concepts of topology, however, there are definitions in terms of limits that prove useful; the following is only a primer.

Continuous Functions

🌐 Continuous function 🔗 | Discontinuities 🔗

- **Continuous function:** a function that does not have any abrupt changes in value.
 - I.e., a function is continuous if and only if arbitrarily small changes in its output can be assured by restricting to sufficiently small changes in its input.
- **Discontinuous:** when a function is not continuous at a point in its domain, leading to a discontinuity; there are three classifications:
 - **Removable:** when both **one-sided limits** [↑] exist, are finite, and are equal, but the actual value of $f(x)$ is not equal to the limit and instead equal to some other value.
 - The discontinuity can be removed to regain continuity.
 - Sometimes the term *removable discontinuity* is mistaken for a *removable singularity*, or a “whole” in the function (the point is not defined elsewhere).
 - **Jump:** when a single limit does not exist because the one-sided limits exist and are finite, but not equal.
 - Points can be defined at the discontinuity, but the function can not be made continuous.
 - **Essential:** when at least one of the two one-sided limits do not exist; can be the result of oscillating or unbounded functions.

Intermediate Value Theorem

🌐 Intermediate value theorem 🔗

- **Intermediate value theorem:** if f is a continuous function whose domain contains the interval $[a, b]$, then it **takes on any given value between $f(a)$ and $f(b)$** at some point within the intervals.
- Relevant deductions, i.e., important corollaries:
 - **Bolzano's theorem:** if a continuous function has values of opposite sign inside an interval, then it **has a root** in that interval.
 - The image of a continuous function over an interval is itself an interval.
- Thus, the image set $f(\mathcal{I})$ (which has no gaps) is also an interval, and it contains:

$$[\min(f(a), f(b)), \max(f(a), f(b))]$$

Limits Involving Infinity

🌐 Thomas (2.6) 📖

- Let $S \subset \mathbb{R}$, $x \in S$ and $f : S \mapsto \mathbb{R}$, then limits of these functions can approach arbitrarily large (\pm) values, providing a connection to asymptotes, and thus, analysis.

Limits at Infinity and Infinite Limits

🌐 Limits involving infinity 🌀

- **Limits at infinity:** limits defined as $f(x) \pm$ infinity are defined much like normal limits:

$$\lim_{x \rightarrow -\infty} f(x) = L \quad \lim_{x \rightarrow \infty} f(x) = L$$

- Formally, for all measures of closeness ε there exists a point c such that $|f(x) - L| < \varepsilon$ whenever $x < c \vee x > c$ (respectively), i.e.,

$$\forall \varepsilon > 0 (\exists c (\forall x \{< \vee >\} c : |f(x) - L| < \varepsilon))$$

- Basic rules for rational functions $f(x) = p(x)q(x)^{-1}$, where p and q are polynomials, where the degree of each is denoted as $\{p \vee q\}^\circ$, and where the leading coefficients are denoted as P, Q , then:

$$\cdot p^\circ > q^\circ \implies \pm L, \text{ depending on the sign of the leading coefficient.}$$

$$\cdot p^\circ = q^\circ \implies L = PQ^{-1}$$

$$\cdot p^\circ < q^\circ \implies L = 0$$

- **Infinite limits:** the usual limit does not exist for a limit that grows out of bounds, however, limits with infinite values can be introduced:

$$\lim_{x \rightarrow c} f(x) = \infty, \quad \text{i.e., } \forall n > 0 (\exists \delta > 0 : f(x) > n \iff 0 < |x - c| < \delta)$$

Asymptotes of functions

🌐 Asymptotes 🌀

- **Asymptote:** a tangent line of a curve at a point at infinity; the distance between the curve and the line approaches zero as a coordinate tends to infinity.
- There are three kinds of asymptotes: *horizontal*, *vertical* and *oblique*; the nature of the asymptote is dependent on a function's relation to infinity.

- **Horizontal asymptote:** a result of **limits at infinity**, i.e., when $x \rightarrow \pm\infty$.

- **Vertical asymptote:** a result of **infinite limits**, i.e., when $x \rightarrow \pm c = \pm\infty$

- **Oblique asymptote:** when a linear asymptote is not parallel to either axis; $f(x)$ is asymptotic to the straight line $y = mx + n$ ($m \neq 0$) if:

$$\lim_{x \rightarrow \pm\infty} [f(x) - (mx + n)] = 0$$

Derivatives



Derivative Fundamentals

🌐 Derivative 🔗 | Thomas (3.2, 3.4) 📖

- **Derivative:** the measure of **sensitivity to change** of the function **value** with respect to some change in its **in argument**.
 - Often described as the **instantaneous rate of change** of a single variable function, since it is the slope of a tangent line at a particular point, when it exists.
 - **Tangent line:** the line through a pair of points on a curve (secant line), except the points are **infinitely close**, hence, it is the rate of change at that “instant”.

Derivative Notation

- Formally, a derivative of the function $f(x)$, with respect to the variable x , is the function f' whose value at x is (provided the limit exists):

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

- Let $z = x + h$, then $h = z - x \wedge h \rightarrow 0 \iff z \rightarrow x$; this leads to an equivalent definition of the derivative (sometimes more convenient):

$$f'(x) = \lim_{z \rightarrow x} \frac{f(z) - f(x)}{z - x}$$

- **Notation:** there are many ways to denote the derivative; different notation can be useful in various contexts, some common notations (for $y = f(x)$):

$$f'(x) = y' = \dot{y} = \frac{dy}{dx} = \frac{d}{dx}f(x) = D(f)(x) = D_x f(x)$$

- **Differentiation:** the process of finding a derivative; if f' exists at a particular point, then f is said to be differentiable at that point.
 - If f' exists at every point on an interval, then f is differentiable on that interval.
 - f' is differentiable on a closed interval $[a, b]$ if both **one-sided limits**[↑] of the function ($h \rightarrow \{0^+:a, 0^-:b\}$) exist at the end points, and it is differentiable on the interior.
 - Not all continuous functions have a derivative, but **functions with a derivative are continuous**; functions with any of the following **do not have derivatives**:
 - **corners** (one-sided derivatives differ at a point),
 - **cusps** (slope approaches alternating $\pm\infty$ on both sides of a point),
 - **discontinuities**, or **vertical tangent lines**.

Differentiation Rules

🌐 Differentiation rules 📖 | Thomas (3.3, 3.5, 3.6) 📖

- Derivatives can be found by computing the limit, but there are several methods that use combinations of simpler functions to make computation easier.

Linear, Product, Chain, Inverse

🌐 Product 📖 | Chain 📖 | Inverse 📖

- Linear:** differentiation of linear functions consists of the constant and sum rules, given the following:

$$\forall (f \wedge g) \wedge \forall (a \wedge b \in \mathbb{R}) \implies \frac{d(af + bg)}{dx} = a \frac{df}{dx} + b \frac{dg}{dx}$$

Constant

$$\frac{d}{dx}(c) = 0$$

Constant factor

$$(af)' = af'$$

Sum / Difference

$$(f + g)' = f' + g'$$

- Product rule:** used for the product of two functions; can be generalized[↓]

$$\frac{d(fg)}{dx} = g \frac{df}{dx} + f \frac{dg}{dx}$$

- Chain rule:** used for the composition of two functions $f(g(x))$; if z depends on y , which is dependent on x , then z depends on x as well, i.e.,

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$$

- The following is used to indicate points of evaluation:

$$\left. \frac{dz}{dx} \right|_x = \left. \frac{dz}{dy} \right|_{y(x)} \cdot \left. \frac{dy}{dx} \right|_x$$

- Outside-Inside rule:** take the derivative of the “outside” function, leave the “inside” alone, and multiply it by the derivative of the “inside.”
- This method must be recursively “chained” when there are further compositions in the inside function, hence the name.
- Inverse function rule:** can be applied if the function f has an inverse function g , i.e., “undoes” the effect of f .

$$\{g(f(x)) = x \wedge f(g(y)) = y\} \implies \frac{dx}{dy} = \left(\frac{dy}{dx} \right)^{-1}$$

- Application of the chain rule on $f^{-1}(y) = x$ in terms of x clearly shows the result, if the derivatives exist and are reciprocal,

$$\frac{dx}{dy} \cdot \frac{dy}{dx} = \frac{dx}{dx} = 1$$

Power, Polynomial, Reciprocal, Quotient

🌐 Power 🌐 | Reciprocal 🌐 | Quotient 🌐

- **Power rule:** used to differentiate functions in the form of $f(x) = x^r$; can be applied to polynomials since differentiation is linear.
 - Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a function satisfying

$$f(x) = x^r, \quad \forall x (r \in \mathbb{R}) \implies \frac{d}{dx} = rx^{r-1}$$

- **Reciprocal rule:** yields the derivative of the reciprocal (multiplicative inverse) of a function f in terms of the derivative of f .
 - Can be used to show that the power rule holds for negative exponents.
 - The product and reciprocal rules can be used to deduce the quotient rule.
 - Let f be differentiable at x and $f(x) \neq 0$, then $g(x) = f(x)^{-1}$ is also differentiable and

$$\frac{d(f^{-1})}{dx} = -f^{-2} \frac{df}{dx} \quad \text{i.e.,} \quad g' = -\frac{f'}{f^2}$$

- **Quotient rule:** used to find the derivative of a function that is a ratio of two differentiable functions.
 - Let f and g be differentiable and $g(x) \neq 0$, then

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

Trigonometric Differentiation

🌐 Trigonometric functions 🌐

- All derivatives of circular trigonometric functions can be found from those of $\sin(x)$ and $\cos(x)$ by means of the quotient rule.

$\sin(x) \rightarrow \cos(x)$	$\arcsin(x) \rightarrow \left(\sqrt{1-x^2}\right)^{-1}$
$\cos(x) \rightarrow -\sin(x)$	$\arccos(x) \rightarrow -\left(\sqrt{1-x^2}\right)^{-1}$
$\tan(x) \rightarrow \sec^2(x)$	$\arctan(x) \rightarrow \left(x^2+1\right)^{-1}$
$\cot(x) \rightarrow -\csc^2(x)$	$\operatorname{arccot}(x) \rightarrow -\left(x^2+1\right)^{-1}$
$\sec(x) \rightarrow \sec(x)\tan(x)$	$\operatorname{arcsec}(x) \rightarrow \left(x \sqrt{x^2-1}\right)^{-1}$
$\csc(x) \rightarrow -\csc(x)\cot(x)$	$\operatorname{arccsc}(x) \rightarrow -\left(x \sqrt{x^2-1}\right)^{-1}$

- Inverse trigonometric functions are found using **implicit differentiation** ↓.

Differentiation Concepts

🌐 Thomas (3.7, 3.8) 📖

Implicit Differentiation

🌐 Implicit differentiation 🔗

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Logarithmic Differentiation

🌐 Logarithmic differentiation 🔗

○

Higher Order Derivatives

🌐 Second derivative 🔗

○

Related Rates

🌐 Related rates 🔗

○

Applications of Derivatives



Stationary Point

Maxima and Minima

-

Extreme Value Theorem

-

Interior Extremum Theorem

-

Mean Value Theorem

Rolle's Theorem

-

Corollaries of the Mean Value Theorem

-

Monotonic Functions

Derivative Tests

First-Derivative Test

-

Second-Derivative Test

-

Concavity

-

Higher-Order Derivative Test

-

Differential Methods

Newton's Method

-

Taylor's Theorem

-

General Leibniz Rule

-

Integrals



Integral Fundamentals

Terminology and Notation

-

Primer: Formal Definitions

-

Definite Integrals

Riemann Integral

-

Integrability

-

Properties of Definite Integrals

-

The Fundamental Theorem of Calculus

Fundamental Theorem, Part 1

-

Fundamental Theorem, Part 2

-

The Integral of a Rate

-

Total Area

-

Integration By Substitution

Indefinite Integrals

-

Definite Integrals

-

Symmetric Functions

-

Area Between Curves

-

Applications of Definite Integrals



Solid of Revolution

Disc Integration

-

Shell Integration

-

Arc Length

Dealing with Discontinuities

-

Differential Arc Length

-

Surface of Revolution

Revolution about the y-Axis

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Transcendental Functions



Inverse Functions

One-to-One Functions

-

Derivative Rule for Inverses

-

Logarithmic Functions

Natural Logarithm

-

Properties of Logarithms

-

Trigonometric Integrals

-

Exponential Functions

Euler's Number

-

Natural Exponential Function

-

Laws of Exponents

-

General Exponential Function

-

Exponential Change

- Separable Differential Equations

-

Examples of Exponential Change

-

Indeterminate Forms

Indeterminate Form 0/0

-

L'Hôpital's Rule

-

Infinite Indeterminate Forms

-

Indeterminate Powers

-

Inverse Trigonometric Functions

Principal Trigonometric Values

-

Inverse Trigonometric Tables

-

Hyperbolic Functions

Hyperbolic Function Tables

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Techniques of Integration



Integration by Parts

Definite Integrals by Parts

○

Trigonometric Integral Methods

Trigonometric Products and Powers

-

Trigonometric Square Roots

-

Trigonometric Substitutions

-

Partial Fraction Decomposition

Partial Fraction Principles

-

General Statement

-

Numerical Integration

Trapezoidal Rule

-

Simpson's Rule

-

Improper Integrals

Indirect Evaluation

-

First-Order Differential Equations



Ordinary Differential Equations

🌐 Differential equations 🔗 | Ordinary DEQ 🔗

- **Solving ODEs**

🌐 Thomas (9.1) 📖 | Rogawski (9.1) 📖

◦

Models Involving $y' = k(y-b)$

🌐 Rogawski (9.2) 📖

◦

First-Order Methods

Euler's Method

🌐 Euler's method [🔗] | Thomas (9.1) [📖]

◦

Slope Fields

🌐 Slope field [🔗] | Rogawski (9.3) [📖]

◦

The Logistic Equation

🌐 Rogawski (9.4) [📖]

◦

First-Order Linear Differential Equations

🌐 Linear differential equation 🔗

Solving LDEs

🌐 Thomas (9.2) 📖 | Rogawski (9.2) 📖

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Infinite Sequences and Series



Parametric Equations and Polar Coordinates



Vectors and Vector-Valued Functions



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Partial Derivatives



Multiple Integrals



Vector Calculus



Second-Order Differential Equations

