Calculus

Limits and Continuity

Limits 5

Limits of a Functions and Sequences 5
Properties of Limits 6
One-Sided Limit 6

Continuity 7

Continuous Functions 7
Intermediate Value Theorem 7

Limits Involving Infinity 8

Limits at Infinity and Infinite Limits 8
Asymptotes of functions 8

Derivatives

Derivative Fundamentals 9

Derivative Notation 9

Differentiation Rules 10

Linear, Product, Chain, Inverse 10
Power, Polynomial, Reciprocal, Quotient 11
Trigonometric Differentiation 11

Differentiation Concepts 12

Implicit Differentiation 12
Logarithmic Differentiation 12
Higher Order Derivatives 12
Related Rates 12

Applications of Derivatives

Stationary Point 13

Maxima and Minima 13
Extreme Value Theorem 13
Interior Extremum Theorem 13

Mean Value Theorem 14

Rolle's Theorem 14
Corollaries of the Mean Value Theorem 14
Monotonic Functions 14

Derivative Tests 15

First-Derivative Test 15
Second-Derivative Test 15

Concavity 15
Higher-Order Derivative Test 15

Differential Methods **16**
Newton's Method 16
Taylor's Theorem 16
General Leibniz Rule 16

Integrals

Integral Fundamentals **17**
Terminology and Notation 17
Primer: Formal Definitions 17

Definite Integrals **18**
Riemann Integral 18
Integrability 18
Properties of Definite Integrals 18

The Fundamental Theorem of Calculus **19**
Fundamental Theorem, Part 1 19
Fundamental Theorem, Part 2 19
The Integral of a Rate 19
Total Area 19

Integration By Substitution **20**
Indefinite Integrals 20
Definite Integrals 20
Symmetric Functions 20
Area Between Curves 20

Applications of Definite Integrals

Solid of Revolution **21**
Disc Integration 21
Shell Integration 21

Arc Length **22**
Dealing with Discontinuities 22
Differential Arc Length 22

Surface of Revolution **23**
Revolution about the y-Axis 23

Transcendental Functions

Inverse Functions **24**
One-to-One Functions 24
Derivative Rule for Inverses 24

Logarithmic Functions **25**
Natural Logarithm 25

Properties of Logarithms . . . 25
Trigonometric Integrals . . . 25
Exponential Functions **26**
Euler's Number . . . 26
Natural Exponential Function . . . 26
Laws of Exponents . . . 26
General Exponential Function . . . 26
Exponential Change **27**
Separable Differential Equations . . . 27
Examples of Exponential Change . . . 27
Indeterminate Forms **28**
Indeterminate Form 0/0 . . . 28
L'Hôpital's Rule . . . 28
Infinite Indeterminate Forms . . . 28
Indeterminate Powers . . . 28
Inverse Trigonometric Functions **29**
Principal Trigonometric Values . . . 29
Inverse Trigonometric Tables . . . 29
Hyperbolic Functions **30**
Hyperbolic Function Tables . . . 30

Techniques of Integration

Integration by Parts **31**
Definite Integrals by Parts . . . 31
Trigonometric Integral Methods **32**
Trigonometric Products and Powers . . . 32
Trigonometric Square Roots . . . 32
Trigonometric Substitutions . . . 32
Partial Fraction Decomposition **33**
Partial Fraction Principles . . . 33
General Statement . . . 33
Numerical Integration **34**
Trapezoidal Rule . . . 34
Simpson's Rule . . . 34
Improper Integrals **35**
Indirect Evaluation . . . 35

Infinite Sequences and Series

First-Order Differential Equations

Parametric Equations and Polar Coordinates

Vectors and Vector-Valued Functions

Partial Derivatives

Multiple Integrals

Vector Calculus

Second-Order Differential Equations

Limits and Continuity

Limits

Limit | Thomas (2.2–2.4)

- **Limit** $\lim\limits_{x \to c}$: the value of a function (or sequence) as the input (or index) approaches some value (note: an informal definition).
  - Limits are used to define continuity, derivatives, and integrals.

Limits of a Functions and Sequences

Limit of a function | Limit of a sequence | Essence of Calculus, Ea

- **Limit of a function**: a fundamental concept in calculus and analysis concerning the behavior of a function near a particular input c, i.e.,

$$\lim_{x \to c} f(x) = L$$

  - Reads as "f of x tends to L as x tends to c"

- ϵ, δ **Limit of a function**: a formalized definition, wherein $f(x)$ is defined on an open interval $\mathcal{I}$, except possibly at c itself, leading to the informal definition, if and only if

$$f : \mathbb{R} \to \mathbb{R}, c, L \in \mathbb{R} \Rightarrow \lim_{x \to c} f(x) = L$$

$$\Updownarrow$$

$$\forall \varepsilon > 0 \left(\exists \delta > 0 : \forall x \in \mathcal{I} \left(0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon\right)\right)$$

- Functions do not have a limit when the function:
  - has a unit step, i.e., it "jumps" at a point;
  - is not bounded, i.e., it tends towards infinity;
  - or it oscillate, i.e., it does not stay close to any single number.

- **Limit of a sequence**: the value that the terms of a sequence $(x_n)_{n \in \mathbb{N}}$ "tends to" (and not to any other) as n approaches infinity (or some other point), i.e.,

$$\lim_{n \to \infty} x_n = x$$

- $\mathcal{E}$ **Limit of a sequence**: for every measure of closeness $\mathcal{E}$, the sequence's x_n term eventually converges to the limit, i.e.,

$$\forall \varepsilon > 0 \left(\exists N \in \mathbb{N} \left(\forall n \in \mathbb{N} \left(n \geq N \Rightarrow |x_n - x| < \varepsilon\right)\right)\right)$$

  - **Convergent**: when a limit of a sequence exists.
  - **Divergent**: a sequence that does not converge.

Properties of Limits

List of limits | Squeeze theorem

- **Operations on a single known limit**: if $\lim_{x\to c} f(x) = L$, then:
  - $\lim_{x\to c} \left[f(x) \pm \alpha\right] = L \pm \alpha$
  - $\lim_{x\to c} \alpha f(x) = \alpha L$
  - $\lim_{x\to c} f(x)^{-1} = L^{-1}, L \neq 0$
  - $\lim_{x\to c} f(x)^{n} = L^{n}, n \in \mathbb{N}$
  - $\lim_{x\to c} f(x)^{n^{-1}} = L^{n^{-1}}$, if $n \in \mathbb{N}_e \Rightarrow L > 0$
- **Operations on two known limits**: if $\lim_{x\to c}$ and $\lim_{x\to c} g(x) = L_2$, then:
  - $\lim_{x\to c} \left[f(x) \pm g(x)\right] = L_1 \pm L_2$
  - $\lim_{x\to c} [f(x)g(x)] = L_1 L_2$
  - $\lim_{x\to c} f(x)g(x)^{-1} = L_1 L_2^{-1}$
- **Squeeze theorem**: used to confirm the limit of a difficult to compute function via comparison with two other functions whose limits are easily known or computed.
  - Let $\mathcal{I}$ be an interval having the point c as a limit point.
  - Let g, f, and h be functions defined on $\mathcal{I}$, except possibly at c itself.
  - Suppose that $\forall x \in \mathcal{I} \wedge x \neq \ \Rightarrow g(x) \leq f(x) \leq h(x)$
  - and $\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$
  - then, $\lim_{x\to c} f(x) = L$
  - Essentially, the hard to compute limit of the "middle function" can be found by finding the limit of two other "easier" functions that that "squeeze" the middle function at a point of interest.

One-Sided Limit

One-Sided Limit

- **One-sided limit**: one of two limits of $f(x)$ as x approaches a specified point from either the left or from the right right.
  - From the left: $\lim_{x\to c^-} = L$
  - From the right: $\lim_{x\to c^+} = L$
- If the left and right limits exist and are equal, then

$$\lim_{x\to c} f(x) = L \Leftrightarrow \lim_{x\to c^-} f(x) = L \wedge \lim_{x\to c^+} f(x) = L$$

- Limits can still exist, even if the function is defined at a different point, as long as both one-sided limits approach the same value near the given input.

Continuity

Thomas (2.5)

- Continuity of functions is one of the core concepts of topology, however, there are definitions in terms of limits that prove useful; the following is only a primer.

Continuous Functions

Continuous function | Discontinuities

- **Continuous function**: a function that does not have any abrupt changes in value.
  - I.e., a function is continuous if and only if arbitrarily small changes in its output can be assured by restricting to sufficiently small changes in its input.
- **Discontinuous**: when a function is not continuous at a point in its domain, leading to a discontinuity; there are three classifications:
  - **Removable**: when both one-sided limits↑ exist, are finite, and are equal, but the actual value of $f(x)$ is not equal to the limit and instead equal to some other value.
    - The discontinuity can be removed to regain continuity.
    - Sometimes the term *removable discontinuity* is mistaken for a *removable singularity*, or a "whole" in the function (the point is not defined elsewhere).
  - **Jump**: when a single limit does not exist because the one-sided limits exist and are finite, but not equal.
    - Points can be defined at the discontinuity, but the function can not be made continuous.
  - **Essential**: when at least one of the two one-sided limits do not exist; can be the result of oscillating or unbounded functions.

Intermediate Value Theorem

Intermediate value theorem

- **Intermediate value theorem**: if f is a continuous function whose domain contains the interval $[a, b]$, then it takes on any given value between $f(a)$ and $f(b)$ at some point within the intervals.
- Relevant deductions, i.e., important corollaries:
  - **Bolzano's theorem**: if a continuous function has values of opposite sign inside an interval, then it has a root in that interval.
  - The image of a continuous function over an interval is itself an interval.
- Thus, the image set $f(\mathcal{I})$ (which has no gaps) is also an interval, and it contains:

$$[\min(f(a), f(b)), \max(f(a), f(b))]$$

Limits Involving Infinity

Thomas (2.6)

- Let $S \subset \mathbb{R}$, $x \in S$ and $f : S \mapsto \mathbb{R}$, then limits of these functions can approach arbitrarily large $(\pm)$ values, providing a connection to asymptotes, and thus, analysis.

Limits at Infinity and Infinite Limits

Limits involving infinity

- **Limits at infinity**: limits defined as $f(x) \pm$ infinity are defined much like normal limits:

$$\lim_{x \to -\infty} f(x) = L \qquad \lim_{x \to \infty} f(x) = L$$

  - Formally, for all measures of closeness ε there exists a point c such that $|f(x) - L| < \varepsilon$ whenever $x < c \vee x > c$ (respectively), i.e.,

$$\forall \varepsilon > 0\, (\exists c\, (\forall x \{< \vee >\} c : |f(x) - L| < \varepsilon))$$

  - Basic rules for rational functions $f(x) = p(x)q(x)^{-1}$, where p and q are polynomials, where the degree of each is denoted as $\{p \vee q\}^{\circ}$, and where the leading coefficients are denoted as P, Q, then:
    - $p^{\circ} > q^{\circ} \Rightarrow \pm L$, depending on the sign of the leading coefficient.
    - $p^{\circ} = q^{\circ} \Rightarrow L = PQ^{-1}$
    - $p^{\circ} < q^{\circ} \Rightarrow L = 0$
- **Infinite limits**: the usual limit does not exist for a limit that grows out of bounds, however, limits with infinite values can be introduced:

$$\lim_{x \to c} f(x) = \infty, \quad \text{i.e.,} \quad \forall n > 0\, (\exists \delta > 0 : f(x) > n \Leftrightarrow 0 < |x - c| < \delta)$$

Asymptotes of functions

Asymptotes

- **Asymptote**: a tangent line of a curve at a point at infinity; the distance between the curve and the line approaches zero as a coordinate tends to infinity.
- There are three kinds of asymptotes: *horizontal, vertical* and *oblique*; the nature of the asymptote is dependent on a function's relation to infinity.
  - **Horizontal asymptote**: a result of limits at infinity, i.e., when $x \to \pm\infty$.
  - **Vertical asymptote**: a result of infinite limits, i.e., when $x \to \pm c = \pm\infty$
  - **Oblique asymptote**: when a linear asymptote is not parallel to either axis; $f(x)$ is asymptotic to the straight line $y = mx + n\ (m \neq 0)$ if:

$$\lim_{x \to \pm\infty} [f(x) - (mx + n)] = 0$$

Derivatives

Derivative Fundamentals

Derivative | Thomas (3.2, 3.4)

- **Derivative**: the measure of sensitivity to change of the function value with respect to some change in its in argument.
  - Often descried as the instantaneous rate of change of a single variable function, since it is the slope of a tangent line at a particular point, when it exists.
    - **Tangent line**: the line through a pair of points on a curve (secant line), except the points are infinitely close, hence, it is the rate of change at that "instant".

Derivative Notation

- Formally, a derivative of the function $f(x)$, with respect to the variable x, is the function f' whose value at x is (provided the limit exists):

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

  - Let $z = x + h$, then $h = z - x \wedge h \to 0 \Leftrightarrow z \to x$; this leads to an equivalent definition of the derivative (sometimes more convenient):

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}$$

- **Notation**: there are many ways to denote the derivative; different notation can be useful in various contexts, some common notations (for $y = f(x)$):

$$f'(x) = y' = \dot{y} = \frac{dy}{dx} = \frac{d}{dx} f(x) = D(f)(x) = D_x f(x)$$

- **Differentiation**: the process of finding a derivative; if f' exists at a particular point, then f is said to be differentiable at that point.
  - If f' exists at every point on an interval, then f is differentiable on that interval.
  - f' is differentiable on a closed interval $[a, b]$ if both one-sided limits↑ of the function ($h \to \{0^+ : a, 0^- : b\}$) exist at the end points, and it is differentiable on the interior.
  - Not all continuous functions have a derivative, but functions with a derivative are continuous; functions with any of the following do not have derivatives:
    - corners (one-sided derivatives differ at a point),
    - cusps (slope approaches alternating $\pm\infty$ on both sides of a point),
    - discontinuities, or vertical tangent lines.

Differentiation Rules

Differentiation rules | Thomas (3.3, 3.5, 3.6)

- Derivatives can be found by computing the limit, but there are several methods that use combinations of simpler functions to make computation easier.

Linear, Product, Chain, Inverse

Product | Chain | Inverse

- **Linear**: differentiation of linear functions consists of the constant and sum rules, given the following:

$$\forall(f \wedge g) \wedge \forall(a \wedge b \in \mathbb{R}) \Rightarrow \frac{d(af + bg)}{dx} = a\frac{df}{dx} + b\frac{dg}{dx}$$

Constant	Constant factor	Sum / Difference
$\frac{d}{dx}(c) = 0$	$(af)' = af'$	$(f+g)' = f' + g'$

- **Product rule**: used for the product of two functions; can be generalized ↓

$$\frac{d(fg)}{dx} = g\frac{df}{dx} + f\frac{dg}{dx}$$

- **Chain rule**: used for the composition of two functions $f(g(x))$; if z depends on y, which is dependent on x, then z depends on x as well, i.e.,

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$$

  - The following is used to indicate points of evaluation:

$$\left.\frac{dz}{dx}\right|_{x} = \left.\frac{dz}{dy}\right|_{y(x)} \cdot \left.\frac{dy}{dx}\right|_{x}$$

  - **Outside-Inside rule**: take the derivative of the “outside” function, leave the “inside” alone, and multiply it by the derivative of the “inside.”
  - This method must be recursively “chained” when there are further compositions in the inside function, hence the name.

- **Inverse function rule**: can be applied if the function f has an inverse function g, i.e., “undoes” the effect of f.

$$\{g(f(x)) = x \wedge f(g(y)) = y\} \Rightarrow \frac{dx}{dy} = \left(\frac{dy}{dx}\right)^{-1}$$

  - Application of the chain rule on $f^{-1}(y) = x$ in terms of x clearly shows the result, if the derivatives exist and are reciprocal,

$$\frac{dx}{dy} \cdot \frac{dy}{dx} = \frac{dx}{dx} = 1$$

Power, Polynomial, Reciprocal, Quotient

Power | Reciprocal | Quotient

- **Power rule**: used to differentiate functions in the form of $f(x) = x^r$; can be applied to polynomials since differentiation is linear.
  - Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a function satisfying

$$f(x) = x^r, \quad \forall x \, (r \in \mathbb{R}) \Rightarrow \frac{d}{dx} = rx^{r-1}$$

- **Reciprocal rule**: yields the derivative of the reciprocal (multiplicative inverse) of a function f in terms of the derivative of f.
  - Can be used to show that the power rule holds for negative exponents.
  - The product and reciprocal rules can be used to deduce the quotient rule.
  - Let f be differentiable at x and $f(x) \neq 0$, then $g(x) = f(x)^{-1}$ is also differentiable and

$$\frac{d(f^{-1})}{dx} = -f^{-2}\frac{df}{dx} \quad \text{i.e.,} \quad g' = -\frac{f'}{f^2}$$

- **Quotient rule**: used to find the derivative of a function that is a ratio of two differentiable functions.
  - Let f and g be differentiable and $g(x) \neq 0$, then

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

Trigonometric Differentiation

Trigonometric functions

- All derivatives of circular trigonometric functions can be found from those of $\sin(x)$ and $\cos(x)$ by means of the quotient rule.

$$\begin{aligned}
\sin(x) &\to \cos(x) & \arcsin(x) &\to \left(\sqrt{1-x^2}\right)^{-1} \\
\cos(x) &\to -\sin(x) & \arccos(x) &\to -\left(\sqrt{1-x^2}\right)^{-1} \\
\tan(x) &\to \sec^2(x) & \arctan(x) &\to \left(x^2+1\right)^{-1} \\
\cot(x) &\to -\csc^2(x) & \operatorname{arccot}(x) &\to -\left(x^2+1\right)^{-1} \\
\sec(x) &\to \sec(x)\tan(x) & \operatorname{arcsec}(x) &\to \left(|x|\sqrt{x^2-1}\right)^{-1} \\
\csc(x) &\to -\csc(x)\cot(x) & \operatorname{arccsc}(x) &\to -\left(|x|\sqrt{x^2-1}\right)^{-1}
\end{aligned}$$

- Inverse trigonometric functions are found using implicit differentiation↓ .

Differentiation Concepts

Thomas (3.7, 3.8)

Implicit Differentiation

Implicit differentiation

-

Logarithmic Differentiation

Logarithmic differentiation

-

Higher Order Derivatives

Second derivative

-

Related Rates

Related rates

-

Applications of Derivatives

Stationary Point

Maxima and Minima

-

Extreme Value Theorem

-

Interior Extremum Theorem

-

Mean Value Theorem

Rolle’s Theorem

- ○

Corollaries of the Mean Value Theorem

- ○

Monotonic Functions

Derivative Tests

First-Derivative Test

-

Second-Derivative Test

-

Concavity

-

Higher-Order Derivative Test

-

Differential Methods

Newton’s Method

-

Taylor’s Theorem

-

General Leibniz Rule

-

Integrals

Integral Fundamentals

Terminology and Notation

-

Primer: Formal Definitions

-

Definite Integrals

Riemann Integral

-

Integrability

-

Properties of Definite Integrals

-

The Fundamental Theorem of Calculus

Fundamental Theorem, Part 1

-

Fundamental Theorem, Part 2

-

The Integral of a Rate

-

Total Area

-

Integration By Substitution

Indefinite Integrals

-

Definite Integrals

-

Symmetric Functions

-

Area Between Curves

-

Applications of Definite Integrals

Solid of Revolution

Disc Integration

-

Shell Integration

-

Arc Length

Dealing with Discontinuities

-

Differential Arc Length

-

Surface of Revolution

Revolution about the y-Axis

-

Transcendental Functions

Inverse Functions

One-to-One Functions

-

Derivative Rule for Inverses

-

Logarithmic Functions

Natural Logarithm

-

Properties of Logarithms

-

Trigonometric Integrals

-

Exponential Functions

Euler’s Number

- ○

Natural Exponential Function

- ○

Laws of Exponents

- ○

General Exponential Function

- ○

Exponential Change

- **Separable Differential Equations**
    -

  Examples of Exponential Change

    -

Indeterminate Forms

Indeterminate Form 0/0

- ◦

L'Hôpital's Rule

- ◦

Infinite Indeterminate Forms

- ◦

Indeterminate Powers

- ◦

Inverse Trigonometric Functions

Principal Trigonometric Values

-

Inverse Trigonometric Tables

-

Hyperbolic Functions

Hyperbolic Function Tables

-

Integration by Parts

Definite Integrals by Parts

-

Trigonometric Integral Methods

Trigonometric Products and Powers

-

Trigonometric Square Roots

-

Trigonometric Substitutions

-

Partial Fraction Decomposition

Partial Fraction Principles

-

General Statement

-

Numerical Integration

Trapezoidal Rule

-

Simpson’s Rule

-

Improper Integrals

Indirect Evaluation