

Calculus III Midterm Review (Solutions)



1. True or false? (with justification)

(a) If a_n is bounded, then it converges.

✗ **False**; sequences can oscillate between values, yielding a bounded sequence that does not converge to a consistent value. E.g., $a_n = (-1)^n$

(b) If a_n converges, then it must be bounded.

✓ **True**; by *THEOREM 5: Convergent Sequences are Bounded*; if a_n converges, then a_n is bounded.

(c) If a_n is not bounded, then it diverges.

✓ **True**; inverse of (b); sequences must be bounded in order to converge, if they aren't then they diverge.

(d) If a_n diverges, then it is not bounded.

✗ **False**; inverse of (a); bounded sequences can diverge. Another example is $a_n = \cos \pi n$

2. A hot anvil with cooling constant $k = 0.02 \text{ s}^{-1}$ is submerged in a large pool of water whose temperature is 10°C . Let $y(t)$ be the anvil's temperature t seconds later.

(a) What is the differential equation satisfied by $y(t)$?

$$\text{Newton's Law of Cooling: } y' = -k(y - T_0)$$

$y(t)$ = temp. of object, T_0 = ambient temp., k = cooling constant.

$$k = 0.02, \quad T_0 = 10 \implies \boxed{y' = -0.02(y - 10)}$$

(b) Find a formula for $y(t)$, assuming the object initial temperature is 100°C .

$$\text{General Solution of } y' = -k(y - b) \rightarrow y = b + Ce^{-kt}$$

$$b = 10, \quad k = 0.02, \implies y = 10 + Ce^{-0.02t}$$

$$P(0) = 100, \implies 100 = 10 + Ce^0$$

$$\implies C = 90$$

$$\implies \boxed{y = 10 + 90e^{-0.02t}}$$

3. As an object cools, its rate of cooling slows. Explain how this follows from Newton's Law of Cooling.

- The difference in temperature between a cooling object and the ambient temperature is decreasing. Hence, the rate of cooling, which is proportional to this difference, is also decreasing in magnitude.

4. In Yellowstone park there were approximately 500 bison in 1970 and 3,000 bison in 1990.

(a) Using the model that the rate of change of the population is proportional to the population itself, set up (do not solve) an initial value problem to model this situation.

Given: $P(t)$ = population of bison at time t

$$1970 \rightarrow t = 0$$

$$P' \propto P$$

$$\implies \boxed{P' = kP \quad P(0) = 500 \wedge P(20) = 3,000}$$

(b) Now find the particular solution satisfying your initial value problem.

Given: $P' = kP$, $y = Ce^{kt}$ (general solution to $y' = ky$)

$$\implies P = Ce^{kt}$$

$$P(0) = 500 \implies 500 = Ce^0$$

$$\implies C = 500, \quad P = 500e^{kt}$$

$$P(3,000) = 500e^{20k} \implies k = \left(\ln \frac{3000/500}{20}\right) \approx 0.0896$$

$$\implies \boxed{P = 500e^{0.0896t}}$$

5. Which of the following are first-order linear equations?

(a) $y' + x^2y = 1$ ✓ true

(c) $x^5y' + y = e^x$ ✓ true

(b) $y' + xy^2 = 1$ ✗ false

(d) $x^5y' + y = e^y$ ✗ false

6. For what function P is the integrating factor $\alpha(x)$ equal to x ? What about e^x ?

$$\boxed{P(x) = x^{-1}} \rightarrow \alpha(x) = x$$

$$\boxed{P(x) = 1} \rightarrow \alpha(x) = e^x$$

7. Find the limit of the sequence $a_n = \frac{n+1}{3n+2}$. Be sure to justify your answer.

Using the Limit Laws:

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{n+1}{3n+2} &= \lim_{x \rightarrow \infty} \frac{x+1}{3x+2} = \lim_{x \rightarrow \infty} \frac{x+1}{3x+2} \cdot \frac{x^{-1}}{x^{-1}} \\ &= \lim_{x \rightarrow \infty} \frac{1+x^{-1}}{3+x^{-1}} \\ &= \lim_{x \rightarrow \infty} \frac{1}{3} + \lim_{x \rightarrow \infty} \frac{x^{-1}}{x^{-1}} \\ &= \frac{1}{3} + \frac{0}{0} = \boxed{\frac{1}{3}}\end{aligned}$$

Using L'Hôpital Rule:

$$f(x) = n+1, \quad g(x) = 3n+2$$

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} &= \frac{\infty}{\infty} \\ \implies \frac{f'(x)}{g'(x)} &= \boxed{\frac{1}{3}}\end{aligned}$$

8. Which of the following sequences converge to zero?

$$\frac{n^2}{n^2+1} \quad 2^n \quad \boxed{\left(-\frac{1}{2}\right)^n}$$

9. Which of the following sequences is defined recursively?

$$a_n = \sqrt{4+n} \quad \boxed{b_n = \sqrt{4+b_{n-1}}}$$

10. Let a_n be the n^{th} decimal approximation to $\sqrt{2}$. I.e., $a_1 = 1$, $a_2 = 1.4$, $a_3 = 1.41$ and so on. What is $\lim_{n \rightarrow \infty} a_n$?

$$\boxed{\lim_{n \rightarrow \infty} a_n = \sqrt{2}}$$

11. Biologists stocked a lake with 400 fish and estimated the carrying capacity to be 10,000. The number of fish tripled in the first year. Assuming the size of the fish population satisfies the logistic equation, find an expression for the size of the population after t years.

- The **logistic equation** and its general non-equilibrium solution ($k > 0 \wedge A > 0$)

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{A}\right) \rightarrow y = \frac{A}{1 - e^{-kt}/B} \equiv \frac{y}{y - A} = Be^{kt}$$

Given:

$$y(0) = 400, \quad y(1) = 1200, \quad A = 10,000$$

$$\frac{400}{400 - 10,000} = Be^0$$

$$B = -\frac{1}{24}$$

$$\Rightarrow y = \frac{10,000}{1 - e^{-kt}/-\frac{1}{24}} = \frac{10,000}{1 + 24e^{-kt}}$$

$$1,200 = \frac{10,000}{1 + 24e^{-k}}$$

$$1 + 24e^{-k} = \frac{10,000}{1,200}$$

$$e^{-k} = \frac{22}{3} \cdot \frac{1}{24} = \frac{2 \cdot 11}{3 \cdot 2 \cdot 12} = \frac{11}{36}$$

$$k = -\ln\left(\frac{11}{36}\right) \approx 1.186$$

$$\Rightarrow \boxed{y = \frac{10,000}{1 + 24e^{-1.186t}}}$$

12. Find the general term, a_n , for the sequence given below. Assume that we start our sequence at $n = 1$.

$$1, \frac{4}{2}, \frac{9}{4}, \frac{16}{8}, \frac{25}{16}, \frac{36}{32} \dots$$

$$\Rightarrow \frac{1^2}{2^{1-1}}, \frac{2^2}{2^{2-1}}, \frac{3^2}{2^{3-1}}, \frac{4^2}{2^{4-1}}, \frac{5^2}{2^{5-1}}, \frac{6^2}{2^{6-1}}, \dots, \frac{n^2}{2^{n-1}}$$

$$\Rightarrow \boxed{a_n = \frac{n^2}{2^{n-1}}}$$

13. Determine whether each of the following series converge or diverge by using either the Comparison Test or the Limit Comparison Test to justify your answer.

(a) $\sum_{n=1}^{\infty} \frac{1}{e^n + n^2}$

$$e^n + n^2 \geq n^2$$

$$\frac{1}{e^n + n^2} \leq \frac{1}{n^2}$$

$$a_n = \frac{1}{e^n + n^2}, \quad b_n = \frac{1}{n^2}$$

Using direct comparison test: if b_n converges, then a_n converges. We know $\frac{1}{n^2}$ converges by the p -series, since $p > 1$. Thus, a_n converges.

(b) $\sum_{n=1}^{\infty} \frac{1}{e^n - n^2}$

$$\text{Let } a_n = \frac{1}{e^n - n^2}, \quad b_n = \frac{1}{e^n}$$

$$\begin{aligned} \Rightarrow \lim_{n \rightarrow \infty} \frac{a_n}{b_n} &= \frac{e^n}{e^n - n^2} \\ &= \lim_{n \rightarrow \infty} \frac{e^n}{e^n - n^2} \cdot \frac{e^{-n}}{e^{-n}} \\ &= \frac{1}{1 - n^2 e^{-n}} = 1 \end{aligned}$$

Thus, since $L > 0$, and $\sum b_n$ converges, then $\sum a_n$ converges.

23. Water is draining from a cylindrical tank with cross sectional area $4m^2$ and height $5m$. Torricelli's Law says that the rate of change of the height of the water in such a cylindrical tank is proportional to the square root of the height of the water in the tank. Suppose that the tank starts full of water and after 30 minutes the height of the water has decreased to $4m$. Set up an initial value problem to model this situation.

24. Find the limit of each of the following sequences. Justify your answer using limit laws, the squeeze theorem, and/or L'Hospital's Rule.

(a) $a_n = \frac{2(-1)^{n+1}}{2n-1}$

(b) $a_n = \frac{n^2}{2^{n-1}}$

25. Does the series $\sum_{n=1}^{\infty} \cos\left(\frac{1}{n}\right)$ converge or diverge? How do you know?

26. Find the sum of each of the following geometric series or state that it diverges. Be sure to explain how you arrived at your solution.

(a) $\frac{1}{6} + \frac{1}{12} + \frac{1}{24} + \frac{1}{48} + \frac{1}{96} + \cdots$

(b) $-2 + \frac{2}{5} - \frac{2}{25} + \frac{2}{125} - \frac{2}{625} + \cdots$

27. What role do partial sums play in defining the sum of an infinite series?

28. Indicate whether or not the reasoning in the following statements are correct:

(a) $\sum_{n=1}^{\infty} \frac{1}{n^2} = 0$ because $\frac{1}{n^2}$ tends to zero.

(b) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ because $\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} = 0$

29. Find an N such that $S_N > 25$ for the series $\sum_{n=1}^{\infty} 2$.

30. Does there exist an N such that $S_N > 25$ for the series $\sum_{n=1}^{\infty} 2^{-n}$? Explain.

31. Give an example of a divergent infinite series whose general term tends to zero.

32. For the series $\sum_{n=1}^{\infty} a_n$, if the partial sums S_N are increasing, then (choose the correct conclusion):

(a) a_n is an increasing sequence.

(b) a_n is a positive sequence.

33. What are the hypotheses of the Integral Test?

34. Which test would you use to determine whether the following series converge?

(a) $\sum_{n=1}^{\infty} n^{-3.2}$

(b) $\sum_{n=1}^{\infty} \frac{1}{2^n + \sqrt{n}}$