

Calculus



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Limits and Continuity



Limits

🌐 Limit 📖 | Thomas's Calculus^{14th} (2.2–2.4) 📖

- **Limit** $\lim_{x \rightarrow c}$: the value of a function (or sequence) approaches as the input (or index) approaches some value (informal definition).
 - Limits are used to define **continuity** ↓, **derivatives** ↓, and **integrals** ↓.

Limits of a Functions and Sequences

🌐 Limit of a function 📖 | Limit of a sequence 📖 | Essence of Calculus, E7 📖

- **Limit of a function**: a fundamental concept in calculus and analysis concerning the behavior of a function near a particular input c , i.e.,

$$\lim_{x \rightarrow c} f(x) = L$$

- Reads as “ f of x tends to L as x tends to c ”
- ϵ, δ **Limit of function**: a formalized definition, wherein $f(x)$ is defined on an open interval \mathcal{I} , except possibly at c itself, leading to above definition, if and only if:
 - For every real measure of **closeness** $\epsilon > 0$, there exists a real **corresponding** $\delta > 0$, such that for all existing further approaches there exist a smaller ϵ , i.e.,

$$f : \mathbb{R} \rightarrow \mathbb{R}, c, L \in \mathbb{R} \Rightarrow \lim_{x \rightarrow c} f(x) = L$$



$$\forall \epsilon > 0 (\exists \delta > 0 : \forall x \in \mathcal{I} (0 < |x - c| < \delta \Rightarrow |f(x) - L| < \epsilon))$$

- Functions **do not have** a limit when the function:
 - has a **unit step**, i.e., it “jumps” at a point;
 - is **not bounded**, i.e., it tends towards infinity;
 - or it **oscillates**, i.e., does not stay close to any single number.
- **Limit of a sequence**: the value that the terms of a sequence (x_n) “tends to” (and not to any other) as n approaches infinity (or some point), i.e.,

$$\lim_{n \rightarrow \infty} x_n = x$$

- ϵ **Limit of sequence**: for every measure of closeness ϵ , the sequence's x_n term eventually converge to the limit, i.e.,

$$\forall \epsilon > 0 (\exists N \in \mathbb{N} (\forall n \in \mathbb{N} (n \geq N \Rightarrow |x_n - x| < \epsilon)))$$

- **Convergent**: when a limit of a sequence **exists**.
- **Divergent**: a sequence that **does not** converge.

Properties of Limits

🌐 List of limits 🌐 | Squeeze theorem 🌐

- **Operations on a single known limit:** if $\lim_{x \rightarrow c} f(x) = L$ then:
 - $\lim_{x \rightarrow c} [f(x) \pm \alpha] = L \pm \alpha$
 - $\lim_{x \rightarrow c} \alpha f(x) = \alpha L$
 - $\lim_{x \rightarrow c} f(x)^{-1} = L^{-1}, L \neq 0$
 - $\lim_{x \rightarrow c} f(x)^n = L^n, n \in \mathbb{N}$
 - $\lim_{x \rightarrow c} f(x)^{n^{-1}} = L^{n^{-1}}, n \in \mathbb{N}, \text{ if } n \in \mathbb{N}_e \Rightarrow L > 0$
- **Operations on two known limits:** if $\lim_{x \rightarrow c} f(x) = L_1$ and $\lim_{x \rightarrow c} g(x) = L_2$
 - $\lim_{x \rightarrow c} [f(x) \pm g(x)] = L_1 \pm L_2$
 - $\lim_{x \rightarrow c} [f(x)g(x)] = L_1 L_2$
 - $\lim_{x \rightarrow c} f(x)g(x)^{-1} = L_1 L_2^{-1}$
- **Squeeze theorem:** used to confirm the limit of a difficult to compute function via comparison with two other functions whose limits are easily known or computed.
 - Let \mathcal{I} be an interval having the point c as a limit point.
 - Let g, f , and h , be functions defined on \mathcal{I} , except possibly at c itself.
 - Suppose that $\forall x \in \mathcal{I} \wedge x \neq c \Rightarrow g(x) \leq f(x) \leq h(x)$
 - And suppose that $\lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} h(x) = L$
 - Then, $\lim_{x \rightarrow c} f(x) = L$
 - Essentially, the hard to compute limit of the “middle function” is found by finding two other easy functions that “squeeze” the middle function at that point.

One-Sided Limit

🌐 One-Sided Limit 🌐

- **One-sided limit:** one of two limits of $f(x)$ as x approaches a specified point from either the left or from the right.

- From the left: $\lim_{x \rightarrow c^-} = L$
- From the right: $\lim_{x \rightarrow c^+} = L$

- If the left and right limits exist and are equal, then

$$\lim_{x \rightarrow c} f(x) = L \Leftrightarrow \lim_{x \rightarrow c^-} f(x) = L \wedge \lim_{x \rightarrow c^+} f(x) = L$$

- Limits can still exist, even if the function is defined at a different point, as long as both one-sided limits approach the same value near the given input.

Continuity

🌐 Continuous function 🌐 | Discontinuities 🌐 | Thomas's Calculus^{14th} (2.5) 📖

- Continuity of functions is one of the core concepts of topology, however, there are definitions in terms of limits that prove useful; the following is only a primer.

Continuous Functions

- **Continuous function:** a function that does not have any abrupt changes in value.
 - I.e., a function is continuous if and only if arbitrarily small changes in its output can be assured by restricting to sufficiently small changes in its input.
- **Discontinuous:** when a function is not continuous at a point in its domain, leading to a discontinuity; there are three classifications:
 - **Removable:** when both **one-sided limits** ↑ exist, are finite, and are equal, but the actual value of $f(x)$ is not equal to the limit and equal to some other value.
 - The discontinuity can be removed to regain continuity.
 - Sometimes the term *removable discontinuity* is mistaken for *removable singularity*, or a “hole” in the function (the point is not defined elsewhere).
 - **Jump:** when a single limit does not exist because the one-sided limits exist and are finite, but not equal.
 - Points can be defined at the discontinuity, but the function can not be made continuous.
 - **Essential:** when at least one of two one-sided limits doesn't exist; can be the result of oscillating or unbounded functions.

Intermediate Value Theorem

🌐 Intermediate value theorem 🌐

- **Intermediate value theorem:** if f is a continuous function whose domain contains the interval $[a, b]$, then it **takes on any given value between $f(a)$ and $f(b)$** at some point within the intervals.
- Relevant deductions, i.e., important corollaries:
 - **Bolzano's theorem:** if a continuous function has values of opposite sign inside an interval, then it **has a root** in that interval.
 - The image of a continuous function over an interval is itself an interval.
- Thus, the image set $f(\mathcal{I})$ (which has no gaps) is also an interval, and it contains:

$$[\min(f(a), f(b)), \max(f(a), f(b))]$$

Limits Involving Infinity

🌐 Limits involving infinity 🐞 | Thomas's Calculus^{14th} (2.6) 📖

- Let $S \subseteq \mathbb{R}$, $x \in S$ and $f : S \mapsto \mathbb{R}$, then limits of these functions can approach arbitrarily large (\pm) values, providing a connection to asymptotes, and thus, analysis.

Limits at Infinity and Infinite Limits

- **Limits at infinity:** limits defined as $f(x) \pm$ infinity are defined much like normal limits:

$$\lim_{x \rightarrow -\infty} f(x) = L \quad \lim_{x \rightarrow \infty} f(x) = L$$

- Formally, for all measures of closeness ϵ there exists a point c such that $|f(x) - L| < \epsilon$ whenever $x < c \vee x > c$ (respectively), i.e.,

$$\forall \epsilon > 0 (\exists c (\forall x \{<, >\} c : |f(x) - L| < \epsilon))$$

- Basic rules for rational functions $f(x) = p(x)q(x)^{-1}$, where p and q are polynomials, the degree of each is denoted as $\{p, q\}^\circ$, and the leading coefficients are denoted as P, Q , then:
 - $p^\circ > q^\circ \Rightarrow L$ is $\{+, -\}$ depending on the sign of the leading coefficients.
 - $p^\circ = q^\circ \Rightarrow L = PQ^{-1}$
 - $p^\circ < q^\circ \Rightarrow L = 0$
- **Infinite limits:** the usual limit does not exist for a limit that grows out of bounds, however, limits with infinite values can be introduced:

$$\lim_{x \rightarrow a} f(x) = \infty, \quad \text{i.e.,} \quad \forall n > 0 (\exists \delta > 0 : f(x) > n \Leftrightarrow 0 < |x - a| < \delta)$$

Asymptotes of functions

🌐 Asymptotes 🐞

- **Asymptote:** a tangent line of a curve at a point at infinity; the distance between the curve and the line approaches zero as a coordinate tends to infinity.
- There are three kinds of asymptotes: *horizontal*, *vertical* and *oblique*; nature of the asymptote is dependent on a function's relation to infinity.
 - **Horizontal asymptotes:** a result of limits at infinity, i.e., when $x \rightarrow \pm\infty$
 - **Vertical asymptotes:** a result of infinite limits, i.e., when $x \rightarrow \pm a = \pm\infty$
 - **Oblique asymptotes:** when a linear asymptote is not parallel to either axis. $f(x)$ is asymptotic to the straight line $y = mx + n (m \neq 0)$ if:

$$\lim_{x \rightarrow \pm\infty} [f(x) - (mx + n)] = 0$$

Derivatives



Derivative Fundamentals

🌐 Derivative 📖 | Thomas's Calculus^{14th} (3.2) 📖

- **Derivative:** the measure of **sensitivity to change** of the function **value** with respect to some change in its **argument**.
 - Often described as the **instantaneous rate of change** of a single variable function, since it is the slope a tangent line at a particular point, when it exists.
 - **Tangent line:** the line through a pair of points on a curve (secant line), except the points are **infinitely close**, thus, it's the rate of change at that "instant".

Definition, Notation

- Formally, a derivative of the function $f(x)$ with respect to the variable x is the function f' whose value at x is (provided the limit exists)

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

- Let $z = x + h$, then $h = z - x \wedge h \rightarrow 0 \Leftrightarrow z \rightarrow x$; this leads to an equivalent definition of the derivative (sometimes more convenient):

$$f'(x) = \lim_{z \rightarrow x} \frac{f(z) - f(x)}{z - x}$$

- **Notation:** there are many ways to denote the derivative; notations can be useful in various contexts, some common notations (for $y = f(x)$):

$$f'(x) = y' = \dot{y} = \frac{dy}{dx} = \frac{d}{dx}f(x) = D(f)(x) = D_x f(x)$$

- **Differentiation:** the process of finding a derivative; if f' exists at a particular point, then f is said to be differentiable at that point.
 - If f' exists at every point on an interval, then f is differentiable on that interval.
 - f' is differentiable on a closed interval $[a, b]$ if both **one-sided limits**[↑] of the function ($h \rightarrow \{0^+ : a, 0^- : b\}$) exist at the end points and is differentiable on the interior.
 - Not all continuous functions have a derivative, but **functions with a derivative are continuous**; functions with any of following **do not have derivatives**:
 - **corners** (one-sided derivatives differ at a point),
 - **cusps** (slope approaches alternating $\pm\infty$ on both sides of a point),
 - **discontinuities**, or **vertical tangent lines**.

Differentiation Rules

📖 Differentiation rules 📖 | Thomas's Calculus^{14th} (3.3, 3.5, 3.7) 📖

- Derivatives can be found by computing its limit, but there are several methods that use of combinations of simpler functions to make computation easier.

Linear, Product, Chain, Inverse

- Linear:** differentiation of linear functions consists of the constant and sum (& subtraction) rules, given the following

$$\forall (f \wedge g) \wedge \forall (a \wedge b \in \mathbb{R}) \Rightarrow \frac{d(af + bg)}{dx} = a \frac{df}{dx} + b \frac{dg}{dx}$$

Constant

$$\frac{d}{dx}(c) = 0$$

Constant factor

$$(af)' = af'$$

Sum (Difference)

$$(f_{+/-} g)' = f'_{+/-} g'$$

- Product rule:** used for the product of two functions; can be generalized[↓]

$$\frac{d(fg)}{dx} = \frac{df}{dx}g + f \frac{dg}{dx}$$

- Chain rule:** used for the composition of two functions $f(g(x))$; if z depends on y , which is dependent on x , then z depends on x as well, i.e.,

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$$

The following is used to indicate which points the derivatives have to evaluated at:

$$\left. \frac{dz}{dx} \right|_x = \left. \frac{dz}{dy} \right|_{y(x)} \cdot \left. \frac{dy}{dx} \right|_x$$

- “Outside-Inside Rule”: take the derivative of the “outside” function, leave “inside” alone, and multiply it by the derivative of the “inside.”
- This method must be recursively “chained” when there are further compositions in the inside function, hence the name.
- **Inverse function rule:** can be applied if the function f has an inverse function g , i.e., a function that “undoes” the effect of f .

$$\{g(f(x)) = x \wedge f(g(y)) = y\} \Rightarrow \frac{dx}{dy} = \left(\frac{dy}{dx} \right)^{-1}$$

- Application of the chain rule on $f^{-1}(y) = x$ in terms of x clearly shows the result if the derivatives exist and are reciprocal,

$$\frac{dx}{dy} \cdot \frac{dy}{dx} = \frac{dx}{dx} = 1$$

Powers, Polynomials, Quotients, Reciprocals

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Exponential, Logarithmic

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Trigonometric, Hyperbolic

-

Differentials and Related Concepts

Differentials

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Linearization

-

Implicit Differentiation

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Related Rates

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Applications of Derivatives



Stationary Point

Maxima and Minima

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Extreme Value Theorem

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Interior Extremum Theorem

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Mean Value Theorem

Rolle's Theorem

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Corollaries of the Mean Value Theorem

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Monotonic Functions

Derivative Tests

First-Derivative Test

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Second-Derivative Test

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Symmetric Functions

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Area Between Curves

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Applications of Definite Integrals



Solid of Revolution

Disc Integration

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Shell Integration

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Arc Length

Dealing with Discontinuities

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Differential Arc Length

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Surface of Revolution

Revolution about the y-Axis

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Transcendental Functions



Inverse Functions

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Definite Integrals by Parts

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First-Order Differential Equations



Parametric Equations and Polar Coordinates



Vectors and Vector-Valued Functions



Partial Derivatives



Multiple Integrals



Vector Calculus



Second-Order Differential Equations



