

Applied Linear Algebra



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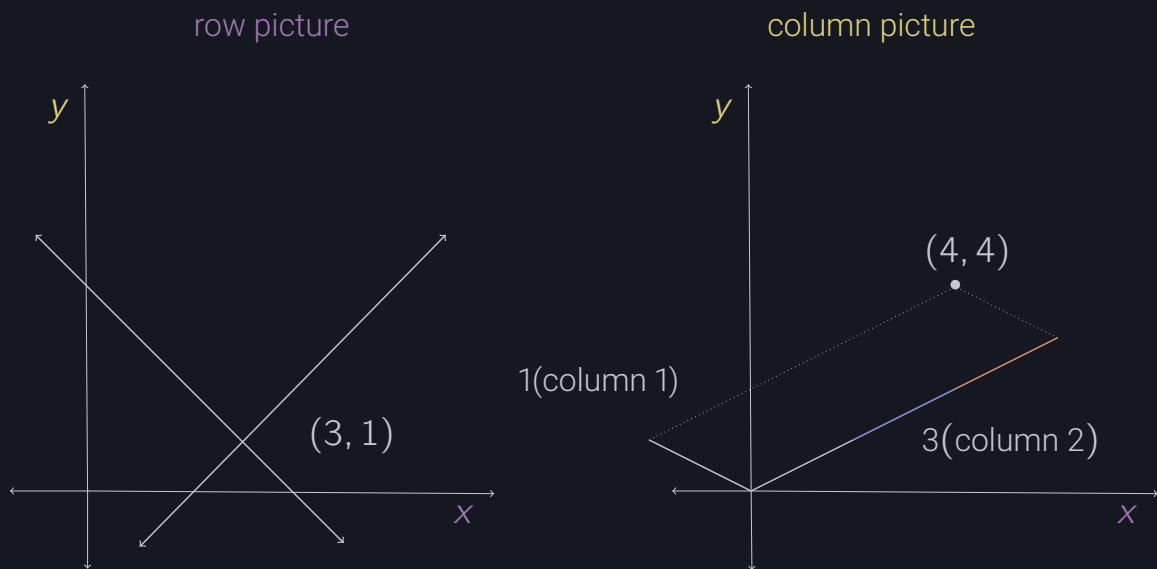
1 Matrices and Gaussian Elimination



1.2 The Geometry of Linear Equations

Problems 1–12

- For the equations $x + y = 4$, $2x - 2y = 4$, draw the row picture (two intersecting lines) and the column picture (combination of two columns equal to the column vector $(4,4)$ on the right side).



1.2.1

- Solve to find a combination of the columns that equals b :

$$u - v - w = b_1$$

$$v + w = b_2$$

$$w = b_3$$

$$\implies w = b_3$$

$$\implies v = b_2 - b_3$$

$$\implies u = b_1 + v + w = b_1 + b_2$$

- Describe the intersection of the three planes $u + v + w + z = 6$ and $u + w + z = 4$ and $u + w = 2$ (all in four-dimensional space). Is it a line or a point or an empty set? What is the intersection if the fourth plane $u = -1$ is included? Find a fourth equation that leaves us with no solution.

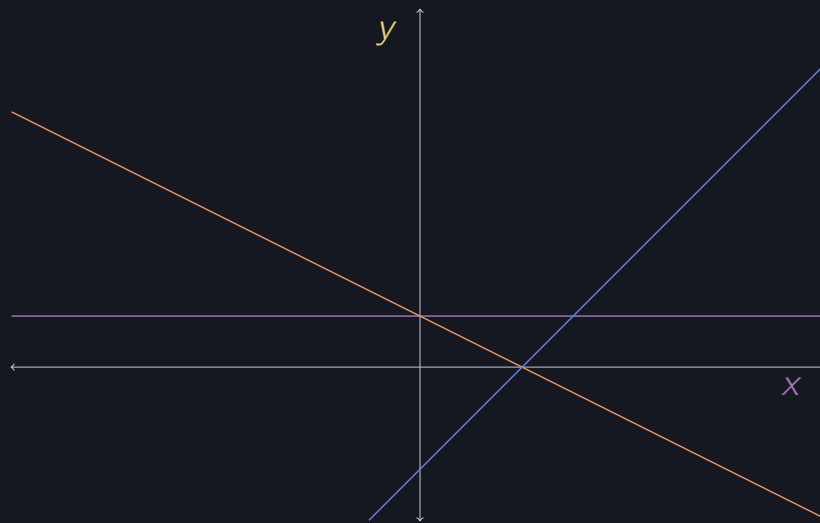
- A line; as $u + w = 2$ is only a line. A fourth plane with $u = -1$ would produce a normally intersecting point. Any addition equation when $u + w \neq 2$ would produce an inconsistent equation.

4. Sketch these three lines and decide if the equations are solvable:

$$x + 2y = 2$$

$$x - y = 2$$

$$y = 1$$



1.2.4

Inconsistent; multiple points of intersect

What happens if all right-hand sides are zero? Is there any nonzero choice of right-hand sides that allows the three lines to intersect at the same point?

- If all the solutions were zero, then it would be a trivial solution.
 - Yes, e.g., $x - y = -1$ would produce a single point of intersection.
5. Find two points on the line of intersection of the three planes $t = 0$ and $z = 0$ and $x + y + z + t = 1$ in four-dimensional space.

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

6. When $b = (2, 5, 7)$, find a solution (u, v, w) to equation (4) different from the solution $(1, 0, 1)$ mentioned in the text.
- Since there are infinite solutions, and if \mathbf{s} vector describing one solution and λ is any scalar, then $\mathbf{s}\lambda$ is also a solution. E.g., $(1, 0, 1)42 = (42, 0, 42)$

8. Explain why the system

$$\begin{aligned}u + v + w &= 2 \\u + 2v + 3w &= 1 \\v + 2w &= 0\end{aligned}$$

is singular by finding a combination of the three equations that adds up to $0 = 1$. What value should replace the last zero on the right side to allow the equations to have solutions—and what is one of the solutions?

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 0 \end{array} \right] \xrightarrow{R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 2 & 0 \end{array} \right] \xrightarrow{R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

- Replacing the last zero with -1 would yield infinite solutions. One solution would be $[3, -1, 0]^T$
9. The column picture for the previous exercise (singular system) is

$$u \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + v \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + w \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = b$$

Show that the three columns on the left lie in the same plane by expressing the third as a combination of the first two. What are all the solutions (u, v, w) if b is the zero vector $(0, 0, 0)$?

$$-1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

- If b is equal to the zero vector $\mathbf{0}$ then the solutions are equal to the kernel² i.e., $-1x_1, 2x_2, 0x_3 = \mathbf{0}$
10. Under what condition on y_1, y_2, y_3 do the points $(0, y_1), (1, y_2), (2, y_3)$ lie on a straight line?
- Question 9 describes the state at which they are collinear, i.e., $y_3 = 2y_2 - y_1$
11. These equations are certain to have the solution $x = y = 0$. For which values of a is there a whole line of solutions?

$$\begin{aligned}ax + 2y &= 0 \\2x + ay &= 0\end{aligned}$$

- Only the scalars that make the lines linearly dependent, i.e., $a = 2, -2$

Problems 17–23

17. The first of these equations plus the second equals the third:

$$\begin{aligned}x + y + z &= 2 \\x + 2y + z &= 3 \\2x + 3y + 2z &= 5\end{aligned}$$

The first two planes meet along a line. The third plane contains that line, because if x, y, z satisfy the first two equations then they also **span all of \mathbb{R}^3** . The equations have infinitely many solutions (the whole line L). Find three solutions.

◦ $\mathbf{v} = (4, 4, 0)$, $\mathbf{w} = (6, 3, 2)$, $\mathbf{u} = 2\mathbf{v} + -1\mathbf{w}$

18. Move the third plane in Problem 17 to a parallel plane $2x + 3y + 2z = 9$. Now the three equations have no solution—*why not*? The first two planes meet along the line L , but the third plane doesn't that **cross** that line.

19. In Problem 17 the columns are $(1, 1, 2)$ and $(1, 2, 3)$ and $(1, 1, 2)$. This is a “singular case” because the third column is **linearly dependent**. Find two combinations of the columns that give $\mathbf{b} = (2, 3, 5)$. This is only possible for $\mathbf{b} = (4, 6, c)$ if $c = 10$

20. Normally 4 “planes” in four-dimensional space meet at a **tensor**. Normally 4 column vectors in four-dimensional space can combine to produce \mathbf{b} . What combination of $(1, 0, 0, 0)$, $(1, 1, 0, 0)$, $(1, 1, 1, 0)$, $(1, 1, 1, 1)$ produces $\mathbf{b} = (3, 3, 3, 2)$? $(1, 0, 0, -2)$? What 4 equations for x, y, z, t are you solving? A **lower triangular matrix**, i.e.,

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3 \\ 1 & 1 & 0 & 0 & 3 \\ 1 & 1 & 1 & 0 & 3 \\ 1 & 1 & 1 & 1 & 1 \end{array} \right]$$

21. When equation 1 is added to equation 2, which of these are changed: the planes in the row picture, the column picture, the coefficient matrix, the solution?

◦ Row operations do not change the solution. Row 2 is changed, thus the second plane is changed. **All columns are changed.**?

1.3 Gaussian Elimination

Problems 6, 7

6. Choose a coefficient b that makes this system singular. Then choose a right-hand side g that makes it solvable. Find two solutions in that singular case.

$$2x + by = 16$$

$$4x + 8y = g$$

$$2x + 4y = 16$$

$$4x + 8y = 32$$

- Since R_2 is just a multiple of R_1 , then solving for x, y , with one variable = 0, in the first equation will yield two solutions, i.e., $(8, 0), (0, 4)$
7. For which numbers a does elimination break down (a) permanently, and (b) temporarily?

$$ax + 3y = -3$$

$$4x + 6y = 6$$

Solve for x and y after fixing the second breakdown by a row exchange.

- Permanently: $a = 2$ (linearly dependent, no solution)
- Temporarily: $a = 0$;

$$\left[\begin{array}{cc|c} 4 & 6 & 6 \\ 0 & 3 & -3 \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -1 \end{array} \right]$$
$$y = -1, \quad x = 3$$

Problems 17, 18, 19

17. Which number q makes this system singular and which right-hand side t gives it infinitely many solutions? Find the solution that has $z = 1$.

$$x + 4y - 2z = 1$$

$$x + 7y - 6z = 6$$

$$3y + qz = t$$

$$x + 4y - 2z = 1$$

$$x + 7y - 6z = 6$$

$$3y + -4z = 5$$

- If $q = -4$, then R_3 would have no pivot
- If $t = 5$, then there would be finite solutions, R_3 would be linearly dependent with R_2

18. It is impossible for a system of linear equations to have exactly two solutions. Explain why.

- If (x, y, z) and (X, Y, Z) are two solutions, what is the other one?
 - There is no other *one*, there would be infinitely many.
- If 25 planes meet at two points, where else do they meet?
 - Every other single point, they would span all of \mathbb{R}^3

19. Three planes can fail to have an intersection point, when no two planes are parallel. The system is singular if row 3 of \mathbf{A} is a **linearly dependent; a combination** of the first two rows. Find a third equation that can't be solved if $x + y + z = 0$ and $x - 2y - z = 1$.

$$x + y + z = 0$$

$$x - 2y - z = 1$$

$$R_1 + R_2 \neq 1 \rightarrow \text{parallel; no solution, e.g.,}$$

$$2x - y = 42$$

Problems 30, 31

30. Use elimination to solve

$$u + v + w = 6$$

$$u + 2v + 2w = 11$$

$$2u + 3v - 4w = 3$$

$$u + v + w = 7$$

$$u + 2v + 2w = 10$$

$$2u + 3v - 4w = 3$$

$$\begin{aligned} \text{rref} \left(\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 2 & 11 \\ 2 & 3 & -4 & 3 \end{array} \right] \right) &\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array} \right] \\ \text{rref} \left(\left[\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 1 & 2 & 2 & 10 \\ 2 & 3 & -4 & 3 \end{array} \right] \right) &\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array} \right] \end{aligned}$$

31. For which three numbers a will elimination fail to give three pivots?

$$ax + 2y + 3z = b_1$$

$$ax + ay + 4z = b_2$$

$$ax + ay + az = b_3$$

- For $a = 0$, multiple failures.
- For $a = 2$, columns 0, 1 would be equal.
- For $a = 4$, rows 1, 2 would be equal.

1.4 Matrix Notation and Matrix Multiplication

Problems 4, 10, 17, 19

4. If an $m \times n$ matrix \mathbf{A} multiplies an n -dimensional vector \mathbf{x} , how many separate multiplications are involved? What if \mathbf{A} multiplies an $n \times p$ matrix \mathbf{B} ?
10. True or false? Give a specific counterexample when false.
- If rows 1 and 3 of \mathbf{B} are the same, so are rows 1 and 3 of \mathbf{AB} .
 - If columns 1 and 3 of \mathbf{B} are the same, so are columns 1 and 3 of \mathbf{AB} .
 - If rows 1 and 3 of \mathbf{A} are the same, so are rows 1 and 3 of \mathbf{AB} .
 - $(\mathbf{AB})^2 = \mathbf{A}^2\mathbf{B}^2$.
17. Which of the following matrices are guaranteed to equal $(\mathbf{A} + \mathbf{B})^2$?

$$\mathbf{A}^2 + 2\mathbf{AB} + \mathbf{B}^2, \quad \mathbf{A}(\mathbf{A} + \mathbf{B}) + \mathbf{B}(\mathbf{A} + \mathbf{B}), \quad (\mathbf{A} + \mathbf{B})(\mathbf{B} + \mathbf{A}), \quad \mathbf{A}^2 + \mathbf{AB} + \mathbf{BA} + \mathbf{B}^2$$

19. A fourth way to multiply matrices is columns of \mathbf{A} times rows of \mathbf{B} :
- $$\mathbf{AB} = (\text{column 1})(\text{row 1}) + \cdots + (\text{column } n)(\text{row } n) = \text{sum of simple matrices.}$$
- Give a $2 \times$ example of this important rule for matrix multiplication.

Problems 29–31

29. ↓

- (a) \mathbf{E}_{21} subtracts row 1 from row 2 and then \mathbf{P}_{23} exchanges rows 2 and 3. What matrix $\mathbf{E} = \mathbf{P}_{23}\mathbf{E}_{21}$ does both steps at once?
- (b) \mathbf{P}_{23} exchanges rows 2 and 3 and then \mathbf{E}_{31} subtracts row 1 from row 3. What matrix $\mathbf{M} = \mathbf{E}_{31}\mathbf{P}_{23}$ does both steps at once? Explain why the \mathbf{M} 's are the same

30. Multiply these matrices:

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \\ 1 & 4 & 0 \end{bmatrix}.$$

31. This 4×4 matrix needs which elimination matrices \mathbf{E}_{21} and \mathbf{E}_{32} and \mathbf{E}_{32} ?

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

Problems 34, 35, 38, 42

34. Multiply these matrices in the orders EF and FE and E^2 :

35. ↓

(a) Suppose all columns of B are the same. Then all columns of EB are the same, because each one is E times $\langle ? \rangle$.

(b) Suppose all rows of B are $[124]$. Show by example that all rows of EB are not $[1, 2, 4]$. It is true that those rows are $\langle ? \rangle$.

38. If $AB = I$ and $BC = I$, use the associative law to prove $A = C$.

42. True or false?

(a) If A^2 is defined then A is necessarily square.

(b) If AB and BA are defined, then A and B are square.

(c) If AB and BA are defined, then AB and BA are square.

(d) If $AB = B$ then $A = I$

1.5 Triangular Factors and Row Exchanges

Problems 1, 6, 7, 8, 12, 18, 19

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Problems 25–30

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Problems 32, 33, 35, 42, 43

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1.6 Inverses and Transposes

Problems 3, 10, 12, 13, 18, 20, 21

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Problems 28–30

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Problems 40–43

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Problems 49–59

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1.7 Special Matrices and Applications

Problems 1, 2, 5, 6

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Problems 7, 8, 10

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