Calculus

Limits and Continuity

Limits

Limit | Thomas (2.2–2.4)

- **Limit** $\lim_{x \to c}$: the value of a function (or sequence) as the input (or index) approaches some value (note: an informal definition).
  - Limits are used to define continuity↓, derivatives↓, and integrals↓.

Limits of a Functions and Sequences

Limit of a function | Limit of a sequence | Essence of Calculus, E7

- **Limit of a function**: a fundamental concept in calculus and analysis concerning the behavior of a function near a particular input c, i.e.,

$$\lim_{x \to c} f(x) = L$$

  - Reads as "f of x tends to L as x tends to c"
- ϵ, δ **Limit of a function**: a formalized definition, wherein $f(x)$ is defined on an open interval $\mathcal{I}$, except possibly at c itself, leading to the informal definition, if and only if

$$f : \mathbb{R} \to \mathbb{R}, c, L \in \mathbb{R} \Rightarrow \lim_{x \to c} f(x) = L$$
$$\Updownarrow$$
$$\forall \varepsilon > 0 \left(\exists \delta > 0 : \forall x \in \mathcal{I} \left(0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon\right)\right)$$

- functions do not have a limit when the function:
  - has a unit step, i.e., it "jumps" at a point;
  - is not bounded, i.e., it tends towards infinity;
  - or it oscillates, i.e., does not stay close to any single number.
- **Limit of a sequence**: the value that the terms of a sequence $(x_n)_{n \in \mathbb{N}}$ "tends to" (and not to any other) as n approaches infinity (or some other point), i.e.,

$$\lim_{n \to \infty} x_n = x$$

- ε **Limit of a sequence**: for every measure of closeness ε, the sequence's x_n term eventually converges to the limit, i.e.,

$$\forall \varepsilon > 0 \left(\exists N \in \mathbb{N} \left(\forall n \in \mathbb{N} \left(n \geq N \Rightarrow |x_n - x| < \varepsilon\right)\right)\right)$$

Properties of Limits

One-Sided Limit

Continuity

Continuous Functions

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Intermediate Value Theorem

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Limits Involving Infinity

Limits at Infinity and Infinite Limits

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Asymptotes of functions

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Derivatives

Derivative Fundamentals

Derivative Notation

- ...

Differentiation Rules

- Linear, Product, Chain, Inverse
  -
- Powers, Polynomials, Quotients, Reciprocals
  -
- Exponential, Logarithmic
  -
- Trigonometric, Hyperbolic
  -

Differentials and Related Concepts

Differentials

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Linearization

-

Implicit Differentiation

-

Related Rates

-

Applications of Derivatives

Stationary Point

Maxima and Minima

-

Extreme Value Theorem

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Interior Extremum Theorem

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Mean Value Theorem

Rolle’s Theorem

-

Corollaries of the Mean Value Theorem

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Monotonic Functions

Derivative Tests

First-Derivative Test

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Second-Derivative Test

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Concavity

-

Higher-Order Derivative Test

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Differential Methods

Newton's Method

-

Taylor's Theorem

-

General Leibniz Rule

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Integrals

Integral Fundamentals

Terminology and Notation

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Primer: Formal Definitions

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Definite Integrals

Riemann Integral

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Integrability

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Properties of Definite Integrals

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The Fundamental Theorem of Calculus

Fundamental Theorem, Part 1

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Fundamental Theorem, Part 2

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The Integral of a Rate

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Total Area

-

Integration By Substitution

Indefinite Integrals

-

Definite Integrals

-

Symmetric Functions

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Area Between Curves

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Applications of Definite Integrals

Solid of Revolution

Disc Integration

-

Shell Integration

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Arc Length

Dealing with Discontinuities

-

Differential Arc Length

-

Surface of Revolution

Revolution about the y-Axis

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Transcendental Functions

Inverse Functions

One-to-One Functions

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Derivative Rule for Inverses

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Logarithmic Functions

Natural Logarithm

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Properties of Logarithms

-

Trigonometric Integrals

-

Logarithmic Differentiation

-

Exponential Functions

Euler's Number

-

Natural Exponential Function

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Laws of Exponents

-

General Exponential Function

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Exponential Change

- Separable Differential Equations
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  Examples of Exponential Change
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Indeterminate Forms

Indeterminate Form 0/0

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L'Hôpital's Rule

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Infinite Indeterminate Forms

-

Indeterminate Powers

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Inverse Trigonometric Functions

Principal Trigonometric Values

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Inverse Trigonometric Tables

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Hyperbolic Functions

Hyperbolic Function Tables

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Integration by Parts

Definite Integrals by Parts

Trigonometric Integral Methods

Trigonometric Products and Powers

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Trigonometric Square Roots

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Trigonometric Substitutions

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Partial Fraction Decomposition

Partial Fraction Principles

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General Statement

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Numerical Integration

Trapezoidal Rule

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Simpson's Rule

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Improper Integrals

Indirect Evaluation

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Infinite Sequences and Series

First-Order Differential Equations

Partial Derivatives

Multiple Integrals

Vector Calculus

Second-Order Differential Equations