Calculus III Exercises

9 Introduction to Differential Equations

9.1 Edfinity: Solving Differential Equations

9.1.5

- Solve $y' = x^5y^2$, using separation of variables, given the initial condition $y\left(0\right) = 9$

$$\frac{\frac{dy}{dx}}{y^2} = x^5$$
$$\int \frac{\frac{dy}{dx}}{y(x)^2} = \int x^5 dx$$
$$-\frac{1}{y(x)} = \frac{x^6}{6} + c_1$$
$$y(x) = -\frac{6}{x^6 + c_1}$$
$$9 = -\frac{6}{c}, \quad c = -\frac{6}{9}$$
$$\boxed{y(x) = -\frac{18}{2x^6 - 2}}$$

9.1.6

- Solve the initial value problem $\frac{dy}{dx} + 3y = 0, \quad y(\ln 4) = 3.$

$$\frac{dy}{dx} = -3y$$
$$\int \frac{dy}{dx} y^{-1} dx = \int -3dx$$
$$\ln|y| = -3x + \lambda$$
$$y = e^{-3x}\lambda$$
$$3 = e^{3(\ln 4)}\lambda \implies \lambda = 192$$
$$y = 192e^{-3x}$$

9.1.7

- Solve $(t^2+36)\frac{dx}{dt}=(x^2+9)$, using separation of variables, given the initial condition $x(0)=3$.

$$\begin{aligned}
\frac{dx}{dt} &= \left(t^2+36\right)^{-1} \\
\frac{dx}{dt}\left(x^2+9\right)^{-1} &= (t^2+36)^{-1} \\
\int \frac{dx}{dt}\left(x^2+9\right)^{-1} &= \int (t^2+36)^{-1}dt \\
\frac{1}{9}\int\left(\frac{x^2}{9}+1\right)^{-1}dx &= \frac{1}{36}\int\left(\frac{t^2}{36}+1\right)^{-1}dt \\
\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) &= \frac{1}{6}\tan^{-1}\left(\frac{t}{6}\right)+\lambda \\
\tan^{-1}\left(\frac{x}{3}\right) &= \frac{1}{2}\tan^{-1}\left(\frac{t}{6}\right)+\lambda \\
\tan^{-1}\left(\frac{3}{3}\right) &= \frac{1}{2}\tan^{-1}\left(\frac{0}{6}\right)+\lambda \\
\frac{\pi}{4} &= \lambda \\
x &= 3\tan\left(\frac{1}{2}\tan^{-1}\left(\frac{t}{6}\right)+\frac{\pi}{4}\right)
\end{aligned}$$

9.1.8

- Solve the initial value problem $\frac{dy}{dx}=(x-7)(y-8),\quad y(0)=4$

$$\begin{aligned}
\frac{dy}{dx}(y-8)^{-1} &= (x-7) \\
\int dy\,(y-8)^{-1} &= \int (x-7)\,dx \\
\ln y-8 &= x^{-2}-7x+\lambda \\
y &= e^{\frac{x^2}{2}-7x}\lambda+8 \\
-4 &= \lambda \\
y &= -4e^{\frac{x^2}{2}-7x}+8
\end{aligned}$$

9.1.9

- Solve the initial value problem $t^2\frac{dy}{dt} - t = 1 + y + ty, \quad y(1) = 7$

$$\begin{aligned}
\int (y+1)\,dy &= \int \frac{1+t}{t^2} dt \\
\ln|1+y| &= -t^{-1} + \ln t + \lambda \\
y &= \lambda e^{-t^{-1}+\ln t} - 1 = \lambda \frac{t}{e^{t^{-1}}} - 1 \\
7 &= \lambda e^{-1} - 1 \\
\lambda &= 8e \\
y &= 8te^{1-t^{-1}} - 1
\end{aligned}$$

9.1.10

- Solve the initial value problem $y' = 2y^2 \sin x, \quad y(0) = 6$

$$\begin{aligned}
\frac{dy}{dx}(2y^2)^{-1} &= \sin x \\
\int \frac{dy}{dx} y^{-2} dx &= \int 2 \sin x dx \\
-y^{-1} &= -2\cos x + \lambda \\
y &= (2\cos x + \lambda)^{-1} \\
6 &= (2\cos 0 + \lambda)^{-1} \\
-\frac{11}{6} &= \lambda \\
y &= \left(2\cos x - \frac{11}{6}\right)^{-1}
\end{aligned}$$

9.2 Edfinity: Models Involving y'=k(y-b)

9.2.2

- Find the general solution of $y' = 5\,(y - 16)$.

$$y(t) = b + Ce^{kt} \qquad\qquad y' = k(y - b)$$

$$\begin{aligned} y(t) &= 16 + Ce^{5t} \\ 30 &= 16 + C \\ C &= 14 \\ y(t) &= 16 + 14e^{5t} \\ 1 &= 16 + C \\ C &= -15 \\ y(t) &= 16 + -15e^{5t} \end{aligned}$$

9.2.3

- A 62 kg skydiver jumps out of an airplane. What is her terminal velocity in miles per hour, assuming that $k = 10\frac{kg}{s}$ for free fall?

$$-\frac{gm}{k} = -\frac{9.8(62)}{10} = -60.76\frac{m}{s} = 199.343\frac{ft}{s} = -134.916 \text{ mph}$$

9.2.4

- A continuous annuity with withdrawal rate $N = \$600\,\text{y}$ and interest rate $r = 5\%$ is funded by an initial deposit P_0
- When will the annuity run out of funds if $P_0 = \$10,000$?

$$\begin{aligned} P(t) = Nr^{-1} + Ce^{rt} = 600(0.05)^{-1} + Ce^{0.05t} &= 12,000 + Ce^{0.05t} \\ 10,000 &= 12,000 + C \\ C &= -2,000 \\ t = 0.05^{-1} \ln \frac{12,000}{2,000} &= 35.83 \approx 38 \text{ years} \end{aligned}$$

- Which initial deposit P_0 yields a constant balance?

$$\begin{aligned} P(t) &= 12,000 + Ce^{0.05^t}, \quad C = 0 \\ P_0 &= 12,000 \end{aligned}$$

9.2.5

- A cup of coffee, cooling off in a room temperature $20\,^\circ\mathrm{C}$, has cooling constant $k = 0.085\ \mathrm{min}^{-1}$.
- How fast is the coffee cooling when its temperature is $T = 70\,^\circ\mathrm{C}$?

$$k(T - T_0)$$
$$0.085(70 - 20) = 4.25\,^\circ\mathrm{C}\,\mathrm{min}^{-1}$$

- Use the Linear Approximation to estimate the change in temperature over the next 4 seconds when $T = 70\,^\circ\mathrm{C}$

$$4.25\,^\circ\mathrm{C}\,\mathrm{min}^{-1}(4\mathrm{s})60\,\mathrm{s}\,\mathrm{min}^{-1} = 0.283\,^\circ\mathrm{C}$$

- The coffee is served at a temperature of $86\,^\circ\mathrm{C}$. How long should you wait before drinking it if the optimal temperature is $65\,^\circ\mathrm{C}$?

$$65 = 20 + 66e^{-0.085t}$$
$$t = -(0.085)^{-1}\ln\left(\frac{45}{66}\right)$$
$$t \approx 4.5\ \mathrm{min}$$

9.3 Edfinity: Graphical and Numerical Methods

4. **9.3.4**

   - User Euler's method with step size 0.2 to estimate $y(1)$, where $y(x)$ is the solution of the initial-value problem.

$$y' = 4x + y^2, \quad y(0) = 1$$

$$y_k = y_{k-1} + hF(t_{k-1}, y_{k-1}), \quad F = 4x + y^2, \quad h = 0.2$$

$$\begin{aligned}
y(0.2) &\approx 1 + 0.2(4(0) + 1^2) = 1.2 \\
y(0.4) &\approx 1.2 + 0.2(4(0.2) + 1.2^2) \approx 1.648 \\
y(0.6) &\approx 1.648 + 0.2(4(0.4) + 1.648^2) \approx 2.511 \\
y(0.8) &\approx 2.511 + 0.2(4(0.6) + 2.511^2) \approx 4.092 \\
y(1) &\approx 4.092 + 0.2(4(0.8) + 4.092^2) \approx 8.578
\end{aligned}$$

9.3.5

- User Euler's method with $\Delta x = 0.1$ to estimate $y(1.4)$.

$$y' = -x - y, \quad y(1) = 1$$

$$\begin{aligned}
y(1) &\approx 1 + 0.1(-1 - 1) = 0.8 \\
y(1.1) &\approx 0.8 + 0.1(-1.1 - 0.8) = 0.61 \\
y(1.2) &\approx 0.61 + 0.1(-1.2 - 0.61) = 0.429 \\
y(1.3) &\approx 0.429 + 0.1(-1.3 - 0.429) = 0.2561 \\
y(1.4) &\approx 0.2561
\end{aligned}$$

9.4 Edfinity: The Logistic Equation

- The logistic equation and general non-equilibrium solution ($k > 0$ and $A > 0$)

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{A}\right), \quad y = \frac{A}{1 - e^{-kt}/B} \quad \Leftrightarrow \quad \frac{y}{y - A} = Be^{kt}$$

- $y = 0 \to$ unstable equilibrium
- $y = A \to$ stable equilibrium
- If the initial value $y_0 = y(0)$ satisfies $y_0 > 0$, then $y(t)$ approach the stable equilibrium $y = A$, i.e., $\lim_{t\to\infty} y(t) = A$

9.4.1

- A population P obeys the logistic model. It satisfies the equation

$$\frac{dP}{dt} = \frac{7}{1100}P(11 - P) \text{ for } P > 0$$
$$\frac{dP}{dt} = \frac{7}{1100}P(1 - \frac{P}{11}) \text{ for } P > 0$$

c. Assume that $P(0) = 4$. Find $P(87)$

$$\frac{y}{y - A} = Be^{kt}$$
$$\frac{4}{4 - 11} = Be^{\frac{7}{1110}0}$$
$$-0.571 \approx B$$
$$\implies P(87) = \frac{11}{1 - e^{-0.06\cdot 87}/ - 0.571} \approx -10.9$$

9.4.2

- Assuming $P \geq 0$, suppose that a population develops according to the logistic equation

$$\frac{dP}{dt} = 0.01P - 5 \times 10^{-5}P^2$$

where t is measured in weeks.

$$\frac{dP}{dt} = 0.01P - 5 \times 10^{-5}P^2$$
$$= 0.01P\left(1 - \frac{5 \times 10^{-5}}{0.01}P\right)$$
$$= 0.01P\left(1 - \frac{P}{200}\right)$$

$$\text{Carrying capacity} = A = 200$$
$$k = 0.01$$
$$\text{increasing} = (0, 200)$$
$$\text{decreasing} = (200, \infty)$$

9.4.3

- A population of squirrels lives in a forest with a carrying capacity of 1600. Assume logistic growth with growth constant $k = 1\,\text{yr}^{-1}$
- Find a formula for the squirrel population $P(t)$, assuming an initial population of 400 squirrels.

$$\begin{aligned}
\frac{dP}{dt} &= 1P\left(1 - \frac{P}{1600}\right) \\
B &= \frac{400}{400 - 1600} = -\frac{1}{3} \\
P(t) &= 1600/1 - \frac{e^{-t}}{-\frac{1}{3}} = \frac{1600}{1 + 3e^{-t}} \\
800 &= \frac{1600}{1 + 3e^{-t}} \\
1 + 3e^{-t} &= 2 \\
e^{-t} &= \frac{1}{3} \\
t &= -\ln\frac{1}{3} = 1.098\,\text{yr}
\end{aligned}$$

9.4.4

- Sunset Lake is stocked with 2700 rainbow trout and after 1 year the population has grown to 7050. Assuming logistic growth with a carrying capacity of 27,000, find the growth constant k, and determine when the population will increase to 13600.

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{27000}\right)\ P(0) = 2700, \quad P(t) = 13600$$

$$\begin{aligned}
B &= \frac{2700}{2700 - 27000} = -\frac{1}{9} \\
7050 &= \frac{27000}{1 + 9e^{-k\cdot 1}} \\
1 + 9e^{-k} &= \frac{27000}{7050} \\
k &= -\ln\frac{\frac{27000}{7050} - 1}{9} = 1.157 \\
13600 &= \frac{27000}{1 + 9e^{1.157t}} \\
e^{1.157t} &= \frac{\frac{27000}{13600} - 1}{9} \\
t &= \ln\left(\frac{\frac{27000}{13600} - 1}{9}\right)(1.157)^{-1} \\
t &\approx 1.911\,\text{yr}
\end{aligned}$$

9.4.5

- Suppose that a population develops according to the logistic equation

$$\frac{dP}{dt} = 0.05P - 6.667 \times 10^{-5}P^2$$

where t is measure in weeks.

$$\frac{dP}{dt} = 0.05P\left(1 - \frac{6.667 \times 10^{-5}}{0.05}P\right) = 0.05P\left(1 - \frac{P}{750}\right)$$

9.5 Edfinity: First-Order Linear Equations

- Hammers:

$$
\begin{aligned}
y' + P(x)y &= Q(x) \\
\alpha(x) &= e^{\int P(x)dx} \\
y &= \alpha(x)^{-1}\left(\int \alpha(x)Q(x)dx + C\right)
\end{aligned}
$$

9.5.1

- Solve $y' + 3x^{-1}y = x^2, \qquad y(1) = -9$
  - Idetify the integrating factor, $\alpha(x)$

$$
\begin{aligned}
\alpha(x) &= e^{\int 3x^{-1}dx} \\
\alpha(x) &= e^{3\ln x} \\
\alpha(x) &= x^3
\end{aligned}
$$

  - Find the general solution, $y(x)$

$$
\begin{aligned}
y &= x^{-3}\left(\int x^3x^2dx + C\right) \\
y &= x^{-3}\left(\frac{x^6}{6} + C\right) \\
y &= \frac{x^3}{6} + Cx^{-3}
\end{aligned}
$$

  - Solve the innitla value problem, $y(1) = -9$

$$
\begin{aligned}
-9 &= \frac{1^3}{6} + C^{-3} \\
1.5 &= C^{-3} \\
C &= -\left(\frac{55}{6}\right)^{3^{-1}} \\
\implies y &= \frac{x^3}{6} - \frac{55}{6}^{\frac{1}{3}}x^{-3} \\
y &= \frac{x^3}{6} - 9.167x^{-3}
\end{aligned}
$$

9.5.2

- Solve $4xy' - 8y = x^{-1}, \qquad y(1) = 6$

$$\implies y' - 2x^{-1}y = \frac{1}{4}x^{-2}$$

  - Idetify the integrating factor, $\alpha(x)$

$$\alpha(x) = e^{\int -2x^{-1}dx}$$
$$\alpha(x) = e^{-2\ln x}$$
$$\alpha(x) = x^{-2}$$

  - Find the general solution, $y(x)$

$$y = x^2\left(\int x^{-2}\frac{1}{4}x^{-2}dx + C\right)$$
$$y = x^2\left(-\frac{1}{12}x^{-3} + C\right)$$
$$y = -\frac{1}{12}x^{-1} + Cx^2$$

  - Solve the innitla value problem, $y(1) = 6$

$$6 = -\frac{1}{12}1^{-1} + C$$
$$C = 6 + \frac{1}{12} = 6.083$$
$$\implies y = \frac{1}{12}x^{-1} + 6.083x^2$$

9.5.3

- Solve $xy' = 2y - 9x, \qquad y(1) = -2$

$$\implies y' - 2x^{-1}y = -9$$

  - Idetify the integrating factor, $\alpha(x)$

$$\alpha(x) = e^{\int -2x^{-1}dx}$$
$$\alpha(x) = e^{-2\ln x}$$
$$\alpha(x) = x^{-2}$$

  - Find the general solution, $y(x)$

$$y = x^2\left(\int -9x^{-2}dx + C\right)$$
$$y = x^2(9x^{-1} + C)$$
$$y = 9x + Cx^2$$

- Solve the innitla value problem, $y(1) = -2$

$$\begin{aligned} -2 &= 9 + C \\ C &= -11 \\ \implies y &= 9x - 11x^2 \end{aligned}$$

9.5.4

◦ Find the general solution of the first-order linear differetial equation

$$y' - (\ln x)y = 2x^x$$

$$\begin{aligned} \alpha(x) &= e^{\int -\ln x dx} \\ \alpha(x) &= e^{x - x\ln x} \\ \alpha(x) &= e^x x^{-x} \\ y &= e^{-x}x^x \left(\int e^x x^{-x} 2x^x dx + C \right) \\ y &= e^{-x}x^x(2e^x + C) \\ y &= 2x^x + Ce^{-x}x^x \end{aligned}$$

9.5.5

◦ Solve the initial value problem $y' + 4y = e^{8x}, \qquad y(0) = -7$

$$\begin{aligned} \alpha(x) &= e^{4x} \\ y &= e^{-4x} \left(\int e^{4x} e^{8x} dx + C \right) \\ y &= e^{-4x} \left(\frac{e^{12x}}{12} + C \right) \\ y &= \frac{e^{8x}}{12} + Ce^{-4x} \\ -7 &= \frac{1}{12} + C \\ C &= -\frac{85}{12} \\ y &= \frac{e^{8x}}{12} + -\frac{85}{12} e^{-4x} \end{aligned}$$

9 Rogawski: Review

Chapter 9 Toolbox

- **Separable first-order**: a differential equation in the form

$$\frac{dy}{dx} = f(x)g(y)$$

  - **General solution**: when $\frac{dy}{dt} = ky$, then $y(t) = De^{kt}$

$$\begin{aligned} y^{-1}dy &= kdt \\ \int y^{-1}dy &= \int kdt \\ \ln|y| &= kt + C \\ |y| &= e^{kt} \\ y &= De^{kt} \end{aligned}$$

  - Exponential decay: $k < 0$; half-life: $(\ln 0.5)k^{-1}$
  - Exponential growth: $k > 0$; doubling: $(\ln 2)k^{-1}$
- **First-order linear constant coefficient**: when a quantity y whose rate of change is proportional to the difference $y - b$, i.e.,

$$\frac{dy}{dt} = k(y - b)$$

  - **General solution**: using separation of variables,

$$y(t) = b + Ce^{kt} \quad \leftrightarrow \quad \frac{d}{dt}(y - b) = k(y - b)$$

  - **Newton's law of Cooling**: where k is the cooling constant (dependent on object) and T_0 is the ambient temperature.

$$\frac{dy}{dt} = -k(y - T_0) \implies y(t) = T_0 + C^{-kt}$$

  - **Newton's Second Law of Motion**: $F = ma = mv' = -mg - kv$, i.e.,

$$\frac{dv}{dt} = -\frac{k}{m}\left(v + \frac{mg}{k}\right) \implies v(t) = -\frac{mg}{k} + Ce^{-\frac{k}{m}t}$$

  - **Annuity/Compound interest**: modeling balance in annuity by the differential equation

$$\frac{dP}{dt} = rP - N = r(P - \frac{N}{r}) \implies P(t) = \frac{N}{r} + C^{rt}$$

- **Slope filed**: when a first-order differential equation $\frac{dy}{dt} = F(t, y)$ is obtained by drawing small segments of slope $F(t, y)$ at points t, y.
  - Test points particular points, often two easy tests are enough to match an equation to graph via elimination of potential options.
- **Euler's Method**: an approximate solution to $\frac{dy}{dt} = F(t, y)$ when given an initial condition $y(t_0) = y_0$ and time step h.
  - Setting $t_k = t_0 + kh$ yields $y_1, y_2, \ldots, y_n$ through recursive application of

$$y_k = y_{k-1} + hF(t_{k-1}, y_{k-1})$$

  - I.e.,

$$\begin{aligned} y_1 &= y_0 + hF(t_0, y_0) \\ &\searrow \\ y_2 &= y_1 + hF(t_{0+1h}, y_1) \\ &\searrow \\ y_3 &= y_2 + hF(t_{0+2h}, y_2) \\ &\searrow \\ &\vdots \end{aligned}$$

  where each y_k is an approximate of $y(t_n)$
- **Logistic differential equation**: where $y(t)$ is the population at time t and A denotes the carrying capacity, yielding a representation of room for growth $A - y(t)$.
  - The assumption is that the $\frac{dy}{dt}$ is proportional to the amount of $y(t)$ present and amount of $A - y(t)$ of room for growth, i.e.,

$$\frac{dy}{dt} = Ky(A - y), \qquad K = \text{ proportionality constant}$$

  - Which an be written as

$$\frac{dy}{dt} = ky(1 - \frac{y}{A}), \qquad k = KA$$

  - **General non-equilibrium solution**: when $k > 0 \wedge A > 0$:

$$y = \frac{A}{1 - \frac{e^{-kt}}{B}} \quad \leftrightarrow \quad \frac{y}{y - A} = Be^{kt}$$

- Two equilibrium constant solutions:
  - $y = 0$; unstable equilibrium.
  - $y = A$; a stable equilibrium.
- If the initial value $y_0 = y(0)$ satisfies $y_0 > 0$, then $\lim_{t \to \infty} y(t) = A$

- **First-Order Linear Equations**: method of solving all first-order linear differential equations, separable or not, as long as the equation can be put in the form:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

  - **Integrating factor**:

$$\alpha(x) = e^{\int P(x)dx}$$

  - **General solution**:

$$y = \alpha(x)^{-1}\left(\int \alpha(x)Q(x)dx + C\right)$$

  - Approach to the problems:
    1. Arrange equation in first-order linear form.
    2. Find the Integrating factor.
    3. Solve general solution.
    4. Solve initial value by finding C in solved general solution, if given $y(t)$.

9.4.9 Spread of Rumor

- One model for the spread of a rumor is that the rate of change of the percent of the population that has heard the rumor is proportional to the product of the percent of the population that has heard the rumor and the percent that has not heard the rumor. Suppose a small town has a population of 1,000 people. At 9 AM, 60 people had heard a rumor. By noon, half of the town had heard it. Set up an initial value problem to model this situation.

$$\begin{aligned}
\frac{y}{y-A} &= Be^{kt}, \qquad y(0) = 60, A = 1000 \\
\frac{60}{60-1000} &= Be^{kt} \\
B &= -\frac{3}{47} \\
y = \frac{1000}{1 - \frac{e^{-kt}}{-\frac{3}{47}}} &= \frac{3000}{3 + 47e^{-kt}}
\end{aligned}$$

$$\begin{aligned}
500 &= \frac{3000}{3 + 47e^{-3k}}, \qquad y(3) = 500 \\
1500 + 23500e^{-3k} &= 3000 \\
e^{-3k} &= \frac{1500}{23500} \\
-3k &= \ln\left(\frac{3}{47}\right) \\
k &= -\frac{1}{3}\ln\left(\frac{3}{47}\right) \approx 0.917
\end{aligned}$$

11 Parametric Equations, Polar Coordinates, and Conic Sections