Calculus

Limits and Continuity

Limits

Limit | Thomas (2.2–2.4)

- **Limit** $\lim\limits_{x \to c}$: the value of a function (or sequence) as the input (or index) approaches some value (note: an informal definition).
  - Limits are used to define continuity, derivatives, and integrals.

Limits of a Functions and Sequences

Limit of a function | Limit of a sequence | Essence of Calculus, Ea

- **Limit of a function**: a fundamental concept in calculus and analysis concerning the behavior of a function near a particular input c, i.e.,

$$\lim_{x \to c} f(x) = L$$

  - Reads as "f of x tends to L as x tends to c"

- ϵ, δ **Limit of a function**: a formalized definition, wherein $f(x)$ is defined on an open interval $\mathcal{I}$, except possibly at c itself, leading to the informal definition, if and only if

$$f : \mathbb{R} \to \mathbb{R},\, c, L \in \mathbb{R} \Rightarrow \lim_{x \to c} f(x) = L$$
$$\Updownarrow$$
$$\forall \varepsilon > 0\, (\exists \delta > 0 : \forall x \in \mathcal{I}\, (0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon))$$

- Functions do not have a limit when the function:
  - has a unit step, i.e., it "jumps" at a point;
  - is not bounded, i.e., it tends towards infinity;
  - or it oscillate, i.e., it does not stay close to any single number.
- **Limit of a sequence**: the value that the terms of a sequence $(x_n)_{n \in \mathbb{N}}$ "tends to" (and not to any other) as n approaches infinity (or some other point), i.e.,

$$\lim_{n \to \infty} x_n = x$$

- $\mathcal{E}$ **Limit of a sequence**: for every measure of closeness $\mathcal{E}$, the sequence's x_n term eventually converges to the limit, i.e.,

$$\forall \varepsilon > 0\, (\exists N \in \mathbb{N}\, (\forall n \in \mathbb{N}\, (n \geq N \implies |x_n - x| < \varepsilon)))$$

  - **Convergent**: when a limit of a sequence exists.
  - **Divergent**: a sequence that does not converge.

Properties of Limits

List of limits | Squeeze theorem

- **Operations on a single known limit**: if $\lim_{x \to c} f(x) = L$, then:
  - $\lim_{x \to c} \left[f(x) \pm \alpha\right] = L \pm \alpha$
  - $\lim_{x \to c} \alpha f(x) = \alpha L$
  - $\lim_{x \to c} f(x)^{-1} = L^{-1}, L \neq 0$
  - $\lim_{x \to c} f(x)^{n} = L^{n}, n \in \mathbb{N}$
  - $\lim_{x \to c} f(x)^{n^{-1}} = L^{n^{-1}}$, if $n \in \mathbb{N}_e \implies L > 0$
- **Operations on two known limits**: if $\lim_{x \to c}$ and $\lim_{x \to c} g(x) = L_2$, then:
  - $\lim_{x \to c} \left[f(x) \pm g(x)\right] = L_1 \pm L_2$
  - $\lim_{x \to c} [f(x)g(x)] = L_1 L_2$
  - $\lim_{x \to c} f(x)g(x)^{-1} = L_1 L_2^{-1}$
- **Squeeze theorem**: used to confirm the limit of a difficult to compute function via comparison with two other functions whose limits are easily known or computed.
  - Let $\mathcal{I}$ be an interval having the point c as a limit point.
  - Let g, f, and h be functions defined on $\mathcal{I}$, except possibly at c itself.
  - Suppose that $\forall x \in \mathcal{I} \wedge x \neq \implies g(x) \leq f(x) \leq h(x)$
  - and $\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L$
  - then, $\lim_{x \to c} f(x) = L$
  - Essentially, the hard to compute limit of the "middle function" can be found by finding the limit of two other "easier" functions that that "squeeze" the middle function at a point of interest.

One-Sided Limit

One-Sided Limit

- **One-sided limit**: one of two limits of $f(x)$ as x approaches a specified point from either the left or from the right right.
  - From the left: $\lim_{x \to c^-} = L$
  - From the right: $\lim_{x \to c^+} = L$
- If the left and right limits exist and are equal, then

$$\lim_{x \to c} f(x) = L \iff \lim_{x \to c^-} f(x) = L \wedge \lim_{x \to c^+} f(x) = L$$

- Limits can still exist, even if the function is defined at a different point, as long as both one-sided limits approach the same value near the given input.

Continuity

Thomas (2.5)

- Continuity of functions is one of the core concepts of topology, however, there are definitions in terms of limits that prove useful; the following is only a primer.

Continuous Functions

Continuous function | Discontinuities

- **Continuous function**: a function that does not have any abrupt changes in value.
  - I.e., a function is continuous if and only if arbitrarily small changes in its output can be assured by restricting to sufficiently small changes in its input.
- **Discontinuous**: when a function is not continuous at a point in its domain, leading to a discontinuity; there are three classifications:
  - **Removable**: when both one-sided limits↑ exist, are finite, and are equal, but the actual value of $f(x)$ is not equal to the limit and instead equal to some other value.
    - The discontinuity can be removed to regain continuity.
    - Sometimes the term *removable discontinuity* is mistaken for a *removable singularity*, or a "whole" in the function (the point is not defined elsewhere).
  - **Jump**: when a single limit does not exist because the one-sided limits exist and are finite, but not equal.
    - Points can be defined at the discontinuity, but the function can not be made continuous.
  - **Essential**: when at least one of the two one-sided limits do not exist; can be the result of oscillating or unbounded functions.

Intermediate Value Theorem

Intermediate value theorem

- **Intermediate value theorem**: if f is a continuous function whose domain contains the interval $[a, b]$, then it takes on any given value between $f(a)$ and $f(b)$ at some point within the intervals.
- Relevant deductions, i.e., important corollaries:
  - **Bolzano's theorem**: if a continuous function has values of opposite sign inside an interval, then it has a root in that interval.
  - The image of a continuous function over an interval is itself an interval.
- Thus, the image set $f(\mathcal{I})$ (which has no gaps) is also an interval, and it contains:

$$[\min(f(a), f(b)), \max(f(a), f(b))]$$

Limits Involving Infinity

Thomas (2.6)

- Let $S \subset \mathbb{R}$, $x \in S$ and $f : S \mapsto \mathbb{R}$, then limits of these functions can approach arbitrarily large $(\pm)$ values, providing a connection to asymptotes, and thus, analysis.

Limits at Infinity and Infinite Limits

Limits involving infinity

- **Limits at infinity**: limits defined as $f(x)\pm$ infinity are defined much like normal limits:

$$\lim_{x\to-\infty} f(x) = L \qquad \lim_{x\to\infty} f(x) = L$$

  - Formally, for all measures of closeness ε there exists a point c such that $|f(x) - L| < \varepsilon$ whenever $x<c \vee x>c$ (respectively), i.e.,

$$\forall\varepsilon > 0\,(\exists c\,(\forall x\{<\vee>\}c : |f(x) - L| < \varepsilon))$$

  - Basic rules for rational functions $f(x) = p(x)q(x)^{-1}$, where p and q are polynomials, where the degree of each is denoted as $\{p \vee q\}^{\circ}$, and where the leading coefficients are denoted as P, Q, then:
    - $p^{\circ} > q^{\circ} \implies \pm L$, depending on the sign of the leading coefficient.
    - $p^{\circ} = q^{\circ} \implies L = PQ^{-1}$
    - $p^{\circ} < q^{\circ} \implies L = 0$
- **Infinite limits**: the usual limit does not exist for a limit that grows out of bounds, however, limits with infinite values can be introduced:

$$\lim_{x\to c} f(x) = \infty, \quad \text{i.e.,} \quad \forall n > 0\,(\exists\delta > 0 : f(x) > n \iff 0 < |x - c| < \delta)$$

Asymptotes of functions

Asymptotes

- **Asymptote**: a tangent line of a curve at a point at infinity; the distance between the curve and the line approaches zero as a coordinate tends to infinity.
- There are three kinds of asymptotes: *horizontal*, *vertical* and *oblique*; the nature of the asymptote is dependent on a function's relation to infinity.
  - **Horizontal asymptote**: a result of limits at infinity, i.e., when $x \to \pm\infty$.
  - **Vertical asymptote**: a result of infinite limits, i.e., when $x \to \pm c = \pm\infty$
  - **Oblique asymptote**: when a linear asymptote is not parallel to either axis; $f(x)$ is asymptotic to the straight line $y = mx + n\,(m \neq 0)$ if:

$$\lim_{x\to\pm\infty} [f(x) - (mx + n)] = 0$$

Derivatives

Derivative Fundamentals

Derivative | Thomas (3.2, 3.4)

- **Derivative**: the measure of sensitivity to change of the function value with respect to some change in its in argument.
  - Often descried as the instantaneous rate of change of a single variable function, since it is the slope of a tangent line at a particular point, when it exists.
    - **Tangent line**: the line through a pair of points on a curve (secant line), except the points are infinitely close, hence, it is the rate of change at that "instant".

Derivative Notation

- Formally, a derivative of the function $f(x)$, with respect to the variable x, is the function f' whose value at x is (provided the limit exists):

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

  - Let $z = x + h$, then $h = z - x \land h \to 0 \iff z \to x$; this leads to an equivalent definition of the derivative (sometimes more convenient):

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}$$

- **Notation**: there are many ways to denote the derivative; different notation can be useful in various contexts, some common notations (for $y = f(x)$):

$$f'(x) = y' = \dot{y} = \frac{dy}{dx} = \frac{d}{dx} f(x) = D(f)(x) = D_x f(x)$$

- **Differentiation**: the process of finding a derivative; if f' exists at a particular point, then f is said to be differentiable at that point.
  - If f' exists at every point on an interval, then f is differentiable on that interval.
  - f' is differentiable on a closed interval $[a, b]$ if both one-sided limits↑ of the function ($h \to \{0^+ : a, 0^- : b\}$) exist at the end points, and it is differentiable on the interior.
  - Not all continuous functions have a derivative, but functions with a derivative are continuous; functions with any of the following do not have derivatives:
    - corners (one-sided derivatives differ at a point),
    - cusps (slope approaches alternating $\pm\infty$ on both sides of a point),
    - discontinuities, or vertical tangent lines.

Differentiation Rules

Differentiation rules | Thomas (3.3, 3.5, 3.6)

- Derivatives can be found by computing the limit, but there are several methods that use combinations of simpler functions to make computation easier.

Linear, Product, Chain, Inverse

Product | Chain | Inverse

- **Linear**: differentiation of linear functions consists of the constant and sum rules, given the following:

$$\forall(f \land g) \land \forall(a \land b \in \mathbb{R}) \implies \frac{d(af + bg)}{dx} = a\frac{df}{dx} + b\frac{dg}{dx}$$

Constant
$\frac{d}{dx}(c) = 0$

Constant factor
$(af)' = af'$

Sum / Difference
$(f + g)' = f' + g'$

- **Product rule**: used for the product of two functions; can be generalized↓

$$\frac{d(fg)}{dx} = g\frac{df}{dx} + f\frac{dg}{dx}$$

- **Chain rule**: used for the composition of two functions $f(g(x))$; if z depends on y, which is dependent on x, then z depends on x as well, i.e.,

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$$

  - The following is used to indicate points of evaluation:

$$\left.\frac{dz}{dx}\right|_{x} = \left.\frac{dz}{dy}\right|_{y(x)} \cdot \left.\frac{dy}{dx}\right|_{x}$$

  - **Outside-Inside rule**: take the derivative of the "outside" function, leave the "inside" alone, and multiply it by the derivative of the "inside."
  - This method must be recursively "chained" when there are further compositions in the inside function, hence the name.

- **Inverse function rule**: can be applied if the function f has an inverse function g, i.e., "undoes" the effect of f.

$$\{g(f(x)) = x \land f(g(y)) = y\} \implies \frac{dx}{dy} = \left(\frac{dy}{dx}\right)^{-1}$$

  - Application of the chain rule on $f^{-1}(y) = x$ in terms of x clearly shows the result, if the derivatives exist and are reciprocal,

$$\frac{dx}{dy} \cdot \frac{dy}{dx} = \frac{dx}{dx} = 1$$

Power, Polynomial, Reciprocal, Quotient

Power | Reciprocal | Quotient

- **Power rule**: used to differentiate functions in the form of $f(x) = x^r$; can be applied to polynomials since differentiation is linear.
  - Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a function satisfying

$$f(x) = x^r, \quad \forall x\,(r \in \mathbb{R}) \implies \frac{d}{dx} = rx^{r-1}$$

- **Reciprocal rule**: yields the derivative of the reciprocal (multiplicative inverse) of a function f in terms of the derivative of f.
  - Can be used to show that the power rule holds for negative exponents.
  - The product and reciprocal rules can be used to deduce the quotient rule.
  - Let f be differentiable at x and $f(x) \neq 0$, then $g(x) = f(x)^{-1}$ is also differentiable and

$$\frac{d(f^{-1})}{dx} = -f^{-2}\frac{df}{dx} \quad \text{i.e.,} \quad g' = -\frac{f'}{f^2}$$

- **Quotient rule**: used to find the derivative of a function that is a ratio of two differentiable functions.
  - Let f and g be differentiable and $g(x) \neq 0$, then

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

Trigonometric Differentiation

Trigonometric functions

- All derivatives of circular trigonometric functions can be found from those of $\sin(x)$ and $\cos(x)$ by means of the quotient rule.

$$\begin{array}{rclrcl}
\sin(x) & \to & \cos(x) & \arcsin(x) & \to & \left(\sqrt{1-x^2}\right)^{-1} \\
\cos(x) & \to & -\sin(x) & \arccos(x) & \to & -\left(\sqrt{1-x^2}\right)^{-1} \\
\tan(x) & \to & \sec^2(x) & \arctan(x) & \to & \left(x^2+1\right)^{-1} \\
\cot(x) & \to & -\csc^2(x) & \operatorname{arccot}(x) & \to & -\left(x^2+1\right)^{-1} \\
\sec(x) & \to & \sec(x)\tan(x) & \operatorname{arcsec}(x) & \to & \left(|x|\sqrt{x^2-1}\right)^{-1} \\
\csc(x) & \to & -\csc(x)\cot(x) & \operatorname{arccsc}(x) & \to & -\left(|x|\sqrt{x^2-1}\right)^{-1}
\end{array}$$

- Inverse trigonometric functions are found using implicit differentiation↓ .

Differentiation Concepts

Thomas (3.7, 3.8)

Implicit Differentiation

Implicit differentiation

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Logarithmic Differentiation

Logarithmic differentiation

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Higher Order Derivatives

Second derivative

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Related Rates

Related rates

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Applications of Derivatives

Stationary Point

Maxima and Minima

-

Extreme Value Theorem

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Interior Extremum Theorem

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Mean Value Theorem

Rolle's Theorem

- ◦

Corollaries of the Mean Value Theorem

- ◦

Monotonic Functions

Derivative Tests

First-Derivative Test

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Second-Derivative Test

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Concavity

-

Higher-Order Derivative Test

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Differential Methods

Newton’s Method

-

Taylor’s Theorem

-

General Leibniz Rule

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Integrals

Integral Fundamentals

Terminology and Notation

-

Primer: Formal Definitions

-

Definite Integrals

Riemann Integral

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Integrability

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Properties of Definite Integrals

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The Fundamental Theorem of Calculus

Fundamental Theorem, Part 1

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Fundamental Theorem, Part 2

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The Integral of a Rate

-

Total Area

-

Integration By Substitution

Indefinite Integrals

-

Definite Integrals

-

Symmetric Functions

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Area Between Curves

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Applications of Definite Integrals

Solid of Revolution

Disc Integration

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Shell Integration

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Arc Length

Dealing with Discontinuities

-

Differential Arc Length

-

Surface of Revolution

Revolution about the y-Axis

-

Inverse Functions

One-to-One Functions

-

Derivative Rule for Inverses

-

Logarithmic Functions

Natural Logarithm

-

Properties of Logarithms

-

Trigonometric Integrals

-

Exponential Functions

Euler's Number

-

Natural Exponential Function

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Laws of Exponents

-

General Exponential Function

-

Exponential Change

- **Separable Differential Equations**
    -

  Examples of Exponential Change

    -

Indeterminate Forms

Indeterminate Form 0/0

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L'Hôpital's Rule

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Infinite Indeterminate Forms

-

Indeterminate Powers

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Inverse Trigonometric Functions

Principal Trigonometric Values

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Inverse Trigonometric Tables

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Hyperbolic Functions

Hyperbolic Function Tables

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Integration by Parts

Definite Integrals by Parts

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Trigonometric Integral Methods

Trigonometric Products and Powers

-

Trigonometric Square Roots

-

Trigonometric Substitutions

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Partial Fraction Decomposition

Partial Fraction Principles

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General Statement

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Numerical Integration

Trapezoidal Rule

-

Simpson’s Rule

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Improper Integrals

Indirect Evaluation

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First-Order Differential Equations

Ordinary Differential Equations

Differential equations | Ordinary DEQ

Solving ODEs

Thomas (9.1) | Rogawski (9.1)

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Models Involving y' = k(y-b)

Rogawski (9.2)

-

Slope Fields

Slope field

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Euler's Method

Euler's method

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First-Order Linear Differential Equations

Linear differential equation |

Solving LDEs

Thomas (9.2) | Rogawski (9.2)

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Partial Derivatives

Vector Calculus