

# Calculus III Exercises



## Introduction to Differential Equations

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# Introduction to Differential Equations



## 9.1 Edfinity: Solving Differential Equations

### 9.1.5

- Solve  $y' = x^5 y^2$ , using separation of variables, given the initial condition  $y(0) = 9$

$$\begin{aligned}\frac{\frac{dy}{dx}}{y^2} &= x^5 \\ \int \frac{\frac{dy}{dx}}{y(x)^2} &= \int x^5 dx \\ -\frac{1}{y(x)} &= \frac{x^6}{6} + c_1 \\ y(x) &= -\frac{6}{x^6 + c_1} \\ 9 &= -\frac{6}{c}, \quad c = -\frac{6}{9} \\ \boxed{y(x) &= -\frac{18}{2x^6 - 2}}\end{aligned}$$

### 9.1.6

- Solve the initial value problem  $\frac{dy}{dx} + 3y = 0$ ,  $y(\ln 4) = 3$ .

$$\begin{aligned}\frac{dy}{dx} &= -3y \\ \int \frac{dy}{dx} y^{-1} dx &= \int -3 dx \\ \ln |y| &= -3x + \lambda \\ y &= e^{-3x} \lambda \\ 3 &= e^{3(\ln 4)} \lambda \implies \lambda = 192 \\ y &= 192e^{-3x}\end{aligned}$$

### 9.1.7

- Solve  $(t^2 + 36)\frac{dx}{dt} = (x^2 + 9)$ , using separation of variables, given the initial condition  $x(0) = 3$ .

$$\begin{aligned}\frac{dx}{dt} &= (t^2 + 36)^{-1} \\ \frac{dx}{dt} (x^2 + 9)^{-1} &= (t^2 + 36)^{-1} \\ \int \frac{dx}{dt} (x^2 + 9)^{-1} &= \int (t^2 + 36)^{-1} dt \\ \frac{1}{9} \int \left( \frac{x^2}{9} + 1 \right)^{-1} dx &= \frac{1}{36} \int \left( \frac{t^2}{36} + 1 \right)^{-1} dt \\ \frac{1}{3} \tan^{-1} \left( \frac{x}{3} \right) &= \frac{1}{6} \tan^{-1} \left( \frac{t}{6} \right) + \lambda \\ \tan^{-1} \left( \frac{x}{3} \right) &= \frac{1}{2} \tan^{-1} \left( \frac{t}{6} \right) + \lambda \\ \tan^{-1} \left( \frac{3}{3} \right) &= \frac{1}{2} \tan^{-1} \left( \frac{0}{6} \right) + \lambda \\ \frac{\pi}{4} &= \lambda \\ x &= 3 \tan \left( \frac{1}{2} \tan^{-1} \left( \frac{t}{6} \right) + \frac{\pi}{4} \right)\end{aligned}$$

### 9.1.8

- Solve the initial value problem  $\frac{dy}{dx} = (x - 7)(y - 8)$ ,  $y(0) = 4$

$$\begin{aligned}\frac{dy}{dx} (y - 8)^{-1} &= (x - 7) \\ \int dy (y - 8)^{-1} &= \int (x - 7) dx \\ \ln y - 8 &= x^{-2} - 7x + \lambda \\ y &= e^{\frac{x^2}{2} - 7x} \lambda + 8 \\ -4 &= \lambda \\ y &= -4e^{\frac{x^2}{2} - 7x} + 8\end{aligned}$$

### 9.1.9

- Solve the initial value problem  $t^2 \frac{dy}{dt} - t = 1 + y + ty$ ,  $y(1) = 7$

$$\int (y + 1) dy = \int \frac{1 + t}{t^2} dt$$

$$\ln |1 + y| = -t^{-1} + \ln t + \lambda$$

$$y = \lambda e^{-t^{-1} + \ln t} - 1 = \lambda \frac{t}{e^{t^{-1}}} - 1$$

$$7 = \lambda e^{-1} - 1$$

$$\lambda = 8e$$

$$y = 8te^{1-t^{-1}} - 1$$

### 9.1.10

- Solve the initial value problem  $y' = 2y^2 \sin x$ ,  $y(0) = 6$

$$\frac{dy}{dx}(2y^2)^{-1} = \sin x$$

$$\int \frac{dy}{dx} y^{-2} dx = \int 2 \sin x dx$$

$$-y^{-1} = -2 \cos x + \lambda$$

$$y = (2 \cos x + \lambda)^{-1}$$

$$6 = (2 \cos 0 + \lambda)^{-1}$$

$$-\frac{11}{6} = \lambda$$

$$y = \left( 2 \cos x - \frac{11}{6} \right)^{-1}$$

## 9.2 Edfinity: Models Involving $y' = k(y-b)$

### 9.2.2

- Find the general solution of  $y' = 5(y - 16)$ .

$$y(t) = b + Ce^{kt}$$

$$y' = k(y - b)$$

$$y(t) = 16 + Ce^{5t}$$

$$30 = 16 + C$$

$$C = 14$$

$$y(t) = 16 + 14e^{5t}$$

$$1 = 16 + C$$

$$C = -15$$

$$y(t) = 16 + -15e^{5t}$$

### 9.2.3

- A 62 kg skydiver jumps out of an airplane. What is her terminal velocity in miles per hour, assuming that  $k = 10 \frac{kg}{s}$  for free fall?

$$-\frac{gm}{k} = -\frac{9.8(62)}{10} = -60.76 \frac{m}{s} = 199.343 \frac{ft}{s} = -134.916 \text{ mph}$$

### 9.2.4

- A continuous annuity with withdrawal rate  $N = \$600$  y and interest rate  $r = 5\%$  is funded by an initial deposit  $P_0$
- When will the annuity run out of funds if  $P_0 = \$10,000$ ?

$$P(t) = Nr^{-1} + Ce^{rt} = 600(0.05)^{-1} + Ce^{0.05t} = 12,000 + Ce^{0.05t}$$

$$10,000 = 12,000 + C$$

$$C = -2,000$$

$$t = 0.05^{-1} \ln \frac{12,000}{2,000} = 35.83 \approx 38 \text{ years}$$

- Which initial deposit  $P_0$  yields a constant balance?

$$P(t) = 12,000 + Ce^{0.05t}, \quad C = 0$$

$$P_0 = 12,000$$

### 9.2.5

- A cup of coffee, cooling off in a room temperature  $20^\circ\text{C}$ , has cooling constant  $k = 0.085 \text{ min}^{-1}$ .
- How fast is the coffee cooling when its temperature is  $T = 70^\circ\text{C}$ ?

$$k(T - T_0)$$
$$0.085(70 - 20) = 4.25^\circ\text{C}\cdot\text{min}^{-1}$$

- Use the Linear Approximation to estimate the change in temperature over the next 4 seconds when  $T = 70^\circ\text{C}$

$$4.25^\circ\text{C}\cdot\text{min}^{-1}(4\text{s})60\text{s}\cdot\text{min}^{-1} = 0.283^\circ\text{C}$$

- The coffee is served at a temperature of  $86^\circ\text{C}$ . How long should you wait before drinking it if the optimal temperature is  $65^\circ\text{C}$ ?

$$65 = 20 + 66e^{-0.085t}$$
$$t = -(0.085)^{-1} \ln \left( \frac{45}{66} \right)$$
$$t \approx 4.5 \text{ min}$$

