1. Determine if the following are 1 to 1, onto, or total. Justify your answer.

- **Total (total)**: $\forall x \in A \implies f(x)$ is defined.
- **1 to 1 (injective)**: $\forall x, y \in X, \quad f(x) = f(y) \implies x = y$.
- **Onto (surjective)**: $f : X \to Y, \quad \forall y \in Y, \quad \exists x \in X \implies f(x) = y$

(a) $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \sin x$

- Total: ✓ true: $\forall x \in \mathbb{R}, \quad f(x) \in \mathbb{R}$
- Injective: ✘ false: $\forall x, y \in \mathbb{R} : x \neq y$, some $f(x) = f(y)$
- Surjective: ✘ false: e.g., $2 \in \mathbb{R}$ but $\notin [-1, 1]$

(b) $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \sqrt{x}$

- Total: ✘ false: e.g., $f(-1) = i, i \notin \mathbb{R}$
- Injective: ✓ true (partial): $\forall x, y \in \mathbb{R}^+ : x = y \implies f(x) = f(y)$
- Surjective: ✓ true (partial): $\forall y \in \mathbb{R}^+, \exists x \in \mathbb{R} : f(x) = y$

(c) $f : \mathbb{N} \to \mathbb{R}^+, \quad f(x) = \sqrt{x}$

- Total: ✓ true: $\forall x \in \mathbb{N}, \quad f(x) \in \mathbb{R}^+$
- Injective: ✓ true: $\forall x, y \in \mathbb{N} : x = y \implies f(x) = f(y)$
- Surjective: ✓ true: $\forall y \in \mathbb{R}^+, \exists x \in \mathbb{N} : f(x) = y$

(d) $f : \mathbb{R}^+ \to \mathbb{N}, \quad f(x) = \sqrt{x}$

- Total: ✘ false: e.g., $\sqrt{42} \notin \mathbb{N}$
- Injective: ✓ true (partial):
  $\forall x, y \in \mathbb{R}^+ : x = y \implies f(x) = f(y) \iff f(x) \in \mathbb{N}$
- Surjective: ✓ true: $\forall y \in \mathbb{N}, \quad \exists x \in \mathbb{R}^+ : f(x) = y$

(e) $f : \mathbb{R} \to \mathbb{R}^+, \quad f(x) = x^2$

- Total: ✓ true: $\forall x \in \mathbb{R}, \quad f(x) \in \mathbb{R}^+$
- Injective: ✘ false: e.g., $f(-2) = 4, f(2) = 4$, i.e, $x \neq y, f(x) = f(y)$
- Surjective: ✓ true: $\forall \in \mathbb{R}^+, \quad \exists x \in \mathbb{R} : f(x) = y$

(f) $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = x^3$

- Total: ✓ true: $\forall x \in \mathbb{R}, \quad f(x) \in \mathbb{R}$
- Injective: ✓ true: $\forall x, y \in \mathbb{R}, : x = y \implies f(x) = f(y)$
- Surjective: ✓ true: $\forall y \in \mathbb{R}, \quad \exists x \in \mathbb{R} : f(x) = y$

2. Determine if the following relations are reflexive, symmetric, antisymmetric, or transitive. For this question

$$a, b \in \mathbb{N}, \quad \frac{p}{q}, \frac{m}{n} \in \mathbb{Q}, \quad s, t \in \Sigma^*$$

Justify your answer.

- **Equivalence**: $\sim, \equiv$, "is equal to"
- **Reflexive**: $\forall a \in X, \quad a \sim a$
- **Symmetric**: $\forall a, b \in X, \quad a \sim b \iff b \sim a$
- **Antisymmetric**:
  $\forall a, b \in X, \quad a \sim b, a \neq b \implies b \nsim a$ ...equiv... $a \sim b, b \sim a \implies a = b$
- **Transitive**: $\forall a, b, c \in X, \quad : a \sim b, b \sim c \implies a \sim c$

(a) $a \sim b$ if $a + b = 10$

- Reflexive: ✘ false: $6 \nsim 6$
- Symmetric: ✓ true: $a = 10 - b, b = 10 - a$
- Antisymmetric: ✘ false: $6 \sim 4, a \neq b \implies b \nsim a$
- Transitive: ✘ false: $4 \sim 6, 5 \sim 5, 4 \nsim 5$

(b) $a \sim b$ if a and b are both even/odd

- Reflexive:
- Symmetric:
- Antisymmetric:
- Transitive:

(c) $\frac{p}{q} \sim \frac{r}{s}$ if $q \leq s$

- Reflexive:
- Symmetric:
- Antisymmetric:
- Transitive:

(d) $s \sim t$ if $s = reverse(t)$

- Reflexive:
- Symmetric:
- Antisymmetric:
- Transitive:

(e) $a \sim b$ if $b = c \cdot a$ for some c

- Reflexive:
- Symmetric:
- Antisymmetric:
- Transitive:

(f) $a \sim b$ if $a^b = b^a$

- Reflexive:
- Symmetric:
- Antisymmetric:
- Transitive:

3. Prove that if $f : B \to C$ is 1 to 1, and $g : A \to B$ is 1 to 1, then $f \circ g$ is also 1 to 1.

4. Prove or disprove:

   (a) for any sets A and B, $P(A \cap B) = P(A) \cap P(B)$

   (b) for any sets A and B, $P(A \cup B) = P(A) \cup P(B)$

5. De Morgan's rule is a logical equivalence $(\neg a) \vee (\neg b) = \neg(a \wedge b)$
   You can verify this equivalence with a truth table.

   Set theory also has a version of De Morgan's rule. Let A and B be sets in universe U. Prove that $A' \cup B' = (A \cap B)'$

6. Mathologer is a YouTube channel that does videos illustrating math concepts. He did a video on the fair division problem

   NWT: Spanner's lemma defeats the rental harmony problem

   It's an interesting proof about coloring triangles. Unfortunately, He doesn't complete the proof. At about the 5 minute mark he says that "there will always be an odd number of doors at the bottom of the triangle", but he never proves this. Show that there will always be an odd number of doors on the bottom of the triangle.

7. In class I said that two functions were equal if they give the same output for every input. So, if

$$f : A \to B \text{ and } g : A \to B, \text{ and } \forall x \in A, f(x) = g(x) \implies f = g$$

Show that this is actually an equivalence relation.

8. In a dictionary all of the words are arranged in a specific order. “a”, “aardvark”, and so on. This is called the dictionary order for words.

   Our ordering here is very simple. Look at the first character of two words w_1 and w_2. If the first character of w_1 comes before w_2 ($w_1[0] < w_2[0]$) then $w_1 < w_2$.

   If the characters are equal, then we move on to the second character. We continue this until we find a different character, or one of the words ends. Show that this dictionary order is a partial order (reflexive, antisymmetric, and transitive).