Calculus III Midterm Review (Solutions)

1. True or false? (with justification)

   (a) If a_n is bounded, then it converges.

   ✘ False; sequences can oscillate between values, yielding a bounded sequence that does not converge to a consistent value. E.g., $a_n = (-1)^n$

   (b) If a_n converges, then is must be bounded.

   ✓ True; by Convergent Sequences theorem, i.e., if a_n converges, then a_n is bounded.

   (c) If a_n is not bounded, then it diverges.

   ✓ True; inverse of (b); sequences must be bounded in order to converge, if they aren't then they diverge.

   (d) If a_n diverges, then it is not bounded.

   ✘ False; inverse of (a); bounded sequences can diverge. Another example is $a_n = \cos \pi n$

2. A hot anvil with cooling constant $k = 0.02\,\mathrm{s}^{-1}$ is submerged in a large pool of water whose temperature is $10\,°\mathrm{C}$. Let $y(t)$ be the anvil's temperature t seconds later.

(a) What is the differential equation satisfied by $y(t)$?

Newton's Law of Cooling: $y' = -k(y - T_0)$

$$y(t) = \text{ temp. Of object}, \quad T_0 = \text{ ambient temp.}, \quad k = \text{ cooling constant.}$$

$$k = 0.02, \qquad T_0 = 10 \implies \boxed{y' = -0.02(y - 10)}$$

(b) Find a formula for $y(t)$, assuming the object initial temperature is $100\,°\mathrm{C}$.

General Solution of $y' = -k(y - b) \rightarrow y = b + Ce^{-kt}$

$$\begin{aligned} b = 10, \quad k = -0.02, &\implies y = 10 + Ce^{-0.02t} \\ P(0) = 100, &\implies 100 = 10 + Ce^{0} \\ &\implies C = 90 \\ &\implies \boxed{y = 10 + 90e^{-0.02t}} \end{aligned}$$

3. As an object cools, its rate of cooing slows. Explain how this follows from Newton's Law of Cooling.

- The difference in temperature between a cooling object and the ambient temperature is decreasing. Hence, the rate of cooling, which is proportional to this difference, is also decreasing in magnitude.

4. In Yellowstone park there were approximately 500 bison in 1970 and 3,000 bison in 1990.

   (a) Using the model that the rate of change of the population is proportional to the population itself, set up (do not solve) an initial value problem to model this situation.

$$
\begin{aligned}
&\text{Given: } P(t) = \text{ population of bison at time } t \\
&1970 \to t = 0 \\
&P' \propto P \\
&\implies \boxed{P' = kP \quad P(0) = 500 \wedge P(20) = 3,000}
\end{aligned}
$$

   (b) Now find the particular solution satisfying your initial value problem.

$$
\begin{aligned}
\text{Given: } P' = kP, \quad y = Ce^{kt} \text{ (general solution to } y' = ky) \\
&\implies P = Ce^{kt} \\
P(0) = 500 &\implies 500 = Ce^0 \\
&\implies C = 500, \quad P = 500e^{kt}
\end{aligned}
$$

$$
\begin{aligned}
P(3,000) = 500e^{20k} &\implies k = (\ln \frac{3000/500}{20}) \approx 0.0896 \\
&\implies \boxed{P = 500e^{0.0896t}}
\end{aligned}
$$

5. Which of the following are first-order linear equations?

   (a) $y' + x^2y = 1$ ✓ true

   (b) $y' + xy^2 = 1$ ✘ false

   (c) $x^5y' + y = e^x$ ✓ true

   (d) $x^5y' + y = e^y$ ✘ false

6. For what function P is the integrating factor $\alpha(x)$ equal to x? What about e^x?

$$\boxed{P(x) = x^{-1}} \to \alpha(x) = x$$

$$\boxed{P(x) = 1} \to \alpha(x) = e^x$$

7. Find the limit of the sequence $a_n = \dfrac{n+1}{3n+2}$. Be sure to justify your answer.

Using the Limit Laws:

$$\begin{aligned}
\lim_{n\to\infty} \frac{n+1}{3n+1} = \lim_{x\to\infty} \frac{x+1}{3x+1} &= \lim_{x\to\infty} \frac{x+1}{3x+1} \cdot \frac{x^{-1}}{x^{-1}} \\
&= \lim_{x\to\infty} \frac{1+x^{-1}}{3+x^{-1}} \\
&= \lim_{x\to\infty} \frac{1}{3} + \lim_{x\to\infty} \frac{x^{-1}}{x^{-1}} \\
&= \frac{1}{3} + \frac{0}{0} = \boxed{\frac{1}{3}}
\end{aligned}$$

Using L'HôpitalRule:

$$f(x) = n+1, \quad g(x) = 3n+2$$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{\infty}{\infty}$$

$$\implies \frac{f'(x)}{g'(x)} = \boxed{\frac{1}{3}}$$

8. Which of the following sequences converge to zero?

$$\frac{n^2}{n^2+1} \qquad 2^n \qquad \boxed{\left(-\frac{1}{2}\right)^n}$$

9. Which of the following sequences is defined recursively?

$$a_n = \sqrt{4+n} \qquad \boxed{b_n = \sqrt{4+b_{n-1}}}$$

10. Let a_n be the n^{th} decimal approximation to $\sqrt{2}$. I.e., $a_1 = 1$, $a_2 = 1.4$, $a_3 = 1.41$ and so on. What is $\lim_{n\to\infty} a_n$?

$$\boxed{\lim_{n\to\infty} a_n = \sqrt{2}}$$

11. Biologists stocked a lake with 400 fish and estimated the carrying capacity to be 10,000. The number of fish tripled in the first year. Assuming the size of the fish population satisfies the logistic equation, find an expression for the size of the population after t years.

- The logistic equation and it general non-equilibrium solution $(k > 0 \wedge A > 0)$

$$\frac{dy}{dt} = ky(1 - \frac{y}{A}) \to y = \frac{A}{1 - e^{-kt}/B} \equiv \frac{y}{y - A} = Be^{kt}$$

Given:

$$y(0) = 400, \quad y(1) = 1200, \quad A = 10,000$$

$$\begin{aligned} \frac{400}{400 - 10,000} &= Be^0 \\ B &= -\frac{1}{24} \\ \implies y &= \frac{10,000}{1 - e^{-kt}/ - \frac{1}{24}} = \frac{10,000}{1 + 24e^{-kt}} \end{aligned}$$

$$\begin{aligned} 1,200 &= \frac{10,000}{1 + 24e^{-k}} \\ 1 + 24e^{-k} &= \frac{10,000}{1,200} \\ e^{-k} &= \frac{22}{3} \cdot \frac{1}{24} = \frac{2 \cdot 11}{3 \cdot 2 \cdot 12} = \frac{11}{36} \\ k &= -\ln\left(\frac{11}{36}\right) \approx 1.186 \end{aligned}$$

$$\implies \boxed{y = \frac{10,000}{1 + 24e^{-1.186t}}}$$

12. Find the general term, a_n, for the sequence given below. Assume that we start our sequence at $n = 1$.

$$1, \frac{4}{2}, \frac{9}{4}, \frac{16}{8}, \frac{25}{16}, \frac{36}{32} \dots$$

$$\implies \frac{1^2}{2^{1-1}}, \frac{2^2}{2^{2-1}}, \frac{3^2}{2^{3-1}}, \frac{4^2}{2^{4-1}}, \frac{5^2}{2^{5-1}}, \frac{6^2}{2^{6-1}}, \dots, \frac{n^2}{2^{n-1}}$$

$$\implies \boxed{a_n = \frac{n^2}{2^{n-1}}}$$

13. Determine whether each of the following series converge or diverge by using either the Comparison Test or the Limit Comparison Test to justify your answer.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{e^n + n^2}$

$$e^n + n^2 \geq n^2$$
$$\frac{1}{e^n + n^2} \leq \frac{1}{n^2}$$

$$a_n = \frac{1}{e^n + n^2}, \quad b_n = \frac{1}{n^2}$$

Using direct compassion test: if b_n converges, then a_n converges. We know $\frac{1}{n^2}$ converges by the p-series, since $p > 1$. Thus, a_n converges.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{e^n - n^2}$

$$\text{Let } a_n = \frac{1}{e^n - n^2}, \quad b_n = \frac{1}{e^n}$$

$$\begin{aligned} \implies \lim_{n\to\infty} \frac{a_n}{b_n} &= \frac{e^n}{e^n - n^2} \\ &= \lim_{n\to\infty} \frac{e^n}{e^n - n^2} \cdot \frac{e^{-n}}{e^{-n}} \\ &= \frac{1}{1 - n^2 e^{-n}} = 1 \end{aligned}$$

Thus, since $L > 0$, and $\sum b_n$ converges, then $\sum a_n$ converges by the Limit Comparison Test.

14. True or false? If $k > 0$, then all solutions of $y' = -k(y - b)$ approach the same limit as $t \to \infty$.

✓ True; $k > 0$ will yield a general solution in the form of $y = b + Ce^{kt}$. Thus, all solutions will approach the same limit as $t \to \infty$.

15. Write a solution to $y' = 4(y - 5)$ that tends to $-\infty$ as $t \to \infty$.

$$\boxed{y(t) = 5 - Ce^{4t}, \quad c > 0}$$

16. Does $y' = -4(y - 5)$ have a solution that tends to ∞ and $t \to \infty$?

✘ No; any solution eventually tend to $-\infty$ with $k = -4$ as $t \to \infty$.

17. Find the general solution $y' = xe^{-\sin x} - y\cos x$.

$$\begin{aligned} y' + P(x)y &= Q(x) && \text{First-order linear DEQ} \\ \alpha(x) &= e^{\int P(x)dx} && \text{Integrating factor} \\ y &= \alpha(x)^{-1}\left(\int \alpha(x)Q(x)dx + C\right) && \text{General solution} \end{aligned}$$

$$\begin{aligned} y' + \cos(x)y &= xe^{-\sin x} && \text{Rewrite equation} \\ \alpha(x) &= e^{\int \cos x dx} = e^{\sin x} && \text{Find integrating factor} \end{aligned}$$

$$\begin{aligned} y &= e^{-\sin x}\left(\int e^{\sin x}xe^{-\sin x}dx + C\right) && \text{Plug in } \alpha(x) \\ &\boxed{y = e^{-\sin x}\left(\frac{x^2}{2} + C\right)} && \text{Simplify, integrate} \end{aligned}$$

18. True or false?

(a) $t\frac{dy}{dt} = 3\sqrt{1+y}$ is a separable differential equation.

✓ True;

$$t\frac{dy}{dt} = 3\sqrt{1+y}$$
$$\Longrightarrow \frac{dy}{dt} = \frac{3\sqrt{1+y}}{t} = 3t^{-1}\cdot\sqrt{1+y}$$

(b) $yy' + x + y = 0$ is a first-order linear differential equation.

✘ False; yy' makes this equation nonlinear. Note: it is still first order.

19. Determine the order of the following differential equations:

(a) $x^5y' = 1 \longrightarrow$ 1st

(b) $(y')^3 + x = 1 \longrightarrow$ 1st

(c) $y''' + x^4y' = 2 \longrightarrow$ 3rd

(d) $\sin(y'') + x = y \longrightarrow$ 2nd

20. Which of the following differential equations are directly integrable?

(a) $y' = x + y$ ✘ false

(b) $x\frac{dy}{dx} = 3$ ✓ true

(c) $\frac{dP}{dt} = 4P + 1$ ✘ false

(d) $\frac{dw}{dt} = \frac{2t}{1+4t}$ ✓ true

(e) $\frac{dx}{dt} = t^2e^{-3t}$ ✓ true

(f) $t^2\frac{dx}{dt} = x - 1$ ✘ false

Note: directly integrable differential equation is in the form $\frac{dy}{dx} = f(x)$.

21. Which of the following differential equations are separable?

(a) $\dfrac{dy}{dx} = x - 2y$ ✘ false

(b) $xy' + 8ye^x = 0$ ✓ true

(c) $y' = x^2y^2$ ✓ true

(d) $y' = 1 - y^2$ ✓ true

(e) $t\dfrac{dy}{dt} = 3\sqrt{1+y}$ ✓ true

(f) $\dfrac{dP}{dt} = \dfrac{P+t}{t}$ ✘ false

22. Which of the following equations are first-order?

(a) $y' = x^2$ ✓ true

(b) $y'' = y^2$ ✘ false

(c) $(y')^3 + yy' = \sin x$ ✓ true

(d) $x^2y' - e^xy = \sin y$ ✓ true

(e) $y'' + 3y' = \dfrac{y}{x}$ ✘ false

(f) $yy' + x + y = 0$ ✓ true

23. Water is draining from a cylindrical tank with cross sectional area $4m^2$ and height $5m$. Torricelli's Law says that the rate of change of the height of the water in such a cylindrical tank is proportional to the square root of the height of the water in the tank. Suppose that the tank starts full of water and after 30 minutes the height of the water has decreased to $4m$. Set up an initial value problem to model this situation.

Let $h(t)$ be the height of the water (meters) at time t (minutes), Then

$$\frac{dy}{dt} = k\sqrt{y}, \quad y(0) = 5, \quad y(30) = 4$$

$$\implies \int \frac{dy}{\sqrt{y}} = k \int dt$$

$$2\sqrt{y} = kt + C$$

$$\implies 2\sqrt{5} = C$$

$$\implies 2\sqrt{30} = k(4) + 2\sqrt{5}$$

$$4k = 2\sqrt{30} - 2\sqrt{5}$$

$$k = \frac{\sqrt{30} - \sqrt{5}}{2} \approx 1.621$$

$$\implies 2\sqrt{y} = 1.621t + 2\sqrt{5}$$

$$\sqrt{y} = 0.8105t + \sqrt{5}$$

$$\boxed{y = (0.8105t)^2 + 5}$$

24. Find the limit of each of the following sequences. Justify your answer using limit laws, the squeeze theorem, and/or L'Hôpital's Rule.

(a) $a_n = \dfrac{2(-1)^{n+1}}{2n-1}$

$$\forall n \geq 1, \qquad -\frac{2}{2n-1} \leq \frac{2(-1)^{n+1}}{2n-1} \leq \frac{2}{2n-1}$$

$$\lim_{n\to\infty} -\frac{2}{2n-1} = \lim_{n\to\infty} \frac{2}{2n-1} = 0$$

$$\implies \lim_{n\to\infty} \frac{2(-1)^{n+1}}{2n-1} = 0, \quad \text{By the squeeze theorem}$$

(b) $a_n = \dfrac{n^2}{2^{n-1}}$

$$\begin{aligned}
\lim_{n\to\infty} \frac{n^2}{2^{n-1}} &= \frac{\infty}{\infty} \\
&\implies \lim_{n\to\infty} \frac{2n}{\ln(2)2^{n-1}} && \text{by L'Hôpital's Rule} \\
&\implies \lim_{n\to\infty} \frac{2}{\ln(2)^2 2^{n-1}} && \text{by L'Hôpital's Rule} \\
&\implies \boxed{a_n = 0}
\end{aligned}$$

25. Does the series $\sum_{n=1}^{\infty} \cos\left(\frac{1}{n}\right)$ converge or diverge? How do you know?

Diverge;

$$\lim_{n\to\infty} \cos n^{-1} = \cos\left(\lim_{n\to\infty} n^{-1}\right) = \cos 0 = 1$$

By the n^{th} Term Divergence test, i.e., if $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Note: a series may still diverge even if a_n tends to zero.

26. Find the sum of each of the following geometric series or state that it diverges. Be sure to explain how you arrived at your solution.

(a) $\frac{1}{6} + \frac{1}{12} + \frac{1}{24} + \frac{1}{48} + \frac{1}{96} + \cdots$

$$\frac{1}{6}\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}\right) \qquad \text{Factor out ratio between terms}$$

$$\implies c = \frac{1}{6}, \quad r = \frac{1}{2} \qquad |r| < 1\text{, thus the series converges}$$

$$\sum_{n=1}^{\infty} cr^n = \frac{c}{1-r} \qquad \text{Sum of a Geometric Series}$$

$$\implies \frac{\frac{1}{6}}{1-\frac{1}{2}} = \boxed{\frac{1}{3}}$$

(b) $-2 + \frac{2}{5} - \frac{2}{25} + \frac{2}{125} - \frac{2}{625} + \cdots$

$$-2\left(1 - \frac{1}{5} + \frac{1}{25} - \frac{1}{125} + \frac{1}{625}\right) \qquad \text{Factor out ratio between terms}$$

$$\implies c = 2, \quad r = -\frac{1}{5} \qquad |r| < 1\text{, thus the series converges}$$

$$\implies \frac{2}{1-\frac{-1}{5}} = \boxed{-\frac{5}{3}}$$

27. What role do partial sums play in defining the sum of an infinite series?

- The sum of an infinite series is defined as the limit of the sequence of partial sums. If the limit of this sequence does not exist, the series is said to diverge.

28. Indicate whether of not the reasoning in the following statements are correct:

(a) $\sum_{n=1}^{\infty} \frac{1}{n^2} = 0$ because $\frac{1}{n^2}$ tends to zero.

✘ False; the infinite sum still tends towards infinity. The series does converge to zero, however.

(b) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ because $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

✘ False; again, same reasoning, the infinite sum will still tend toward infinity. This series does not converge, however.

29. Find an N such that $S_N > 25$ for the series $\sum_{n=1}^{\infty} 2$.

$$\frac{25}{2} = 12.5, \text{ thus, } N = 13$$

30. Does there exist an N such that $S_N > 25$ for the series $\sum_{n=1}^{\infty} 2^{-n}$? Explain.

✘ No; the series converges to 1 and is increasing, thus $S_N \leq 1$ for all N.

31. For the series $\sum_{n=1}^{\infty} a_n$, if the partial sums S_N are increasing, then (choose the correct conclusion):

(a) $\{a_n\}$ is an increasing sequence. ✘ false

(b) $\{a_n\}$ is a positive sequence. ✓ true

32. Which test would you use to determine whether the following series converge?

(a) $\sum_{n=1}^{\infty} n^{-3.2}$

p-Series $\left(\frac{1}{n^{3.2}}\right)$, or the Integral Test.

(b) $\sum_{n=1}^{\infty} \frac{1}{2^n + \sqrt{n}}$

Direct Comparison Test; as it is easy to choose a_n and b_n