Definitions

(1) $a|b \iff \exists c \in \mathbb{Z} \implies b = ca$

(2) $a\%b = r \impliedby \frac{a}{b}$ has remainder r

(3) $a \equiv b \bmod n \iff n|b-a$

Theorem 4.1: if n is even then n^2 is even. lecture

Theorem 4.2: $a|b \wedge a|c \implies a|b+c$ notes_4.pdf

Theorem 4.3: $a \equiv a\%n \bmod n$ notes_4.pdf

1. Prove that $n^2 \neq 2 \bmod 3, \quad \forall n \in \mathbb{Z}$

   Proof.

$$\forall n \in E, 2|n \implies 2|n^2 \quad \textit{by theorem 4.1}$$
$$2|n^2 = 2 \bmod 0 \neq 2 \bmod 3 \qquad \blacksquare$$

2. Fermat's Last theorem is a famous theorem in Math that was unproven for 200 years. The theorem says $\forall n > 2,\ a, b, c \in \mathbb{N} \implies a^n + b^n \neq c^n$. Another way to state this is $a^n + b^n = c^n$ has no integer solutions for n larger than 2. Use this theorem to prove that $\sqrt[n]{2}$ is irrational for n larger than 2.

   Proof.

$$\begin{aligned} \sqrt[n]{2} \in \mathbb{Q} &\implies \exists a, b \in \mathbb{Z} : \gcd(a, b) = 1 \\ &\implies \sqrt[n]{2} = \frac{a}{b} \implies a^n = 2b^n \\ &\implies a^n = b^n + b^n \qquad \blacksquare \end{aligned}$$

   Note: this is essentially zscoder's proof. No real credit here; I couldn't figure it out myself at first. It's pretty simple though, so I couldn't formulate something else that was better without adding unnecessary steps (originally completed in hwy).

3. Prove $\forall a, b, c \in \mathbb{Z} : a|b \land a|c \implies a|bx + cy \quad \forall x, y \in \mathbb{Z}$

Proof.

$$b = qa, \quad c = qa \quad \textit{by definition 1}$$
$$\implies a|qax + qay = a|a(qx + qy) = a|qa \qquad \blacksquare$$

4. Prove $\forall n, a, b \in \mathbb{Z}, n|a - b \iff a\%n = b\%n$

Proof.

$$a\%n = b\%n \iff \exists q \in \mathbb{Z} : \frac{a}{n} = \frac{qb}{n}$$
$$\implies a = qb$$
$$\implies n|qb - b = n|b(q - 1) \qquad \blacksquare$$

Proof by contradiction.

$$a\%n \neq b\%n \implies \exists q \notin \mathbb{Z} : \frac{a}{n} = \frac{qb}{n}$$
$$\implies a \neq qb \qquad \blacksquare$$

Thus, if $a\%n = b\%n$ then one integer is guaranteed to be a multiple of the other, which must be true for $a - b$ to be divisible by n. Alternatively, a contradiction arises because every integer should be able to be represented as a multiple of some other integer.

5. Let $a, b \in \mathbb{Z}, n \in \mathbb{N}$. Prove that

$$a \sim b \Longleftarrow a \equiv b \bmod n$$

is an equivalence relation for any n.

Proof.

$$a \sim a \wedge b \sim b \quad \textit{by theorem 4.3} \implies \checkmark \text{ reflexive}$$

$$\begin{aligned} n|b-a = n|a-b &\implies a \sim b \Longleftrightarrow b \sim a \quad \textit{by definition 3} \\ &\implies \checkmark \text{ symmetric} \end{aligned}$$

$$\begin{aligned} n|b-a &\implies a \sim b \wedge n|c-b \\ \implies b \sim c \implies n|c-a &\implies a \sim c \\ &\implies \checkmark \text{ transitive} \quad \blacksquare \end{aligned}$$

6. The greatest common divisor of natural numbers a, b; $\gcd(a, b)$, is the largest number δ such that $\delta|a \wedge \delta|b$

(a) Let $\delta = \gcd(b, a\%b)$, prove that $\delta|a \wedge \delta|b$

Proof.

$$\begin{aligned} a\%b = 0 &\implies a|b, \ \gcd(b, 0) = b \\ &\implies b = \delta, \ b = ca \\ &\implies \delta|b, \ \delta|ca \\ &\implies \delta|b \wedge \delta|a \end{aligned}$$

$$\begin{aligned} a\%b \neq 0 &\implies a\%b = r \quad \textit{by definition 2} \\ &\implies r|b-a \quad \textit{by definition 3} \\ &\implies r|a \wedge r|b \quad \textit{by question 3} \\ \delta|r \quad \textit{by definition of gcd} &\implies \delta|a \wedge \delta|b \quad \blacksquare \end{aligned}$$

(b) Use (a) to show that $\gcd(a, b) = \gcd(b, a\%b)$

Proof.

$$\begin{aligned} a\%b = 0 &\implies a \leq b, \delta = \max(a, b) = b \quad \textit{by part (a)} \\ a\%b \neq 0 &\implies \delta|r, 0 < r < a \leq b \quad \textit{by part (a)} \\ &\implies \delta = \max(b, r) = b \quad \blacksquare \end{aligned}$$

7. We defined the identity function

$$\mathrm{id} : A \to A, \quad \mathrm{id}(x) = x, \quad \text{has property: } \forall f : A \to A,\ \mathrm{id} \circ f = f \circ \mathrm{id} = f$$

Prove that id is the only function that can have this property.

Proof by contradiction.

$$g \neq \mathrm{id}, \forall g : A \to A \implies \forall a \in A : g(a) \notin A \wedge \mathrm{id}(a) \in A \qquad \blacksquare$$

I.e., there is no other distinct function that can map an element to itself that isn't already mapped to itself by the identity function.