1. Compute the following summations

(a) $\sum_{i=1}^{n} 6 = 6n$

(b) $\sum_{i=1}^{n} 5i = 5\left(\frac{n(n+1)}{2}\right)$

(c) $\sum_{i=1}^{n} 3i + 2 + 2^i = 3\left(\frac{n(n+1)}{2}\right) + 2n + 2^{n+1} - 1$

(d) $\sum_{i=1}^{n} 3^i = \sum_{i=0}^{n-1} 3^i = \frac{3^n - 1}{2}$

(e) $\sum_{i=1}^{n} \frac{1}{3^i} - \sum_{i=1}^{n} \frac{1}{3^i + 1} = \sum_{i=1}^{n} \frac{1}{3^i} - \frac{1}{3^i + 1} = -\frac{1}{3^i + 1} - \frac{3^{-n}}{2} + \frac{1}{2}$

(f) $\sum_{i=1}^{n} \sum_{j=1}^{n} j + ij = \sum_{j=1}^{n} j + \sum_{j=1}^{n} ij = \sum_{i=1}^{n} \frac{n(n+1) + in(n+1)}{2} = \frac{n^4 + 4n^3 + 3n^2}{4}$

(g) $\sum_{k=0}^{n} \binom{n}{k} 2^k = (1+2)^n = 3^n$ *by binomial theorem* $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$

2. Complete the following:

(a) You have 10 people, how many ways are there to split into two teams of 5?

$$\binom{10-1}{5-1} = 126$$

(b) If you have a red die and green die, how many different ways are there to roll a 7?

$$S_7 = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\} \implies |S_7| = 6$$

(c) How many ways are there to draw 5 cards from a deck of 52?

$$\binom{52}{5} = 2,598,960$$

(d) How many possible full house hands are there in poker?

$$13\binom{4}{3}12\binom{4}{2} = 3,744$$

(e) Out of a bag of red, blue and green marbles, how many ways are there to draw three marbles where the order doesn't matter?

$$\binom{3+2}{3} = 10$$

3. There are 38 people in this class. Prove that at least 4 people were born in the same month

   Proof.
   There are 12 possible months.
   In the most distributed case, then there are 12 months with 3 people.
   Therefore, by the pigeon hole principle, at least 1 of the months must have 4 or more people born in that month. ■

4. A round robin tournament is one in which every team players every other team. Assuming that there are no ties, and every team wins at least one game, prove that two teams won the same number of games.

   Proof.
   Assume there are n teams.
   Each team can have 1 to $n - 1$ wins since every team must have at least one win.
   $n > n - 1$, thus, by the pigeon hole principle, at least two teams must have the same number of wins. ■

5. We saw a lot of summation properties in class. Fortunately we don't need to prove all of them separately. We can prove several of them at the same time.

   Use induction to prove that summations are linear:

$$\sum_{i=m}^{n} c \cdot a_i + d \cdot b_i = c \cdot \sum_{i=m}^{n} a_i + d \cdot \sum_{i=m}^{n} b_i$$

Proof.

Base case: $\sum_{i=1}^{1} (ca_i + db_i) = c \cdot \sum_{i=1}^{1} a_i + d \cdot \sum_{i=1}^{1} b_i$

Inductive hypothesis: $\sum_{i=m}^{k} (ca_i + db_i) = c \cdot \sum_{i=m}^{k} a_i + d \cdot \sum_{i=m}^{k} b_i$

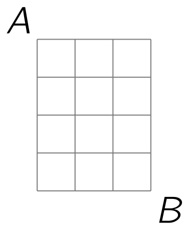
Inductive case:

$$\begin{aligned}
\sum_{i=1}^{k+1} (ca_i + db_i) &= \sum_{i=1}^{k} (ca_i + db_i) + (ca_{k+1} + db_{k+1}) \\
&= c \sum_{i=m}^{k} a_i + d \sum_{i=m}^{k} b_i + (ca_{k+1} + db_{k+1}) && \text{by I.H.} \\
&= \left(\sum_{i=m}^{k} ca_i + ca_{k+1} \right) + \left(\sum_{i=m}^{k} db_i + db_{k+1} \right) \\
&= c \sum_{i=1}^{k+1} a_i + d \sum_{i=1}^{k+1} b_i
\end{aligned}$$

■

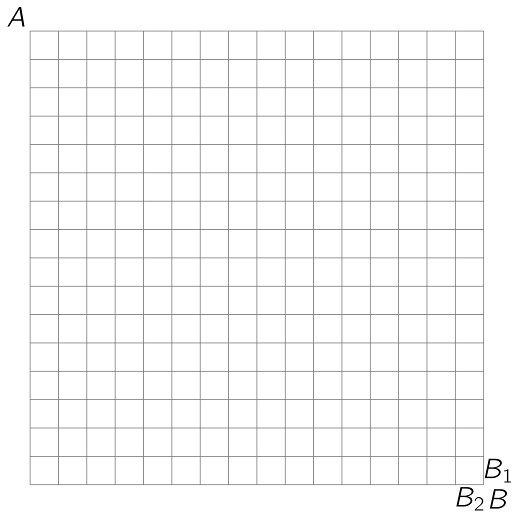
6. Complete the following:

(a) How many paths are there from A to B in the following grid if we're only allowed to move right and down.



$$m = 4, n = 3, \implies \binom{7}{4} = 35$$

(b) Now we have a much larger grid. This would be very difficult to count the paths by hand. Instead, let's try to come up with a recursive solution. Let's have the number of paths from A to B be P, the number of paths from A to B_1 be P_1, and the number of paths from A to B_2 be P_2. Now give a formula for P in terms of P_1 and P_2



(c) Now given a grid of m rows and n columns, give a recursive formula for the number of paths from A to B

$$P(m, n) = \binom{m+n}{m}$$