

Applied Linear Algebra



1 Matrices and Gaussian Elimination

1.2 The Geometry of Linear Equations	2
Problems 1–12.....	2
Problems 17–23	5
1.3 Gaussian Elimination	7
Problems 6, 7	7
Problems 17, 18, 19	7
Problems 30, 31	8
1.4 Matrix Notation and Matrix Multiplication	9
Problems 4, 6, 10, 17, 19.....	9
Problems 24, 29, 30, 31	9
Problems 34, 35, 38, 42.....	9
1.5 Triangular Factors and Row Exchanges	10
Problems 1, 6, 7, 8, 12, 18, 19	10
Problems 25–30.....	10
Problems 32, 33, 35, 42, 43	10
1.6 Inverses and Transposes	11
Problems 3, 10, 12, 13, 18, 20, 21	11
Problems 28–30.....	11
Problems 40–43.....	11
Problems 49–59.....	11
1.7 Special Matrices and Applications	12
Problems 1, 2, 5, 6	12
Problems 7, 8, 10.....	12

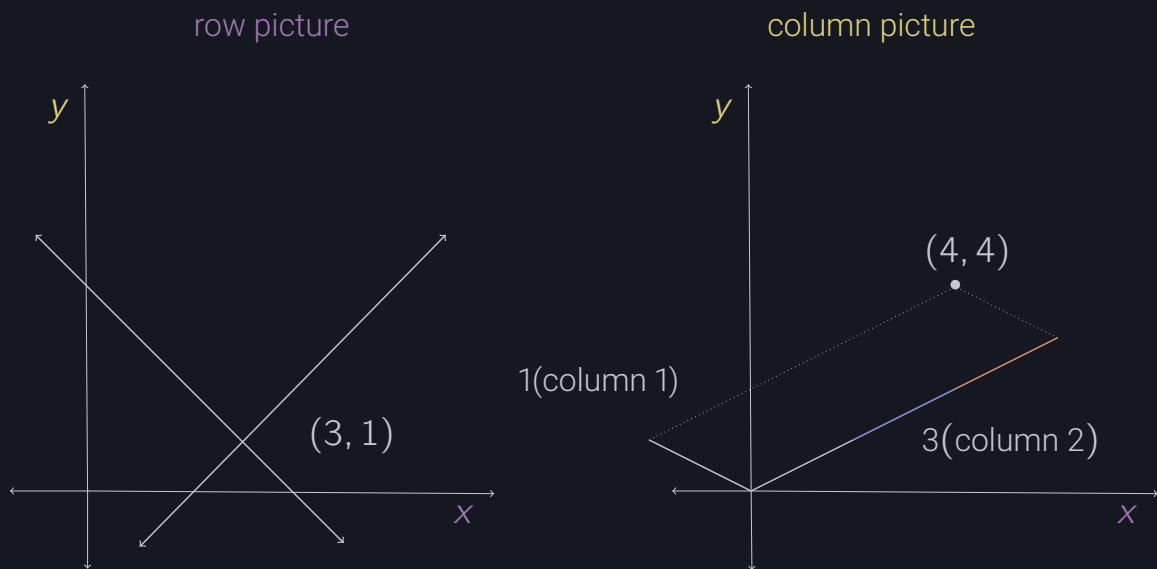
1 Matrices and Gaussian Elimination



1.2 The Geometry of Linear Equations

Problems 1–12

- For the equations $x + y = 4$, $2x - 2y = 4$, draw the row picture (two intersecting lines) and the column picture (combination of two columns equal to the column vector $(4, 4)$ on the right side).



1.2.1

- Solve to find a combination of the columns that equals b :

$$u - v - w = b_1$$

$$v + w = b_2$$

$$w = b_3$$

$$\implies w = b_3$$

$$\implies v = b_2 - b_3$$

$$\implies u = b_1 + v + w = b_1 + b_2$$

- Describe the intersection of the three planes $u + v + w + z = 6$ and $u + w + z = 4$ and $u + w = 2$ (all in four-dimensional space). Is it a line or a point or an empty set? What is the intersection if the fourth plane $u = -1$ is included? Find a fourth equation that leaves us with no solution.

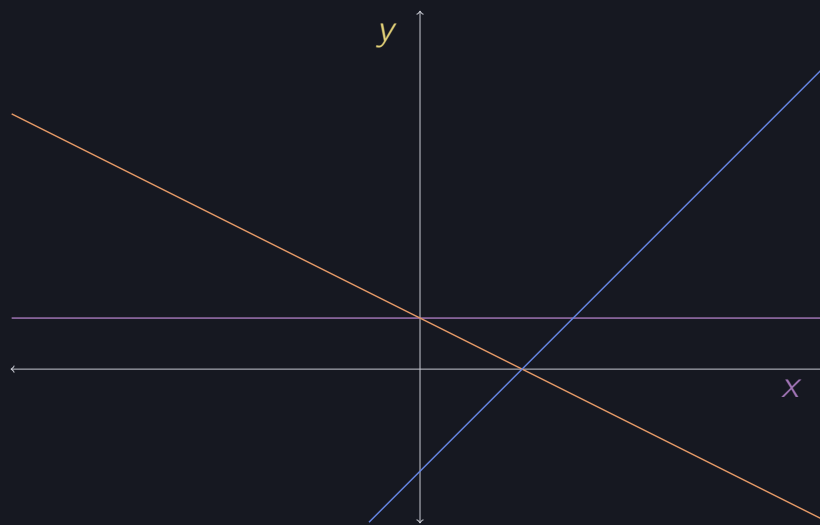
- A line; as $u + w = 2$ is only a line[?]. A fourth plane with $u = -1$ would produce a normally intersecting point. Any addition equation when $u + w \neq 2$ would produce an inconsistent equation.

4. Sketch these three lines and decide if the equations are solvable:

$$x + 2y = 2$$

$$x - y = 2$$

$$y = 1$$



1.2.4

Inconsistent; multiple points of intersect

What happens if all right-hand sides are zero? Is there any nonzero choice of right-hand sides that allows the three lines to intersect at the same point?

- If all the solutions were zero, then it would be a trivial solution.
- Yes, e.g., $x - y = -1$ would produce a single point of intersection.

5. Find two points on the line of intersection of the three planes $t = 0$ and $z = 0$ and $x + y + z + t = 1$ in four-dimensional space.

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

6. When $b = (2, 5, 7)$, find a solution (u, v, w) to equation (4) different from the solution $(1, 0, 1)$ mentioned in the text.

- Since there are infinite solutions, and if \mathbf{s} vector describing one solution and λ is any scalar, then $\mathbf{s}\lambda$ is also a solution. E.g., $(1, 0, 1) 42 = (42, 0, 42)$

8. Explain why the system

$$\begin{aligned}u + v + w &= 2 \\u + 2v + 3w &= 1 \\v + 2w &= 0\end{aligned}$$

is singular by finding a combination of the three equations that adds up to $0 = 1$. What value should replace the last zero on the right side to allow the equations to have solutions—and what is one of the solutions?

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 0 \end{array} \right] \xrightarrow{R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 2 & 0 \end{array} \right] \xrightarrow{R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

- Replacing the last zero with -1 would yield infinite solutions. One solution would be $[3, -1, 0]^T$

9. The column picture for the previous exercise (singular system) is

$$u \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + v \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + w \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = b$$

Show that the three columns on the left lie in the same plane by expressing the third as a combination of the first two. What are all the solutions (u, v, w) if b is the zero vector $(0, 0, 0)$?

$$-1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

- If b is equal to the zero vector $\mathbf{0}$ then the solutions are equal to the kernel² i.e., $-1x_1, 2x_2, 0x_3 = \mathbf{0}$
10. Under what condition on y_1, y_2, y_3 do the points $(0, y_1), (1, y_2), (2, y_3)$ lie on a straight line?
- Question 9 describes the state at which they are collinear, i.e., $y_3 = 2y_2 - y_1$
11. These equations are certain to have the solution $x = y = 0$. For which values of a is there a whole line of solutions?

$$\begin{aligned}ax + 2y &= 0 \\2x + ay &= 0\end{aligned}$$

- Only the scalars that make the lines linearly dependent, i.e., $a = 2, -2$

Problems 17–23

17. The first of these equations plus the second equals the third:

$$x + y + z = 2$$

$$x + 2y + z = 3$$

$$2x + 3y + 2z = 5$$

The first two planes meet along a line. The third plane contains that line, because if x, y, z satisfy the first two equations then they also **span all of \mathbb{R}^3** . The equations have infinitely many solutions (the whole line **L**). Find three solutions.

◦ $\mathbf{v} = (4, 4, 0)$, $\mathbf{w} = (6, 3, 2)$, $\mathbf{u} = 2\mathbf{v} + -1\mathbf{w}$

18. Move the third plane in Problem 17 to a parallel plane $2x + 3y + 2z = 9$. Now the three equations have no solution—*why not*? The first two planes meet along the line **L** , but the third plane doesn't that **cross** that line.

19. In Problem 17 the columns are $(1, 1, 2)$ and $(1, 2, 3)$ and $(1, 1, 2)$. This is a “singular case” because the third column is **linearly dependent**. Find two combinations of the columns that give $\mathbf{b} = (2, 3, 5)$. This is only possible for $\mathbf{b} = (4, 6, c)$ if $c = 10$

20. Normally 4 “planes” in four-dimensional space meet at a **tensor**. Normally 4 column vectors in four-dimensional space can combine to produce \mathbf{b} . What combination of $(1, 0, 0, 0)$, $(1, 1, 0, 0)$, $(1, 1, 1, 0)$, $(1, 1, 1, 1)$ produces $\mathbf{b} = (3, 3, 3, 2)$? **$(1, 0, 0, -2)$** ? What 4 equations for x, y, z, t are you solving? A **lower triangular matrix**, i.e.,

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3 \\ 1 & 1 & 0 & 0 & 3 \\ 1 & 1 & 1 & 0 & 3 \\ 1 & 1 & 1 & 1 & 1 \end{array} \right]$$

21. When equation 1 is added to equation 2, which of these are changed: the planes in the row picture, the column picture, the coefficient matrix, the solution?

◦ Row operations do not change the solution. Row 2 is changed, thus the second plane is changed. **All columns are changed.**?

1.3 Gaussian Elimination

Problems 6, 7

6. Choose a coefficient b that makes this system singular. Then choose a right-hand side g that makes it solvable. Find two solutions in that singular case.

$$2x + by = 16$$

$$4x + 8y = g$$

7. For which numbers a does elimination break down (a) permanently, and (b) temporarily?

$$ax + 3y = -3$$

$$4x + 6y = 6$$

Solve for x and y after fixing the second breakdown by a row exchange.

Problems 17, 18, 19

17. Which number q makes this system singular and which right-hand side t gives it infinitely many solutions? Find the solution that has $z = 1$.

$$x + 4y - 2z = 1$$

$$x + 7y - 6z = 6$$

$$3y + qz = t$$

18. It is impossible for a system of linear equations to have exactly two solutions. Explain why.
- If (x, y, z) and (X, Y, Z) are two solutions, what is the other one?
 - If 25 planes meet at two points, where else do they meet?
19. Three planes can fail to have an intersection point, when no two planes are parallel. The system is singular if row 3 of \mathbf{A} is a $\langle ? \rangle$ of the first two rows. Find a third equation that can't be solved if $x + y + z = 0$ and $x - 2y - z = 1$.

Problems 30, 31

30. Use elimination to solve

$$\begin{array}{rcl} u + v + w & = & 6 \\ u + 2v + 2w & = & 11 \\ 2u + 3v - 4w & = & 3 \end{array} \quad \text{and} \quad \begin{array}{rcl} u + v + w & = & 7 \\ u + 2v + 2w & = & 10 \\ 2u + 3v - 4w & = & 3 \end{array}$$

31. For which three numbers a will elimination fail to give three pivots?

$$\begin{array}{rcl} ax + 2y + 3z & = & b_1 \\ ax + ay + 4z & = & b_2 \\ ax + ay + az & = & b_3 \end{array}$$

1.4 Matrix Notation and Matrix Multiplication

Problems 4, 6, 10, 17, 19

○

Problems 24, 29, 30, 31

○

Problems 34, 35, 38, 42

○

1.5 Triangular Factors and Row Exchanges

Problems 1, 6, 7, 8, 12, 18, 19

-

Problems 25–30

-

Problems 32, 33, 35, 42, 43

-

1.6 Inverses and Transposes

Problems 3, 10, 12, 13, 18, 20, 21

-

Problems 28–30

-

Problems 40–43

-

Problems 49–59

-

1.7 Special Matrices and Applications

Problems 1, 2, 5, 6

-

Problems 7, 8, 10

-