

CALCULUS III FINAL REVIEW

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FINAL REVIEW QUESTIONS

CONVERGENCE: 10.3–10.5

Convergence Notes

- Let $\sum_{n=1}^{\infty} a_n$ be given and note for which series convergence is known, i.e.:

Geometric: let $c \neq 0$, if $|r| < 1$, then

p-Series: converges if $p > 1$.

$$\sum_{n=0}^{\infty} cr^n = \frac{c}{1-r}$$

$$\sum_{n=0}^{\infty} \frac{1}{n^p}$$

$|r| > 1 \implies$ diverges

$p < 1 \implies$ diverges

- The n^{th} Term Divergence Test:** a relatively easy test that can be used to quickly determine if a test diverges if the $\lim_{n \rightarrow \infty} a_n \neq 0$. If $\lim_{n \rightarrow \infty} a_n = 0$, then the test is inconclusive and other tests must be applied.

Tests for Positive Series

- Direct Comparison Test:** use if dropping terms from the denominator or numerator gives a series b_n wherein convergence is easily found, then compare to the original series a_n as follows:

$$\sum_{n=1}^{\infty} b_n \text{ converges} \implies \sum_{n=1}^{\infty} a_n \text{ converges} \quad \leftarrow 0 \leq a_n \leq b_n$$

$$\sum_{n=1}^{\infty} b_n \text{ diverges} \implies \sum_{n=1}^{\infty} a_n \text{ diverges} \quad \leftarrow 0 \leq b_n \leq a_n$$

- Limit Comparison Test:** use when the direct comparison test isn't convenient or when comparing two series. One can take the dominant term in the numerator and denominator from a_n to form a new positive sequence b_n if needed.

Assuming the following limit $L = \lim_{n \rightarrow \infty} \frac{a_n}{b_n}$ exists, then:

$$L > 0 \implies \sum_{n=1}^{\infty} a_n \text{ converges} \iff \sum_{n=1}^{\infty} b_n \text{ converges}$$

$$L = 0 \text{ and } \sum_{n=1}^{\infty} b_n \text{ converges} \implies \sum_{n=1}^{\infty} a_n \text{ converges}$$

$$L = \infty \text{ and } \sum_{n=1}^{\infty} a_n \text{ converges} \implies \sum_{n=1}^{\infty} b_n \text{ converges}$$

- **Ratio Test:** often used in the presence of a factorial ($n!$) or when the are constants raised to the power of n (c^n).

Assuming the following limit $\rho = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$ exists, then

$$\rho < 1 \implies \sum a_n \text{ converges absolutely}$$

$$\rho > 1 \implies \sum a_n \text{ diverges}$$

$$\rho = 1 \implies \text{test is inconclusive}$$

- **Root Test:** used when there is a term in the form of $f(n)^{g(n)}$.

Assuming the following limit $C = \lim_{n \rightarrow \infty} |a_n|^{\frac{1}{n}}$ exists, then

$$C < 1 \implies \sum a_n \text{ converges absolutely}$$

$$C > 1 \implies \sum a_n \text{ diverges}$$

$$C = 1 \implies \text{test is inconclusive}$$

- **Integral Test:** if the other tests fail and $a_n = f(n)$ is a decreasing function, then one can use the improper integral $\int_1^\infty f(x)dx$ to test for convergence.

Let $a_n = f(n)$ be a positive, decreasing, and continuous function $\forall x \geq 1$, then:

$$\int_1^\infty f(x)dx \text{ converges} \implies \sum_{n=1}^\infty a_n \text{ converges}$$

$$\int_1^\infty f(x)dx \text{ diverges} \implies \sum_{n=1}^\infty a_n \text{ diverges}$$

Tests for Non-Positive Series

- **Alternating Series Test:** used for series in the form $\sum_{n=0}^\infty (-1)^n a_n$

Converges if $|a_n|$ decreases monotonically ($|a_{n+1}| \leq |a_n|$) and if $\lim_{n \rightarrow \infty} a_n = 0$

- **Absolute Convergence:** used if the series $\sum a_n$ is not alternating (if it is alternating, use the alternating test in conjunction); simply test if $\sum |a_n|$ converges using the test for positive series.

$\sum a_n$ converges **conditionally** if $\sum a_n$ converges, but $\sum |a_n|$ diverges.

$\sum a_n$ converges **absolutely** if $\sum |a_n|$ converges.

Convergence Problems

10.5 Exercises

Determine convergence or divergence using any method.

$$1. \sum_{n=1}^{\infty} \frac{2^n + 4^n}{7^n}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{2^n}{7^n} + \sum_{n=1}^{\infty} \frac{4^n}{7^n}$$

Separate into two geometric series[†]

$$\Rightarrow r = \frac{2}{7} < 1, \quad r = \frac{4}{7} < 1$$

Both geometric series converge, thus the original series **converges**.

$$2. \sum_{n=1}^{\infty} \frac{n^3}{n!}$$

$$\Rightarrow \rho = \lim_{n \rightarrow \infty} \left| \frac{(n+1)^3}{(n+1)!} \cdot \frac{n!}{n^3} \right|$$

Apply the ratio test[†]

$$= \lim_{n \rightarrow \infty} \frac{n^3 + 3n^2 + 3n + 1}{(n+1)n!} \cdot \frac{n!}{n^3}$$

$$= \lim_{n \rightarrow \infty} \frac{n^3 + 3n^2 + 3n + 1}{n^4 + n^3}$$

$$= \lim_{n \rightarrow \infty} \frac{n^3 + 3n^2 + 3n + 1}{n^4 + n^3} \cdot \frac{n^{-4}}{n^{-4}}$$

$$= \lim_{n \rightarrow \infty} \frac{n^{-1} + 3n^{-2} + 3n^{-3} + n^{-4}}{1 + n^{-1}} = 0$$

$\rho = 0 < 1$, thus the series **converges**.

$$3. \sum_{n=1}^{\infty} \frac{n}{2n+1}$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{n}{2n+1}$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{n}{2n+1} = \frac{1}{2}$$

Apply the n^{th} term test[†]

By L'Hôpital's Rule

$\lim_{n \rightarrow \infty} a_n \neq 0$, thus the series **diverges**.

$$4. \sum_{n=1}^{\infty} 2^{\frac{1}{n}}$$

$$\Rightarrow \lim_{n \rightarrow \infty} 2^{\frac{1}{n}} = 2^0 = 1$$

Apply the n^{th} term test[†]

$\lim_{n \rightarrow \infty} a_n \neq 0$, thus the series **diverges**.

$$5. \sum_{n=1}^{\infty} \frac{\sin n}{n^2}$$

$$0 \leq \sin n \leq 1$$

$$\leftarrow \forall n \geq 1$$

$$0 \leq \frac{\sin n}{n^2} \leq \frac{1}{n^2}$$

Apply the **direct comparison test**[†]

$$b_n = \frac{1}{n^2} \rightarrow \text{converges}$$

by **p -series**[†]

The larger (b_n) series converges, thus the smaller (a_n) **converges**.

$$6. \sum_{n=1}^{\infty} \frac{n!}{(2n)!}$$

$$\Rightarrow \rho = \lim_{n \rightarrow \infty} \left| \frac{(n+1)!}{(2n+2)!} \cdot \frac{(2n)!}{n!} \right|$$

Apply the **ratio test** \uparrow

$$= \lim_{n \rightarrow \infty} \frac{(n+1)n!}{(2n+2)(2n+1)2n!} \cdot \frac{(2n)!}{n!}$$

$$= \lim_{n \rightarrow \infty} \frac{n+1}{(2n+2)(2n+1)} = \frac{n+1}{4n^2 + 6n + 2}$$

$$= \lim_{n \rightarrow \infty} \frac{1}{8n+6} = 0$$

By L'Hôpital's Rule

$\rho = 0 < 1$, thus the series **converges**.

$$7. \sum_{n=1}^{\infty} \frac{1}{n + \sqrt{n}}$$

$$0 \leq n \leq n + \sqrt{n} \quad \leftarrow \forall n \geq 1$$

$$0 \leq \frac{1}{n + \sqrt{n}} \leq \frac{1}{n}$$

Apply the **direct comparison test** \uparrow

$$b_n = \frac{1}{n} \rightarrow \text{diverges}$$

The smaller (b_n) series diverges, thus the larger a_n original series **diverges**.

$$8. \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$$

f is positive, decreasing, and continuous for $x \geq 2$ Apply the **integral test**[†]

$$\Rightarrow \int_2^{\infty} f(x) dx = \lim_{R \rightarrow \infty} \int_2^R \frac{1}{x(\ln x)^3} dx \quad \ln x = u, \quad x du = dx$$

$$\begin{aligned} \Rightarrow \lim_{R \rightarrow \infty} \int_2^R \frac{1}{x(u)^3} x du &= \int_2^R \frac{1^3}{u} du \\ &= -\frac{1}{2(u)^2} \\ &= -\frac{1}{2 \ln^2(x)} + C \Big|_2^{\infty} \end{aligned}$$

$$\Rightarrow 0 - \left(-\frac{1}{2 \ln^2(2)} \right) = \frac{1}{2 \ln^2(2)}$$

The improper integral converges, thus the original series **converges**.

$$9. \sum_{n=1}^{\infty} \frac{n^3}{5^n}$$

$$\begin{aligned} \Rightarrow \rho &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)^3}{5^{n+1}} \cdot \frac{5^n}{n^3} \right| \\ &= \lim_{n \rightarrow \infty} \frac{n^3 + 1}{5^n + 5^1} \cdot \frac{5^n}{n^3} = \frac{1}{5} \end{aligned}$$

Apply the **ratio test**[†]

$\rho = \frac{1}{5} < 1$, thus the series **converges**.

$$10. \sum_{n=2}^{\infty} \frac{1}{\sqrt{n^3 - n^2}}$$

$$L = \lim_{n \rightarrow \infty} \frac{a_n}{b_n}, \quad b_n = \frac{1}{\sqrt{n^3}}$$

Apply the [limit comparison test](#) [†]

$$\Rightarrow L = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n^3 - n^2}} \cdot \frac{\sqrt{n^3}}{1}$$

$$= \lim_{n \rightarrow \infty} \sqrt{\frac{n^3}{n^3(1 - n^{-1})}}$$

$$= \sqrt{\frac{1}{1(1 - 0)}} = 1$$

$L > 0$, thus a_n converges if b_n converges.

b_n converges by the p -series test, as $\frac{3}{2} > 1$, thus a_n [converges](#).

$$11. \sum_{n=1}^{\infty} \frac{n^2 + 4n}{3n^4 + 9}$$

$$L = \lim_{n \rightarrow \infty} \frac{a_n}{b_n}, \quad b_n = \frac{1}{n^2}$$

Apply the [limit comparison test](#) [†]

$$= \lim_{n \rightarrow \infty} \frac{n^2 + 4n}{3n^4 + 9} \cdot n^2$$

$$= \lim_{n \rightarrow \infty} \frac{n^4 + 4n^3}{3n^4 + 9} \cdot \frac{n^{-4}}{n^{-4}}$$

$$= \lim_{n \rightarrow \infty} \frac{1 + 4n^{-1}}{3 + 9n^{-4}} = \frac{1}{3}$$

$L > 0$, thus a_n converges if b_n converges.

b_n converges by the p -series test, as $2 > 1$, thus a_n [converges](#).

$$12. \sum_{n=1}^{\infty} (0.8)^{-n} n^{-0.8}$$

$$\begin{aligned} \rho &= \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{(0.8)^{-(n+1)} (n+1)^{-0.8}}{(0.8)^{-n} n^{-0.8}} \right| \\ &= \lim_{n \rightarrow \infty} \frac{(0.8)^n n^{0.8}}{(0.8)^{n+1} (n+1)^{0.8}} \\ &= \lim_{n \rightarrow \infty} \frac{1}{0.8} = 1.25 \end{aligned}$$

Apply the **ratio test** \uparrow

$\rho = 1.25 > 1$, thus a_n **diverges**.

$$13. \sum_{n=1}^{\infty} 4^{-2n+1}$$

$$\begin{aligned} \rho &= \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \\ &= \lim_{n \rightarrow \infty} \frac{4^{-2(n+1)+1}}{4^{-2n+1}} \\ &= \lim_{n \rightarrow \infty} \frac{4^{-2n-1}}{4^{-2n+1}} \\ &= \lim_{n \rightarrow \infty} \frac{4^{-2n} 4^{-1}}{4^{-2n} 4} = \frac{1}{16} \end{aligned}$$

Apply the **ratio test** \uparrow

$\rho = \frac{1}{16} < 1$, thus a_n **converges**.

$$14. \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$$

$$\sum_{n=1}^{\infty} |a_n|$$

Apply the **Absolute convergence test** [†]

$$\Rightarrow \sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$$

$|a_n|$ diverges by the p -series, as $\frac{1}{2} < 1$, meaning a_n **converges conditionally** since $|a_n|$ decreases monotonically and $\lim_{n \rightarrow \infty} a_n = 0$

$$15. \sum_{n=1}^{\infty} \sin \frac{1}{n^2}$$

$$L = \lim_{n \rightarrow \infty} \frac{a_n}{b_n}, \quad b_n = \frac{1}{n^2}$$

Apply the **limit comparison test** [†]

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{\sin(n^{-2})}{n^{-2}} = \frac{0}{0}$$

$$= \lim_{n \rightarrow \infty} \frac{\cos(n^{-2})(-2n^{-3})}{-2n^{-3}}$$

by L'Hôpital's Rule

$$= \lim_{n \rightarrow \infty} \cos(n^{-2}) = 1$$

$L > 0$, thus a_n converges if b_n converges.

b_n converges by the p -series test, as $2 > 1$, thus a_n **converges**.

$$16. \sum_{n=1}^{\infty} (-1)^n \cos n^{-1}$$

$$\sum_{n=1}^{\infty} (-1)^n a_n, \quad a_n = \cos(n^{-1})$$

Apply the **alternating series test** [†]

$$\Rightarrow L = \lim_{n \rightarrow \infty} \cos(n^{-1}) = 1$$

$L \neq 0$, thus the series **diverges**

$$17. \sum_{n=1}^{\infty} \frac{(-2)^n}{\sqrt{n}}$$

$$\sum_{n=1}^{\infty} (-1)^n a_n, \quad a_n = \frac{2^n}{\sqrt{n}}$$

Apply the **alternating series test** [†]

$$\begin{aligned} \Rightarrow L &= \lim_{n \rightarrow \infty} \frac{2^n}{\sqrt{n}} = \frac{\infty}{\infty} \\ &= \frac{2^n \ln 2}{\frac{1}{2} n^{-\frac{1}{2}}} = 2^n \ln 2 \cdot 2\sqrt{n} \quad \text{By L'Hôpital's Rule} \\ &= 2 \lim_{n \rightarrow \infty} 2^n \ln(2) \sqrt{n} = \infty \end{aligned}$$

$L \neq 0$, thus the series **diverges**

$$18. \sum_{n=1}^{\infty} \left(\frac{n}{n+12} \right)^n$$

$$L = \lim_{n \rightarrow \infty} a_n \neq 0 \rightarrow \text{diverges} \quad \text{Apply the } n^{\text{th}} \text{ term test}^{\dagger}$$

$$\Rightarrow L = \lim_{n \rightarrow \infty} \left(\frac{n}{n+12} \right)^n$$

$$= \lim_{n \rightarrow \infty} e^{-12}$$

$$\text{By common limit } \left(\frac{x}{x+k} \right)^x = e^{-k}$$

$L \neq 0$, thus the series **diverges**.

POWER/TAYLOR SERIES: 10.6–10.8

Power/Taylor Series Notes

Power Series

- **Power series:** a infinite series in the form:

$$F(x) = \sum_{n=0}^{\infty} a_n(x - c)^n$$

Where the constant c is the *center* of the power series $F(x)$.

- **Radius of convergence R :** the range of values of the variable x whereby the power series $F(x)$ converges.
 - Every power series converges at $x = c$, as $(x - c)^0 = 1$, though the series may diverge for other values of x .
 - $F(x)$ converges for $|x - c| < R$ and diverges for $|x - c| > R$
 - $F(x)$ may converge or diverge at endpoints $c - R$ and $c + R$
 - **Interval of convergence:** the open interval $(c - R, c + R)$ and possibly one of both of the endpoints, each must be tested.
 - In most cases, the **ratio test** [†] can be used to find R .
 - If $R > 0$, then F is differentiable over the interval of convergence; the derivative and antiderivative can be obtained using the following:

$$F'(x) = \sum_{n=1}^{\infty} n a_n (x - c)^{n-1} \qquad F(x) dx = C + \sum_{n=0}^{\infty} \frac{a_n}{n+1} (x - c)^{n+1}$$

- **Useful Power Series:** the following power series (more examples: **Taylor series** [↓]) can be used to drive expansions of other related functions via substitution, integration, or differentiation:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \leftarrow |x| < 1 \qquad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Taylor Series

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Power/Taylor Series Problems

10.6 Exercises

1.

10.8 Exercises

1.

PARAMETRIC EQUATIONS: 11.1

Parametric Equations Notes

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Parametric Problems

11.1 Exercises

1.

ARC LENGTH, POLAR COORDINATES: 11.2–11.4

11.2–11.4 Notes

Arc Length and Speed

-

Polar Coordinates

-

Area and Arc Length in Polar Coordinates

-

Polar Coordinate Problems

11.2 Exercises

1.

11.3 Exercises

1.

11.4 Exercises

1.

QUIZ QUESTIONS

Quiz 3

1. For each statement below, indicate whether it is **True** or **False** and provide a brief justification.

(a) The series $\sum_{n=1}^{\infty} (-1)^n \cos\left(\frac{1}{n}\right)$ converges.

(b) If the radius of converges of the power series $\sum_{n=0}^{\infty} a_n x^n$ is $R = 5$, then the series must converge for $x = -3$ and $x = -4$.

2. Determine whether the following series converge absolutely/conditionally, or diverge.

(a)
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}n}{2+5}$$

(b)
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{2\sqrt{n}-1}$$

3. Find a power series expansion with the center $c = 0$ for

$$f(x) = \frac{1}{1+x^3}$$

and find the interval of convergence. Hint: use $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \leftarrow |x| < 1$

4. Find the radius of convergence of the power series given by

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+1}}{2^n n}$$

Quiz 4

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FINAL REVIEW QUESTIONS

Note: these questions were taken from a provided review sheet; they focus on sections 10.6–11.4. Some questions already exist on the quizzes, but will be duplicated here.

1. Find the interval of convergence of the following power series.

(a) $\sum_{n=1}^{\infty} \frac{5^n}{n} x^n$

(b) $\frac{(x-2)^n}{n^2+1}$

2. Find the Taylor series of the following functions $f(x)$ centered at the given value of c using the definition.

(a) $f(x) = e^x, \quad c = 2$

(b) $f(x) = \sqrt{x}, \quad c = 1$

3. Find the Maclaurin series of the following functions using substitution and/or multiplication.

(a) $f(x) = x \cos(2x)$

(b) $f(x) = \frac{x^3}{1+x}$

4. Express the following integral as a power series, first by finding the Maclaurin series of the integrand, then integrating this series term-by-term:

$$\int_0^1 e^{-x^2} dx$$

5. Find the parametric equations for the following curves.

(a) The line through $(3, 6)$ and $(-2, 0)$.

(b) The circle of radius 5 centered at $(1, 7)$.

(c) The ellipse

$$\left(\frac{x-1}{2}\right)^2 + \frac{y+1}{3} = 1$$

6. Find the equation of the tangent line to the curve

$$x = \sin(2t) + \cos(t), \quad y = \cos(2t) - \sin(t), \quad \leftarrow t = \pi$$

7. Find the arc length of the curve

$$x = \frac{2}{3}t^2, \quad y = t^2 - 2, \quad \leftarrow 0 \leq t \leq 2$$

8. Find the surface area obtained by rotating the following around the x-axis;

$$x = e^t - t, \quad y = 4e^{\frac{t}{2}}, \quad \leftarrow 0 \leq t \leq 1$$

9. Match each equation in rectangular coordinates with its equation in polar coordinates.

(a) $x^2 + y^2 = 4$

(i) $r^2(1 - 2 \sin^2 \theta) = 4$

(b) $x^2 + (y - 1)^2 = 1$

(ii) $r(\cos \theta + \sin \theta) = 4$

(c) $x^2 - y^2 = 4$

(iii) $r = \sin \theta$

(d) $x + y = 4$

(iv) $r = 2$

10. Find the area enclosed by one loop of the curve

$$r^2 \cos 2\theta$$