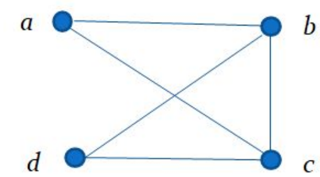
Graphs and Trees
กราฟและต้นไม้

Graphs

- **Definition**: A **graph** $G = (V, E)$ consists of a nonempty set V of **vertices** (or **nodes**) and a set E of **edges**.
- Each edge has either one or two vertices associated with it, called its endpoints. An edge is said to connect its endpoints.
- A graph with an infinite vertex set is called an ***infinite graph***. A graph with a finite vertex set is called a ***finite graph***. This chapter focuses on finite graphs.
- Example: A graph with 4 vertices and 5 edges.



Graphs

- There are different kinds of graphs: directed or undirected, and loops.
- A graph represents connections between objects.
  - Acquaintance between people
  - Collaboration between working people
  - Links between websites
  - Computer network and the Internet
  - Road and rail network

Chapter Summary

1. Graphs and Graph Models
2. Graph Terminology and Special Types of Graphs
3. Representing Graphs
4. Connectivity
5. Euler and Hamiltonian Graphs
6. Graph Coloring

1. Graphs and Graph Models

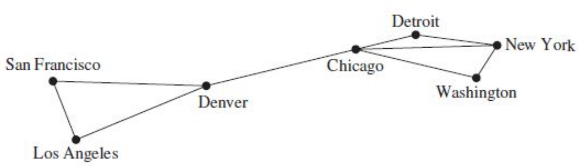
Terminology

- In a ***simple graph*** each edge connects two different vertices and no two edges connect the same pair of vertices.
- ***Multigraphs*** may have multiple edges connecting the same two vertices.
  - When m different edges connect the vertices *u* and *v*, we say that {*u*, *v*} is an edge of multiplicity m.
- An edge that connects a vertex to itself is called a ***loop***.

Graph Models: Computer Networks

- A network is made up of data centers and communication links between computers.
- When we build a graph model, we use the appropriate type of graph to capture the important features of the application.
- We illustrate this process using graph models of different types of computer networks.
- In all these graph models, the vertices represent data centers and the edges represent communication links.

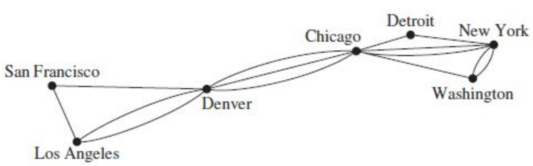
Graph Models: Computer Networks



Simple Graph

- A simple graph can be used to model a computer network where we are only concerned whether two data centers are connected by a communications link.
- Only interest in whether two data centers are directly linked (and not how many links there may be) and all communications links work in both directions.

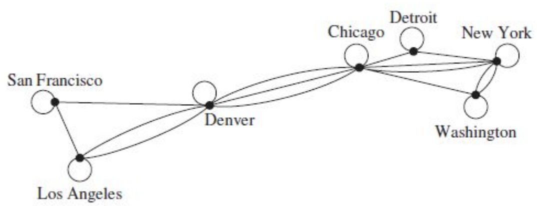
Graph Models: Computer Networks



Multigraph

- A multigraph can be used to model a computer network where we care about the number of links between data centers.

Graph Models: Computer Networks



Multigraphs with loops

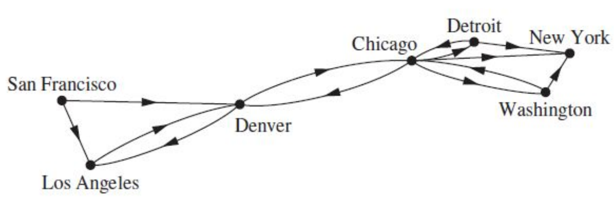
- To model a computer network with diagnostic links at data centers, we use a pseudograph, as feedback loops are needed.

Directed Graph

Definition: A ***directed graph*** (or ***digraph***) $G = (V, E)$ consists of a nonempty set V of vertices (or nodes) and a set E of directed edges (or arcs).

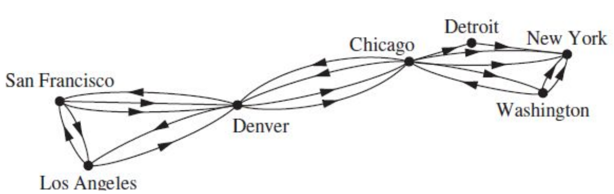
- Each edge is associated with an ordered pair of vertices.
- The directed edge associated with the ordered pair $(u\ v)$ is said to start at u and end at v.
- Graphs where the endpoints of an edge are not ordered are said to be *undirected graphs*.

Graph Models: Computer Networks



Directed Graph

- To model a network with direction.

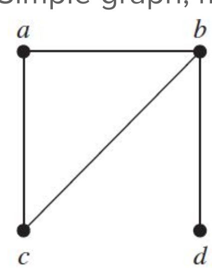


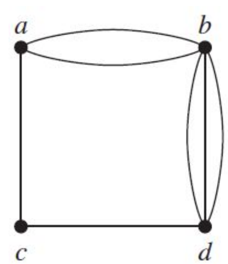
Directed Multigraphs

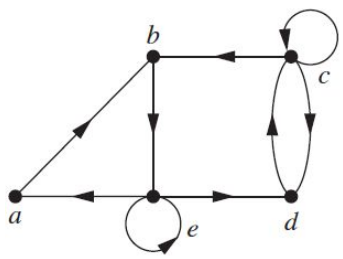
- To model a network with multiple one-way links.

Type of Graph Exercise

Simple graph, multigraph, directed?







Other Applications of Graphs

- Social networks
- Communications networks (Call graphs)
- Information networks (Web pages and links)
- Software design (Modules/Library dependencies)
- Transportation networks (Airline routes and road networks)
- Biological networks (Species and protein interaction network)
- Semantic Networks (NLP)
- Tournaments

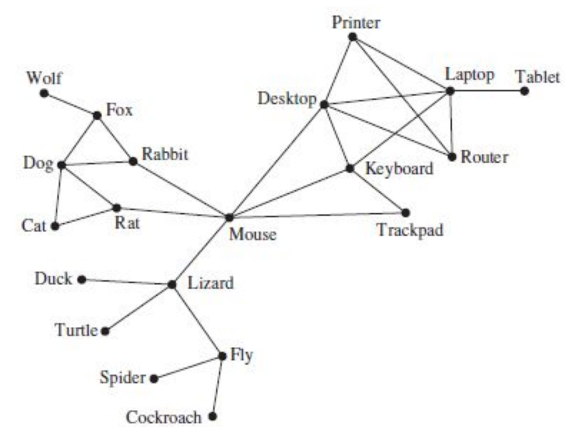
Graph Models: Social Networks

- Graphs can be used to model social structures based on different kinds of relationships between people or groups.
- In a *social network*, vertices represent individuals or organizations and edges represent relationships between them.
- Useful graph models of social networks include:
  - *friendship graphs* - undirected graphs where two people are connected if they are friends (in the real world, on Facebook, or in a particular virtual world, and so on.)
  - *collaboration graphs* - undirected graphs where two people are connected if they collaborate in a specific way.
  - *influence graphs* - directed graphs where there is an edge from one person to another if the first person can influence the second person.

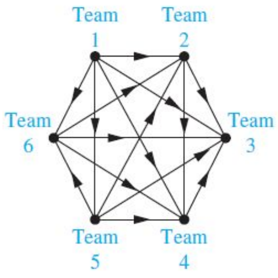
Example of Collaboration Graphs

- The ***Hollywood graph*** models the collaboration of actors in films.
  - We represent actors by vertices and we connect two vertices if the actors they represent have appeared in the same movie.
  - We will study the Hollywood Graph when we discuss Kevin Bacon numbers.

Example of Semantic Network

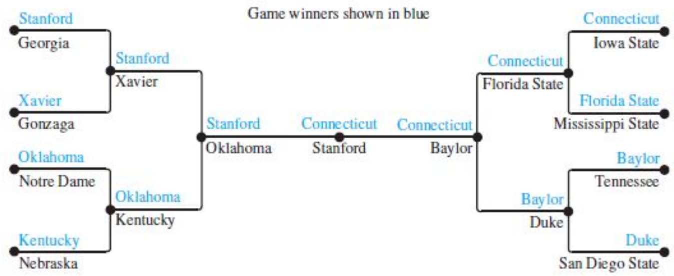


Example of Round-Robin Tournaments



- Each team plays every other team exactly once and no ties are allowed.
- (a, b) is an edge if team a beats team b.
- Simple directed graph, no loops or multiple directed edges.
- Whose team is undefeated?
- Whose team never win?

Example of Single-Elimination Tournaments



- Each contestant is eliminated after one loss.

Graph Exercise

Exercise 1: Draw graph models, stating the type of graph used, to represent airline routes where every day there are

- 4 flights from Bangkok to Chiang Mai,
- 2 flights from Chiang Mai to Bangkok,
- 3 flights from Chiang Mai to Krabi,
- 2 flights from Krabi to Chiang Mai,
- 1 flight from Chiang Mai to Loei,
- 2 flights from Loei, to Chiang Mai,
- 3 flights from Chiang Mai to Samui,
- 2 flights from Samui to Chiang Mai, and
- 1 flight from Samui to Krabi, with (*continued in next page*)

Graph Exercise

a) An edge between vertices representing cities that have a flight between them (in either direction).

Graph Exercise

b) An edge between vertices representing cities for each flight that operates between them (in either direction).

Graph Exercise

c) An edge from a vertex representing a city where a flight starts to the vertex representing the city where it ends.

Graph Exercise

d) An edge for each flight from a vertex representing a city where the flight begins to the vertex representing the city where the flight ends.

2. Graph Terminology and Special Types of Graphs

Terminology

Definition 1: Two vertices u, v in an undirected graph G are called ***adjacent*** (or ***neighbors***) in G if there is an edge e between u and v.

- Such an edge e is called *incident* with the vertices u and v and e is said to connect u and v.

Definition 2. The set of all neighbors of a vertex v of $G = (V, E)$, denoted by $N(v)$, is called the ***neighborhood*** of v.

- If A is a subset of V, we denote by $N(A)$ the set of all vertices in G that are adjacent to at least one vertex in A. So, $N(A) = \bigcup_{v \in A} N(v)$.

2. Graph Terminology and Special Types of Graphs

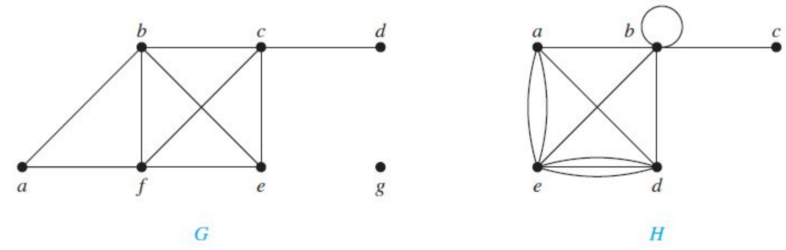
Terminology

Definition 3: The ***degree of a vertex*** in a *undirected graph* is the number of edges incident with it, except that a loop at a vertex contributes two to the degree of that vertex.

- The degree of the vertex v is denoted by deg(v).

Degrees and Neighborhoods of Vertices

Example: What are the degrees and neighborhoods of the vertices in the graphs G and H?



Degrees of Vertices

Theorem 1 (*Handshaking Theorem*): If $G = (V, E)$ is an undirected graph with m edges, then

$$2m = \sum_{v \in V} \deg(v)$$

Proof:

Each edge contributes twice to the degree count of all vertices. Hence, both the left-hand and right-hand sides of this equation equal twice the number of edges.

- Think about the graph where vertices represent the people at a party and an edge connects two people who have shaken hands.

Handshaking Theorem

Example 1: How many edges are there in a graph with 10 vertices of degree six?

Example 2: If a graph has 5 vertices, can each vertex have degree 3?

Degrees of Vertices

Theorem 2: An undirected graph has an even number of vertices of odd degree.

Proof: Let V_1 be the vertices of even degree and V_2 be the vertices of odd degree in an undirected graph $G = (V, E)$ with m edges. Then

$$2m = \sum_{v \in V} \deg(v) = \sum_{v \in V_1} \deg(v) + \sum_{v \in V_2} \deg(v).$$

Directed Graph

Definition 4: When (u, v) is an edge of the graph G with directed edges, u is said to be ***adjacent*** to v and v is said to be ***adjacent*** from u.

- The vertex u is called the *initial vertex* of (u, v), and v is called the *terminal* or *end vertex* of (u, v).
- The initial vertex and terminal vertex of a loop are the same.

Directed Graph

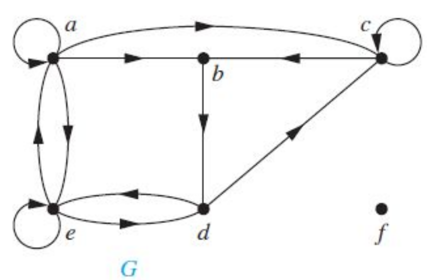
Definition 5: In a graph with directed edges the ***in-degree of a vertex v***, denoted by $\deg^{-}(v)$, is the number of edges with v as their *terminal* vertex.

The ***out-degree of v,*** denoted by $\deg^{+}(v)$, is the number of edges with v as their *initial* vertex.

- Note that a loop at a vertex contributes 1 to both the in-degree and the out-degree of this vertex.

Directed Graph Example

Example: Find the in-degree and out-degree of each vertex in the graph G with directed edges.



Directed Graph

Theorem 3: Let $G = (V, E)$ be a graph with directed edges. Then

$$|E| = \sum_{v \in V} deg^-(v) = \sum_{v \in V} deg^+(v).$$

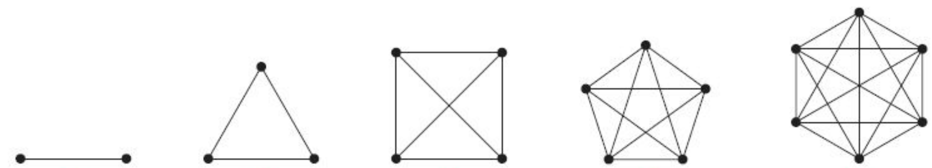
Proof:

- Because each edge has an initial vertex and a terminal vertex, the sum of the in-degrees and the sum of the out-degrees of all vertices in a graph with directed edges are the same.
- Both of these sums are the number of edges in the graph.

Special Types of Simple Graphs

A ***complete graph*** on n vertices, denoted by K_n, is a simple graph that contains exactly one edge between each pair of distinct vertices.

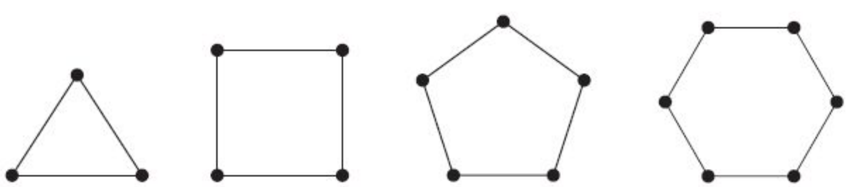
A simple graph for which there is at least one pair of distinct vertex not connected by an edge is called **noncomplete**.



K_2 K_3 K_4 K_5 K_6

Special Types of Simple Graphs

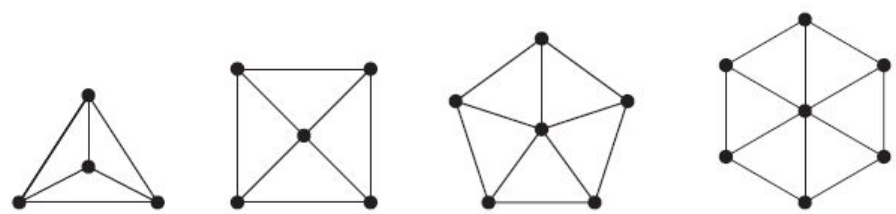
A **cycle** C_n, $n \geq 3$, consists of n vertices $v_1, v_2, \ldots, v_n$ and edges $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}$, and $\{v_n, v_1\}$.



C_3 C_4 C_5 C_6

Special Types of Simple Graphs

A **wheel** W_n is obtained by adding an additional vertex to a cycle C_n, for $n \geq 3$, and connect this new vertex to each of the n vertices in C_n, by new edges.

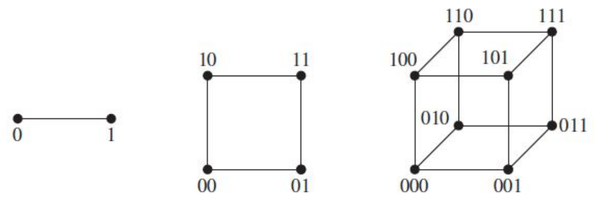


W_3 W_4 W_5 W_6

Special Types of Simple Graphs

An **n-dimensional hypercube**, or ***n*-cube**, denoted by $\boldsymbol{Q_n}$, is a graph that has vertices representing the 2^n bit strings of length n.

Two vertices are adjacent if and only if the bit strings that they represent differ in exactly one bit position.



Q_1 Q_2 Q_3

3. Representing Graphs

Adjacency Lists

Definition 1: An adjacency list can be used to represent a graph with no multiple edges by specifying the vertices that are adjacent to each vertex of the graph.

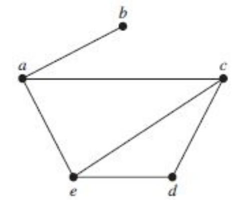
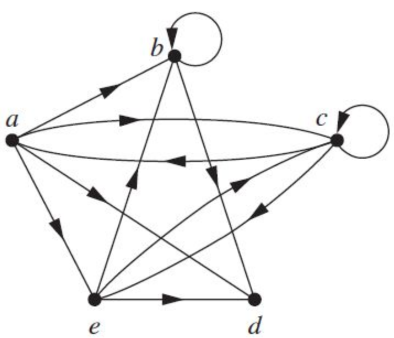


FIGURE 1 A simple graph.

TABLE 1 An Adjacency List for a Simple Graph.

Vertex	*Adjacent Vertices*
a	b, c, e
b	a
c	a, d, e
d	c, e
e	a, c, d

Adjacency Lists Exercise



Adjacency Matrices

Definition 2: Suppose that $G = (V, E)$ is a simple graph where $|V| = n$. Suppose that the vertices of G are listed arbitrarily as $v_1, v_2, \ldots, v_n$.

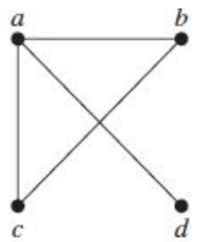
The adjacency matrix $\boldsymbol{A}_G$ of G, with respect to the listing of vertices, is the

$n \times n$ zero-one matrix with 1 as its (i, j)th entry when v_i and v_j are adjacent, and 0 as its (i, j)th entry when they are not adjacent.

- In other words, if the graphs adjacency matrix is $\boldsymbol{A}_G = [a_{ij}]$, then

$$a_{ij} = \begin{cases} 1 & \text{if } \{v_i, v_j\} \text{ is an edge of } G, \\ 0 & \text{otherwise.} \end{cases}$$

Adjacency Matrices Examples



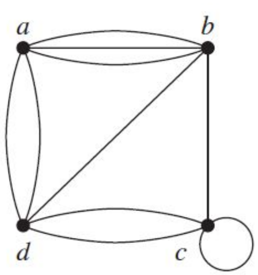
$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

The ordering of vertices is a, b, c, d.

Adjacency Matrices

- Adjacency matrices can also be used to represent undirected graphs with loops and multiple edges.
- A loop at the vertex v_i is represented by a 1 at the (i, i)th position of the matrix.
- When multiple edges connect the same pair of vertices v_i and v_j, (or if multiple loops are present at the same vertex), the (i, j)th entry equals the number of edges connecting the pair of vertices.
- All undirected graphs, including multigraphs, have symmetric adjacency matrices.
    - For directed graph, the matrix does not have to be symmetric (last week's lecture).

Adjacency Matrices Examples



The ordering of vertices is a, b, c, d.

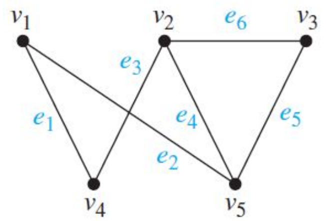
Incidence Matrices

Definition 3: Let $G = (V, E)$ be an undirected graph. Suppose that v_1, $v_2,\ldots, v_n$ are the vertices and $e_1, e_2,\ldots, e_m$ are the edges of G. Then the incidence matrix with respect to this ordering of V and E is the $n \times m$ matrix $\mathbf{M} = [m_{ij}]$, where

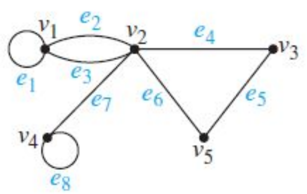
$$m_{ij} = \begin{cases} 1 & \text{when edge } e_j \text{ is incident with } v_i, \\ 0 & \text{otherwise.} \end{cases}$$

- Incidence matrices can also be used to represent undirected graphs with loops and multiple edges.

Incidence Matrices Examples



$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}.$$



4. Connectivity

Paths

Informal Definition: A *path* is a sequence of edges that begins at a vertex of a graph and travels from vertex to vertex along edges of the graph.

As the path travels along its edges, it visits the vertices along this path, that is, the endpoints of these.

Applications: Numerous problems can be modeled with paths formed by traveling along edges of graphs such as:

- determining whether a message can be sent between two computers.
- efficiently planning routes for mail delivery, or garbage pickup.

Paths in Undirected Graph

Definition 1: Let n be a nonnegative integer and G an undirected graph. A **path** of length n from u to v in G is a sequence of n edges $e_1,\ldots, e_n$ of G for which there exists a sequence $x_0 = u, x_1,\ldots, x_{n-1}, x_n = v$ of vertices such that e_i has, for $i = 1,\ldots, n$, the endpoints x_i–1 and x_i.

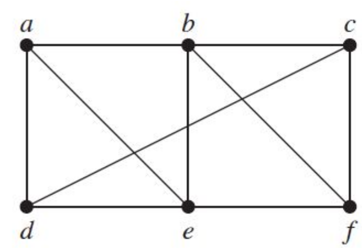
When the graph is simple, we denote this path by its vertex sequence $x_0, x_1,\ldots, x_n$ (because listing these vertices uniquely determines the path).

The path is a ***circuit*** if it begins and ends at the same vertex, that is, if $u = v$, and has length greater than zero.

The path or circuit is said to ***pass through*** the vertices $x_1, x_2,\ldots, x_{n-1}$ or ***traverse*** the edges $e_1, e_2,\ldots, e_n$.

A path or circuit is ***simple*** if it does not contain the same edge more than once.

Path Example



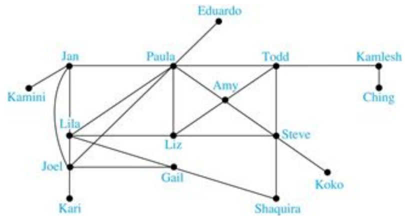
- a, d, c, f , e is a simple path of length 4, because {a, d}, {d, c}, {c, f }, and {f, e} are all edges.
- d, e, c, a is not a path, because {e, c} is not an edge.
- b, c, f , e, b is a circuit of length 4 because {b, c}, {c, f }, { f, e}, and {e, b} are edges, and this path begins and ends at b.
- The path a, b, e, d, a, b, which is of length 5, is not simple because it contains the edge {a, b} twice.

Six Degrees of Separation

Paths in Acquaintanceship Graphs

In an acquaintanceship graph there is a path between 2 people if there is a chain of people linking these people, where 2 people adjacent in the chain know one another.

For example, there is a chain of six people linking Kamini and Ching.



Six Degrees of Separation

- Many social scientists have speculated that almost every pair of people in the world are linked by a small chain of no more than six, or fewer.
- This would mean that almost every pair of vertices in the acquaintanceship graph containing all people in the world is linked by a path of length not exceeding five.

Erdős numbers

Paths in Collaboration Graphs

- In a collaboration graph, two people *a* and *b* are connected by a path when there is a sequence of people starting with *a* and ending with *b* such that the endpoints of each edge in the path are people who have collaborated.
- In the academic collaboration graph of people who have written papers in mathematics, the **Erdős number** of a person *m* is the length of the shortest path between *m* and the prolific mathematician Paul Erdős.

TABLE 1 **The Number of Mathematicians with a Given Erdős Number (as of early 2006).**

Erdős Number	*Number of People*
0	1
1	504
2	6,593
3	33,605
4	83,642
5	87,760
6	40,014
7	11,591
8	3,146
9	819
10	244
11	68
12	23
13	5

Bacon Number

- In the Hollywood graph, two actors *a* and *b* are linked when there is a chain of actors linking *a* and *b*, where every two actors adjacent in the chain have acted in the same movie.
- The ***Bacon number*** of an actor c is defined to be the length of the shortest path connecting c and the well-known actor Kevin Bacon.
- (Note that we can define a similar number by replacing Kevin Bacon by a different actor.)
- https://oracleofbacon.org/movielinks.php

TABLE 2 **The Number of Actors with a Given Bacon Number (as of August 2017).**

Bacon Number	*Number of People*
0	1
1	3,452
2	401,636
3	1,496,104
4	390,878
5	4,388
6	631
7	131
8	9
9	1

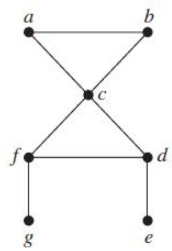
Connectedness in Undirected Graphs

Definition 2: An undirected graph is called ***connected*** if there is a path between every pair of distinct vertices of the graph.

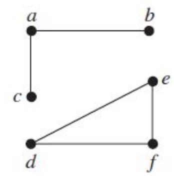
An undirected graph that is not connected is called ***disconnected***.

We say that we ***disconnect*** a graph when we remove vertices or edges, or both, to produce a disconnected subgraph.

For instance, any 2 computers on a network can communicate if and only if the graph is connected.



G_1

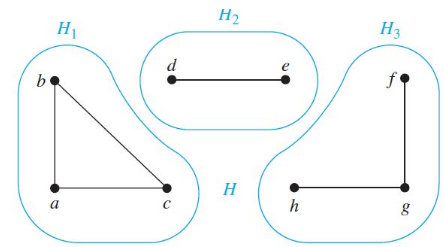


G_2

Connected Components

Definition 3: A **connected component** of a graph *G* is a connected subgraph of *G* that is not a proper subgraph of another connected subgraph of *G*.

A graph *G* that is not connected has two or more connected components that are disjoint and have *G* as their union.



Cut Vertices and Cut Edges

Suppose that a graph represents a computer network. A connected graph indicates that any 2 computers on the network can communicate.

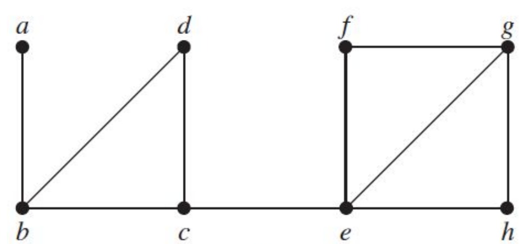
However, how reliable this network is? If a communication link fails, will it still be possible for all computers to communicate?

The removal of a vertex from a graph and all incident edges produces a subgraph with more connected components (i.e. a subgraph that is not connected).

- Such vertices are **cut vertices** (or **articulation points**). Example: Router
- Such edge is called a **cut edge** or **bridge**. Example: Link

Cut Vertices and Cut Edges

Example: Find the cut vertices and cut edges in the graph G_1

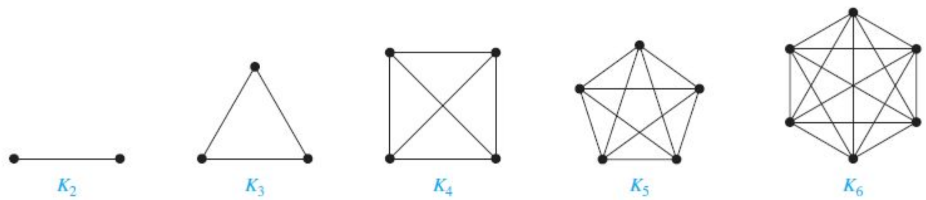


G_1

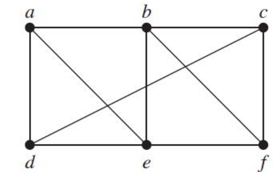
Vertex Connectivity

For the complete graph K_n, where $n \geq 3$, removing a vertex from K_n and all edges incident to it, the resulting subgraph is the complete graph K_{n-1}, a connected graph.

A subset V' of the vertex set V of $G = (V, E)$ is a **vertex cut**, or **separating set**, if $G - V'$ is disconnected.



Vertex Connectivity



A subset V' of the vertex set V of $G = (V, E)$ is a **vertex cut**, or **separating set**, if $G - V'$ is disconnected.

- For instance, the set {b, c, e} is a vertex cut with 3 vertices.

We define the **vertex connectivity** of a noncomplete graph G, denoted by $\kappa(G)$, as the <u>minimum number of vertices</u> in a vertex cut.

- When G is a complete graph, removing any subset of its vertices and all incident edges still leaves a complete graph.
- Instead, we set $\kappa(K_n) = n - 1$, the number of vertices needed to be removed to produce a graph with a single vertex.

Vertex Connectivity

For every graph G, $\kappa(G)$ is minimum number of vertices that can be removed

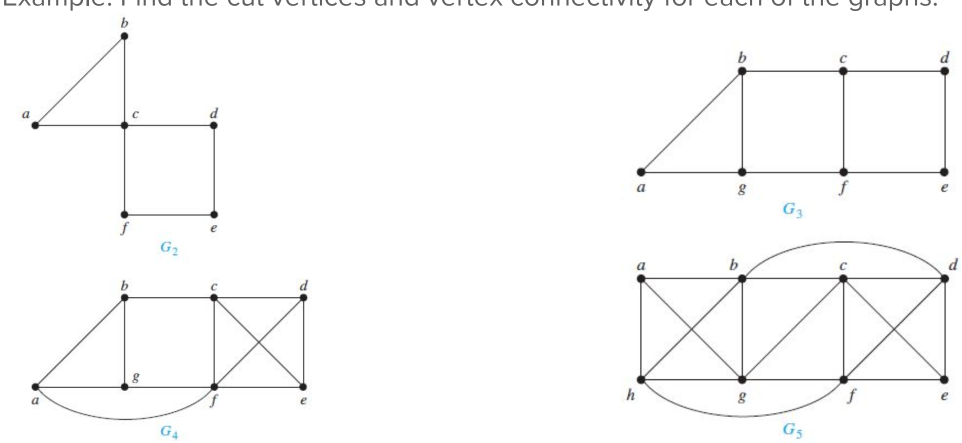
from G to either disconnect G or produce a graph with a single vertex.

- If G has n vertices, $0 \leq \kappa(G) \leq n - 1$.
- $\kappa(G) = 0$ if and only if G is disconnected.

The larger $\kappa(G)$ is, the more connected G to be.

Vertex Connectivity Example

Example: Find the cut vertices and vertex connectivity for each of the graphs.



Edge Connectivity

We can also measure the connectivity of a connected graph G = (V, E) in terms of the <u>minimum number of edges</u> that we can remove to disconnect it.

If a graph has a **cut edge**, then we need only remove it to disconnect *G*.

If *G* does not have a cut edge, we look for the smallest set of edges that can be removed to disconnect it.

A set of edges E' is called an **edge cut** of *G* if the subgraph $G - E'$ is disconnected.

The **edge connectivity** of a graph *G*, denoted by $\lambda(G)$, is the minimum number of edges in an edge cut of *G*.

Edge Connectivity

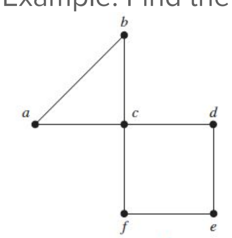
$\lambda(G) = 0$ if *G* is not connected.

- Also, $\lambda(G) = 0$ if *G* is a graph consisting of a single vertex.

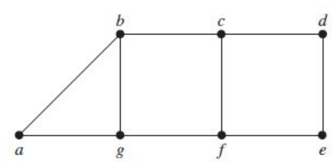
if *G* is a graph with *n* vertices, then $0 \le \lambda(G) \le n - 1$.

Edge Connectivity Example

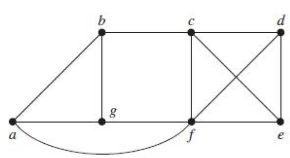
Example: Find the cut edges and edge connectivity for each of the graphs.



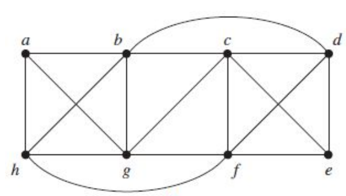
G_2



G_3



G_4



G_5

An Inequality for Vertex Connectivity and Edge Connectivity.

When $G = (V, E)$ is a noncomplete connected graph with at least 3 vertices, the minimum degree of a vertex of *G* is an upper bound for both the vertex connectivity of G and the edge connectivity of G.

Then, it follows that

$$\kappa(G) \le \lambda(G) \le \min_{v \in V} \deg(v).$$

Note that $\kappa(G) = \lambda(G) = 0$ when *G* is a disconnected graph.

Applications of Vertex and Edge Connectivity

Graph connectivity plays an important role in reliability of networks.

- A data network can be modeled using vertices to represent **routers** and edges to represent **links** between them.
- The vertex connectivity of the resulting graph equals the minimum number of routers that disconnect the network when they are out of service.
  - If fewer routers are down, data transmission between every pair of routers is still possible.
- The edge connectivity represents the minimum number of fiber optic links that can be down to disconnect the network.
  - If fewer links are down, it will still be possible for data to be transmitted between every pair of routers.

Counting Paths between Vertices

Adjacency matrix can be used to find the number of paths between 2 vertices in the graph.

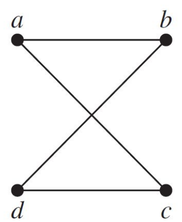
Theorem: Let G be a graph with adjacency matrix **A** with respect to the ordering $v_1, v_2, \ldots, v_n$ of the vertices of the graph (with directed or undirected edges, with multiple edges and loops allowed).

The number of different paths of length r from v_i to v_j, where r is a positive integer, equals the (i, j)th entry of $\mathbf{A}^r$.

- Note that $\mathbf{A}^r$ denotes the rth power of **A**, i.e. **A** multiplied by itself r times.

Counting Paths between Vertices

Example: How many paths of length 4 are there from a to d in the simple graph?



5. Euler and Hamiltonian Graphs

Can we travel along the edges of a graph starting at a vertex and returning to it by traversing each edge of the graph exactly once?

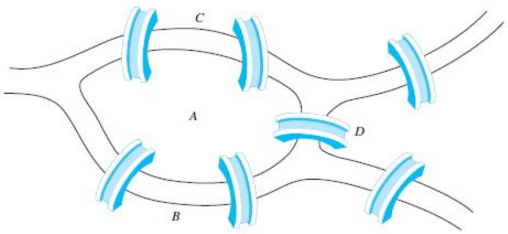
- Does a graph has an ***Euler circuit***?
- Can be easily answered by examining the degrees of vertices of the graph.

Similarly, can we travel along the edges of a graph starting at a vertex and returning to it while visiting each vertex of the graph exactly once?

- Does a graph has an ***Hamilton circuit***?
- Difficult to solve for most graphs

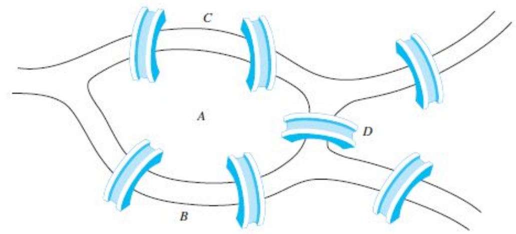
History of Graph

- The town of Königsberg, Prussia (now called Kaliningrad and part of the Russian republic), was divided into 4 sections by the branches of the Pregel River.
- In the 18th century, 7 bridges of Königsberg connected these regions.



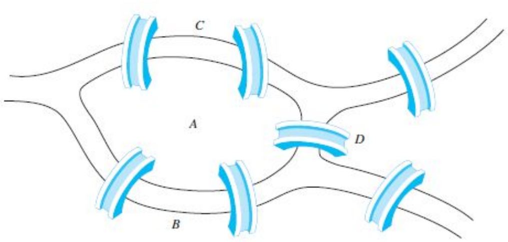
History of Graph

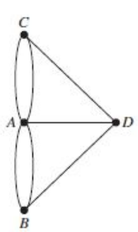
- The townspeople wondered whether it was possible to start at some location in the town, travel across all the bridges once without crossing any bridge twice, and return to the starting point.
- Possible?



History of Graph

- The Swiss mathematician Leonhard Euler proved that no such path exists in 1736.
- This result is often considered to be the first theorem ever proved in graph theory.





Euler Circuits and Paths

The problem of traveling across every bridge without crossing any bridge more than once can be rephrased in terms of multigraph model.

- The question becomes: Is there a simple circuit in this multigraph that contains every edge?

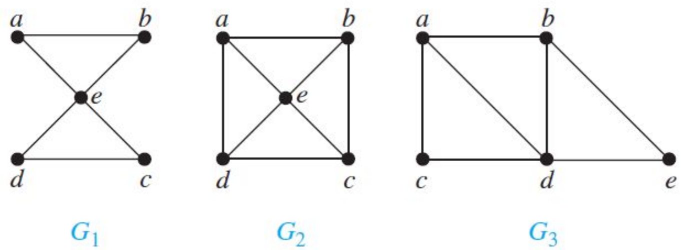
Definition 1: An ***Euler circuit*** in a graph *G* is a simple circuit containing every edge of *G*.

An ***Euler path*** in *G* is a simple path containing every edge of *G*.

- Recall that a path or circuit is simple it contains each edge exactly once.

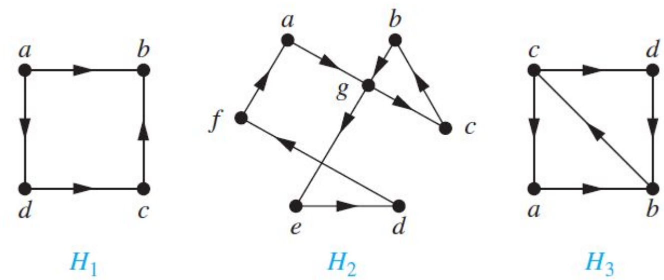
Euler Circuits and Paths

Example 1: Which of the undirected graphs have an Euler circuit? Of those that do not, which have an Euler path?



Euler Circuits and Paths

Example 2: Which of the directed graphs have an Euler circuit? Of those that do not, which have an Euler path?



Necessary Conditions for Euler Circuits and Paths

Simple criteria for determining whether a multigraph has an Euler circuit or path.

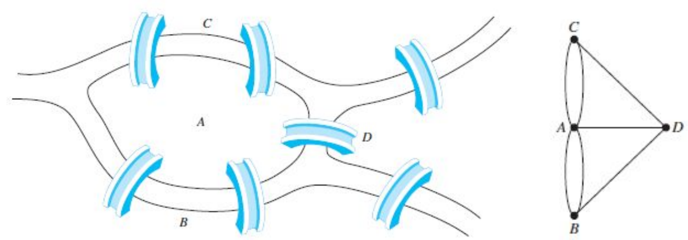
- Euler discovered necessary and sufficient conditions when he solved the famous Königsberg bridge problem.
- Necessary condition: Every vertex must have even degree.

Theorem 1: A connected multigraph with ≥ 2 vertices has an Euler circuit if and only if each of its vertices has even degree.

Königsberg Bridge Problem

Because the multigraph representing these bridges has 4 vertices of odd degree, by theorem 1, it does not have an Euler circuit.

- There is no way to start at a given point, cross each bridge exactly once, and return to the starting point.

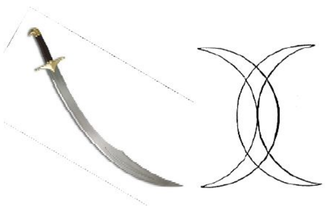


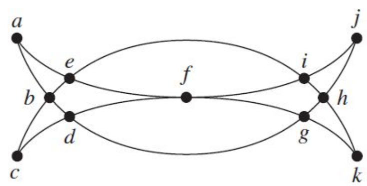
Mohammed’s Scimitars Problem

Some puzzles ask you to draw a picture in a continuous motion without lifting a pencil so that no part of the picture is retraced.

We can solve such puzzles using Euler circuits and paths.

For example, can Mohammed’s scimitars be drawn in this way, where the drawing begins and ends at the same point?

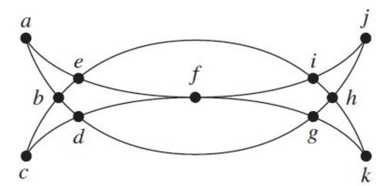




G

Mohammed’s Scimitars Problem

For example, can Mohammed’s scimitars be drawn in this way, where the drawing begins and ends at the same point?



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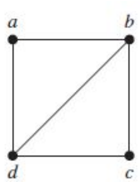
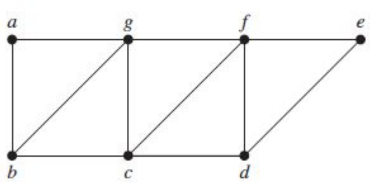
Necessary Conditions for Euler Circuits and Paths

Theorem 2: A connected multigraph has an Euler path but not an Euler circuit if and only if it has exactly 2 vertices of odd degree.

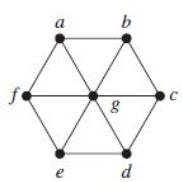
Example: Which graphs have an Euler path?



G_1



G_2

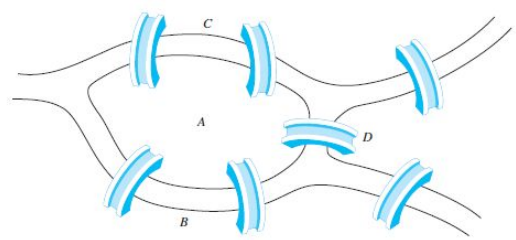


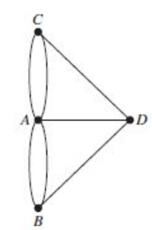
G_3

Königsberg Bridge Problem

Is it possible to start at some point in the town, travel across all the bridges, and end up at some other point in town?

Because there are 4 vertices of odd degree in this multigraph, there is no Euler path, so such a trip is impossible.





Applications of Euler Paths and Circuits

Euler paths and circuits can be used to solve many practical problems such as finding a path or circuit that traverses each

- street in a neighborhood,
- road in a transportation network,
- connection in a utility grid,
- link in a communications network, exactly once.

Other applications are found in the

- layout of circuits,
- network multicasting,
- molecular biology, where Euler paths are used in the sequencing of DNA.

Hamilton Circuits and Paths

Euler paths and circuits contained every edge only once.

Now we look at paths and circuits that contain every vertex exactly once.

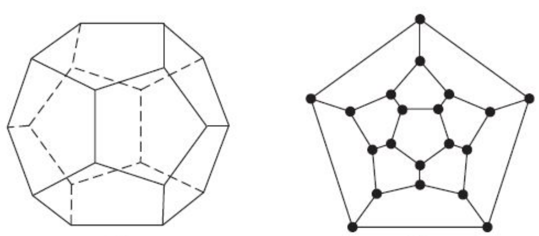
Definition 2: A simple path in a graph G that passes through every vertex exactly once is called a ***Hamilton path***, and a simple circuit in a graph G that passes through every vertex exactly once is called a ***Hamilton circuit***.

That is, the simple path $x_0, x_1, \ldots, x_{n-1}, x_n$ in the graph $G = (V, E)$ is a Hamilton path if $V = \{x_0, x_1, \ldots, x_{n-1}, x_n\}$ and $x_i \neq x_j$ for $0 \leq i < j \leq n$, and the simple circuit $x_0, x_1, \ldots, x_{n-1}, x_n, x_0$ (with $n > 0$) is a Hamilton circuit if $x_0, x_1, \ldots, x_{n-1}, x_n$ is a Hamilton path.

Hamilton Circuits and Paths

Hamilton terminology comes from a game, called the **Icosian puzzle**, invented in 1857 by the Irish mathematician Sir William Rowan Hamilton.

It consisted of a wooden dodecahedron [a polyhedron with 12 regular pentagons as faces], with a peg at each vertex of the dodecahedron, and string.



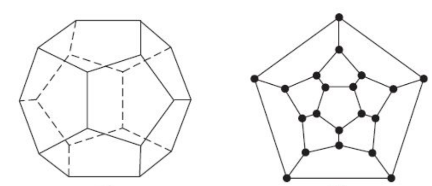
(a) (b)

Hamilton Circuits and Paths

The 20 vertices of the dodecahedron were labeled with different cities in the world.

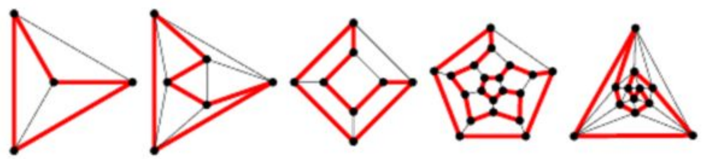
String was used to used to plot a circuit visiting 20 cities exactly once.

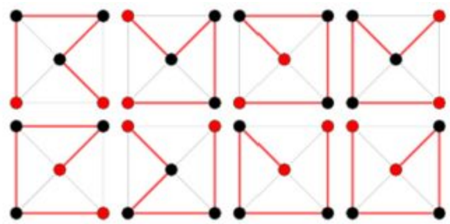
The object of the puzzle was to start at a city and travel along the edges of the dodecahedron, visiting each of the other 19 cities exactly once, and end back at the first city.



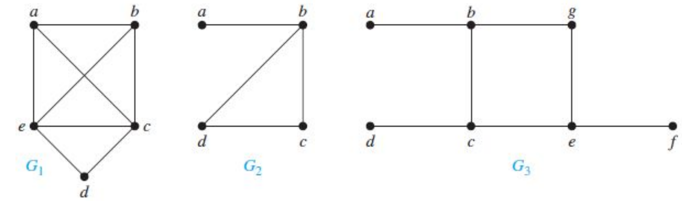
(a) (b)

Hamilton Circuits and Paths Examples





Hamilton Circuits and Paths Exercise



Exercise 1: Which of the simple graphs have a Hamilton circuit or, if not, a Hamilton path?

Necessary Conditions for Hamilton Circuits

Unlike for an Euler circuit, no simple necessary and sufficient conditions are known for the existence of a Hamilton circuit.

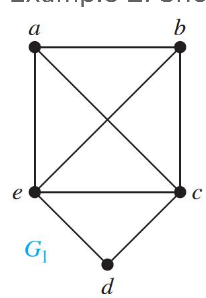
However, there are some useful necessary conditions.

Dirac's Theorem: If *G* is a simple graph with $n \geq 3$ vertices such that the degree of every vertex in *G* is $\geq n/2$, then *G* has a Hamilton circuit.

Ore's Theorem: If *G* is a simple graph with $n \geq 3$ vertices such that $\deg(u) + \deg(v) \geq n$ for every pair of nonadjacent vertices, then *G* has a Hamilton circuit.

Hamilton Circuits and Paths Example

Example 2: Show that a graph has a Hamilton circuit.



n =

deg(a) =

deg(b) =

deg(c) =

deg(d) =

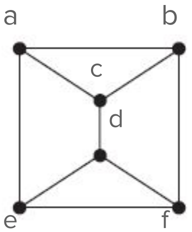
deg(e) =

Dirac's Theorem: $n \geq 3$ vertices and each node has degree $\geq n/2$

Ore's Theorem: $n \geq 3$ vertices such that $\deg(u) + \deg(v) \geq n$ for

Hamilton Circuits and Paths Exercise

Exercise: Show that a graph has a Hamilton circuit.



n =

deg(a) =

deg(b) =

deg(c) =

deg(d) =

deg(e) =

deg(f) =

Dirac's Theorem: $n \geq 3$ vertices and each node has degree $\geq n/2$

Ore's Theorem: $n \geq 3$ vertices such that $\deg(u) + \deg(v) \geq n$ for

Applications of Hamilton Paths and Circuits

Hamilton paths and circuits can be used to solve many practical problems such as finding a path or circuit that visits each

- road intersection,
- place pipelines in a utility grid,
- node in a communications network, exactly once.

The famous **traveling salesman problem** asks for the shortest route a traveling salesperson should take to visit a set of cities.

- A solution involves finding a Hamilton circuit in a complete graph such that the total weight of its edges is as small as possible.

6. Graph Coloring

Problems related to the coloring of maps of regions, such as maps of parts of the world, have generated many results in graph theory.

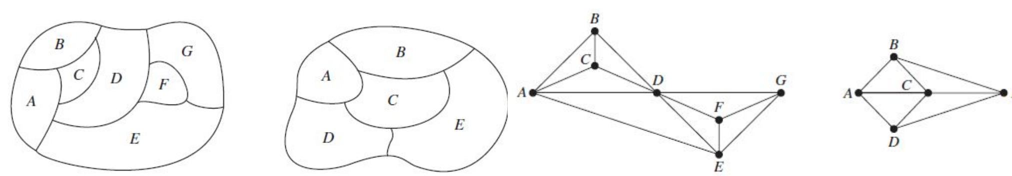
When a map is colored, two regions with a common border are customarily assigned different colors.

Graph Coloring

Consider the problem of determining the least number of colors that can be used to color a map so that adjacent regions never have the same color.

Each map in the plane can be represented by a graph.

- Region is represented by a vertex.
- Edges connect 2 vertices if the regions have a common border.



Graph Coloring

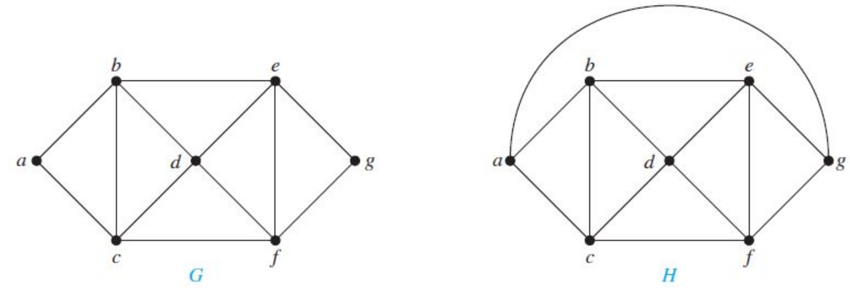
Definition 1: A coloring of a simple graph is the assignment of a color to each vertex of the graph so that no 2 adjacent vertices are assigned the same color.

It is possible to use fewer colors than the number of vertices in the graph.

Definition 2: The chromatic number of a graph is the least number of colors needed for a coloring of this graph. The chromatic number of a graph *G* is denoted by $\chi(G)$. (Here χ is the Greek letter chi.)

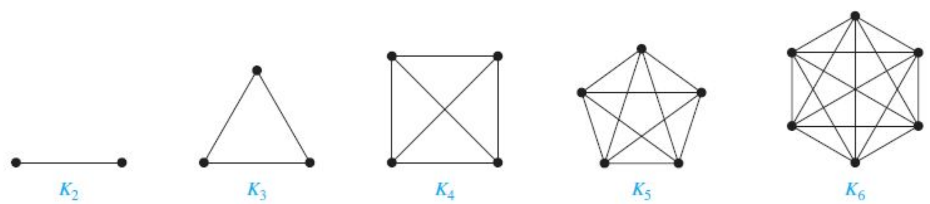
Graph Coloring Example

Example 1: What are the chromatic numbers of the graphs?



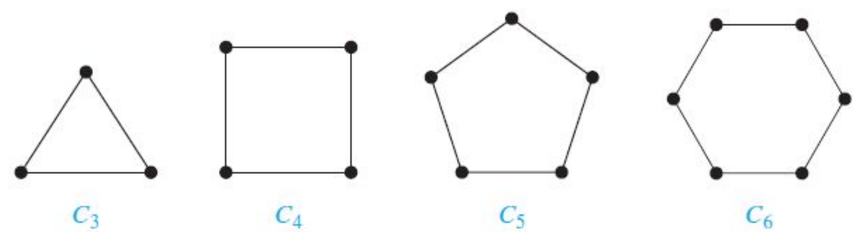
Graph Coloring Example

Example 2: What are the chromatic numbers of the following <u>complete</u> graphs?



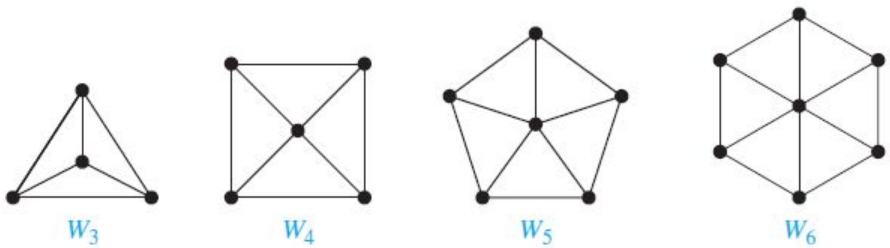
Graph Coloring Example

Example 3: What are the chromatic numbers of the following <u>cycle</u> graphs?



Graph Coloring Example

Example 3: What are the chromatic numbers of the following wheel graphs?



Applications of Graph Colorings

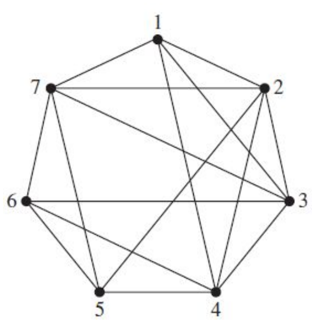
Graph coloring can be applied to problems involving scheduling and assignments.

Example 1: How can the final exams at a university be scheduled so that no student has 2 exams at the same time?

Suppose there are 7 exams to be scheduled. Suppose that the following pairs of courses have common students:

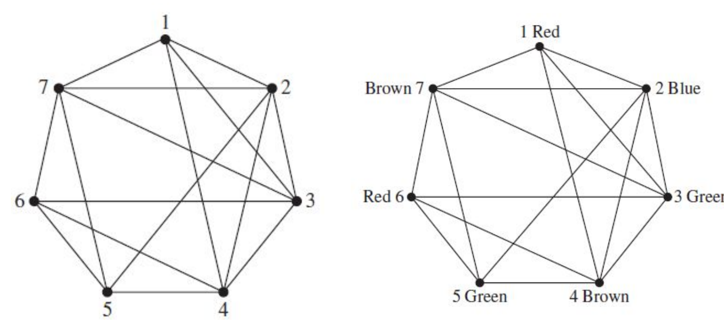
- 1 and 2, 1 and 3, 1 and 4, 1 and 7,
- 2 and 3, 2 and 4, 2 and 5, 2 and 7,
- 3 and 4, 3 and 6, 3 and 7,
- 4 and 5, 4 and 6,
- 5 and 6, 5 and 7, and
- 6 and 7

Applications of Graph Colorings



The chromatic number of this graph is 4, which means 4 time slots are needed.

Applications of Graph Colorings



Time Period	Courses
I	1, 6
II	2
III	3, 5
IV	4, 7

Applications of Graph Colorings Exercise

Exercise: The IT department has 6 committees, each meeting once a month.

How many different meeting times must be used to ensure that no member is scheduled to attend 2 meetings at the same time if the committees are

- C1 = {Arlinghaus, Brand, Zaslavsky},
- C2 = {Brand, Lee, Rosen},
- C3 = {Arlinghaus, Rosen, Zaslavsky},
- C4 = {Lee, Rosen, Zaslavsky},
- C5 = {Arlinghaus, Brand}, and
- C6 = {Brand, Rosen, Zaslavsky} ?

Shortest-Path Problems

Graphs that have a number assigned to each edge are called **weighted graphs**.

- Weighted graphs are used to model computer networks.
- Communications costs (such as the monthly cost of leasing a telephone line), the response times of the computers over these lines, or the distance between computers, can all be studied using weighted graphs.

The question is: What is a **shortest path**,that is, a path of least length, between two given vertices?

Trees

- **Definition**: A **tree** is a connected undirected graph with no simple circuits
- Example: File system, directories, and organisation structures.
- Tree Traversal Algorithms
  - Preorder, Inorder, Postorder.
  - Depth-First Search, and Breadth-First Search
- Spanning Trees
  - **Definition**: A subtree of *G* that contains every vertex of *G*.
- Minimum Spanning Trees
  - **Definition**: A spanning tree with minimised total sum of weights.