Discrete Mathematics

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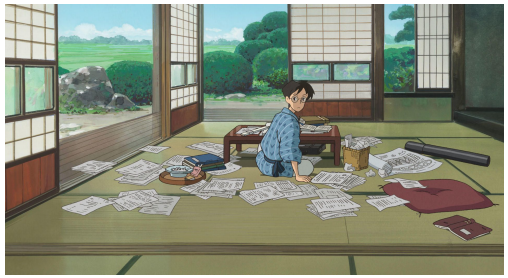
Course Materials

- https://github.com/noswolf/DM_DSBA_2020
- Lecture Slides and Homework



Grading (Final)

- Final 40%
- Homework 10%

Course Outline

1)	Number Theory	02/10/2020
2)	Number Theory and Cryptography	09/10/2020
3)	Counting	16/10/2020
4)	Relations	23/10/2020 (TBA)
5)	Graphs & Trees (covered in Data Structures)	30/10/2020
6)	Boolean Algebra	06/11/2020
7)	Modeling Computation	13/11/2020
8)	Finite State Automata	20/11/2020 (TBA)

Number Theory
ทฤษฎีจำนวน

Number Theory (or Arithmetic)

- A branch of pure mathematics (independent of application)
- Dedicated to the study of integers and their properties.
- Key ideas: Division and Prime numbers.
- Number theory has important applications to computer science and cryptography.

Chapter Summary

1. Divisibility and Modular Arithmetic
2. Integer Representations and Algorithms
3. Primes and Greatest Common Divisors
4. Applications of Congruences
5. Cryptography

1. Divisibility and Modular Arithmetic

b / a

- Division of an integer (b) by a nonzero integer ($a \neq 0$)
  - Dividend or numerator (b) = ตัวตั้ง
  - Divider, divisor, or denominator (a) = ตัวหาร

1.1 Division

b / a

- **Definition**:
  - If *a* and *b* are integers with $a \neq 0$, *a* divides *b* (or *b* is divided by *a*) if there is an integer *c* such that $b = ac$.
  - a | b denotes *a* divides *b*.
  - If a | b, then b / a is an **integer**.
  - a $\nmid$ b means *a* **does not** divide *b*.
    - There is no integer *c* such that $b = ac$.

Division

b / a -> a | b -> b = ac

- Example:
  - 4 | 8 because 8 / 4 = 2, which is an integer
    - In other words, there is c = 2 such that 8 = 4(2)
  - -3 | 9 because 9 / (-3) = -3
  - 10 | 0 because 0 / 10 = 0
  - 5 $\nmid$ 7 because 7 / 5 = 1.4, which is **not** an integer

Some Properties of Division

- **Theorem 1**: Let *a*, *b*, and *c* be integers, where $a \neq 0$. Then
  - if *a* | *b* and *a* | *c*, then *a* | (*b* + *c*);
- Direct Proof
  - Suppose that *a* | *b* and *a* | *c*.
  - From the definition of divisibility, it follows that there are integers *s* and *t* with $b = as$ and $c = at$.
  - Thus, $b + c = as + at = a(s + t)$
- Example: 3 | 6 and 3 | 12, then 3 | (6 + 12) = 3 | 18

Some Properties of Division

- **Theorem 1**: Let *a*, *b*, and *c* be integers, where $a \neq 0$. Then
  - if *a* | *b* and *a* | *c*, then *a* | (*b* + *c*);
  - if *a* | *b* and *a* | *c*, then *a* | *mb* + nc where m and n are integers;
  - if *a* | *b* then *a* | *bc* for all integers c;
  - if *a* | *b* and *b* | *c* then *a* | *c*;
  - if *a* | *b* then $|a| \leq |b|$

1.2 Division Algorithm/Theorem

- Division of an integer (a) by a positive integer, there is a quotient and a remainder.

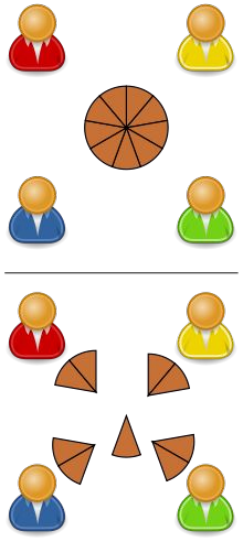
  a / d = q remainder r

- **Theorem 2**: Let *a* be an integer and *d* a positive integer.
  - Then there are unique integers *q* and *r*, with $0 \leq r < d$, such that $a = dq + r$.
- Also known as **Euclidean Division**.

Division Algorithm/Theorem

a = dq + r
a / d = q remainder r

- Definition
  - Dividend or numerator (a) = ตัวตั้ง
  - Divider, divisor, or denominator (d) = ตัวหาร
  - Quotient (q) = ผลหาร
    - *q* = *a* **div** *d*
  - Remainder (r) = เศษเหลือ
    - *r* = *a* **mod** *d*



Division Algorithm/Theorem

a = dq + r
a / d = q remainder r
q = a **div** d
r = a **mod** d

- Example: Find the quotient and remainder.
  - 101 / 11 -> 101 = (11 * 9) + 2
    - Quotient (q) -> 9 = 101 **div** 11
    - Remainder (r) -> 2 = 101 **mod** 11

Division Algorithm/Theorem

a = dq + r
a / d = q remainder r
q = a **div** d
r = a **mod** d

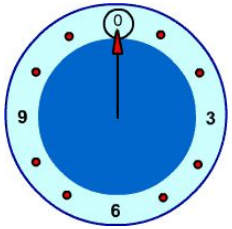
- Example: Find the quotient and remainder.
  - -13 / 4 -> -13 = 4(-4) + 3
    - Quotient (q) -> -4 = -13 **div** 4
    - Remainder (r) -> 3 = -13 **mod** 4
  - -13 / 4 -> -13 = 4(-3) -1 ?

1.3 Modular Arithmetic

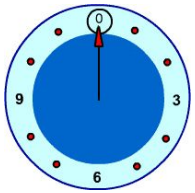
- In certain circumstances, we are only interested about the remainder of an integer when divided by some positive integer.
  - For these cases, there is an operator called the **modulo** operator.
  - a **mod** d = r -> a modulo d is equal to r
    - d is known as a modulus.
  - Example: 20 mod 3 = 2

Modular Arithmetic

- Real-life example:
  - 12-hour clock
  - Circular Queue (Data Structure)

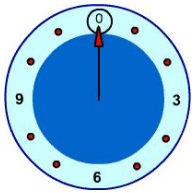


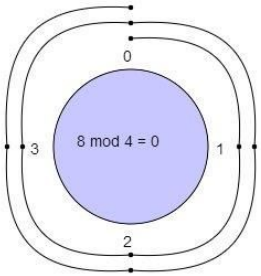
Modular Arithmetic

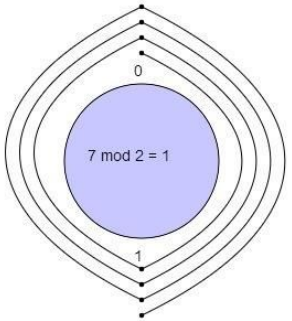


- Visualising modulo operator using circles.
  1. Write 0 at the top of a circle and continuing clockwise writing integers 1, 2, ... up to one less than the modulus (d).
  2. Start at 0 and move around the clock *a* steps
     - If *a* is positive, step clockwise
     - If *a* is negative, step counter-clockwise.

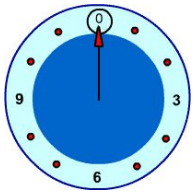
Modular Arithmetic

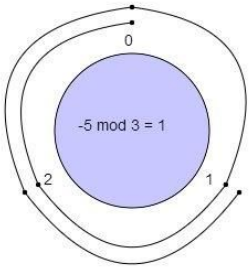






Modular Arithmetic

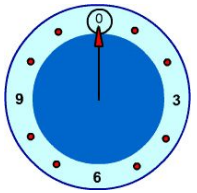




Alternatively, the formula for modulo operation when *a* is negative is:

a **mod** d = a - (floor (a/d) * d)

Modular Arithmetic Exercise



1. -17 **mod** 7
2. -100 **mod** 8
3. (((29 **mod** 17) **mod** 7) **mod** 3)
4. (109 **mod** 12) + (-109 **mod** 12)
5. (((-29 **mod** 17) **mod** 7) **mod** 3)

Congruence Modulo

- **Definition**: Two integers (*a* and *b*) have the same remainder when they are divided by the positive integer *c*.
  - *a* is congruent(สมภาค) to *b* modulo *c* if *c* divides *a* - *b* (i.e. *c* | *a* - *b*).
  - *a* ≡ *b*(mod *c*).
- **Theorem 3**: Let *a* and *b* be integers, and *c* be a positive integer.
  - Then *a* ≡ *b* (mod *c*) if and only if *a* **mod** *c* = *b* **mod** *c*.

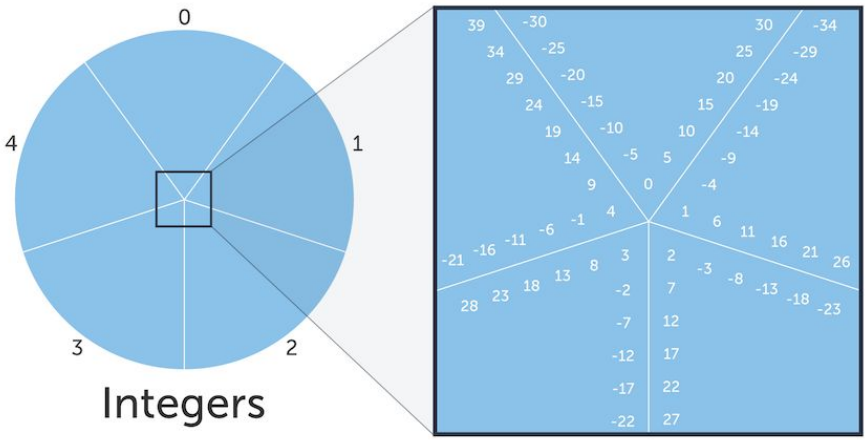
Congruence Modulo

- Example: 20 ≡ 8(mod 12)
  - 8 mod 12 = 8 and 20 mod 12 = 8
- The use of “mod” in *a* ≡ *b* (mod *c*) and *a* **mod** *c* = *b* are different.
  - *a* ≡ *b* (mod c) is a relation on the set of integers (congruent mod).
  - In a **mod** c = b, the notation **mod** denotes a function.

Congruence Modulo

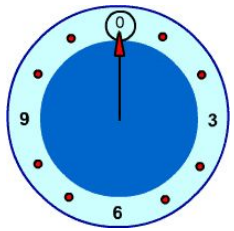
- **Theorem 4**: Let c be a positive integer. The integers a and b are congruent modulo c if and only if there is an integer k such that $a = b + kc$.
- Proof:
  - If $a \equiv b \pmod{c}$, then (by the definition of congruence) $c \mid a - b$.
  - Thus, there is an integer k such that $a - b = kc$, so that $a = b + kc$.

Congruence Modulo



Congruence Modulo Example

- 12-hour clock
  - Currently it is 8:00am
  - 6 Hours later is 8 + 6 = 14 (pm)
  - (8+6) mod 12, means 14:00 and 2:00pm
  - 14 is **congruent** to 2 modulo 12
    - 14 ≡ 2 (mod 12)



Congruence Properties

- Equivalent statements
  - $a \equiv b \pmod{c}$
  - a **mod** $c = b$ **mod** c
  - $c \mid (a - b)$
  - $a = b + kc$
- Example: $11 \equiv 21 \bmod 5$
  - $11 \equiv 21 \pmod{5}$
  - $11 \bmod 5 = 21 \bmod 5$
  - $5 \mid (11 - 21)$
  - $11 = 21 + 5k$
    - $k = -2$

Congruence Modulo Exercise

- Suppose that a and b are integers,
  - $a \equiv 4 \pmod{13}$
  - $b \equiv 9 \pmod{13}$
  - Find the integer c with $0 \leq c \leq 12$ such that
    - $c \equiv 9a \pmod{13}$
    - $c \equiv 11b \pmod{13}$
    - $c \equiv a + b \pmod{13}$

Congruence Modulo is an Equivalence Relation

- Every pair of values in a slice are related to each other.
- No any value is in more than one slice.
- A pie contains all slices, which have **all** of the **integer** values.
- A pie has **equivalence relation**, which defines how to partition set of values (**mod**) into **congruence** classes (slices).

Equivalence Relation Properties

If congruence modulo c is an equivalence relation for (mod c).

1. **Reflexive**: a is related to a.
   - $a \equiv a \pmod{c}$
2. **Symmetric**: if a is related to b, then b is related to a.
   - if $a \equiv b \pmod{c}$ then $b \equiv a \pmod{c}$
3. **Transitive**: if a is related to b and b is related to d, then a is related to d.
   - if $a \equiv b \pmod{c}$ and $b \equiv d \pmod{c}$ then $a \equiv d \pmod{c}$

Equivalence Relation Properties (Example)

If congruence modulo c is an equivalence relation for (mod c).

1. **Reflexive**: a is related to a.
   - $2 \equiv 2 \pmod{5}$
2. **Symmetric**: if a is related to b, then b is related to a.
   - if $2 \equiv 7 \pmod{5}$ then $7 \equiv 2 \pmod{5}$
3. **Transitive**: if a is related to b and b is related to d, then a is related to d.
   - if $2 \equiv 7 \pmod{5}$ and $7 \equiv 17 \pmod{5}$ then $2 \equiv 17 \pmod{5}$

Congruence Properties (Sums and Products)

- **Theorem 5**: Let c be a positive integer.
- If $a \equiv b \pmod{c}$ and $d \equiv e \pmod{c}$, then
  - $a + d \equiv b + e \pmod{c}$ and
  - $ad \equiv be \pmod{c}$
- Example: $16 \equiv 2 \pmod{7}$ and $8 \equiv 1 \pmod{7}$
  - $16 + 8 \equiv (2 + 1) \pmod{7}$ -> $24 \equiv 3 \pmod{7}$
  - $16 * 8 \equiv (2 * 1) \pmod{7}$ -> $128 \equiv 2 \pmod{7}$

Modular Operations

- Addition
  - $(a + b)$ **mod** $c = ((a$ **mod** $c) + (b$ **mod** $c))$ **mod** c
- Multiplication
  - ab **mod** $c = ((a$ **mod** $c)(b$ **mod** $c))$ **mod** c

Modular Operations Exercise

- $(14 + 17)$ **mod** 5 and $((14$ **mod** $5) + (17$ **mod** $5))$ **mod** 5
- $((-7) + (-10))$ **mod** 7
- 71 * 122 **mod** 11
- (46 * 71 * 122) **mod** 11
- (-45 * 77) **mod** 17

Modular Operations

- Exponentiation
  - a^b **mod** $c = (a$ **mod** $c)^b$ **mod** c
  - b can get very large, making a^b massively large and taking a long time to calculate the mod.
- For quicker calculation, divide and conquer can be used.
- Examples: 2^{90} **mod** 13
  - $(2^{50} * 2^{40})$ **mod** 13
  - $(2^{50}$ **mod** 13 * 2^{40} **mod** 13) **mod** 13-> (4 * 3) **mod** 13

2. Integer Representations

- In the modern world, we use ***decimal***, or ***base 10***, notation to represent integers. For example when we write 965, we mean $9*10^2 + 6*10^1 + 5*10^0$.
- We can represent numbers using any base *b*, where *b* is a positive integer greater than 1.
- The bases *b* = 2 (binary), *b* = 8 (octal) , and *b* = 16 (hexadecimal) are important for computing and communications.
- The ancient Mayans (2000 BC) used base 20.
- The ancient Babylonians (1895 BC – 539 BC) used base 60.

Base *b* Representations

- We can use positive integer *b* greater than 1 as a base
- **Theorem 1**: Let *b* be a positive integer greater than 1. Then if *n* is a positive integer, it can be expressed uniquely in the form:

  $$n = a_k b^k + a_{k-1} b^{k-1} + ... + a_1 b + a_0$$

  where *k* is a nonnegative integer, $a_0, a_1, ..., a_k$ are nonnegative integers less than *b*, and $a_k \neq 0$. The a_j, j = 0,...,*k* are called the base-*b* digits of the representation.
- The representation of n given in Theorem 1 is called the *base b expansion of n* and is denoted by $(a_k a_{k-1} .... a_1 a_0)_b$.

Binary Expansions (base 2)

- Most computers represent integers and do arithmetic with binary (base 2) expansions of integers. In these expansions, the only digits used are 0 and 1.
- Example 1: What is the decimal expansion of the integer that has $(1\ 0101\ 1111)_2$ as its binary expansion?
- Example 2: What is the decimal expansion of the integer that has $(11011)_2$ as its binary expansion?

Octal Expansions (base 8)

- The octal expansion (base 8) uses the digits {0, 1, 2, 3, 4, 5, 6, 7}.
- Example 1: What is the decimal expansion of the number with octal expansion $(7016)_8$?
- Example 2: What is the decimal expansion of the number with octal expansion $(111)_8$?

Hexadecimal Expansions (base 16)

- The hexadecimal expansion needs 16 digits, but our decimal system provides only 10. So letters are used for the additional symbols.
- The hexadecimal system uses the digits {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F}. The letters A through F represent the decimal numbers 10 through 15.
- Example 1: What is the decimal expansion of the number with hexadecimal expansion $(2AE0B)_{16}$?
- Example 2: What is the decimal expansion of the number with hexadecimal expansion $(E5)_{16}$?

Base Conversion Algorithm

To construct the base b expansion of an integer n:

- Divide n by b to obtain a quotient and remainder.

  $n = bq_0 + a_0 \quad 0 \le a_0 \le b$

- The remainder, a_0 , is the rightmost digit in the base b expansion of n.
- Next, divide q_0 by b.

  $q_0 = bq_1 + a_1 \quad 0 \le a_1 \le b$

- The remainder, a_1, is the second digit from the right in the base b expansion of n.
- Continue by successively dividing the quotients by b, obtaining the additional base b digits as the remainder. The process terminates when the quotient is 0.

Base Conversion Algorithm

Algorithm base_b_expansion(n, b: positive integers with b > 1)
q = n, k = 0
while (q ≠ 0):
    a_k = q **mod** b
    q = q **div** b
    k = k + 1
return(a_{k-1} ,..., a_1,a_0) where {(a_{k-1} ... a_1a_0)$_b$ is base b expansion of n}

- q represents the quotient obtained by successive divisions by b, starting with $q = n$.
- The digits in the base b expansion are the remainders of the division given by q **mod** b.

Base Conversion Example 1

Example: Find the octal expansion of $(12345)_{10}$

Solution: Successively dividing by 8 gives:

12345 = 8 * 1543 + 1

1543 = 8 * 192 + 7

192 = 8 * 24 + 0

24 = 8 * 3 + 0

3 = 8 * 0 + 3

The remainders are the digits from right to left yielding $(30071)_8$.

Base Conversion Example 2

Example: Find the hexadecimal expansion of $(177130)_{10}$

Base Conversion Example 3

Example: Find the binary expansion of $(241)_{10}$

Base Conversion

TABLE 1 **Hexadecimal, Octal, and Binary Representation of the Integers 0 through 15.**

Decimal	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Hexadecimal	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
Octal	0	1	2	3	4	5	6	7	10	11	12	13	14	15	16	17
Binary	0	1	10	11	100	101	110	111	1000	1001	1010	1011	1100	1101	1110	1111

Each octal digit corresponds to a block of 3 binary digits.

Each hexadecimal digit corresponds to a block of 4 binary digits.

So, conversion between binary, octal, and hexadecimal is easy.

Base Conversion Example 4

Example 1: Find the octal and hexadecimal expansions of $(11\ 1110\ 1011\ 1100)_2$.

Example 2: Find the binary expansions of $(765)_8$ and $(A8D)_{16}$.

2.1 Integer Operations Algorithms

- Algorithms for performing operations with integers using their binary expansions are extremely important as computer chips work with binary numbers. Each digit is called a bit.
- Suppose the binary expressions of a and b are

  $a = (a_{n-1},a_{n-2},\ldots,a_0)_2$ and $b = (b_{n-1},b_{n-2},\ldots,b_0)_2$

  where a and b each have n bits.

Binary Addition Algorithm

To add two integers a and b in binary notation:

- First, add their rightmost bits.

  $a_0 + b_0 = c_0 * 2 + s_0$
- The rightmost bit, s_0 is the result of $a + b$ and c_0 is the **carry**.
- Next, add the next pair of bits and the carry.

  $a_1 + b_1 + c_0 = c_1 * 2 + s_1$
- s_1 is next bit from the right.
- Continue by successively adding the corresponding bits and the carry. The leading bit of the sum is $s_n = c_{n-1}$.

Binary Addition Algorithm Example

Example: Add $a = (1110)_2$ and $b = (1011)_2$.

```
 1 1 1
   1 1 1 0
+  1 0 1 1
-------------
 1 1 0 0 1
```

Binary Addition Algorithm

Algorithm binary_add(a, b: positive integers)
# where a = $(a_{n-1},a_{n-2},\ldots,a_0)_2$ and b = $(b_{n-1},b_{n-2},\ldots,b_0)_2$
c = 0 # carry
for j in range(0, n):
  d = ⌊$(a_j + b_j + c)/2$⌋
  $s_j = a_j + b_j + c - 2d$
  c = d
s_n = c
return($s_0,s_1,\ldots, s_n$) where {the binary expansion of the sum is $(s_n,s_{n-1},\ldots,s_0)_2$}

Binary Multiplication Algorithm

To multiply two integers a and b in binary notation, distributive law can be used:

$$ab = a(b_0 2^0 + b_1 2^1 + \cdots + b_{n-1} 2^{n-1})$$
$$= a(b_0 2^0) + a(b_1 2^1) + \cdots + a(b_{n-1} 2^{n-1})$$

- Note that $ab_j = a$ if $b_j = 1$ and $ab_j = 0$ if $b_j = 0$.
- Each time when multiply, the binary expansion is shifted one place to the left and add a zero at the tail end of the expression.

Binary Multiplication Algorithm Example

Example: Multiply $a = (110)_2$ and $b = (101)_2$.

$$\begin{array}{r} 1\ 1\ 0 \\ \times\ 1\ 0\ 1 \\ \hline 1\ 1\ 0 \\ 0\ 0\ 0 \\ 1\ 1\ 0 \\ \hline 1\ 1\ 1\ 1\ 0 \end{array}$$

Binary Multiplication Algorithm

Algorithm binary_multiply(a, b: positive integers)
where a = $(a_{n-1},a_{n-2},\ldots,a_0)_2$ and b = $(b_{n-1},b_{n-2},\ldots,b_0)_2$
for j in range(0, n):
if $b_j = 1$ **then** c_j = a shifted j places to the left and add 0s at the tail
else $c_j = 0$
{$c_o,c_1,\ldots, c_{n-1}$ are the partial products = ตัวทด}
p = 0
for j in range (0, n): # Add
p = p + c_j
return p where {p is the value of ab}

Integer Operations Algorithms

- The number of additions of bits used by the algorithm to add two n-bit integers is $O(n)$.
- The number of additions of bits used by the algorithm to multiply two n-bit integers is $O(n^2)$.

2.2 Modular Operations

- Exponentiation
  - a^b **mod** c = (a **mod** c)^b **mod** c
  - b can get very large, making a^b massively large and taking a long time to calculate the mod.
- For quicker calculation, divide and conquer can be used.
- Examples: 2^{90} **mod** 13
  - (2^{50} * 2^{40}) **mod** 13
  - (2^{50} **mod** 13 * 2^{40} **mod** 13) **mod** 13-> (4 * 3) **mod** 13

Modular Operations

- Method #1: **Divide and conquer**
  - a^b **mod** c = (a **mod** c)^b **mod** c
- If b is an even number,
  - a^b **mod** c = $(a^2)^{b/2}$ **mod** c
    = (a^2 **mod** c)$^{b/2}$ **mod** c
- If b is an odd number,
  - a^b **mod** c = (aa^{b-1}) **mod** c
    = (a * $(a^2)^{(b-1)/2}$) **mod** c
    = (a * a^2 **mod** c)$^{(b-1)/2}$ **mod** c

Modular Exponentiation Example

Example: Compute 5^{50} **mod** 19 using divide and conquer strategy.

Binary Modular Exponentiation

- In cryptography, it is important to be able to find b^n **mod** c efficiently without using an excessive amount of memory, where b, n, and c are large integers.
- Method # 2: Use the **binary** expansion of n, $n = (a_{k-1},\ldots,a_1,a_0)_2$, to compute b^n.

$$b^n = b^{a_{k-1}\cdot 2^{k-1}+\cdots+a_1\cdot 2+a_0} = b^{a_{k-1}\cdot 2^{k-1}} \cdots b^{a_1\cdot 2} \cdot b^{a_0}$$

- Therefore, to compute b^n, we need only compute the values of b, b^2, $(b^2)^2 = b^4$, $(b^4)^2 = b^8$, ..., b^{2^k} and the multiply the terms b^{2^j} in this list, where $a_j = 1$.

Binary Modular Exponentiation Example

Example: Compute 3^{11} using binary modular exponentiation.

Solution: $11 = (1011)_2$

so that $3^{11} = 3^8\ 3^2\ 3^1 = ((3^2)^2)^2\ 3^2\ 3^1$

$= (9^2)^2\ 9^3 = (81)^2\ 9^3$

$= 6561 * 9^3$

$= 117{,}147.$

Fast Modular Exponentiation Algorithm

- The algorithm successively finds b **mod** c, b^2 **mod** c, b^4 **mod** c, ..., $b^{2^{k-1}}$ **mod** c, and multiplies together the terms b^{2^j} **mod** c where $a_j = 1$.

Algorithm modular_expo(b: integer, n = $(a_{k-1}a_{k-2}...a_1a_0)_2$, c: positive integers)
x = 1
power = b **mod** c
for i in range(0, k):
    if a_i = 1 **then** x = (x*power) **mod** c
    power = (power*power) **mod** c
return x where {x equals b^n mod c }

- $O((\log c)^2 \log n)$ bit operations are used to find b^n mod c.

Fast Modular Exponentiation Example

Example: Compute 3^{644} **mod** 645 using fast (binary) modular exponentiation.

Fast Modular Exponentiation Exercise

Example: Compute 5^{600} **mod** 13 using fast (binary) modular exponentiation.