Counting
การนับ

Counting

- Suppose that a password consists of 6, 7, or 8 characters
- Each of these characters must be a digit or a letter of the alphabet.
- Each password must contain at least 1 digit.
- How many such passwords are there?

Counting

- **Combinatorics** or the study of objects' arrangements.
- **Enumeration** is the counting of objects with certain properties.
  - Counting **combinations** and **permutations** are two examples.
- Counting can be used to solve various types of problems.
  - Determine probabilities of discrete events (ex. Success outcome vs all possible outcomes)
  - Determine the (time) complexity of algorithms.
  - Determine the required numbers of IP addresses to meet demand.
  - Determine how many phone numbers or passwords are possible.

Chapter Summary

1. The Basics of Counting
2. The Pigeonhole Principle
3. Permutations and Combinations
4. Pascal's Identity and Triangle
5. Generalised Permutations and Combinations
6. Advanced Counting: Recurrence Relations
7. Inclusion-Exclusion

1. The Basics of Counting

The Product Rule: A procedure can be broken down into a sequence of two tasks/events.

- There are n_1 ways to do the first task and n_2 ways to do the second task.
- Then there are $n_1 * n_2$ ways to do the procedure.

Example 1: How many bit strings of length 7 are there?

The Product Rule Example

Example 2: A company with two employees, A and B, rents a floor of a building with 12 offices. How many ways are there to assign different offices?

Example 3: The chairs of an auditorium are to be labeled with an uppercase English letter followed by a positive integer not exceeding 100. How many ways to label number of chairs differently?

The Product Rule Example

Example 4: There are 32 computers in a data center in the cloud. Each of these computers has 24 ports. How many different computer ports are there in this data center?

Example 5: How many different possible license plates are there in Thailand?

Counting Functions

Counting Functions: How many functions are there from a set with m elements to a set with n elements?

Solution: For each of m elements, it corresponds to a set with n elements. Thus, by the product rule, there are $n * n * ... * n = n^m$.

For example, from a set with 3 elements to a set with 5 elements, there are $5^3 = 125$ different functions.

Counting One-to-One Functions

Counting One-to-One Functions: How many 1-to-1 functions are there from a set with m elements to a set with n elements?

Solution: Let $m <= n$. Suppose the elements in the domain are $a_1, a_2, \ldots, a_m$.

There are n ways to choose the value of a_1.

There are $n-1$ ways to choose a_2, because the value used for a_1 cannot be used again.

By the product rule, there are $n*(n-1)*(n-2)*\cdots*(n-m+1)$ 1-to-1 functions.

Example: From a set with 3 elements to a set with 5 elements, there are 5 * 4 * 3 = 60 1-to-1 functions.

Thai Mobile Phone Numbering Plan

Example: In Thailand, the National Broadcasting and Telecommunication Commission (NBTC) specifies that a mobile phone number consists of 10 digits. There are certain restrictions on the digits (as of 2018).

- Let X denote a digit that is 0.
- Let Y denote a digit that is 6, 8 or 9.
- Let Z denote a digit from 0 through 9.

In the current plan, the format is XYZ-ZZZ-ZZZZ.

How many different mobile numbers are possible under the current plan?

Counting Loop

Example: What is the value of *k* after the following code has been executed?

```
k = 0
for i_1 in range(0, n_1):
    for i_2 in range(0, n_2):
        ....
        for i_m in range(0, n_m):
            k = k + 1
```

Counting Subsets of a Finite Set

Counting Subsets of a Finite Set: Use the product rule to show that the number of different subsets of a finite set *S* is $2^{|S|}$.

Solution: Let S be a finite set.

When the elements of *S* are listed in an arbitrary order, there is 1-to-1 correspondence between subsets of *S* and bit strings of length $|S|$.

When the *i*th element is in the subset, the bit string has a 1 in the *i*th position and a 0 otherwise.

By the product rule, there are $2^{|S|}$ bit strings of length $|S|$, thus, $2^{|S|}$ subsets.

Product Rule in Terms of Sets

- If $A_1, A_2, \ldots, A_m$ are finite sets, then the number of elements in the **Cartesian product** of these sets is the product of the number of elements of each set.
- The task of choosing an element in the Cartesian product $A_1 \times A_2 \times \cdots \times A_m$ is done by choosing an element in A_1, an element in A_2, ..., and an element in A_m.
- By the product rule, it follows that:

$$|A_1 \times A_2 \times \cdots \times A_m| = |A_1| * |A_2| * \cdots * |A_m|.$$

Basic Counting Principles: The Sum Rule

The Sum Rule: If a task can be done either in one of n_1 ways or in one of n_2, where none of the set of n_1 ways is the same as any of the n_2 ways.

- Then there are $n_1 + n_2$ ways to do the procedure.

Basic Counting Principles: The Sum Rule

Example 1: Suppose that the IT Faculty must choose either a student or a faculty member as a representative to a university committee.

How many different choices are there for this representative if there are 37 members of the IT faculty and 54 data science major students and no one is both a faculty member and a student?

Basic Counting Principles: The Sum Rule

The Sum Rule can be extended to more than 2 tasks.

Example: A student can choose a computer project from 1 of three lists. The three lists contain 23, 15, and 19 possible projects, respectively. No project is on more than one list.

How many possible projects are there to choose from?

Counting Loop

Example: What is the value of *k* after the following code has been executed?

```
k = 0
for i_1 in range(0, n_1):
    k = k + 1
for i_2 in range(0, n_2):
    k = k + 1
    ....
for i_m in range(0, n_m):
    k = k + 1
```

The Sum Rule in Terms of Sets

- If $A_1, A_2, \dots, A_m$ are pairwise disjoint finite sets, then the number of elements in the **union** of these sets is the sum of the numbers of elements in the sets.
- Because the sets are pairwise disjoint, when select an element from one of the sets A_i, we do not select an element from a different set A_j. In other words, cannot select an element from two of these sets at the same time.
- By the sum rule, the number of ways to choose an element from one of the sets is the number of elements in the union:

$|A_1 \cup A_2 \cup \cdots \cup A_m| = |A_1| + |A_2| + \cdots + |A_m|$ when $A_i \cap A_j = \varnothing$ for all *i*, *j*.

Combining the Sum and Product Rule

Example: Suppose a variable in a programming language is a string of one or two characters. It can be either a single letter or it begins with a letter then followed by a digit.

How many possible variable names are there?

Counting Passwords

Example: Each user on a computer system has a password, which is 6 to 8 characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?

Partial Solution: Let P be the total number of passwords, and let P_6, P_7, and P_8 be the passwords of length 6, 7, and 8.

By the sum rule, $P = P_6 + P_7 + P_8$.

By the product rule, the number of strings of 6 characters is 36^6 and the number of strings with no digits is 26^6.

Internet Protocol Addresses (IPv4)

- Each computer in the network or the Internet is assigned an *IP address*.
- Version 4 of the Internet Protocol uses 32 bits whereas IPv6 uses 128 bit.
- IP address starts with a *network number* (*netid*), followed by a *host number* (*hostid*), which identifies a computer as a member of a particular network.

Bit Number	0	1	2	3	4	8	16	24	31
Class A	0	netid					hostid		
Class B	1	0	netid					hostid	
Class C	1	1	0	netid					hostid
Class D	1	1	1	0	Multicast Address				
Class E	1	1	1	1	0	Address			

Internet Protocol Addresses (IPv4)

3 forms of addresses

- **Class A addresses**: used for the largest networks, consists of 0, followed by a 7-bit netid and a 24-bit hostid.
- **Class B addresses**: used for the medium-sized networks, consists of 10, followed by a 14-bit netid and a 16-bit hostid.
- **Class C addresses**: used for the smallest networks, consists of 110, followed by a 21-bit netid and a 8-bit hostid.

Bit Number	0	1	2	3	4	8	16	24	31
Class A	0	netid					hostid		
Class B	1	0	netid					hostid	
Class C	1	1	0	netid					hostid
Class D	1	1	1	0	Multicast Address				
Class E	1	1	1	1	0	Address			

Internet Protocol Addresses (IPv4)

- Neither Class D (for multicasting) nor Class E (reserved for future use) addresses are assigned as the address of a computer on the internet. Only Classes A, B, and C are available.
- 1111111 is not available as the netid of a Class A network.
- Hostids consisting of all 0s and all 1s are not available in any network.

Bit Number	0	1	2	3	4	8	16	24	31
Class A	0	netid					hostid		
Class B	1	0	netid				hostid		
Class C	1	1	0	netid				hostid	
Class D	1	1	1	0	Multicast Address				
Class E	1	1	1	1	0	Address			

Counting IP Addresses (IPv4)

Example: How many different IPv4 addresses are available for computers on the internet?

Partial Solution: Let x be the number of available addresses.

Let x_A, x_B, and x_C denote the number of Class A, B, and C addresses.

By the sum rule, $x = x_A + x_B + x_C$

To find x_A: $2^7 - 1 = 127$ netids where 1 is 1111111.

For each netid, there are $2^{24} - 2 = 16{,}777{,}214$ hostids where 2 represents the hostids of all 0s and all 1s.

Consequently, by the product rule, $x_A = 127 * 16{,}777{,}214 = 2{,}130{,}706{,}178$.

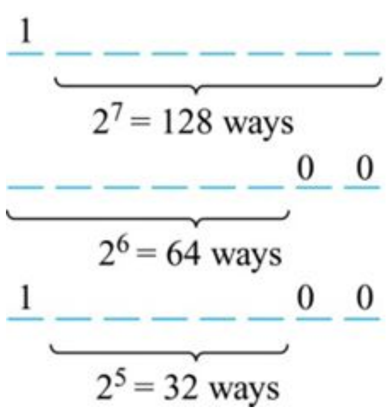
Basic Counting Principles: Subtraction Rule

Subtraction Rule: If a task can be done either in one of n_1 ways or in one of n_2 ways, then the total number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.

- Suppose that a task can be done in one of two ways, but some of the ways to do it are common to both ways.
- If add the number of ways to do the tasks in these two ways, we get an overcount of the total number of ways to do it, because the ways to do the task that are common to the two are counted twice.
- To correctly count, must **subtract** the number of ways that are counted twice.

Counting Bit Strings

Example: How many bit strings of length 8 either start with a 1 bit **or** end with the two bits 00?



Basic Counting Principles: Subtraction Rule

Subtraction Rule is also known as the **principle of inclusion-exclusion**.

- Suppose that A and B are sets.
- Then, there are |A| ways to select an element from A and |B| ways to select an element from B.
- The number of ways to select an element from A or from B is the number of ways to select an element from their union:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Basic Counting Principles: Division Rule

Division Rule: There are n/d ways to do a task if it can be done using a procedure that can be carried out in *n* ways, and for every way *w*, exactly d of the *n* ways correspond to way *w*.

- Restated in terms of sets: If the finite set A is the union of n pairwise disjoint subsets each with d elements, then $n = |A|/d$.
- In terms of functions: If f is a function from A to B, where both are finite sets, and for every value $y \in B$ there are exactly d values $x \in A$ such that $f(x) = y$, then $|B| = |A|/d$.

Basic Counting Principles: Division Rule

Division Rule: There are n/d ways to do a task if it can be done using a procedure that can be carried out in *n* ways, and for every way *w*, exactly d of the *n* ways correspond to way *w*.

- Put it simply, a task can be done in *n* different ways, but it turns out that for each way of doing the task, there are *d* equivalent ways of doing it.
    - Under these circumstances, we can conclude that there are *n/d* inequivalent ways of doing the task.

Counting Cows' legs

Example: Suppose that you have developed an automated system that counts the legs of cows in a farm.

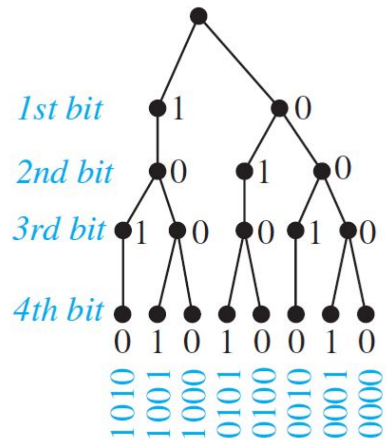
It determined that there are exactly 572 legs in the farm.

How many cows are there in this farm, assuming that each cow has 4 legs and that there are no other animals present?

Tree Diagrams

Counting problems can be solved using **tree diagrams**.

- A branch represents a possible choice and the leaves represent possible outcomes.
- Example 1: How many bit strings of length 4 do not have 2 consecutive 1s?



Tree Diagrams

Counting problems can be solved using **tree diagrams**.

- A branch represents a possible choice and the leaves represent possible outcomes.
- Example 2: A playoff between 2 teams consists of at most 5 games. The first team that wins 3 games wins the playoff. In how many ways can the play off occur?

Tree Diagrams

- Example 3: Suppose that a Japanese brand t-shirt comes in 5 different sizes: S, M, L, XL, and XXL. Each size comes in 4 colours (white, red, green, and black), except XL, which comes only in red, green, and black, and XXL, which comes only in green and black.
- How many different shirts that a store needs to stock to have one of each size and colour available?

2. The Pigeonhole Principle

- Suppose that a flock of 20 pigeons flies into a set of 19 pigeonholes to roost.
    - One of the pigeonholes must have more than 1 pigeon.
- **Pigeonhole Principle**: If k is a positive integer and $k + 1$ or more objects are placed into k boxes, then at least one box contains two or more objects.
- **Proof**: By contraposition (p -> q :: ~q -> ~p).
    - Suppose none of the k boxes has more than one object (~q). Then the total number of objects would be at most k (~p). This contradicts the statement that we have $k + 1$ objects.

Pigeonhole Principle Example

- Example 1: Among any group of 367 people, there must be at least two with the same birthday, because there are only 366 possible birthdays.
- Example 2: In any group of 27 English words, there must be at least two that begin with the same letter, because there are 26 letters in the English alphabet.
- Example 3: How many students must be in a class to guarantee that at least two students receive the same score on the final exam, if the exam is graded on a scale from 0 to 100 points?

The Generalised Pigeonhole Principle

- **The Generalised Pigeonhole Principle**: If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.
- **Proof**: By contraposition (p -> q :: ~q -> ~p).
  - Suppose none of the k boxes contains more than $\lceil N/k \rceil - 1$ objects (~q). Then the total number of objects would be at most

    $$k\left(\left\lceil \frac{N}{k} \right\rceil - 1\right) < k\left(\left(\frac{N}{k} + 1\right) - 1\right) = N,$$

    where the inequality $\lceil N/k \rceil < (N/k) + 1$ has been used (~p).
  - Thus, the total number of objects is less than N. This contradicts the statement that we have N objects.

The Generalised Pigeonhole Principle

Example 1: Among 100 people, there are at least $\lceil 100/12 \rceil = 9$ who were born in the same month.

Example 2: What is the minimum number of students required in a discrete mathematics class to be sure that at least six will receive the same grade, if there are five possible grades, A, B, C, D, and F?.

The Generalised Pigeonhole Principle

Example 3: How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen?

The Generalised Pigeonhole Principle

Example 4: How many must be selected from a standard deck of 52 cards to guarantee that at least three hearts are selected?

3. Permutations and Combinations

Many counting problems can be solved by

- finding the number of ways to **arrange** a specified number of distinct elements of a set of a particular size, where the order of these elements matters.

Many other counting problems can be solved by

- Finding the number of ways to **select** a particular number of elements from a set of a particular size, where the order of the elements selected does not matter.

Permutations

Example: In how many ways can we select 3 students from a group of 5 students to stand in line for a picture?

In this case, the <u>order</u> in which the students are selected <u>matters</u>.

- 5 ways to select the first student at the start of the line.
  - 4 ways to select the second student.
    - 3 ways to select the third student.
- By product rule, there are 5 * 4 * 3 = 60 ways to select 3 students from a group of 5 students to stand in line for a picture.

Permutations

Example: In how many ways can we arrange all 5 students to stand in line for a picture?

In this case, the <u>order</u> in which the students are selected <u>matters</u>.

- 5 ways to select the first student at the start of the line.
  - 4 ways to select the second student.
    - 3 ways to select the third student.
      - 2 ways to select the fourth student.
        - 1 way to select the fifth student.
- By product rule, there are 5 * 4 * 3 * 2 * 1 = 120 ways to select all 5 students to stand in line for a picture.

Permutations

Definition: A ***permutation*** of a set of distinct objects is an ordered arrangement of these objects.

An ordered arrangement of *r* elements of a set is called an ***r*-permutation**.

Example: Let $S = \{1, 2, 3\}$.

- The ordered arrangement 3, 1, 2 is a permutation of *S*.
- The ordered arrangement 3, 2 is a 2-permutation of *S*.

The number of *r*- permutations of a set with *n* elements is denoted by $P(n, r)$.

- The 2-permutations of S = {1, 2, 3} are {1, 2}, {1, 3}, {2, 1}, {2, 3}, {3, 1}, {3, 2}.
- Therefore, $P(3, 2) = 3 * 2 = 6$.

Combinations

Definition: An ***r*-combination** of elements of a set is an unordered selection of *r* elements from the set. Thus, an *r*-combination is simply a subset of the set with *r* elements.

The number of *r*-combinations of a set with *n* distinct elements is denoted by $C(n, r)$. The notation $\binom{n}{r}$ is also used and is called a **binomial coefficient**.

Example: Let $S = \{1, 2, 3, 4\}$.

- {1, 2, 3} and {2, 3, 1} are the 3-combination from *S*.
- The order in which the elements of a set are listed does not matter.
- C(4, 2) = 6 because the 2-combinations of {1, 2, 3, 4} are the six subsets {1, 2}, {1, 3}, {1, 4}, {2, 3}, {2, 4}, and {3, 4}.

Permutations and Combinations Formulas

- Permutations without repetition

$$P(n,r) = \frac{n!}{(n-r)!}$$

- Combinations without repetition

$$C(n,r) = \frac{n!}{(n-r)!r!}.$$

- **Proof**: By the product rule *P*(*n*, *r*) = *C*(*n*, *r*) * *P*(*r*, *r*).
  - Therefore, $C(n,r) = \frac{P(n,r)}{P(r,r)} = \frac{n!/(n-r)!}{r!/(r-r)!} = \frac{n!}{(n-r)!r!}$.

Permutations Example

Example 1: How many ways are there to select a first-prize winner, a second-prize winner, and a third-prize winner from 100 different people who have entered a contest?

Permutations Example

Example 3: Suppose that a saleswoman has to visit 8 different cities. She must begin her trip in a specified city, but she can visit the other 7 cities in any order she wishes.

How many possible orders can the saleswoman use when visiting these cities?

Permutations Example

Example 4: How many permutations of the letters *ABCDEFGH* contain the string *ABC*?

Combinations Example

Example 1: How many ways are there to select 47 cards from a standard deck of 52 cards?

$$C(52,47) = \frac{52!}{47!5!} = C(52,5) = 2,598,960.$$

Combinations Example

Example 2: How many ways are there to select 5 players from a 10-member tennis team to make a trip to a match at another school?

$$C(10,5) = \frac{10!}{5!5!} = 252.$$

Combinations Example

Example 3: A group of 30 people have been trained as astronauts to go on the first mission to Mars.

How many ways are there to select a crew of 6 people to go on this mission?

$$C(30,6) = \frac{30!}{6!24!} = \frac{30 \cdot 29 \cdot 28 \cdot 27 \cdot 26 \cdot 25}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 593,775\ .$$

Combinations Example

Example 4: Suppose that there are 9 faculty members in the mathematics department and 11 in the computer science department.

How many ways are there to select a committee to develop a discrete mathematics course at a school if the committee is to consist of 3 faculty members from the mathematics department and 4 from the computer science department?

4. Pascal’s Identity

Pascal’s Identity: If *n* and *k* are integers with $n \geq k \geq 0$, then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

It is the basis for a geometric arrangement of the **binomial coefficients** in a triangle.

Adding two adjacent binomial coefficients results in the binomial coefficient in the next row between these two coefficients.

Pascal’s Triangle

The *n*th row in the triangle consists of the binomial coefficients $\binom{n}{k}$

$k = 0, 1, \ldots, n$.

$$\begin{array}{c}
\binom{0}{0} \\
\binom{1}{0}\ \binom{1}{1} \\
\binom{2}{0}\ \binom{2}{1}\ \binom{2}{2} \\
\binom{3}{0}\ \binom{3}{1}\ \binom{3}{2}\ \binom{3}{3} \\
\binom{4}{0}\ \binom{4}{1}\ \binom{4}{2}\ \binom{4}{3}\ \binom{4}{4} \\
\binom{5}{0}\ \binom{5}{1}\ \binom{5}{2}\ \binom{5}{3}\ \binom{5}{4}\ \binom{5}{5} \\
\binom{6}{0}\ \binom{6}{1}\ \binom{6}{2}\ \binom{6}{3}\ \binom{6}{4}\ \binom{6}{5}\ \binom{6}{6} \\
\binom{7}{0}\ \binom{7}{1}\ \binom{7}{2}\ \binom{7}{3}\ \binom{7}{4}\ \binom{7}{5}\ \binom{7}{6}\ \binom{7}{7} \\
\binom{8}{0}\ \binom{8}{1}\ \binom{8}{2}\ \binom{8}{3}\ \binom{8}{4}\ \binom{8}{5}\ \binom{8}{6}\ \binom{8}{7}\ \binom{8}{8} \\
\cdots
\end{array}$$

(a)

By Pascal's identity:

$$\binom{6}{4} + \binom{6}{5} = \binom{7}{5}$$

$$\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \\
1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1 \\
\cdots
\end{array}$$

(b)

5. Generalised Permutations and Combinations

Many counting problems may use elements **repeatedly**.

- A letter or digit may be used > 1 time on a license plate.

Some counting problems involve **indistinguishable** elements.

- Count the number of ways the letters of the word *SUCCESS* can be rearranged.

Some counting problems of the ways distinguishable elements can be placed in boxes.

Permutations with Repetition

Theorem 1: The number of r-permutations of a set of n objects with repetition allowed is n^r.

Proof: There are n ways to select an element of the set for each of the r positions in the r-permutation when repetition is allowed.

Hence, by the product rule there are n^r r-permutations with repetition.

Example 1: How many strings of length r can be formed from the uppercase letters of the English alphabet?

Combinations with Repetition

Example 1: How many ways are there to select 5 bills from a cash box containing \$1, \$2, \$5, \$10, \$20, \$50, and \$100? Assume the order does not matter.

Place the selected bills in the appropriate position of a cash box illustrated below:

Combinations with Repetition

$$C(11,5) = \frac{11!}{5!6!} = 462$$

$| \, | \, | * * | \, | \, | * * *$

$* | * | * * | \, | * | \, |$

$* | \, | \, | * * | \, | * | *$

Combinations with Repetition

Theorem 2: The number of *r*-combinations from a set with *n* elements when repetition of elements is allowed is

$$C(n + r - 1, r) = C(n + r - 1, n - 1).$$

Combinations with Repetition

Example 2: Suppose that a cookie shop has 4 different kinds of cookies. How many different ways can 6 cookies be chosen? Assume that only the type of cookie, and not the individual cookies or the order in which they are chosen, matters.

Summary

TABLE 1 Combinations and Permutations With and Without Repetition.

Type	*Repetition Allowed?*	*Formula*
r-permutations	No	$\frac{n!}{(n-r)!}$
r-combinations	No	$\frac{n!}{r!\,(n-r)!}$
r-permutations	Yes	n^r
r-combinations	Yes	$\frac{(n+r-1)!}{r!\,(n-1)!}$

Permutations with Indistinguishable Objects

Example: Count the number of ways the letters of the word SUCCESS can be rearranged.

Permutations with Indistinguishable Objects

Theorem 3: The number of different permutations of n objects, where there are n_1 indistinguishable objects of type 1, n_2 indistinguishable objects of type 2, ….., and n_k indistinguishable objects of type k, is:

$$\frac{n!}{n_1!n_2!\cdots n_k!}\,.$$

Permutations with Indistinguishable Objects

Proof: The n_1 objects of type one can be placed in the n positions in $C(n, n_1)$ ways, leaving $n - n_1$ positions

- Then the n_2 objects of type two can be placed in the $n - n_1$ positions in $C(n - n_1, n_2)$ ways, leaving $n - n_1 - n_2$ positions free.
- Continue placing, until n_k objects of type k are placed in $C(n - n_1 - n_2 - \cdots - n_k, n_k)$ ways.
- Thus, by the product rule the total number of permutations is

$$C(n, n_1)\, C(n - n_1, n_2) \cdots C(n - n_1 - n_2 - \cdots - n_k, n_k)$$

$$\frac{n!}{n_1!(n-n_1)!}\frac{(n-n_1)!}{n_2!(n-n_1-n_2!)}\cdots\frac{(n-n_1-\cdots-n_{k-1})!}{n_k!0!} = \frac{n!}{n_1!n_2!\cdots n_k!}\,.$$

6. Recurrence Relations

Definition: A *recurrence relation* for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer.

A sequence is called a *solution* of a recurrence relation if its terms satisfy the recurrence relation.

The *initial conditions* for a sequence specify the terms that precede the first term where the recurrence relation takes effect.

Rabbits and the Fibonacci Numbers

Example: A young pair of rabbits (one of each gender) is placed on an island.

A pair of rabbits does not breed until they are 2 months old.

After they are 2 months old, each pair of rabbits produces another pair each month.

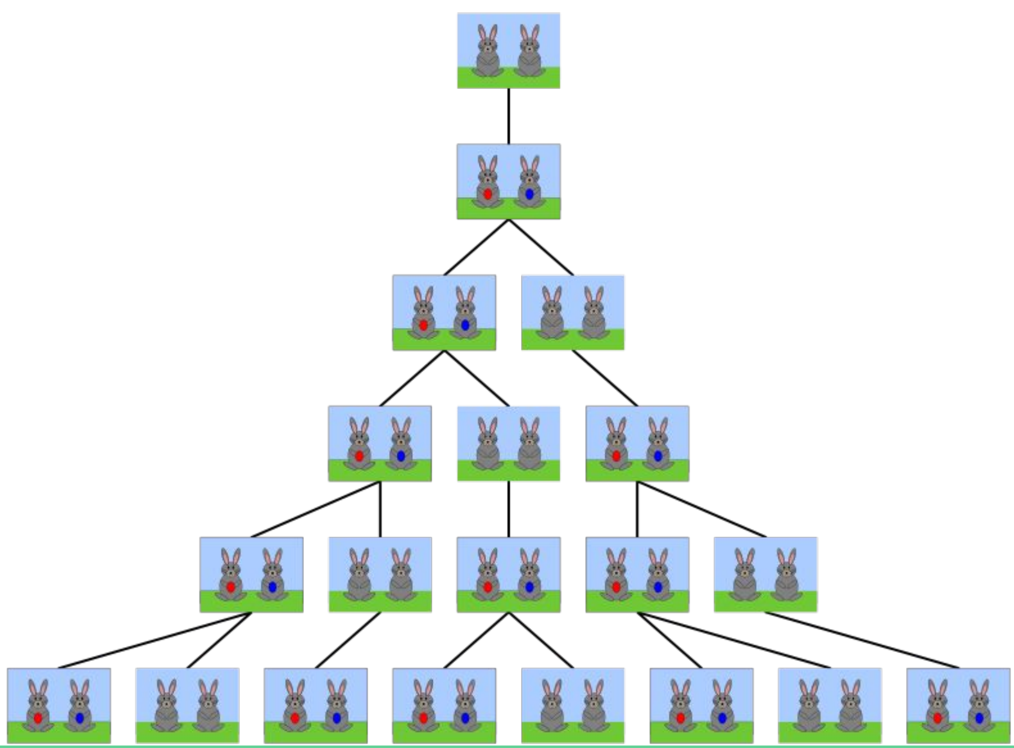
Find a recurrence relation for the number of pairs of rabbits on the island after n months, assuming that rabbits never die.

- This is the original problem considered by Leonardo Pisano (Fibonacci) in the thirteenth century.

Rabbits and the Fibonacci Numbers

Reproducing pairs (at least two months old)	Young pairs (less than two months old)	Month	Reproducing pairs	Young pairs	Total pairs
		1	0	1	1
		2	0	1	1
		3	1	1	2
		4	1	2	3
		5	2	3	5
		6	3	5	8

Rabbits and the Fibonacci Numbers



The Tower of Hanoi

In the late 19th century, the French mathematician Édouard Lucas invented a puzzle consisting of three pegs on a board with disks of different sizes. Initially all of the disks are on the first peg in order of size, with the largest on the bottom.

Rules: You are allowed to move the disks one at a time from one peg to another as long as a larger disk is never placed on a smaller.

Goal: Using allowable moves, end up with all the disks on the 2nd peg in order of size with largest on the bottom.

The Tower of Hanoi

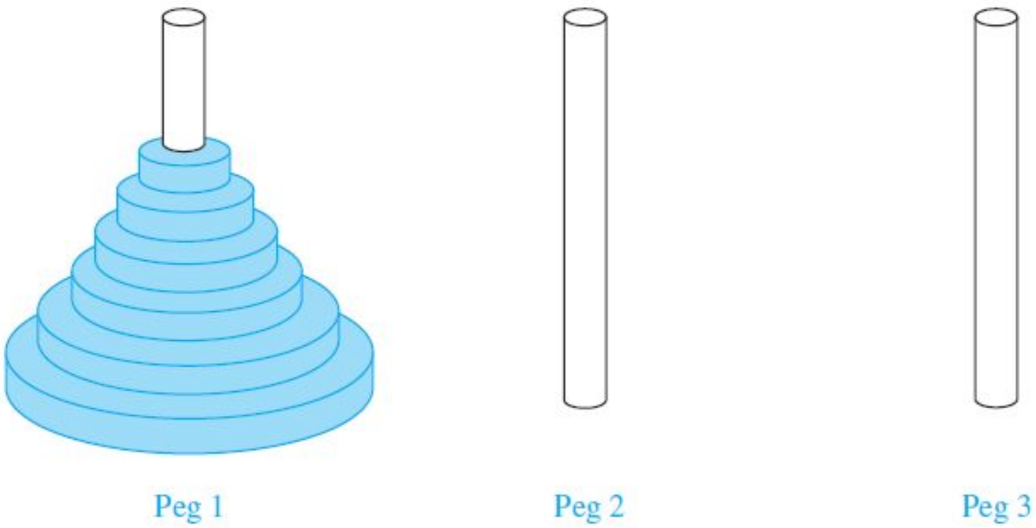


FIGURE 2 The initial position in the Tower of Hanoi.

Counting Bit Strings

Example: Find a recurrence relation and give initial conditions for the number of bit strings of length *n* without two consecutive 0s. How many such bit strings are there of length 5?

7. Inclusion-Exclusion

How many elements are in the union of two finite sets?

$$|A \cup B| = |A| + |B| - |A \cap B|$$

- The number of elements in the union of the two sets A and B is the sum of the numbers of elements in the sets minus the number of elements in their intersection.

Two Finite Sets

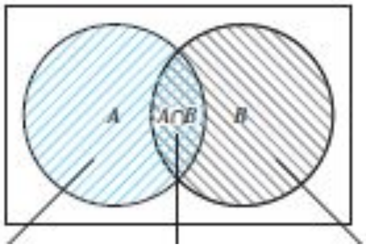
Example 1: In a discrete mathematics class every student like coding or mathematics or both.

The number of students who like coding (possibly along with maths) is ?

The number of students who like maths (possibly along with coding) is ?

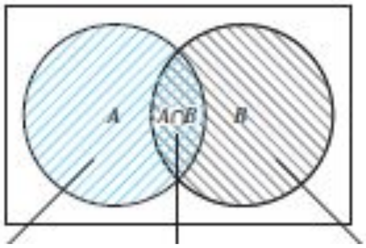
The number of students who like both is ?

How many students are in the class?



Two Finite Sets

Example 2: How many positive integers not exceeding 1000 are divisible by 7 or 11?



Two Finite Sets

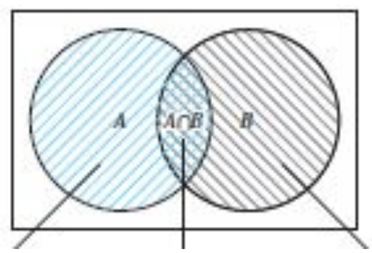
Example 3: Suppose there are 500 students in IT Faculty.

123 are taking discrete maths class.

46 are taking a course in web programming.

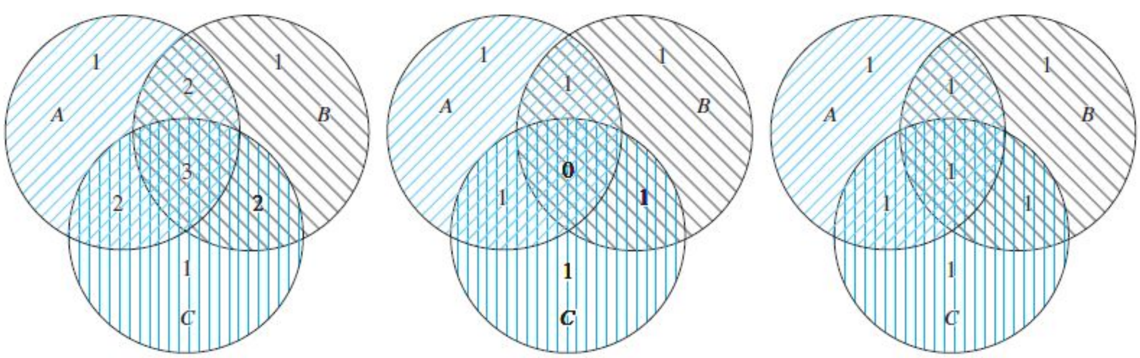
25 are taking courses in both discrete maths and web programming.

How many are not taking a course either in discrete maths or in web programming?



Three Finite Sets

$$|A \cup B \cup C| =$$
$$|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$



(a) Count of elements by $|A| + |B| + |C|$

(b) Count of elements by $|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C|$

(c) Count of elements by $|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

Three Finite Sets

Example 4: A total of 1232 students have taken a course in Spanish.

879 have taken a course in French

114 have taken a course in Russian.

Further, 103 have taken courses in both Spanish and French

23 have taken courses in both Spanish and Russian

14 have taken courses in both French and Russian.

If 2092 students have taken a course in at least one of Spanish French and Russian, how many students have taken a course in all 3 languages.