Faculty of Information Technology
King Mongkut's Institute of Technology Ladkrabang

Chapter 11: Graph Algorithms Part 1

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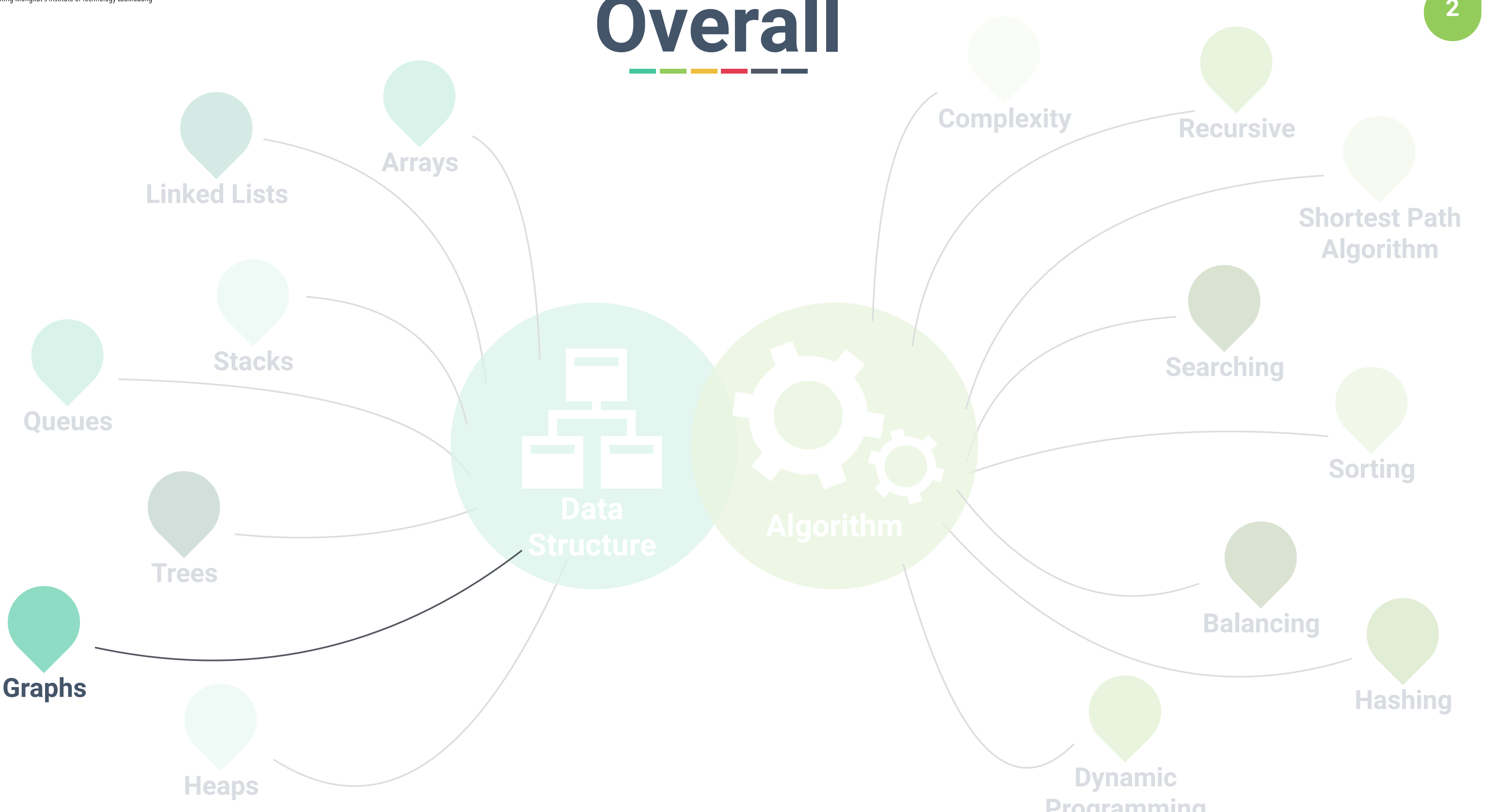
Overall



Outline

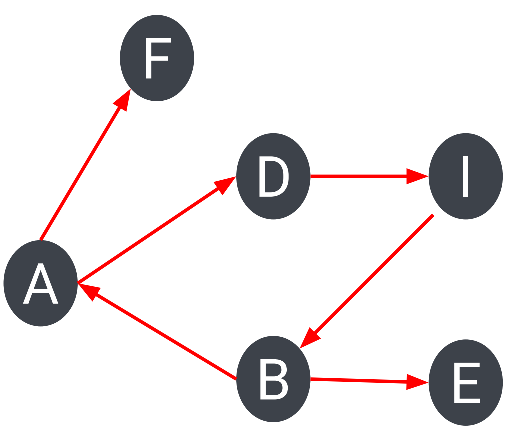
Graphs

- Definition, elements and types

Graph Algorithms

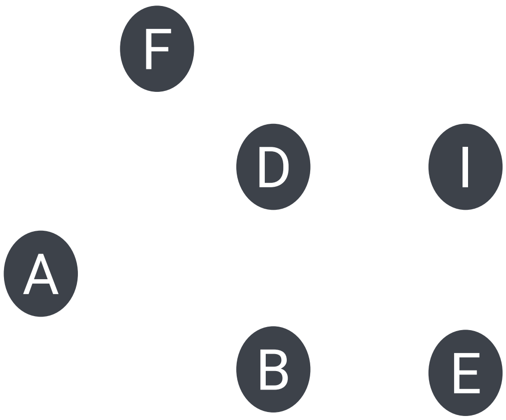
- Traversal
- Minimum Spanning Tree (next week)
- Shortest Path

Graphs



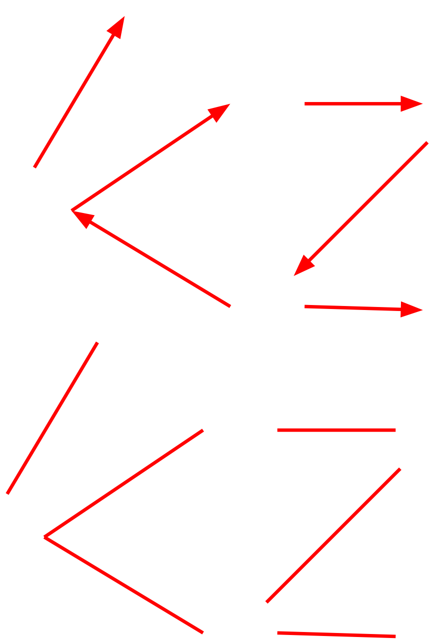
- A **graph** is a set of objects, called vertices or **nodes**, where the actual data is stored and a collection of connections between them, called **edges** or arcs.
- A graph can be used to represent relationships between pairs of objects.
- Applications that require efficient processing between networks such as mapping, transportation and computer networks (Internet).

The Basic Elements of Graph



- Formally, a graph *G* is a set *V* of vertices and a collection *E* of pairs of vertices, called edges.
- V = {A, B, D, E, F, I}

The Basic Elements of Graph



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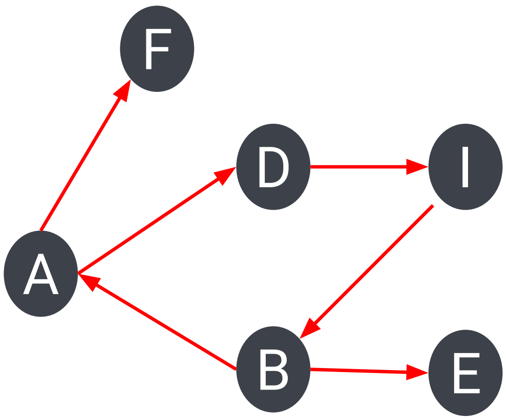
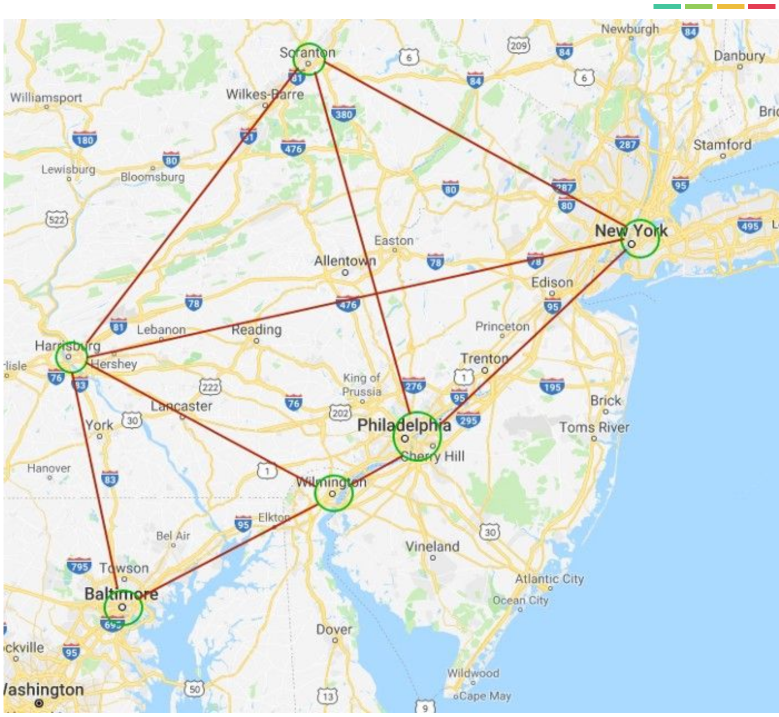
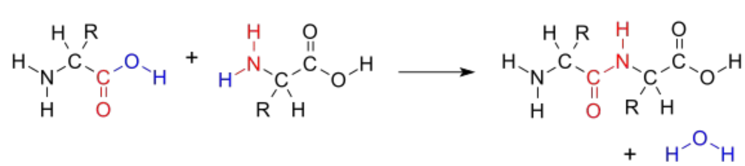
The Basic Elements of Graph



- Formally, a graph *G* is a set *V* of vertices and a collection *E* of pairs of vertices, called edges.
- V = {A, B, D, E, F, I}
- E = {(A, F), (A, D), (B, A), (D, I), (I, B), (B, E)}

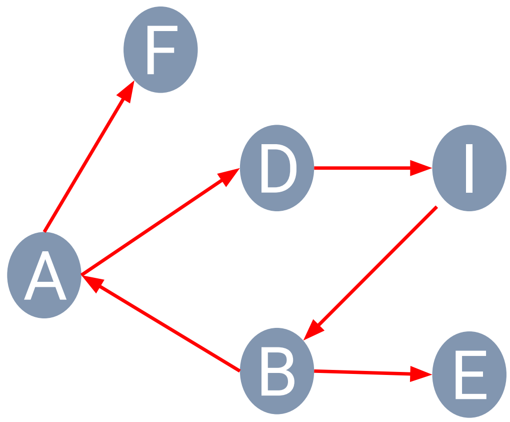
Graphs



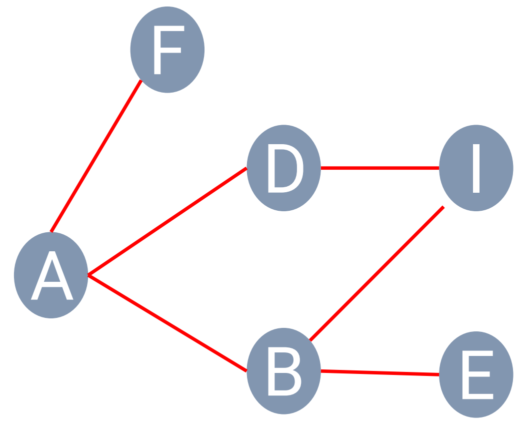


Ref: https://miro.medium.com/max/700/1*2jL0bKbT8ttskAlEVDv1yg.jpeg

Graph Types



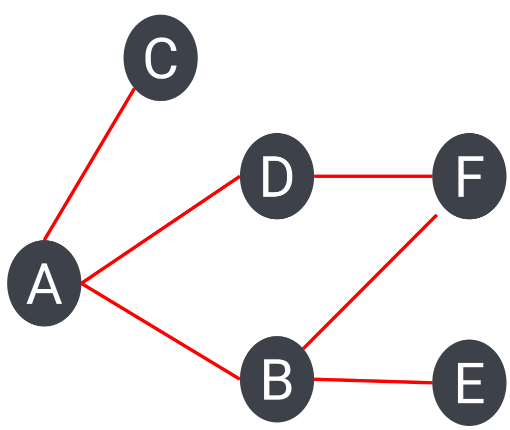
Directed



Undirected

- An edge (u, v) is directed from u to v if the pair (u, v) is ordered.
- An edge (u, v) is undirected if the pair (u, v) is not ordered.

Graph Representation

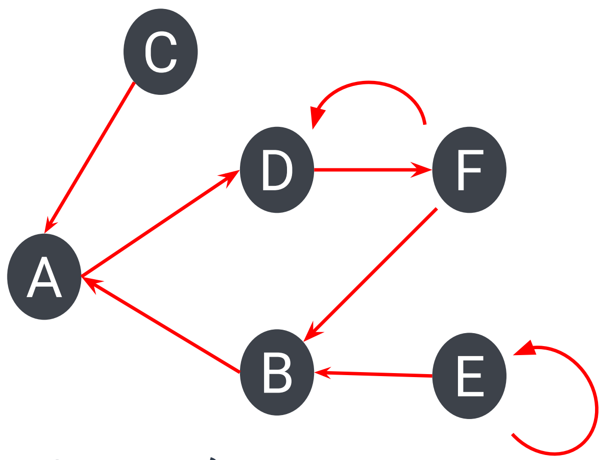


- V = {A, B, C, D, E, F}
- E = {(A, C), (A, D), (A, B), (D, F), (B, E), (B, F)}

	A	B	C	D	E	F
A	0	1	1	1	0	0
B	1	0	0	0	1	1
C	1	0	0	0	0	0
D	1	0	0	0	0	1
E	0	1	0	0	0	0
F	0	1	0	1	0	0

Adjacency Matrix

Graph Representation

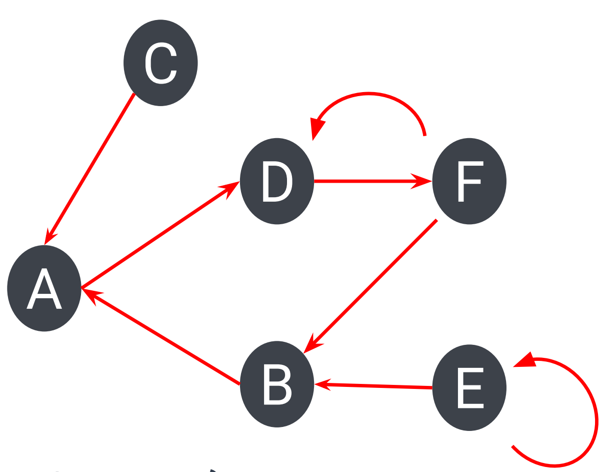


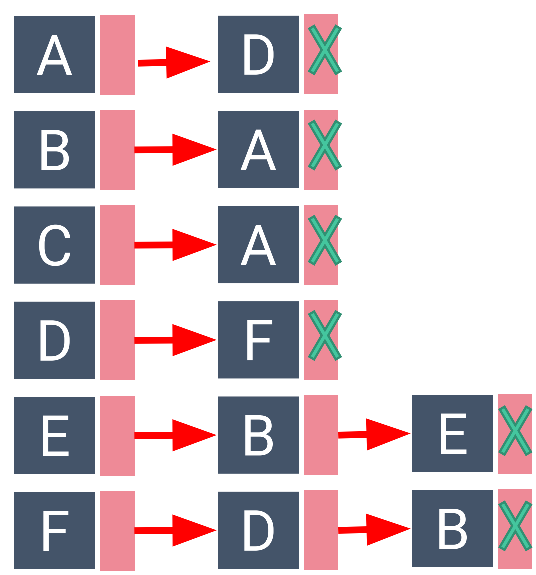
- V = {A, B, C, D, E, F}
- E = {(C, A), (A, D), (B, A), (D, F), (F, D), (E, B), (F, B), (E, E)}

	A	B	C	D	E	F
A	0	0	0	1	0	0
B	1	0	0	0	0	0
C	1	0	0	0	0	0
D	0	0	0	0	0	1
E	0	1	0	0	1	0
F	0	1	0	1	0	0

Adjacency Matrix

Graph Representation





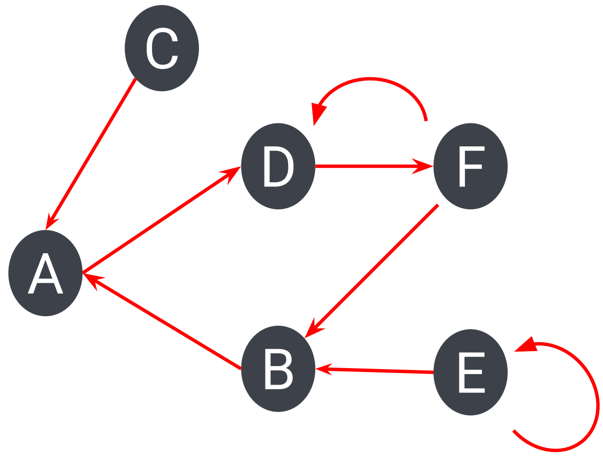
- V = {A, B, C, D, E, F}
- E = {(C, A), (A, D), (B, A), (D, F), (F, D), (E, B), (F, B), (E, E)}

Adjacency List

Graph Representation

Node Representation

Node	Name	Phone
A	Able	
B	Baker	
C	Charlie	
D	Denver	
E	Ethan	
F	Fred	

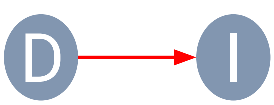


Edge Representation

	A	B	C	D	E	F
A	0	0	0	1	0	0
B	1	0	0	0	0	0
C	1	0	0	0	0	0
D	0	0	0	0	0	1
E	0	1	0	0	1	0
F	0	1	0	1	0	0

Adjacency Matrix

Graph Notations

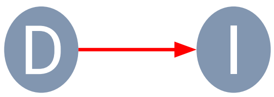


V = {D, I}

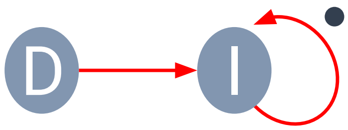
E = {(D, I)}

- **Endpoints**: Two nodes (*u*, *v*) that are joined by an edge.
  - These two nodes are **adjacent**.
- **Origin**: First endpoint (*u*) on a directed edge.
- **Destination**: Second endpoint (*v*) on a directed edge.

Graph Notations

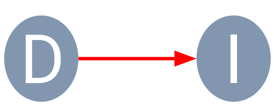


- A **path** is a sequence of nodes and edges that starts at a node and ends at a node such that each node is adjacent to the next one.
- Formally, a path is a sequence of nodes $V_1, V_2, V_3, \ldots, V_n$ where $(V_1, V_2), (V_2, V_3), \ldots, (V_{n-1}, V_n) \in E$.



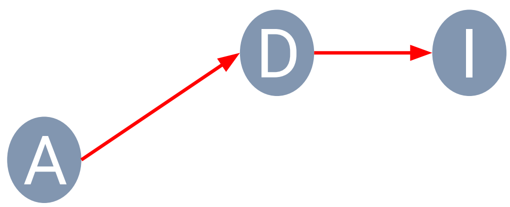
- A **loop** is a special case of path where two endpoints are the same.
  - An edge that starts and ends with the same node.

Graph Notations



- A **cycle** is a path that starts and ends at the same node, having at least one edge.
- A **simple path** is when each node in the path is distinct.
- A **simple cycle** is when each node in the cycle is distinct, except for the first and last one.

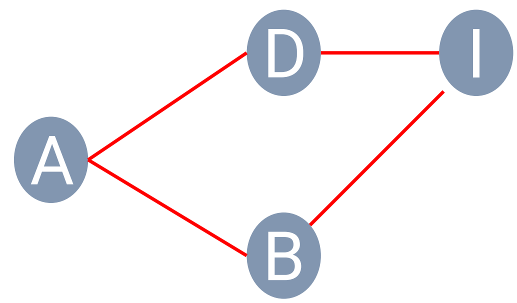
Graph Properties



A, D, I

Simple Path

Acyclic (No Cycle)

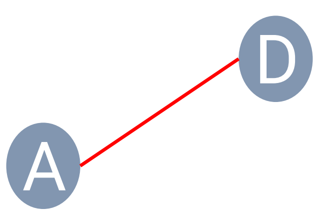


A, D, I, B, A

Simple Path

Undirected Simple Cycle

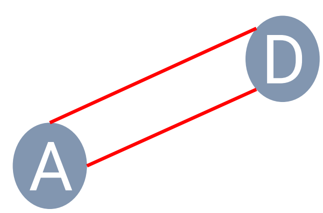
Graph Properties



A, B, A

Non Simple Path

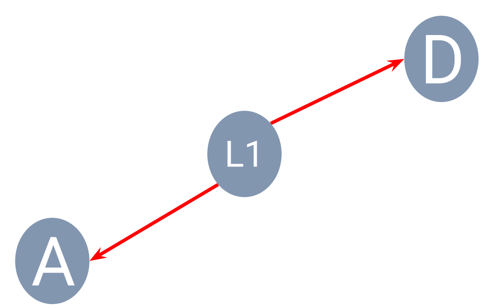
Acyclic (No Cycle)

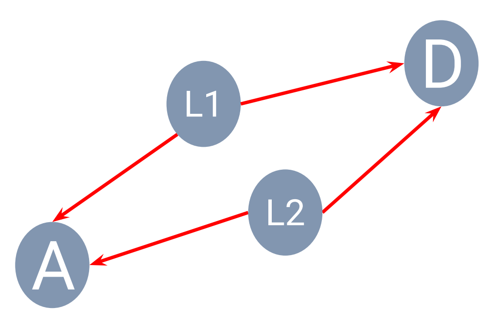


A, D, A

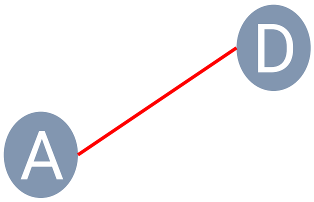
Non Simple Path

Acyclic (No Cycle)





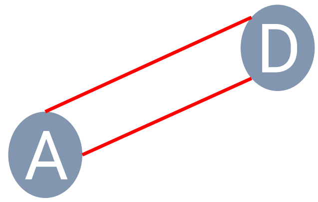
Graph Properties



A, B, A

Non Simple Path

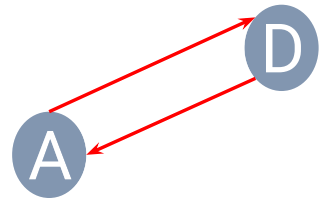
Acyclic (No Cycle)



A, D, A

Non Simple Path

Acyclic (No Cycle)

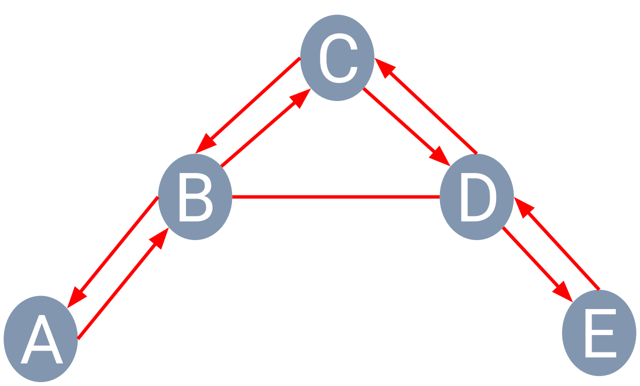


A, D, A

Non Simple Path

Directed cycle

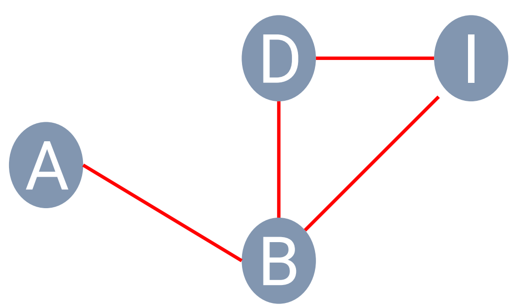
Graph Properties



A, B, C, D, E, D, B, A

Non Simple Path

Cycle

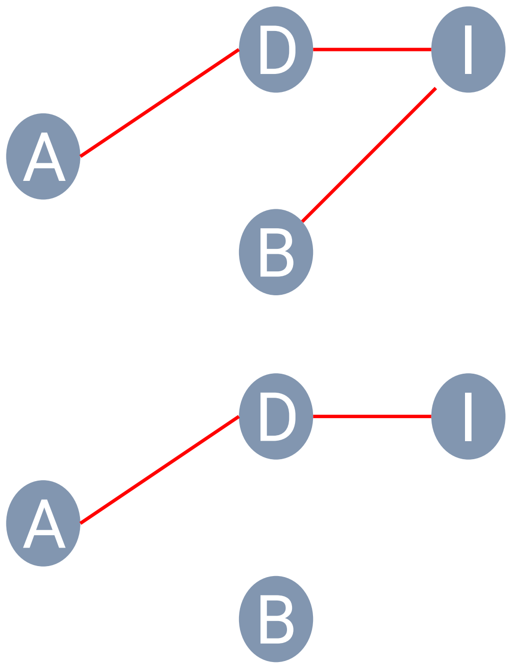


A, B, I, D, B, A

Non Simple Path

Undirected Cycle

Graph Notations



- A graph is **connected** if, for any two nodes, there is a path between them.
- The **in-degree** of a node v is the number of the incoming edges of v.
- The **out-degree** of a node v is the number of the outgoing edges of v.
- For instance, node D has in-degree = 2, out-degree = 2.

Graph Algorithms

- Traversals
- Minimum Spanning Tree
- Shortest Path

Graph Traversals

- A **traversal** is a systematic procedure for exploring a graph by examining all of its nodes and edges.
- Graph traversal algorithms are key to answering many fundamental questions about graphs involving the notion of **reachability**, that is, in determining how to travel from one node to another while following paths of a graph.
- Two efficient graph traversal algorithms: **depth-first search** and **breadth-first search**.

Depth-First Search (DFS)

- Imagine wandering in a labyrinth/maze with a string and a can of paint.
- Starts at node *s* in graph *G*, fix one end of the string to *s* and paint *s* as "visited".
  - The node *s* is now a "current" node, which is a current node *u*.
  - Painting is analogous to putting a visited node in a stack.
- Then, traverse *G* by considering an edge (*u*, *v*) that is connected to *u*.
  - If the edge (*u*, *v*) leads to a node *v* that is already existed/painted, ignore.
  - Otherwise, if (*u*, *v*) leads to an unvisited node *v*, then unroll the string and go to *v*. Then painted *v* as "visited" and make it the current node *u*.
- Repeat the step above until reaching a "dead end", that is, a current node *v* such that all the edges connected to *v* lead to nodes already visited.

Depth-First Search (DFS)

- To get out of the “dead end”, roll the string back up, backtracking along the edge to a previously visited node *u*.
- Then make *u* the current node and repeat the traversal step for any edges connected to *u* that have not yet considered.
- The traversal is continued until the process terminates when the backtracking leads back to the start node *s*, and there are no more unexplored edges connected to *s*.
- DFS can have many solutions, however, for consistency, when there are multiple edges available, the node with fewer values should be selected first.

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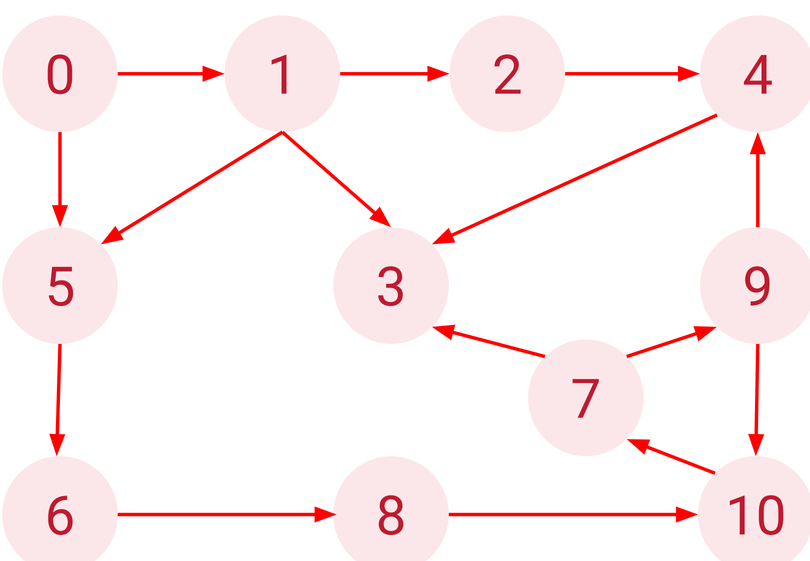
Depth-First Search (DFS)



Stack

“Parent ⇨ Children ⇨ Grandchild”

Output : ?

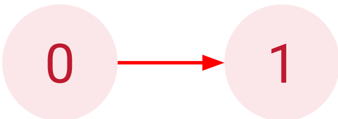
Depth-First Search (DFS)

Output : 0

Depth-First Search (DFS)

Output : 0

Depth-First Search (DFS)



1
0
Stack

Output : 0 1

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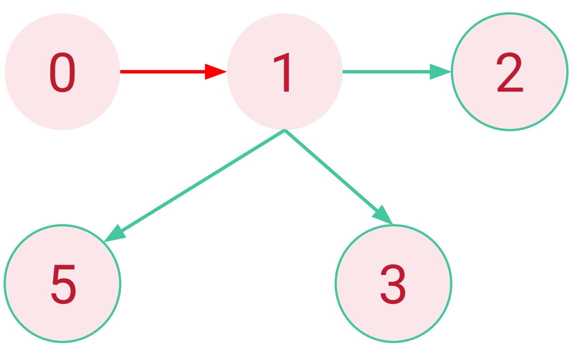
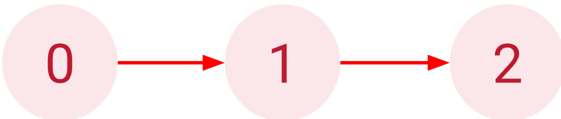
Depth-First Search (DFS)



1
0
Stack

Output : 0 1

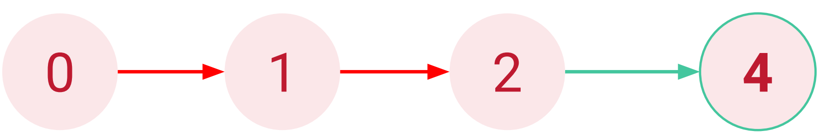
Depth-First Search (DFS)



2
1
0
Stack

Output : 0 1 2

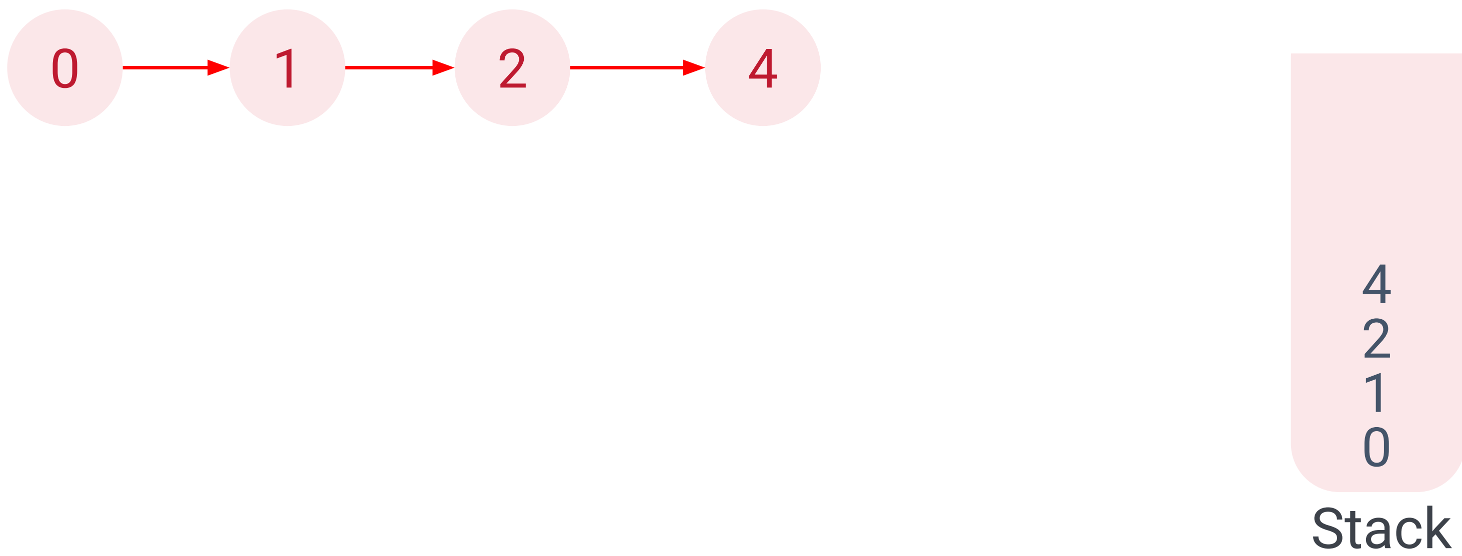
Depth-First Search (DFS)



2
1
0
Stack

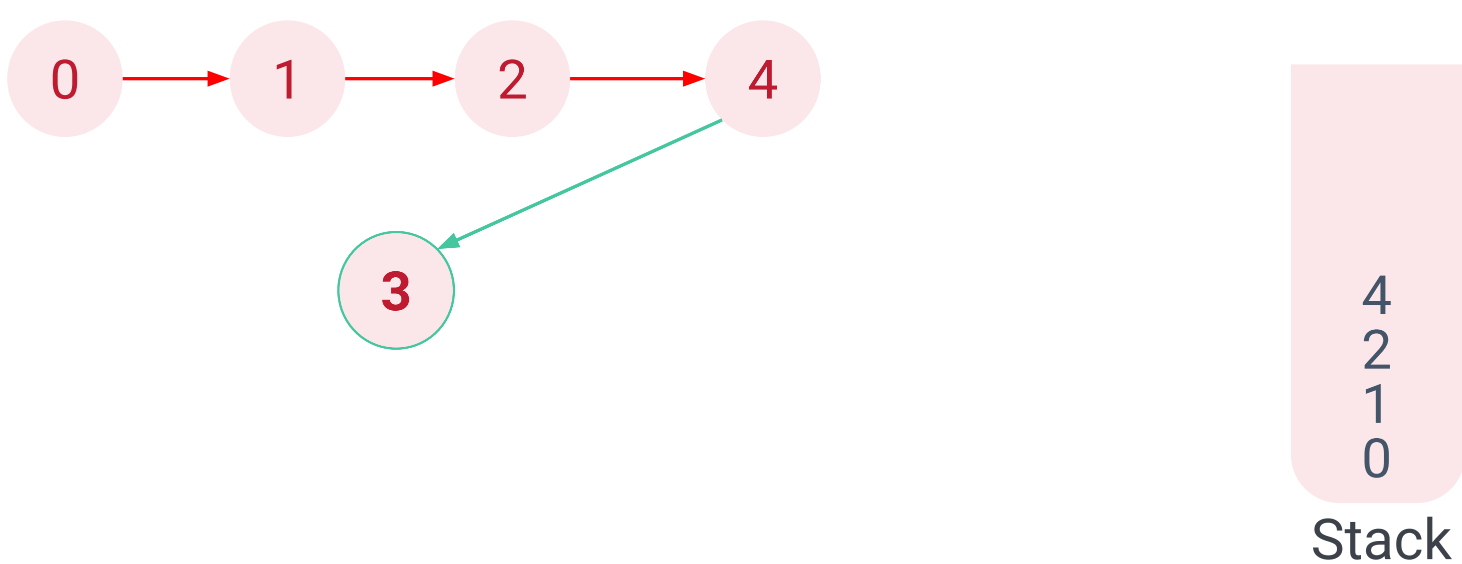
Output : 0 1 2

Depth-First Search (DFS)



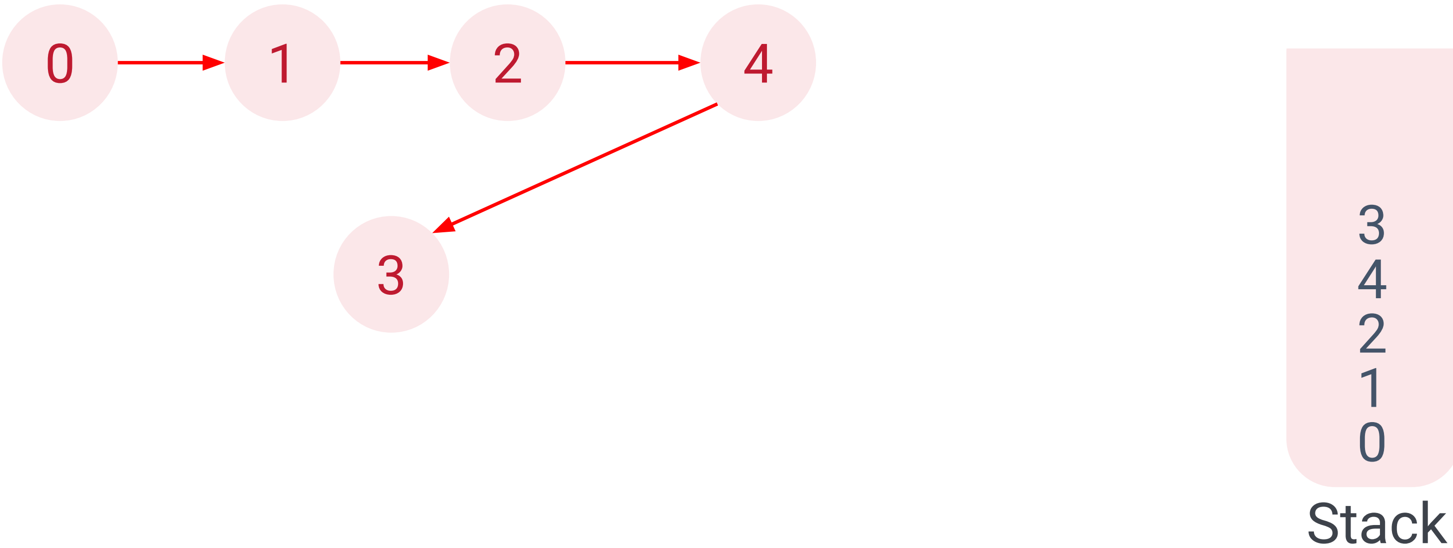
Output : 0 1 2 4

Depth-First Search (DFS)



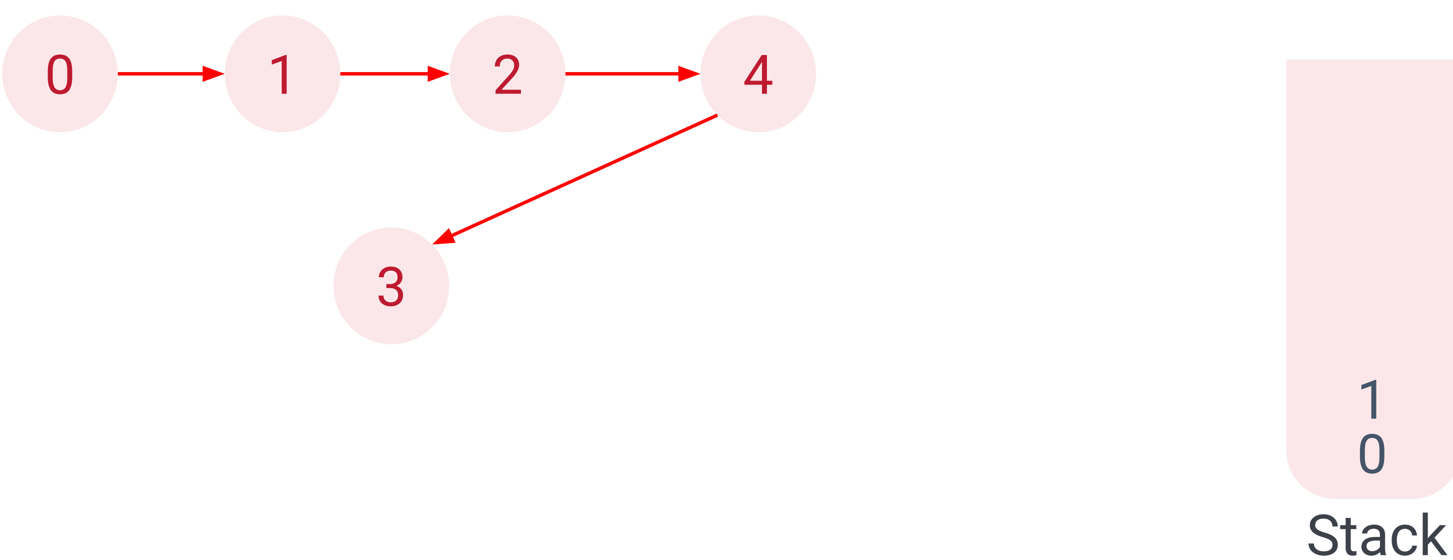
Output : 0 1 2 4

Depth-First Search (DFS)



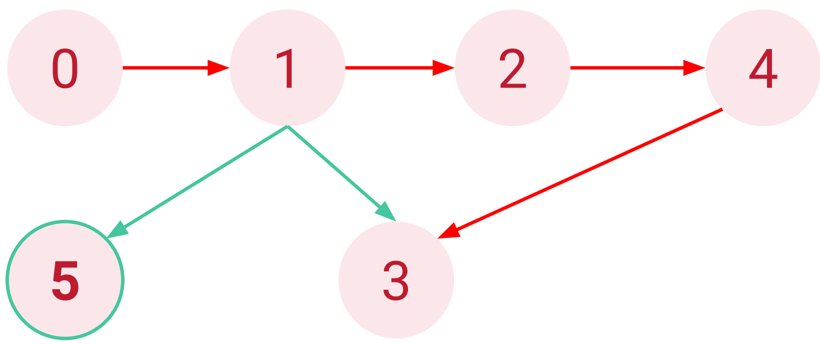
Output : 0 1 2 4 3

Depth-First Search (DFS)



Output : 0 1 2 4 3

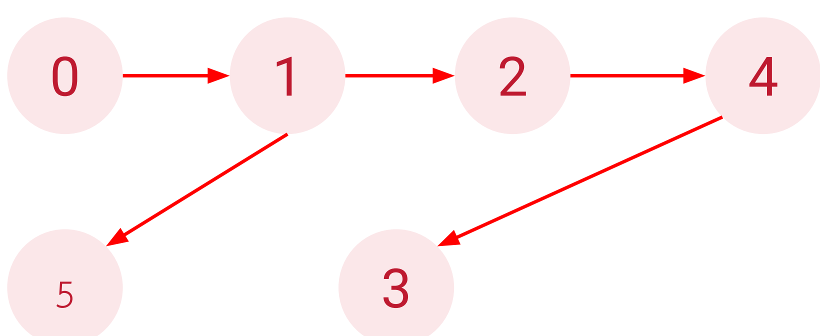
Depth-First Search (DFS)



Stack

Output : 0 1 2 4 3

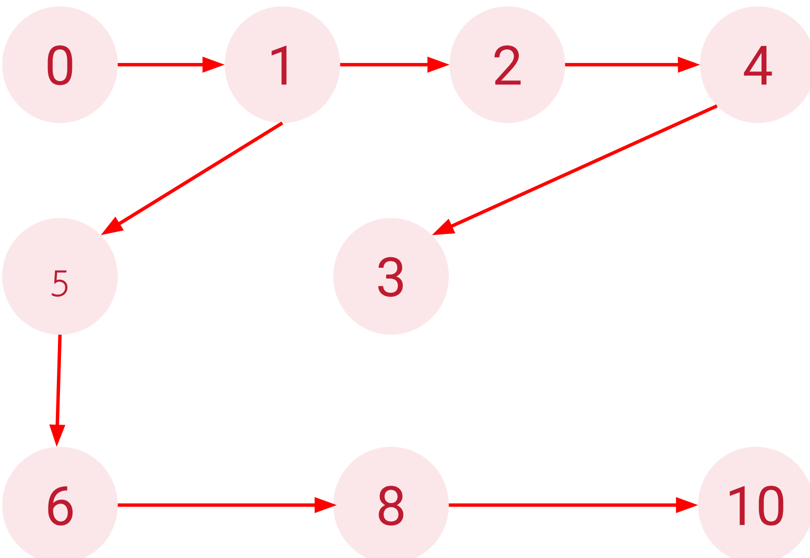
Depth-First Search (DFS)



Stack

Output : 0 1 2 4 3 5

Depth-First Search (DFS)

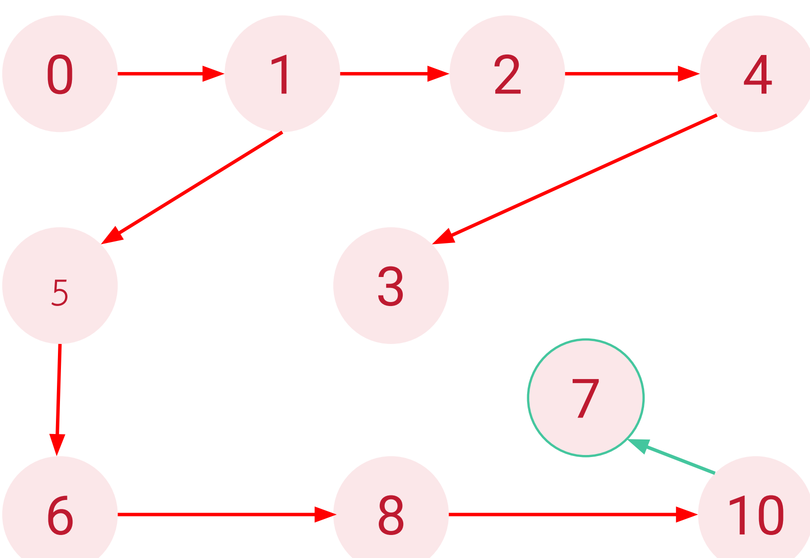


10
8
6
5
1
0

Stack

Output : 0 1 2 4 3 5 6 8 10

Depth-First Search (DFS)

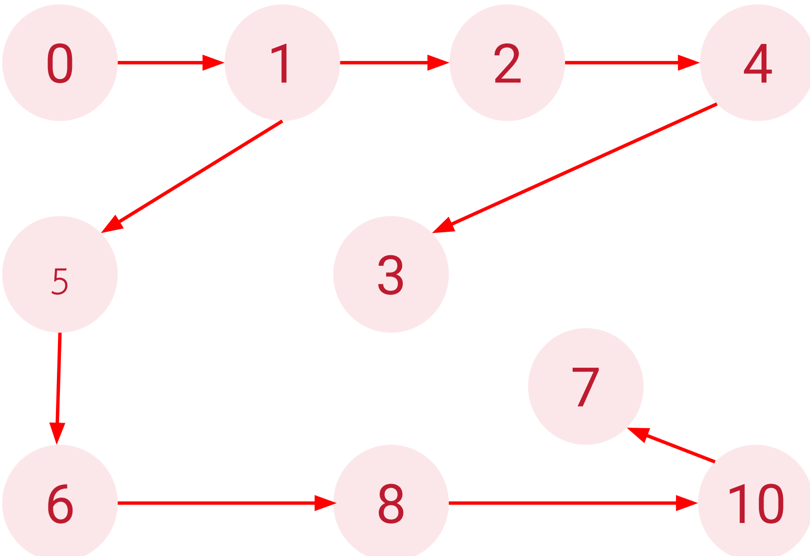


10
8
6
5
1
0

Stack

Output : 0 1 2 4 3 5 6 8 10

Depth-First Search (DFS)



7
10
8
6
5
1
0

Stack

Output : 0 1 2 4 3 5 6 8 10 7

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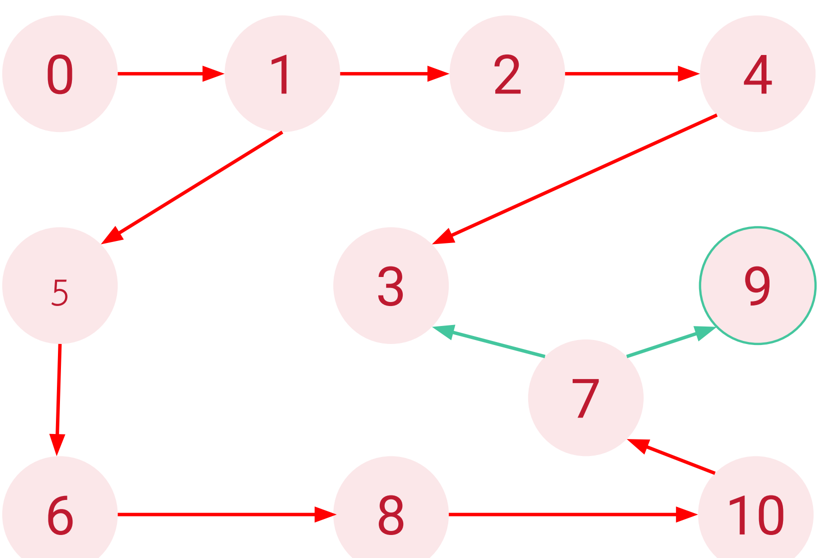
Depth-First Search (DFS)

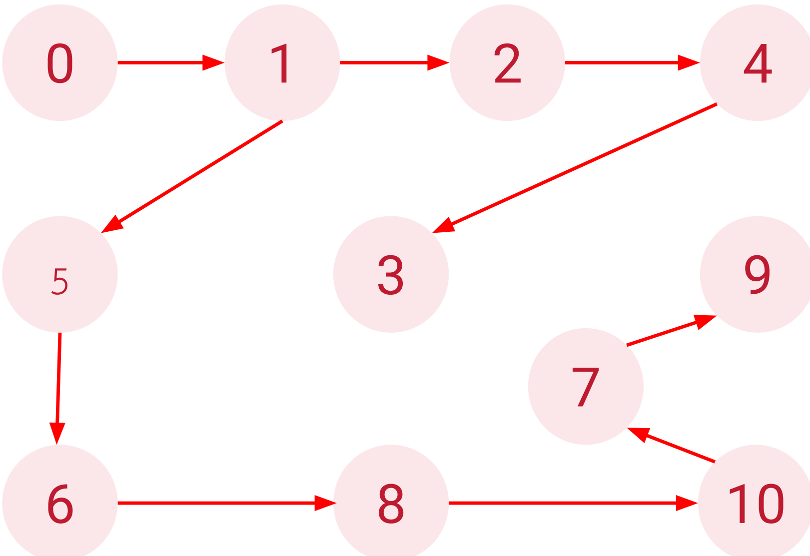


7
10
8
6
5
1
0

Stack

Output : 0 1 2 4 3 5 6 8 10 7

Depth-First Search (DFS)



9
7
10
8
6
5
1
0
Stack

Output : 0 1 2 4 3 5 6 8 10 7 9

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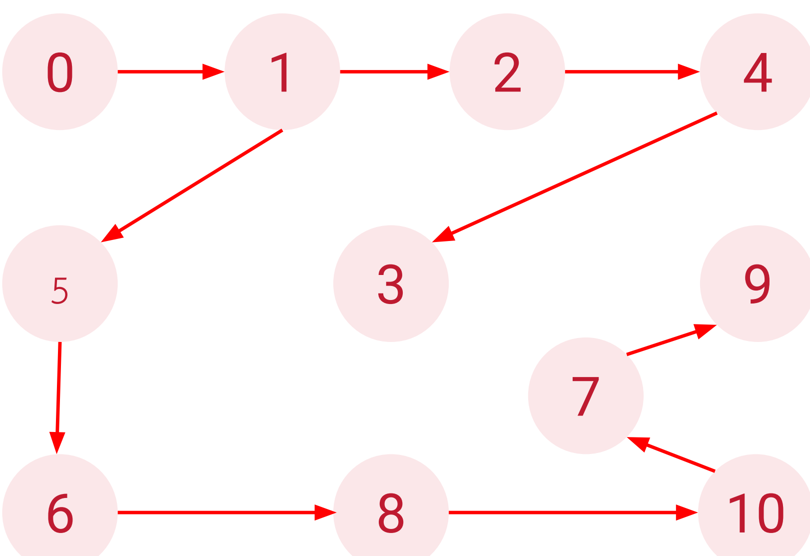
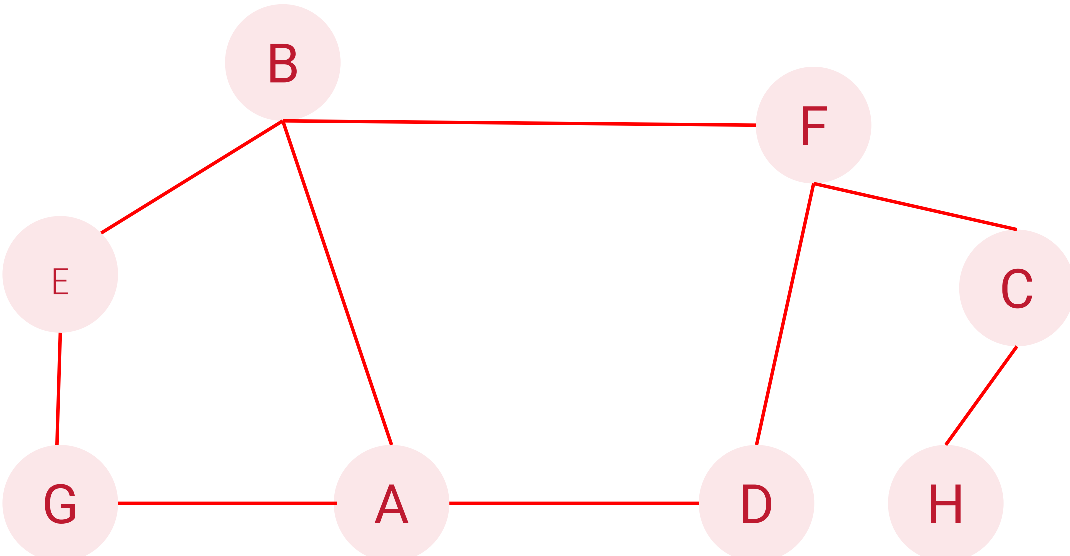
Depth-First Search (DFS)



0
Stack

Output : 0 1 2 4 3 5 6 8 10 7 9

Depth-First Search (DFS) Exercise



Output : ?

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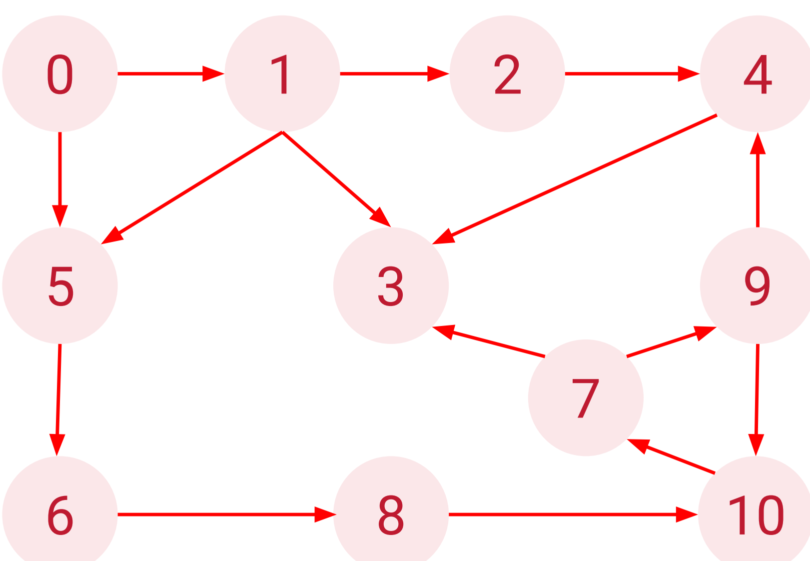
Breadth-First Search (BFS)

- DFS is similar to sending a single person to navigate the graph.
- BFS is more akin to sending out many people in all directions to traverse a graph in coordinated fashion.
- A BFS proceeds in rounds and subdivides the nodes into **levels**.
- Starts at node *s*, which is level 0.
  - 1st Round: paint all nodes adjacent to node *s* as “visited” and placed into level 1.
  - 2nd Round: All nodes adjacent to level 1’s nodes are placed into level 2 and marked as “visited”.
  - This process continues until no new nodes are found in a level.

Breadth-First Search (BFS)

- First, starts with the initial node anywhere on the graph, which is the starting point.
  - Usually starts with a node with lowest value.
- From there, add (enqueue) every node that is directly connected to the current node into a queue.
- Then dequeue the current node and check the next node in queue.

Breadth-First Search (BFS)

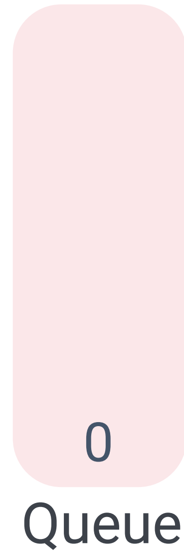


Queue

“Parent ➪ Children ➪ Grandchild”

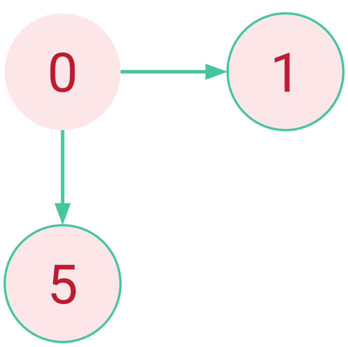
Output : ?

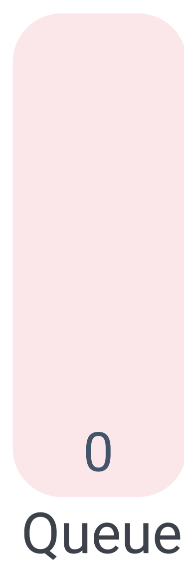
Breadth-First Search (BFS)



Output : 0

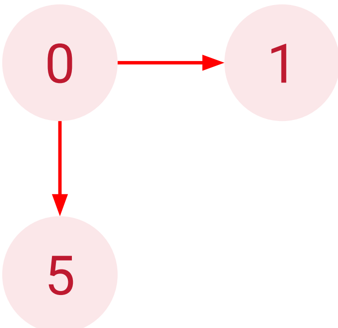
Breadth-First Search (BFS)





Output : 0

Breadth-First Search (BFS)



5
1
Queue

Output : 0 1 5

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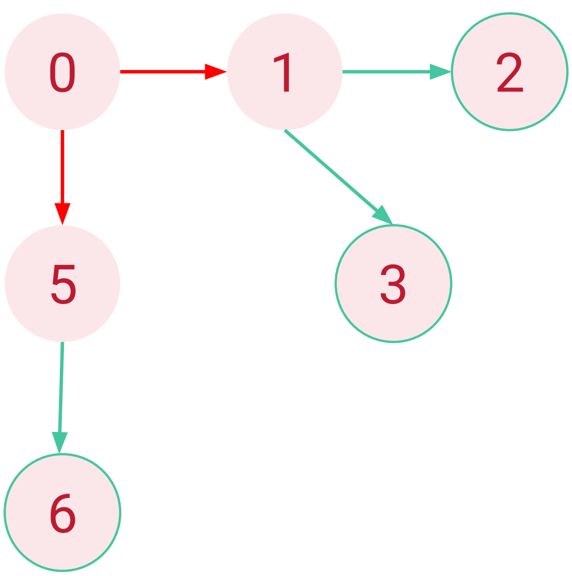
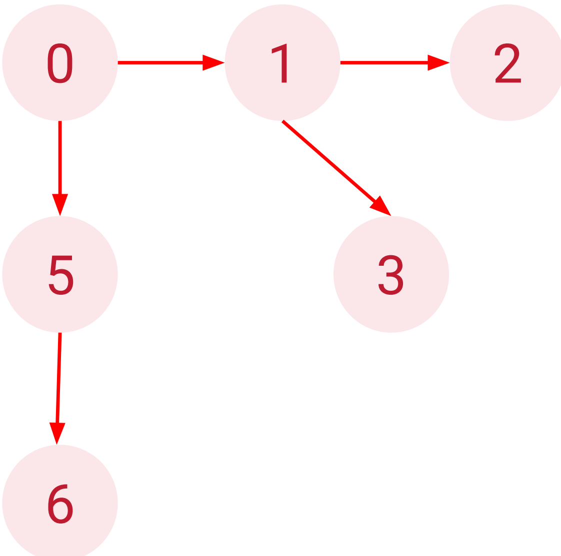
Breadth-First Search (BFS)

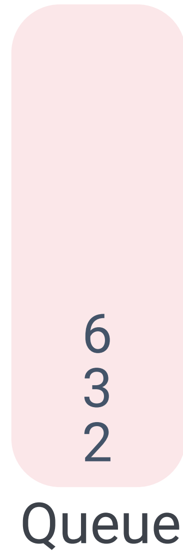


5
1
Queue

Output : 0 1 5

Breadth-First Search (BFS)





Output : 0 1 5 2 3 6

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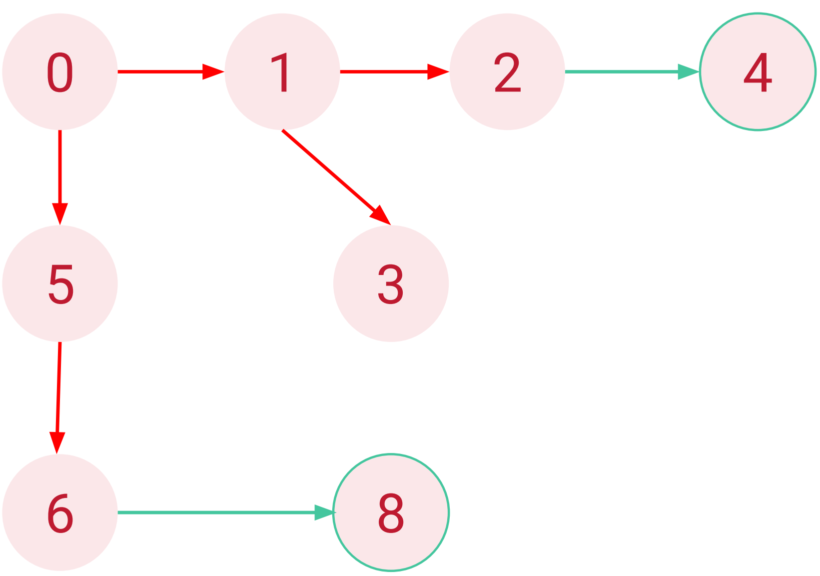
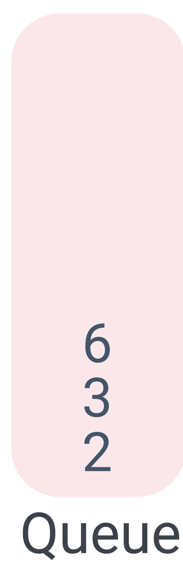
Breadth-First Search (BFS)





Output : 0 1 5 2 3 6

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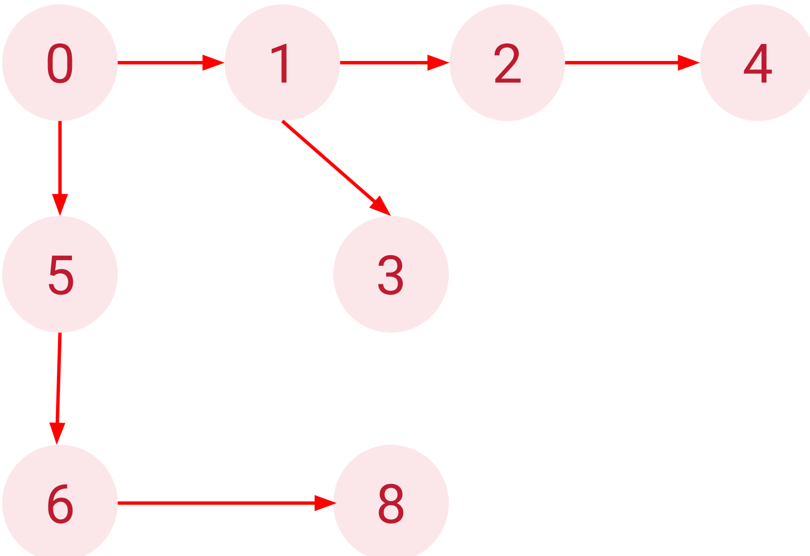
Breadth-First Search (BFS)



8
4
Queue

Output : 0 1 5 2 3 6 4 8

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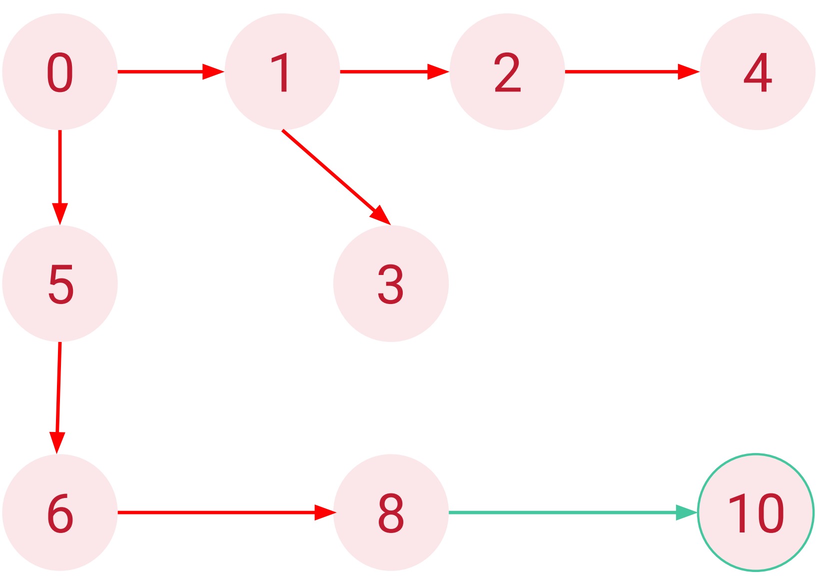
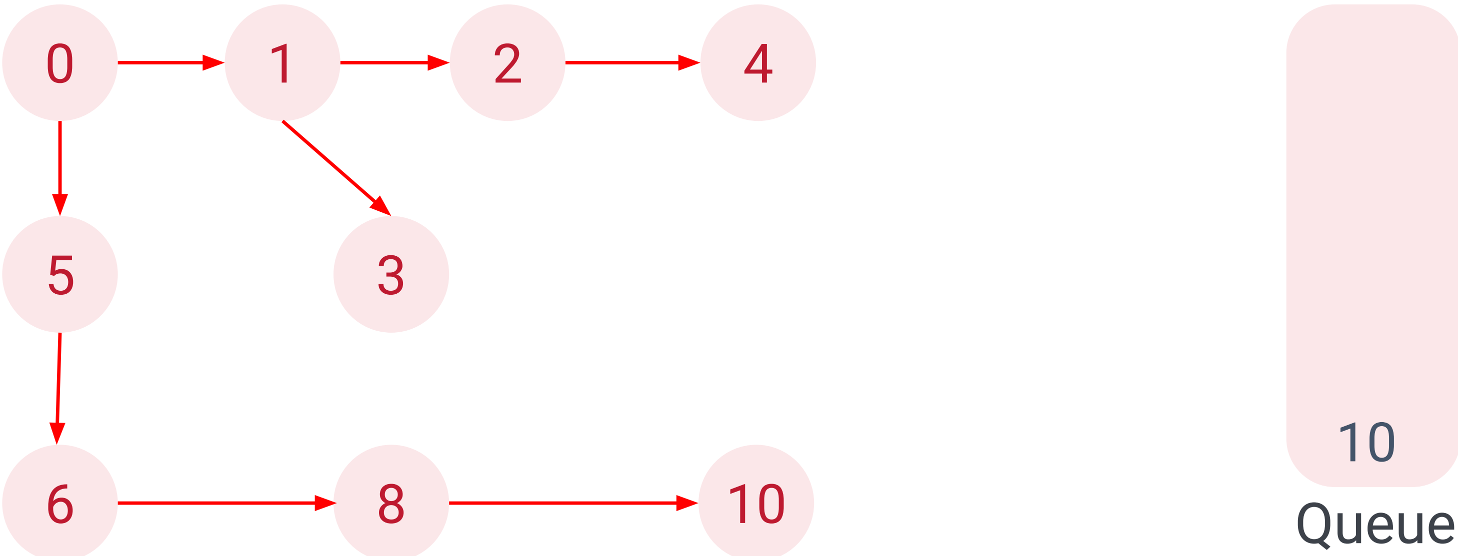
Breadth-First Search (BFS)



8
4
Queue

Output : 0 1 5 2 3 6 4 8

Breadth-First Search (BFS)



Output : 0 1 5 2 3 6 4 8 10

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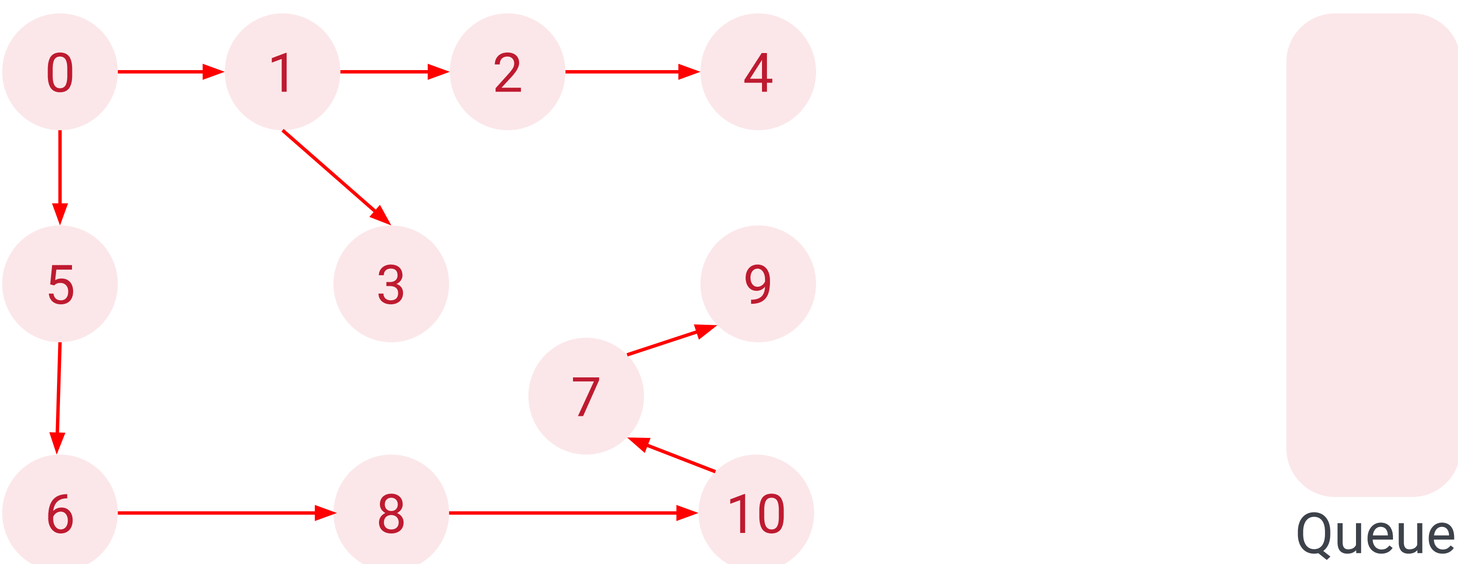
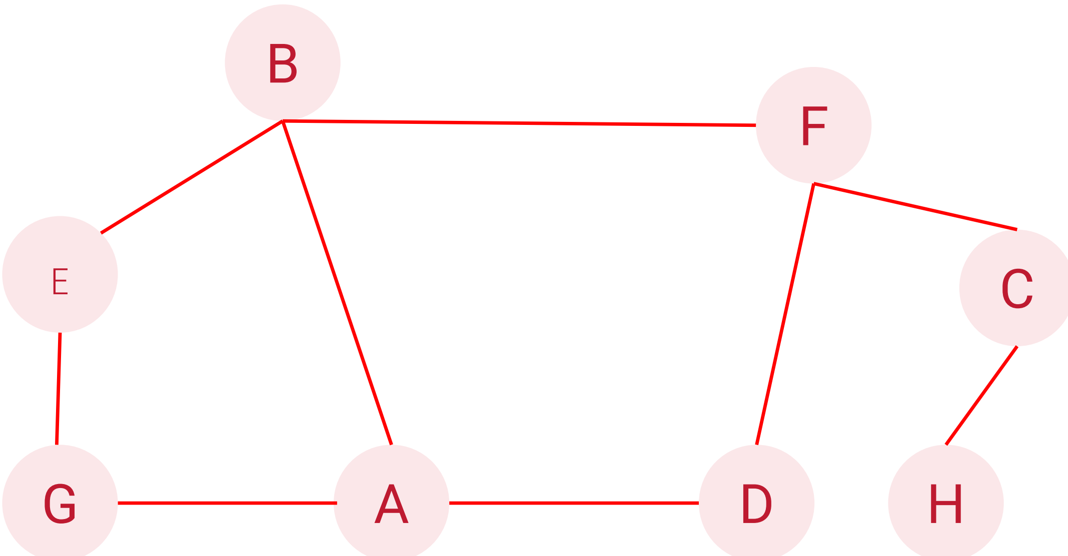
Breadth-First Search (BFS)



Output : 0 1 5 2 3 6 4 8 10 7 9

Breadth-First Search (BFS) Exercise



Output : ?

Graph Traversals

- On undirected and directed graph with n nodes and m edges.
  - A DFS traversal can be performed in $O(n + m)$ time.
  - A BFS traversal can be conducted in $O(n + m)$ time

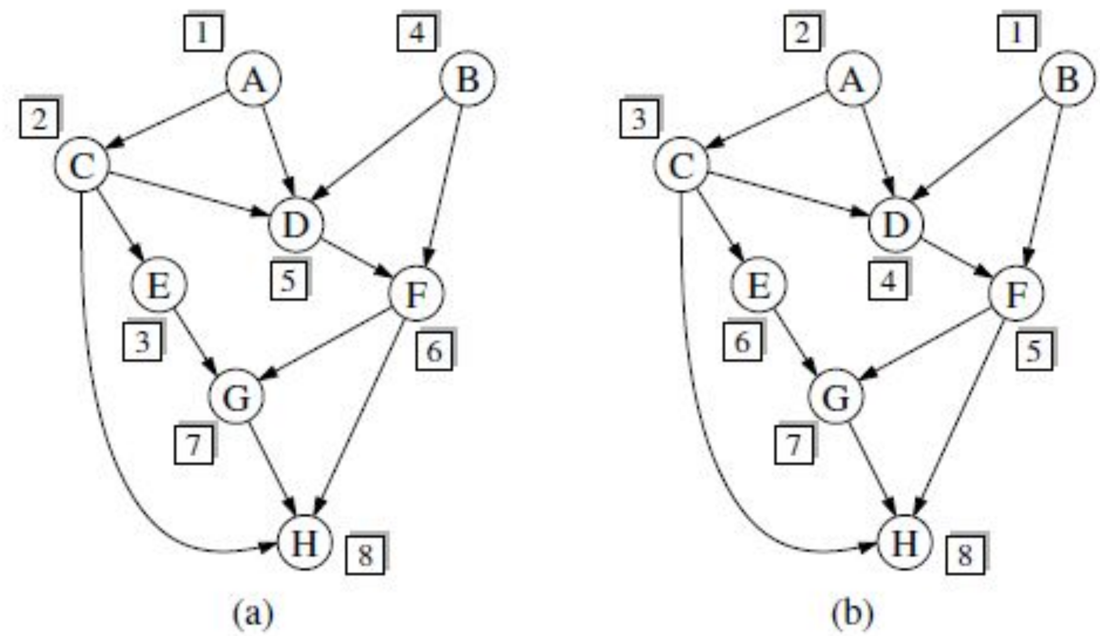
Directed Acyclic Graphs (DAG)

- Directed graph without directed cycles is a directed acyclic graph (DAG).
- Applications of such graphs, for instance, are:
  - Prerequisites between courses of a degree program.
  - Scheduling constraints between the tasks of a project.
    - Task *a* must be completed before task *b* is started.

Topological Ordering

- A topological ordering of graph G is an ordering of the nodes $(v_1, \ldots, v_n)$ such that for every edge (v_i, v_j) of G, the condition $i < j$ must be preserved.
- In other words, if there is a part from v_i to v_j, v_j must be behind v_i in the ordering.

Topological Ordering

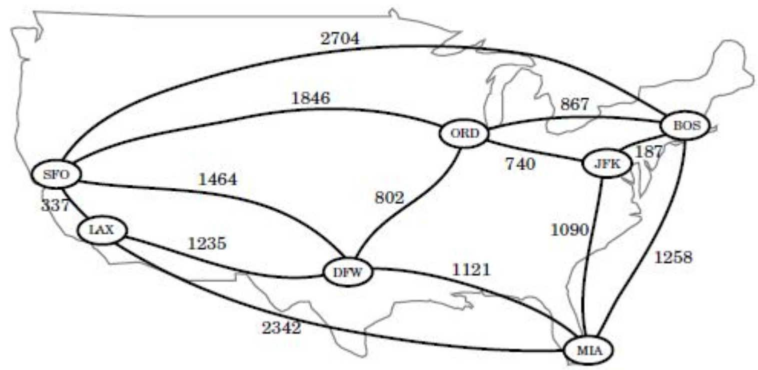


Shortest Paths

- The BFS strategy can be used to find a **shortest path** from some starting node to every other node in a connected graph.
- This approach is suitable in cases where each edge is equal to others.
- However, for other situations, this approach is not efficient.
- For example, a graph representing the road network between cities, and we would like to find the fastest way to travel from A to B.
- It is natural, therefore, to consider graphs whose edges are <u>not weighted equally</u>.

Weighted Graphs

- A **weight graph** is a graph that has a numeric label *w*(*e*) associated with each edge *e*, called the **weight** of edge *e*.
- For *e* = (*u*, *v*), *w*(*u*, *v*) = *w*(*e*).
- Such weights might represent:
  - Costs
  - Lengths
  - Capacities
  - etc.



Defining Shortest Paths in a Weighted Graph

- Let *G* be a weight graph.
- The **length** (or **weight**) of a **path** is the sum of the weights of the edges of *P*.
  - P = $((v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k))$
  - Length of P, denoted w(P) is defined as $$w(P) = \sum_{i=0}^{k-1} w(v_i, v_{i+1}).$$
- The distance from a node *u* to a node *v* in *G*, denoted *d*(*u*, *v*) is the length of a minimum-length path (also called **shortest path**) from *u* to *v*.

Shortest Paths Algorithms

- Shortest path with all equal weights (=1) can be solved with BFS traversal algorithm.
- Distances cannot be arbitrarily low negative numbers.
  - For instance, the weight of edges represent the cost to travel between cities. If someone pay you to go between the cities, the cost would be negative.
  - Edge weights in *G* should be nonnegative (that is, *w*(*e*) >= 0) for each edge.

Dijkstra's Algorithm

- Apply **greedy method** to solve the problem by repeatedly selecting the best choice from among those available in each iteration.
  - Useful for optimising cost function over a collection of objects.
- "Weight" breadth-first search starting at the source node *s*.
- *D*[*v*] keeps the length of the best path so far from the source node *s* to each node *v* in the graph.
  - Initially, *D*[*s*] = 0 and *D*[*v*] = Inf for each *v* != s
- *Q* is a set of all the unvisited nodes, called the unvisited set.
- Array *prev* is used to keep track of the shortest path.

Dijkstra's Algorithm

1. Set the source node as current node.
2. For the current node, consider all of its unvisited neighbors and calculate their distances through the current node.
3. Compare the distances and select the unvisited neighbor node (*v*) with the smallest distance through the current node.
4. Mark the current node as visited and remove it from the unvisited set Q.
    a. A visited node will never be checked again.
5. If the destination node has been marked visited, then stop.
6. If the smallest distance among the nodes in the unvisited set is infinity, then stop.
    a. Occurs when there is no connection between the source node and remaining unvisited nodes
7. Otherwise, set the unvisited node with the smallest distance as the new "current node" and repeat step 3.

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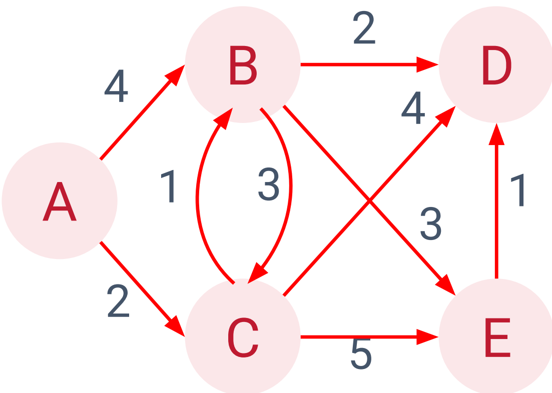
Dijkstra's Algorithm

```
function Dijkstra(Graph, source):

    create vertex set Q

    for each vertex v in Graph:
        dist[v] ← INFINITY
        prev[v] ← UNDEFINED
        add v to Q
    dist[source] ← 0

    while Q is not empty:
        u ← vertex in Q with min dist[u]

        remove u from Q

        for each neighbor v of u:           // only v
that are still in Q
            alt ← dist[u] + length(u, v)
            if alt < dist[v]:
                dist[v] ← alt
                prev[v] ← u

    return dist[], prev[]
```

Dijkstra’s Algorithm

Round #0

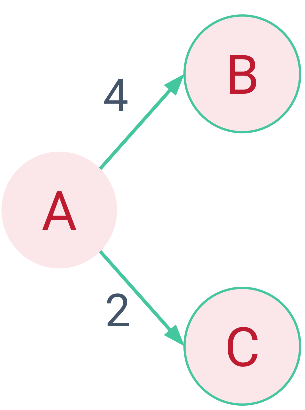


Node	Cumulative weight
A	-
B	-
C	-
D	-
E	-

Q = {‘A’, ‘B’, ‘C’, ‘D’, ‘E’}
D[‘A’] = 0, D[‘B’] = D[‘C’] = D[‘D’] = D[‘E’] = Inf
prev[‘A’] = prev[‘B’] = prev[‘C’] = prev[‘D’] = prev[‘E’] = None

Dijkstra’s Algorithm

Round #1

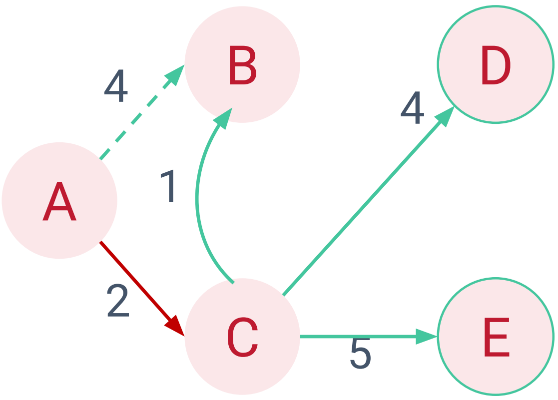


Node	Cumulative weight
A	-
B	4
C	2
D	-
E	-

Q = {“B’, ‘C’, ‘D’, ‘E’}
D[‘C’] = 2
prev[‘C’] = ‘A’

Dijkstra’s Algorithm

Round #2

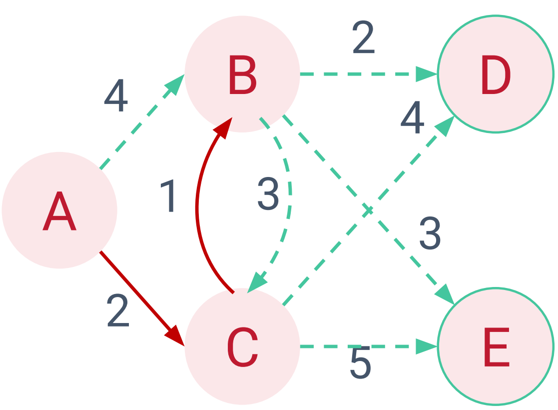


Node	Cumulative weight
A	-
B	4 or 2 + 1 = 3
C	2
D	2 + 4 = 6
E	2 + 5 = 7

Q = { ‘B’, ‘D’, ‘E’}
D[‘C’] = 2 D[‘B’] = 3
prev[‘C’] = ‘A’ prev[‘B’] = ‘C’

Dijkstra’s Algorithm

Round #3

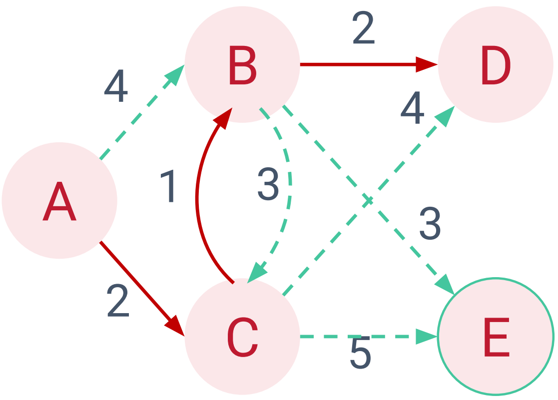


Node	Cumulative weight
A	-
B	3
C	2
D	(ACBD) 3 + 2 = 5 or (ACD) 2 + 4 = 6
E	(ACE) 7 or (ACBE) 3 + 3 = 6

Q = { ‘D’, ‘E’}
D[‘C’] = 2 D[‘B’] = 3 D[‘D’] = 5
prev[‘C’] = ‘A’ prev[‘B’] = ‘C’ prev[‘D’] = ‘B’

Dijkstra's Algorithm

Round #4



Node	Cumulative weight
A	-
B	3
C	2
D	5
E	(ACE) 7 or (ACBE) 3 + 3 = 6

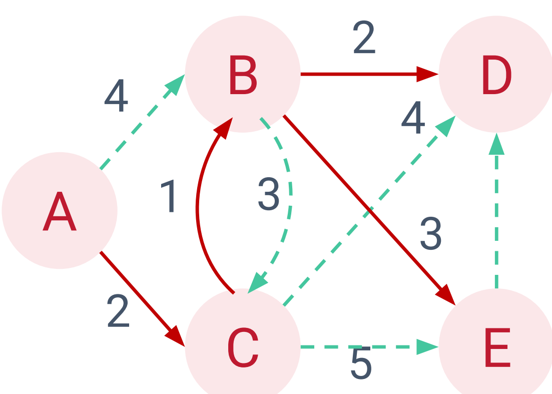
Q = { 'E'}

D['C'] = 2 D['B'] = 3 D['D'] = 5 D['E'] = 6

prev['C'] = 'A' prev['B'] = 'C' prev['D'] = 'B' prev['E'] = 'B'

Dijkstra's Algorithm

Round #4



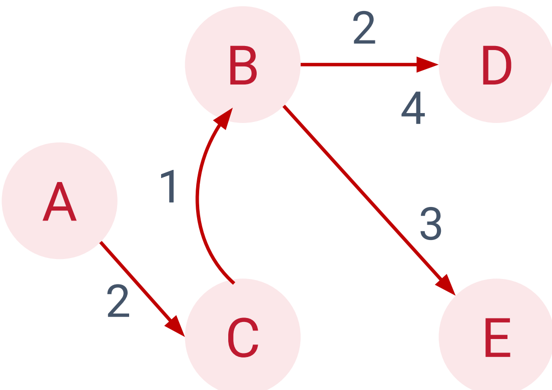
Node	Cumulative weight
A	-
B	3
C	2
D	5
E	6

Q = {}

D['C'] = 2 D['B'] = 3 D['D'] = 5 D['E'] = 6

prev['C'] = 'A' prev['B'] = 'C' prev['D'] = 'B' prev['E'] = 'B'

Dijkstra’s Algorithm



Round #4

Node	Cumulative weight
A	-
B	3
C	2
D	5
E	6

Shortest Path

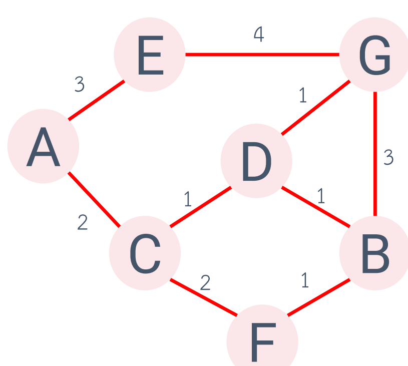
Route	Cumulative weight
A	0
A > C > B	3
A > C	2
A > C > B > D	5
A > C > B > E	6

Q = {}

D[‘C’] = 2 D[‘B’] = 3 D[‘D’] = 5 D[‘E’] = 6

prev[‘C’] = ‘A’ prev[‘B’] = ‘C’ prev[‘D’] = ‘B’ prev[‘E’] = ‘B’

Dijkstra’s Algorithm Exercise



Find the shortest path for each node from node A

Find the distance from A to F