High-frequency trading in a limit order book

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1 Trading Order Book Dynamics

1.1 Basics of Trading Order Book (TOB)

- Buyers/Sellers express their intent to trade by submitting bids/asks
- These are Limit Orders (LO) with a price P and size N
- Buy LO (P, N) states willingness to buy N shares at a price $\leq P$
- Sell LO (P, N) states willingness to sell N shares at a price $\geq P$
- Trading Order Book aggregates order sizes for each unique price
- So we can represent with two sorted lists of (Price, Size) pairs

$$\begin{array}{l} \text{Bids: } \left[\left(P_i^{(b)}, N_i^{(b)}\right) \mid 1 \leq i \leq m\right], P_i^{(b)} > P_j^{(b)} \text{ for } i < j \\ \text{Asks: } \left[\left(P_i^{(a)}, N_i^{(a)}\right) \mid 1 \leq i \leq n\right], P_i^{(a)} < P_j^{(a)} \text{ for } i < j \end{array}$$

- We call $P_1^{(b)}$ as simply Bid, $P_1^{(a)}$ as Ask, $\frac{P_1^{(a)}+P_1^{(b)}}{2}$ as Mid
- We call $P_1^{(a)} - P_1^{(b)}$ as Spread, $P_n^{(a)} - P_m^{(b)}$ as Market Depth
- A Market Order (MO) states intent to buy/sell N shares at the best possible price(s) available on the TOB at the time of MO submission

Trading Order Book (TOB) Activity

- A new Sell LO (P, N) potentially removes best bid prices on the TOB Removal:

$$\left[\left(P_i^{(b)}, \min\left(N_i^{(b)}, \max\left(0, N - \sum_{j=1}^{i-1} N_j^{(b)}\right)\right)\right) \mid \left(i : P_i^{(b)} \geq P\right)\right]$$

- After this removal, it adds the following to the asks side of the TOB

$$\left(P, \max\left(0, N - \sum_{i:P_i^{(b)} \geq P} N_i^{(b)}\right)\right)$$

- A new Buy MO operates analogously (on the other side of the TOB)
- A Sell Market Order N will remove the best bid prices on the TOB Removal:

$$\left[\left(P_i^{(b)}, \min\left(N_i^{(b)}, \max\left(0, N - \sum_{j=1}^{i-1} N_j^{(b)}\right)\right)\right) \mid 1 \leq i \leq m\right]$$

- A new Buy MO operates analogously (on the other side of the TOB)
- A Sell Market Order N will remove the best bid prices on the TOB Removal:

$$\left[\left(P_i^{(b)}, \min\left(N_i^{(b)}, \max\left(0, N - \sum_{j=1}^{i-1} N_j^{(b)}\right)\right)\right) \mid 1 \leq i \leq m\right]$$

- A Buy Market Order N will remove the best ask prices on the TOB

$$\text{Removal: } \left[\left(P_i^{(a)}, \min\left(N_i^{(a)}, \max\left(0, N - \sum_{i=1}^{i-1} N_j^{(a)}\right)\right)\right) \mid 1 \leq i \leq n\right]$$

2 Notation for Optimal Market-Making Problem

TOB Dynamics and Market-Making

- Modeling TOB Dynamics involves predicting arrival of MOs and LOs
- Market-makers are liquidity providers (providers of Buy and Sell LOs)
- Other market participants are typically diquidity takers (MOs)
- But there are also other market participants that trade with LOs
- Complex interplay between market-makers & other mkt participants
- Hence, TOB Dynamics tend to be quite complex
- We view the TOB from the perspective of a single market-maker who aims to gain with Buy/Sell LOs of appropriate width/size
- By anticipating TOB Dynamics & dynamically adjusting Buy/Sell LOs
- Goal is to maximize Utility of Gains at the end of a suitable horizon
- If Buy/Sell LOs are too narrow, more frequent but small gains
- If Buy/Sell LOs are too wide, less frequent but large gains
- Market-maker also needs to manage potential unfavorable inventory (long or short) buildup and consequent unfavorable liquidation

Notation for Optimal Market-Making Problem

- We simplify the setting for ease of exposition
- Assume finite time steps indexed by $t = 0, 1, \ldots, T$
- Denote $W_t \in \mathbb{R}$ as Market-maker's trading PnL at time t
- Denote $I_t \in \mathbb{Z}$ as Market-maker's inventory of shares at time t $(I_0 = 0)$
- $S_t \in \mathbb{R}^+$ is the TOB Mid Price at time t (assume stochastic process)
- $P_t^{(b)} \in \mathbb{R}^+, N_t^{(b)} \in \mathbb{Z}^+$ are market maker's Bid Price, Bid Size at time t

- $P_t^{(a)} \in \mathbb{R}^+, N_t^{(a)} \in \mathbb{Z}^+$ are market-maker's Ask Price, Ask Size at time t
- Assume market-maker can add or remove bids/asks costlessly
- Denote $\delta_t^{(b)} = S_t - P_t^{(b)}$ as Bid Spread, $\delta_t^{(a)} = P_t^{(a)} - S_t$ as Ask Spread
- Random var $X_t^{(b)} \in \mathbb{Z}_{\geq 0}$ denotes bid-shares "hit" up to time t
- Random var $X_t^{(a)} \in \mathbb{Z}_{\geq 0}$ denotes ask-shares "lifted" up to time t $W_{t+1} = W_t + P_t^{(a)} \cdot \left(X_{t+1}^{(a)} - X_t^{(a)}\right) - P_t^{(b)} \cdot \left(X_{t+1}^{(b)} - X_t^{(b)}\right), I_t = X_t^{(b)} - X_t^{(a)}$
- Goal to maximize $\mathbb{E}\left[U\left(W_T + I_T \cdot S_T\right)\right]$ for appropriate concave $U(\cdot)$

3 Derivation of Avellaneda-Stoikov Analytical Solution

3.1 Avellaneda-Stoikov Continuous Time Formulation

- Adapt our discrete-time notation to their continuous-time setting
- $X_t^{(b)}, X_t^{(a)}$ are Poisson processes with hit/lift-rate means $\lambda_t^{(b)}, \lambda_t^{(a)}$

$$dX_t^{(b)} \sim \text{Poisson}\left(\lambda_t^{(b)} \cdot dt\right), dX_t^{(a)} \sim \text{Poisson}\left(\lambda_t^{(a)} \cdot dt\right)$$

$\lambda_t^{(b)} = f^{(b)}\left(\delta_t^{(b)}\right), \lambda_t^{(a)} = f^{(a)}\left(\delta_t^{(a)}\right)$ for decreasing functions $f^{(b)}, f^{(a)}$

$$dW_t = P_t^{(a)} \cdot dX_t^{(a)} - P_t^{(b)} \cdot dX_t^{(b)}, I_t = X_t^{(b)} - X_t^{(a)} \text{ (note: } I_0 = 0)$$

- Since infinitesimal Poisson random variables $dX_t^{(b)}$ (shares hit in time dt) and $dX_t^{(a)}$ (shares lifted in time dt) are Bernoulli (shares hit/lifted in time dt are 0 or 1), $N_t^{(b)}$ and $N_t^{(a)}$ can be assumed to be 1
- This simplifies the Action at time t to be just the pair: $\left(\delta_t^{(b)}, \delta_t^{(a)}\right)$
- TOB Mid Price Dynamics: $dS_t = \sigma \cdot dz_t$ (scaled brownian motion)
- Utility function $U(x) = -e^{-\gamma x}$ where $\gamma > 0$ is coeff. of risk-aversion

3.2 Hamilton-Jacobi-Bellman (HJB) Equation

- We denote the Optimal Value function as $V^*\left(t, S_t, W_t, l_t\right)$

$$V^*\left(t, S_t, W_t, I_t\right) = \max_{\delta_t^{(b)}, \delta_t^{(a)}} \mathbb{E}\left[-e^{-\gamma \cdot \left(W_T + l_t \cdot S_T\right)}\right]$$

- $V^*\left(t, S_t, W_t, I_t\right)$ satisfies a recursive formulation for $0 \leq t < t_1 < T$:

$$V^*\left(t, S_t, W_t, I_t\right) = \max_{\delta_t^{(b)}, \delta_t^{(a)}} \mathbb{E}\left[V^*\left(t_1, S_{t_1}, W_{t_1}, l_{t_1}\right)\right]$$

- Rewriting in stochastic differential form, we have the HJB Equation

$$\begin{gathered} \max_{\delta_t^{(b)}, \delta_t^{(a)}} \mathbb{E}\left[dV^*\left(t, S_t, W_t, I_t\right)\right] = 0 \text{ for } t < T \\ V^*\left(T, S_T, W_T, I_T\right) = -e^{-\gamma \cdot \left(W_T + I_T \cdot S_T\right)} \end{gathered}$$

3.3 Converting HJB to a Partial Dierential Equation

- Change to $V^*(t, S_t, W_t, l_t)$ is comprised of 3 components:
  - Due to pure movement in time t
  - Due to randomness in TOB Mid-Price S_t
  - Due to randomness in hitting/lifting the Bid/Ask
- With this, we can expand $dV^*(t, S_t, W_t, l_t)$ and rewrite HJB as:

$$\begin{aligned}\max_{\delta_t^{(b)}, \delta_t^{(a)}} \{ & \frac{\partial V^*}{\partial t} dt + \mathbb{E}\left[\sigma \frac{\partial V^*}{\partial S_t} dz_t + \frac{\sigma^2}{2} \frac{\partial^2 V^*}{\partial S_t^2} (dz_t)^2\right] \\ & + \lambda_t^{(b)} \cdot dt \cdot V^*\left(t, S_t, W_t - S_t + \delta_t^{(b)}, I_t + 1\right) \\ & + \lambda_t^{(a)} \cdot dt \cdot V^*\left(t, S_t, W_t + S_t + \delta_t^{(a)}, I_t - 1\right) \\ & + \left(1 - \lambda_t^{(b)} \cdot dt - \lambda_t^{(a)} \cdot dt\right) \cdot V^*(t, S_t, W_t, l_t) \\ & - V^*(t, S_t, W_t, I_t)\} = 0\end{aligned}$$

- We can simplify this equation with a few observations:
  - $\mathbb{E}[dz_t] = 0$
  - $\mathbb{E}\left[(dz_t)^2\right] = dt$
  - Organize the terms involving $\lambda_t^{(b)}$ and $\lambda_t^{(a)}$ better with some algebra
  - Divide throughout by dt

$$\begin{aligned}\max_{\delta_t^{(b)}, \delta_t^{(a)}} \{ & \frac{\partial V^*}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 V^*}{\partial S_t^2} \\ & + \lambda_t^{(b)} \cdot \left(V^*\left(t, S_t, W_t - S_t + \delta_t^{(b)}, I_t + 1\right) - V^*(t, S_t, W_t, I_t)\right) \\ & + \lambda_t^{(a)} \cdot \left(V^*\left(t, S_t, W_t + S_t + \delta_t^{(a)}, I_t - 1\right) - V^*(t, S_t, W_t, I_t)\right)\Big\} = 0\end{aligned}$$

- Next, note that $\lambda_t^{(b)} = f^{(b)}\left(\delta_t^{(b)}\right)$ and $\lambda_t^{(a)} = f^{(a)}\left(\delta_t^{(a)}\right)$, and apply the max only on the relevant terms

$$\begin{aligned}& \frac{\partial V^*}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 V^*}{\partial S_t^2} \\ & + \max_{\delta_t^{(b)}} \left\{f^{(b)}\left(\delta_t^{(b)}\right) \cdot \left(V^*\left(t, S_t, W_t - S_t + \delta_t^{(b)}, l_t + 1\right) - V^*(t, S_t, W_t, I_t)\right)\right\} \\ & + \max_{\delta_t^{(a)}} \left\{f^{(a)}\left(\delta_t^{(a)}\right) \cdot \left(V^*\left(t, S_t, W_t + S_t + \delta_t^{(a)}, I_t - 1\right) - V^*(t, S_t, W_t, I_t)\right)\right\} = 0\end{aligned}$$

- This combines with the boundary condition:

$$V^*(T, S_T, W_T, I_T) = -e^{-\gamma \cdot (W_T + I_T \cdot S_T)}$$

- We make an "educated guess" for the structure of $V^*(t, S_t, W_t, I_t)$

$$V^*(t, S_t, W_t, I_t) = -e^{-\gamma (W_t + \theta(t, S_t, I_t))} \tag{1}$$

and reduce the problem to a PDE in terms of $\theta(t, S_t, l_t)$

- Substituting this into the above PDE for $V^*(t, S_t, W_t, I_t)$ gives:

$$
\begin{aligned}
&\tfrac{\partial \theta}{\partial t} + \tfrac{\sigma^2}{2}\left(\tfrac{\partial^2 \theta}{\partial S_t^2} - \gamma\left(\tfrac{\partial \theta}{\partial S_t}\right)^2\right) \\
&+\max_{\delta_t^{(b)}}\left\{\frac{f^{(b)}\left(\delta_t^{(b)}\right)}{\gamma}\cdot\left(1-e^{-\gamma\left(\delta_t^{(b)}-S_t+\theta(t,S_t,l_t+1)-\theta(t,S_t,I_t)\right)}\right)\right\} \\
&+\max{}_{\delta_t^{(a)})}\ \frac{f^{(a)}\left(\delta_t^{(a)}\right)}{\gamma}\cdot\left(1-e^{-\gamma\left(\delta_t^{(a)}+S_t+\theta(t,S_t,I_t-1)-\theta(t,S_t,I_t)\right)}\right)\Bigg\} = 0
\end{aligned}
$$

- The boundary condition is:

$$\theta(T, S_T, I_T) = I_T \cdot S_T$$

3.4 Indifference Bid/Ask Price

- It turns out that $\theta(t, S_t, I_t+1) - \theta(t, S_t, I_t)$ and $\theta(t, S_t, I_t) - \theta(t, S_t, l_t-1)$ are equal to financially meaningful quantities known as Indifference Bid and Ask Prices
- Indifference Bid Price $Q^{(b)}(t, S_t, I_t)$ is defined as:

$$V^*\left(t, S_t, W_t - Q^{(b)}(t, S_t, I_t), I_t+1\right) = V^*(t, S_t, W_t, l_t) \tag{2}$$

- $Q^{(b)}(t, S_t, l_t)$ is the price to buy a share with guarantee of immediate purchase that results in Optimum Expected Utility being unchanged
- Likewise, Indifference Ask Price $Q^{(a)}(t, S_t, I_t)$ is defined as:

$$V^*\left(t, S_t, W_t + Q^{(a)}(t, S_t, I_t), I_t-1\right) = V^*(t, S_t, W_t, I_t) \tag{3}$$

- $Q^{(a)}(t, S_t, l_t)$ is the price to sell a share with guarantee of immediate sale that results in Optimum Expected Utility being unchanged
- We abbreviate $Q^{(b)}(t, S_t, I_t)$ as $Q_t^{(b)}$ and $Q^{(a)}(t, S_t, I_t)$ as $Q_t^{(a)}$

Indifference Bid/Ask Price in the PDE for θ

- Express $V^*\left(t, S_t, W_t - Q_t^{(b)}, l_t+1\right) = V^*(t, S_t, W_t, I_t)$ in terms of θ :

$$
\begin{gathered}
-e^{-\gamma\left(W_t - Q_t^{(b)} + \theta(t, S_t, l_t+1)\right)} = -e^{-\gamma(W_t + \theta(t, S_t, I_t))} \\
\Rightarrow Q_t^{(b)} = \theta(t, S_t, l_t+1) - \theta(t, S_t, l_t)
\end{gathered}
\tag{4}
$$

- Likewise for $Q_t^{(a)}$, we get:

$$Q_t^{(a)} = \theta(t, S_t, I_t) - \theta(t, S_t, I_t-1) \tag{5}$$

- Using equations (4) and (5), bring $Q_t^{(b)}$ and $Q_t^{(a)}$ in the PDE for θ

$$
\begin{gathered}
\tfrac{\partial \theta}{\partial t} + \tfrac{\sigma^2}{2}\left(\tfrac{\partial^2 \theta}{\partial S_t^2} - \gamma\left(\tfrac{\partial \theta}{\partial S_t}\right)^2\right) + \max{}_{\delta_t^{(b)}}\, g\left(\delta_t^{(b)}\right) + \max{}_{\delta_t^{(a)}}\, h\left(\delta_t^{(b)}\right) = 0 \\
\text{where } g\left(\delta_t^{(b)}\right) = \frac{f^{(b)}\left(\delta_t^{(b)}\right)}{\gamma}\cdot\left(1-e^{-\gamma\left(\delta_t^{(b)}-S_t+Q_t^{(b)}\right)}\right) \\
\text{and } h\left(\delta_t^{(a)}\right) = \frac{f^{(a)}\left(\delta_t^{(a)}\right)}{\gamma}\cdot\left(1-e^{-\gamma\left(\delta_t^{(a)}+S_t-Q_t^{(a)}\right)}\right)
\end{gathered}
$$

3.5 Optimal Bid Spread and Optimal Ask Spread

- To maximize $g\left(\delta_t^{(b)}\right)$, differentiate g with respect to $\delta_t^{(b)}$ and set to 0

$$
\begin{aligned}
e^{-\gamma\left(\delta_t^{(b)^*}-S_t+Q_t^{(b)}\right)} \cdot \left(\gamma \cdot f^{(b)}\left(\delta_t^{(b)^*}\right) - \frac{\partial f^{(b)}}{\partial \delta_t^{(b)}}\left(\delta_t^{(b)^*}\right)\right) + \frac{\partial f^{(b)}}{\partial \delta_t^{(b)}}\left(\delta_t^{(b)^*}\right) = 0 \\
\Rightarrow \delta_t^{(b)^*} = S_t - Q_t^{(b)} + \frac{1}{\gamma} \cdot \ln\left(1 - \gamma \cdot \frac{f^{(b)}\left(\delta_t^{(b)^*}\right)}{\frac{\partial f^{(b)}}{\partial \delta_t^{(b)}}\left(\delta_t^{(b)^*}\right)}\right)
\end{aligned} \tag{6}
$$

- To maximize $g\left(\delta_t^{(a)}\right)$, differentiate g with respect to $\delta_t^{(a)}$ and set to 0

$$
\begin{aligned}
e^{-\gamma\left(\delta_t^{(a)^*}+S_t-Q_t^{(a)}\right)} \cdot \left(\gamma \cdot f^{(a)}\left(\delta_t^{(a)^*}\right) - \frac{\partial f^{(a)}}{\partial \delta_t^{(a)}}\left(\delta_t^{(a)^*}\right)\right) + \frac{\partial f^{(a)}}{\partial \delta_t^{(a)}}\left(\delta_t^{(a)^*}\right) = 0 \\
\Rightarrow \delta_t^{(a)^*} = Q_t^{(a)} - S_t + \frac{1}{\gamma} \cdot \ln\left(1 - \gamma \cdot \frac{f^{(a)}\left(\delta_t^{(a)^*}\right)}{\frac{\partial f^{(a)}}{\partial \delta_t^{(a)}}\left(\delta_t^{(a)^*}\right)}\right)
\end{aligned} \tag{7}
$$

- (6) and (7) are implicit equations for $\delta_t^{(b)^*}$ and $\delta_t^{(a)^*}$ respectively

Solving for θ and for Optimal Bid/Ask Spreads

- Let us write the PDE in terms of the Optimal Bid and Ask Spreads

$$
\begin{aligned}
&\frac{\partial \theta}{\partial t} + \frac{\sigma^2}{2}\left(\frac{\partial^2 \theta}{\partial S_t^2} - \gamma\left(\frac{\partial \theta}{\partial S_t}\right)^2\right) \\
&+\frac{f^{(b)}\left(\delta_t^{(b)^*}\right)}{\gamma} \cdot \left(1 - e^{-\gamma\left(\delta_t^{(b)^*}-S_t+\theta(t,S_t,I_t+1)-\theta(t,S_t,l_t)\right)}\right) \\
&+\frac{f^{(a)}\left(\delta_t^{(a)^*}\right)}{\gamma} \cdot \left(1 - e^{-\gamma\left(\delta_t^{(a)^*}+S_t+\theta(t,S_t,I_t-1)-\theta(t,S_t,I_t)\right)}\right) = 0
\end{aligned} \tag{8}
$$

- with boundary condition $\theta\left(T, S_T, I_T\right) = I_T \cdot S_T$
- First we solve PDE (8) for θ in terms of $\delta_t^{(b)^*}$ and $\delta_t^{(a)^*}$
- In general, this would be a numerical PDE solution
- Using (4) and (5), we have $Q_t^{(b)}$ and $Q_t^{(a)}$ in terms of $\delta_t^{(b)^*}$ and $\delta_t^{(a)^*}$
- Substitute above-obtained $Q_t^{(b)}$ and $Q_t^{(a)}$ in equations (6) and (7)
- Solve implicit equations for $\delta_t^{(b)^*}$ and $\delta_t^{(a)^*}$ (in general, numerically)

Building Intuition

- Define Indifference Mid Price $Q_t^{(m)} = \frac{Q_t^{(b)}+Q_t^{(a)}}{2}$
- To develop intuition for Indifference Prices, consider a simple case where the market-maker doesn't supply any bids or asks

$$
V^*\left(t, S_t, W_t, l_t\right) = \mathbb{E}\left[-e^{-\gamma\left(W_t+l_t \cdot S_T\right)} \mid\right]
$$

- Combining this with the diffusion $dS_t = \sigma \cdot dz_t$, we get:

$$V^*(t, S_t, W_t, I_t) = -e^{-\gamma\left(W_t + l_t \cdot S_t - \frac{\gamma \cdot l_t^2 \cdot \sigma^2 (T-t)}{2}\right)}$$

- Combining this with equations (2) and (3), we get:

$$Q_t^{(b)} = S_t - (2I_t + 1)\frac{\gamma\sigma^2(T-t)}{2}, Q_t^{(a)} = S_t - (2l_t - 1)\frac{\gamma\sigma^2(T-t)}{2}$$
$$Q_t^{(m)} = S_t - I_t\gamma\sigma^2(T-t), Q_t^{(a)} - Q_t^{(b)} = \gamma\sigma^2(T-t)$$

- These results for the simple case of no-market-making serve as approximations for our problem of optimal market-making
- Think of $Q_t^{(m)}$ as inventory-risk-adjusted mid-price (adjustment to S_t)
- If market-maker is long inventory $(I_t > 0)$, $Q_t^{(m)} < S_t$ indicating inclination to sell than buy, and if market-maker is short inventory, $Q_t^{(m)} > S_t$ indicating inclination to buy than sell
- Armed with this intuition, we come back to optimal market-making, observing from eqns (6) and (7) : $P_t^{(b)^*} < Q_t^{(b)} < Q_t^{(m)} < Q_t^{(a)} < P_t^{(a)^*}$
- Think of $\left[P_t^{(b)^*}, P_t^{(a)^*}\right]$ as "centered" at $Q_t^{(m)}$ (rather than at S_t), i.e., $\left[P_t^{(b)^*}, P_t^{(a)^*}\right]$ will (together) move up/down in tandem with $Q_t^{(m)}$ moving up/down (as a function of inventory position I_t)

$$\begin{aligned} Q_t^{(m)} - P_t^{(b)^*} &= \frac{Q_t^{(a)} - Q_t^{(b)}}{2} + \frac{1}{\gamma} \cdot \ln\left(1 - \gamma \cdot \frac{f^{(b)}\left(\delta_t^{(b)^*}\right)}{\frac{\partial f^{(b)}}{\partial \delta_t^{(b)}}\left(\delta_t^{(b)^*}\right)}\right) \\ P_t^{(a)^*} - Q_t^{(m)} &= \frac{Q_t^{(a)} - Q_t^{(b)}}{2} + \frac{1}{\gamma} \cdot \ln\left(1 - \gamma \cdot \frac{f^{(a)}\left(\delta_t^{(a)^*}\right)}{\frac{\partial f(a)}{\partial \delta_t^{(a)}}\left(\delta_t^{(a)^*}\right)}\right) \end{aligned} \tag{9,10}$$

3.6 Simple Functional Form for Hitting/Lifting Rate Means

- The PDE for θ and the implicit equations for $\delta_t^{(b)^*}, \delta_t^{(a)^*}$ are messy
- We make some assumptions, simplify, derive analytical approximations
- First we assume a fairly standard functional form for $f^{(b)}$ and $f^{(a)}$

$$f^{(b)}(\delta) = f^{(a)}(\delta) = c \cdot e^{-k \cdot \delta}$$

- This reduces equations (6) and (7) to:

$$\begin{aligned} \delta_t^{(b)^*} &= S_t - Q_t^{(b)} + \frac{1}{\gamma}\ln\left(1 + \frac{\gamma}{k}\right) \\ \delta_t^{(a)^*} &= Q_t^{(a)} - S_t + \frac{1}{\gamma}\ln\left(1 + \frac{\gamma}{k}\right) \end{aligned} \tag{11,12}$$

$$\Rightarrow P_t^{(b)^*} and P_t^{(a)^*} \text{ are equidistant from } Q_t^{(m)}$$

- Substituting these simplified $\delta_t^{(b)^*}, \delta_t^{(a)^*}$ in (8) reduces the PDE to:

$$\frac{\partial \theta}{\partial t} + \frac{\sigma^2}{2}\left(\frac{\partial^2 \theta}{\partial S_t^2} - \gamma\left(\frac{\partial \theta}{\partial S_t}\right)^2\right) + \frac{c}{k + \gamma}\left(e^{-k \cdot \delta_t^{(b)^*}} + e^{-k \cdot \delta_t^{(a)^*}}\right) = 0 \tag{13}$$

- with boundary condition $\theta(T, S_T, I_T) = I_T \cdot S_T$

3.7 Simplifying the PDE with Approximations

- Note that this PDE (13) involves $\delta_t^{(b)^*}$ and $\delta_t^{(a)^*}$
- However, equations $(11), (12), (4), (5)$ enable expressing $\delta_t^{(b)^*}$ and $\delta_t^{(a)^*}$ in terms of $\theta\left(t, S_t, I_t-1\right), \theta\left(t, S_t, I_t\right), \theta\left(t, S_t, I_t+1\right)$
- This would give us a PDE just in terms of θ
- Solving that PDE for θ would not only give us $V^*\left(t, S_t, W_t, l_t\right)$ but also $\delta_t^{(b)^*}$ and $\delta_t^{(a)^*}$ (using equations $(11), (12), (4), (5)$)
- To solve the PDE, we need to make a couple of approximations
- First we make a linear approx for $e^{-k \cdot \delta_t^{(b)^*}}$ and $e^{-k \cdot \delta_t^{(a)^*}}$ in PDE (13):

$$\frac{\partial \theta}{\partial t}+\frac{\sigma^2}{2}\left(\frac{\partial^2 \theta}{\partial S_t^2}-\gamma\left(\frac{\partial \theta}{\partial S_t}\right)^2\right)+\frac{c}{k+\gamma}\left(1-k \cdot \delta_t^{(b)^*}+1-k \cdot \delta_t^{(a)^*}\right)=0 \tag{14}$$

- Equations $(11), (12), (4), (5)$ tell us that:

$$\delta_t^{(b)^*}+\delta_t^{(a)^*}=\frac{2}{\gamma} \ln \left(1+\frac{\gamma}{k}\right)+2 \theta\left(t, S_t, I_t\right)-\theta\left(t, S_t, I_t+1\right)-\theta\left(t, S_t, I_t-1\right)$$

3.8 Asymptotic Expansion of θ in I_t

- With this expression for $\delta_t^{(b)^*}+\delta_t^{(a)^*}$, PDE (14) takes the form:

$$\begin{aligned} \frac{\partial \theta}{\partial t}+\frac{\sigma^2}{2}\left(\frac{\partial^2 \theta}{\partial S_t^2}-\gamma\left(\frac{\partial \theta}{\partial S_t}\right)^2\right)+\frac{c}{k+\gamma}&\left(2-\frac{2 k}{\gamma} \ln \left(1+\frac{\gamma}{k}\right)\right. \\ &\left.-k\left(2 \theta\left(t, S_t, I_t\right)-\theta\left(t, S_t, l_t+1\right)-\theta\left(t, S_t, I_t-1\right)\right)\right)=0 \end{aligned} \tag{15}$$

- To solve PDE (15), we consider this asymptotic expansion of θ in I_t :

$$\theta\left(t, S_t, I_t\right)=\sum_{n=0}^{\infty} \frac{l_t^n}{n!} \cdot \theta^{(n)}\left(t, S_t\right)$$

- So we need to determine the functions $\theta^{(n)}\left(t, S_t\right)$ for all $n=0,1,2, \ldots$
- For tractability, we approximate this expansion to the first 3 terms:

$$\theta\left(t, S_t, I_t\right) \approx \theta^{(0)}\left(t, S_t\right)+I_t \cdot \theta^{(1)}\left(t, S_t\right)+\frac{I_t^2}{2} \cdot \theta^{(2)}\left(t, S_t\right)$$

3.9 Approximation of θ in I_t

- We note that the Optimal Value Function V^* can depend on S_t only through the current Value of the Inventory (i.e., through $I_t \cdot S_t$), i.e., it cannot depend on S_t in any other way
- This means $V^*\left(t, S_t, W_t, 0\right)=-e^{-\gamma\left(W_t+\theta^{(0)}\left(t, S_t\right)\right)}$ is independent of S_t
- This means $\theta^{(0)}\left(t, S_t\right)$ is independent of S_t
- So, we can write it as simply $\theta^{(0)}(t)$, meaning $\frac{\partial \theta^{(0)}}{\partial S_t}$ and $\frac{\partial^2 \theta^{(0)}}{\partial S_t^2}$ are 0
- Therefore, we can write the approximate expansion for $\theta\left(t, S_t, I_t\right)$ as:

$$\theta\left(t, S_t, I_t\right)=\theta^{(0)}(t)+I_t \cdot \theta^{(1)}\left(t, S_t\right)+\frac{I_t^2}{2} \cdot \theta^{(2)}\left(t, S_t\right) \tag{16}$$

3.10 Solving the PDE

- Substitute this approximation (16) for $\theta\left(t, S_t, I_t\right)$ in PDE (15)

$$\frac{\partial \theta^{(0)}}{\partial t}+I_t \frac{\partial \theta^{(1)}}{\partial t}+\frac{l_t^2}{2} \frac{\partial \theta^{(2)}}{\partial t}+\frac{\sigma^2}{2}\left(I_t \frac{\partial^2 \theta^{(1)}}{\partial S_t^2}+\frac{l_t^2}{2} \frac{\partial^2 \theta^{(2)}}{\partial S_t^2}\right)$$
$$-\frac{\gamma \sigma^2}{2}\left(I_t \frac{\partial \theta^{(1)}}{\partial S_t}+\frac{l_t^2}{2} \frac{\partial \theta^{(2)}}{\partial S_t}\right)^2+\frac{c}{k+\gamma}\left(2-\frac{2 k}{\gamma} \ln \left(1+\frac{\gamma}{k}\right)+k \cdot \theta^{(2)}\right)=0$$

- with boundary condition:

$$\theta^{(0)}(T)+I_T \cdot \theta^{(1)}\left(T, S_T\right)+\frac{I_T^2}{2} \cdot \theta^{(2)}\left(T, S_T\right)=I_T \cdot S_T \tag{17}$$

- We will separately collect terms involving specific powers of I_t, each yielding a separate PDE:
  - Terms devoid of I_t (i.e., I_t^0)
  - Terms involving I_t (i.e., I_t^1)
  - Terms involving I_t^2
- We start by collecting terms involving I_t

$$\frac{\partial \theta^{(1)}}{\partial t}+\frac{\sigma^2}{2} \cdot \frac{\partial^2 \theta^{(1)}}{\partial S_t^2}=0 \text{ with boundary condition } \theta^{(1)}\left(T, S_T\right)=S_T$$

- The solution to this PDE is:

$$\theta^{(1)}\left(t, S_t\right)=S_t \tag{18}$$

- Next, we collect terms involving l_t^2

$$\frac{\partial \theta^{(2)}}{\partial t}+\frac{\sigma^2}{2} \cdot \frac{\partial^2 \theta^{(2)}}{\partial S_t^2}-\gamma \sigma^2 \cdot\left(\frac{\partial \theta^{(1)}}{\partial S_t}\right)^2=0 \text{ with boundary } \theta^{(2)}\left(T, S_T\right)=0$$

- Noting that $\theta^{(1)}\left(t, S_t\right)=S_t$, we solve this PDE as:

$$\theta^{(2)}\left(t, S_t\right)=-\gamma \sigma^2(T-t) \tag{19}$$

- Finally, we collect the terms devoid of I_t

$$\frac{\partial \theta^{(0)}}{\partial t}+\frac{c}{k+\gamma}\left(2-\frac{2 k}{\gamma} \ln \left(1+\frac{\gamma}{k}\right)+k \cdot \theta^{(2)}\right)=0 \text{ with boundary } \theta^{(0)}(T)=0$$

- Noting that $\theta^{(2)}\left(t, S_t\right)=-\gamma \sigma^2(T-t)$, we solve as:

$$\theta^{(0)}(t)=\frac{c}{k+\gamma}\left(\left(2-\frac{2 k}{\gamma} \ln \left(1+\frac{\gamma}{k}\right)\right)(T-t)-\frac{k \gamma \sigma^2}{2}(T-t)^2\right) \tag{20}$$

- This completes the PDE solution for $\theta\left(t, S_t, l_t\right)$ and hence, for $V^*\left(t, S_t, W_t, l_t\right)$
- Lastly, we derive formulas for $Q_t^{(b)}, Q_t^{(a)}, Q_t^{(m)}, \delta_t^{(b)^*}, \delta_t^{(a)^*}$

3.11 Formulas for Prices and Spreads

- Using equations (4) and (5), we get:

$$\begin{aligned} Q_t^{(b)} &= \theta^{(1)}(t, S_t) + (2I_t + 1) \cdot \theta^{(2)}(t, S_t) = S_t - (2I_t + 1)\tfrac{\gamma\sigma^2(T-t)}{2} \\ Q_t^{(a)} &= \theta^{(1)}(t, S_t) + (2I_t - 1) \cdot \theta^{(2)}(t, S_t) = S_t - (2I_t - 1)\tfrac{\gamma\sigma^2(T-t)}{2} \end{aligned} \tag{21,22}$$

- Using equations (11) and (12), we get:

$$\delta_t^{(b)^*} = \frac{(2I_t + 1)\,\gamma\sigma^2(T-t)}{2} + \frac{1}{\gamma}\ln\left(1 + \frac{\gamma}{k}\right) \tag{23}$$

$$\delta_t^{(a)^*} = \frac{(1 - 2I_t)\,\gamma\sigma^2(T-t)}{2} + \frac{1}{\gamma}\ln\left(1 + \frac{\gamma}{k}\right) \tag{24}$$

- Optimal Bid-Ask Spread

$$\delta_t^{(b)^*} + \delta_t^{(a)^*} = \gamma\sigma^2(T-t) + \frac{2}{\gamma}\ln\left(1 + \frac{\gamma}{k}\right) \tag{25}$$

$$\text{Optimal "Mid" } Q_t^{(m)} = \frac{Q_t^{(b)} + Q_t^{(a)}}{2} = \frac{P_t^{(b)^*} + P_t^{(a)^*}}{2} = S_t - I_t\gamma\sigma^2(T-t) \tag{26}$$

4 Summary

- Think of $Q_t^{(m)}$ as inventory-risk-adjusted mid-price (adjustment to S_t)
- If market-maker is long inventory $(I_t > 0)$, $Q_t^{(m)} < S_t$ indicating inclination to sell than buy, and if market-maker is short inventory, $Q_t^{(m)} > S_t$ indicating inclination to buy than sell
- Think of $\left[P_t^{(b)^*}, P_t^{(a)^*}\right]$ as "centered" at $Q_t^{(m)}$ (rather than at S_t), i.e., $\left[P_t^{(b)^*}, P_t^{(a)^*}\right]$ will (together) move up/down in tandem with $Q_t^{(m)}$ moving up/down (as a function of inventory position I_t)
- Note from equation (25) that the Optimal Bid-Ask Spread $P_t^{(a)^*} - P_t^{(b)^*}$ is independent of inventory I_t
- Useful view: $P_t^{(b)^*} < Q_t^{(b)} < Q_t^{(m)} < Q_t^{(a)} < P_t^{(a)^*}$, with these spreads:
  - Outer Spreads $P_t^{(a)^*} - Q_t^{(a)} = Q_t^{(b)} - P_t^{(b)^*} = \frac{1}{\gamma}\ln\left(1 + \frac{\gamma}{k}\right)$
  - Inner Spreads $Q_t^{(a)} - Q_t^{(m)} = Q_t^{(m)} - Q_t^{(b)} = \frac{\gamma\sigma^2(T-t)}{2}$