

# High-frequency trading in a limit order book

MARCO AVELLANEDA and SASHA STOIKOV\*

## 1 Trading Order Book Dynamics

### 1.1 Basics of Trading Order Book (TOB)

- Buyers/Sellers express their intent to trade by submitting bids/asks
- These are Limit Orders (LO) with a price  $P$  and size  $N$
- Buy LO  $(P, N)$  states willingness to buy  $N$  shares at a price  $\leq P$
- Sell LO  $(P, N)$  states willingness to sell  $N$  shares at a price  $\geq P$
- Trading Order Book aggregates order sizes for each unique price
- So we can represent with two sorted lists of (Price, Size) pairs

$$\begin{aligned} \text{Bids: } & \left[ \left( P_i^{(b)}, N_i^{(b)} \right) \mid 1 \leq i \leq m \right], P_i^{(b)} > P_j^{(b)} \text{ for } i < j \\ \text{Asks: } & \left[ \left( P_i^{(a)}, N_i^{(a)} \right) \mid 1 \leq i \leq n \right], P_i^{(a)} < P_j^{(a)} \text{ for } i < j \end{aligned}$$

- We call  $P_1^{(b)}$  as simply Bid,  $P_1^{(a)}$  as Ask,  $\frac{P_1^{(a)} + P_1^{(b)}}{2}$  as Mid
- We call  $P_1^{(a)} - P_1^{(b)}$  as Spread,  $P_n^{(a)} - P_m^{(b)}$  as Market Depth
- A Market Order (MO) states intent to buy/sell  $N$  shares at the best possible price(s) available on the TOB at the time of MO submission

#### Trading Order Book (TOB) Activity

- A new Sell LO  $(P, N)$  potentially removes best bid prices on the TOB Removal:

$$\left[ \left( P_i^{(b)}, \min \left( N_i^{(b)}, \max \left( 0, N - \sum_{j=1}^{i-1} N_j^{(b)} \right) \right) \right) \mid (i : P_i^{(b)} \geq P) \right]$$

- After this removal, it adds the following to the asks side of the TOB

$$\left( P, \max \left( 0, N - \sum_{i: P_i^{(b)} \geq P} N_i^{(b)} \right) \right)$$

- A new Buy MO operates analogously (on the other side of the TOB)
- A Sell Market Order  $N$  will remove the best bid prices on the TOB Removal:

$$\left[ \left( P_i^{(b)}, \min \left( N_i^{(b)}, \max \left( 0, N - \sum_{j=1}^{i-1} N_j^{(b)} \right) \right) \right) \mid 1 \leq i \leq m \right]$$

- A new Buy MO operates analogously (on the other side of the TOB)
- A Sell Market Order  $N$  will remove the best bid prices on the TOB Removal:

$$\left[ \left( P_i^{(b)}, \min \left( N_i^{(b)}, \max \left( 0, N - \sum_{j=1}^{i-1} N_j^{(b)} \right) \right) \right) \mid 1 \leq i \leq m \right]$$

- A Buy Market Order  $N$  will remove the best ask prices on the TOB

$$\text{Removal: } \left[ \left( P_i^{(a)}, \min \left( N_i^{(a)}, \max \left( 0, N - \sum_{j=1}^{i-1} N_j^{(a)} \right) \right) \right) \mid 1 \leq i \leq n \right]$$

## 2 Notation for Optimal Market-Making Problem

### TOB Dynamics and Market-Making

- Modeling TOB Dynamics involves predicting arrival of MOs and LOs
- Market-makers are liquidity providers (providers of Buy and Sell LOs)
- Other market participants are typically liquidity takers (MOs)
- But there are also other market participants that trade with LOs
- Complex interplay between market-makers & other mkt participants
- Hence, TOB Dynamics tend to be quite complex
- We view the TOB from the perspective of a single market-maker who aims to gain with Buy/Sell LOs of appropriate width/size
- By anticipating TOB Dynamics & dynamically adjusting Buy/Sell LOs
- Goal is to maximize Utility of Gains at the end of a suitable horizon
- If Buy/Sell LOs are too narrow, more frequent but small gains
- If Buy/Sell LOs are too wide, less frequent but large gains
- Market-maker also needs to manage potential unfavorable inventory (long or short) buildup and consequent unfavorable liquidation

### Notation for Optimal Market-Making Problem

- We simplify the setting for ease of exposition
- Assume finite time steps indexed by  $t = 0, 1, \dots, T$
- Denote  $W_t \in \mathbb{R}$  as Market-maker's trading PnL at time  $t$
- Denote  $I_t \in \mathbb{Z}$  as Market-maker's inventory of shares at time  $t$  ( $I_0 = 0$ )
- $S_t \in \mathbb{R}^+$  is the TOB Mid Price at time  $t$  (assume stochastic process)
- $P_t^{(b)} \in \mathbb{R}^+, N_t^{(b)} \in \mathbb{Z}^+$  are market maker's Bid Price, Bid Size at time  $t$

- $P_t^{(a)} \in \mathbb{R}^+, N_t^{(a)} \in \mathbb{Z}^+$  are market-maker's Ask Price, Ask Size at time  $t$
- Assume market-maker can add or remove bids/asks costlessly
- Denote  $\delta_t^{(b)} = S_t - P_t^{(b)}$  as Bid Spread,  $\delta_t^{(a)} = P_t^{(a)} - S_t$  as Ask Spread
- Random var  $X_t^{(b)} \in \mathbb{Z}_{\geq 0}$  denotes bid-shares "hit" up to time  $t$
- Random var  $X_t^{(a)} \in \mathbb{Z}_{\geq 0}$  denotes ask-shares "lifted" up to time  $t$   $W_{t+1} = W_t + P_t^{(a)} \cdot (X_{t+1}^{(a)} - X_t^{(a)}) - P_t^{(b)} \cdot (X_{t+1}^{(b)} - X_t^{(b)})$ ,  $I_t = X_t^{(b)} - X_t^{(a)}$
- Goal to maximize  $\mathbb{E}[U(W_T + I_T \cdot S_T)]$  for appropriate concave  $U(\cdot)$

### 3 Derivation of Avellaneda-Stoikov Analytical Solution

#### 3.1 Avellaneda-Stoikov Continuous Time Formulation

- Adapt our discrete-time notation to their continuous-time setting
- $X_t^{(b)}, X_t^{(a)}$  are Poisson processes with hit/lift-rate means  $\lambda_t^{(b)}, \lambda_t^{(a)}$

$$dX_t^{(b)} \sim \text{Poisson}(\lambda_t^{(b)} \cdot dt), dX_t^{(a)} \sim \text{Poisson}(\lambda_t^{(a)} \cdot dt)$$

$$\lambda_t^{(b)} = f^{(b)}(\delta_t^{(b)}), \lambda_t^{(a)} = f^{(a)}(\delta_t^{(a)}) \text{ for decreasing functions } f^{(b)}, f^{(a)}$$

$$dW_t = P_t^{(a)} \cdot dX_t^{(a)} - P_t^{(b)} \cdot dX_t^{(b)}, I_t = X_t^{(b)} - X_t^{(a)} \text{ (note: } I_0 = 0)$$

- Since infinitesimal Poisson random variables  $dX_t^{(b)}$  (shares hit in time  $dt$ ) and  $dX_t^{(a)}$  (shares lifted in time  $dt$ ) are Bernoulli (shares hit/lifted in time  $dt$  are 0 or 1),  $N_t^{(b)}$  and  $N_t^{(a)}$  can be assumed to be 1
- This simplifies the Action at time  $t$  to be just the pair:  $(\delta_t^{(b)}, \delta_t^{(a)})$
- TOB Mid Price Dynamics:  $dS_t = \sigma \cdot dz_t$  (scaled brownian motion)
- Utility function  $U(x) = -e^{-\gamma x}$  where  $\gamma > 0$  is coeff. of risk-aversion

#### 3.2 Hamilton-Jacobi-Bellman (HJB) Equation

- We denote the Optimal Value function as  $V^*(t, S_t, W_t, I_t)$

$$V^*(t, S_t, W_t, I_t) = \max_{\delta_t^{(b)}, \delta_t^{(a)}} \mathbb{E} \left[ -e^{-\gamma \cdot (W_T + I_T \cdot S_T)} \right]$$

- $V^*(t, S_t, W_t, I_t)$  satisfies a recursive formulation for  $0 \leq t < t_1 < T$ :

$$V^*(t, S_t, W_t, I_t) = \max_{\delta_t^{(b)}, \delta_t^{(a)}} \mathbb{E} [V^*(t_1, S_{t_1}, W_{t_1}, I_{t_1})]$$

- Rewriting in stochastic differential form, we have the HJB Equation

$$\max_{\delta_t^{(b)}, \delta_t^{(a)}} \mathbb{E} [dV^*(t, S_t, W_t, I_t)] = 0 \text{ for } t < T$$

$$V^*(T, S_T, W_T, I_T) = -e^{-\gamma \cdot (W_T + I_T \cdot S_T)}$$

### 3.3 Converting HJB to a Partial Dierential Equation

- Change to  $V^*(t, S_t, W_t, l_t)$  is comprised of 3 components:

- Due to pure movement in time  $t$
- Due to randomness in TOB Mid-Price  $S_t$
- Due to randomness in hitting/lifting the Bid/Ask

- With this, we can expand  $dV^*(t, S_t, W_t, l_t)$  and rewrite HJB as:

$$\begin{aligned} \max_{\delta_t^{(b)}, \delta_t^{(a)}} \{ & \frac{\partial V^*}{\partial t} dt + \mathbb{E} \left[ \sigma \frac{\partial V^*}{\partial S_t} dz_t + \frac{\sigma^2}{2} \frac{\partial^2 V^*}{\partial S_t^2} (dz_t)^2 \right] \\ & + \lambda_t^{(b)} \cdot dt \cdot V^*(t, S_t, W_t - S_t + \delta_t^{(b)}, I_t + 1) \\ & + \lambda_t^{(a)} \cdot dt \cdot V^*(t, S_t, W_t + S_t + \delta_t^{(a)}, I_t - 1) \\ & + (1 - \lambda_t^{(b)} \cdot dt - \lambda_t^{(a)} \cdot dt) \cdot V^*(t, S_t, W_t, l_t) \\ & - V^*(t, S_t, W_t, I_t) \} = 0 \end{aligned}$$

- We can simplify this equation with a few observations:

- $\mathbb{E}[dz_t] = 0$
- $\mathbb{E}[(dz_t)^2] = dt$
- Organize the terms involving  $\lambda_t^{(b)}$  and  $\lambda_t^{(a)}$  better with some algebra
- Divide throughout by  $dt$

$$\begin{aligned} \max_{\delta_t^{(b)}, \delta_t^{(a)}} \{ & \frac{\partial V^*}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 V^*}{\partial S_t^2} \\ & + \lambda_t^{(b)} \cdot (V^*(t, S_t, W_t - S_t + \delta_t^{(b)}, I_t + 1) - V^*(t, S_t, W_t, I_t)) \\ & + \lambda_t^{(a)} \cdot (V^*(t, S_t, W_t + S_t + \delta_t^{(a)}, I_t - 1) - V^*(t, S_t, W_t, I_t)) \} = 0 \end{aligned}$$

- Next, note that  $\lambda_t^{(b)} = f^{(b)}(\delta_t^{(b)})$  and  $\lambda_t^{(a)} = f^{(a)}(\delta_t^{(a)})$ , and apply the max only on the relevant terms

$$\begin{aligned} & \frac{\partial V^*}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 V^*}{\partial S_t^2} \\ & + \max_{\delta_t^{(b)}} \left\{ f^{(b)}(\delta_t^{(b)}) \cdot (V^*(t, S_t, W_t - S_t + \delta_t^{(b)}, I_t + 1) - V^*(t, S_t, W_t, I_t)) \right\} \\ & + \max_{\delta_t^{(a)}} \left\{ f^{(a)}(\delta_t^{(a)}) \cdot (V^*(t, S_t, W_t + S_t + \delta_t^{(a)}, I_t - 1) - V^*(t, S_t, W_t, I_t)) \right\} = 0 \end{aligned}$$

- This combines with the boundary condition:

$$V^*(T, S_T, W_T, I_T) = -e^{-\gamma \cdot (W_T + I_T \cdot S_T)}$$

- We make an "educated guess" for the structure of  $V^*(t, S_t, W_t, I_t)$

$$V^*(t, S_t, W_t, I_t) = -e^{-\gamma(W_t + \theta(t, S_t, I_t))} \quad (1)$$

and reduce the problem to a PDE in terms of  $\theta(t, S_t, l_t)$

- Substituting this into the above PDE for  $V^*(t, S_t, W_t, I_t)$  gives:

$$\begin{aligned} & \frac{\partial \theta}{\partial t} + \frac{\sigma^2}{2} \left( \frac{\partial^2 \theta}{\partial S_t^2} - \gamma \left( \frac{\partial \theta}{\partial S_t} \right)^2 \right) \\ & + \max_{\delta_t^{(b)}} \left\{ \frac{f^{(b)}(\delta_t^{(b)})}{\gamma} \cdot \left( 1 - e^{-\gamma(\delta_t^{(b)} - S_t + \theta(t, S_t, I_t + 1) - \theta(t, S_t, I_t))} \right) \right\} \\ & + \max_{\delta_t^{(a)}} \left\{ \frac{f^{(a)}(\delta_t^{(a)})}{\gamma} \cdot \left( 1 - e^{-\gamma(\delta_t^{(a)} + S_t + \theta(t, S_t, I_t - 1) - \theta(t, S_t, I_t))} \right) \right\} = 0 \end{aligned}$$

- The boundary condition is:

$$\theta(T, S_T, I_T) = I_T \cdot S_T$$

### 3.4 Indifference Bid/Ask Price

- It turns out that  $\theta(t, S_t, I_t + 1) - \theta(t, S_t, I_t)$  and  $\theta(t, S_t, I_t) - \theta(t, S_t, I_t - 1)$  are equal to financially meaningful quantities known as Indifference Bid and Ask Prices
- Indifference Bid Price  $Q^{(b)}(t, S_t, I_t)$  is defined as:

$$V^*(t, S_t, W_t - Q^{(b)}(t, S_t, I_t), I_t + 1) = V^*(t, S_t, W_t, I_t) \quad (2)$$

- $Q^{(b)}(t, S_t, I_t)$  is the price to buy a share with guarantee of immediate purchase that results in Optimum Expected Utility being unchanged
- Likewise, Indifference Ask Price  $Q^{(a)}(t, S_t, I_t)$  is defined as:

$$V^*(t, S_t, W_t + Q^{(a)}(t, S_t, I_t), I_t - 1) = V^*(t, S_t, W_t, I_t) \quad (3)$$

- $Q^{(a)}(t, S_t, I_t)$  is the price to sell a share with guarantee of immediate sale that results in Optimum Expected Utility being unchanged
- We abbreviate  $Q^{(b)}(t, S_t, I_t)$  as  $Q_t^{(b)}$  and  $Q^{(a)}(t, S_t, I_t)$  as  $Q_t^{(a)}$

#### Indifference Bid/Ask Price in the PDE for $\theta$

- Express  $V^*(t, S_t, W_t - Q_t^{(b)}, I_t + 1) = V^*(t, S_t, W_t, I_t)$  in terms of  $\theta$ :

$$\begin{aligned} -e^{-\gamma(W_t - Q_t^{(b)} + \theta(t, S_t, I_t + 1))} &= -e^{-\gamma(W_t + \theta(t, S_t, I_t))} \\ \Rightarrow Q_t^{(b)} &= \theta(t, S_t, I_t + 1) - \theta(t, S_t, I_t) \end{aligned} \quad (4)$$

- Likewise for  $Q_t^{(a)}$ , we get:

$$Q_t^{(a)} = \theta(t, S_t, I_t) - \theta(t, S_t, I_t - 1) \quad (5)$$

- Using equations (4) and (5), bring  $Q_t^{(b)}$  and  $Q_t^{(a)}$  in the PDE for  $\theta$

$$\begin{aligned} & \frac{\partial \theta}{\partial t} + \frac{\sigma^2}{2} \left( \frac{\partial^2 \theta}{\partial S_t^2} - \gamma \left( \frac{\partial \theta}{\partial S_t} \right)^2 \right) + \max_{\delta_t^{(b)}} g(\delta_t^{(b)}) + \max_{\delta_t^{(a)}} h(\delta_t^{(a)}) = 0 \\ & \text{where } g(\delta_t^{(b)}) = \frac{f^{(b)}(\delta_t^{(b)})}{\gamma} \cdot \left( 1 - e^{-\gamma(\delta_t^{(b)} - S_t + Q_t^{(b)})} \right) \\ & \text{and } h(\delta_t^{(a)}) = \frac{f^{(a)}(\delta_t^{(a)})}{\gamma} \cdot \left( 1 - e^{-\gamma(\delta_t^{(a)} + S_t - Q_t^{(a)})} \right) \end{aligned}$$

### 3.5 Optimal Bid Spread and Optimal Ask Spread

- To maximize  $g\left(\delta_t^{(b)}\right)$ , differentiate  $g$  with respect to  $\delta_t^{(b)}$  and set to 0

$$\begin{aligned} e^{-\gamma\left(\delta_t^{(b)*} - S_t + Q_t^{(b)}\right)} \cdot \left( \gamma \cdot f^{(b)}\left(\delta_t^{(b)*}\right) - \frac{\partial f^{(b)}}{\partial \delta_t^{(b)}}\left(\delta_t^{(b)*}\right) \right) + \frac{\partial f^{(b)}}{\partial \delta_t^{(b)}}\left(\delta_t^{(b)*}\right) &= 0 \\ \Rightarrow \delta_t^{(b)*} = S_t - Q_t^{(b)} + \frac{1}{\gamma} \cdot \ln \left( 1 - \gamma \cdot \frac{f^{(b)}\left(\delta_t^{(b)*}\right)}{\frac{\partial f^{(b)}}{\partial \delta_t^{(b)}}\left(\delta_t^{(b)*}\right)} \right) \end{aligned} \quad (6)$$

- To maximize  $g\left(\delta_t^{(a)}\right)$ , differentiate  $g$  with respect to  $\delta_t^{(a)}$  and set to 0

$$\begin{aligned} e^{-\gamma\left(\delta_t^{(a)*} + S_t - Q_t^{(a)}\right)} \cdot \left( \gamma \cdot f^{(a)}\left(\delta_t^{(a)*}\right) - \frac{\partial f^{(a)}}{\partial \delta_t^{(a)}}\left(\delta_t^{(a)*}\right) \right) + \frac{\partial f^{(a)}}{\partial \delta_t^{(a)}}\left(\delta_t^{(a)*}\right) &= 0 \\ \Rightarrow \delta_t^{(a)*} = Q_t^{(a)} - S_t + \frac{1}{\gamma} \cdot \ln \left( 1 - \gamma \cdot \frac{f^{(a)}\left(\delta_t^{(a)*}\right)}{\frac{\partial f^{(a)}}{\partial \delta_t^{(a)}}\left(\delta_t^{(a)*}\right)} \right) \end{aligned} \quad (7)$$

- (6) and (7) are implicit equations for  $\delta_t^{(b)*}$  and  $\delta_t^{(a)*}$  respectively

#### Solving for $\theta$ and for Optimal Bid/Ask Spreads

- Let us write the PDE in terms of the Optimal Bid and Ask Spreads

$$\begin{aligned} \frac{\partial \theta}{\partial t} + \frac{\sigma^2}{2} \left( \frac{\partial^2 \theta}{\partial S_t^2} - \gamma \left( \frac{\partial \theta}{\partial S_t} \right)^2 \right) \\ + \frac{f^{(b)}\left(\delta_t^{(b)*}\right)}{\gamma} \cdot \left( 1 - e^{-\gamma\left(\delta_t^{(b)*} - S_t + \theta(t, S_t, I_t + 1) - \theta(t, S_t, I_t)\right)} \right) \\ + \frac{f^{(a)}\left(\delta_t^{(a)*}\right)}{\gamma} \cdot \left( 1 - e^{-\gamma\left(\delta_t^{(a)*} + S_t + \theta(t, S_t, I_t - 1) - \theta(t, S_t, I_t)\right)} \right) = 0 \end{aligned} \quad (8)$$

- with boundary condition  $\theta(T, S_T, I_T) = I_T \cdot S_T$
- First we solve PDE (8) for  $\theta$  in terms of  $\delta_t^{(b)*}$  and  $\delta_t^{(a)*}$
- In general, this would be a numerical PDE solution
- Using (4) and (5), we have  $Q_t^{(b)}$  and  $Q_t^{(a)}$  in terms of  $\delta_t^{(b)*}$  and  $\delta_t^{(a)*}$
- Substitute above-obtained  $Q_t^{(b)}$  and  $Q_t^{(a)}$  in equations (6) and (7)
- Solve implicit equations for  $\delta_t^{(b)*}$  and  $\delta_t^{(a)*}$  (in general, numerically)

#### Building Intuition

- Define Indifference Mid Price  $Q_t^{(m)} = \frac{Q_t^{(b)} + Q_t^{(a)}}{2}$
- To develop intuition for Indifference Prices, consider a simple case where the market-maker doesn't supply any bids or asks

$$V^*(t, S_t, W_t, l_t) = \mathbb{E} \left[ -e^{-\gamma(W_t + l_t \cdot S_T)} \mid \right]$$

- Combining this with the diffusion  $dS_t = \sigma \cdot dz_t$ , we get:

$$V^*(t, S_t, W_t, I_t) = -e^{-\gamma \left( W_t + I_t \cdot S_t - \frac{\gamma \cdot I_t^2 \cdot \sigma^2 (T-t)}{2} \right)}$$

- Combining this with equations (2) and (3), we get:

$$\begin{aligned} Q_t^{(b)} &= S_t - (2I_t + 1) \frac{\gamma \sigma^2 (T-t)}{2}, Q_t^{(a)} = S_t - (2I_t - 1) \frac{\gamma \sigma^2 (T-t)}{2} \\ Q_t^{(m)} &= S_t - I_t \gamma \sigma^2 (T-t), Q_t^{(a)} - Q_t^{(b)} = \gamma \sigma^2 (T-t) \end{aligned}$$

- These results for the simple case of no-market-making serve as approximations for our problem of optimal market-making
- Think of  $Q_t^{(m)}$  as inventory-risk-adjusted mid-price (adjustment to  $S_t$ )
- If market-maker is long inventory ( $I_t > 0$ ),  $Q_t^{(m)} < S_t$  indicating inclination to sell than buy, and if market-maker is short inventory,  $Q_t^{(m)} > S_t$  indicating inclination to buy than sell
- Armed with this intuition, we come back to optimal market-making, observing from eqns (6) and (7) :  $P_t^{(b)*} < Q_t^{(b)} < Q_t^{(m)} < Q_t^{(a)} < P_t^{(a)*}$
- Think of  $[P_t^{(b)*}, P_t^{(a)*}]$  as "centered" at  $Q_t^{(m)}$  (rather than at  $S_t$ ), i.e.,  $[P_t^{(b)*}, P_t^{(a)*}]$  will (together) move up/down in tandem with  $Q_t^{(m)}$  moving up/down (as a function of inventory position  $I_t$ )

$$\begin{aligned} Q_t^{(m)} - P_t^{(b)*} &= \frac{Q_t^{(a)} - Q_t^{(b)}}{2} + \frac{1}{\gamma} \cdot \ln \left( 1 - \gamma \cdot \frac{f^{(b)}(\delta_t^{(b)*})}{\frac{\partial f^{(b)}}{\partial \delta_t^{(b)}}(\delta_t^{(b)*})} \right) \\ P_t^{(a)*} - Q_t^{(m)} &= \frac{Q_t^{(a)} - Q_t^{(b)}}{2} + \frac{1}{\gamma} \cdot \ln \left( 1 - \gamma \cdot \frac{f^{(a)}(\delta_t^{(a)*})}{\frac{\partial f^{(a)}}{\partial \delta_t^{(a)}}(\delta_t^{(a)*})} \right) \end{aligned} \quad (9,10)$$

### 3.6 Simple Functional Form for Hitting/Lifting Rate Means

- The PDE for  $\theta$  and the implicit equations for  $\delta_t^{(b)*}, \delta_t^{(a)*}$  are messy
- We make some assumptions, simplify, derive analytical approximations
- First we assume a fairly standard functional form for  $f^{(b)}$  and  $f^{(a)}$

$$f^{(b)}(\delta) = f^{(a)}(\delta) = c \cdot e^{-k \cdot \delta}$$

- This reduces equations (6) and (7) to:

$$\begin{aligned} \delta_t^{(b)*} &= S_t - Q_t^{(b)} + \frac{1}{\gamma} \ln \left( 1 + \frac{\gamma}{k} \right) \\ \delta_t^{(a)*} &= Q_t^{(a)} - S_t + \frac{1}{\gamma} \ln \left( 1 + \frac{\gamma}{k} \right) \\ \Rightarrow P_t^{(b)*} \text{ and } P_t^{(a)*} &\text{ are equidistant from } Q_t^{(m)} \end{aligned} \quad (11,12)$$

- Substituting these simplified  $\delta_t^{(b)*}, \delta_t^{(a)*}$  in (8) reduces the PDE to:

$$\frac{\partial \theta}{\partial t} + \frac{\sigma^2}{2} \left( \frac{\partial^2 \theta}{\partial S_t^2} - \gamma \left( \frac{\partial \theta}{\partial S_t} \right)^2 \right) + \frac{c}{k + \gamma} \left( e^{-k \cdot \delta_t^{(b)*}} + e^{-k \cdot \delta_t^{(a)*}} \right) = 0 \quad (13)$$

- with boundary condition  $\theta(T, S_T, I_T) = I_T \cdot S_T$

### 3.7 Simplifying the PDE with Approximations

- Note that this PDE (13) involves  $\delta_t^{(b)*}$  and  $\delta_t^{(a)*}$
- However, equations (11), (12), (4), (5) enable expressing  $\delta_t^{(b)*}$  and  $\delta_t^{(a)*}$  in terms of  $\theta(t, S_t, I_t - 1)$ ,  $\theta(t, S_t, I_t)$ ,  $\theta(t, S_t, I_t + 1)$
- This would give us a PDE just in terms of  $\theta$
- Solving that PDE for  $\theta$  would not only give us  $V^*(t, S_t, W_t, I_t)$  but also  $\delta_t^{(b)*}$  and  $\delta_t^{(a)*}$  (using equations (11), (12), (4), (5))
- To solve the PDE, we need to make a couple of approximations
- First we make a linear approx for  $e^{-k \cdot \delta_t^{(b)*}}$  and  $e^{-k \cdot \delta_t^{(a)*}}$  in PDE (13):

$$\frac{\partial \theta}{\partial t} + \frac{\sigma^2}{2} \left( \frac{\partial^2 \theta}{\partial S_t^2} - \gamma \left( \frac{\partial \theta}{\partial S_t} \right)^2 \right) + \frac{c}{k + \gamma} \left( 1 - k \cdot \delta_t^{(b)*} + 1 - k \cdot \delta_t^{(a)*} \right) = 0 \quad (14)$$

- Equations (11), (12), (4), (5) tell us that:

$$\delta_t^{(b)*} + \delta_t^{(a)*} = \frac{2}{\gamma} \ln \left( 1 + \frac{\gamma}{k} \right) + 2\theta(t, S_t, I_t) - \theta(t, S_t, I_t + 1) - \theta(t, S_t, I_t - 1)$$

### 3.8 Asymptotic Expansion of $\theta$ in $I_t$

- With this expression for  $\delta_t^{(b)*} + \delta_t^{(a)*}$ , PDE (14) takes the form:

$$\begin{aligned} \frac{\partial \theta}{\partial t} + \frac{\sigma^2}{2} \left( \frac{\partial^2 \theta}{\partial S_t^2} - \gamma \left( \frac{\partial \theta}{\partial S_t} \right)^2 \right) + \frac{c}{k + \gamma} \left( 2 - \frac{2k}{\gamma} \ln \left( 1 + \frac{\gamma}{k} \right) \right. \\ \left. - k(2\theta(t, S_t, I_t) - \theta(t, S_t, I_t + 1) - \theta(t, S_t, I_t - 1)) \right) = 0 \end{aligned} \quad (15)$$

- To solve PDE (15), we consider this asymptotic expansion of  $\theta$  in  $I_t$ :

$$\theta(t, S_t, I_t) = \sum_{n=0}^{\infty} \frac{I_t^n}{n!} \cdot \theta^{(n)}(t, S_t)$$

- So we need to determine the functions  $\theta^{(n)}(t, S_t)$  for all  $n = 0, 1, 2, \dots$
- For tractability, we approximate this expansion to the first 3 terms:

$$\theta(t, S_t, I_t) \approx \theta^{(0)}(t, S_t) + I_t \cdot \theta^{(1)}(t, S_t) + \frac{I_t^2}{2} \cdot \theta^{(2)}(t, S_t)$$

### 3.9 Approximation of $\theta$ in $I_t$

- We note that the Optimal Value Function  $V^*$  can depend on  $S_t$  only through the current Value of the Inventory (i.e., through  $I_t \cdot S_t$ ), i.e., it cannot depend on  $S_t$  in any other way
- This means  $V^*(t, S_t, W_t, 0) = -e^{-\gamma(W_t + \theta^{(0)}(t, S_t))}$  is independent of  $S_t$
- This means  $\theta^{(0)}(t, S_t)$  is independent of  $S_t$
- So, we can write it as simply  $\theta^{(0)}(t)$ , meaning  $\frac{\partial \theta^{(0)}}{\partial S_t}$  and  $\frac{\partial^2 \theta^{(0)}}{\partial S_t^2}$  are 0
- Therefore, we can write the approximate expansion for  $\theta(t, S_t, I_t)$  as:

$$\theta(t, S_t, I_t) = \theta^{(0)}(t) + I_t \cdot \theta^{(1)}(t, S_t) + \frac{I_t^2}{2} \cdot \theta^{(2)}(t, S_t) \quad (16)$$



### 3.10 Solving the PDE

- Substitute this approximation (16) for  $\theta(t, S_t, I_t)$  in PDE (15)

$$\begin{aligned} & \frac{\partial \theta^{(0)}}{\partial t} + I_t \frac{\partial \theta^{(1)}}{\partial t} + \frac{l_t^2}{2} \frac{\partial \theta^{(2)}}{\partial t} + \frac{\sigma^2}{2} \left( I_t \frac{\partial^2 \theta^{(1)}}{\partial S_t^2} + \frac{l_t^2}{2} \frac{\partial^2 \theta^{(2)}}{\partial S_t^2} \right) \\ & - \frac{\gamma \sigma^2}{2} \left( I_t \frac{\partial \theta^{(1)}}{\partial S_t} + \frac{l_t^2}{2} \frac{\partial \theta^{(2)}}{\partial S_t} \right)^2 + \frac{c}{k+\gamma} \left( 2 - \frac{2k}{\gamma} \ln \left( 1 + \frac{\gamma}{k} \right) + k \cdot \theta^{(2)} \right) = 0 \end{aligned}$$

- with boundary condition:

$$\theta^{(0)}(T) + I_T \cdot \theta^{(1)}(T, S_T) + \frac{I_T^2}{2} \cdot \theta^{(2)}(T, S_T) = I_T \cdot S_T \quad (17)$$

- We will separately collect terms involving specific powers of  $I_t$ , each yielding a separate PDE:

- Terms devoid of  $I_t$  (i.e.,  $I_t^0$ )
- Terms involving  $I_t$  (i.e.,  $I_t^1$ )
- Terms involving  $I_t^2$

- We start by collecting terms involving  $I_t$

$$\frac{\partial \theta^{(1)}}{\partial t} + \frac{\sigma^2}{2} \cdot \frac{\partial^2 \theta^{(1)}}{\partial S_t^2} = 0 \text{ with boundary condition } \theta^{(1)}(T, S_T) = S_T$$

- The solution to this PDE is:

$$\theta^{(1)}(t, S_t) = S_t \quad (18)$$

- Next, we collect terms involving  $l_t^2$

$$\frac{\partial \theta^{(2)}}{\partial t} + \frac{\sigma^2}{2} \cdot \frac{\partial^2 \theta^{(2)}}{\partial S_t^2} - \gamma \sigma^2 \cdot \left( \frac{\partial \theta^{(1)}}{\partial S_t} \right)^2 = 0 \text{ with boundary } \theta^{(2)}(T, S_T) = 0$$

- Noting that  $\theta^{(1)}(t, S_t) = S_t$ , we solve this PDE as:

$$\theta^{(2)}(t, S_t) = -\gamma \sigma^2 (T - t) \quad (19)$$

- Finally, we collect the terms devoid of  $I_t$

$$\frac{\partial \theta^{(0)}}{\partial t} + \frac{c}{k+\gamma} \left( 2 - \frac{2k}{\gamma} \ln \left( 1 + \frac{\gamma}{k} \right) + k \cdot \theta^{(2)} \right) = 0 \text{ with boundary } \theta^{(0)}(T) = 0$$

- Noting that  $\theta^{(2)}(t, S_t) = -\gamma \sigma^2 (T - t)$ , we solve as:

$$\theta^{(0)}(t) = \frac{c}{k+\gamma} \left( \left( 2 - \frac{2k}{\gamma} \ln \left( 1 + \frac{\gamma}{k} \right) \right) (T - t) - \frac{k\gamma\sigma^2}{2} (T - t)^2 \right) \quad (20)$$

- This completes the PDE solution for  $\theta(t, S_t, l_t)$  and hence, for  $V^*(t, S_t, W_t, l_t)$

- Lastly, we derive formulas for  $Q_t^{(b)}$ ,  $Q_t^{(a)}$ ,  $Q_t^{(m)}$ ,  $\delta_t^{(b)*}$ ,  $\delta_t^{(a)*}$

### 3.11 Formulas for Prices and Spreads

- Using equations (4) and (5), we get:

$$\begin{aligned} Q_t^{(b)} &= \theta^{(1)}(t, S_t) + (2I_t + 1) \cdot \theta^{(2)}(t, S_t) = S_t - (2I_t + 1) \frac{\gamma \sigma^2 (T-t)}{2} \\ Q_t^{(a)} &= \theta^{(1)}(t, S_t) + (2I_t - 1) \cdot \theta^{(2)}(t, S_t) = S_t - (2I_t - 1) \frac{\gamma \sigma^2 (T-t)}{2} \end{aligned} \quad (21,22)$$

- Using equations (11) and (12), we get:

$$\delta_t^{(b)*} = \frac{(2I_t + 1) \gamma \sigma^2 (T-t)}{2} + \frac{1}{\gamma} \ln \left( 1 + \frac{\gamma}{k} \right) \quad (23)$$

$$\delta_t^{(a)*} = \frac{(1 - 2I_t) \gamma \sigma^2 (T-t)}{2} + \frac{1}{\gamma} \ln \left( 1 + \frac{\gamma}{k} \right) \quad (24)$$

- Optimal Bid-Ask Spread

$$\delta_t^{(b)*} + \delta_t^{(a)*} = \gamma \sigma^2 (T-t) + \frac{2}{\gamma} \ln \left( 1 + \frac{\gamma}{k} \right) \quad (25)$$

$$\text{Optimal "Mid" } Q_t^{(m)} = \frac{Q_t^{(b)} + Q_t^{(a)}}{2} = \frac{P_t^{(b)*} + P_t^{(a)*}}{2} = S_t - I_t \gamma \sigma^2 (T-t) \quad (26)$$

## 4 Summary

- Think of  $Q_t^{(m)}$  as inventory-risk-adjusted mid-price (adjustment to  $S_t$ )
- If market-maker is long inventory ( $I_t > 0$ ),  $Q_t^{(m)} < S_t$  indicating inclination to sell than buy, and if market-maker is short inventory,  $Q_t^{(m)} > S_t$  indicating inclination to buy than sell
- Think of  $[P_t^{(b)*}, P_t^{(a)*}]$  as "centered" at  $Q_t^{(m)}$  (rather than at  $S_t$ ), i.e.,  $[P_t^{(b)*}, P_t^{(a)*}]$  will (together) move up/down in tandem with  $Q_t^{(m)}$  moving up/down (as a function of inventory position  $I_t$ )
- Note from equation (25) that the Optimal Bid-Ask Spread  $P_t^{(a)*} - P_t^{(b)*}$  is independent of inventory  $I_t$
- Useful view:  $P_t^{(b)*} < Q_t^{(b)} < Q_t^{(m)} < Q_t^{(a)} < P_t^{(a)*}$ , with these spreads:
  - Outer Spreads  $P_t^{(a)*} - Q_t^{(a)} = Q_t^{(b)} - P_t^{(b)*} = \frac{1}{\gamma} \ln \left( 1 + \frac{\gamma}{k} \right)$
  - Inner Spreads  $Q_t^{(a)} - Q_t^{(m)} = Q_t^{(m)} - Q_t^{(b)} = \frac{\gamma \sigma^2 (T-t)}{2}$