Optimal Execution of Portfolio Transactions

Robert Almgren and Neil Chriss - December 2000

1 Introduction

1.1 Portfolio Liquidation

Financial problem

- We want to sell a large quantity of a stock (or of several stocks) in one day.
- How to choose the transaction times?

1.2 Strategies (1)

Naive strategies

- 2 extreme strategies:
  - Sell everything right now $\rightarrow$ huge transaction cost since we need to "eat" a lot in the order book. However this cost is known.
  - Sell regularly in the day small amounts of assets $\rightarrow$ small transaction costs (volumes are much smaller) but the final profit is unknown because of the daily price fluctuations : Volatility risk.

1.3 Strategies (2)

Optimization

- We need to optimize between transaction costs and volatility risk.
- To do so, we use the Almgren and Chriss framework which takes into account the market impact phenomenon and emphasizes the importance of having good statistical estimators of market parameters.

2 Almgren and Chriss model

2.1 Trading strategy

Setup

- We consider we are selling one asset. We have X shares of this assets at $t_0 = 0$
- We want everything to be sold at $t = T$.
- We split $[0, T]$ into N intervals of length $\tau = T/N$ and set $t_k = k\tau, k = 0, \dots, N$
- A trading strategy is a vector $(x_0, \dots, x_N)$, with x_k the number of shares we still have at time t_k.
- $x_0 = X, x_N = 0$ and $n_k = x_{k-1} - x_k$ is the number of assets sold between t_{k-1} and t_k, decided at time t_{k-1}.

2.2 Price decomposition

Price components

- The price we have access to moves because of :
  - The drift $\rightarrow$ negligible at the intraday level.
  - The volatility.
  - The market impact.

2.3 Permanent market impact

Permanent impact component

- Market participants see us selling large quantities.
- Thus they revise their prices down.
- Therefore, the "equilibrium price" of the asset is modified in permanent way.
- Let S_k be the equilibrium price at time t_k :

$$S_k = S_{k-1} + \sigma\tau^{1/2}\xi_k - \tau g\left(n_k/\tau\right)$$

- with ξ_k iid standard Gaussian and n_k/τ the average trading rate between t_{k-1} and t_k.

2.4 Temporary market impact

Temporary impact component

- It is due to the transaction costs: we are liquidity taker since we "eat" the order book.
- If we sell a large amount of shares, our price per share is significantly worse than when selling only one share.
- We assume this effect is temporary and the liquidity comes back after each period.
- Let $\tilde{S}_k = \left(\sum n_{k,i}p_i\right)/n_k$, with $n_{k,i}$ the number of shares sold at price p_i between t_{k-1} and t_k. We set

$$\tilde{S}_k = S_{k-1} - h\left(n_k/\tau\right)$$

- The term $h\left(n_k/\tau\right)$ does not influence the next equilibrium price S_k.

2.5 Profit and Loss

Cost of trading

- The result of the sell of the asset is

$$\sum_{k=1}^{N} n_k\tilde{S}_k$$
$$= XS_0 + \sum_{k=1}^{N}\left(\sigma\tau^{1/2}\xi_k - \tau g\left(n_k/\tau\right)\right)x_k - \sum_{k=1}^{N} n_k h\left(n_k/\tau\right)$$

- The trading cost $\mathcal{C} = XS_0 - \sum_{k=1}^{N} n_k\tilde{S}_k$ is equal to Vol. cost + Perm. Impact cost + Temp. Impact cost.

2.6 Mean-Variance analysis

Moments

- Consider a static strategy (fully known in t_0), which is in fact optimal in this framework. We have

$$\mathbb{E}[\mathcal{C}] = \sum_{k=1}^{N} \tau x_k g\left(n_k/\tau\right) + \sum_{k=1}^{N} n_k h\left(n_k/\tau\right), \quad \operatorname{Var}[\mathcal{C}] = \sigma^2 \sum_{k=1}^{N} \tau x_k^2$$

- In order to build optimal trading trajectories, we will look for strategies minimizing

$$\mathbb{E}[\mathcal{C}] + \lambda \operatorname{Var}[\mathcal{C}]$$

with λ a risk aversion parameter.

3 Naive strategies

3.1 Assumptions (1)

Permanent impact

- Linear permanent impact: $g(v) = \gamma v$.
- If we sell n shares, the price per share decreases by γn. Thus

$$S_k = S_0 + \sigma \sum_{j=1}^{k} \tau^{1/2} \xi_j - \gamma\left(X - x_k\right)$$

- and in $\mathbb{E}[\mathcal{C}]$, the permanent impact component satisfies

$$\sum_{k=1}^{N} \tau x_k g\left(n_k/\tau\right) = \gamma \sum_{k=1}^{N} x_k\left(x_{k-1} - x_k\right) = \frac{1}{2}\gamma X^2 - \frac{1}{2}\gamma \sum_{k=1}^{N} n_k^2$$

3.2 Assumptions (2)

Temporary impact

- Affine temporary impact: $h\left(n_k/\tau\right) = \varepsilon + \eta\left(n_k/\tau\right)$.
- ε represents a fixed cost : fees + bid ask spread.
- Let $\tilde{\eta} = \eta - \frac{1}{2}\gamma\tau$, we get

$$\mathbb{E}[\mathcal{C}] = \frac{1}{2}\gamma X^2 + \varepsilon X + \frac{\tilde{\eta}}{\tau} \sum_{k=1}^{N} n_k^2$$

3.3 Regular liquidation

Regular strategy

- Take $n_k = X/N, x_k = (N-k)X/N, k = 1, \ldots, N$.
- We easily get

$$\begin{aligned}\mathbb{E}[\mathcal{C}] &= \tfrac{1}{2}\gamma X^2 + \varepsilon X + \tilde{\eta}\tfrac{X^2}{T} \\ \operatorname{Var}[\mathcal{C}] &= \tfrac{\sigma^2}{3} X^2 T\left(1 - \tfrac{1}{N}\right)\left(1 - \tfrac{1}{2N}\right).\end{aligned}$$

- We can show this strategy has the smallest expectation. However the variance can be very big if T is large.

3.4 Immediate selling

Selling everything at t0

- Take $n_1 = X, n_2 = \ldots = n_N = 0, x_1 = \ldots = x_N = 0$
- We get

$$\begin{array}{l} \mathbb{E}[\mathcal{C}] = \varepsilon X + \frac{\eta X^2}{\tau} \\ \mathrm{Var}[\mathcal{C}] = 0 \end{array}$$

- This strategy has the smallest variance. However, if τ is small, the expectation can be very large.

4 Optimal strategies

4.1 Optimization (1)

Optimization program

- - The trader wants to minimize

$$U(\mathcal{C}) = \mathbb{E}[\mathcal{C}] + \lambda \, \mathrm{Var}[\mathcal{C}].$$

- $U(\mathcal{C})$ is equal to

$$\frac{1}{2}\gamma X^2 + \varepsilon X + \frac{\tilde{\eta}}{\tau}\sum_{k=1}^{N}(x_{k-1} - x_k)^2 + \lambda\sigma^2\sum_{k=1}^{N}\tau x_k^2$$

4.2 Optimization (2)

Derivation

- For $j = 1, \ldots, N-1$,

$$\frac{\partial U}{\partial x_j} = 2\tau\left(\lambda\sigma^2 x_j - \tilde{\eta}\frac{(x_{j-1} - 2x_j + x_{j+1})}{\tau^2}\right)$$

- Therefore

$$\frac{\partial U}{\partial x_j} = 0 \Leftrightarrow \frac{(x_{j-1} - 2x_j + x_{j+1})}{\tau^2} = \tilde{K}x_j$$

  with $\tilde{K} = \lambda\sigma^2/\tilde{\eta}$.

4.3 Optimization (3)

Solution

- It is shown that the solution can be written $x_0 = X$ and for $j = 1, \ldots, N$:

$$\begin{array}{l} x_j = \frac{\sinh(K(T-t_j))}{\sinh(KT)}X \\ n_j = \frac{2\sinh(K\tau/2)}{\sinh(KT)}\cosh(K(T - j\tau + \tau/2)) \end{array}$$

  where K satisfies $\frac{2}{\tau^2}(\cosh(K\tau) - 1) = \tilde{K}$

- If $\lambda = 0$, then $\tilde{K} = K = 0$ and so $n_j = \tau/T = X/N$. We retrieve the strategy with minimal expected cost.

4.4 Remarks on this approach

Remarks

- It is easy to show that the solution is time homogenous: if we compute the optimal strategy in t_k, we obtain the value between t_k and T of the optimal strategy computed in t_0.
- In this approach, we obtain an efficient frontier of trading.
- The optimal trajectories are very sensitive to the volatility parameter. It is therefore important to obtain accurate volatility estimates.
- The Almgren and Chriss framework can be extended in dimension n (if we sell several assets). In that case, correlation parameters come into the picture.

4.5 Results

Optimal Trajectory for a Single-Asset Portfolio

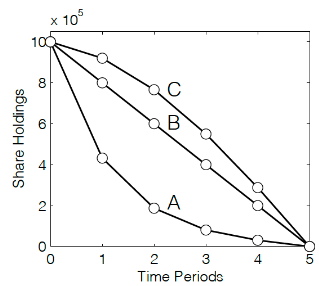

Figure 1: Optimal trajectories. The trajectories corresponding to the points shown in Figure 1. (A) $\lambda = 2 \times 10^{-6}$, (B) $\lambda = 0$, (C) $\lambda = -2 \times 10^{-7}$. [1, p.18]

Optimal Trajectories for the liquidation of a Two-Asset Portfolio

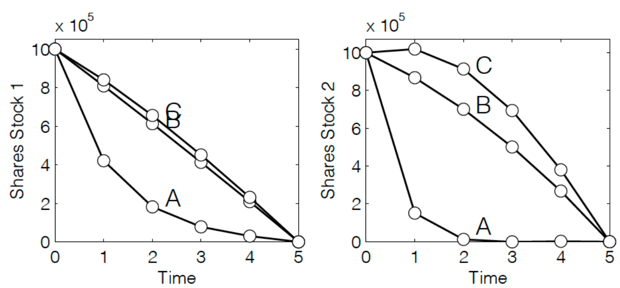

Figure 2: Optimal trajectories for two securities. As in Figure 5 , for (A) $\lambda = 2 \times 10^{-6}$, (B) the naïve strategy with $\lambda = 0$, (C)$\lambda = -5 \times 10^{-8}$. [1, p.41]

References

[1] R. Almgren and N. Chriss. Optimal execution of portfolio transactions. Journal of Risk, 2000.