Finite Differences Schemes for Pricing of European and American Options

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Abstract

Starting with the Black-Scholes model, we use finite differences schemes to value European and American options. Two approaches, resulting from transformations of the Black-Scholes equation, will be considered. An application to real data, using the Local Volatility Model to determine a volatility surface consistent with the Black-Scholes equation will be presented. Applying this local volatility surface to the finite difference method, it is possible to calibrate the Black-Scholes model, obtaining a reasonable approximation of the real market values.

1 Introduction

Derivatives are nowadays widely used in financial markets. A derivative is a financial instrument whose value depends on the price of the underlying variable and can be used to hedge risk, speculation and arbitrage. The value of a financial asset is uncertain and invest directly in them comprehends a big risk. Derivatives arise as a way to invest in actives with less risk. However, by continuing to depend on the value of the underlying asset, there is a great difficulty in determining their value. In this paper we will propose methods for option pricing (European and American options, Vanilla and Barrier type).

The rest of the paper is organized as follows. In the next section we discuss the Black-Scholes model on which we rely for option pricing. We will see how this model can be applied to vanilla options (European and American) and we present two transformations of this equation that will facilitate its resolution through the application of finite difference methods. In section 3 we will present the finite difference schemes for solving the problem of valuation of vanilla and barrier options (European and American). Then in section 4, we show how to calibrate the Black-Scholes Model, with an application to real data. Finally, in Section 5 we present some conclusions and discuss the future work.

2 Black-Scholes Model

The Black-Scholes Model describes the behaviour of options on assets that follow a Geometric Brownian Motion that satisfies the following stochastic differential equation (cf. [7])

$$\frac{dS}{S} = \mu dt + \sigma dW_t \tag{1}$$

In 1973, Black and Scholes [1], defined the following assumptions of the model:

1. *The short-term interest rate is known and is constant through time.*
2. *The stock price follows a random walk in continuous time with variance rate proportional to the square of the stock price. Thus the distribution of possible stock prices at the end of any finite interval is log-normal. The variance rate of return on the stock is constant.*
3. *The stock pays no dividends or other distributions.*
4. *The option is European, that is, it can only be exercised at maturity.*
5. *There are no transaction costs in buying or selling the stock or the option.*
6. *It is possible to borrow any fraction of the price of a security to buy it or hold it, at the short-term interest rate.*
7. *There are no penalties to short selling.*

An option depends on the value of the underlying variable, S, over a period of time, t. Considering that V is the value of an option, we want to find $V(S,t)$. Applying Itô's Lemma (cf. [7]), the desired function is described by

$$dV(S,t) = \Big(\frac{\partial V}{\partial S}\mu S + \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\Big)dt + \frac{\partial V}{\partial S}\sigma S dW(t) \tag{2}$$

The aim now is to eliminate the randomness produced by the parameter dW_t. For this, we choose a portfolio, Π, consisting of a short position on a derivative and a long position on a quantity $\frac{\partial V(S,t)}{\partial S}$ of the underlying asset.

$$\Pi = V(S,t) - \frac{\partial V(S,t)}{\partial S}S \qquad \Longrightarrow \qquad d\Pi = dV(S,t) - \frac{\partial V(S,t)}{\partial S}dS \tag{3}$$

Replacing dS and $dV(S,t)$ by the equations 1 and 2, respectively, and assuming that there is no arbitrage (that is, $r\Pi dt = d\Pi$), we obtain the Black-Scholes differential equation that describes the behaviour of a European option.

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0 \tag{4}$$

2.1 Options

Notation 1. *From now on, we will use S_0 to represent the present value of the underlying asset; K represents the strike price (the price at which the asset can be bought or sold); T is the maturity date of the contract; r is the continuously compound interest-rate; and σ will represent the volatility (a measure of uncertainty of the return realized on an asset)*

The options can be divided in call and put options, depending on whether the contract is for the purchase or the sale of an asset. They can still be divided into European and American options depending on the date at which they can be exercised. European options can only be exercised at maturity while American options can be exercised at any time until the maturity date.

2.1.1 European Options

The value of a European option can be found by solving the equation of Black-Scholes. However, the equation only has unique solution if the boundary conditions are defined. In this case, we know the value of an option, $V(S,t)$, at maturity, T (date where it is decided whether the option is exercised or not) and when the asset value is zero or is so big that it can be considered infinite.

Boundary Conditions for European

$$
\begin{array}{ll}
\textbf{Call Options} & \textbf{Put Options} \\
 & \\
V(S,T) = max\{S-K,0\} & V(S,T) = max\{K-S,0\} \\
V(0,t) = 0 & V(0,t) = Ke^{-r(T-t)} \\
\lim_{S\to+\infty} V(S,t) = S & \lim_{S\to+\infty} V(S,t) = 0
\end{array} \tag{5}
$$

2.1.2 American Options

Since American options allow early exercise, the arbitrage argument used for the European option no longer leads to a unique value for the return on the portfolio, only to a inequality. We can only say that the value of the portfolio can not be greater than the value of a riskless investment on the same value. Therefore, the Black-Scholes equation only can be used in the form of an inequality:

$$
\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV \leq 0 \tag{6}
$$

Assuming again that there are no-arbitrage opportunities, we can also impose the condition[1] $V(S,t) \leq g(S,t)$, where $g(S,t)$ represents the payoff of the option.

From these two conditions we can define the problem of valuation of American options through the following complementarity linear problem.

$$
\begin{array}{c}
\Big(\mathcal{L}_{BS}V(S,t)\Big)\Big(g(S,t) - V(S,t)\Big) = 0 \\
\mathcal{L}_{BS}V(S,t) \leq 0 \quad \text{e} \quad g(S,t) - V(S,t) \leq 0
\end{array} \tag{7}
$$

Similarly to European options, it is also necessary to define the boundary conditions. Again, for American options, we have the boundary conditions at maturity ($t = T$) and when $S = 0$ or $S \to \infty$

Boundary Conditions for American

$$
\begin{array}{ll}
\textbf{Call Options} & \textbf{Put Options} \\
 & \\
V(S,T) = max\{S-K,0\} & V(S,T) = max\{K-S,0\} \\
V(0,t) = 0 & V(0,t) = K \\
\lim_{S\to+\infty} V(S,t) = S & \lim_{S\to+\infty} V(S,t) = 0
\end{array} \tag{8}
$$

2.2 Black-Scholes Equation Transformations

2.2.1 Heat Equation Transformation

One of the approaches used to solve the equation of Black-Scholes is through its transformation into an heat equation ($\partial_t u = \partial_x^2 u$).The transformation used by Wilmott et al. in [3] is presented below.

$$
\begin{cases}
S = Ke^x \\
t = T - \frac{\tau}{\sigma^2/2} \\
V(S,t) = Ke^{\alpha x + \beta\tau}u(x,\tau)
\end{cases}
\quad \text{with} \quad
\begin{cases}
\alpha = -\frac{1}{2}(c-1) \\
\beta = -\frac{1}{4}(c+1)^2 \\
c = \frac{r}{\sigma^2/2}
\end{cases}
$$

[1] Otherwise, one could buy this option and sell it, just profiting from it.

2.2.2 Logarithmic Transformation of the Asset Value

In this section we introduce a not so radical way, as to transform the Black-Scholes equation into an heat equation, of solving the problem. It avoids problems that arise from the way the spacing is done, through the simple change of variable $z = log(S)$, and at the same time preserves the presence of r and σ in the equation.

3 Finite Difference Methods for Option Pricing

In this section we present the finite difference method to solve the previous transformations of the Black-Scholes Equation. This is a method that uses finite difference equations to approximate derivatives. First, we discretize the domain of the problem and then, knowing its boundary points, it is possible to retrieve the value of the internal points. The finite difference schemes vary depending on what type of the derivative approximation is made. In our work we studied the Explicit, Fully-Implicit and Crank-Nicolson Schemes.

3.1 European Options

3.1.1 Methods Based on Reduction to the Heat Equation

Recalling the transformation of Black-Scholes equation into the heat equation given in section 2.2.1, we have that the function $u(x, \tau)$ is defined in $-\infty < x < \infty$ and $0 \leq \tau \leq T\sigma^2/2$. Where T is the maturity date of the option and σ is the market volatility. With no loss of generality, consider that the region Ω where we apply the finite difference schemes is $[x_{N^-} = -A, x_{N^+} = A] \times [\tau_0 = 0, \tau_M = T\sigma^2/2]$ where A is a sufficiently large number. The domain discretization is done with equal spacing between the points (x_n, τ_m). Considering $h_x = \frac{x_{N^+} - x_{N^-}}{2N}$ and $h_\tau = \frac{\tau_M - \tau_0}{M}$, respectively, the spacing in x and in τ, we have the points defined by

$$\begin{array}{ll} x_n = x_{N^-} + nh_x, & n = N^-, N^- - 1, \ldots, 0, \ldots, N^+ - 1, N^+ \\ \tau_m = \tau_0 + mh_\tau, & m = 0, 1, \ldots, M-1, M \end{array}$$

Knowing the value of boundary points, we just need to aproximate the values within the grid. From this point forward, we will use $u_{n,m}$ to denote the approximation of $u(x_n, \tau_m)$. The Dirichlet problem for a European option is defined below.

$$\begin{cases} \partial_\tau u(x, \tau) = \partial_x^2 u(x, \tau), & (x, \tau) \in (x_{N^-}, x_{N^+}) \times (\tau_0, \tau_M) \quad (i) \\ u(x, 0) = u_0(x), & x \in (x_{N^-}, x_{N^+}) \quad (ii) \\ u(-A, \tau) = u_{N^-}(\tau), & \tau \in (\tau_0, \tau_M) \quad (iii) \\ u(A, \tau) = u_{N^+}(\tau), & \tau \in (\tau_0, \tau_M) \quad (iv) \end{cases} \tag{9}$$

Note that the boundary conditions defined by the equations (9)-$(ii) - (iv)$ vary according to the type of European option that we are approximating. Up to a transformation, these boundary conditions are those defined in section 2.1.1. The objective will be to replace the partial derivatives by approximations defined for each scheme. These approaches can be found in [3] or in the main work, for Explicit, Fully-Implicit and Crank-Nicolson Schemes.

3.1.2 Methods Based on $z = log(S)$ Transformation

For this transformation, the function $v(z, t)$ is set in $-\infty < z < \infty$ and $0 \leq t \leq T$. Again, we consider that the region, Ω, is $[z_{N^-} = -A, z_{N^+} = A] \times [t_0 = 0, t_M = T]$, where A is a sufficiently large number. The discretization of the domain is again made with equal spacing between points.

Considering $h_z = \frac{z_{N^+} - z_{N^-}}{2N}$ and $h_t = \frac{t_M - t_0}{M}$, respectively, the spacing in z and in t, we have the points defined by

$$\begin{array}{ll} z_n = z_{N^-} + nh_z, & n = N^-, N^- - 1, \ldots, 0, \ldots, N^+ - 1, N^+ \\ t_m = t_0 + mh_t, & m = 0, 1, \ldots, M-1, M \end{array}$$

In this section, $v_{n,m}$ denotes the approximation of the values $v(z_n, t_m)$. The problem for this approach is defined by,

$$\begin{cases} \partial_t v(z,t) + (r - \frac{\sigma^2}{2})\frac{\partial v(z,t)}{\partial z} + \frac{1}{2}\sigma^2 \frac{\partial^2 v(z,t)}{\partial z^2} - rv(z,t) = 0, & (x,t) \in (z_{N^-}, z_{N^+}) \times (t_0, t_M) \quad (i) \\ v(z,T) = v_M(z), & z \in (z_{N^-}, z_{N^+}) \quad (ii) \\ v(-A,t) = v_{N^-}(t), & t \in (t_0, t_M) \quad (iii) \\ v(A,t) = v_{N^+}(t), & t \in (t_0, t_M) \quad (iv) \end{cases} \tag{10}$$

The boundary conditions (10)-$(ii)-(iv)$ should be replaced according to the type of option (cf. section 2.1.1). Also for this problem, we studied three finite differences schemes. In this paper, we only present the Crank-Nicolson Scheme, the others can be found on the main work, as well as the derivation of the stability condition of the explicit method (the only scheme studied which is not stable for any choice of h_z and h_t).

Crank-Nicolson Scheme

$$\begin{array}{l} \partial_t v(z_n, t_{m+\frac{1}{2}}) = \frac{v_{n,m+1} - v_{n,m}}{h_t} + O(h_t^2) \\ \partial_z v(z_n, t_{m+\frac{1}{2}}) = \frac{1}{2}(\frac{v_{n+1,m+1} - v_{n-1,m+1}}{2h_z}) + \frac{1}{2}(\frac{v_{n+1,m} - v_{n-1,m}}{2h_z}) + O(h_z) \\ \partial_z^2 v(z_n, t_{m+\frac{1}{2}}) = \frac{1}{2}(\frac{v_{n+1,m+1} - 2v_{n,m+1} + v_{n-1,m+1}}{h_z^2}) + \frac{1}{2}(\frac{v_{n+1,m} - 2v_{n,m} + v_{n-1,m}}{h_z^2}) + O(h_z^2) \end{array} \tag{11}$$

Replacing the approximations of the derivatives in the equation 10-(i), we obtain the Crank-Nicolson scheme.

$$av_{n+1,m+1} + dv_{n,m+1} + cv_{n-1,m+1} = a^* v_{n+1,m} + d^* v_{n,m} + c^* v_{n-1,m} \tag{12}$$

with,

$$\begin{array}{ll} a = \frac{1}{4}(\frac{\sigma^2}{h_z^2} + \frac{(r - \frac{\sigma^2}{2})}{h_z}) & a^* = -a \\ d = \frac{1}{h_t} - \frac{1}{2}\frac{\sigma^2}{h_z^2} & d^* = \frac{1}{h_t} + r + \frac{1}{2}\frac{\sigma^2}{h_z^2} \\ c = \frac{1}{4}(\frac{\sigma^2}{h_z^2} - \frac{(r - \frac{\sigma^2}{2})}{h_z}) & c^* = -c \end{array} \tag{13}$$

Assuming $Z_{n,m+1} = av_{n+1,m+1} + dv_{n,m+1} + cv_{n-1,m+1}$ we find the desired points by solving the following linear system,

$$\begin{bmatrix} d^* & a^* & \ddots & 0 \\ c^* & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & a^* \\ 0 & \ddots & c^* & d^* \end{bmatrix} \begin{bmatrix} v_{N^-+1,m} \\ \\ \vdots \\ \\ v_{N^+-1,m} \end{bmatrix} = \begin{bmatrix} Z_{N^-+1,m+1} \\ \\ \vdots \\ \\ Z_{N^+-1,m+1} \end{bmatrix} - \begin{bmatrix} c^* v_{N^-,m} \\ 0 \\ \vdots \\ 0 \\ a^* v_{N^+,m} \end{bmatrix}$$

3.1.3 Numerical Results

In the finite difference methods, the aproximation errors must take into account all points of the domain. However, to avoid the influences on this quantity, made by the wide variation of $V(S,t)$, and because the value that we are looking for is the point $V(S_0, t_0)$, the aproximation errors that will be presented in this and the following sections will be errors regarding the approximation of the function V at point (S_0, t_0).

Figure 1 shows the convergence of three finite difference schemes for call and put European options, obtained through the transformation into the heat equation and the logarithmic transformation of the asset value, $z = log(S)$. For both approaches, we aproximated options with the following parameters: $S_0 = 1000$, $K = 1500$, $T = 10$, $r = 0.1$ and $\sigma = 0.4$. The graphs show the evolution of relative error with the decreasing spacing of the grid (increase of divisions axis).

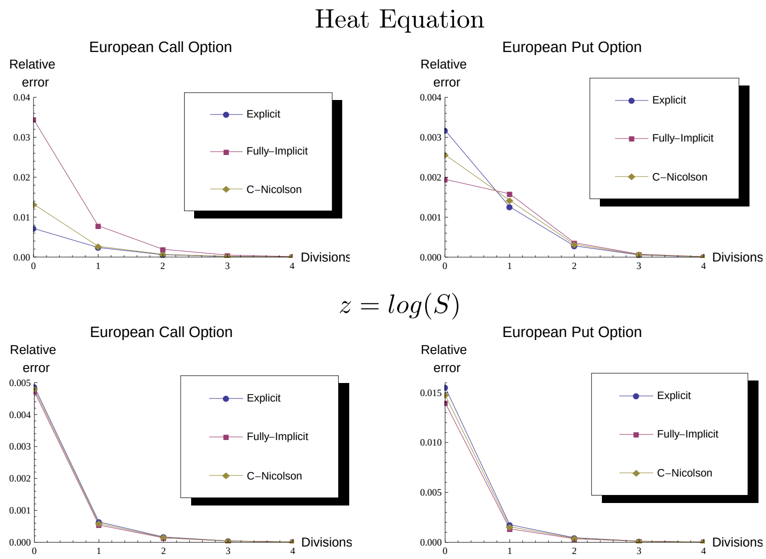

Figure 1: Convergence schemes obtained for European options.

3.2 American Options

3.2.1 Methods Based on Reduction to the Heat Equation

Applying the change of variables proposed in section 2.2.1 to the linear complementarity problem defined on eq. 7, we obtain

$$\begin{array}{ll} \left(\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2}\right) \geq 0 & (i) \\ (u(x,\tau) - g(x,\tau)) \geq 0 & (ii) \\ \left(\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2}\right)(u(x,\tau) - g(x,\tau)) = 0 & (iii) \end{array} \tag{14}$$

By defining the appropriate boundary conditions, it is possible to solve this problem through the Projected SOR method, described in the main work.

3.2.2 Methods Based on $z = log(S)$ Transformation

For this transformation the linear complementarity problem that describes American options value is given by

$$\begin{array}{ll} \Big(\partial_t v(z,t) + (r - \frac{\sigma^2}{2})\frac{\partial v(z,t)}{\partial z} + \frac{1}{2}\sigma^2 \frac{\partial^2 v(z,t)}{\partial z^2} - rv(z,t)\Big)\Big(g(S,t) - V(S,t)\Big) = 0 & (i) \\ \partial_t v(z,t) + (r - \frac{\sigma^2}{2})\frac{\partial v(z,t)}{\partial z} + \frac{1}{2}\sigma^2 \frac{\partial^2 v(z,t)}{\partial z^2} - rv(z,t) \leq 0 & (ii) \\ g(S,t) - V(S,t) \leq 0 & (iii) \end{array} \tag{15}$$

Again, the problem should be defined with appropriate boundary conditions. In the following paragraph, there is the resolution of the problem using the Crank-Nicolson Scheme (other schemes can be consulted in the main work).

Crank-Nicolson Scheme

Considering the approximations of the derivatives as defined in equation (11), the Crank-Nicolson Scheme for American options is given by

$$av_{n+1,m+1} + dv_{n,m+1} + cv_{n-1,m+1} \leq a^* v_{n+1,m} + d^* v_{n,m} + c^* v_{n-1,m}$$

with a, d, c, a^*, d^* e c^* as defined in eq. (13).

Assuming $Z_{n,m+1} = av_{n+1,m+1} + dv_{n,m+1} + cv_{n-1,m+1}$, we define A and b_{m+1} as

$$A = \begin{bmatrix} d^* & a^* & \ddots & 0 \\ c^* & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & a^* \\ 0 & \ddots & c^* & d^* \end{bmatrix} \qquad b_{m+1} = \begin{bmatrix} Z_{N^-+1,m+1} \\ \\ \vdots \\ \\ Z_{N^+-1,m+1} \end{bmatrix} - \begin{bmatrix} c^* g_{N^-,m} \\ 0 \\ \vdots \\ 0 \\ a^* g_{N^+,m} \end{bmatrix} \tag{16}$$

Applying the Projected SOR method, we obtain the equations that through iterations will give us a good approximation of the required points, $v_{n,m}$.

$$y_{n,m}^{k+1} = \frac{1}{d}(b_{n,m+1} - cv_{n-1,m}^{k+1} - av_{n+1,m}^{k})$$

$$v_{n,m}^{k+1} = max\{v_{n,m}^{k} + \omega(y_{n,m}^{k+1} - v_{n,m}^{k}), g_{n,m}\}$$

3.2.3 Numerical Results

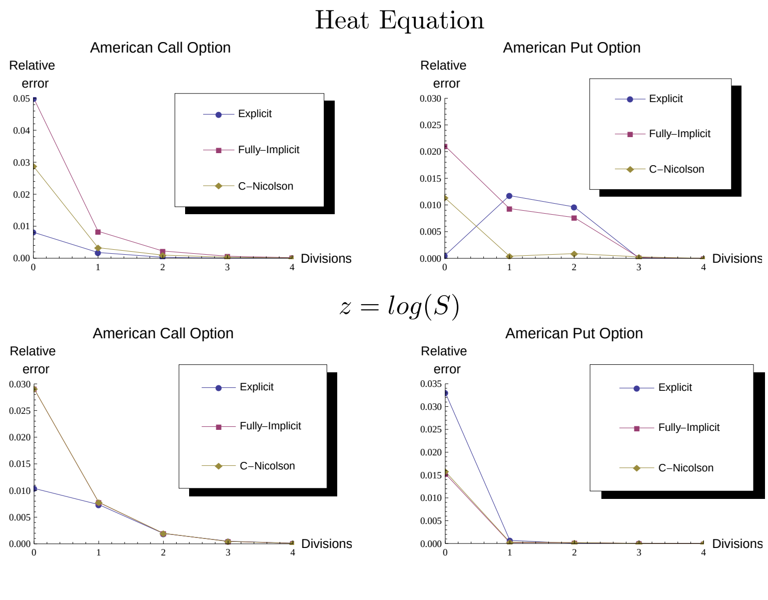

Figure 2: Convergence schemes obtained for American options.

Figure 2 shows the convergence of the three finite difference schemes for call and put American options, obtained through the transformation to the heat equation and the transformation $z = log(S)$. For both approaches, we approximate options with $S_0 = 1000$, $K = 1500$, $T = 10$, $r = 0.1$ and $\sigma = 0.4$. The graphs show the evolution of relative error with the decreasing spacing of the grid.

3.3 Barrier Options

Untill now we have considered vanilla options. In this section we will see how we can make the valuation of barrier options. The barrier options can be American and European, and what distinguishes them from the vanilla options, is the fact that a barrier B is introduced. When the asset value, S, reaches this barrier, the option is enabled (in the case of a Knock-In option) or ceases to exist (if it is a Knock-Out option).

Due to the way that this type of options is presented, it is quite easy to adapt the methods of finite differences used for vanilla options, it is enough to put one of the boundary conditions on the value of the barrier.

There is also the division in up-and-out and down-and-out options to distinguish whether the barrier is, respectively, above or below the present asset value, S_0. Here, we present the case of a Knock-Out-Barrier (when the asset value reaches B, option value turns to zero). Considering that $K(S,t)$ is the value of a barrier option and $V(S,t)$ is the value of the respective vanilla option, the boundary conditions for a Knock-Out are given by,

$$\begin{array}{ll} \textbf{Up-and-Out} & \textbf{Down-and-Out} \\ K(S,T) = V(S,T) & K(S,T) = V(S,T) \\ K(B,t) = 0 & K(B,t) = 0 \\ K(0,t) = V(0,t) & \lim_{S\to+\infty} K(S,t) = V(S,t) \end{array} \tag{17}$$

Figure 3 shows the graphs convergence of the approximation obtained for call and put down-and-out options. Both with the same characteristics: $S_0 = 1000$, $K = 1500$, $T = 10$ and $B = 500$. The interest-rate was fixed at $r = 10\%$.

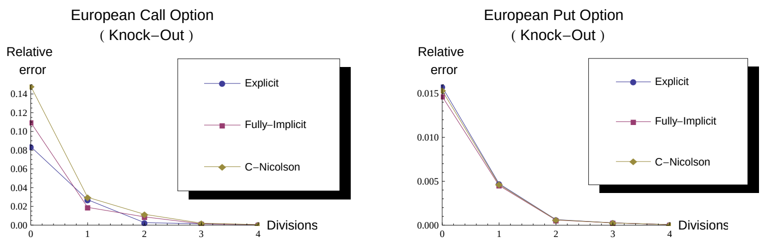

Figure 3: Convergence schemes obtained for Down-and-Out options.

4 Calibration of the Models

One of the assumptions of the Black-Scholes Model is that the variance of the process followed by the asset value is withdrawn from the market, assuming that is known and constant for any instant of time. Market volatility is not constant and can not be known *a priori*. The aim in this section is to adapt the Black-Scholes Model to market reality, that is, with variable volatility.

4.1 Local Volatility Model

The Local Volatility Model, states that from the option prices, $V(S,t)$ for all strike prices, K, and for all maturities, T, it is possible to determine a function $\sigma(S,t)$ such that, used in the Black-Scholes equation in place of σ, will be able to produce the initial $V(S,t)$.

In 1994, Dupire [4], Derman and Kani [2] showed that if the asset value follows the process

$$\frac{dS}{S} = rdt + \sigma(S,t)dW_t \tag{18}$$

then it can be determined $\sigma_L(K,T)$ from European options values for all K and T. The function $\sigma_L(K,T)$ is given by the Formula of Dupire (cf. [4] and [2]):

$$\sigma_L(K,T|S_0,t_0) = \sqrt{\frac{\frac{\partial C}{\partial T} + rK\frac{\partial C}{\partial K}}{\frac{1}{2}K^2\frac{\partial^2 C}{\partial K^2}}} \tag{19}$$

However this formula can produce some numerical instabilities and therefore in this paper we used a transformation of this formula that depends on the implied volatilities (cf. [5]).

Gatheral [5], also shows that the local volatility $\sigma(K,T)$ is the expected value of the instantaneous volatility conditional on the value $S = K$ when $t = T$. Thus, making a relabelling of variables, we obtain

$$\sigma(S,t|S_0,t_0) = \sqrt{\frac{\sigma_{imp}^2 + 2(t-t_0)\sigma_{imp}\frac{\partial\sigma_{imp}}{\partial t} + 2(r-D)S(t-t_0)\sigma_{imp}\frac{\partial\sigma_{imp}}{\partial S}}{\left(1 + Sd_1\sqrt{t-t_0}\frac{\partial\sigma_{imp}}{\partial S}\right)^2 + S^2(t-t_0)\sigma_{imp}\left(\frac{\partial^2\sigma_{imp}}{\partial S^2} - d_1(\frac{\partial\sigma_{imp}}{\partial S})^2\sqrt{t-t_0}\right)}} \tag{20}$$

with

$$d_1 = \frac{log(S_0/S) + \left(r + \frac{1}{2}\sigma_{imp}^2\right)\left(t-t_0\right)}{\sigma_{imp}\sqrt{t-t_0}} \tag{21}$$

4.2 Real data aplication

In this section we present an example of how to do the calibration of the BlackScholes Model. Were taken from the CBOE[2] options data on the index $S\&P$ 500, whose asset value in August 2004 was $S_0 = 1099.15$ and the volume of transactions was more than 50 options (cf. data table in main work).

Then we calculated implied volatilities of this data through Newton-Raphson's iterative method

Because the available data did not cover the domain required for the finite differences method, we made some interpolation and extrapolation of the implied volatility by fitting a multiquadric RBF expansion (cf. [6]). Thus, we generated a suitable implied volatility surface.

Based on the implied volatilities, we applied the formula of Dupire depending on the implied volatilities 20. The figure 4.2 shows the implied and local volatility surfaces.

Implied Volatility $\longrightarrow$ Local Volatility

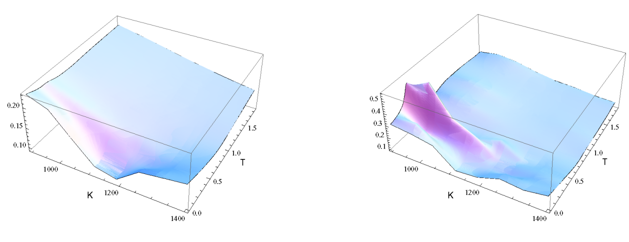

Figure 4: Implied and local volatility surfaces

Determined the local volatility surface with the same domain as needed in the finite differences method, we can solve the equation of Black-Scholes model for σ variable, replacing σ by $\sigma(S,t)$, and find the value of the options we want.

Table 1 shows the approximated values of some call and put options and their relative errors.

[2]http://www.marketdataexpress.com/

Scheme	Strike Price	Maturity Date	Call Option	Relative Error (C)	Put Option	Relative Error (P)
Explicit				0.00334635		0.130117
Fully-Implicit	950	3/12	153.6	0.00322917	4.4277	0.125901
C-Nicolson				0.00328776		0.127999
Explicit				0.518668		0.00105554
Fully-Implicit	1400	7/12	0.5	0.50404	301.268	0.0010323
C-Nicolson				0.511136		0.00104226
Explicit				0.00419035		0.0093985
Fully-Implicit	1005	16/12	151.3	0.00393919	57.02	0.0093985
C-Nicolson				0.00406477		0.00973693

Table 1: Legenda

5 Conclusions and future work

In this work we present two ways of resolution of the problem for European and American option pricing (vanilla and barrier options) through the use of finite differences method: transformation of the Black-Scholes equation into the heat equation and by $z = log(S)$. However, we conclude that the first approach is not suitable when we need to apply this method to real situations. Finally, we proposed a way to calibrate the Black-Scholes Model with a real data application.

As future work we pretend to use Adaptative Mesh Model to make the refinement of the grid in the most interesting points, thereby improving the use of the finite difference method. The study of Simulated Annealing Method or similar methods could be considered in future work as a way to find the parameter needed to apply the RBF approach.

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