CAPM (SML) 의 도출

weights $\vec{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ $(\Sigma x_i = 1)$, mean returns $\vec{R} = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_n \end{bmatrix}$, $V = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{12} & \sigma_{22} & & \\ \vdots & & \ddots & \\ \sigma_{1n} & & & \sigma_{nn} \end{bmatrix}$ (non-singular)

<Mean-Variance Portfolio Selection>

$$\min_{\vec{X}} \sigma_p^2 = \vec{X}^T V \vec{X}$$

$$\text{s.t.} \begin{cases} \vec{X}^T \vec{R} = R_p \\ \vec{X}^T \vec{1} = 1 \end{cases}$$

sol) $\min_{\vec{X}, \lambda_1, \lambda_2} \mathcal{L} = \vec{X}^T V \vec{X} + \lambda_1 (R_p - \vec{X}^T \vec{R}) + \lambda_2 (1 - \vec{X}^T \vec{1})$

FOC) $\partial \mathcal{L} / \partial \vec{X} = 2V\vec{X} - \lambda_1 \vec{R} - \lambda_2 \vec{1} = \vec{0}$ ... ①

$\partial \mathcal{L} / \partial \lambda_1 = R_p - \vec{X}^T \vec{R} = 0$ ... ②

$\partial \mathcal{L} / \partial \lambda_2 = 1 - \vec{X}^T \vec{1} = 0$ ... ③

$\begin{bmatrix} R_p \\ 1 \end{bmatrix} = [\vec{R} \ \vec{1}]^T \vec{X}$

from ①, $\vec{X} = \frac{1}{2} V^{-1} [\vec{R} \ \vec{1}] \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$ ... ①′

$$[\vec{R} \ \vec{1}]^T \vec{X} = \frac{1}{2} \underbrace{[\vec{R} \ \vec{1}]^T V^{-1} [\vec{R} \ \vec{1}]}_{\equiv A} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$$

$$A = \begin{bmatrix} \vec{R}^T V^{-1} \vec{R} & \vec{R}^T V^{-1} \vec{1} \\ \vec{R}^T V^{-1} \vec{1} & \vec{1}^T V^{-1} \vec{1} \end{bmatrix} = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

$\therefore \begin{bmatrix} R_p \\ 1 \end{bmatrix} = \frac{1}{2} A \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$

$\because$ symmetric, pos. def $\Big)$ invertible

$A^{-1} \begin{bmatrix} R_p \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$

from ①′, $\vec{X}^* = V^{-1} [\vec{R} \ \vec{1}] A^{-1} \begin{bmatrix} R_p \\ 1 \end{bmatrix}$: Minimum Variance PRT

<MV efficient frontier>

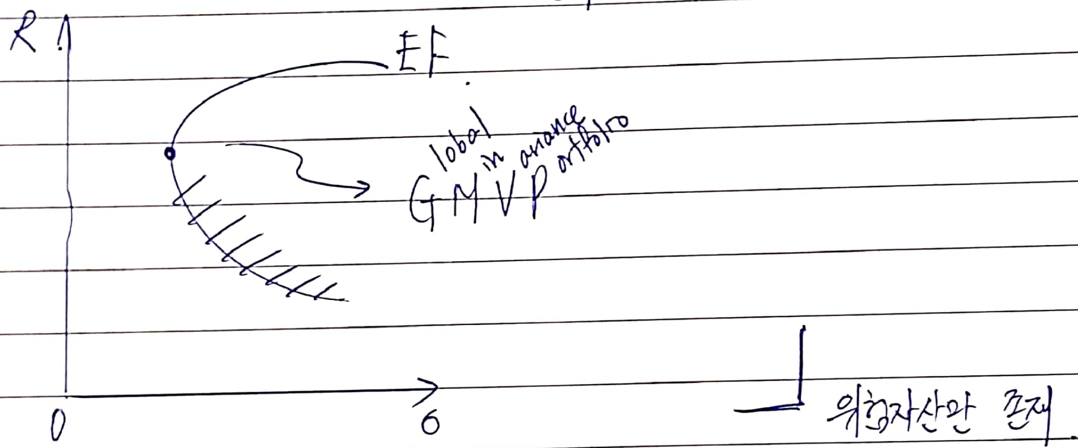
$$\sigma_p^2 = \vec{x}^{*T} V \vec{x}^{*} = \left[[R_p \ \ 1]\underbrace{(A^{-1})^T}_{=A^{-1}}[\vec{R} \ \ \vec{1}]^T\underbrace{(V^{-1})^T}_{=V^{-1}}\right] \cancel{V} \left[V^{-1}[\vec{R} \ \ \vec{1}]A^{-1}\begin{bmatrix} R_p \\ 1 \end{bmatrix}\right]$$

$$\underbrace{[\vec{R} \ \ \vec{1}]^T V^{-1} [\vec{R} \ \ \vec{1}]}_{= \cancel{A}}$$

$$= [R_p \ \ 1] A^{-1} \begin{bmatrix} R_p \\ 1 \end{bmatrix}$$

$$= [R_p \ \ 1] \frac{1}{ac-b^2} \begin{bmatrix} c & -b \\ -b & a \end{bmatrix} \begin{bmatrix} R_p \\ 1 \end{bmatrix}$$

$$\therefore \sigma_p^2 = \frac{a - 2bR_p + cR_p^2}{ac-b^2}$$ (쌍곡선 형태)



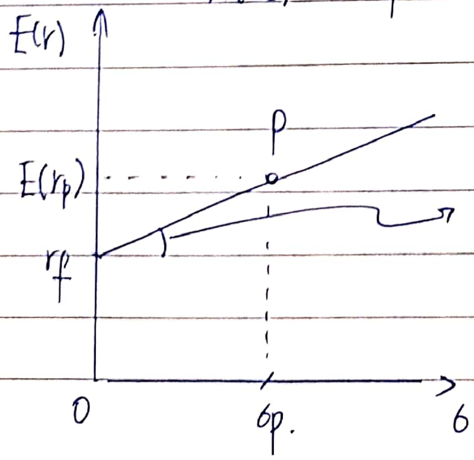
<Capital Allocation> (btw risky & risk-free)

complete PRT C = risky P + risk-free F

→ $r_c = \underset{\text{비중}}{y} \cdot r_p + (1-y) r_F$

$$\begin{cases} E(r_c) = r_f + y[E(r_p) - r_f] \\ \sigma_c = y \cdot r_p \end{cases}$$

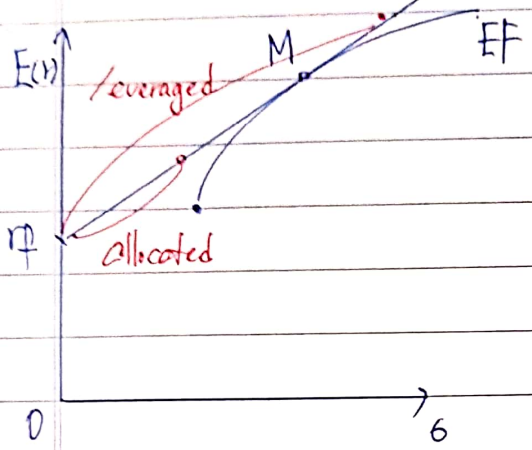
$$\therefore E(r_c) = r_f + [E(r_p) - r_f] \cdot \frac{\sigma_c}{\sigma_p}$$: CAL (capital allocation line)



slope = $\frac{E(r_p) - r_f}{\sigma_p}$: Sharpe Ratio.

apital orket ine

< CML > : Optimal CAL, CAL(M) : highest Sharpe Ratio.



< SML 의 도출 >

p = risk-free F + risky i + market M

$$E(r_p) = w_f \cdot r_f + w_i \cdot E(r_i) + w_M E(r_M) \quad (w_f + w_i + w_M = 1)$$

$$\sigma_p^2 = w_i^2 \sigma_i^2 + w_M^2 \sigma_M^2 + 2 w_i w_M Cov(r_i, r_M)$$

$$E(r_p) = (1 - w_i - w_M) r_f + w_i E(r_i) + w_M E(r_M)$$

$$= r_f + w_i [E(r_i) - r_f] + w_M [E(r_M) - r_f]$$

risk-return tradeoff)

$$\frac{\partial E(r_p) / \partial w_i}{\partial \sigma_p^2 / \partial w_i} = \frac{E(r_i) - r_f}{2 w_i \sigma_i^2 + 2 w_M Cov(r_i, r_M)}$$

At equilibrium total borrowing = total lending ($\Leftrightarrow w_M = 1$)

$$\rightarrow \text{tradeoff} = \frac{E(r_i) - r_f}{2 Cov(r_i, r_M)}$$

tradeoff for portfolio M)

$$\frac{\partial E(r_p) / \partial w_M}{\partial \sigma_p^2 / \partial w_M} = \frac{E(r_M) - r_f}{2 w_M \sigma_M^2 + 2 w_i Cov(r_i, r_M)} \rightarrow \frac{E(r_M) - r_f}{2 \sigma_M^2}$$

Tradeoff must be the same for all assets

$$\therefore \frac{E(r_i) - r_f}{\cancel{2} Cov(r_i, r_M)} = \frac{E(r_M) - r_f}{\cancel{2} \sigma_M^2}, \quad E(r_i) - r_f = \underbrace{\frac{Cov(r_i, r_M)}{\sigma_M^2}}_{:\ \beta} [E(r_M) - r_f]$$