Partial differentiation

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Q1. if $u = x^2y + y^2z + z^2x$ then prove that,

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = (x+y+z)^2.$$

$\Rightarrow$ $u = x^2y + y^2z + z^2x$ — (1)

Partially diff. eqn (1) w.r.t. x by keeping y & z as const.

$$\frac{\partial u}{\partial x} = (2x)y + 0 + z^2 \cdot 1$$

$$\frac{\partial u}{\partial x} = 2xy + z^2 \quad \text{— (A)}$$

Partially diff. eqn (1) w.r.t. y by keeping x & z as const.

$$\frac{\partial u}{\partial y} = x^2 \cdot 1 + 2y \cdot z + 0.$$

$$\frac{\partial u}{\partial y} = x^2 + 2yz \quad \text{— (B)}$$

Partially diff. eqn (1) w.r.t. z by keeping x & y as const.

$$\frac{\partial u}{\partial z} = 0 + y^2 \cdot 1 + 2z \cdot x$$

$$\frac{\partial u}{\partial z} = y^2 + 2zx \quad \text{— (C)}$$

Adding eqn (A), (B) and (C)

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$$

$$= (x+y+z)^2$$

Hence proved

Chain Rule :

Q1. If $u = x^2+y^2$, $x = s+3t$, $y = 2s-t$

Then find the values of $\frac{\partial u}{\partial s}, \frac{\partial u}{\partial t}, \frac{\partial^2 u}{\partial s^2}, \frac{\partial^2 u}{\partial t^2}$

Soln: Given

$u = x^2+y^2$
$x = s+3t$
$y = 2s-t$

from these eqns we can say that $u = f(x,y) \Rightarrow u$ is a fn of x & y. and x & y are the fn of s & t.

$$u \rightarrow x, y \rightarrow s, t$$

By chainrule:

$$u \rightarrow x, y \rightarrow s, t$$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial s} \quad \text{--- (1)}$$

Similarly,

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial t} \quad \text{--- (2)}$$

$u = x^2+y^2$	$x = s+3t$	$y = 2s-t$
$\frac{\partial u}{\partial x} = 2x+0$	$\frac{\partial x}{\partial s} = 1.$	$\frac{\partial y}{\partial x} = 2$
$\frac{\partial u}{\partial y} = 2y+0$	$\frac{\partial x}{\partial t} = 3$	$\frac{\partial y}{\partial t} = -1$

Eq (1) $\Rightarrow$ $\frac{\partial u}{\partial s} = (2x \times 1) + (2y \times 2)$

$$\frac{\partial u}{\partial s} = 2x + 4y \quad \text{--- (A)}$$

Eqn (2) $\Rightarrow \frac{\partial u}{\partial t} = 2x \times 3 + 2y(-1)$

$$\frac{\partial u}{\partial t} = 6x - 2y \quad \text{(B)}$$

partially diff eq (A) w.r.t. s, we get:

now,

$$\frac{\partial^2 u}{\partial s^2} = 2 \cdot \frac{\partial x}{\partial s} + 4 \cdot \frac{\partial y}{\partial s}$$

$$= 2 \times 1 + 4 \times 2$$

$$\frac{\partial^2 u}{\partial s^2} = 10$$

partially diff eq (B) wrt. t, we get:

$$\frac{\partial^2 u}{\partial t^2} = 6 \frac{\partial x}{\partial t} - 2 \frac{\partial y}{\partial t}$$

$$= 6 \times 3 - 2(-1)$$

$$= 18 + 2 = 20$$

$$\frac{\partial^2 u}{\partial t^2} = 20$$

Note: If $z = f(x,y)$

$x = \phi(u,v)$

$y = \psi(u,v)$

$$\rightarrow \frac{\partial z}{\partial u} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$\rightarrow \frac{\partial z}{\partial v} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial v}$$

Q. If $z = f(x,y)$, $x = e^{2u} + e^{-2v}$

$y = e^{-2u} + e^{2v}$

then prove that

$$\frac{\partial f}{\partial u} - \frac{\partial f}{\partial v} = 2\left[x\frac{\partial f}{\partial x} - y\frac{\partial f}{\partial y}\right]$$

$$\frac{\partial z}{\partial u} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$= \frac{\partial f}{\partial x} \cdot (2 \cdot e^{2u}) + \frac{\partial f}{\partial y} \cdot (-2e^{-2u})$$

$$\frac{\partial z}{\partial u} = 2\left[e^{2u} \cdot \frac{\partial f}{\partial x} - e^{-2u} \cdot \frac{\partial f}{\partial y}\right] \quad \text{--- (1)}$$

and $$\frac{\partial z}{\partial v} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial v}$$

$$= \frac{\partial f}{\partial x}\left[(-2)e^{-2v}\right] + \frac{\partial f}{\partial y} \cdot \left(2 \cdot e^{2v}\right]$$

$$\frac{\partial z}{\partial v} = 2\left[-\frac{\partial f}{\partial x}e^{-2v} + \frac{\partial f}{\partial y}e^{2v}\right] \quad \text{—— (2)}$$

Now, subtract (1)-(2):

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = 2\left[(e^{2u} + e^{-2v})\frac{\partial f}{\partial x} - (e^{-2v} + e^{2v})\frac{\partial f}{\partial y}\right]$$

$$\frac{\partial f}{\partial u} - \frac{\partial f}{\partial v} = 2\left[x\frac{\partial f}{\partial x} - y\cdot\frac{\partial f}{\partial y}\right]$$

Hence proved

—— x ——

Q. Let f be a composite function of u and v, and u, v be the function of x, y given by.

$$u = x^2 - y^2$$
$$v = 2xy.$$

Prove that:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 4(x^2+y^2)\left(\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2}\right)$$

Soln:

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\cdot(2x) + \frac{\partial f}{\partial v}\cdot(2y + 2x(0)) \quad \text{—— (1)}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$$= \frac{\partial f}{\partial u}\cdot(-2y) + \frac{\partial f}{\partial v}\cdot(2x) \quad \text{—— (2)}$$

$$\frac{\partial f}{\partial y} = 2\left[x\cdot\frac{\partial f}{\partial v} - y\,\frac{\partial f}{\partial u}\right] \Bigg| \frac{\partial}{\partial y} = 2\left(x\cdot\frac{\partial}{\partial v} - y\frac{\partial}{\partial u}\right)$$

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$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\cdot(2x) + \frac{\partial f}{\partial v}(2y)$$

$$\frac{\partial f}{\partial x} = 2\left[x\cdot\frac{\partial f}{\partial u} + y\cdot\frac{\partial f}{\partial v}\right]$$

$$\Rightarrow \frac{\partial}{\partial x} = 2\left[x\cdot\frac{\partial}{\partial u} + y\cdot\frac{\partial}{\partial v}\right]$$

and also,

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \quad ***$$

$$\frac{\partial^2 f}{\partial x^2} = \left[2\left(x\cdot\frac{\partial}{\partial u} + y\cdot\frac{\partial}{\partial v}\right)\right]\left[2\left(x\cdot\frac{\partial f}{\partial u} + y\cdot\frac{\partial f}{\partial v}\right)\right]$$

$$= 4\left\{x\cdot\frac{\partial}{\partial u}\left(x\frac{\partial f}{\partial u} + y\frac{\partial f}{\partial v}\right) + y\cdot\frac{\partial}{\partial v}\left(x\cdot\frac{\partial f}{\partial u} + y\cdot\frac{\partial f}{\partial v}\right)\right.$$

$$= 4\left\{x^2\left[\left(\frac{\partial^2 f}{\partial u^2}\right) + x\cdot y\left(\frac{\partial^2 f}{\partial u\cdot\partial v}\right)\right] + yx\frac{\partial^2 f}{\partial v\cdot\partial u} + y^2\cdot\frac{\partial^2 f}{\partial v^2}\right.$$

$$\left\{\frac{\partial^2 f}{\partial x^2} = 4x^2\frac{\partial^2 f}{\partial u^2} + 8xy\frac{\partial^2 f}{\partial u\cdot\partial v} + 4y^2\frac{\partial^2 f}{\partial v^2}\right\} \quad \text{--- (3)}$$

from eq (2) & (*):

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

$$= 2\left(x\cdot\frac{\partial}{\partial v} - y\frac{\partial}{\partial u}\right)2\left(x\cdot\frac{\partial f}{\partial v} - y\frac{\partial f}{\partial u}\right)\Big]$$

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$$= 4\left\{x\cdot\frac{\partial}{\partial v}\left(x\cdot\frac{\partial f}{\partial v} - y\frac{\partial f}{\partial u}\right) - y\cdot\frac{\partial}{\partial u}\left(x\cdot\frac{\partial f}{\partial v} - y\cdot\frac{\partial f}{\partial u}\right)\right\}$$

$$= 4\left\{x^2\left(\frac{\partial^2 f}{\partial v^2} - xy\frac{\partial^2 f}{\partial v\cdot\partial u}\right] - \left(y^2\frac{\partial^2 f}{\partial u\cdot\partial v} + y^2\cdot\frac{\partial^2 f}{\partial u^2}\right)\right\}$$

$$\frac{\partial^2 f}{\partial y^2} = 4x^2\frac{\partial^2 f}{\partial v^2} - 8xy\frac{\partial^2 f}{\partial u\cdot\partial v} + 4y^2\cdot\frac{\partial^2 f}{\partial u^2} \quad \text{--- (4)}$$

now, Add eq (3) & (4):

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 4(x^2+y^2)\left[\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2}\right]$$

Hence proved.

Exam imp. Q Transform the equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$ into polar co-ordinates r, θ.

$x = r\cos\theta$

$y = r\sin\theta$

$x^2+y^2 = r^2$

$\theta = \tan^{-1}(y/x)$

$$2r\cdot\frac{\partial r}{\partial x} = 2x$$

$$\frac{\partial r}{\partial x} = x/r = \cos\theta$$

and $$2r\cdot\frac{\partial r}{\partial y} = 2y$$

$$\frac{\partial r}{\partial y} = y/r = \sin\theta.$$

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$\cancel{= \cos^2\theta}$

$$\cancel{= \cos\theta \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\cdot\cos\theta + \left(\frac{1}{1+(y/x)^2}\right)\cdot\frac{\partial u}{\partial \theta}\right)}$$

$$\cancel{+ \frac{1}{1+(y/x)^2}\frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial x}\cdot\cos\theta + \frac{1}{1+(y/x)^2}\cdot\frac{\partial u}{\partial \theta}\right)}$$

as

$$\theta = \tan^{-1}(y/x)$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1+(y/x)^2}\left(-y/x^2\right) = -\frac{y}{x^2+y^2} = -\frac{r\sin\theta}{r^2}$$

$$\frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial x}$$

$$= \frac{\partial u}{\partial r}\cdot\cos\theta + \frac{\partial u}{\partial \theta}\left(-\frac{\sin\theta}{r}\right)$$

$$\frac{\partial}{\partial x} = \cos\theta\cdot\frac{\partial}{\partial x} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \left(\cos\theta\frac{\partial}{\partial x} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\cos\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right)$$

$$= \cos\theta\frac{\partial}{\partial r}\left(\cos\theta\cdot\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right) - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right)$$

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$$= \cos\theta\left(\frac{\partial^2 u}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial u}{\partial \theta} - \frac{\sin\theta}{r}\frac{\partial^2 u}{\partial r \partial \theta}\right)$$

$$- \frac{\sin\theta}{r}\left[-\sin\theta\frac{\partial u}{\partial r} + \cos\theta\frac{\partial^2 u}{\partial\theta\cdot\partial r} - \frac{\cos\theta}{r}\frac{\partial u}{\partial\theta} - \frac{\sin\theta}{r}\frac{\partial^2 u}{\partial\theta^2}\right]$$

$$= \left(\cos\theta\frac{\partial^2 u}{\partial r^2} + \frac{\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial\theta} - \frac{\sin\theta\cos\theta}{r}\cdot\frac{\partial^2 u}{\partial r\partial\theta}\right) +$$

$$\left(\frac{\sin^2\theta}{r}\frac{\partial u}{\partial r} - \frac{\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial\theta\cdot\partial x} + \frac{\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial\theta} + \frac{\sin^2\theta}{r^2}\cdot\frac{\partial^2 u}{\partial\theta^2}\right)$$

$$\frac{\partial^2 u}{\partial x^2} = \left[\cos\theta\frac{\partial^2 u}{\partial r^2} - \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial^2 u}{\partial r\cdot\partial\theta} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{2\sin\theta\cos\theta}{r}\frac{\partial u}{\partial\theta} + \frac{\sin^2\theta}{r}\frac{\partial u}{\partial r}\right] \quad \text{--- (1)}$$

now,

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial\theta}\cdot\frac{\partial\theta}{\partial y}$$

as $\theta = \tan^{-1}(y/x)$

$$\frac{\partial\theta}{\partial y} = \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right)$$

$$= \frac{x}{x^2+y^2} = \frac{\cos\theta}{r}$$

$r^2 = x^2+y^2$

diff wrt 'y' taking 'x' as const.

$2r\cdot\frac{\partial r}{\partial y} = 2y$

$\frac{\partial r}{\partial y} = y/r \sin\theta$

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$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial y}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}(\sin\theta) + \frac{\partial u}{\partial \theta}\left(\frac{\cos\theta}{r}\right) \quad \text{—①}$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial r}(\sin\theta) + \frac{\partial}{\partial \theta}\left(\frac{\cos\theta}{r}\right) \quad \text{—②}$$

and

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$$

$$= \left(\frac{\partial}{\partial r}\sin\theta + \frac{\partial}{\partial \theta}\left(\frac{\cos\theta}{r}\right)\right)\left(\frac{\partial u}{\partial r}\sin\theta + \frac{\partial u}{\partial \theta}\left(\frac{\cos\theta}{r}\right)\right)$$

$$= \frac{\partial}{\partial r}\sin\theta\left(\frac{\partial u}{\partial r}\sin\theta + \frac{\partial u}{\partial \theta}\left(\frac{\cos\theta}{r}\right)\right) + \frac{\partial}{\partial \theta}\left(\frac{\cos\theta}{r}\right)\left(\frac{\partial u}{\partial r}\sin\theta + \frac{\partial u}{\partial \theta}\left(\frac{\cos\theta}{r}\right)\right)$$

$$= \sin\theta\left\{\sin\theta\frac{\partial^2 u}{\partial r^2} - \frac{\cos\theta}{r^2}\frac{\partial u}{\partial \theta} + \frac{\cos\theta}{r}\frac{\partial^2 u}{\partial r\partial \theta}\right\} + \frac{\cos\theta}{r}\left\{\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta} + \frac{\cos\theta}{r}\frac{\partial^2 u}{\partial \theta^2}\right\}$$

$$= \sin^2\theta\frac{\partial^2 u}{\partial r^2} - \frac{\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial \theta} + \frac{\cos\theta\sin\theta}{r}\frac{\partial^2 u}{\partial r\partial \theta} +$$

$$+ \frac{\cos^2\theta}{r}\frac{\partial u}{\partial r} - \frac{\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial \theta} + \frac{\cos^2\theta}{r^2}\cdot\frac{\partial^2 u}{\partial \theta^2}.$$

$$\frac{\partial^2 u}{\partial y^2} = \sin^2\theta \frac{\partial^2 u}{\partial r^2} - \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial \theta} + \frac{\cos\theta\sin\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta} + \frac{\cos^2\theta}{r}\frac{\partial u}{\partial r}$$

$$+ \frac{\cos^2\theta}{r^2}\frac{\partial^2 u}{\partial \theta^2} \quad \text{--- (iii)}$$

now adding $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$, we get:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \left(\sin^2\theta + \cos^2\theta\right)\frac{\partial^2 u}{\partial r^2} + \frac{\sin^2\theta + \cos^2\theta}{r^2}\frac{\partial^2 u}{\partial \theta^2} +$$

$$+ \frac{\sin^2\theta + \cos^2\theta}{r}\frac{\partial u}{\partial r}$$

$$= \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r}\frac{\partial u}{\partial r}$$

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Engineering Math -I

Differentials:

$$\frac{\Delta y}{\Delta x} = \frac{dy}{dx} \text{ (Approximately)}$$

$$\Delta y = \frac{dy}{dx} \cdot \Delta x$$

$$dy = f'(x)\Delta x$$ → differential approximation in one variable.

if $z = f(x,y)$

$x = \phi(t)$

$y = \psi(t)$

the $$\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

$$df = \frac{\partial f}{\partial x} \cdot dx + \frac{\partial f}{\partial y} \cdot dy$$

Total differential

Q. find the total differentials of the following functions

① $z = \tan^{-1}(x/y)$, $(x,y) \neq (0,0)$

$$\frac{\partial z}{\partial x} = \frac{1}{1+\frac{x^2}{y^2}}(1/y) = \frac{y}{x^2+y^2}$$

$$\frac{\partial z}{\partial y} = \frac{1}{1+\frac{x^2}{y^2}}\left(\frac{-x}{y^2}\right) = \frac{-x}{x^2+y^2}$$

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$$dz = \frac{\partial f}{\partial x} \cdot dx + \frac{\partial f}{\partial y} dy$$

$$dz = \frac{1}{x^2+y^2} (y\,dx - x\,dy)$$

Ans

(2) $u = \left(xz + \frac{x}{z}\right)^y$, $z \neq 0$.

Soln

$$u = f(x, y, z)$$

$$du = \frac{\partial f}{\partial x} \cdot dx + \frac{\partial f}{\partial y} \cdot dy + \frac{\partial f}{\partial z} \cdot dz.$$

$$\frac{\partial f}{\partial x} = y\left(xz + \frac{x}{z}\right)^{y-1} \left(z + 1/z\right)$$

$$\frac{\partial f}{\partial y} = \left(xz + \frac{x}{z}\right)^y \log_e\left(xz + x/z\right)$$

$$\frac{\partial f}{\partial z} = y\left(xz + \frac{x}{z}\right)^{y-1} \left(x + \left(\frac{-x}{z^2}\right)\right)$$

$$du = \left[y\left(xz + \frac{x}{z}\right)^{y-1}\left(z + \frac{1}{z}\right)\right] dx + \left[\left(xz + \frac{x}{z}\right)^y \log_e\left(xz + \frac{x}{z}\right)\right] dy$$

$$+ \left[y\left(xz + \frac{x}{z}\right)^{y-1}\left(x + \left(\frac{-x}{z^2}\right)\right)\right] dz$$

$$du = \left(xz+\frac{x}{2}\right)^{y-1}\left[y\left(z+\frac{1}{2}\right)dx + xy\left(1-\frac{1}{z^2}\right)dz\right] + \left(xz+\frac{x}{2}\right)^{y}\log\left(xz+\frac{x}{2}\right)$$

Ans

25/11/22 (Eng. Math-I)

Change in Variables:

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

$$\begin{array}{ccccccc} & x & & y & & 1 & \\ b_1 & & c_1 & & a_1 & & b_1 \\ b_2 & & c_2 & & a_2 & & b_2 \end{array}$$

Jacobian :

i) if u and v are functions of two independent variable x and y then the determinant :

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

is called the jacobian of u, v w.r.t x, y.

And is denoted by :

$$J\left(\frac{u,v}{x,y}\right) \quad \text{or} \quad \frac{\partial(u,v)}{\partial(x,y)}$$

ii) Similarly Jacobian of u, v, w w.r.t x, y, z :

$$J\left(\frac{u,v,w}{x,y,z}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

Cramer's Rule

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \quad \Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

$$\Delta_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}, \quad \Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

then $x = \dfrac{\Delta_1}{\Delta}$

$y = \dfrac{\Delta_2}{\Delta}$

and $z = \dfrac{\Delta_3}{\Delta}$

Q If $z = f(x, y)$

$x = r\cos\theta$

$y = r\sin\theta$

Then show that:

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = \left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial f}{\partial \theta}\right)^2$$

Soln: $\dfrac{\partial f}{\partial x} = \dfrac{1}{J}\left[\dfrac{\partial(f, y)}{\partial(u, v)}\right]$, $\dfrac{\partial f}{\partial y} = \dfrac{1}{J}\left(\dfrac{\partial(f, x)}{\partial(u, v)}\right)$

$x = r\cos\theta$
$y = r\sin\theta$

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$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$r^2 = x^2 + y^2$

$2r\frac{\partial y}{\partial x} = 2x$

$\frac{\partial y}{\partial x} = \frac{x}{r} = \cos\theta,$

$\frac{\partial r}{\partial y} = y/r = \sin\theta$

$\theta = \tan^{-1}(y/x)$

$\frac{\partial \theta}{\partial x} = \frac{1}{1+(y/x)^2}\left(\frac{y}{-x^2}\right)$

$= \frac{-y}{x^2+y^2} = \frac{-\sin\theta}{r}$

Similarly,

$\frac{\partial \theta}{\partial y} = \frac{1}{1+(y/x)^2}(1/x)$

$= \frac{x}{x^2+y^2} = \frac{\cos\theta}{r}$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\left[\cos^2\theta + \sin^2\theta\right]$$

$= r.$

$$\frac{\partial(f,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial f}{\partial r} & \partial f/\partial\theta \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \frac{\partial f}{\partial r} & \partial f/\partial\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos\theta\,\frac{\partial f}{\partial \theta} - \sin\theta\cdot\frac{\partial f}{\partial \theta}$$

$$\frac{f(f,x)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial f}{\partial r} & \partial f/\partial\theta \\ \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \frac{\partial f}{\partial r} & \frac{\partial f}{\partial \theta} \\ \cos\theta & -r\sin\theta \end{vmatrix} = -r\sin\theta\frac{\partial f}{\partial r} - \cos\theta\frac{\partial f}{\partial \theta}.$$

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$$\frac{\partial f}{\partial x} = \frac{1}{J}\left[\frac{\partial(f,y)}{\partial(r,\theta)}\right]$$

$$= \frac{1}{r}\left[r\cos\theta\frac{\partial f}{\partial r} - \sin\theta\frac{\partial f}{\partial \theta}\right]$$

$$= \cos\theta\frac{\partial f}{\partial r} - \frac{\sin\theta}{r}\frac{\partial f}{\partial \theta}$$

$$\frac{\partial f}{\partial y} = \frac{\partial(f,x)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial f}{\partial r} & \frac{\partial f}{\partial \theta} \\ \frac{\partial x}{\partial r} & \frac{\partial x}{\partial y} \end{vmatrix}$$

$$= \sin\theta\frac{\partial f}{\partial r} + \frac{\cos\theta}{r}\cdot\frac{\partial f}{\partial \theta}$$

28/11/22

Engineering Math

Maximum, minimum value of the function:

$z = f(x,y)$ is defined and continuous in some close and bounded region R. Let (a,b) an interior pt. of R and $(a+h, b+k)$ be a point in its neighborhood and lies inside R,

Then (i) The pt (a,b) is called a point of relative (local) min,

if $f(a+h, b+k) \geq f(a,b)$ $\forall (h,k)$

(ii) $f(a+h, b+k) \leq f(a,b)$ max.

⇒ Second derivative test:

$$\left.\begin{aligned} p &= \frac{\partial f}{\partial x} = 0 \\ q &= \frac{\partial f}{\partial y} = 0 \end{aligned}\right\} \text{Solve}$$

$$r = \frac{\partial^2 f}{\partial x^2}$$

$$t = \frac{\partial^2 f}{\partial y^2}$$

$$s = \frac{\partial^2 f}{\partial x \cdot \partial y}$$

then,

$$\left.\begin{aligned} rt - s^2 &> 0 \\ r &< 0 \end{aligned}\right\} \text{then } (a,b) \text{ is the point of max.}$$

and,

$$\left.\begin{aligned} rt - s^2 &> 0 \\ r &> 0 \end{aligned}\right\} \text{then } (a,b) \text{ is the point of min}$$

Extremum : Maximum & minimum Collectively called as extremum.

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and if $rt - s^2 < 0$ } (a,b) is called the saddle point.

—— x ——

Q. find the local extreme values of the function

$$f(x,y) = xy - x^2 - y^2 - 2x - 2y + 4.$$

$$\frac{\partial f}{\partial x} = y - 2x - 2$$

$$\frac{\partial f}{\partial y} = x - 2y - 2$$

for maxima & minima

$$\frac{\partial f}{\partial x} = 0 \quad , \quad \frac{\partial f}{\partial y} = 0.$$

$$\Rightarrow \quad -2x + y - 2 = 0$$
$$x - 2y - 2 = 0$$

$$\frac{x}{\cancel{-4+}\ -2-4} = \frac{-y}{(4+2)} = \frac{1}{+4-1}$$

$$\frac{x}{-6} = \frac{y}{-6} = \frac{1}{3}$$

$$x = -2$$
$$y = -2$$

$$r = \frac{\partial^2 f}{\partial x^2} = -2$$

$$s = \frac{\partial^2 f}{\partial x \cdot \partial y} = 1$$

$$t = \frac{\partial^2 f}{\partial y^2} = -2$$

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$$rt - s^2 = (-2)(-2) - 1^2$$
$$= 4 - 1 = 3 > 0$$
$$r = -2 < 0$$

$\therefore$ $(-2, -2)$ is point of maxima

and maximum value is

$$f(-2,-2) = \cancel{4} - \cancel{4} - \cancel{4} + \cancel{4} + 4 + 4$$
$$= \underline{8}$$

$\Rightarrow$ <u>Lagrange's method of undermined multipliers</u>

$z = f(x, y)$

$$u = f(x, y, z) \quad -(1)$$

$$\phi(x, y, z) = 0 \quad -(2)$$

$$\frac{\partial f}{\partial x} + \lambda \cdot \frac{\partial \phi}{\partial x} = 0 \quad -(3)$$

$$\frac{\partial f}{\partial y} + \lambda \cdot \frac{\partial \phi}{\partial y} = 0 \quad -(4)$$

$$\frac{\partial f}{\partial z} + \lambda \cdot \frac{\partial \phi}{\partial z} = 0 \quad -(5)$$

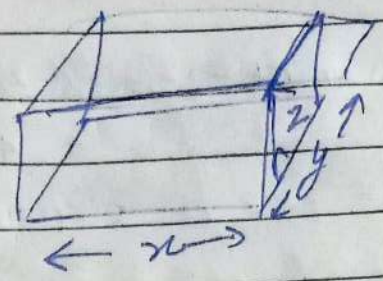
Q. A rectangular box open at the top is to have a volume of 32 cubic ft. find the dimension of the box requiring least material for its construction.

Date

Soln:

$$xyz = 32$$

$$z = \frac{32}{xy}$$



$$S = xy + 2yz + 2zx$$

$$= xy + 2\cancel{y}\left(\frac{32}{x\cancel{y}}\right) + 2\cancel{x}\left(\frac{32}{\cancel{x}y}\right)$$

$$S = xy + \frac{64}{x} + \frac{64}{y}$$

$$\frac{\partial S}{\partial x} = y - \frac{64}{x^2}$$

$$\frac{\partial S}{\partial y} = x - \frac{64}{y^2}$$

and now, for maxima & minima

$$\frac{\partial S}{\partial x} = 0 \quad , \frac{\partial S}{\partial y} = 0.$$

$$y - \frac{64}{x^2} = 0 \quad , x - \frac{64}{y^2} = 0.$$

$$x^2y - 64 = 0 \quad , xy^2 - 64 = 0.$$

$$x = y = 4 \Big\} \text{ critical points}$$

and $z = \frac{\cancel{xy}}{\cancel{2}} \frac{32}{xy} = \frac{32}{16}$

$$z = 2$$

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$$\frac{\partial^2 S}{\partial x^2} = \frac{128}{x^3}, \quad \left(\frac{\partial^2 S}{\partial x^2}\right)_{(4,4)} = 2 = r$$

$$\frac{\partial^2 S}{\partial y^2} = 128/y^3, \quad \left(\frac{\partial^2 S}{\partial y^2}\right)_{(4,4)} = 2 = t$$

$$\frac{\partial^2 S}{\partial x \partial y} = 1 = s$$

and $rt - s^2 > 0$

$2 \times 2 - 1^2 = 3 > 0$

$r = 2 > 0$

point of min.

and minimum value of f.

$$f(4,4) = xy + 2yz + 2zx.$$

$$= 4 \times 4 + 2 \times 4 \times 2 + 2 \times 2 \times 4$$

$$= 16 + 16 + 16 = 48 \text{ sq ft}$$

Method 2:

Lagrange's method:

$$S(x,y,z) = xy + 2yz + 2zx \quad \text{—(1)}$$

$$\phi(x,y,z) = xyz - 32 = 0 \quad \text{—(2)}$$

$$(y + 2z) + \lambda(yz) = 0 \quad \text{—(3)}$$

$$(x + 2z) + \lambda(xz) = 0 \quad \text{—(4)}$$

$$(2y + 2x) + \lambda(xy) = 0 \quad \text{—(5)}$$

multiplying ③ by x and ④ by y and subtracting, we get:

$$2zx - 2zy = 0$$

$$2z(x-y) = 0$$

either $2z = 0$ or $(x-y) = 0$ ✓

not possible as z is height (dimension) of rectangle.

$x = y$

Again multiplying ① by y and ⑤ by z and subtracting, we get:

$$xy + \cancel{2zy} - \cancel{2zy} - 2zx = 0$$

$$x(y - 2z) = 0$$

as $x \neq 0$, so $(y - 2z) = 0$

$$y = 2z$$

$$z = y/2$$ ✓

$xyz = 32$ (given)

$$x \cdot x \cdot \frac{x}{2} = 32$$

$$x^3 = 64$$

$x = 4$ similarly $y = 4$ & $z = 2$

Q. find the minimum value of $\overset{f(x,y,z)=}{x^2+y^2+z^2}$, subject to the condition $\phi(x,y,z) = xyz - a^3 = 0$.

Soln: By Lagrange method:

$$\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0 \Rightarrow 2x + \lambda(yz) = 0.$$

$$\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0 \Rightarrow 2y + \lambda(xz) = 0$$

$$\frac{\partial f}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 0 \Rightarrow 2z + \lambda(xy) = 0$$

$\lambda yz = -2x$ → mult. by x $\quad$ $\lambda xyz = -2x^2$

$\lambda xz = -2y$ → mult. by y $\quad$ $\lambda xyz = -2y^2$

$\lambda xy = -2z$ → mult by z $\quad$ $\lambda xyz = -2z^2$

$\therefore$ $x^2 = y^2 = z^2 = K$ (const.) — (1)

and

$xyz = a^3$ (given)

on sq. both side we get:

$$x^2 y^2 z^2 = a^6$$

$$K^3 = a^6$$

$$K = a^2$$

from (1) $x^2 = a^2$, $y^2 = a^2$, $z^2 = a^2$

$x = \pm a$, $y = \pm a$, $z = \pm a$

$K = a^2$

max possible co-ordinates : $(-a,-a,a)$
$(a,-a,-a)$
$(-a,a,-a)$
(a,a,a)

now, $x^2+y^2+z^2$
$=(3a^2)$

and $A.M = \dfrac{x^2+y^2+z^2}{3} = \dfrac{3a^2}{3} = a^2$

$G.M. = (x^2y^2z^2)^{1/3}$
$= a^2$

Since $A.M \geq G.M$

$$\frac{x^2+y^2+z^2}{3} \geq a^2$$

$$x^2+y^2+z^2 \geq 3a^2$$

⇒ <u>Differentiation under integral sign</u> :

$$I(\alpha) = \int_a^b f(x,\alpha)\,dx$$

$$I'(\alpha) = \frac{\partial}{\partial\alpha} I'(\alpha) = \frac{\partial}{\partial\alpha}\int_a^b f(x,\alpha)\,dx$$

$$= \int_a^b \frac{\partial}{\partial\alpha} f(x,\alpha)\,dx.$$

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Evaluate

Q. $\phi(a) = \int_0^\infty \frac{e^{-x}}{x}(1-e^{-ax}) \cdot dx$

$$\phi'(a) = \int_0^\infty \frac{\partial}{\partial a}\left(\frac{e^{-x}}{x}\right)(1-e^{-ax}) \cdot dx$$

$$= \int_0^\infty \frac{\partial}{\partial a}\left(\frac{e^{-x}}{x}\right)(1-e^{-ax}) \, dx$$

$$= \int_0^\infty \frac{e^{-x}}{\cancel{x}} \cdot \left(+e^{-ax}(\cancel{-x})\right) dx$$

$$= +\int_0^\infty e^{-(1+a)x} \cdot dx$$

$$= +\left(\frac{+e^{-(1+a)x}}{(1+a)}\right)_0^\infty = \frac{1}{1+a}$$

$$\frac{d\phi}{da} = \frac{1}{1+a}$$

$$\frac{d\phi}{da} = \frac{d\phi}{1+a}$$

$$\phi = \log(1+a) + c$$

Put $a=0$

$$\phi(0) = \underbrace{\log 1}_{0} + c$$

$$c = \phi(0)$$

$\rightarrow \phi(0) = 0$

$c = 0$

$\phi(a) = \log(1+a)$

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Leibnitz Rule:

→ Differentiation under Integral sign when the limits are the function if the parameter.

$$I(\alpha) = \int_{a(\alpha)}^{b(\alpha)} f(x,\alpha)\,dx$$

$$\boxed{\frac{dI}{d\alpha} = \int_{a(\alpha)}^{b(\alpha)} \frac{\partial}{\partial\alpha}(f(x,\alpha)dx + f(b(\alpha),\alpha)\cdot\frac{d}{d\alpha}(b(\alpha)) - f(a(\alpha),\alpha)\cdot\frac{d}{d\alpha}(a(\alpha))}$$

Q. $I(\alpha) = \int_0^{\alpha} \frac{\tan^{-1}(\alpha x)}{1+x^2}\,dx$

$$\frac{dI}{d\alpha} = \int_0^{\alpha} \frac{\partial}{\partial\alpha}\frac{\tan^{-1}(\alpha x)}{1+x^2}dx + f$$

Q. $I(\alpha) = \int_0^{\alpha} \frac{\log(1+\alpha x)}{(1+x^2)}\,dx$ — ①

$$\frac{dI}{d\alpha} = \int_0^{\alpha} \frac{\partial}{\partial\alpha}\left(\frac{\log(1+\alpha x)}{1+x^2}\right)dx + \frac{\log(1+\alpha^2)}{1+\alpha^2}\cdot 1 - 0$$

$$= \int_0^{\alpha} \frac{x}{(1+\alpha x)(1+x^2)}\,dx + \frac{\log(1+\alpha^2)}{1+\alpha^2} \quad \text{— ②}$$

$$\frac{x}{(1+\alpha x)(1+x^2)} = \frac{A}{1+\alpha x} + \frac{(Bx+C)}{1+x^2}$$

$$x = A(1+x^2) + (Bx+C)(1+\alpha x)$$

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put $1+\alpha x=0$

$x = \frac{-1}{\alpha}$

$x = A(1+x^2) + (Bx+C)(1+\alpha x)$

$\frac{-1}{\alpha} = A\left(1+\frac{1}{\alpha^2}\right)$

$$\boxed{A = -\frac{\alpha}{1+\alpha^2}}$$

$$B = \frac{1}{1+\alpha^2}, \quad C = \frac{\alpha}{1+\alpha^2}$$

$$\int_0^\alpha \frac{x}{(1+\alpha x)(1+x^2)}\cdot dx = -\frac{\alpha}{1+\alpha^2}\int_0^\alpha \frac{dx}{1+\alpha x} + \frac{1}{(1+\alpha^2)}\int_0^\alpha \frac{x}{1+x^2}dx$$

$$+ \frac{\alpha}{\alpha^2+1}\int_0^\alpha \frac{dx}{1+x^2}$$

$$= \frac{-1}{(1+\alpha^2)}\log(1+\alpha^2) + \frac{1}{2(1+\alpha^2)}\log(1+\alpha^2) + \frac{\alpha(\tan^{-1}\alpha)}{\alpha^2+1}$$

Substituting this value in.

$$\frac{dI}{d\alpha} = \cancel{\frac{-1}{(1+\alpha^2)}\log(1+\alpha^2)} + \frac{1}{2(1+\alpha^2)}\log(1+\alpha^2) + \frac{\alpha(\tan^{-1}\alpha)}{\alpha^2+1} + \cancel{\frac{\log(1+\alpha^2)}{(1+\alpha^2)}}$$

$$\frac{dI}{d\alpha} = \frac{1}{2(1+\alpha^2)}\log(1+\alpha^2) + \frac{\alpha}{1+\alpha^2}\tan^{-1}\alpha$$

$$\int dI = \frac{1}{2}\left[\int \frac{\log(1+x^2)}{1+\alpha^2}d\alpha\right] + \int \frac{\alpha}{1+\alpha^2}\tan^{-1}\alpha\, d\alpha + c$$

Now partial Integrating:

$$I = \frac{1}{2}\left[\log(1+\alpha^2)\tan^{-1}\alpha - \int \frac{2\alpha}{1+\alpha^2}\tan^{-1}\alpha\, dx\right] + \int \frac{\alpha}{1+\alpha^2}\tan^{-1}\alpha\, dx + c$$

$$= \frac{1}{2}\log(1+\alpha^2)\tan^{-1}\alpha - \int \frac{\alpha}{1+\alpha^2}\tan^{-1}\alpha \, d\alpha + \int \frac{\alpha}{1+\alpha^2}\tan^{-1}\alpha \, d\alpha + C$$

$$I(\alpha) = \frac{1}{2}\log(1+\alpha^2)\tan^{-1}\alpha + c$$

now, put $\alpha = 0$

$$I(0) = \frac{1}{2}\log(1+0)\tan^{-1}0 + c$$

$$c = 0$$

$$\therefore \boxed{I(\alpha) = \frac{1}{2}\log(1+\alpha^2)\tan^{-1}\alpha}$$

Same q. part

(b) prove that

$$\int_0^1 \frac{\log(1+x)}{1+x^2}\,dx = \frac{\pi}{8}\log 2$$

$x = \tan\theta$.

$dx = \sec^2\theta \, d\theta$

$x = 0, \ \theta = 0$.

$x = 1, \ \theta = \pi/4$

$$\int_0^{\pi/4} \frac{\log(1+\tan\theta)}{1+\tan^2\theta} \cdot \sec^2\theta \, d\theta.$$

$$I = \int_0^{\pi/4} \log(1+\tan\theta)\, d\theta$$

Q. $(x-h)^2 + (y-k)^2 = r^2$

diff w.r.t. x.

$$2(x-h) + 2(y-k)\frac{dy}{dx} = 0$$

$$(x-h) + (y-k)\frac{dy}{dx} = 0 \quad \text{—①}$$

again diff. w.r.t. x

$$1 + (y-k)\left(\frac{d^2y}{dx^2}\right) + \left(\frac{dy}{dx}\right)^2 = 0.$$

$$(y-k) = -\left(\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}\right) \quad \text{—②}$$

from ① & ②

$$(x-h) + \left(-\left(\frac{1+\left(\frac{dy}{dx}\right)^2}{d^2y/dx^2}\right)\right)\frac{dy}{dx} = 0$$

$$(x-h) - \left(\frac{1+\left(\frac{dy}{dx}\right)^2}{d^2y/dx^2}\right)\frac{dy}{dx} = 0$$

$$(x-h) = \left(\frac{1+\left(\frac{dy}{dx}\right)^2}{d^2y/dx^2}\right)\left(\frac{dy}{dx}\right) \quad \text{—③}$$

now put ② & ③ in given eqn.

$$\left(\frac{1+\left(\frac{dy}{dx}\right)^2}{d^2y/dx^2}\left(\frac{dy}{dx}\right)\right)^2 + \left(-\left(\frac{1+\left(\frac{dy}{dx}\right)^2}{d^2y/dx^2}\right)\right)^2 = r^2$$

$$\frac{\left(1+\left(\frac{dy}{dx}\right)^2\right)^2}{\left(\frac{d^2y}{dx^2}\right)^2}\left[1+\left(\frac{dy}{dx}\right)^2\right] = r^2$$

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$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = r^2 \frac{d^2y}{dx^2}$$

— x —

Variable Seperable:

Q Solve: $\sqrt{1+x^2+y^2+x^2y^2} + xy \cdot \frac{dy}{dx} = 0.$

$$\Rightarrow \sqrt{(1+x^2)(1+y^2)} + xy \cdot \frac{dy}{dx} = 0.$$

$$\frac{\sqrt{1+x^2}}{x}dx + \frac{y}{\sqrt{1+y^2}}dy = 0.$$

$$\int \frac{\sqrt{1+x^2}}{x}dx + \int \frac{y}{\sqrt{1+y^2}}dy = c.$$

$$\sqrt{1+x^2} + \frac{1}{2}\log\left|\frac{\sqrt{x^2+1}-1}{\sqrt{x^2+1}+1}\right| + \sqrt{1+y^2} = C$$

Ans.

formula

$$\int \frac{f'(x)}{f(x)} = \log(f(x)) + c$$

$$\int \frac{f'(x)}{\sqrt{f(x)}}dx = 2\sqrt{f(x)} + c.$$

$$\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c$$

$$\int \frac{\sqrt{1+x^2}}{x}dx = \int \frac{x\sqrt{1+x^2}}{x^2}dx$$

$$1+x^2 = t^2$$

$$x\,dx = t\,dt$$

$$\int \frac{t^2\,dt}{t^2-1} = \int\left(1+\frac{1}{t^2-1}\right)dt = t + \frac{1}{2}\log\left|\frac{t-1}{t+1}\right|$$

$$= \sqrt{1+x^2} + \frac{1}{2}\log\left|\frac{\sqrt{x^2+1}-1}{\sqrt{x^2+1}+1}\right|$$

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Reducible to variable form:

Q) $(x+y+)^2 \frac{dy}{dx} = 1$

Soln: let $x+y+1 = z$

$\Downarrow$

$1 + \frac{dy}{dz} = \frac{dz}{dx} \Rightarrow \frac{dy}{dx} = \frac{dz}{dx} - 1$

$z^2\left(\frac{dz}{dx} - 1\right) = 1 \Rightarrow \frac{dz}{dx} - 1 = \frac{1}{z^2} \Rightarrow \frac{dz}{dx} = 1 + \frac{1}{z^2}$

$$\begin{array}{r|l} z^2+1 & z^2 \\ & -z^2+1 \\ & -1 \end{array}$$

$\int \frac{z^2 dz}{z^2+1} = \int dx.$

$\Rightarrow \int \left(1 - \frac{1}{z^2+1}\right) dz = \int dx..$

$\Rightarrow z - \tan^{-1} z = x + c \Rightarrow x+y+1 - \tan^{-1}(x+y+1)$

$\Rightarrow y - \tan^{-1}(x+y+1) = c - 1$

$y = \tan^{-1}(x+y+1) + \underbrace{c'}_{\text{Ans}}$

Q) $\sin^{-1}\left(\frac{dy}{dx}\right) = x+y$

Soln: $\frac{dy}{dx} = \sin(x+y)$

put $x+y = z \Rightarrow 1 + \frac{dy}{dx} = \frac{dz}{dx}.$

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$$\frac{dy}{dx} = \left(\frac{dz}{dx} - 1\right)$$

$$\frac{dz}{dx} - 1 = \sin z$$

$$\frac{dz}{dx} = 1 + \sin z$$

$$\int \frac{dz}{1+\sin z} = \int dx.$$

$$\int \frac{dz}{1+\sin z} \cdot \frac{1-\sin z}{1-\sin z} = \int \frac{(1-\sin z)\,dz}{(1+\sin z)(1-\sin z)}$$

$$= \int \frac{(1-\sin z)\,dx}{1-\sin^2 z} = \int \frac{(1-\sin z)\,dz}{\cos^2 z}$$

$$= \int \sec^2 z\,dz - \int \tan z \sec z\,dz$$

$$= \tan z - \sec z + c \quad \text{—(1)}$$

as $\int \frac{dz}{1+\sin z} = \int dx$

$$\int \frac{dz}{1+\sin z} = \int dx$$

$$\int \frac{dz}{1+\cos\left(\frac{\pi}{2} - z\right)} = x + c$$

$$\int \frac{dz}{2\cos^2\left(\frac{x}{4} - z/2\right)} = x + c$$

$$\frac{1}{2}\int \sec^2\left(\pi/4 - z/2\right) dz = x + c$$

$$= \cancel{\frac{1}{2}(-2)} \tan\left(\frac{\pi}{4} - \frac{z}{2}\right) = x + c$$

$$= -\tan\left(\frac{\pi}{4} - \frac{z}{2}\right) = x + c$$

$$= -\tan\left(\frac{\pi}{4} - \frac{(x+y)}{}\right) = x + c.$$

$$\tan\left(\frac{\pi - 4(x+y)}{4}\right) = -x - c$$

$$\tan\left(\frac{\pi - (2x+2y)}{4}\right) + x = c$$

Ans

or method 2

$$\int \frac{dx}{a + b\sin x} \quad \text{or} \quad \int \frac{dx}{a + b\cos x}$$

$$\sin x = \frac{2\tan x/2}{1 + \tan^2 x/2} \qquad \cos x = \frac{1 - \tan^2 x/2}{1 + \tan x/2}$$

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⇒ Homogeneous differential Eqn:

$$\frac{dy}{dx} = f(x,y)$$

$$\frac{dy}{dx} = \lambda^n (f(x,y))$$

Q. $y\,dx - x\,dy = \sqrt{x^2+y^2}\cdot(dx)$

Soln divide by dx.

$$y - x\frac{dy}{dx} = \sqrt{x^2+y^2}$$

$$x\frac{dy}{dx} = y - \sqrt{x^2+y^2}$$

$$\frac{y-\sqrt{x^2+y^2}}{x} = \frac{dy}{dx}$$

$$y/x - \sqrt{1+(y/x)^2} = \frac{dy}{dx}$$

$y = vx$

$\cancel{y/x = y}$

$y/x = v$

$dy/dx = v + x\frac{dv}{dx}$

$$\cancel{v} - \sqrt{1+v^2} = \cancel{v} + x\frac{dv}{dx}$$

$$-\frac{dx}{x} = \frac{dv}{\sqrt{1+v^2}}$$

$$-\log x + \log c = \log\left|v+\sqrt{1+v^2}\right|$$

$$\log\left(\frac{c}{x}\right) = \log\left|\frac{y+\sqrt{x^2+y^2}}{x}\right|$$

$$y + \sqrt{x^2+y^2} = c$$

Ans

Reducible to homogeneous form:

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$$

if $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

$x = X + h$, $dx = dX$
$y = Y + K$, $dy = dY$

Q. $\frac{dy}{dx} = \frac{2x + 3y - 4}{4x + y - 3}$

here $\frac{a_1}{a_2} = \frac{2}{4} = \frac{1}{2} \neq \frac{b_1}{b_2} = \frac{3}{1}$

$\therefore$ $x = X + h$, $y = Y + K$

$$\frac{dY}{dX} = \frac{2(X+h) + 3(Y+K) - 4}{4(X+h) + (Y+K) - 3}$$

$$= \frac{2X + 2h + 3Y + 3K - 4}{4X + 4h + Y + K - 3} \quad \text{— (1)}$$

To make it homog. we need to eliminate h & K terms, so put it equal to zero.

$2h + 3K - 4 = 0$
$4h + K - 3 = 0$.

$$\frac{h}{-9+4} = \frac{-K}{-6+16} = \frac{1}{2-12}$$

$$\frac{h}{-5} = \frac{-K}{+10} = \frac{1}{-10}$$

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$h = 1/2$ & $K = 1$

now,

$$\frac{dy}{dx} = \frac{2x+3y}{4x+y}$$

put $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = \frac{2x+3vx}{4x+vx}$$

$$= \frac{2+3v}{4+v}$$

$$x\frac{dv}{dx} = \frac{2+3v}{4+v} - v$$

$$= \frac{2+3v-4v-v^2}{4+v} = \frac{2-v-v^2}{4+v}$$

$$\frac{(v+4)\,dv}{v^2+v-2} = -\frac{dx}{x}$$

$$\Rightarrow \int \frac{v+4}{(v-1)(v+2)}\,dv = \frac{-dx}{x}$$

After solving

$X = x - 1/2$

$Y = y - 1$

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$$(y+2x-2)^2 = (2x-2y+1)^5$$

$$(2x+y+1)dx + (4x+2y+1)dy = 0.$$

$$\frac{dy}{dx} = -\frac{2x+y+1}{4x+2y+1} = -\frac{2x+y+1}{2(2x+y)+1}$$

$$\cancel{\frac{a_1}{a_2} = \frac{b_1}{b_2} = \lambda}$$

$$2x+y = t$$

$$2 + \frac{dy}{dx} = \frac{dt}{dx}$$

$$\frac{dt}{dx} = 2 - \frac{t+1}{2t+1}$$

$$\frac{dy}{dx} = \frac{dt}{dx} - 2$$

$$= \frac{4t+2-t-1}{2t+1} = \frac{3t+1}{2t+1} \qquad \left(\frac{dt}{dx} - 2 = -\frac{t+1}{2(t)+1}\right)$$

$$\int \frac{2t+1}{3t+1}\,dt = dx \Rightarrow \int\left(\frac{2}{3} + \frac{1/3}{3t+1}\right)dt = x+c$$

$$= \frac{2}{3}t + \log|3t+1| = x+c$$

$$= \frac{2}{3}(2x+y) + \log|3(2x+y)+1)| = x+c$$

Ans

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Q. ⇒ Linear differential Equation :

$$\left(\frac{dy}{dx}\right)' + P(y) = Q$$

$$I.f. = e^{\int P.dx}$$

(integrating factor)

$$y \cdot (I.f.) = \int Q \cdot (I.f) \cdot dx.$$

$$I.f. = e^{\int P dy}.$$

$$\boxed{x(I.f.) = \int Q.(If)\, dy}$$

$$(1+y^2)dx + (x - e^{\tan^{-1} y})\, dy = 0.$$

$$(1+y^2)\frac{dx}{dy} + x - \tan^{-1} y = 0$$

$$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1} y}{1+y^2}$$

$$P = \frac{1}{1+y^2}, \quad Q = \frac{\tan^{-1} y}{1+y^2}$$

$$I.f. = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}$$

$$x \cdot e^{\tan^{-1} y} = \int \frac{(\tan^{-1} y)^2}{1+y^2} \cdot e^{+\tan^{-1} y} \cdot dy$$

$$x \cdot e^{\tan^{-1} y} = \int \frac{e^{\tan^{-1} y}}{1+y^2} \cdot e^{\tan^{-1} y} \cdot dy$$

let $\tan^{-1}y = t$

$\frac{1}{1+y^2}\,dy = dt$,

$$x e^{\tan^{-1}y} = \int e^{2t}\,dt = \frac{e^{2t}}{2} + c$$

$$x e^{\tan^{-1}y} = \frac{1}{2} e^{2(\tan^{-1}y)}$$

$$x = \frac{1}{2} e^{\tan^{-1}y} + c\, e^{-\tan^{-1}y}.$$

—— x ——

Equation Reducible to linear differential equation.

$$\frac{dy}{dx} + Py = Qy^n$$

$$y^{-n}\frac{dy}{dx} + Py^{1-n} = Q$$

$$y^{1-n} = z$$

$$y^{-n}\cdot\frac{dy}{dx} = \frac{1}{1-n}\frac{dz}{dx}$$

$$\frac{dy}{dx} - x^3y^3 + xy = 0.$$

$$\frac{dy}{dx} + xy = x^3y^3$$

$$y^{-3}\cdot\frac{dy}{dx} + xy^{-2} = x^3$$

put $y^{-2} = t$

$$-2y^{-3}\frac{dy}{dx} = \frac{dt}{dx}$$

$$y^{-3}\frac{dy}{dx} = \frac{-1}{2}\frac{dt}{dx}$$

$$-\frac{1}{2}\left(\frac{dt}{dx}\right) + xt = x^3$$

$$\frac{dt}{dx} - 2xt = -2x^3$$

$$I.f. = e^{\int -2x\,dx}$$

$$= e^{-x^2}$$

$$t \cdot e^{-x^2} = -2\int x^2 \cdot e^{-x^2}(x\,dx)$$

put $x^2 = u$

$x \cdot dx = \frac{1}{2}du$

$$= -\int u e^{-u}\,du$$

$$= -\left[\frac{u e^{-u}}{-1} + \int 1 \cdot \frac{e^{-u}}{+1}\,du\right] + c$$

$$= -\left[\frac{u e^{-u}}{-1} - e^{-u}\right] + c = (u+1)e^{-u} + c$$

$$= (x^2+1)e^{-x^2} + c$$

$$\Rightarrow \frac{dy}{dx} - x^3y^3 + xy = 0.$$

$$t e^{-x^2} = (x^2+1)e^{-x^2} + c$$

$$\frac{e^{-x^2}}{y^2} = (x^2+1)e^{-x^2} + c$$

$$1/y = (x^2+1) + ce^{x^2}$$

Exact differential equation:

$$M\,dx + N\,dy = 0$$

$$u = f(x,y) = c$$

$$du = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$$

$$M = \frac{\partial f}{\partial x}, \quad N = \frac{\partial f}{\partial y}$$

$$\frac{\partial M}{\partial y} = \frac{\partial^2 f}{\partial y\,\partial x}, \quad \frac{\partial N}{\partial y\,\partial x} = \frac{\partial^2 f}{\partial y\,\partial x}$$

By the continuity

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\int M\,dx + \int N(\text{terms free from } x)\,dy = c$$

$$\underbrace{(2xy^2 + 3y)}_{M}dx + \underbrace{(3x^2y^2 + 3x)}_{N}dy = 0.$$

$$\frac{\partial M}{\partial y} = 6xy^2 + 3$$

$$\frac{\partial N}{\partial x} = 6xy^2 + 3$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

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$$\int (2xy^3 + 3y)\,dx + \int 0 \cdot dy = c$$

$$\underline{x^3y^3 + 3xy = c}$$

Reducible to exact form:

If $Mdx + Ndy = 0$ is homogeneous equation in x and y

$Mdx + Ndy = 0$ $\qquad$ $I.f. = \dfrac{1}{Mx + Ny}$.

Q) $\underbrace{(y^3 - 3xy^2)}_{M}dx + \underbrace{(2x^2y - xy^2)}_{N}dy = 0$

$$\frac{dy}{dx} = f(x,y)$$
$$x \to \lambda x$$
$$y \to \lambda y$$
$$\frac{dy}{dx} = \lambda^n f(x,y)$$

$$I.F. = \frac{1}{\cancel{xy^3} - 3x^2y^2 + 2x^2y^2 - \cancel{xy^3}}$$

$$= \frac{1}{-x^2y^2} \quad \text{--- (1)}$$

~~Now divide the given eqn by (1)~~

$$\left(\frac{3}{x} - \frac{y}{x^2}\right)dx + \left(\frac{1}{x} - \frac{2}{y}\right)dy$$

$$3\log|x| + \frac{y}{x} - 2\log|y| = \log c.$$

$$\log\left|\frac{x^3}{cy^2}\right| = -y/x$$

$$x^3 = cy^2 e^{-y/x}.$$

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Rule 2: if $f_1(xy)y\,dx + f_2(xy)x\,dy = 0$.

Q. $(\underbrace{x^2y^3 + 2y}_{M})dx + (\underbrace{2x - 2x^3y^2}_{N})dy = 0$

making this eqn in the form of rule 2

$$(x^2y^2 + 2)y\,dx + (2 - 2x^2y^2)x\,dy = 0.$$

$$I.F. = \frac{1}{Mx - Ny}, \quad Mx \neq Ny.$$

$$IF = \frac{1}{x^3y^3 + \cancel{2xy} - \cancel{2xy} + 2x^3y^3}$$

$$= \frac{1}{3x^3y^3}$$

multiplying 1 by $\frac{1}{x^3y^3}$

$$\left(\frac{1}{x} + \frac{2}{x^3y^2}\right)dx + \left(\frac{2}{x^2y^3} - \frac{2}{y}\right)dy = 0$$

$$= \log|x| - \frac{1}{x^2y^2} - 2\log|y| = \log c.$$

$$= \log\left|\frac{x}{cy^2}\right| = \frac{1}{x^2y^2}$$

$$x = cy^2 \cdot e^{1/x^2y^2}$$

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Rule 3: $Mdx + Ndy = 0$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$$ is a function of x alone

$g(x)$ say.

$$I.F. = e^{\int g(x)dx}$$

Q^n → $(xy^3 - e^{1/x^3}) - x^2y\,dy = 0$.

$$\frac{\partial M}{\partial y} = 3xy^2, \quad \frac{\partial N}{\partial x} = -2xy$$

now,

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{3xy^2 + 2xy}{-x^2y} = -\frac{4\cancel{xy}}{x^{\cancel{2}}\cancel{y}} = \frac{-4}{x},$$

$$I.f. = e^{-4\int 1/x\,dx} = e^{-4\log x} = \frac{1}{x^4}$$

$$\left(\frac{y^2}{x^3} - \frac{e^{1/x^3}}{x^4}\right)dx + \left(\frac{-y}{x^2}\right)dy = 0.$$

$$= -\frac{y^2}{2x^2} + \frac{1}{3}e^{1/x^3} = c$$

$$3y^2 - 2x^2e^{1/x^3} = -cx^2$$

Rule 4:- $$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = f(y) \qquad I.f. = e^{\int f(y)dy}$$

Ex. $x \cdot e^x (dx - dy) + e^x dx + y \cdot e^y dy = 0$

$$\underbrace{(x+1)e^x dx}_{M} + \underbrace{(y \cdot e^y - x e^x)}_{N} dy = 0.$$

$$\frac{\partial M}{y} = 0 \;,\; -(x+1)e^x.$$

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = f(y) = \frac{-(x+1)e^x - 0}{(x+1)e^x}$$

$$= -1.$$

$$I.f. = e^{\int f(x) \cdot dy}$$

$$= e^{-\int dy} = e^{-y}$$

$$(e^{-y}(x+1) \cdot e^x) dx + (y - x e^{x-y}) dy = 0.$$

integrating by part we get:

$$= x e^{x-y} + \frac{y^2}{2} = c.$$

Rule 5:-

$$x^a y^b (mydx + nxdy) + x^{a'} y^{b'} (m'y dx + n' x dy) = 0$$

I.f. $x^h y^k$

for h&k value: $\frac{a+h+1}{m} = \frac{b+k+1}{n}$ — (1)

$$\frac{a'+h+1}{m'} = \frac{b'+k+1}{n'} \quad \text{— (2)}$$

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Q. $x(3y\,dx + 2x\,dy) + 8y^4(y\,dx + 3x\,dy) = 0$

here $a=1,\ b=0,\ m=3,\ n=2$

here $\dfrac{a+h+1}{m} = \dfrac{b+k+1}{n}$

$\dfrac{1+h+1}{3} = \dfrac{k+1}{2}$

$(h+2)2 = (k+1)3$

$2h - 3k + 1 = 0$ — (1)

$a'=0,\ b'=4,\ m=1,\ n=3$

here $\dfrac{a'+h+1}{m'} = \dfrac{b'+k+1}{n'}$

$\dfrac{0+h+1}{1} = \dfrac{4+k+1}{3}$

$3h+3 = k+5$

$3h - k - 2 = 0$ — (2)

$[3h - k - 2 = 0 \text{ — (2)}] \times 3$

$+ \quad +$

$-7h + 0 + 7 = 0$

$h = -1$

$-3 - k - 2 = 0$

$k = -5$

$2h - 3k + 1 = 0$

$3h - k - 2 = 0$

$$-\frac{h}{6+1} = \frac{-k}{-4-3} = \frac{1}{-2+9}$$

$$\frac{h}{7} = \frac{k}{7} = \frac{1}{7}$$

$h = 1$ & $k = 1$

now,

I.f. $x^h y^k = x^1 y^1 = xy$

$(3x^2y^2dx + 2x^2y\,dy) + (8xy^6dx + 24x^2y^4dy) = 0.$

$(3x^2y^2 + 8xy^6)dx + (2x^2y + 24x^2y^5)dy = 0.$

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$$6x^2y + 48xy^5 = \frac{\partial M}{\partial y}$$

$$6x^2y + 48xy^5 = \frac{\partial N}{\partial y}$$

as $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial y}$

integrating by part we get:

$$x^3y^2 + 4x^2y^6 = c$$

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Math

Differential operator :

A.E ⇒ Auxiliary Eq^n

C.F ⇒ Complimenty function.

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx} = D$$

$$\frac{dy}{dx} = \frac{d}{dx}(y)$$

$$= Dy$$

$$\frac{d^2y}{dx^2} = D^2y$$

$$\Rightarrow a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_n y = X$$

$$\Rightarrow (a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_n) y = X$$

A.E. (Auxiliary Eq^n) $= a_0 D^n + a_1 D^{n-1} + \cdots + a_n = 0$.

$$(D-m_1)(D-m_2)(D-m_3) \cdots = 0.$$

$$D = m_1, m_2, m_3$$

(real ~~and~~ distant roots)

Case ① → Roots are Real & distant

$$\therefore C.F = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x}$$

(~~coefficient factor~~) (Compilementory fn)

Case ②: Roots are Real & repeated

$$D = m, m$$

$$C.F = (c_1 + c_2 x) e^{mx}$$

$$D = m, m, m$$

$$C.F = (c_1 + c_2 x + c_3 x^2) e^{mx}$$

Case ③: if complex roots:

$$D = \alpha + i\beta$$

$$c.f. = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$$

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Q. $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$

→ A.E. ⇒ $D^2 + 5D + 6 = 0$

$(D+2)(D+3) = 0.$

$D = -2, -3.$

$C.F. = c_1 e^{-2x} + c_2 e^{-3x}$

Q $\frac{d^2y}{dx^2} + 4y = 0$

→ A.E ⇒ $D^2 + 4 = 0.$

$D = 0 \pm 2i$

$C.F. = e^{0x}[c_1 \cos 2x + c_2 \sin 2x].$

Q. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0.$

→ A.E. ⇒ $D^2 - D + 1 = 0.$

$(D-1)^2 = 0.$

$D = 1, 1$

$C.F. = (C_1 + C_2 x)e^{(1)x}.$

Q. $(D-2)^3 y = 0$

$\rightarrow$ A.E $\Rightarrow (D-2)^3 = 0$

$D = 2, 2, 2$

$C.F. = (c_1 + c_2 x + c_3 x^2) e^{2x}$

Particular Integral (P.I.):

$X = e^{ax}$

$$\boxed{(PI) \Rightarrow \frac{1}{f(D)} e^{ax} = \frac{e^{ax}}{f(a)}}, \; f(a) \neq 0$$

if $f(a) = 0$

$$\boxed{PI \Rightarrow \frac{1}{f(D+a)} e^{ax}}$$

Q. $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{4x}$

A.E $\Rightarrow D^2 - 5D + 6 = 0$

$(D-2)(D-3) = 0$

$D = 2, 3$

$C.f. = c_1 e^{2x} + c_2 e^{3x}$

and $P.I. = \frac{1}{f(D)} e^{ax} = \frac{e^{ax}}{f(a)}$

$= \frac{e^{4x}}{D^2 - 5D + 6} = \frac{e^{4x}}{16 - 2a + 6} = \frac{e^{4x}}{2}$

$P.I = \frac{e^{4x}}{2}$

$C.S. = C.F. + P.I.$

$$\boxed{C.S. = c_1 e^{2x} + c_2 e^{3x} + \frac{e^{4x}}{2}}$$ Ans

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Q. $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{2x} = e^{ax}$ $\therefore a=2$

$D^2 - 5D + 6 = 0 \Rightarrow (D-2)(D-3) = 0 \Rightarrow D = 2, 3$

$C.F. = c_1 e^{2x} + c_2 e^{3x}$

$P.I. = \frac{e^{2x}}{D^2 - 5D + 6}$

$= e^{2x} \frac{1}{D^2 - 5D + 6}$

$= e^{2x} \frac{1}{(D+2)^2 - 5(D+2) + 6}$

$= e^{2x} \cdot \frac{1}{D^2 - 4D}$

$= e^{2x} \cdot \frac{e^{0x}}{D(D-4)}$

$= \frac{-1}{4} e^{2x} \cdot \frac{1}{D}$

$P.I. = \frac{-e^{2x} \cdot x}{4}$

$y = c_1 e^{2x} + c_2 e^{3x} - \frac{xe^{2x}}{4}$ Ans

here
$f(a) = D^2 - 5D + 6$
$f(2) = (2)^2 - (5 \times 2) + 6$
$= 4 - 10 + 6 = 0$
$\therefore f(a) = 0$
So use
$P.I. = \frac{e^{ax}}{f(D+a)}$

$\frac{1}{D} = \int$

Case (II): if $X = \sin(ax+b)$ or $\cos(ax+b$

$$P.I = \frac{\sin(ax+b)}{f(D^2)} = \frac{\sin(ax+b)}{f(-a^2)}$$

$\rightarrow f(-a^2) \neq 0.$

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Q. $\frac{d^2y}{dx^2} + 4y = \sin 3X$

$A.E = D^2 + 4 = 0$

$D = 0 \pm 2i$

$C.F. = e^{0X}[C_1 \cos 2X + C_2 \sin 2x]$

$$P.I. = \frac{\sin 3x}{D^2+4} = \frac{\sin 3X}{-9+4} = \frac{-\sin 3X}{5}$$

Q. $\frac{d^2y}{dx^2} + 4y = \sin 2x.$

$A.E = D^2 + 4 = 0$

$D = 0 \pm 2i$

$C.F = e^{0X}[C_1 \cos 2X + C_2 \sin 2X]$

$$P.I. = \frac{\sin 2X}{D^2+4}$$

as $f(-a^2) = \underline{-4+4=0}.$

$$= x\frac{\sin 2X}{2D}$$

$\rightarrow \frac{1}{D} = \int$

$$= \frac{X}{2}\cdot\left(\frac{-\cos 2X}{2}\right)$$

$$= -\frac{X}{4}\cos 2X$$

So, we multiply P.I by X and differentiate denominator term

$$\frac{\sin 2X}{D^2+4} = x\frac{\sin 2X}{\frac{d}{dD}(D^2+4)}$$

$$= \frac{X \sin 2X}{2D}$$

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→ Case III: $X = x^m$, $m > 0$ and integer

$$\frac{x^m}{f(D)} = [f(D)]^{-1} x^m$$

$(x+a)^n = x^n + {}^nC_1 x^{n-1} a^1 + {}^nC_2 x^{n-2} a^2 + \cdots$

$x = 1, a = x$

$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \cdots$

$|x| < 1$

Q. $y'' + 16y = 64x^2$

A.E. $= D^2 + 16 = 0$.

$D = 0 \pm 4i$

$C.F. = e^{0x}(c_1 \cos 4x + c_2 \sin 4x)$

$$P.I = \frac{x^m}{f(D)} = \frac{64x^2}{D^2+16} = \frac{\cancel{64}^{4} x^2}{\cancel{16}\left(\frac{D^2}{16}+1\right)}$$

$$= 4\frac{x^2}{\left(1+\frac{D^2}{16}\right)} = 4\left(1+\frac{D^2}{16}\right)^{-1} x^2 = 4\left(1-\frac{D^2}{16}\right)x^2$$

$$= 4\left[x - \frac{1}{8}\right] = 4x - \frac{1}{2}$$

$C.S = CF + PI.$

$= (c_1 \cos 4x + c_2 \sin 4x) + 4x - \frac{1}{2}$.

Q. $y'' + 16y = 64x^2$

$$P.E = \frac{64x^2}{D^2+16} = 4\frac{x^2}{1+D^2/16} = 4\cdot\left(1+\frac{D^2}{16}\right)^{-1} x^2$$

$$= 4\left(1 - \frac{D^2}{16} + \frac{\cancel{2}}{\cancel{2!}}\frac{D^4}{256}\right)x^4 = 4\left[1 - \frac{12x^2}{16} + \frac{24}{256}\right] = 4 - 3x^2 + \frac{3}{8}$$

$= \frac{25}{8} - 3x^2$ Ans.

→ Case IV: $X = e^{ax} \cdot V$ (where $V \to$ some another fn of x)

$$P.I. = \frac{e^{ax} \cdot V}{f(D)}$$

$$= \boxed{e^{ax} \cdot \frac{1}{f(D+a)} \cdot V}$$

Q. $(D^3 - 3D - 2)y = x^2 \cdot e^{-x}$

A.E. $\Rightarrow D^3 - 3D - 2 = 0$

$$(D+1)(D^2 - D - 2) = 0$$

$$(D+1)(D+1)(D-2) = 0$$

$$(D+1)^2 (D-2) = 0$$

$$D = -1, -1, 2$$

$$C.F. = (c_1 + c_2 x)e^{-x} + c_3 e^{2x})$$

now,

$$P.I. = \frac{e^{-x} \cdot x^2}{D^3 - 3D - 2} \quad \to f(-1) = 0$$

$$= e^{-x} \cdot \frac{x^2}{(D-1)^3 - 3(D-1) - 2} = \frac{e^{-x} \cdot x^2}{D^3 - D^2}$$

$$= e^{-x} \cdot \frac{x^2}{-3D^2\left(1 - \frac{D}{3}\right)} = -\frac{e^{-x} \cdot x^2}{3D^2}\left(1 - \frac{D}{3}\right)^{-1}$$

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$$= -e^{-x} \cdot \frac{1}{3D^2}\left(1 + \frac{D}{3} + \frac{D^2}{9}\right)x^2$$

$$= -e^{-x} \cdot \frac{1}{3D^2}\left(x^2 + \frac{2}{3}x + \frac{2}{9}\right)$$

$$= -e^{-x} \frac{1}{3D}\left(\frac{x^3}{3} + \frac{x^2}{3} + \frac{2}{9}x\right)$$

$$= \frac{-e^{-x}}{3}\left(\frac{x^4}{12} + \frac{x^3}{9} + \frac{1}{9}x^2\right)$$

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→ Case V: ⇒ $X = x \cdot v$

$$P.I. = \underbrace{\frac{x \cdot v}{f(D)}}_{\text{Apply product rule and diff. w.r.t. } D.} = x \cdot \frac{v}{f(D)} - \frac{f'(D)}{[f(D)]^2} \cdot v$$

Imp. Q

Q. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x \cdot e^x \sin x$

$A.E. = D^2 - 2D + 1 = 0$

$C.F. = (c_1 + c_2 x) e^x$

$$P.I. = \frac{e^x \cdot (x \sin x)}{(D-1)^2}$$

$$= e^x \cdot \frac{1}{(D+1-1)^2} (x \sin x)$$

$$= e^x \cdot \frac{1}{D^2} (x \sin x)$$

$$= e^x \cdot \frac{1}{D} \left[-x \cos x + \sin x \right] \Rightarrow e^x \left[-x \sin x - \int (-1) \underbrace{\sin x \, dx}_{-\cos x} \right]$$

$$\cancel{= e^x \left[-x \sin x + \int \sin x \, dx \right] \cos x}$$

$$\cancel{= e^x \left[-x \sin x + 2 \cos x \right]}$$

$$= e^x \left[-x \sin x - 2 \cos x \right]$$

→ **General method:**

$$\frac{1}{D-a}x = e^{ax}\int e^{-ax}\, x\,dx$$

A.E. = -

$$C.F = c_1\cos ax + c_2\sin ax$$

$$P.I. = \frac{1}{D^2+a^2}\tan x$$

$$\frac{1}{D-a}x = e^{ax}\int e^{-ax}\, x\,dx$$

$$\frac{1}{(D-ia)(D+ia)} = \frac{A}{D-ia} + \frac{B}{D+ia}.$$

$$\frac{1}{D-ia}\tan x = e^{iax}\int e^{-iax}\tan ax\,dx$$

$$= e^{iax}\int(\cos ax - i\sin ax)\frac{\sin ax}{\cos ax}\,dx$$

$$= e^{iax}\int\left(\sin ax - \left(\frac{\sin^2 ax}{\cos ax}\right)\right)dx$$

from De'moivre's theorem

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

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$$= e^{iax}\int\left[\sin ax - i\left(\frac{1-\cos^2 ax}{\cos ax}\right)\right]dx.$$

$$= e^{iax}\int(\sin ax - i\sec ax + i\cos ax)\,dx$$

$$\frac{1}{D-ia}\tan ax = e^{iax}\left[-\frac{\cos ax}{a} - \frac{i}{a}\log|\sec ax + \tan ax| + \frac{i}{a}\sin ax\right] \quad \text{—(1)}$$

Changing i to $-i$ in eq (1):

$$\frac{1}{D+ia}\tan x = e^{-iax}\left[-\frac{\cos ax}{a} + \frac{i}{a}\log|\sec ax + \tan ax| - \frac{i}{a}\sin ax\right] \quad \text{—(2)}$$

$$= \frac{1}{2ia}\left[\left\{-\frac{e^{iax}\cdot\cos ax}{a} - \frac{i}{a}e^{\cos x}\log|\sec ax + \tan ax| + \frac{i}{a}\sin ax\right\}\right.$$

$$\left.-\left\{-\frac{e^{-iax}}{a}\cos ax + \frac{e^{-iax}}{a}i\log(\sec ax + \tan x) - e^{-iax}\frac{i}{a}\cdot\sin ax\right\}\right]$$

$$= \frac{1}{2ia}\left[-(e^{iax} - e^{-iax})\frac{\cos ax}{a} - \frac{i}{a}(e^{iax} + e^{-iax})\log|\sec ax + \tan ax|\right.$$

$$\left. + \frac{i}{a}(e^{iax} + e^{-iax})\sin ax\right]$$

$$= \left[\frac{-\cos ax}{a^2}\left(\frac{e^{iax} - e^{-iax}}{2i}\right) - \frac{1}{a^2}\left(\frac{e^{iax} + e^{-iax}}{2}\right)\log|\sec ax + \tan ax|\right.$$

$$\left. + \frac{\sin ax}{a^2}\left(\frac{e^{iax} + e^{-iax}}{2}\right)\right]$$

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$$= \cancel{\frac{-1}{a^2}\cos ax \sin ax} - \frac{1}{a^2}\cos ax \log|\sec ax + \tan ax| + \cancel{\frac{\sin ax \cos ax}{a^2}}$$

$$= \frac{-1}{a^2}\cos^2 ax \log|\sec ax + \tan ax|$$

Variation of parameters:

Q ⇒ $(D^2+a^2)y = \tan x$

Sol^n ⇒

$$C.F = c_1 \cos ax + c_2 \sin ax$$

let $\cos ax = y_1$
$\sin ax = y_2$

using Wronskian method:

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix} = a$$

$$P.I. = uy_1 + vy_2$$

$$u = -\int \frac{Y_2 X}{W} dx$$

$$= -\int \frac{\sin ax}{a} \tan ax \cdot dx$$

$$= \frac{-1}{a}\int \frac{\sin^2 ax}{\cos ax} dx$$

$$= -\frac{1}{a}\int \frac{1-\cos^2 ax}{\cos ax} dx$$

$$u = \frac{-1}{a^2}\log|\sec ax + \tan ax| + \frac{1}{a^2}\sin ax$$

Similarly

$$v = \int \frac{Y_1 X}{W} dx = \int \frac{\cos ax \cdot \tan ax}{a} dx$$

$$\boxed{v = \frac{-\cos ax}{a^2}}$$

$$P.I. = uy_1 + vy_2$$

$$= \left(-\frac{1}{a^2}\log\left|\sec ax + \tan ax\right| + \frac{1}{a^2}\sin ax\right)\cos ax$$

$$-\frac{\cos ax}{a^2}\cdot\sin ax$$

$$= -\frac{1}{a^2}\cos ax\log\left|\sec ax + \tan ax\right|$$

Cauchy - Eular equation:

$$\frac{x^n \cdot d^n y}{dx^n} + x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_n y = X$$

$x = e^t \quad x > 0 \quad \Big| \frac{d}{dt} = D$

$t = \ln(x)$

$$\boxed{\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt}}$$

$$x\frac{dy}{dx} = Dy.$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}\left(\frac{1}{x}\cdot\frac{dy}{dt}\right)$$

$$= \frac{1}{x}\cdot\frac{d^2y}{dt^2}\cdot\frac{dt}{dx} - \frac{1}{x^2}\frac{dy}{dt}$$

$$= \frac{1}{x^2}\left[\frac{d^2y}{dx^2} - \frac{dy}{dt}\right]$$

$$x^2\cdot\frac{d^2y}{dx^2} = (D^2 - D)y - D(D-1)y.$$

$$x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)y$$

Q. $x^3 \frac{d^3y}{dx^3} - 4x^2 \frac{d^2y}{dx^2} + 8x \frac{dy}{dx} - 8y = 4\ln x.$

from Cachey-Euler's theorem.

$$x^3 \frac{d^3y}{dx^3} = D(D-1)(D-2)y$$

$$x^2 \cdot \frac{d^2y}{dx^2} = D(D-1)y.$$

$$D(D-1)(D-2) - 4D(D-1) + 8(D-8)y = e^{4t}$$

$$A.E. = D(D^2-3D+2) - 4D^2+4D+8D-8=0.$$

$$= D^3 - 3D^2 + 2D - 4D^2 + 4D - 8 = 0.$$

$$= D^3 - 7D^2 + 14D - 8 = 0.$$

$$= \cancel{8} - 28 + \cancel{28} - \cancel{8} = 0$$

$$D - 2 = 0$$

$$D^2(D-2) - 5D(D-2) + 4(D-2) = 0$$

$$(D-2)(D^2-5D+4) = 0.$$

$$(D^2 - 4D - D + 4) = 0$$

$$D(D-4) - 1(D-4) = 0$$

$$(D-1)(D-4) = 0.$$

$$\therefore D = 2, 1, 4$$

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$$C.f. = c_1 e^{t} + c_2 e^{2t} + c_3 e^{4t}$$

$$P.I = 4 \frac{t}{D^2 - 7D^2 + 14D - 8}$$

$$= \frac{-1}{2}\left(1 - (D^2 - 7D^2 + 14D)\right)^{-1} \cdot t$$

using

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots$$

$$= -\frac{1}{2}\left(t + 14\right) = -7 - \frac{t}{2} = -\left[7 + \frac{1}{2}t\right]$$

$$= -\left[7 + \frac{1}{2}\ln x\right]$$

eg: $y' - y = 0$

Let $y = \sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$

$$y' = \sum_{m=1}^{\infty} m a_m x^{m-1}$$

$$\sum_{m=1}^{\infty} m a_m x^{m-1} - \sum_{m=0}^{\infty} a_m x^m = 0$$

$$\sum (m+1) a_{m+1} x^m - \sum_{m=0}^{\infty} a_m x^m = 0.$$

Equating to zero the coefficient of x^m

$$(m+1) a_{m+1} - a_m = 0.$$

$$a_{m+1} = \frac{a_m}{m+1}$$

Put $m=0$

$$a_1 = a_0.$$

Put $m=1$

$$a_2 = \frac{a_1}{2} = \frac{a_0}{2} = \frac{a_0}{2!}$$

Put $m=3$

$$a_3 = \frac{a_2}{3} = \frac{a_0}{6} = \frac{a_0}{3!}$$

Put $m=4$

$$a_4 = \frac{a_3}{4} = \frac{a_0}{24} = \frac{a_0}{4!}$$

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$$\therefore\ y = \sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

$$= a_0 + a_0 x + \frac{a_0}{2!} x^2 + \frac{a_0}{3!} x^3 + \frac{a_0}{4!} x^4 + \cdots$$

$$= a_0\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) = a_0 e^x$$

$$\left\{ \text{as } \frac{1}{1-x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \right\}$$

eg: $y'' + y = 0$

let $y = \sum_{m=0}^{\infty} a_m x^m$

$$y' = \sum_{m=1}^{\infty} a_m x^{m-1}$$

$$y'' = \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2}$$

$$\sum_{m=2}^{\infty} m(m-1) a_m x^{m-2} + \sum_{m=0}^{\infty} a_m x^m = 0.$$

$$\sum (m+2)(m+1) a_{(m+2)} x^m + \sum_{m=0}^{\infty} a_m x^m = 0.$$

Equating to zero the coefficient of x^m

$$(m+2)(m+1) a_{m+2} + a_m = 0$$

$$a_{m+2} = \frac{-a_m}{(m+1)(m+2)} \quad \text{--- (1)}$$

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now in eq ①

Put $m=0$

$$a_2 = \frac{-a_0}{2}$$

Put $m=1$

$$a_3 = \frac{-a_1}{6}$$

Put $m=2$

$$a_4 = \frac{-a_2}{3\cdot 4}$$

Put $m=3$

$$a_5 = \frac{-a_3}{4\cdot 5}$$

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

$$= a_0 + a_1 x - \frac{a_0}{2} x^2 - \frac{a_1}{6} x^3 + + \frac{a_0}{24} x^4 + \cdots$$

$$= a_0\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + a_1\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)$$

$$= a_0 \cos x + a_1 \sin x$$

$$\left\{ \begin{array}{l} \text{as} \\ \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \\ \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \end{array} \right\}$$

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Q. Solve $y'' + xy = 0$ — (1)

Let $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots$

$$= \sum_{m=0}^{\infty} a_m x^m$$

$$y' = \sum_{m=1}^{\infty} m a_m x^{m-1}$$

$$y'' = \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2}$$

Substituting these values in eq (1), we get:

$$\sum_{m=2}^{\infty} m(m-1) a_m x^{m-2} + x\left(\sum_{m=0}^{\infty} a_m x^m\right) = 0$$

Now make the same power of (x) to add simply:

$$\sum_{m=2}^{\infty} m(m-1) a_m x^{m-2} + \sum_{m=3}^{\infty} a_{m-3} x^{m-2} = 0$$

$$2\cdot(1) a_2 + \sum_{m=3}^{\infty} \left(m(m-1) a_m + a_{m-3}\right) x^{m-2} = 0.$$

$2a_2 = 0$ Equating to zero the coefficient of x^{m-2}

$\Rightarrow a_2 = 0$

$$a_m = \frac{-a_{m-3}}{m(m-1)} \quad \text{— (2)}$$

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now, In eq (2)

put $m=3$

$$a_3 = \frac{-a_0}{3\cdot 2}$$

Put $m=4$

$$a_4 = \frac{-a_1}{4\cdot 3}$$

Put $m=5$

$$a_5 = \frac{-a_2}{5\cdot 4} = 0.$$

now,

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots$$

$$= a_0 + a_1 x + 0\cdot x^2 - \frac{a_0 x^3}{3\cdot 2} - \frac{a_1}{4\cdot 3} x^4 + 0\cdot x^5 + \frac{a_0 x^6}{6\cdot 5\cdot 3\cdot 2} + \cdots$$

$$= a_0\left(1 - \frac{x^3}{3\cdot 2} + \frac{x^6}{6\cdot 5\cdot 3\cdot 2} \cdots\right) + a_1\left(x - \frac{x^4}{4\cdot 3} + \frac{x^7}{7\cdot 6\cdot 4\cdot 3} \cdots\right)$$

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Real analytic function: A real function $f(x)$ is called analytic at point $x = x_0$, if it can be represented by a power series in power of $(x - x_0)$ with radius of convergence $R > 0$.

Q. $(1-x^2)y'' - 2xy' + 2y = 0.$ — (1)

let $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots$ — (2)

$$= \sum_{m=0}^{\infty} a_m x^m.$$

$$y' = \sum_{m=1}^{\infty} m \cdot a_m \cdot x^{m-1}$$

$$y'' = \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2}$$

Substituting these values in eq (1) we get:

$$\left\{(1-x^2)\sum_{m=2}^{\infty} m(m-1) a_m x^{m-2} - 2x\left(\sum_{m=1}^{\infty} m a_m x^{m-1}\right) + 2\left(\sum_{m=0}^{\infty} a_m x^m\right)\right\} = 0.$$

$$\sum_{m=2}^{\infty} m(m-1) a_m x^{m-2} - \sum_{m=2}^{\infty} m(m-1) a_m x^m - 2\sum_{m=1}^{\infty} m a_m x^m + 2\sum_{m=0}^{\infty} a_m x^m = 0$$

$$\sum m(m-1) a_m x^{m-2} - \sum (m(m-1) + 2m - 2) a_m x^m = 0.$$

$$\sum m(m-1) a_m x^{m-2} - \sum (m^2 + m - 2) a_m x^m = 0.$$

Equating the coeff of x^m to zero.

$$(m+2)(m+1)a_{m+2} - (m^2+m-2)a_m = 0.$$

$$a_{m+2} = \frac{\cancel{(m+2)}(m-1)}{(m+1)\cancel{(m+2)}} a_m \quad \text{--- (3)}$$

now in eq (3)

put $m=0$ $\quad a_2 = \frac{-2a_0}{2} = -a_0$

put $m=1$ $\quad a_3 = 0$

put $m=2$ $\quad a_4 = \frac{1}{3}a_2 = \frac{-a_0}{3}$

put $m=3$ $\quad a_5 = \frac{2}{4} \cdot a_3 = 0$.

put $m=4$ $\quad a_6 = \frac{5}{7}a_4 = -\frac{5a_0}{7 \cdot 3}$

now, from eq (2):

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots$$

$$= a_0 + a_1x - a_0x^2 + 0 \cdot x^3 - \frac{a_0}{3}x^4 + 0 \cdot x^5 - \frac{1}{6}a_0x^6$$

$$= a_0\left(1 - x^2 - \frac{x^4}{3} - \frac{1}{5}x^6 \cdots\right) + a_1x.$$

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Consider the second order homogeneous linear differential eqn:

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)\,y = 0 \quad \text{--- (1)}$$

$$\Rightarrow a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \dots$$

$$= \sum_{m=0}^{\infty} a_m (x-x_0)^m$$

<u>Normalized form</u>:

$$\frac{d^2y}{dx^2} + \frac{a_1(x)}{a_0(x)}\frac{dy}{dx} + \frac{a_2(x)}{a_0(x)}\,y = 0$$

$x = 0$ is an ordinary pt. of (1) if $P(x)$ and $q(x)$ both are analytic.

let $P(x) = \dfrac{a_1(x)}{a_0(x)}$

$q(x) = \dfrac{a_2(x)}{a_0(x)}$

Q. $\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + (x^2+2)y = 0$

here $p(x) = x$, $q(x) = x^2+2$ } both are analytic

$\therefore$ at $x=0$ this a regular analytic functions.

Q. $(x-1)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + \dfrac{1}{x}y = 0$.

Normalised form: $\dfrac{d^2y}{dx^2} + \dfrac{x}{x-1}\dfrac{dy}{dx} + \dfrac{1}{x(x-1)}y = 0$

$P(x) = \dfrac{x}{x-1}$ } Analytic except $x=1$

$q(x) = \dfrac{1}{x(x-1)}$ } Analytic except $x=0, 1$

Frobinian Method:

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Regular Single pt, $x = x_0$

$$\left.\begin{array}{l}(x-x_0)\,p(x)\\(x-x_0)^2\,q(x)\end{array}\right\}$$

eg: $2x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + (x-5)y = 0$.

Normalised form: $\dfrac{d^2y}{dx^2} - \dfrac{1}{2x}\dfrac{dy}{dx} + \dfrac{x-5}{2x^2}y = 0$.

$$\left.\begin{array}{l}p = \dfrac{-1}{2x},\ xp(x) = -1/2\\ q = \dfrac{x-5}{2x^2},\ x^2q(x) = \dfrac{x-5}{3}\\ \therefore\ x=0 \text{ is a regular singular point.}\end{array}\right\}\text{both are analytic.}$$

Frobinius Method:

Q $2x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + (x-5)y = 0$ — (1)

Here $x=0$ is a regular singular pt.

$$y = \sum_{m=0}^{\infty} a_m x^{m+r}$$

$$y' = \sum_{m=0}^{\infty} (m+r)a_m x^{m+r-1}$$

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$$y'' = \sum_{m=0}^{\infty} (m+r)(m+r-1)\, a_m\, x^{m+r-1}$$

now, ~~$2x^2$~~ put these values in eq (1)

$$2x^2 \sum_{m=0}^{\infty} (m+r)(m+r-1) a_m x^{m+r-2} - x \sum_{m=0}^{\infty} (m+x)\, a_m x^{m+r-1}$$

$$+ (x-5) \sum_{n=0}^{\infty} a_m x^{m+r} = 0.$$

$$\sum_{m=0}^{\infty} 2(m+r)(m+r-1) a_m x^{m+r} - \sum_{m=0}^{\infty} (m+r) a_m x^{m+r} + \sum_{m=0}^{\infty} a_m x^{m+r+1}$$

$$- \sum 5 a_m x^{m+r} = 0.$$

$$\sum \left[2(m+r)(m+r-1) - (m+r) - 5 \right] a_m x^{m+r} + \sum_{m=1}^{\infty} a_{m-1} x^{m+r} = 0$$

Put $m=0$

$$\left(2r(r-1) - r - 5 \right) = 0$$

Indicial eq^n $2r^2 - 3r - 5 = 0.$

$$r = 5/2, -1 \text{ (Real and unequal)}$$

$$2(m+r)(m+r-1) - (m+r) - 5) a_m + a_{m-1} = 0.$$

Put $r = 5/2$

$$2(m+5/2)(m+5/2-1) - (m+5/2) - 5) a_m + a_{m-1} = 0.$$

$$m(2m+7) a_m + a_{m-1} = 0.$$

$$a_m = -\frac{a_{m-1}}{m(2m+7)}$$

Put $m=1$

$$a_1 = a_0/9$$

Put $m=2$ $\quad a_2 = \frac{a_0}{198}$

Put $m=3$ $\quad a_3 = \frac{-a_0}{7722}$

$$y = a_0\left(x^{5/2} - \frac{1}{9}x^{7/2} + \frac{1}{198}x^{9/2} - \frac{1}{7722}x^{11/2} + \dots\right)$$

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⇒ When the roots of the indicial eqn are equal.

$$\frac{x^2 d^2y}{dx^2} + (1+x)\frac{dy}{dx} + 2y = 0.$$

$$\frac{d^2y}{dx^2} + \frac{(1+x)}{x}\frac{dy}{dx} + \frac{2}{x}y = 0.$$

$x=0$ is a regular singular point

$$y = \sum_{m=0}^{\infty} a_m x^{m+r}$$

$$y' = \sum_{m=0}^{\infty} (m+r) a_m x^{m+r-1}$$

$$y'' = \sum (m+r)(m+r-1) a_m x^{m+r-2}$$

$$= x\sum (m+r)(m+r-1) a_m x^{m+r-2} + (1+x)\sum (m+r) a_m x^{m+r-1} + 2\sum a_m x^m = 0.$$

$$= \sum (m+r)(m+r-1) a_m x^{m+r-1} + \sum (m+r) a_m x^{m+r-1} + \sum (m+r) a_m x^{m+r} + 2\sum a_m x^m = 0.$$

$$= (m+r)^2 a_m x^{m+r-1} + \sum (m+r+2) a_m x^{m+r} = 0$$

To get the indicial equation equate the coeff of lowest degree term x^{r-1} to zero.

$r^2 a_0 = 0 \quad a_0 \neq 0$

$r^2 = 0$

$r = 0, 0$

To get this recurrence formula, equate the coeff. of x^{m+r} to zero.

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$$(m+r+1)^2 a_{m+1} + (m+r+2) a_m = 0.$$

$$a_{m+1} = \frac{-(m+r+2) a_m}{(m+r+1)^2}$$

$$a_1 = -\frac{(r+2)}{(r+1)^2} a_0$$

$$a_2 = \frac{-(r+3)}{(r+2)^2} a_1$$

$$= \frac{r+3}{(r+1)^2 (r+2)} a_0$$

$$a_3 = \frac{-(r+4)}{(r+1)^2 (r+2)(r+3)} a_0$$

$$a_0 x^2 \left[1 - \frac{r+2}{(r+1)^2} x + \frac{r+3}{(r+1)^2 (r+2)} \frac{x^2}{} - \frac{x+4}{(r+1)^2} \frac{x^2}{(r+2)(r+3)} + \cdots \right] \quad (1)$$

put $r=0$

$$y_1(x) = a_0 \left[1 - 2x + \frac{3x^2}{2!} + \frac{4x^3}{3!} + \cdots \right]$$

differentiating (1) partially w.r.to r.

$$\frac{\partial y}{\partial r} = a_0 x^r \log x \left[1 - \frac{r+2r}{(r+1)^2} + \frac{(r+3)}{(r+1)^2 (r+2)} x^2 - \frac{r+4}{(r+1)^2 (r+2)(r+3)} (x^3) \right]$$

$$+ a_0 x^r \left[0 - \frac{r+2}{(r+1)^2} \right.$$

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$$\left(\frac{1}{r+2} - \frac{2}{r+1}\right)x + \frac{r+3}{(r+1)^2(r+2)}\left(\frac{1}{r+3} - \frac{2}{r+1} - \frac{1}{r+2}\right)x^2$$

$$y = \frac{r+2}{(r+1)^2}$$

$$\log y = \log(r+2) - 2\log(r+1)$$

$$\frac{1}{y}\frac{dy}{dr} = \frac{1}{r+2} - \frac{2}{r+1}$$

$$\frac{dy}{dr} = y\left(\frac{1}{r+2} - \frac{2}{r+1}\right)$$

$$= \frac{r+2}{(r+1)^2}\left(\frac{1}{r+2} - \frac{2}{r+1}\right)$$

$$\left(\frac{\partial y}{\partial r}\right)_{r=0} = a_0 \log x\left[1 - 2x + \frac{3}{2!}x^2 - \frac{4}{3!}x^3 - \cdots\right]$$

$$+ a_0\left[-2\left(\frac{1}{2} - \frac{2}{1}\right) + \frac{3}{2}\left(\frac{1}{3} - \frac{2}{1} - \frac{1}{2}\right)x^2\right]$$

$$y = c_1 y_1(x) + c_2 y_2(x)$$

$$= (A + B\log x)\left(1 - 2x + \frac{3}{2!}x^2 - \frac{4}{3!}x^3 - \cdots\right)$$

$$+ B\left(3x - \frac{3}{2!}\left(2 + \frac{1}{2} - \frac{1}{3}\right)x^2 + \right.$$

$$\left.\frac{4}{3!}\left(2 + \frac{1}{2} + \frac{1}{3} - \frac{1}{4}\right)x^3\right.$$

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The Beta function:

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \beta(n,m).$$

$x = 1-y$
$dx = -dy$
$x=0, y=1$
$x=1, y=0$

$$\int_1^0 (1-y)^{m-1} y^{n-1}(-dy) = \int_0^1 y^{n-1}(1-y)^{m-1}dy$$

$$= \beta(n,m)$$

→ $x = \sin^2\theta$
$dx = 2\sin\theta\cos\theta\, d\theta.$
$x=0, \theta=0$
$x=1, \theta=\pi/2.$

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta\, d\theta$$ } Euler's integral of first kind.

GAMMA function:

$$\Gamma n = \int_0^\infty e^{-x} \cdot x^{n-1}dx$$

$$\Gamma n+1 = \int_0^\infty e^{-x} x^n\, dx$$

$$\Gamma n+1 = n\Gamma n$$
$$= n\cdot(n-1)\Gamma n-1$$
$$= n\cdot(n-1)(n-2)\Gamma n-2$$
$$= n\cdot(n-1)(n-2) \cdots (n-(n-1))\Gamma n-n$$
$$= n!$$

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Q $\int_0^\infty e^{-x} \cdot x^{-1/2} dx$

$\Rightarrow \Gamma\frac{1}{2} = \int_0^\infty e^{-x} \cdot x^{-1/2} dx.$

$x = y^2$

$dx = 2y \cdot dy$

$= 2\int_0^\infty e^{-y^2} dy$ — ①

$\Gamma\frac{1}{2} = 2\int_0^\infty e^{-x^2} dx$ — ②

$\left(\Gamma\frac{1}{2}\right)^2 = 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\, dy$

$x^2+y^2 = r^2$

$= 4\int_0^{\pi/2}\int_0^\infty (e^{-r^2} \cdot r dr) d\theta.$

$= 2\int_0^{\pi/2} d\theta = \pi$

$\left(\Gamma\frac{1}{2}\right)^2 = \pi$

$\Gamma\frac{1}{2} = \sqrt{\pi}$

gamma $(\frac{1}{2})$ = root (π)

Relation between GAMMA and BETA function :

$$\boxed{\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}}$$

proof :

$$\Gamma m = \int_0^\infty e^{-t} \cdot t^{m-1} \, dt$$

now $t = x^2$

$dt = 2x\,dx$

$$\Gamma m = \int_0^\infty e^{-x^2} (x^2)^{m-1} \, 2x\,dx.$$

$$= 2\int_0^\infty e^{-x^2} \cdot x^{2m-1} \, dx \quad \text{—} (1)$$

Similarly,

$$\Gamma n = 2\int_0^\infty -y^2 \, y^{2n-1} \cdot dy. \quad \text{—} (2)$$

mult. (1) & (2) :

now,

$$\Gamma m \, \Gamma n = 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} \, x^{2m-1} \cdot y^{2n-1} \, dx\,dy.$$

$x = r\cos\theta$

$y = r\sin\theta$

$x^2+y^2 = r^2$

$\theta = \tan^{-1} y/x$

$$= 4\int_0^{\pi/2}\int_0^\infty e^{-r^2} \, r^{2m-1} \cdot \cos^{2m-1}\theta \cdot r^{2n-1} \cdot \sin^{2n-1}\theta \; r\,dr\,d\theta.$$

$$= 2\underbrace{\int_0^{\pi/2} \cos^{2m-1}\theta \cdot \sin^{2n-1}\theta \, d\theta}_{\beta(m,n)} \cdot \underbrace{2\int_0^\infty e^{-r^2} \, r^{2(m+n)-1} \, dr}_{\Gamma m+n}$$

$$\Gamma m \, \Gamma n = \beta(m,n) \, (\Gamma m+n)$$

$$\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n} \longrightarrow \text{Hence proved}$$

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Corolinily eqn

$$\int_0^{\pi/2} \sin^p\theta \cdot \cos^q\theta \, d\theta = \frac{\Gamma\frac{p+1}{2} \cdot \Gamma\frac{q+1}{2}}{2\,\Gamma\frac{p+q+2}{2}}$$

—x—

Q. $\Gamma n = \int_0^1 \left(\log\left(\frac{1}{y}\right)\right)^{n-1} dy$ Proved

$\log\left(\frac{1}{y}\right) = x.$

$\frac{1}{y} = e^x$

$y = e^{-x}$

$dy = -e^{-x} dx.$

$\begin{cases} y=0, & x=\infty \\ y=1, & x=0. \end{cases}$

$\int_\infty^0 (x)^{n-1}(-e^{-x} dx)$

due to (-ve) sign limit changes.

$= \int_0^\infty e^{-x} \cdot x^{n-1} dx$

hence proved.

—x—

Q. $\int_0^\infty \frac{x^c}{c^x} \cdot dx.$

We can write

$c^x = e^{\log c^x} = e^{x\log c}.$

$= \int_0^\infty \frac{x^c}{e^{x\log c}} \cdot dx$

put $x\log c = t$

$dx = \frac{1}{\log c} dt$

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$$= \int_0^\infty \frac{\left(\frac{t}{\log c}\right)^c}{e^t} \cdot \frac{1}{\log c}\, dt$$

$$= \int_0^\infty \left(\frac{t}{\log c}\right)^c \cdot e^{-t} \frac{1}{\log c} \cdot dt.$$

$$= \frac{1}{(\log c)^{c+1}} \int_0^\infty e^{-t} \cdot t^c \cdot dt$$

$$= \frac{\Gamma c+1}{(\log c)^{c+1}}$$

Ans

Q. convert $\int_0^\infty a^{-b} x^2 dx$ into GAMMA fn.

$$= \int_0^\infty e^{-bx^2 \log a}\, dx.$$

Put $(b\log a)\, x^2 = t$ $\quad x = \sqrt{\frac{t}{b\log a}}$

$$2x\,dx = \frac{dt}{b\log a} \Big\} \rightarrow dx = \frac{dt}{2\sqrt{\frac{t}{b\log a}}\,(b\log a)}$$

$$= \frac{1}{2\sqrt{b\log a}} \underbrace{\int_0^\infty e^{-t}\, t^{-1/2}\, dt}_{\Gamma 1/2}$$

$$= \frac{\Gamma 1/2}{2\sqrt{b\log a}} = \frac{\sqrt{\pi}}{2\sqrt{b\log a}}$$

Integ^n Rule: $\int_0^1 x^m(1-x^n)^p\,dx$ $\quad$ $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

$\hookrightarrow x^n = t$

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Q. convert $\int_0^1 x^{3/2}(1-\sqrt{x})^{1/2}\,dx$ into GAMMA fn.

$m = 3/2,\ n = 1/2,\ p = 1/2$

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \beta(m,n)$$

using this

put $\sqrt{x} = t$

$x = t^2$

$dx = 2t\,dt.$

$$\int_0^1 (t^2)^{3/2}(1-t)^{1/2}\,2t\,dt.$$

$$= 2\int_0^1 e^4(1-t)^{1/2}\,dt.$$

$$= 2\int_0^1 e^4(1-t)^{1/2}\,dt.$$

$$= 2\beta(5, 3/2)$$

$m = 5$

$n = 3/2$

$$\beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma m+n}$$

$$2\beta(5,3/2) = \frac{\cancel{2}\,\Gamma 5\,\Gamma 3/2}{\cancel{2}\,\Gamma 5+3/2}$$

$$= \frac{24\cdot\frac{1}{2}\sqrt{1/2}}{\Gamma\frac{15}{3}}$$