

# 12: Support Vector Machines (SVMs)

[Previous](#) [Next](#) [Index](#)

## Support Vector Machine (SVM) - Optimization objective

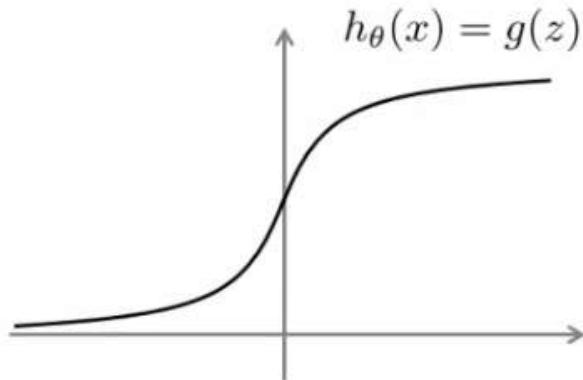
- So far, we've seen a range of different algorithms
  - With supervised learning algorithms - performance is pretty similar
    - What matters more often is;
      - The amount of training data
      - Skill of applying algorithms
- One final supervised learning algorithm that is widely used - **support vector machine (SVM)**
  - Compared to both logistic regression and neural networks, a SVM sometimes gives a cleaner way of learning non-linear functions
  - Later in the course we'll do a survey of different supervised learning algorithms

### An alternative view of logistic regression

- Start with logistic regression, see how we can modify it to get the SVM
  - As before, the logistic regression hypothesis is as follows

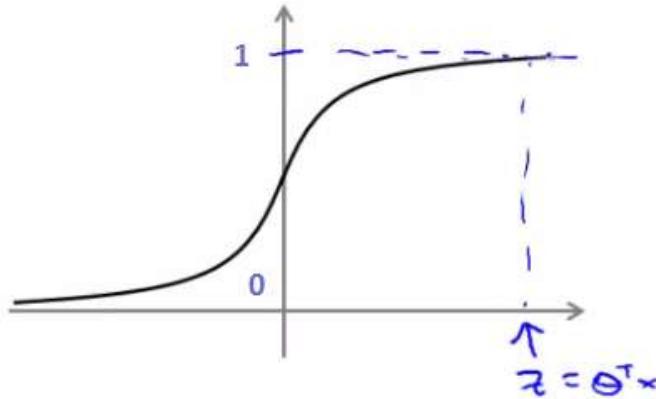
$$h_{\theta}(x) = \frac{1}{1 + e^{-\theta^T x}}$$

- And the sigmoid activation function looks like this



- In order to explain the math, we use  $z$  as defined above
- What do we want logistic regression to do?
  - We have an example where  $y = 1$ 
    - Then we hope  $h_{\theta}(x)$  is close to 1

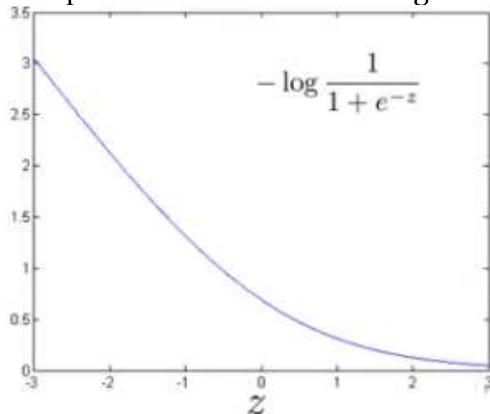
- With  $h_\theta(x)$  close to 1,  $(\theta^T x)$  must be **much larger** than 0



- Similarly, when  $y = 1$ 
  - Then we hope  $h_\theta(x)$  is close to 1
  - With  $h_\theta(x)$  close to 1,  $(\theta^T x)$  must be **much less** than 0
- This is our classic view of logistic regression
  - Let's consider another way of thinking about the problem
- Alternative view of logistic regression
  - If you look at cost function, each example contributes a term like the one below to the overall cost function
$$-(y \log h_\theta(x) + (1 - y) \log(1 - h_\theta(x)))$$
  - For the overall cost function, we sum over all the training examples using the above function, and have a  $1/m$  term
- If you then plug in the hypothesis definition ( $h_\theta(x)$ ), you get an expanded cost function equation;

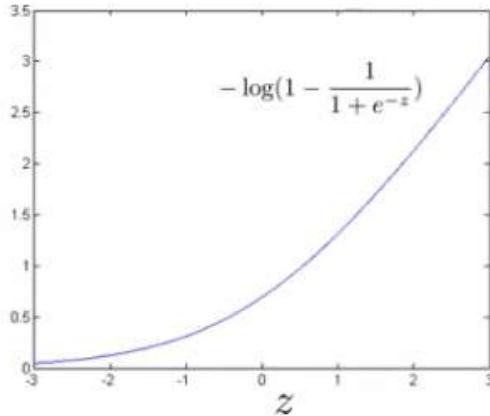
$$= -y \log \frac{1}{1 + e^{-\theta^T x}} - (1 - y) \log\left(1 - \frac{1}{1 + e^{-\theta^T x}}\right)$$

- So each training example contributes that term to the cost function for logistic regression
- If  $y = 1$  then only the first term in the objective matters
  - If we plot the functions vs.  $z$  we get the following graph



- This plot shows the cost contribution of an example when  $y = 1$  given  $z$ 
  - So if  $z$  is big, the cost is low - this is good!
  - But if  $z$  is 0 or negative the cost contribution is high
  - This is why, when logistic regression sees a positive example, it tries to set  $\theta^T x$  to be a very large term
- If  $y = 0$  then only the second term matters

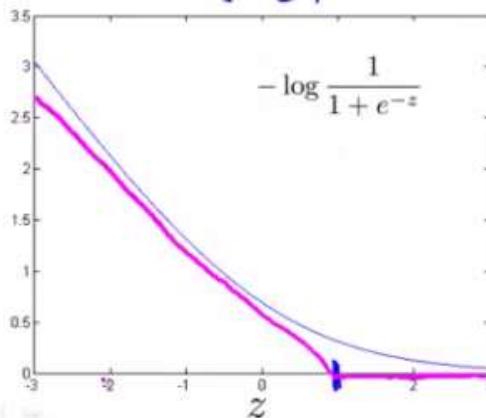
- We can again plot it and get a similar graph



- Same deal, if  $z$  is small then the cost is low
  - But if  $z$  is large then the cost is massive

## SVM cost functions from logistic regression cost functions

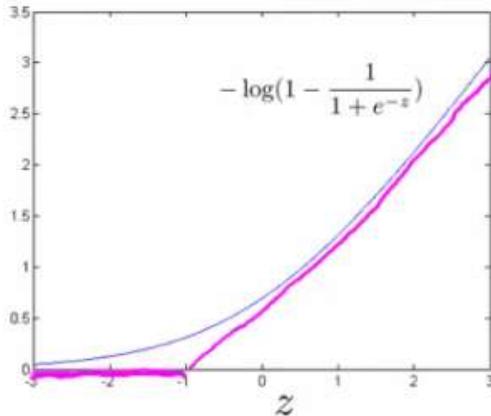
- To build a SVM we must redefine our cost functions
  - When  $y = 1$ 
    - Take the  $y = 1$  function and create a new cost function
    - Instead of a curved line create two straight lines (magenta) which acts as an approximation to the logistic regression  $y = 1$  function



- Take point (1) on the  $z$  axis
  - Flat from 1 onwards
  - Grows when we reach 1 or a lower number
- This means we have two straight lines
  - Flat when cost is 0
  - Straight growing line after 1
- So this is the new  $y=1$  cost function
  - Gives the SVM a computational advantage and an easier optimization problem
  - We call this function **cost<sub>1</sub>(z)**

- Similarly
  - When  $y = 0$

- Do the equivalent with the  $y=0$  function plot



- We call this function  $\text{cost}_0(z)$

- So here we define the two cost function terms for our SVM graphically
  - How do we implement this?

## The complete SVM cost function

- As a comparison/reminder we have logistic regression below

$$\min_{\theta} \frac{1}{m} \left[ \sum_{i=1}^m y^{(i)} \left( -\log h_{\theta}(x^{(i)}) \right) + (1 - y^{(i)}) \left( (-\log(1 - h_{\theta}(x^{(i)}))) \right) \right] + \frac{\lambda}{2m} \sum_{j=1}^n \theta_j^2$$

- If this looks unfamiliar its because we previously had the - sign outside the expression

- For the SVM we take our two logistic regression  $y=1$  and  $y=0$  terms described previously and replace with

- $\text{cost}_1(\theta^T x)$
- $\text{cost}_0(\theta^T x)$

- So we get

$$\min_{\theta} \frac{1}{m} \sum_{i=1}^m \left[ y^{(i)} \text{cost}_1(\theta^T x^{(i)}) + (1 - y^{(i)}) \text{cost}_0(\theta^T x^{(i)}) \right] + \frac{\lambda}{2m} \sum_{i=1}^n \theta_j^2$$

## SVM notation is slightly different

- In convention with SVM notation we rename a few things here

- 1) Get rid of the  $1/m$  terms

- This is just a slightly different convention
- By removing  $1/m$  we should get the same optimal values for
  - $1/m$  is a constant, so should get same optimization
  - e.g. say you have a minimization problem which minimizes to  $u = 5$ 
    - If your cost function \* by a constant, you still generates the minimal value
    - That minimal value is different, but that's irrelevant

- 2) For logistic regression we had two terms;

- Training data set term (i.e. that we sum over  $m$ ) = **A**
- Regularization term (i.e. that we sum over  $n$ ) = **B**
  - So we could describe it as  $A + \lambda B$
  - Need some way to deal with the trade-off between regularization and data set terms
  - Set different values for  $\lambda$  to parametrize this trade-off
- Instead of parameterization this as  $A + \lambda B$ 
  - For SVMs the convention is to use a different parameter called **C**
  - So do  $CA + B$

- If  $C$  were equal to  $1/\lambda$  then the two functions ( $CA + B$  and  $A + \lambda B$ ) would give the same value
- So, our overall equation is

$$\min_{\theta} C \sum_{i=1}^m \left[ y^{(i)} \text{cost}_1(\theta^T x^{(i)}) + (1 - y^{(i)}) \text{cost}_0(\theta^T x^{(i)}) \right] + \frac{1}{2} \sum_{j=1}^n \theta_j^2$$

- Unlike logistic,  $h_{\theta}(x)$  doesn't give us a probability, but instead we get a direct prediction of 1 or 0
  - So if  $\theta^T x$  is equal to or greater than 0  $\rightarrow h_{\theta}(x) = 1$
  - Else  $\rightarrow h_{\theta}(x) = 0$

## Large margin intuition

- Sometimes people refer to SVM as **large margin classifiers**
  - We'll consider what that means and what an SVM hypothesis looks like
  - The SVM cost function is as above, and we've drawn out the cost terms below



If  $y = 1$ , we want  $\theta^T x \geq 1$  (not just  $\geq 0$ )

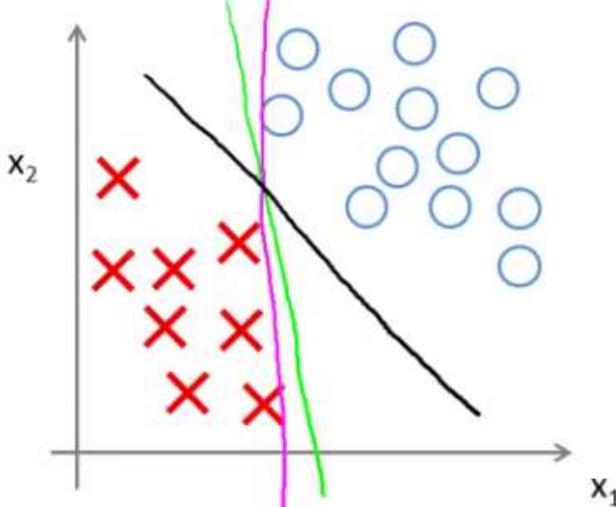
If  $y = 0$ , we want  $\theta^T x \leq -1$  (not just  $< 0$ )

- Left is  $\text{cost}_1$  and right is  $\text{cost}_0$
- What does it take to make terms small
  - If  $y = 1$ 
    - $\text{cost}_1(z) = 0$  only when  $z \geq 1$
  - If  $y = 0$ 
    - $\text{cost}_0(z) = 0$  only when  $z \leq -1$
- Interesting property of SVM
  - If you have a positive example, you only really *need*  $z$  to be greater or equal to 0
    - If this is the case then you predict 1
  - SVM wants a bit more than that - doesn't want to \*just\* get it right, but have the value be quite a bit bigger than zero
    - Throws in an extra safety margin factor
- Logistic regression does something similar
- What are the consequences of this?
  - Consider a case where we set  $C$  to be huge
    - $C = 100,000$
    - So considering we're minimizing  $CA + B$ 
      - If  $C$  is huge we're going to pick an  $A$  value so that  $A$  is equal to zero
      - What is the optimization problem here - how do we make  $A = 0$ ?
    - Making  $A = 0$ 
      - If  $y = 1$ 
        - Then to make our "A" term 0 need to find a value of  $\theta$  so  $(\theta^T x)$  is greater than or equal to 1
      - Similarly, if  $y = 0$ 
        - Then we want to make "A" = 0 then we need to find a value of  $\theta$  so  $(\theta^T x)$  is equal to or less than -1

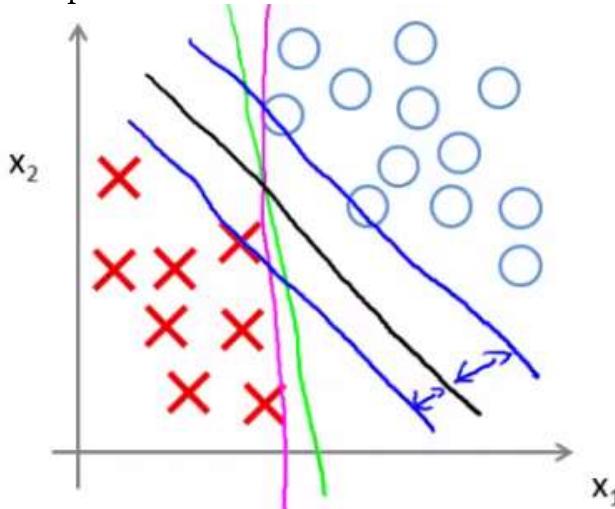
- So - if we think of our optimization problem a way to ensure that this first "A" term is equal to 0, we re-factor our optimization problem into just minimizing the "B" (regularization) term, because
  - When  $A = 0 \rightarrow A^T C = 0$
- So we're minimizing B, under the constraints shown below

$$\begin{aligned} \min \quad & \frac{1}{2} \sum_{j=1}^n \theta_j^2 \\ \text{s.t.} \quad & \theta^T x^{(i)} \geq 1 \quad \text{if } y^{(i)} = 1 \\ & \theta^T x^{(i)} \leq -1 \quad \text{if } y^{(i)} = 0 \end{aligned}$$

- Turns out when you solve this problem you get interesting decision boundaries

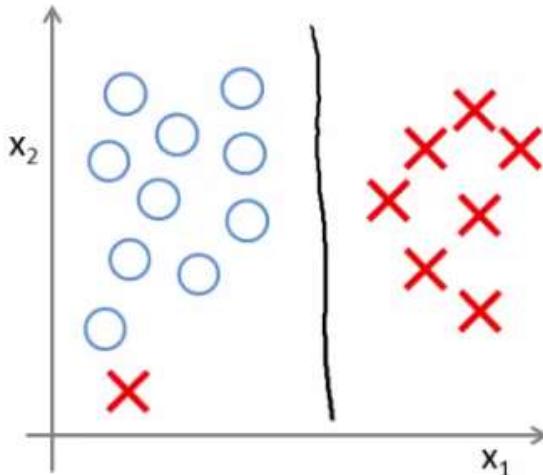


- The green and magenta lines are functional decision boundaries which could be chosen by logistic regression
  - But they probably don't generalize too well
- The black line, by contrast is the the chosen by the SVM because of this safety net imposed by the optimization graph
  - More robust separator
- Mathematically, that black line has a larger minimum distance (margin) from any of the training examples

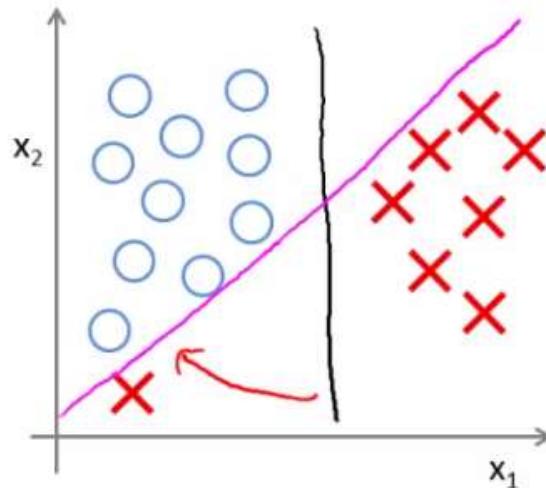


- By separating with the largest margin you incorporate robustness into your decision making process
- We looked at this at when C is very large
  - SVM is more sophisticated than the large margin might look

- If you were just using large margin then SVM would be very sensitive to outliers



- You would risk making a ridiculous huge impact your classification boundary
  - A single example might not represent a good reason to change an algorithm
  - If  $C$  is very large then we *do* use this quite naive maximize the margin approach



- So we'd change the black to the magenta
- But if  $C$  is reasonably small, or not too large, then you stick with the black decision boundary
- What about non-linearly separable data?
  - Then SVM still does the right thing if you use a normal size  $C$
  - So the idea of SVM being a large margin classifier is only really relevant when you have no outliers and you can easily linearly separate data
- Means we ignore a few outliers

## Large margin classification mathematics (optional)

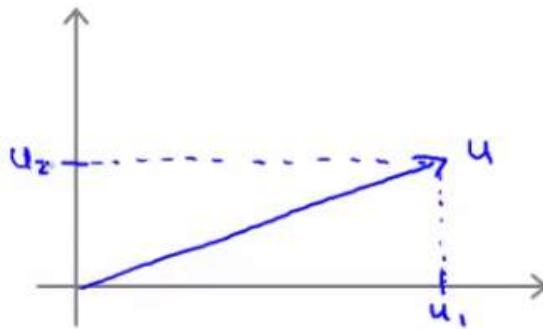
### Vector inner products

- Have two (2D) vectors  $u$  and  $v$  - what is the inner product ( $u^T v$ )?

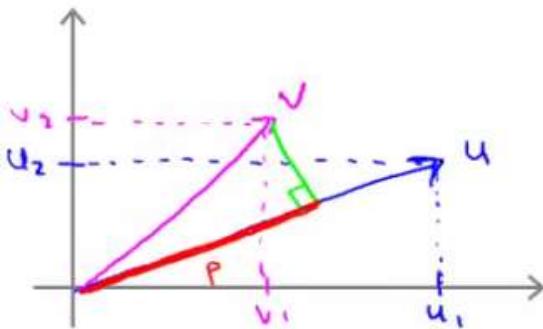
$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

- Plot  $u$  on graph

- i.e  $u_1$  vs.  $u_2$

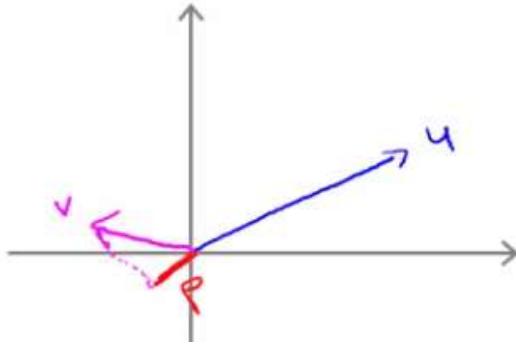


- One property which is good to have is the **norm** of a vector
  - Written as  $\|u\|$ 
    - This is the Euclidean length of vector  $u$
  - So  $\|u\| = \sqrt{u_1^2 + u_2^2}$  = real number
    - i.e. length of the arrow above
    - Can show via Pythagoras
- For the inner product, take  $v$  and orthogonally project down onto  $u$ 
  - First we can plot  $v$  on the same axis in the same way ( $v_1$  vs  $v_1$ )
  - Measure the length/magnitude of the projection



- So here, the green line is the projection
  - $p$  = length along  $u$  to the intersection
  - $p$  is the magnitude of the projection of vector  $v$  onto vector  $u$
- Possible to show that
  - $u^T v = p * \|u\|$ 
    - So this is one way to compute the inner product
  - $u^T v = u_1 v_1 + u_2 v_2$ 
    - So therefore
      - $p * \|u\| = u_1 v_1 + u_2 v_2$
      - This is an important rule in linear algebra
  - We can reverse this too
    - So we could do
      - $v^T u = v_1 u_1 + v_2 u_2$
      - Which would obviously give you the same number

- p can be negative if the angle between them is 90 degrees or more



- So here p is negative
- Use the vector inner product theory to try and understand SVMs a little better

### SVM decision boundary

$$\min_{\theta} \frac{1}{2} \sum_{j=1}^n \theta_j^2$$

$$\text{s.t. } \begin{aligned} \theta^T x^{(i)} &\geq 1 & \text{if } y^{(i)} = 1 \\ \theta^T x^{(i)} &\leq -1 & \text{if } y^{(i)} = 0 \end{aligned}$$

- For the following explanation - two simplification
  - Set  $\theta_0 = 0$  (i.e. ignore intercept terms)
  - Set n = 2 -  $(x_1, x_2)$ 
    - i.e. each example has only 2 features
- Given we only have two parameters we can simplify our function to

$$\frac{1}{2} (\theta_1^2 + \theta_2^2)$$

- And, can be re-written as

$$\frac{1}{2} (\sqrt{\theta_1^2 + \theta_2^2})^2$$

- Should give same thing
- We may notice that

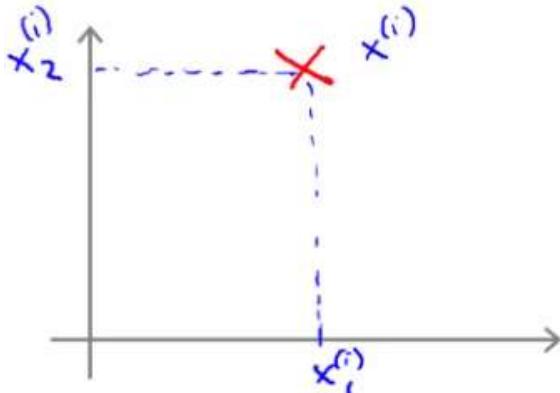
$$\frac{1}{2} (\underbrace{\sqrt{\theta_1^2 + \theta_2^2}}_{= \|\theta\|})^2$$

- The term in red is the norm of  $\theta$ 
  - If we take  $\theta$  as a  $2 \times 1$  vector
  - If we assume  $\theta_0 = 0$  its still true
- So, finally, this means our optimization function can be re-defined as

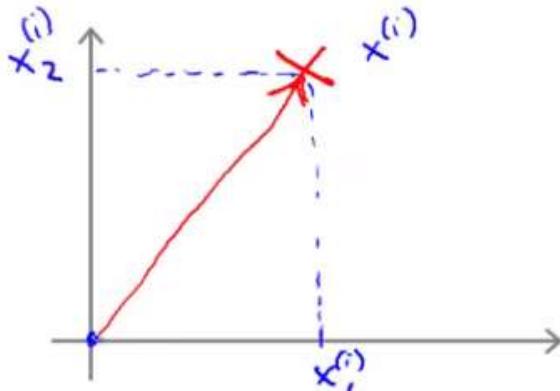
$$= \frac{1}{2} \|\theta\|^2$$

- So the SVM is minimizing the squared norm
- Given this, what are the  $(\theta^T x)$  parameters doing?
  - Given  $\theta$  and given example  $x$  what is this equal to
    - We can look at this in a comparable manner to how we just looked at  $u$  and  $v$

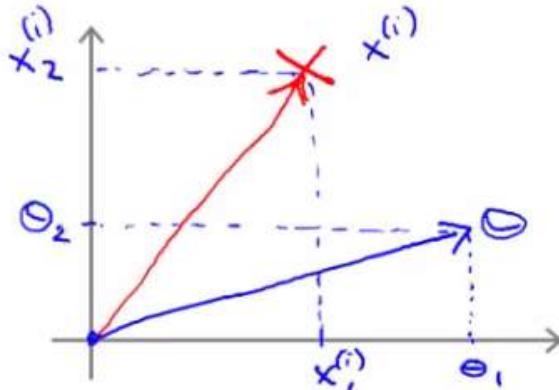
- Say we have a single positive training example (red cross below)



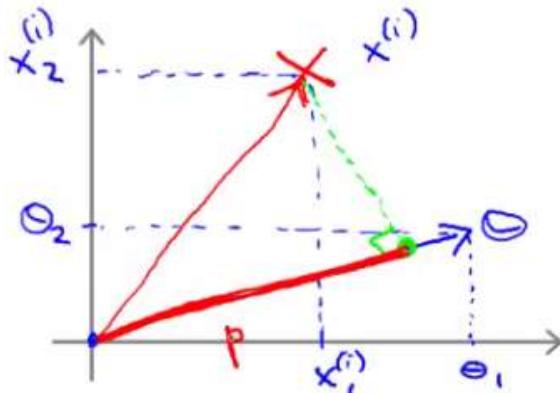
- Although we haven't been thinking about examples as vectors it can be described as such



- Now, say we have our parameter vector \$\theta\$ and we plot that on the same axis



- The next question is what is the inner product of these two vectors



- \$p\_i\$ is in fact \$p^i\$, because it's the length of \$p\$ for example \$i\$
  - Given our previous discussion we know
- $$\begin{aligned} (\theta^T x^i) &= p^i * ||\theta|| \\ &= \theta_1 x_1^i + \theta_2 x_2^i \end{aligned}$$

- So these are both equally valid ways of computing  $\theta^T x^i$

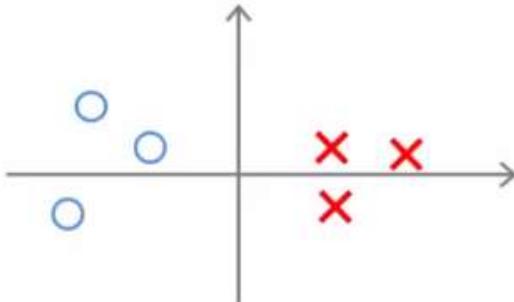
- What does this mean?
  - The constraints we defined earlier
    - $(\theta^T x) \geq 1$  if  $y = 1$
    - $(\theta^T x) \leq -1$  if  $y = 0$
  - Can be replaced/substituted with the constraints
    - $p^i * \|\theta\| \geq 1$  if  $y = 1$
    - $p^i * \|\theta\| \leq -1$  if  $y = 0$
  - Writing that into our optimization objective

$$\min_{\theta} \frac{1}{2} \sum_{j=1}^n \theta_j^2 = \frac{1}{2} \|\theta\|^2$$

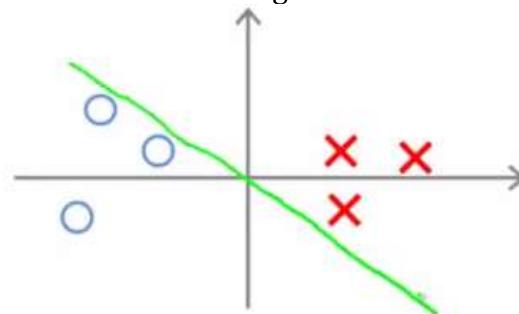
s.t.  $p^{(i)} \cdot \|\theta\| \geq 1 \quad \text{if } y^{(i)} = 1$

$p^{(i)} \cdot \|\theta\| \leq -1 \quad \text{if } y^{(i)} = 0$

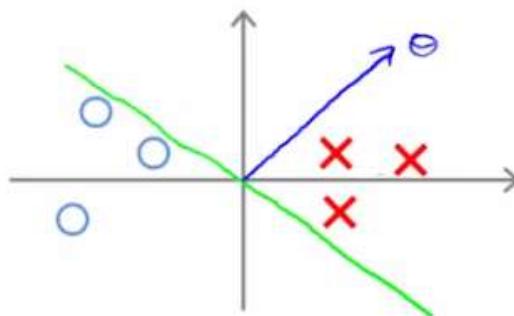
- So, given we've redefined these functions let us now consider the training example below



- Given this data, what boundary will the SVM choose? Note that we're still assuming  $\theta_0 = 0$ , which means the boundary has to pass through the origin (0,0)
  - Green line - small margins



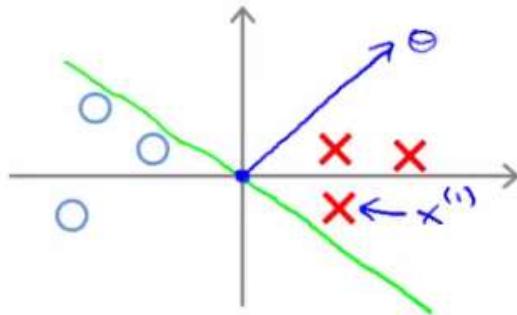
- SVM would not chose this line
  - Decision boundary comes very close to examples
  - Lets discuss *why* the SVM would **not** chose this decision boundary
- Looking at this line
  - We can show that  $\theta$  is at 90 degrees to the decision boundary



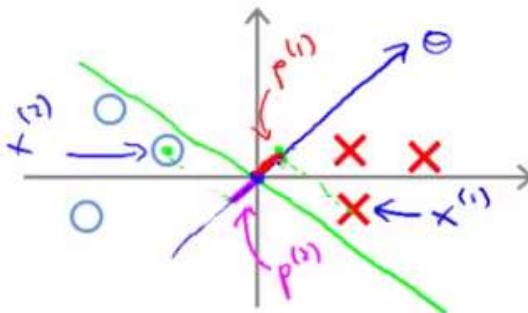
- **$\theta$  is always at 90 degrees to the decision boundary** (can show with linear algebra, although we're not going to!)

- So now lets look at what this implies for the optimization objective

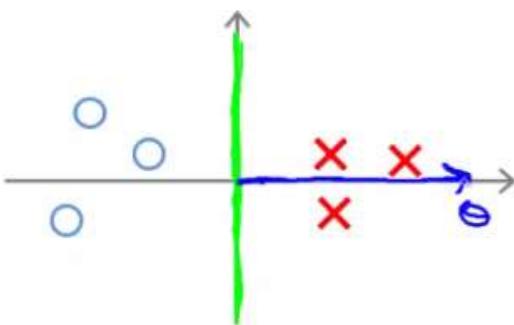
- Look at first example ( $x^1$ )



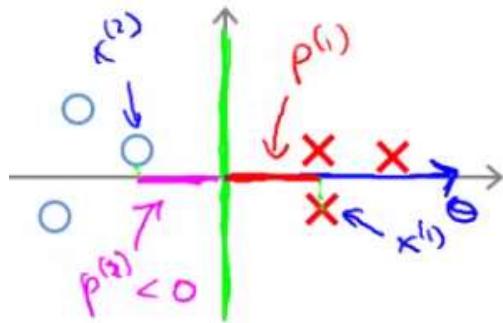
- Project a line from  $x^1$  on to the  $\theta$  vector (so it hits at 90 degrees)
    - The distance between the intersection and the origin is ( $p^1$ )
  - Similarly, look at second example ( $x^2$ )
    - Project a line from  $x^2$  into to the  $\theta$  vector
    - This is the magenta line, which will be **negative** ( $p^2$ )
  - If we overview these two lines below we see a graphical representation of what's going on;



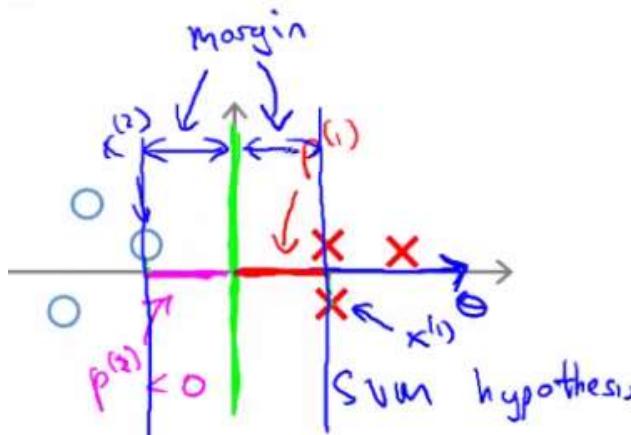
- We find that both these  $p$  values are going to be pretty small
  - If we look back at our optimization objective
    - We know we need  $p^1 * ||\theta||$  to be bigger than or equal to 1 for positive examples
      - If  $p$  is small
        - Means that  $||\theta||$  must be pretty large
    - Similarly, for negative examples we need  $p^2 * ||\theta||$  to be smaller than or equal to -1
      - We saw in this example  $p^2$  is a small negative number
        - So  $||\theta||$  must be a large number
  - Why is this a problem?
    - The optimization objective is trying to find a set of parameters where the norm of theta is small
      - So this doesn't seem like a good direction for the parameter vector (because as  $p$  values get smaller  $||\theta||$  must get larger to compensate)
        - So we should make  $p$  values larger which allows  $||\theta||$  to become smaller
  - So lets chose a different boundary



- Now if you look at the projection of the examples to  $\theta$  we find that  $p^1$  becomes large and  $||\theta||$  can become small
- So with some values drawn in



- This means that by choosing this second decision boundary we can make  $||\theta||$  smaller
  - Which is why the SVM chooses this hypothesis as better
  - This is how we generate the large margin effect

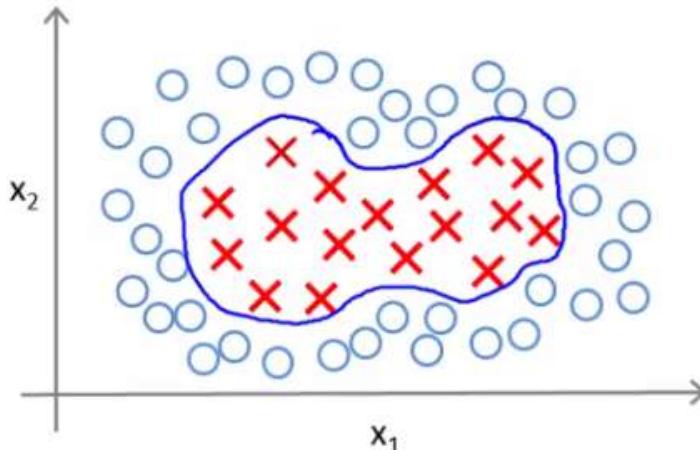


- The magnitude of this margin is a function of the  $p$  values
  - So by maximizing these  $p$  values we minimize  $||\theta||$
- Finally, we did this derivation assuming  $\theta_0 = 0$ ,
  - If this is the case we're entertaining only decision boundaries which pass through (0,0)
  - If you allow  $\theta_0$  to be other values then this simply means you can have decision boundaries which cross through the  $x$  and  $y$  values at points other than (0,0)
  - Can show with basically same logic that this works, and even when  $\theta_0$  is non-zero when you have optimization objective described above (when  $C$  is very large) that the SVM is looking for a large margin separator between the classes

## Kernels - 1: Adapting SVM to non-linear classifiers

- What are kernels and how do we use them
  - We have a training set

- We want to find a non-linear boundary

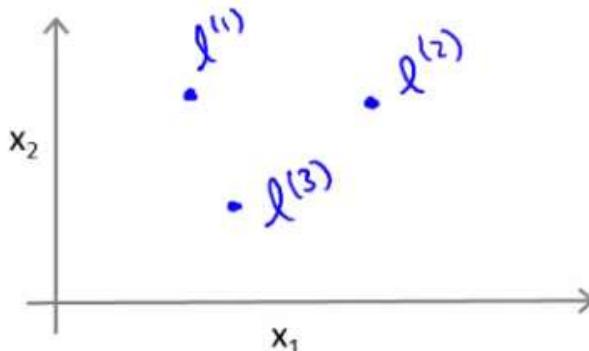


- Come up with a complex set of polynomial features to fit the data

- Have  $h_\theta(x)$  which
  - Returns 1 if the combined weighted sum of vectors (weighted by the parameter vector) is less than or equal to 0
  - Else return 0
- Another way of writing this (new notation) is
  - That a hypothesis computes a decision boundary by taking the sum of the parameter vector multiplied by a **new feature vector  $f$** , which simply contains the various high order  $x$  terms
  - e.g.
    - $h_\theta(x) = \theta_0 + \theta_1 f_1 + \theta_2 f_2 + \theta_3 f_3$
    - Where
      - $f_1 = x_1$
      - $f_2 = x_1 x_2$
      - $f_3 = \dots$
      - i.e. not specific values, but each of the terms from your complex polynomial function
  - Is there a better choice of feature  $f$  than the high order polynomials?
    - As we saw with computer imaging, high order polynomials become computationally expensive

- New features

- Define three features in this example (ignore  $x_0$ )
- Have a graph of  $x_1$  vs.  $x_2$  (don't plot the values, just define the space)
- Pick three points in that space



- These points  $l^1$ ,  $l^2$ , and  $l^3$ , were chosen manually and are called **landmarks**
  - Given  $x$ , define  $f_1$  as the similarity between  $(x, l^1)$

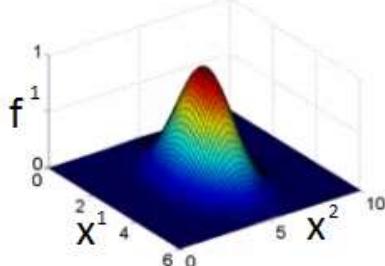
- $= \exp(-(\|x - l^1\|^2) / 2\sigma^2)$
- $$= \exp\left(-\frac{\|x - l^{(1)}\|^2}{2\sigma^2}\right)$$
- $\|x - l^1\|$  is the euclidean distance between the point  $x$  and the landmark  $l^1$  squared
  - Disussed more later
- If we remember our statistics, we know that
  - $\sigma$  is the **standard deviation**
  - $\sigma^2$  is commonly called the **variance**
- Remember, that as discussed

$$\|x - l^{(1)}\|^2 = \sum_{j=1}^n (x_j - l_j^{(1)})^2$$

- So,  $f_2$  is defined as
  - $f_2 = \text{similarity}(x, l^1) = \exp(-(\|x - l^1\|^2) / 2\sigma^2)$
- And similarly
  - $f_3 = \text{similarity}(x, l^2) = \exp(-(\|x - l^2\|^2) / 2\sigma^2)$
- This similarity function is called a **kernel**
  - This function is a **Gaussian Kernel**
- So, instead of writing similarity between  $x$  and  $l$  we might write
  - $f_1 = k(x, l^1)$

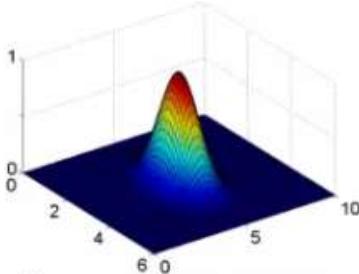
## Diving deeper into the kernel

- So lets see what these kernels do and why the functions defined make sense
  - Say  $x$  is close to a landmark
    - Then the squared distance will be  $\sim 0$ 
      - So
    - Which is basically  $e^{-0}$ 
      - Which is close to 1
  - Say  $x$  is far from a landmark
    - Then the squared distance is big
      - Gives  $e^{-\text{large number}}$ 
        - Which is close to zero
    - Each landmark defines a new features
- If we plot  $f_1$  vs the kernel function we get a plot like this
  - Notice that when  $x = [3,5]$  then  $f_1 = 1$
  - As  $x$  moves away from  $[3,5]$  then the feature takes on values close to zero
  - So this measures how close  $x$  is to this landmark

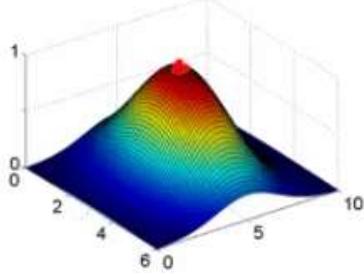


## What does $\sigma$ do?

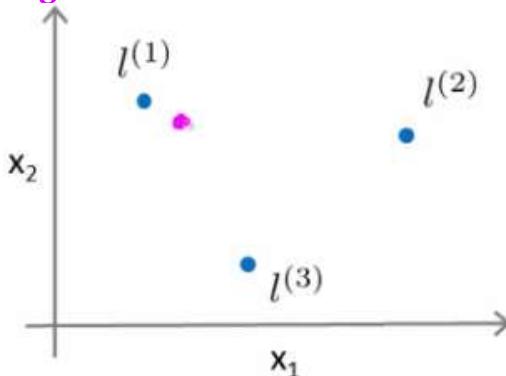
- $\sigma^2$  is a parameter of the Gaussian kernel
  - Defines the steepness of the rise around the landmark
- Above example  $\sigma^2 = 1$
- Below  $\sigma^2 = 0.5$



- We see here that as you move away from 3,5 the feature  $f_1$  falls to zero much more rapidly
- The inverse can be seen if  $\sigma^2 = 3$

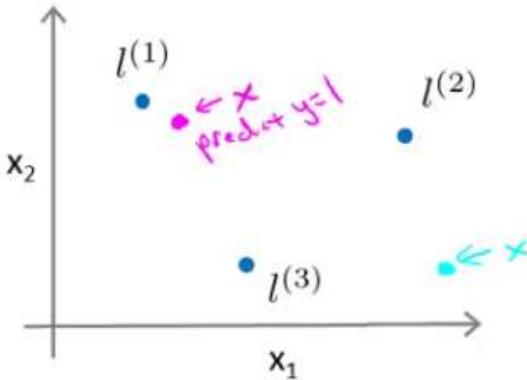


- Given this definition, what kinds of hypotheses can we learn?
  - With training examples  $x$  we predict "1" when
  - $\theta_0 + \theta_1 f_1 + \theta_2 f_2 + \theta_3 f_3 \geq 0$ 
    - For our example, let's say we've already run an algorithm and got the
      - $\theta_0 = -0.5$
      - $\theta_1 = 1$
      - $\theta_2 = 1$
      - $\theta_3 = 0$
    - Given our placement of three examples, what happens if we evaluate an example at the magenta dot below?

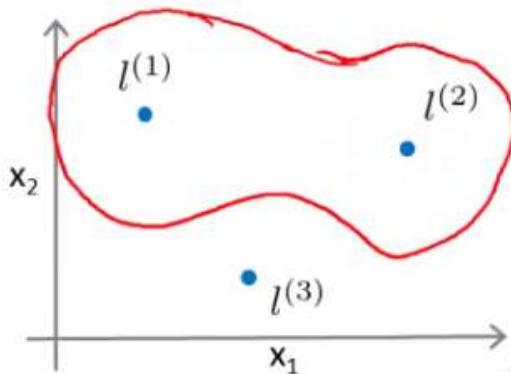


- Looking at our formula, we know  $f_1$  will be close to 1, but  $f_2$  and  $f_3$  will be close to 0
  - So if we look at the formula we have
    - $\theta_0 + \theta_1 f_1 + \theta_2 f_2 + \theta_3 f_3 \geq 0$
    - $-0.5 + 1 + 0 + 0 = 0.5$
    - 0.5 is greater than 1

- If we had **another point** far away from all three



- This equates to -0.5
  - So we predict 0
- Considering our parameter, for points near  $l^1$  and  $l^2$  you predict 1, but for points near  $l^3$  you predict 0
- Which means we create a non-linear decision boundary that goes a lil' something like this;



- Inside we predict  $y = 1$
- Outside we predict  $y = 0$

- So this show how we can create a non-linear boundary with landmarks and the kernel function in the support vector machine
  - But
    - How do we get/choose the landmarks
    - What other kernels can we use (other than the Gaussian kernel)

## Kernels II

- Filling in missing detail and practical implications regarding kernels
- Spoke about picking landmarks manually, defining the kernel, and building a hypothesis function
  - Where do we get the landmarks from?
  - For complex problems we probably want lots of them

### Choosing the landmarks

- Take the training data
- For each example place a landmark at exactly the same location
- So end up with  $m$  landmarks
  - One landmark per location per training example
  - Means our features measure how close to a training set example something is
- Given a new example, compute all the  $f$  values
  - Gives you a feature vector  $f$  ( $f_0$  to  $f_m$ )
    - $f_0 = 1$  always
- A more detailed look at generating the  $f$  vector
  - If we had a training example - features we compute would be using  $(x^i, y^i)$

- So we just cycle through each landmark, calculating how close to that landmark actually  $x^i$  is
  - $f_1^i = k(x^i, l^1)$
  - $f_2^i = k(x^i, l^2)$
  - ...
  - $f_m^i = k(x^i, l^m)$
- Somewhere in the list we compare  $x$  to itself... (i.e. when we're at  $f_i^i$ )
  - So because we're using the Gaussian Kernel this evaluates to 1
- Take these  $m$  features ( $f_1, f_2 \dots f_m$ ) group them into an  $[m+1 \times 1]$  dimensional vector called  $f$ 
  - $f^i$  is the  $f$  feature vector for the  $i$ th example
  - And add a 0th term = 1
- Given these kernels, how do we use a support vector machine

## SVM hypothesis prediction with kernels

- Predict  $y = 1$  if  $(\theta^T f) \geq 0$ 
  - Because  $\theta = [m+1 \times 1]$
  - And  $f = [m+1 \times 1]$
- So, this is how you make a prediction assuming you already have  $\theta$ 
  - How do you get  $\theta$ ?

## SVM training with kernels

- Use the SVM learning algorithm

$$\min_{\theta} C \sum_{i=1}^m y^{(i)} \text{cost}_1(\theta^T f^{(i)}) + (1 - y^{(i)}) \text{cost}_0(\theta^T f^{(i)}) + \frac{1}{2} \sum_{j=1}^n \theta_j^2$$

- Now, we minimize using  $f$  as the feature vector instead of  $x$
  - By solving this minimization problem you get the parameters for your SVM
  - In this setup,  $m = n$ 
    - Because number of features is the number of training data examples we have
  - One final mathematic detail (not crucial to understand)
    - If we ignore  $\theta_0$  then the following is true
- $$\sum_{j=1}^n \theta_j^2 = \theta^T \theta$$
- What many implementations do is
- $$\theta^T M \theta$$
- Where the matrix  $M$  depends on the kernel you use
  - Gives a slightly different minimization - means we determine a rescaled version of  $\theta$
  - Allows more efficient computation, and scale to much bigger training sets
  - If you have a training set with 10 000 values, means you get 10 000 features
    - Solving for all these parameters can become expensive
    - So by adding this in we avoid a for loop and use a matrix multiplication algorithm instead
  - You can apply kernels to other algorithms
    - But they tend to be very computationally expensive
    - But the SVM is far more efficient - so more practical
  - Lots of good off the shelf software to minimize this function

### SVM parameters (C)

- Bias and variance trade off

- Must chose C
  - C plays a role similar to 1/LAMBDA (where LAMBDA is the regularization parameter)
- Large C gives a hypothesis of **low bias high variance** --> overfitting
- Small C gives a hypothesis of **high bias low variance** --> underfitting
- **SVM parameters ( $\sigma^2$ )**
  - Parameter for calculating f values
    - Large  $\sigma^2$  - f features vary more smoothly - higher bias, lower variance
    - Small  $\sigma^2$  - f features vary abruptly - low bias, high variance

## **SVM - implementation and use**

- So far spoken about SVM in a very abstract manner
- What do you need to do this
  - Use SVM software packages (e.g. liblinear, libsvm) to solve parameters  $\theta$
  - Need to specify
    - Choice of parameter C
    - Choice of kernel

### **Choosing a kernel**

- We've looked at the **Gaussian kernel**
  - Need to define  $\sigma$  ( $\sigma^2$ )
    - Discussed  $\sigma^2$
  - When would you chose a Gaussian?
    - If n is small and/or m is large
      - e.g. 2D training set that's large
  - If you're using a Gaussian kernel then you may need to implement the kernel function
    - e.g. a function  
 $f_i = \text{kernel}(x_1, x_2)$ 
      - Returns a real number
    - Some SVM packages will expect you to define kernel
    - Although, some SVM implementations include the Gaussian and a few others
      - Gaussian is probably most popular kernel
  - NB - make sure you perform **feature scaling** before using a Gaussian kernel
    - If you don't features with a large value will dominate the f value
- Could use no kernel - **linear kernel**
  - Predict  $y = 1$  if  $(\theta^T x) \geq 0$ 
    - So no f vector
    - Get a standard linear classifier
  - Why do this?
    - If n is large and m is small then
      - Lots of features, few examples
      - Not enough data - risk overfitting in a high dimensional feature-space
- Other choice of kernel
  - Linear and Gaussian are most common
  - Not all similarity functions you develop are valid kernels
    - Must satisfy **Mercer's Theorem**
    - SVM use numerical optimization tricks
      - Mean certain optimizations can be made, but they must follow the theorem
  - **Polynomial Kernel**
    - We measure the similarity of x and l by doing one of
      - $(x^T l)^2$
      - $(x^T l)^3$
      - $(x^T l+1)^3$

- General form is
  - $(x^T l + \text{Con})^D$
- If they're similar then the inner product tends to be large
- Not used that often
- Two parameters
  - Degree of polynomial (D)
  - Number you add to l (Con)
- Usually performs worse than the Gaussian kernel
- Used when x and l are both non-negative
- **String kernel**
  - Used if input is text strings
  - Use for text classification
- **Chi-squared kernel**
- **Histogram intersection kernel**

## Multi-class classification for SVM

- Many packages have built in multi-class classification packages
- Otherwise use one-vs all method
- Not a big issue

## Logistic regression vs. SVM

- When should you use SVM and when is logistic regression more applicable
- If n (features) is large vs. m (training set)
  - e.g. text classification problem
    - Feature vector dimension is 10 000
    - Training set is 10 - 1000
    - Then use logistic regression or SVM with a linear kernel
- If n is small and m is intermediate
  - n = 1 - 1000
  - m = 10 - 10 000
  - Gaussian kernel is good
- If n is small and m is large
  - n = 1 - 1000
  - m = 50 000+
    - SVM will be slow to run with Gaussian kernel
  - In that case
    - Manually create or add more features
    - Use logistic regression or SVM with a linear kernel
- Logistic regression and SVM with a linear kernel are pretty similar
  - Do similar things
  - Get similar performance
- A lot of SVM's power is using different kernels to learn complex non-linear functions
- For all these regimes a well designed NN should work
  - But, for some of these problems a NN might be slower - SVM well implemented would be faster
- SVM has a convex optimization problem - so you get a global minimum
- It's not always clear how to choose an algorithm
  - Often more important to get enough data
  - Designing new features
  - Debugging the algorithm
- SVM is widely perceived a very powerful learning algorithm