CHAPTER 8
Combinatorial Analysis

8.1 COUNTING PRINCIPLE, FACTORIAL NOTATION

The following basic principle of counting is applied throughout this chapter:

Fundamental Principle of Counting: If some event can occur in n_1 different ways, and if, following this event, a second event can occur in n_2 different ways, and, following this second event, a third event can occur in n_3 different ways, . . . , then the number of ways the events can occur in the order indicated is $n_1 \cdot n_2 \cdot n_3 \cdots$.

8.1 Suppose a license plate contains two letters followed by three digits with the first digit not zero. How many different license plates can be printed?

Each letter can be printed in twenty-six different ways, the first digit in nine ways and each of the other two digits in ten ways. Hence $26 \cdot 26 \cdot 9 \cdot 10 \cdot 10 = 608\,400$ different plates can be printed.

8.2 Find the number n of license plates that can be made where each plate contains two distinct letters followed by three different digits.

Here $n = 26 \cdot 25 \cdot 10 \cdot 9 \cdot 8 = 468\,000$. That is, there are twenty-six choices for the first letter, but only twenty-five choices for the second letter which must be different from the first letter. Similarly, the choices for the digits are 10, 9, and 8 since the digits must be distinct.

8.3 Solve Problem 8.2 if the first digit cannot be 0.

Here $n = 26 \cdot 25 \cdot 9 \cdot 9 \cdot 8 = 421\,200$ which is the same as in Problem 8.2 except that now there are only nine choices for the first digit.

8.4 Find the number n of ways that an organization consisting of twenty-six members can elect a president, treasurer, and secretary (assuming no person is elected to more than one position).

The president can be elected in twenty-six different ways; following this, the treasurer can be elected in twenty-five different ways (since the person chosen president is not eligible to be treasurer); and, following this, the secretary can be elected in twenty-four different ways. Thus, by the above principle of counting, there are $n = 26 \cdot 25 \cdot 24 = 15\,600$ different ways in which the organization can elect the officers.

8.5 There are four bus lines between A and B; and three bus lines between B and C. Find the number of ways a person can travel: **(a)** by bus from A to C by way of B; **(b)** roundtrip by bus from A to C by way of B.

(a) There are four ways to go from A to B and three ways to go from B to C; hence there are $4 \cdot 3 = 12$ ways to go from A to C by way of B.

(b) There are twelve ways to go from A to C by way of B, and 12 ways to return. Hence there are $12 \cdot 12 = 144$ ways to travel roundtrip.

8.6 Suppose the person in Problem 8.5 does not want to use a bus line more than once. In how many ways can the roundtrip by bus be taken from A to C by way of B?

The person will travel from A to B to C to B to A. Enter these letters with connecting arrows as follows:

$$A \to B \to C \to B \to A$$

The person can travel four ways from A to B and three ways from B to C, but only two ways from C to B and three ways from B to A since a bus line cannot be used more than once. Enter these numbers above the corresponding arrows as follows:

$$A \xrightarrow{4} B \xrightarrow{3} C \xrightarrow{2} B \xrightarrow{3} A$$

Thus there are $4 \cdot 3 \cdot 2 \cdot 3 = 72$ ways to travel roundtrip without using the same bus line more than once.

8.7 A student can take one of four mathematics sections and one of five English sections. Find the number n of ways he can register for the two courses.

▌ Consider all triplets of nonnegative integers (n_1, n_2, n_3) for which $n_1 + n_2 + n_3 = 3$. Thus

$$(a+b+c)^3 = \binom{3}{3,0,0}a^3b^0b^0 + \binom{3}{2,1,0}a^2b^1c^0 + \binom{3}{2,0,1}a^2b^0c^1$$
$$+ \binom{3}{1,1,1}a^1b^1c^1 + \binom{3}{0,3,0}a^0b^3c^0 + \binom{3}{0,2,1}a^0b^2c^1$$
$$+ \binom{3}{1,2,0}a^1b^2c^0 + \binom{3}{0,0,3}a^0b^0c^3 + \binom{3}{1,0,2}a^1b^0c^2 + \binom{3}{0,1,2}a^0b^1c^2$$
$$= a^3 + 3a^2b + 3a^2c + 6abc + b^3 + 3b^2c + 3ab^2 + c^3 + 3ac^2 + 3bc^2$$

8.46 Show that $\binom{n}{n_1, n_2} = \binom{n}{n_1} = \binom{n}{n_2}$.

▌ Observe that the expression $\binom{n}{n_1, n_2}$ implicitly implies that $n_1 + n_2 = n$ or $n_2 = n - n_1$. Hence

$$\binom{n}{n_2} = \binom{n}{n_1} = \frac{n!}{n_1!\,(n-n_1)} = \frac{n!}{n_1!\,n_2!} = \binom{n}{n_1, n_2}$$

8.47 Find the term in the expansion of $(2x^3 - 3xy^2 + z^2)^6$ which contains x^{11} and y^4.

▌ The general term of the expansion is

$$\binom{6}{a,b,c}(2x^3)^a(-3xy^2)^b(z^2)^c = \binom{6}{a,b,c}2^a x^{3a}(-3)^b x^b y^{2b} z^{2c}$$
$$= \binom{6}{a,b,c}2^a(-3)^b x^{3a+b} y^{2b} z^{2c}$$

Thus the term containing x^{11} and y^4 has $3a + b = 11$ and $2b = 4$ or $b = 2$ and $a = 3$. Also, since $a + b + c = 6$, we have $c = 1$. Substituting in the above gives

$$\binom{6}{3,2,1}2^3(-3)^2x^{11}y^4z^2 = -\frac{6!}{3!\,2!\,1!}8\cdot 9x^{11}y^4z^2 = -4320x^{11}y^4z^2$$

8.3 PERMUTATIONS

The number of permutations of n objects taken r at a time will be denoted by

$$P(n, r)$$

Some texts use the notation ${}_nP_r$, $P_{n,r}$, or $(n)_r$.

8.48 Discuss permutations using the set $S = \{a, b, c, d\}$ as an example.

▌ Any arrangement of a set of n objects in a given order is called a *permutation* of the objects (taken all at a time). Any arrangement of any $r \le n$ of these objects in a given order is called an *r-permutation* or a *permutation of the n objects taken r at a time*. Thus, for the given set S,

(i) *bdca*, *dcba*, and *acdb* are permutations of the four letters (taken all at a time);
(ii) *bad*, *adb*, *cbd*, and *bca* are permutations of the four letters taken three at a time;
(iii) *ad*, *cb*, *da*, and *bd* are permutations of the four letters taken two at a time.

8.49 Find the number of permutations of six objects, say a, b, c, d, e, f, taken three at a time. In other words, find the number of "three-letter words" using the given six letters without repeating any letter in a given word.

▌ Let the general three-letter word be represented by the following three boxes:

$$\square\square\square$$

Now the first letter can be chosen in six different ways; following this, the second letter can be chosen in five different ways; and, following this, the last letter can be chosen in four different ways. Write each number in its appropriate box as follows:

$$\boxed{6}\boxed{5}\boxed{4}$$

Thus by the fundamental principle of counting there are $6 \cdot 5 \cdot 4 = 120$ possible three-letter words without repetitions from the six letters, or there are 120 permutations of six objects taken three at a time, or by our notation, $P(6, 3) = 120$.

Theorem 8.4: $P(n, r) = \dfrac{n!}{(n-r)!} = n(n-1)(n-2)\cdots(n-r+1)$.

8.50 Prove Theorem 8.4.

▌ The derivation of the formula for the number of permutations of n objects taken r at a time, or the number of r-permutations of n objects, $P(n, r)$, follows the procedure in Problem 8.49. The first element in an r-permutation of n objects can be chosen in n different ways; following this, the second element in the permutation can be chosen in $n-1$ ways; and, following this, the third element in the permutation can be chosen in $n-2$ ways. Continuing in this manner, we have that the rth (last) element in the r-permutation can be chosen in $n-(r-1) = n-r+1$ ways. Thus, by the fundamental principle of counting, we have

$$P(n, r) = n(n-1)(n-2)\cdots(n-r+1)$$

Thus by Problem 8.14,

$$P(n, r) = n(n-1)(n-2)\cdots(n-r+1) = \frac{n(n-1)(n-2)\cdots(n-r+1)(n-r)!}{(n-r)!} = \frac{n!}{(n-r)!}$$

Corollary 8.5: There are $n!$ permutations of n objects (taken all at a time).

8.51 Prove Corollary 8.5.

▌ Using $r = n$ in Theorem 8.4 yields

$$P(n, n) = n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1 = n!$$

8.52 Find all the permutations of the three letters a, b, and c.

▌ There are $3! = 1 \cdot 2 \cdot 3 = 6$ such permutations. These are abc, acb, bac, bca, cab, cba.

8.53 Discuss sampling with and without replacement.

▌ Many problems in combinatorial analysis and, in particular, probability, are concerned with choosing a ball from an urn containing n balls (or a card from a deck, or a person from a population). When we choose one ball after the other from the urn, say r times, we call each choice an ordered sample of size r. There are two important cases:

(i) *Sampling with replacement.* Here the ball is replaced in the urn before the next ball is chosen. Now since there are n different ways to choose each ball, by the fundamental principle of counting there are

$$\overbrace{n \cdot n \cdot n \cdots n}^{r \text{ times}} = n^r$$

Simple permutation

different ordered samples with replacement of size r.

(ii) *Sampling without replacement.* Here the ball is not replaced in the urn after it is chosen. Thus there are no repetitions in the ordered sample. In other words, an ordered sample of size r without replacement is simply an r-permutation of the objects in the urn. Accordingly, there are

$$P(n, r) = n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}$$

r-permutation

different ordered samples of size r without replacement from a population of n objects.

8.54 Suppose repetitions are not permitted. (*a*) How many three-digit numbers can be formed from the six digits 2, 3, 4, 5, 7, and 9? (*b*) How many of these numbers are less that 400? (*c*) How many are even? (*d*) How many are odd? (*e*) How many are multiples of 5?

▌ In each case, draw three boxes $\square\square\square$ to represent an arbitrary number and then write in each box the number of digits that can be placed there.

(*a*) The box on the left can be filled in six ways; following this, the middle box can be filled in five ways; and, lastly, the box on the right can be filled in four ways: $\boxed{6}\boxed{5}\boxed{4}$. Thus there are $6 \cdot 5 \cdot 4 = 120$ numbers.

8.74 Solve Problem 8.73 if they sit at a round table.

▌ The four nationalities can be arranged in a circle in 3! ways (see Problem 8.58 on circular permutations). In

ways, and the two Italians in 2! ways. Thus, altogether, there are $3!\,3!\,4!\,4!\,2! = 41\,472$ arrangements.

8.75 Find the total number of positive integers that can be formed from the digits 1, 2, 3, and 4 if no digit is repeated in any one integer.

▌ Note that no integer can contain more than four digits. Let s_1, s_2, s_3, and s_4 denote the number of integers containing one, two, three, and four digits respectively. We compute each s_i separately.

Since there are four digits, there are four integers containing exactly one digit, i.e., $s_1 = 4$. Also, since there are four digits, there are $4 \cdot 3 = 12$ integers containing two digits, i.e., $s_2 = 12$. Similarly, there are $4 \cdot 3 \cdot 2 = 24$

altogether, there are $s_1 + s_2 + s_3 + s_4 = 4 + 12 + 24 + 24 = 64$ integers. L1 + L2 + L3 + L4

8.76 Find the number n of four-letter words that can be formed from the word NUMERICAL. [A word need not make sense.]

▌ Since there are nine letters, $n = P(9, 4) = 9 \cdot 8 \cdot 7 \cdot 6 = 3024$.

8.77 Solve Problem 8.76 if the words are to begin and end in a consonant.

▌ There are only five consonants. Thus there are five choices for the first letter, four choices for the last letter,

8.78 Solve Problem 8.76 if the words must contain the letter R.

▌ There are four places to put R in the word. The other three places can be chosen in eight, seven, and six

8.79 Solve Problem 8.76 if the words must contain the letter M and end in a vowel.

▌ There are four vowels, and so there are four choices for the last letter. There are three places to put the

$n = 4 \cdot 3 \cdot 7 \cdot 6 = 504$.

8.80 Find the number n of ways that five large books, four medium-sized books, and three small books can be placed on a shelf so that all books of the same size are together.

▌ The three blocks of books can be arranged on the shelf in 3! ways. In each case, the large books can be arranged in 5! ways, the medium-sized books in 4! ways, and the small books in 3! ways. Thus $n = 3!\,5!\,4!\,3! = 103\,680$.

▌ Note $P(n, r) = n(n-1)(n-2)\cdots(n-r+1)$ contains r factors.
(a) $P(7, 3) = 7 \cdot 6 \cdot 5 = 210$.
(b) $P(12, 2) = 12 \cdot 11 = 132$.

8.82 Find: **(a)** $P(5, 7)$, and **(b)** $P(8, 3)$.

▌ **(a)** $P(5, 7)$ is not defined since the second integer cannot exceed the first integer.
(b) $P(8, 3) = 8 \cdot 7 \cdot 6 = 336$.

8.83 Find: **(a)** $P(19, 1)$, and **(b)** $P(6, -2)$.

▌ **(a)** $P(19, 1) = 19$.
(b) $P(6, -2)$ is not defined since $P(n, r)$ is not defined for negative integers.

8.84 Find n if **(a)** $P(n, 2) = 72$, and **(b)** $P(n, 4) = 42P(n, 2)$.

▌ **(a)** $P(n, 2) = n(n-1) = n^2 - n$; hence $n^2 - n = 72$ or $n^2 - n - 72 = 0$ or $(n-9)(n+8) = 0$. Since n must be positive, the only answer is $n = 9$.

8.74 Solve Problem 8.73 if they sit at a round table.

The four nationalities can be arranged in a circle in 3! ways (see Problem 8.58 on circular permutations). In each case, the three Americans can be seated in 3! ways, the four Frenchmen in 4! ways, the four Danes in 4! ways, and the two Italians in 2! ways. Thus, altogether, there are 3! 3! 4! 4! 2! = 41 472 arrangements.

.75 Find the total number of positive integers that can be formed from the digits 1, 2, 3, and 4 if no digit is repeated in any one integer.

Note that no integer can contain more than four digits. Let s_1, s_2, s_3, and s_4 denote the number of integers containing one, two, three, and four digits respectively. We compute each s_i separately.

Since there are four digits, there are four integers containing exactly one digit, i.e., $s_1 = 4$. Also, since there are four digits, there are $4 \cdot 3 = 12$ integers containing two digits, i.e., $s_2 = 12$. Similarly, there are $4 \cdot 3 \cdot 2 = 24$ integers containing three digits and $4 \cdot 3 \cdot 2 \cdot 1 = 24$ integers containing four digits, i.e., $s_3 = 24$ and $s_4 = 24$. Thus, altogether, there are $s_1 + s_2 + s_3 + s_4 = 4 + 12 + 24 + 24 = 64$ integers.

76 Find the number n of four-letter words that can be formed from the word NUMERICAL. [A word need not make sense.]

Since there are nine letters, $n = P(9, 4) = 9 \cdot 8 \cdot 7 \cdot 6 = 3024$.

77 Solve Problem 8.76 if the words are to begin and end in a consonant.

There are only five consonants. Thus there are five choices for the first letter, four choices for the last letter, and then seven and six choices for the second and third letters, respectively. Thus $n = 5 \cdot 7 \cdot 6 \cdot 4 = 840$.

78 Solve Problem 8.76 if the words must contain the letter R.

There are four places to put R in the word. The other three places can be chosen in eight, seven, and six ways respectively. Thus $n = 4 \cdot 8 \cdot 7 \cdot 6 = 1344$.

79 Solve Problem 8.76 if the words must contain the letter M and end in a vowel.

There are four vowels, and so there are four choices for the last letter. There are three places to put the letter M in the word. The remaining two places can be chosen in seven and six ways, respectively. Thus $n = 4 \cdot 3 \cdot 7 \cdot 6 = 504$.

80 Find the number n of ways that five large books, four medium-sized books, and three small books can be placed on a shelf so that all books of the same size are together.

The three blocks of books can be arranged on the shelf in 3! ways. In each case, the large books can be arranged in 5! ways, the medium-sized books in 4! ways, and the small books in 3! ways. Thus $n = 3!\,5!\,4!\,3! = 103\,680$.

1 Find: **(a)** $P(7, 3)$, and **(b)** $P(12, 2)$.

Note $P(n, r) = n(n-1)(n-2)\cdots(n-r+1)$ contains r factors.

(a) $P(7, 3) = 7 \cdot 6 \cdot 5 = 210$.

(b) $P(12, 2) = 12 \cdot 11 = 132$.

2 Find: **(a)** $P(5, 7)$, and **(b)** $P(8, 3)$.

(a) $P(5, 7)$ is not defined since the second integer cannot exceed the first integer.

(b) $P(8, 3) = 8 \cdot 7 \cdot 6 = 336$.

3 Find: **(a)** $P(19, 1)$, and **(b)** $P(6, -2)$.

(a) $P(19, 1) = 19$.

(b) $P(6, -2)$ is not defined since $P(n, r)$ is not defined for negative integers.

4. Find n if **(a)** $P(n, 2) = 72$, and **(b)** $P(n, 4) = 42P(n, 2)$.

(a) $P(n, 2) = n(n-1) = n^2 - n$; hence $n^2 - n = 72$ or $n^2 - n - 72 = 0$ or $(n-9)(n+8) = 0$. Since n must be positive, the only answer is $n = 9$.

(b) $P(n, 4) = n(n-1)(n-2)(n-3)$ and $P(n, 2) = n(n-1)$. Hence

$$n(n-1)(n-2)(n-3) = 42n(n-1) \quad \text{or,} \quad \text{if } n \neq 0,\ n \neq 1,\ (n-2)(n-3) = 42$$
$$\text{or} \quad n^2 - 5n + 6 = 42 \quad \text{or} \quad n^2 - 5n - 36 = 0 \quad \text{or} \quad (n-9)(n+4) = 0$$

Since n must be positive, the only answer is $n = 9$.

8.85 Find n if $2P(n, 2) + 50 = P(2n, 2)$.

$P(n, 2) = n(n-1) = n^2 - n$ and $P(2n, 2) = 2n(2n-1) = 4n^2 - 2n$. Hence

$$2(n^2 - n) + 50 = 4n^2 - 2n \quad \text{or} \quad 2n^2 - 2n + 50 = 4n^2 - 2n \quad \text{or} \quad 50 = 2n^2 \quad \text{or} \quad n^2 = 25$$

Since n must be positive, the only answer is $n = 5$.

8.86 Show that $P(n, r) = r! \binom{n}{r}$.

Recall that $\binom{n}{r} = \dfrac{n!}{r!\,(n-r)!}$ and $P(n, r) = \dfrac{n!}{(n-r)!}$. Multiply $P(n, r)$ by $\dfrac{r!}{r!}$ to obtain

$$P(n, r) = \frac{n!}{(n-r)!} = \frac{r!}{r!}\frac{n!}{(n-r)!} = r!\,\frac{n!}{r!\,(n-r)!} = r!\binom{n}{r}$$

Permutations with Repetitions

8.87 Find the number m of all possible five-letter "words" using the letters from the word "DADDY".

Note first that there are $5! = 120$ permutations of the objects D_1, A, D_2, D_3, Y, where the three D's are distinguished. Furthermore, the following six permutations

$$D_1D_2D_3AY,\ D_2D_1D_3AY,\ D_3D_1D_2AY,\ D_1D_3D_2AY,\ D_2D_3D_1AY, \text{ and } D_3D_2D_1AY$$

produce the same word when the subscripts are removed. The 6 comes from the fact that there are $3! = 3 \cdot 2 \cdot 1 = 6$ different ways of placing the three D's in the first three positions in the permutation. This is true for each set of three positions in which the D's can appear. Accordingly,

$$m = \frac{5!}{3!} = \frac{120}{6} = 20$$

Theorem 8.6: The number of permutations of n objects of which n_1 are alike, n_2 are alike, . . . , n_r are alike is

$$\boxed{\frac{n!}{n_1!\,n_2!\cdots n_r!}}$$

8.88 Prove Theorem 8.6.

The derivation of the formula follows the procedure in Problem 8.87. That is, there are $n!$ permutations when all the n objects are distinguished. We must then divide the $n!$ by $n_1!$ to account for the fact that the n_1 objects which are alike will identify $n_1!$ of these permutations for any given set of positions of the n_1 objects in the permutation. Similarly, we must divide $n!$ by $n_2!, \ldots, n_r!$, which are the number of permutations of the corresponding alike objects. Thus the theorem is proved.

8.89 Find the number m of seven-letter "words" that can be formed using the letters of the word "BENZENE".

We seek the number of permutations of seven objects of which three are alike (the three E's) and two are alike (the two N's). By Theorem 8.6,

no. of permutation.

$$m = \frac{7!}{3!\,2!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1 \cdot 2 \cdot 1} = 420$$

8.90 How many different signals, each consisting of eight flags hung in a vertical line, can be formed from a set of four indistinguishable red flags, three indistinguishable white flags, and a blue flag?

We seek the number of permutations of eight objects of which four are alike and three are alike. There are

$$\frac{8!}{4!\,3!} = \frac{8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{4\cdot3\cdot2\cdot1\cdot3\cdot2\cdot1} = 280$$

different signals.

8.91 Find the number of distinct permutations that can be formed from all the letters of each word: **(a)** THEM, and **(b)** THAT.

(a) $4! = 24$, since there are four letters and no repetitions.

(b) $\frac{4!}{2!} = 12$, since there are four letters of which two are T.

8.92 Find the number of distinct permutations that can be formed from all the letters of each word: **(a)** RADAR, and **(b)** UNUSUAL.

(a) $\frac{5!}{2!\,2!} = 30$, since there are five letters of which two are R and two are A.

(b) $\frac{7!}{3!} = 840$, since there are seven letters of which three are U.

8.93 How many different signals, each consisting of six flags hung in a vertical line, can be formed from four identical red flags and two identical blue flags?

This problem concerns permutations with repetitions. There are $\frac{6!}{4!\,2!} = 15$ signals since there are six flags of which four are red and two are blue.

8.94 Find the number m of permutations that can be formed from all the letters of the word MISSISSIPPI.

There are eleven letters of which four are I, four are S, and two are P; hence $m = \frac{11!}{4!\,4!\,2!} = 34\,650$.

8.95 Solve Problem 8.94 if the words are to begin with an I.

Now there are ten positions left to fill where three are I, four are S, and two are P. Thus $m = \frac{10!}{3!\,4!\,2!} = 12\,600$.

8.96 Solve Problem 8.94 if the words are to begin and end in S.

Now there are nine positions left to fill where four are I, two are S, and two are P. Hence $m = \frac{9!}{4!\,2!\,2!} = 7560$.

8.97 Solve Problem 8.94 if the two P's are to be next to each other.

There are ten ways to place the two P's, the first and second letter, or the second and third letters, . . . , or the tenth and eleventh letters. In each case, there are nine positions left to fill where four are I and four are S. Thus $m = 10\frac{9!}{4!\,4!} = 6300$.

8.98 Solve Problem 8.94 if the four S's are to be next to each other.

Consider the four S's as one letter. Then there are eight letters of which four are I and two are P. Thus $m = \frac{8!}{4!\,2!} = 840$. Alternatively, there are eight ways to place the four S's and, in each case, there are seven positions left to fill where four are I and two are P. Then $m = 8\frac{7!}{4!\,2!} = 840$.

8.99 Find the number m of permutations that can be formed from all the letters of the word ELEVEN.

▌ There are six letters of which three are E; hence $m = \dfrac{6!}{3!} = 120$.

8.100 Solve Problem 8.99 if the words are to begin with L.

▌ Now there are five positions left to fill where three are E. Thus $m = \dfrac{5!}{3!} = 20$.

8.101 Solve Problem 8.99 if the words are to begin and end in E.

▌ Now there are only four positions to fill with four distinct letters; hence $m = 4! = 24$.

8.102 Solve Problem 8.99 if the words are to begin with E and end in N.

▌ Now there are four positions left to fill where two are E. Hence $m = \dfrac{4!}{2!} = 12$.

8.103 Find the number m of permutations that can be formed from all the letters of the word BASEBALL.

▌ There are eight letters of which two are B, two are A, and two are L. Thus $m = \dfrac{8!}{2!\,2!\,2!} = 5040$.

8.104 Solve Problem 8.103 if the two B's are to be next to each other.

▌ Consider the two B's as one letter. Then there are seven letters of which two are A and two are L. Hence $m = \dfrac{7!}{2!\,2!} = 1260$.

8.105 Solve Problem 8.103 if the words are to begin and end in a vowel.

▌ There are three possibilities, the words begin and end in A, the words begin in A and end in E, or the words begin in E and end in A. In each case there are six positions left to fill where two are B and two are L. Hence $m = 3\dfrac{6!}{2!\,2!} = 540$.

8.4 COMBINATIONS

Given a set of n objects, a *combination* of these n objects taken r at a time is any selection of r of the objects where order does not count. In other words, an r-combination of a set of n objects is any subset of r elements. Thus the following are combinations of the set S of Problem 8.48 taken three at a time:

$$\{a, b, c\},\ \{a, b, d\},\ \{a, c, d\},\ \{b, c, d\} \qquad \text{or simply} \qquad abc,\ abd,\ acd,\ bcd$$

Observe that the following combinations are equal:

$$abc,\ acb,\ bac,\ bca,\ cab \quad \text{and} \quad cba$$

That is, each denotes the same set $\{a, b, c\}$.

The number of combinations of n objects taken r at a time or, in other words, the number of r-element subsets of a set with n elements will be denoted by

$$C(n, r)$$

Some texts use the notation ${}_nC_r$ or $C_{n,r}$.

8.106 Find the number of combinations of four objects, say, a, b, c, d, taken three at a time, that is, find $C(4, 3)$.

▌ *Method 1.* List all possible subsets of $S = \{a, b, c, d\}$ with three elements obtaining:

$$\{a, b, c\},\ \{a, b, d\},\ \{a, c, d\},\ \{b, c, d\} \qquad \text{or simply} \qquad abc,\ abd,\ acd,\ bcd$$

Thus $C(4, 3) = 4$.

Method 2. Each combination consisting of three objects determines $3! = 6$ permutations of the objects in the combination as indicated in Fig. 8-2. Thus the number $C(4, 3)$ of combinations multiplied by $3!$ equals the

Combination	Permutations
abc	*abc, acb, bac, bca, cab, cba*
abd	*abd, adb, bad, bda, dab, dba*
acd	*acd, adc, cad, cda, dac, dca*
bcd	*bcd, bdc, cbd, cdb, dbc, dcb*

Fig. 8-2

number $P(4, 3)$ of permutations, that is,

$$C(4, 3) \cdot 3! = P(4, 3) \qquad \text{or} \qquad C(4, 3) = \frac{P(4, 3)}{3!}$$

But $P(4, 3) = 4 \cdot 3 \cdot 2 = 24$ and $3! = 6$; hence $C(4, 3) = 4$ as noted above.

Theorem 8.7: $C(n, r) = \dfrac{P(n, r)}{r!} = \dfrac{n!}{r!\,(n-r)!}$.

Remark: Recall that the binomial coefficient $\binom{n}{r}$ was defined to be $\dfrac{n!}{r!\,(n-r)!}$; hence

$$C(n, r) = \binom{n}{r}$$

Accordingly, we shall use $C(n, r)$ and $\binom{n}{r}$ interchangeably.

8.107 Prove Theorem 8.7.

Any combination of n objects taken r at a time determines $r!$ permutations of the objects in the combination as illustrated in Fig. 8-2 for $r = 3$. Accordingly,

$$P(n, r) = r!\,C(n, r)$$

Dividing by $r!$ gives us our result.

8.108 Find the number m of committees of three that can be formed from eight people.

Each committee is, essentially, a combination of the eight people taken three at a time. Thus

$$m = C(8, 3) = \binom{8}{3} = \frac{8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3} = 56$$

8.109 A farmer buys three cows, two pigs, and four hens from a man who has six cows, five pigs, and eight hens. How many choices does the farmer have?

The farmer can choose the cows in $\binom{6}{3}$ ways, the pigs in $\binom{5}{2}$ ways, and the hens in $\binom{8}{4}$ ways. Hence altogether the farmer can choose the animals in

$$\binom{6}{3}\binom{5}{2}\binom{8}{4} = \frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3}\,\frac{5 \cdot 4}{1 \cdot 2}\,\frac{8 \cdot 7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3 \cdot 4} = 20 \cdot 10 \cdot 70 = 14\,000 \text{ ways}$$

8.110 A class consists of seven men and five women. Find the number m of committees of five that can be selected from the class.

Each committee is a combination of the twelve people taken five at a time. Thus

$$m = C(12, 5) = \binom{12}{5} = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 5544$$

8.111 Solve Problem 8.110 if the committee is to consist of three men and two women.

$\binom{10}{4} = 210$ ways. If both C and D go, then the delegation can be chosen in $\binom{10}{2} = 45$ ways. Altogether, the delegation can be chosen in $m = 210 + 45 = 255$ ways.

8.119 A student is to answer eight out of ten questions on an exam. Find the number m of ways that the student can choose the eight questions.

The eight questions can be selected in $m = \binom{10}{8} = \binom{10}{2} = \frac{10 \cdot 9}{1 \cdot 2} = 45$ ways.

8.120 Solve Problem 8.119 if the student must answer the first three questions.

If the first three questions are answered, then the student can choose the other five questions from the last seven questions in $m = \binom{7}{5} = \binom{7}{2} = \frac{7 \cdot 6}{1 \cdot 2} = 21$ ways.

8.121 Solve Problem 8.119 if the student must answer at least four out of the first five questions.

If the student answers all the first five questions, then the other three questions can be chosen from the last five in $\binom{5}{3} = 10$ ways. On the other hand, if the student answers only four of the first five questions, then these four can be chosen in $\binom{5}{4} = \binom{5}{1} = 5$ ways, and the other four questions from the last five in $\binom{5}{4} = \binom{5}{1} = 5$ ways; hence the student can choose the eight questions in $5 \cdot 5 = 25$ ways. Thus there will be a total of $m = 10 + 25 = 35$ choices.

8.122 There are twelve points $A, B, \ldots$ in a given plane, no three on the same line. **(a)** How many lines are determined by the points? **(b)** How many lines pass through the point A?

(a) Since two points determine a line, there are $\binom{12}{2} = \frac{12 \cdot 11}{1 \cdot 2} = 66$ lines.

(b) To determine a line through A, one other point must be chosen; hence there are eleven lines through A.

8.123 There are twelve points $A, B, \ldots$ in a given plane, no three on the same line. **(a)** How many triangles are determined by the points? **(b)** How many of these triangles contain the point A as a vertex?

(a) Since three points determine a triangle, there are $\binom{12}{3} = \frac{12 \cdot 11 \cdot 10}{1 \cdot 2 \cdot 3} = 220$ triangles.

(b) *Method 1.* To determine a triangle with vertex A, two other points must be chosen; hence there are $\binom{11}{2} = \frac{11 \cdot 10}{1 \cdot 2} = 55$ triangles with A as a vertex.

Method 2. There are $\binom{11}{3} = \frac{11 \cdot 10 \cdot 9}{1 \cdot 2 \cdot 3} = 165$ triangles without A as a vertex. Thus $220 - 165 = 55$ of the triangles do have A as a vertex.

8.124 How many committees of five with a given chairperson can be selected from twelve persons?

The chairperson can be chosen in twelve ways and, following this, the other four on the committee can be chosen from the eleven remaining in $\binom{11}{4}$ ways. Thus there are $12 \cdot \binom{11}{4} = 12 \cdot 330 = 3960$ such committees.

8.125 Find the number of subsets of a set X containing n elements.

Method 1. The number of subsets of X with $r \le n$ elements is given by $\binom{n}{r}$. Hence, altogether there are

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n}$$

subsets of X. The above sum is equal to 2^n (Problem 8.41), and so there are 2^n subsets of X.

The three men can be chosen from the seven men in $\binom{7}{3}$ ways, and the two women can be chosen from the five women in $\binom{5}{2}$ ways. Hence

$$m = \binom{7}{3}\binom{5}{2} = \frac{7\cdot6\cdot5}{1\cdot2\cdot3}\,\frac{5\cdot4}{1\cdot2} = 350$$

8.112 Solve Problem 8.110 if the committee is to consist of at least one man and at least one woman.

By Problem 8.110, there are $C(12, 5) = 5544$ possible committees. Among these possible committees, there is $C(5, 5) = 1$ committee consisting of the five women, and $C(7, 5) = 21$ consisting of five men. Eliminating these from all possible committees yields $m = 5544 - 21 - 1 = 5522$.

8.113 A bag contains five red marbles and six white marbles. Find the number m of ways that four marbles can be drawn from the bag.

The four marbles (of any color) can be chosen from the eleven marbles in

$$m = \binom{11}{4} = \frac{11\cdot10\cdot9\cdot8}{1\cdot2\cdot3\cdot4} = 330 \text{ ways}$$

8.114 Solve Problem 8.113 if two of the marbles must be red and two of the marbles must be white.

The two red marbles may be chosen in $\binom{5}{2}$ ways and the two white marbles may be chosen in $\binom{6}{2}$ ways. Thus

$$m = \binom{5}{2}\binom{6}{2} = \frac{5\cdot4}{2\cdot1}\,\frac{6\cdot5}{2\cdot1} = 150$$

8.115 Solve Problem 8.113 if the four marbles must be of the same color.

There are $\binom{6}{4} = 15$ ways of drawing four white marbles, and $\binom{5}{4} = 5$ ways of drawing four red marbles. Thus there are $15 + 5 = 20$ ways of drawing four marbles of the same color.

8.116 There are twelve students who are eligible to attend the National Student Association annual meeting. Find the number m of ways a delegation of four students can be selected from the twelve eligible students.

The four students can be chosen from the twelve students in $m = \binom{12}{4} = \frac{12\cdot11\cdot10\cdot9}{1\cdot2\cdot3\cdot4} = 495$ ways.

8.117 Solve Problem 8.116 if two of the eligible students will not attend the meeting together.

Let A and B denote the students who will not attend the meeting together.

Method 1. If neither A nor B is included, then the delegation can be chosen in $\binom{10}{4} = \frac{10\cdot9\cdot8\cdot7}{1\cdot2\cdot3\cdot4} = 210$ ways. If either A or B, but not both, is included, then the delegation can be chosen in $2\cdot\binom{10}{3} = 2\cdot\frac{10\cdot9\cdot8}{1\cdot2\cdot3} = 240$ ways. Altogether, the delegation can be chosen in $m = 210 + 240 = 450$ ways.

Method 2. If A and B are both included, then the other two members of the delegation can be chosen in $\binom{10}{2} = 45$ ways. Thus there are $m = 495 - 45 = 450$ ways the delegation can be chosen if A and B are not both included.

8.118 Solve Problem 8.116 if two of the eligible students are married and will only attend the meeting together.

Let C and D denote the married students. If C and D do not go, then the delegation can be chosen in

3.124 Determine whether or not a constant function can be **(a)** one-to-one, **(b)** an onto function.

(a) A constant function is one-to-one if and only if the domain consists of exactly one element.
(b) A constant function is an onto function if and only if the codomain consists of exactly one element.

3.125 On which sets A will the identity function $1_A: A \to A$ be **(a)** one-to-one? **(b)** an onto function?

For any set A, the identity function 1_A is both one-to-one and onto (and hence invertible).

3.126 Find the "largest" interval D on which the formula $f(x) = x^2$ defines a one-to-one function.

As long as the interval D contains either positive or negative numbers, but not both, the function will be one-to-one. Thus D can be the infinite interval

$$[0, \infty) = \{x : x \geq 0\} \qquad \text{or} \qquad (-\infty, 0] = \{x : x \leq 0\}$$

There can be other intervals on which f will be one-to-one, but they will be subsets of one of these two intervals.

3.127 Describe the relationship between the graph of a function $y = f(x)$ and the graph of the inverse function $y = f^{-1}(x)$.

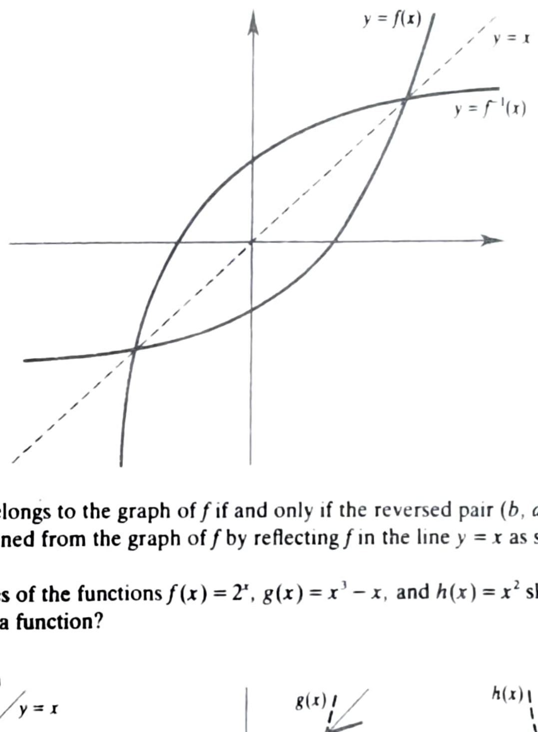

Fig. 3-27

The ordered pair (a, b) belongs to the graph of f if and only if the reversed pair (b, a) belongs to f^{-1}. Thus the graph of f^{-1} may be obtained from the graph of f by reflecting f in the line $y = x$ as shown in Fig. 3-27.

3.128 Find the graphs of the inverses of the functions $f(x) = 2^x$, $g(x) = x^3 - x$, and $h(x) = x^2$ sketched in Fig. 3-24. Which of these graphs define a function?

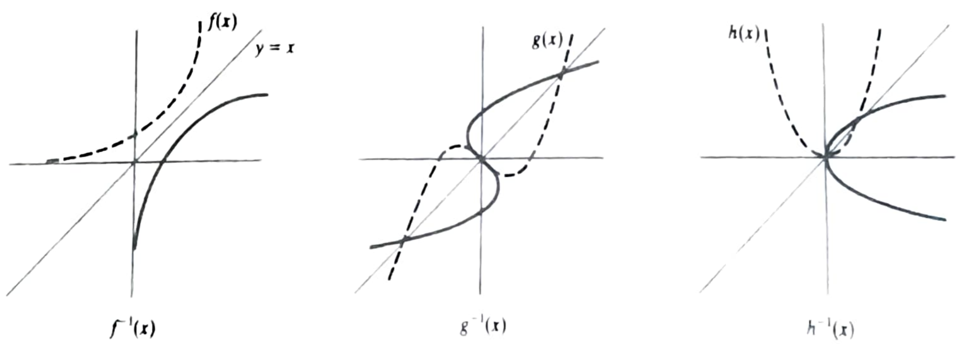

Fig. 3-28

Reflect each graph in the line $y = x$ as in Fig. 3-28. The graphs g^{-1} and h^{-1} are not functions since there are vertical lines which intersect the graph in more than one point. However, as noted in Problem 3.108, f^{-1} does define a function with domain $D = \{x : x > 0\}$.