5.2 PRINCIPLE OF MATHEMATICAL INDUCTION

Let P (n) be a statement or proposition defined by the set of positive integers, I_+ such that it is either true or false for all $n \in I_+$. For the given statement P (n), if we can prove that

(*a*) P (n) is true for $n = n_0$ (the smallest integer), and

(*b*) P (n) is true for $n = k + 1$, assuming that it is true for $n = k$ $(k \geq n_0)$.

then we can conclude that P (n) is true for all natural numbers $n \geq n_0$. Here the first proposition (*a*) is usually referred to as the *basis of induction*. Since assuming that P $(k + 1)$ would be true if P (k) is true is not the same as assuming that P (k) is true for some value of k, therefore the proposition (*b*) is usually referred to as the *induction step*. Also the assumption that P (n) is true for $n = k$ in (*b*) is referred to as the induction hypothesis.

very Easy

The steps of the principle of mathematical induction are summarised as follows:

1. **Basis of induction** Assuming the validity of the statement for the smallest integral value of n (i.e. $n = 1, 2, 3$) for which it is true.
2. **Induction step** If the statement is true for $n = k$, where k denotes any value of n, then it is also true $n = k + 1$.
3. **Conclusion** The statement is true for all integral values of n equal to or greater than that for which it was verified in Step 1.

Illustration One of the methods of finding a square root on an ordinary computing or adding machine is based on the fact that the sum of n odd integers is equal to n^2, specifically,

$$\begin{aligned} 1 &= 1 = 1^2 \\ 1 + 3 &= 4 = 2^2 \\ 1 + 3 + 5 &= 9 = 3^2 \\ 1 + 3 + 5 + 7 &= 16 = 4^2 \end{aligned}$$

Example 1 Prove by the principle of mathematical induction

$$\text{P}(n) : 1 + 3 + 6 + \ldots + \frac{n(n+1)}{2} = \frac{n(n+1)(n+2)}{6}$$

Solution *Basis of induction:* For $n = 1$, the left hand side of statement, P (n) is $\frac{1(1+1)}{2} = 1$ and right hand side of P (n) is $\frac{1(1+1)(1+2)}{6} = 1$. Hence, P (n) is true for $n = 1$.

$n = 2$

Induction step: Assuming that P (n) is true for $n = k$. Thus, we get

$$\text{P}(k) : 1 + 3 + 6 + \ldots + \frac{k(k+1)}{2} = \frac{k(k+1)(k+2)}{6}$$

Adding the term $\frac{(k+1)(k+2)}{2}$ to both sides of P (k), we get

$$\begin{aligned} 1 + 3 + 6 + \ldots + \frac{k(k+1)}{2} + \frac{(k+1)(k+2)}{2} &= \frac{k(k+1)(k+2)}{6} + \frac{(k+1)(k+2)}{2} \\ &= \frac{(k+1)(k+2)(k+3)}{6} \end{aligned}$$

This shows that if P (n) is true for $n = k$, then it is also true for $n = k + 1$. Hence, by mathematical induction it is true for every value of n.

Example 2 Prove by the principle of mathematical induction

$$P(n): 1 \cdot 2 + 2 \cdot 2^2 + \ldots + n \cdot 2^n = (n-1)2^{n+1} + 2$$

Solution *Basis of induction:* If $n = 1$, then left hand side of the statement P (n) is $1 \cdot 2^1 = 2$ and right hand side of P (n) is $(1-1)2^{1+1} + 2 = 2$. Hence, P (n) is true for $n = 1$.

Induction step: Assuming that P (n) is true for $n = k$. Thus, we get

$$P(k): 1 \cdot 2 + 2 \cdot 2^2 + \ldots + k \cdot 2^k = (k-1)2^{k+1} + 2$$

Adding the term $(k+1)2^{k+1}$ to both sides of P (k), we get

$$\begin{aligned} 1 \cdot 2 + 2 \cdot 2^2 + \ldots + k \cdot 2^k + (k+1) \cdot 2^{k+1} &= (k-1) \cdot 2^{k+1} + 2 + (k+1) \cdot 2^{k+1} \\ &= 2^{k+1}\{k - 1 + k + 1\} + 2 \\ &= 2^{k+1} \cdot 2k + 2 = k \cdot 2^{k+1+1} + 2 \\ &= [(k+1) - 1] \cdot 2^{k+1+1} + 2 \end{aligned}$$

This shows that if P (n) is true for $n = k$, then it is also true for $n = k + 1$. Hence, by mathematical induction P (n) is true for every integral value of n.

Example 3 Prove by the principle of mathematical induction

$$P(n): \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \ldots + \frac{1}{n(n+1)} = \frac{1}{n+1}$$

Solution *Basis of induction:* If $n = 1$, then left hand side of P (n) is $\frac{1}{1(1+1)} = \frac{1}{2}$ and right hand side is $\frac{1}{1+1} = \frac{1}{2}$. Hence, P (n) is true for $n = 1$.

Induction step: Assuming that P (n) is true for $n = k$. Thus, we get

$$P(k): \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \ldots + \frac{1}{k(k+1)} = \frac{1}{k+1}$$

Adding the term $\frac{1}{(k+1)(k+2)}$ to both sides of P (k), we get

$$\begin{aligned} \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \ldots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} &= \frac{1}{(k+1)} + \frac{1}{(k+1)(k+2)} \\ &= \frac{k^2 + 2k + 1}{(k+1)(k+2)} = \frac{(k+1)(k+1)}{(k+1)(k+2)} \\ &= \frac{k+1}{k+2} = \frac{k+1}{(k+1)+1} \end{aligned}$$

This shows that if P (n) is true for $n = k$, then it is also true for $n = k + 1$. Hence, by mathematical induction P (n) is true for every integral value of n.

Example 4 Prove by the principle of mathematical induction the inequality

$$P(n) : (a + 1)^n \geq 1 + na, \text{ for } a > -1;\ n = 2, 3, 4$$

Solution *Basis of induction:* If $n = 2$, then left hand side of the given is inequality is $(a + 1)^2 = a^2 + 2a + 1$ and right hand side is $1 + 2a$.

Since left hand side is more than the right hand side, therefore inequality is ture for $n = 2$.

Induction step: Assuming that the inequality is true for $n = k$. Thus, we get

$$(a + 1)^k \geq 1 + ka, \text{ for } a > -1 \qquad (i)$$

Since $$(a + 1)^{k+1} = (a + 1)^k (a + 1)$$

$$\geq (1 + ka)(a + 1); \quad \text{From } (i)$$

$$\geq 1 + (k + 1)a + ka^2$$

$$\geq 1 + (k + 1)a, \text{ since } ka^2 > 0$$

This shows that if the given inequality is true for $n = k$, then it is also true for $n = k + 1$. Hence, by mathematical induction the inequality is true for every integral value of n.

Example 5 Prove by the principle of mathematical induction that

$$P(n) : 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$ [UPTU, MCA, 2006]

Solution *Basis of induction:* If $n = 1$, then left hand side of the statement $P(n)$ is $1^2 = 1$ and right hand side is $\frac{1(1+1)(2 \cdot 1+1)}{6} = 1$. Hence $P(n)$ is true for $n = 1$.

Induction step: Assuming that $P(k)$ is true for $n = k$. Then, we get

$$P(k) : 1^2 + 2^2 + 3^2 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

Adding the term $(k + 1)^2$ to both sides of $P(k)$, we get

$$1^2 + 2^2 + 3^2 + \ldots + k^2 + (k + 1)^2 = \frac{k(k+1)(2k+1)}{6} + (k + 1)^2$$

$$= \frac{(k+1)}{6} \{k(2k + 1) + 6(k + 1)\}$$

$$= \frac{(k+1)}{6} \{2k^2 + 7k + 6\}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

$$= \frac{(k+1)\{(k+1)+1\}\{(2k+1)+1\}}{2}$$

This shows that if $P(n)$ is true for $n = k$, then it is also true for $n = k + 1$. Hence, by mathematical $P(n)$ is true for every integral value of n.

Example 6 Using the principle of mathematical induction, prove that

$$P(n): 1 + 3 + 5 + \ldots + (2n - 1) = n^2$$

Solution *Basis of induction:* If $n = 1$, the left hand side of $P(n)$ is 1 and right hand side is 1. Hence, $P(n)$ is true for $n = 1$.

Induction step: Assuming that $P(n)$ is true for $n = k$. Then we get

$$P(k): 1 + 3 + 5 + \ldots + (2k - 1) = k^2$$

Adding the term $(2k + 1)$ to both sides of $P(k)$, we get

$$1 + 3 + 5 + \ldots + (2k - 1) + (2k + 1) = k^2 + (2k + 1) = (k + 1)^2$$

This shows that if $P(n)$ is true for $n = k$, then it is also true for $n = k + 1$. Hence, by mathematical induction $P(n)$ is true for every integral value of n.

Example 7 Using the principle of mathematical induction, prove that

$$P(n): 1\cdot 3 + 2\cdot 4 + 3\cdot 5 + \ldots + n(n + 2) = \frac{n}{6}(n + 1)(2n + 7)$$

Solution *Basis of induction:* If $n = 1$, then left hand side of $P(n)$ is $1\cdot(1 + 2) = 3$ and right hand side is $\frac{1}{6}(1 + 1)(2\cdot 1 + 7) = 3$. Hence, $P(n)$ is true for $n = 1$.

Induction step: Assuming that $P(n)$ is true for $n = k$. Then, we get

$$P(k): 1\cdot 3 + 2\cdot 4 + 3\cdot 5 + \ldots + k\cdot(k + 2) = \frac{k}{6}(k + 1)(2k + 7)$$

Adding the term $(k + 1)(k + 3)$ to both sides of $P(k)$, we get

$$\begin{aligned} 1\cdot 3 + 2\cdot 4 + 3\cdot 5 + \ldots + k(k + 2) + (k + 1)(k + 3) &= \frac{k}{6}(k + 1)(2k + 7) + (k + 1)(k + 3) \\ &= (k + 1)\left\{\frac{k}{6}(2k + 7) + (k + 3)\right\} \\ &= (k + 1)\left\{\frac{2k^2 + 13k + 18}{6}\right\} \\ &= \frac{(k + 1)(k + 2)(2k + 9)}{6} \end{aligned}$$

This shows that if $P(n)$ is true for $n = k$, then it is also true for $n = k + 1$. Hence, by mathematical induction $P(n)$ is true for every integral value of n.

Example 8 Prove by the principle of mathematical induction that

$$P(n): \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$$

Solution *Basis of induction:* If $n = 1$, then left hand side of $P(n)$ is $\frac{1}{2}$ and right hand side is $1 - \frac{1}{2} = \frac{1}{2}$. Hence $P(n)$ is true for $n = 1$.

Induction step: Assuming that $P(n)$ is true for $n = k$. Then, we get

$$P(k): \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^k} = 1 - \frac{1}{2^k}$$

Adding the term $\frac{1}{2^{k+1}}$ to both sides of P (k), we get

$$\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^k}+\frac{1}{2^{k+1}}=1-\frac{1}{2^k}+\frac{1}{2^{k+1}}=1-\left[\frac{1}{2^k}-\frac{1}{2^{k+1}}\right]=1-\frac{1}{2^{k+1}}$$

This shows that if P (n) is true for $n=k$, then it is also true for $n=k+1$. Hence, by mathematical induction P (n) is true for every integral value of n.

Example 9 Use the principle of mathematical induction to prove that

$$\text{P}(n):\sum_{n=1}^{n}\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1},\text{ for all } n\in \text{N}$$

or
$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\ldots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$$

Solution *Basis of induction:* If $n=1$, the LHS of P (n) is $\sum_{n=1}^{1}\frac{1}{(2n-1)(2n+1)}=\frac{1}{1\cdot 3}=\frac{1}{3}$ and RHS is $\frac{1}{2\cdot 1+1}=\frac{1}{3}$. Hence, P (n) is true for $n=1$

Induction step: Assuming that P (n) is true for $n=k$. Then, we get

$$\text{P}(k):\sum_{r=1}^{k}\frac{1}{(2k-1)(2k+1)}=\frac{k}{2k+1}$$

Adding the term $\frac{1}{(2k+1)(2k+3)}$ to both sides of P (k), we get

$$\sum_{r=1}^{k}\frac{1}{(2k-1)(2k+1)}+\frac{1}{(2k+1)(2k+3)}=\frac{k}{2k+1}+\frac{1}{(2k+1)(2k+3)}$$

or
$$\sum_{r=1}^{k+1}\frac{1}{(2k-1)(2k+1)}=\frac{k(2k+3)+1}{(2k+1)(2k+3)}=\frac{(k+1)(2k+1)}{(2k+1)(2k+3)}=\frac{k+1}{2k+3}$$

This shows that if P (n) is true for $n=k$, then it is also true for $n=k+1$. Hence, by mathematical induction P (n) is true for every given value of n.

Example 10 By the principle of induction, show that $3^{4n+2}+5^{2n+1}$ is a multiple of 14, for all positive integral value of n including zero.

Solution Let the given expression P (n) : $3^{4n+2}+5^{2n+1}$ be the multiple of 14.

Basis of induction: If $n=1$, then we have P (1) : $3^{4\cdot 1+2}+5^{2\cdot 1+1}=854=14\times 61$ which is a multiple of 14. Similarly, for $n=2$,

$$\text{P}(2):3^{4\cdot 2+2}+5^{2\cdot 2+1}=62174=14\times 4441$$

is a multiple of 14.

Induction step: Assuming that the result is true for $n=k$, then

$$\text{P}(k)=3^{4k+2}+5^{2k+1}=14\times t\,;\,t\in \text{I}$$

is multiple of 14.

Replacing k by $k + 1$ in P (k), we get

$$\begin{aligned}3^{4(k+1)+2} + 5^{2(k+1)+1} &= 3^{4k+6} + 5^{2k+3}\\ &= 3^{4k+2}\cdot 3^4 + 5^{2k+1}\cdot 5^2\\ &= 3^{4k+2}(11+70) + 5^{2k+1}(11+14)\\ &= 11(3^{4k+2} + 5^{2k+1}) + 70\cdot 3^{4k+2} + 14\cdot 5^{2k+1}\\ &= 11\cdot 14\,t + 14\{5\cdot 3^{4k+2} + 5^{2k+1}\}\\ &= 14\{11\,t + 5\cdot 3^{4k+2} + 5^{2k+1}\}\end{aligned}$$

which is a multiple of 14. Hence, the result is true for $n = k + 1$.

Moreover, for $n = 0$, we have

$$P(0): 3^{4\cdot 0+2} + 5^{2\cdot 0+1} = 14\cdot 1$$

which is a multiple of 14. Hence, P (n) also holds true for $n = 0$.

Example 11 (a) If nth term of A.P. is $a + (n - 1)\,d$, then show by the principle of mathematical induction that the sum of n terms of A.P. is $\frac{n}{2}\{2a + (n-1)\,d\}$. That is, by the principle of mathematical induction, prove that

$$P(n):\ a + (a+d) + (a+2d) + \ldots + \{a + (n-1)\,d\} = \frac{n}{2}\{2a + (n-1)\,d\}$$

(b) Prove by the principle of mathematical induction the result

$$P(n):\ a + ar + ar^2 + \ldots + ar^{n-1} = a\cdot\frac{r^n - 1}{r-1},\ \text{if } r \neq 1$$

Solution (a) *Basis of induction:* For $n = 1$, LHS of P (n) is a and RHS is $\frac{1}{a}\{2a + (1-1)d\} = a$. Hence, P ($n$) is true for $n = 1$.

Assuming that P (n) is true for $n = k$. Then, we get

$$P(k):\ a + (a+d) + (a+2d) + \ldots + \{a + (k-1)d\} = \frac{k}{2}\{2a + (k-1)\,d\}$$

Adding the term $(a + kd)$ to both sides of P (k), we get

$$\begin{aligned}&a + (a+d) + (a+2d) + \ldots + \{a + (k-1)d\} + (a+kd)\\ &= \frac{k}{2}\{2a + (k-1)d\} + (a+kd)\\ &= ak + \frac{k(k-1)}{2}\cdot d + a + kd = a(k+1) + \frac{k(k-1+2)}{2}\cdot d\\ &= a(k+1) + \frac{k(k+1)}{2}\cdot d = \frac{k(k+1)}{2}\{2a + (k+1-1)\cdot d\}\end{aligned}$$

This shows that if P (n) is true for $n = k$, then it is also true for $n = k + 1$. Hence, by mathematical induction P (n) is true for every positive value of n.

(b) *Basis of induction:* If $n = 1$, then LHS of P (n) is a and RHS is $a\left(\frac{r-1}{r-1}\right) = a$. Hence, P ($n$) is true for $n = 1$.

Induction step: Assuming that P (n) is true for $n = k$. Then we get

$$P(k) : a + ar + ar^2 + \ldots + ar^{k-1} = a \cdot \frac{r^k - 1}{r - 1}, r \neq 1.$$

Adding the term ar^k to both sides of P (k), we get

$$a + ar + ar^2 + \ldots + ar^{k-1} + ar^k = a \cdot \frac{r^k - 1}{r - 1} + ar^k$$

$$= a \cdot \frac{(r^k - 1 + r^{k+1} - r^k)}{r - 1} = \frac{a(r^{k+1} - 1)}{r - 1}, r \neq 1$$

This shows that if P (n) is true for $n = k$, then it is also true for $n = k + 1$. Hence, by mathematical induction P (n) is true for every integral value of n.

Example 12 Using principle of mathematical induction, show that

$$P(n) : 1 + p + p^2 + \ldots + p^{n-1} = \frac{p^n - 1}{p - 1} \; (p > 1)$$

for all natural numbers n.

Solution *Basis of induction:* For $n = 1$, LHS of P (n) is 1 and RHS is also 1. Hence, P (n) is true for $n = 1$.

Induction step: Assuming that P (n) is true for $n = k$. Then we have

$$P(k) : 1 + p + p^2 + \ldots + p^{k-1} = \frac{p^k - 1}{p - 1} \; (p > 1)$$

Adding the term p^k to both sides of P (k), we get

$$1 + p + p^2 + \ldots + p^{k-1} + p^k = \frac{p^k - 1}{p - 1} + p^k = \frac{p^k - 1 + p^{k+1} - p^k}{p - 1} = \frac{p^{k+1} - 1}{p - 1} \; (p > 1)$$

This shows that if P (n) is true for $n = k$, then it is also true for $n = k + 1$. Hence, by mathematical induction P (n) is true for every positive integral value of n.

Example 13 Let $\mathbf{A} = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ be a matrix. Prove by mathematical induction that $\mathbf{A}^n = \begin{bmatrix} 1 & an \\ 0 & 1 \end{bmatrix}$.

Solution *Basis of induction:* For $n = 2, 3$, by actual multiplication of two matrices, we have

$$\mathbf{A}^2 = \mathbf{A} \cdot \mathbf{A} = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2a \\ 0 & 1 \end{bmatrix}$$

and

$$\mathbf{A}^3 = \mathbf{A}^2 \cdot \mathbf{A} = \begin{bmatrix} 1 & 2a \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3a \\ 0 & 1 \end{bmatrix}.$$

Hence, the result is true for $n = 2, 3$.

Induction step: Assuming that $\mathbf{A}^n$ is true for $n = k$, so that $\mathbf{A}^k = \begin{bmatrix} 1 & ka \\ 0 & 1 \end{bmatrix}$.

Multiplying both sides of $\mathbf{A}^k$ with matrix $\mathbf{A}$ once again, we get

$$\mathbf{A}^k \cdot \mathbf{A} = \mathbf{A}^{k+1} = \begin{bmatrix} 1 & ka \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a(k+1) \\ 0 & 1 \end{bmatrix}.$$

This shows that if $\mathbf{A}^n$ is true for $n = k$, then it is also true for $n = k + 1$. Hence, by mathematical induction $\mathbf{A}^n$ is true for every positive integral value of n.

Example 14 Prove by the principle of mathematical induction that P (n): $10^n + 3 \cdot 4^{n+2} + 5$ is divisible by 9.

Solution *Basis of induction:* For $n = 1$, we have

$$10^1 + 3 \cdot 4^{1+2} + 5 = 207 = 9 \times 23$$

This is divisible by 9.

Induction step: Assuming that P (n) is true for $n = k$, so that $10^k + 3 \cdot 4^{k+2} + 5$ is divisible by 9.

Replacing k by $k + 1$ in P (k), we get

$$\begin{aligned}
10^{k+1} + 3 \cdot 4^{k+1+2} + 5 &= 10 \cdot 10^k + 3 \cdot 4 \cdot 4^{k+2} + 5 \\
&= (9 + 1) \cdot 10^k + 3\,(3 + 1) \cdot 4^{k+2} + 5 \\
&= 9 \cdot 10^k + 10^k + 3 \cdot 3 \cdot 4^{k+2} + 3 \cdot 4^{k+2} + 5 \\
&= 9 \cdot 10^k + 9 \cdot 4^{k+2} + (10^k + 3 \cdot 4^{k+2} + 5) \\
&= 9 \cdot 10^k + 9 \cdot 4^{k+2} + 9m \qquad (\text{where } 9m = 10^k + 3.4^{k+2} + 5) \\
&= 9(10^k + 4^{k+2} + m) = 9t \text{ ; for } t = 10^k + 4^{k+2} + m
\end{aligned}$$

which is divisible by 9.

This shows that if P (n) is true for $n = k$, then it is also true for $n = k + 1$. Hence, P (n) is true for all positive integral values of n.

Example 15 Prove by the principle of mathematical induction

$$P(n) : 1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2 (n+1)^2}{4}$$

Solution *Basis of induction:* For $n = 1$, the LHS of P (n) is $1^3 = 1$ and RHS is also $\dfrac{1^2 (1+1)^2}{4} = 1$. Hence, P (n) is true for $n = 1$.

Induction step: Assuming that P (n) is true for $n = k$. Then we get

$$P(k) : 1^3 + 2^3 + 3^3 + \ldots + k^3 = \frac{k^2 (k+1)^2}{4} \qquad \text{...(ii)}$$

Adding the term $(k + 1)^3$ to both sides of P (k), we get

$$\begin{aligned}
1^3 + 2^3 + 3^3 + \ldots + k^3 + (k + 1)^3 &= \frac{k^2 (k+1)^2}{4} + (k + 1)^3 \\
&= \frac{(k+1)^2}{4} \{k^2 + 4\,(k + 1)\} \\
&= \frac{(k+1)^2 (k+2)^2}{4} = \frac{(k+1)^2 \{(k+1)+1\}^2}{4}
\end{aligned}$$

This shows that if P (n) is true for $n = k$, then it is also true for $n = k + 1$. Hence, by mathematical induction P (n) is true for every positive integral value of n.

Example 16 Show that $\sqrt{2}$ is an irrational number.

Solution Suppose $\sqrt{2}$ is not an irrational number. Then let $\sqrt{2} = p/q$, where p and q are integers having no common factor. Since $\sqrt{2} = p/q$, therefore $p^2 = 2q^2$. This implies that p^2 is even, i.e. p is even (as square of an odd number is always an odd number).

Let $p = 2n$, n being an integer. Then $2q^2 = p^2 = (2n)^2 = 4n^2$ or $q^2 = 2n^2 \Rightarrow q^2$ is even number. So q is even number. Since p and q both are even numbers, they will have common factor 2 which contradicts the assumption of p and q have no common factors. Therefore the assumption is false and $\sqrt{2}$ is an irrational number.

5.3 PRINCIPLE OF STRONG MATHEMATICAL INDUCTION

Let P(n) be a statement (or proposition) which may be either true or false for each integral value of n. The P(n) is true for all positive integers provided for $n_0, n_1 \in I_+$, where $n_1 \geq n_0$. Thus two cases may arise:

(a) $P(n_0), P(n_0 + 1), \ldots, P(n_1 - 1), P(n_1)$ are all true.

(b) For some $k \in I_+$, where $k \geq n_1$, the assumption that $P(n_0), P(n_0 + 1), \ldots, P(k - 1), P(k)$ are true implies that $P(k + 1)$ is also true. Then P(n) is true for all $n \geq n_0$.

Example 17 Show that for Fibonacci sequence defined as

$$F_1 = 1,\ F_2 = 1 \text{ and } F_n = F_{n-1} + F_{n-2}, \text{ for all } n \geq 3,$$

the nth Fibonacci number $F_n < 2^m$ of each positive integer n.

Solution Let the statement P(n) be given by: $P(n) : F_n < 2^n$.

Basis of induction: For $n = 1$, the RHS of P(n) is $F_1 = 1(< 2) = 2^1$, and hence, P(n) is true for $n = 1$.

Induction Step: Assuming that P(n) is true for $n = k$. Then P(1), P(2), P(3), ..., P(k) are true for some $k \in I_+$ where $k \geq 2$.

To show that if P(n) is true for $n = 1, 2, \ldots, k$, then $P(k + 1)$ is also true, i.e. $F_{m+1} < 2^{m+1}$. Now

$$\begin{aligned} F_{m+1} &= F_m + F_{m-1} < 2^m + 2^{m-1} \\ &= 2^{m-1}(2 + 1) < 2^{m-1}(2 + 2) \\ &= 2^{m-1} \cdot 2^2 < 2^{m+1} \end{aligned}$$

i.e., $F_{m+1} < 2^{m+1}$. This shows that if P(n) is true for $n = 1, 2, 3, \ldots, k$, then it is also true for $n = k + 1$

Hence, by the principle of strong mathematical induction P(n) is true for all positive integer n.

Example 18 If F_n is the nth Fibonacci number, prove that

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

for all positive integral values of $n \geq 1$.

Solution *Basis of induction:* For $n = 1$, the LHS for F_n is 1, i.e. $F_1 = 1$ and RHS is

$$\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)\right] = 1.$$

Hence, F_n is true for $n = 1$.

Induction Step: Assuming that F_n is true for $n = k$. Then, we get

$$F_k = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right]$$

Substituting k for $k + 1$ to both side of this equation, we get

$$F_{k+1} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}\right]$$

Suppose $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. Then

$$F_{k+1} = F_k + F_{k-1} \qquad \text{[by definition of Fibonacci sequence]}$$

$$= \frac{1}{\sqrt{5}}[\alpha^k - \beta^k] + \frac{1}{\sqrt{5}}[a^{k-1} - \beta^{k-1}]$$

$$= \frac{1}{\sqrt{5}}[\alpha^{k-1}(\alpha+1) - \beta^{k-1}(\beta+1)]$$

where $\alpha + 1 = \frac{1+\sqrt{5}}{2} + 1 = \frac{3+\sqrt{5}}{2} = \alpha^2$, and $\beta + 1 = \frac{1-\sqrt{5}}{2} + 1 = \frac{3-\sqrt{5}}{2} = \beta^2$

Putting these values in F_{k+1}, we get

$$F_{k+1} = \frac{1}{\sqrt{5}}[\alpha^{k-1}.\alpha^2 - \beta^{k-1}.\beta^2] = \frac{1}{\sqrt{5}}[\alpha^{k+1} - \beta^{k+1}]$$

$$F_{k+1} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}\right]$$

This Shows that if F_n is ture for $n = k$, then it also true for $n = k + 1$.

Hence, by the principle of mathematical induction, F_n is true for all integer value of $n \geq 1$.

Example 19 If n is an integer greater than 1, then show that n can be written as the product of primes.

Solution Let the statement or proposition is defined as: $P(n) = n$, where n can be written as the product of primes.

Basic of induction: Since $n > 1$ already, taking $n = 2$. As 2 can be written as the product of one prime (itself). Thus P(2) is true for $n = 2$.

Induction step: Assuming that $P(n)$ is true for $n = k$, i.e. k can be written as product of primes where k is a positive integer, $k \leq n$.

Let $P(n)$ is true for all positive integers k, such that $k \leq n$. Then to prove that the result is also true for $P(n + 1)$ we need to show that $(n + 1)$ can be written as product of primes. Following two cases may arise.

(*a*) $(n + 1)$ is prime, and (*b*) $(n + 1)$ is composite.