CHAPTER 3
Functions

3.1 FUNCTIONS, MAPPINGS

3.1 Define a function from a set A into a set B.

▌ Suppose that to each element of A there is assigned a unique element of B; the collection of such assignments is called a *function* (or *mapping* or *map*) from A into B. We denote a function f from A into B by

$$f: A \to B$$

We write $f(a)$, read "f of a", for the element of B that f assigns to $a \in A$; it is called the *value* of f at a or the *image* of a under f.

Remark: The terms function and mapping are frequently used synonymously, although some texts reserve the word function for a real-valued or complex-valued mapping, that is, which maps a set into **R** or **C**.

3.2 What is the (*a*) domain, (*b*) codomain, (*c*) image of a function $f: A \to B$?

(*a*) The set A is the *domain* of f.

(*b*) The set B is the *codomain* of f.

(*c*) The set of all image values of f is called the *image* (or *range*) of f and is denoted by $\operatorname{Im} f$ or $f(A)$. That is,

$$\operatorname{Im} f = \{b \in B: \text{there exists } a \in A \text{ such that } f(a) = b\}$$

[Observe that $\operatorname{Im} f$ is a subset (perhaps a proper subset) of B.]

3.3 Consider a function $f: A \to B$. (*a*) Let S be a subset of A. Define the image of S under f, denoted by $f(S)$. (*b*) Let T be a subset of B. Define the inverse image or *preimage* of T under f, denoted by $f^{-1}(T)$.

▌ (*a*) Here $f(S) = \{f(a): a \in S\} = \{b \in B: \exists a \in S \text{ such that } f(a) = b\}$. In other words, $f(S)$ consists of all images of the elements in S. (Here $\exists$ is short for "there exists".)

(*b*) Here $f^{-1}(T) = \{a \in A: f(a) \in T\}$. In other words, $f^{-1}(T)$ consists of the elements of A whose images belong to T.

3.4 Define the equality of functions.

▌ Two functions $f: A \to B$ and $g: A \to B$ are defined to be equal, written $f = g$, if $f(a) = g(a)$ for every $a \in A$. The negation of $f = g$ is written $f \neq g$ and is the statement: There exists an $a \in A$ for which $f(a) \neq g(a)$.

3.5 Define the graph of a function $f: A \to B$.

▌ To each function $f: A \to B$ there corresponds the subset of $A \times B$ given by $\{(a, f(a)): a \in A\}$. We call this set the *graph* of f. We note that two functions $f: A \to B$ and $g: A \to B$ are equal if and only if they have the same graph. Thus we do not distinguish between a function and its graph.

3.6 Consider the function f from $A = \{a, b, c, d\}$ into $B = \{x, y, z, w\}$ defined by Fig. 3-1. Find: (*a*) the image of each element of A; (*b*) the image of f; and (*c*) the graph of f, i.e., write f as a set of ordered pairs.

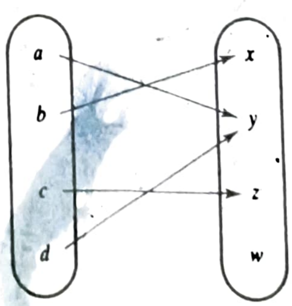


Fig. 3-1

(a) The arrow indicates the image of an element. Thus

$$f(a) = y, \qquad f(b) = x, \qquad f(c) = z, \qquad f(d) = y$$

(b) The image $f(A)$ of f consists of all image values. Only x, y, z appear as image values; hence $f(A) = \{x, y, z\}$.

(c) The ordered pairs $(a, f(a))$, where $a \in A$ form the graph of f. Thus $f = \{(a, y), (b, x), (c, z), (d, y)\}$.

3.7 Consider the function f defined by Fig. 3-1. Find: **(a)** $f(S)$ where $S = \{a, b, d\}$; **(b)** $f^{-1}(T)$ where $T = \{y, z\}$; and **(c)** $f^{-1}(w)$.

(a) $f(S) = f(\{a, b, d\}) = \{f(a), f(b), f(d)\} = \{y, x, y\} = \{x, y\}$.

(b) The elements a, c, and d have images in T, hence $f^{-1}(T) = \{a, c, d\}$.

(c) No element has the image w under f; hence $f^{-1}(w) = \emptyset$, the empty set.

3.8 State whether or not each diagram in Fig. 3-2 defines a function from $A = \{a, b, c\}$ into $B = \{x, y, z\}$

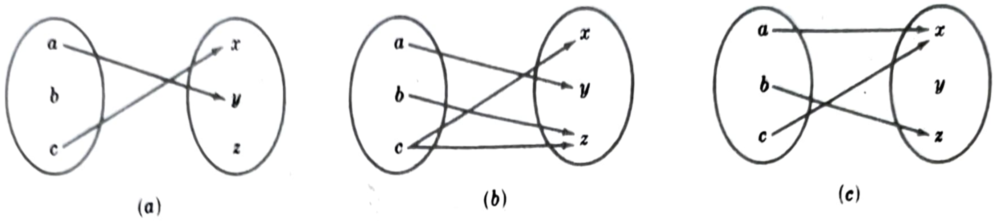


Fig. 3-2

(a) No. There is no element of B assigned to the element $b \in A$.

(b) No. Two elements, x and z, are assigned to $c \in A$.

(c) Yes, since each element of A is assigned a unique element of B.

3.9 State whether or not each diagram of Fig. 3-3 defines a function from $C = \{1, 2, 3\}$ into $D = \{4, 5, 6\}$.

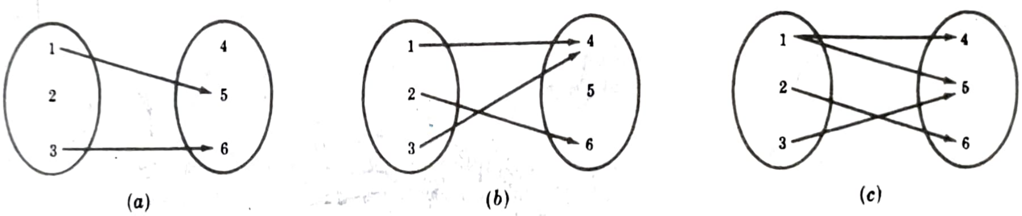


Fig. 3-3

(a) No. There is no element of D assigned to the element $2 \in C$.

(b) Yes, since each element of C is assigned a unique element of D.

(c) No. Two elements, 4 and 5, are assigned to $1 \in C$.

3.10 Let A be the set of students in a school. Determine which of the following assignments defines a function on A. **(a)** To each student assign his or her age. **(b)** To each student assign his or her teacher. **(c)** To each student assign his or her sex. **(d)** To each student assign his or her spouse.

A collection of assignments is a function on A providing each element $a \in A$ is assigned exactly one element. Thus:

(a) Yes, because each student has one and only one age.

(b) Yes, if each student has only one teacher; no, if any student has more than one teacher.

(c) Yes.

(d) No, if any student is not married.

3.11 Consider the set $A = \{1, 2, 3, 4, 5\}$ and the function $f: A \to A$ defined by Fig. 3-4. Find: **(a)** the image of each element of A, and **(b)** the image $f(A)$ of the function f.

(a) The arrow indicates the image of an element; thus $f(1) = 3$, $f(2) = 5$, $f(3) = 5$, $f(4) = 2$, $f(5) = 3$.

(b) The image $f(A)$ of f consists of all the image values. Now only 2, 3, and 5 appear as the image of any elements of A; hence $f(A) = \{2, 3, 5\}$.

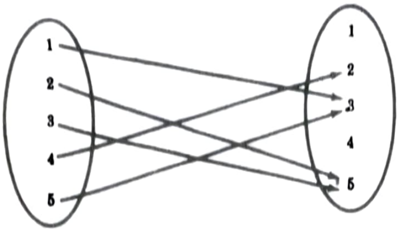


Fig. 3-4

3.12 Find the graph of the function f defined by Fig. 3-4, i.e., write f as a set of ordered pairs.

▌ The ordered pairs $(a, f(a))$, where $a \in A$ form the graph of f. Thus

$$f = \{(1, 3), (2, 5), (3, 5), (4, 2), (5, 3)\}$$

3.13 Consider the function f defined by Fig. 3-4. Find: **(a)** $f(S)$ where $S = \{1, 3, 5\}$; **(b)** $f^{-1}(T)$ where $T = \{1, 2\}$; and **(c)** $f^{-1}(3)$.

▌ **(a)** $f(S) = f(\{1, 3, 5\}) = \{f(1), f(3), f(5)\} = \{3, 5, 3\} = \{3, 5\}$.
(b) Only 4 has its image in $T = \{1, 2\}$. Thus $f^{-1}(T) = \{4\}$.
(c) The elements 1 and 5 have image 3; hence $f^{-1}(3) = \{1, 5\}$.

3.14 Let f be a subset of $A \times B$. When does f define a function from A into B?

▌ A subset f of $A \times B$ is a function $f: A \rightarrow B$ if and only if each $a \in A$ appears as the first coordinate in exactly one ordered pair in f.

3.15 Let $X = \{1, 2, 3, 4\}$. Determine whether each of the following relations on X (set of ordered pairs) is a function from X into X.
(a) $f = \{(2, 3), (1, 4), (2, 1), (3, 2), (4, 4)\}$ **(c)** $h = \{(2, 1), (3, 4), (1, 4), (2, 1), (4, 4)\}$
(b) $g = \{(3, 1), (4, 2), (1, 1)\}$

▌ Recall that a subset f of $X \times X$ is a function $f: X \rightarrow X$ if and only if each $a \in X$ appears as the first coordinate in exactly one ordered pair in f.
(a) No. Two different ordered pairs (2, 3) and (2, 1) in f have the same number 2 as their first coordinate.
(b) No. The element $2 \in X$ does not appear as the first coordinate in any ordered pair in g.
(c) Yes. Although $2 \in X$ appears as the first coordinte in two ordered pairs in h, these two ordered pairs are equal.

3.16 Let $W = \{a, b, c, d\}$. Determine whether each of the following sets of ordered pairs is a function from W into W.
(a) $\{(b, a), (c, d), (d, a), (c, d), (a, d)\}$ **(c)** $\{(a, b), (b, b), (c, b), (d, b)\}$
(b) $\{(d, d), (c, a), (a, b), (d, b)\}$ **(d)** $\{(a, a), (b, a), (a, b), (c, d), (d, a)\}$

▌ **(a)** Yes. Although c appears as the first coordinate in two ordered pairs, these two ordered pairs are equal.
(b) No. The element b does not appear as the first coordinate in any ordered pair.
(c) Yes, since each element of W appears as the first coordinate in exactly one ordered pair.
(d) No. The element a appears as the first coordinate in two different ordered pairs.

3.17 Let $f: \mathbf{R} \rightarrow \mathbf{R}$ be the function which assigns to each real number x its square x^2. Describe different ways of defining f.

▌ The function f may be described by any of the following:

$$f(x) = x^2 \quad \text{or} \quad x \mapsto x^2 \quad \text{or} \quad y = x^2$$

Here the barred arrow $\mapsto$ is read "goes into". In the last notation, x is called the *independent variable* and y is called the *dependent variable* since the value of y will depend on the value that x takes.

Remark: Whenever a function f is given by a formula using the independent variable x, as in Problem 3.17, we assume unless otherwise stated or implied, that f is a function from $\mathbf{R}$ (or the largest subset of $\mathbf{R}$ for which f has meaning) into $\mathbf{R}$. (See Section 3.2.)

3.18 Consider the above function $f(x) = x^2$ in Problem 3.17. Find: **(a)** the value of f at 5, -4, and 0; **(b)** $f(y + 2)$ and $f(x + h)$; and **(c)** $[f(x + h) - f(x)]/h$.

(a) $f(5) = 5^2 = 25$, $f(-4) = (-4)^2 = 16$, and $f(0) = 0^2 = 0$.
(b) $f(y+2) = (y+2)^2 = y^2 + 4y + 4$, and $f(x+h) = (x+h)^2 = x^2 + 2xh + h^2$.
(c) $[f(x+h) - f(x)]/h = (x^2 + 2xh + h^2 - x^2)/h = (2xh + h^2)/h = 2x + h$.

3.19 Consider the function $f(x) = x^2$ in Problem 3.17. Find Im f, the image of f.

Every nonnegative real number a is the square of $\sqrt{a}$, and the square of any number cannot be negative. Hence $\mathrm{Im}\, f = \{x : x \geq 0\}$, that is, the set of nonnegative real numbers.

3.20 Let f assign to each country in the world its capital city. Find: **(a)** the domain of f, and **(b)** f(France), f(Canada), f(Japan).

(a) The domain of f is the set of countries of the world.
(b) Here f(France) = Paris since Paris is the capital of France. Similarly, f(Canada) = Ottawa, and f(Japan) = Tokyo.

3.21 Let g assign to each word in the English language the number of distinct letters needed to spell the word. Find g(letter), g(mathematics), and g(amour).

Here g(letter) = 4 since there are four letters, l, e, t, and r, required to spell "letter". Similarly, g(mathematics) = 8. However, g(amour) is not defined since the domain of g is the set of English words and "amour" is a French word.

3.22 Let A be the set of polygons in the plane. Let $h: A \to \mathbf{N}$ assign to each polygon P its number of sides. Find h(triangle), h(square), h(hexagon), and h(trapezoid).

Here h(triangle) = 3 since a triangle has three sides. Also, h(square) = 4, h(hexagon) = 6, and h(trapezoid) = 4.

3.23 Let $X = \{a, b\}$ and $Y = \{1, 2, 3\}$. Find the number n of functions: **(a)** from X into Y, and **(b)** from Y into X.

(a) There are three choices, 1, 2, or 3, for the image of a and there are the same three choices for the image of b. Thus there are $n = 3 \cdot 3 = 3^2 = 9$ possible functions from X into Y.
(b) There are two choices, a or b, for each of the three elements of Y. Thus there are $n = 2 \cdot 2 \cdot 2 = 2^3 = 8$ possible functions from Y into X.

3.24 Suppose X has $|X|$ elements and Y has $|Y|$ elements. Show that there are $|Y|^{|X|}$ functions from X into Y. (For this reason, one frequently writes Y^X for the collection of all functions from X into Y.)

There are $|Y|$ choices for the image of each of the $|X|$ elements of X; hence there are $|Y|^{|X|}$ possible functions from X into Y.

3.25 Let A be any nonempty set. **(a)** Define the identity mapping on A, denoted by 1_A or 1. **(b)** Find $1_A(3)$, $1_A(6)$, $1_A(8)$ where $A = \{1, 2, 3, \ldots, 9\}$.

(a) The identity map on A is the function $1_A: A \to A$ defined by $1_A(x) = x$ for every $x \in A$.
(b) Under the identity map, the image of an element is the element itself; so $1_A(3) = 3$, $1_A(6) = 6$, $1_A(8) = 8$.

3.26 Define a constant map.

Let f be a function with domain A. Then f is a constant map if every $a \in A$ is assigned the same element.

3.27 Given sets A and B, how many constant maps are there from A into B?

Each $b \in B$ defines the constant map $f(x) = b$ for every $x \in A$. Hence there are $|B|$ constant maps where $|B|$ denotes the number of elements in B.

3.28 Let S be a subset of A and let $f: A \to B$. Define the restriction of f to S.

The *restriction* of f to S is the mapping $\hat{f}: S \to B$ defined by $\hat{f}(s) = f(s)$ for every $s \in S$. One usually writes $f|_S$ to denote the restriction of f to S.

3.29 Let $f: \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = x^2$. Let $\hat{f}: \mathbf{Z} \to \mathbf{R}$ be the restriction of f to $\mathbf{Z}$, that is, let $\hat{f} = f|_Z$. Find $f(4)$, $f(-3)$, and $f(1/2)$.

By definition, $\hat{f}(n) = f(n)$ for every $n \in \mathbf{Z}$. Thus $\hat{f}(4) = f(4) = 4^2 = 16$ and $\hat{f}(-3) = f(-3) = (-3)^2 = 9$. However, $\hat{f}(1/2)$ is not defined since $1/2$ is not in the domain of $\hat{f}$.

3.30 Let S be a subset of A. Define the inclusion map from S into A.

The *inclusion* map from S into A, denoted by $i: S \hookrightarrow A$, is defined by $i(x) = x$ for every $x \in S$. In other words, the inclusion map of S into A is the restriction of the identity map on A to S.

3.31 Consider the inclusion map $i: \mathbf{N} \hookrightarrow \mathbf{R}$. Find $i(4)$, $i(8)$, $i(23)$, $i(-6)$.

The inclusion map sends each element into itself. Thus $i(4) = 4$, $i(8) = 8$ and $i(23) = 23$. However, $i(-6)$ is not defined since -6 does not belong to $\mathbf{N}$ and hence -6 is not in the domain of $i: \mathbf{N} \hookrightarrow \mathbf{R}$.

3.2 REAL-VALUED FUNCTIONS

This section covers real-valued functions, that is, functions f which map sets into $\mathbf{R}$. Frequently, the domain of f is $\mathbf{R}$ or an interval subset of $\mathbf{R}$ and hence the function f can be plotted in the coordinate plane $\mathbf{R} \times \mathbf{R} = \mathbf{R}^2$. In particular, when the functions are piecewise continuous and differentiable, such as polynomial, rational, trigonometric, exponential, and logarithmic functions, the graph of such a function f can be approximated by first plotting some of its points and then drawing a smooth curve through these points. The points are usually obtained from a table where various values are assigned to x and the corresponding values $f(x)$ computed.

The following notation is also used for intervals from a to b where a and b are real numbers such that $a < b$:

$[a, b] = \{x : a \le x \le b\}$, called the closed interval from a to b,

$[a, b) = \{x : a \le x < b\}$, called a half-open interval from a to b,

$(a, b] = \{x : a < x \le b\}$, called a half-open interval from a to b,

$(a, b) = \{x : a < x < b\}$, called the open interval from a to b.

3.32 What is the domain D of a real-valued function $f(x)$ (where x is a real variable) when $f(x)$ is given by a formula?

The domain D consists of the largest subset of $\mathbf{R}$ for which $f(x)$ has meaning and is real, unless otherwise specified.

3.33 Find the domain D of each of the following functions:
(a) $f(x) = 1/(x-2)$, **(b)** $g(x) = x^2 - 3x - 4$, **(c)** $h(x) = \sqrt{25 - x^2}$

(a) f is not defined for $x - 2 = 0$, i.e., for $x = 2$; hence $D = \mathbf{R}\backslash\{2\}$.
(b) g is defined for every real number; hence $D = \mathbf{R}$.
(c) h is not defined when $25 - x^2$ is negative; hence $D = [-5, 5] = \{x : -5 \le x \le 5\}$.

3.34 Find the domain D of the function $f(x) = x^2$ where $0 \le x \le 2$.

Although the formula for f is meaningful for every real number, the domain of f is explicitly given as $D = \{x : 0 \le x \le 2\}$.

3.35 Use a formula to define each of the following functions from $\mathbf{R}$ into $\mathbf{R}$:
(a) To each number let f assign its cube.
(b) To each number let g assign the number 5.
(c) To each positive number let h assign its square, and to each nonpositive number let h assign the number 6.

(a) Since f assigns to any number x its cube x^3, we can define f by $f(x) = x^3$.
(b) Since g assigns 5 to any number x, we can define g by $g(x) = 5$.
(c) Two different rules are used to define h as follows: $h(x) = \begin{cases} x^2 & \text{if } x > 0 \\ 6 & \text{if } x \le 0 \end{cases}$

3.36 Consider the functions f, g, and h of Problem 3.35. Find: **(a)** $f(4)$, $f(-2)$, $f(0)$; **(b)** $g(4)$, $g(-2)$, $g(0)$; **(c)** $h(4)$, $h(-2)$, $h(0)$.

(a) Now $f(x) = x^3$ for every number x; hence $f(4) = 4^3 = 64$, $f(-2) = (-2)^3 = -8$, and $f(0) = 0^3 = 0$.
(b) The image of every number is 5, so $g(4) = 5$, $g(-2) = 5$, and $g(0) = 5$.
(c) Since $4 > 0$, $h(4) = 4^2 = 16$. On the other hand, $-2, 0 \le 0$, and so $h(-2) = 6$, $h(0) = 6$.

3.37 Use a formula to define each of the following functions from **R** into **R**:

(a) To each number let f assign its square plus 3.
(b) To each number let g assign its cube plus twice the number.
(c) To each number greater than or equal to 3 let h assign the number squared, and to each number less than 3 let h assign the number -2.

(a) $f(x)=x^3+3$. **(b)** $g(x)=x^3+2x$. **(c)** Two different rules are used to define h; $h(x)=\begin{cases} x^2 & \text{if } x\geq 3 \\ -2 & \text{if } x<3 \end{cases}$.

3.38 Let $g\colon \mathbf{R}\to\mathbf{R}$ be defined by $g(x)=\begin{cases} x^2-3x & \text{if } x\geq 2 \\ x+2 & \text{if } x<2 \end{cases}$. Find: **(a)** $g(5)$, **(b)** $g(0)$, and **(c)** $g(-2)$.

(a) Since $5\geq 2$, $g(5)=5^2-3(5)=25-15=10$.
(b) Since $0<2$, $g(0)=0+2=2$.
(c) Since $-2<2$, $g(-2)=-2+2=0$.

3.39 Consider the function $f(x)=x^2-3x+2$. Find: **(a)** $f(-3)$, **(b)** $f(2)-f(-4)$, **(c)** $f(y)$, and **(d)** $f(a^2)$.

The function assigns to any element the square of the element minus 3 times the element plus 2.
(a) $f(-3)=(-3)^2-3(-3)+2=9+9+2=20$
(b) $f(2)=(2)^2-3(2)+2=0$, $f(-4)=(-4)^2-3(-4)+2=30$. Then

$$f(2)-f(-4)=0-30=-30$$

(c) $f(y)=(y)^2-3(y)+2=y^2-3y+2$
(d) $f(a^2)=(a^2)^2-3(a^2)+2=a^4-3a^2+2$

3.40 Given the function $f(x)$ of Problem 3.39, find: **(a)** $f(x^2)$, **(b)** $f(y-z)$, **(c)** $f(x+3)$, and **(d)** $f(2x-3)$.

(a) $f(x^2)=(x^2)^2-3(x^2)+2=x^4-3x^2+2$
(b) $f(y-z)=(y-z)^2-3(y-z)+2=y^2-2yz+z^2-3y+3z+2$
(c) $f(x+3)=(x+3)^2-3(x+3)+2=(x^2+6x+9)-3x-9+2=x^2+3x+2$
(d) $f(2x-3)=(2x-3)^2-3(2x-3)+2=4x^2-12x+9-6x+9+2=4x^2-18x+20$

3.41 Given the function $f(x)$, of Problem 3.39, find: **(a)** $f(x+h)$, **(b)** $f(x+h)-f(x)$, **(c)** $[f(x+h)-f(x)]/h$.

(a) $f(x+h)=(x+h)^2-3(x+h)+2=x^2+2xh+h^2-3x-3h+2$
(b) Using **(a)**, we obtain

$$f(x+h)-f(x)=(x^2+2xh+h^2-3x-3h+2)-(x^2-3x+2)=2xh+h^2-3h$$

(c) Using **(b)**, we have

$$[f(x+h)-f(x)]/h=(2xh+h^2-3h)/h=2x+h-3$$

3.42 Determine which of the graphs in Fig. 3-5 are functions from **R** into **R**.

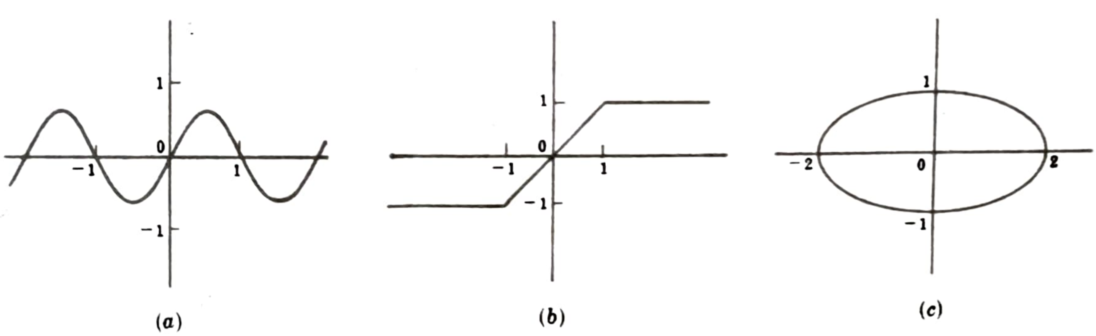


Fig. 3-5

Geometrically speaking, a set of points on a coordinate diagram is a function if and only if every vertical line contains exactly one point of the set. **(a)** Yes. **(b)** Yes. **(c)** No.

Determine which of the graphs in Fig. 3-6 are functions from **R** into **R**.

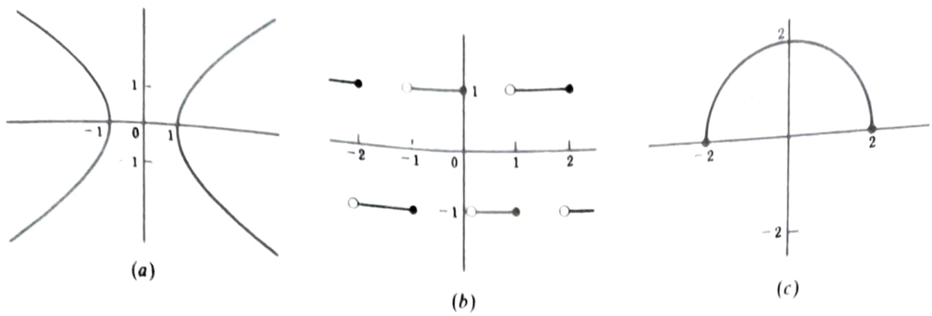
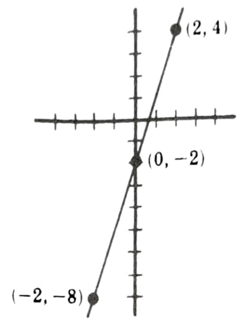


Fig. 3-6

▌ **(a)** No. **(b)** Yes. **(c)** No; however the graph does define a function from D into **R** where $D = \{x: -2 \le x \le 2\}$.

3.44 Sketch the graph of $f(x) = 3x - 2$.

x	$f(x)$
-2	-8
0	-2
2	4



Graph of f **Fig. 3-7**

▌ Since f is linear, only two points (three as a check) are needed to sketch its graph. Set up a table with three values of x, say, $x = -2, 0, 2$ and find the corresponding values of $f(x)$:

$$f(-2) = 3(-2) - 2 = -8, \qquad f(0) = 3(0) - 2 = -2, \qquad f(2) = 3(2) - 2 = 4$$

Draw the line through these points as in Fig. 3-7.

3.45 Consider the function $f: \mathbf{R} \to \mathbf{R}$ defined by $f(x) = x^3$. Find: **(a)** $f(3)$ and $f(-5)$, **(b)** $f(y)$ and $f(y+1)$, **(c)** $f(x+h)$, **(d)** $[f(x+h) - f(x)]/h$.

▌ **(a)** $f(3) = 3^3 = 27, \quad f(-5) = (-5)^3 = -125$
(b) $f(y) = (y)^3 = y^3, \quad f(y+1) = (y+1)^3 = y^3 + 3y^2 + 3y + 1$
(c) $f(x+h) = (x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$
(d) $[f(x+h) - f(x)]/h = (x^3 + 3x^2h + 3xh^2 + h^3 - x^3)/h = (3x^2h + 3xh^2 + h^3)/h = 3x^2 + 3xh + h^2$

3.46 Sketch the graph of the function in Problem 3.45.

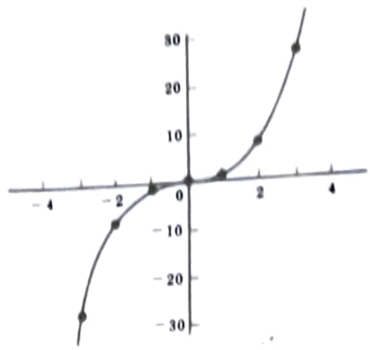
▌ Since f is a polynomial function, it can be sketched by first plotting some points of its graph and then drawing a smooth curve through these points as in Fig. 3-8.

3.47 Sketch the graph of the function $g(x) = x^2 + x - 6$.

▌ Set up a table of values for x and then find the corresponding values of the function. Plot the points in a coordinate diagram, and then draw a smooth continuous curve through these points as in Fig. 3-9.

3.48 Given the function of Problem 3.47, find **(a)** $g^{-1}(14)$, **(b)** $g^{-1}(-8)$.

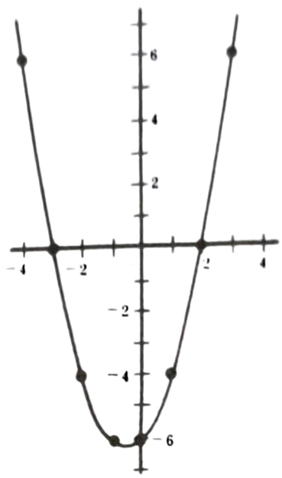
x	$f(x)$
−3	−27
−2	−8
−1	−1
0	0
1	1
2	8
3	27



Graph of $f(x) = x^3$

Fig. 3-8

x	$g(x)$
−4	6
−3	0
−2	−4
−1	−6
0	−6
1	−4
2	0
3	6



Graph of g

Fig. 3-9

▮ **(a)** Set $g(x) = 14$ and solve for x:

$$x^2 + x - 6 = 14 \quad \text{or} \quad x^2 + x - 20 = 0 \quad \text{or} \quad (x+5)(x-4) = 0$$

Thus $x = -5$ and $x = 4$. In other words, $g^{-1}(-4) = \{-5, 4\}$.

(b) Set $g(x) = -8$ and solve for x: $x^2 + x - 6 = -8$ or $x^2 + x + 2 = 0$. Using the quadratic formula, the discriminant $D = b^2 - 4ac = 1^2 - 4(1 \cdot 2) = -7$ is negative and hence there are no real solutions. Thus $g^{-1}(-8) = \emptyset$, the empty set.

3.49 Sketch the graph of $h(x) = x^3 - 3x^2 - x + 3$.

▮ Draw a smooth curve through some of the points of the graph of h as in Fig. 3-10.

3.50 Consider the function $h(x) = x^3 - 3x^2 - x + 3$ (Problem 3.49). **(a)** Find $h(\mathbf{R})$, the image of h. **(b)** How many real roots does h have? **(c)** Find $h^{-1}(A)$ where $A = [-15, 15]$.

▮ Use the graph of h in Fig. 3-10.

(a) Since every horizontal line intersects the graph of h, every real number is an image value. Thus $f(\mathbf{R}) = \mathbf{R}$.

(b) Since the graph crosses the x axis in three points, h has three real roots. That is, $x^3 - 3x^2 - x + 3 = 0$ has three real roots.

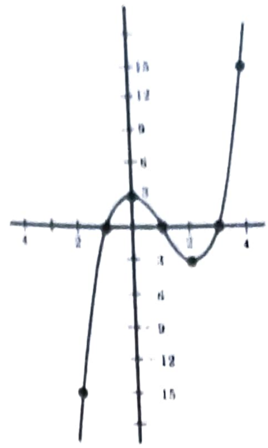
(c) The graph indicates that the image of every x-value between -2 and 4, and only these x-values, lies between -15 and 15. Thus $f^{-1}(A) = [-2, 4]$.

3.51 Sketch the graph of $f(x) = 2$.

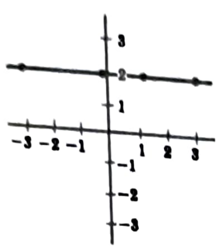
▮ For any value of x, we have $f(x) = 2$. Thus, for example, $(-3, 2)$, $(0, 2)$, $(1, 2)$, $(3, 2)$ lie on the graph of f given by the horizontal line through $y = 2$ as shown in Fig. 3-11.

3.52 Sketch the graph of $g(x) = (1/2)x - 1$.

x	$h(x)$
−2	−15
−1	0
0	3
1	0
2	−3
3	0
4	15

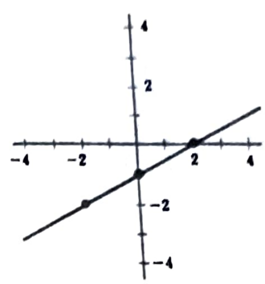


Graph of h **Fig. 3-10**



Graph of f **Fig. 3-11**

x	$g(x)$
−2	−2
0	−1
2	0



Graph of g **Fig. 3-12**

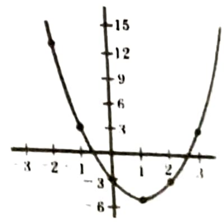
I Since g is linear, only two points (three as a check) are needed to sketch its graph. Set up a table with three values of x, say, $x = -2, 0, 2$ and find the corresponding values of $g(x)$:

$$g(-2) = -1 - 1 = -2, \qquad g(0) = 0 - 1 = -1, \qquad g(2) = 1 - 1 = 0.$$

Draw the line through these points as in Fig. 3-12.

3.53 Sketch the graph of the function $h(x) = 2x^2 - 4x - 3$.

x	$h(x)$
−2	13
−1	3
0	−3
1	−5
2	−3
3	3



Graph of h **Fig. 3-13**

I Draw a smooth continuous curve through some of the points of the graph of h as in Fig. 3-13.

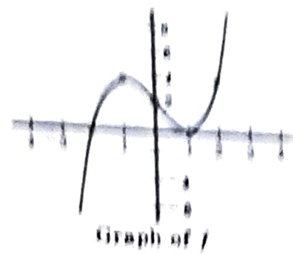
3.54 Sketch the graph of the function $f(x) = x^3 - 3x + 2$.

I Draw a smooth continuous curve through some of the points of the graph of f as in Fig. 3-14.

3.55 Sketch the graph of the function $g(x) = x^4 - 10x^2 + 9$.

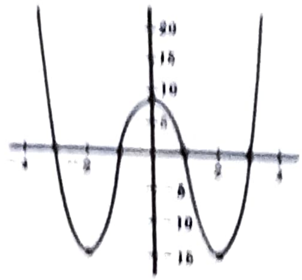
I Draw a smooth continuous curve through some of the points of the graph of g as in Fig. 3-15.

x	$f(x)$
-3	16
-2	0
-1	4
0	2
1	0
2	4
3	20



Graph of f **Fig. 3-14**

x	$g(x)$
-4	105
-3	0
-2	-15
-1	0
0	9
1	0
2	-15
3	0
4	105



Graph of g **Fig. 3-15**

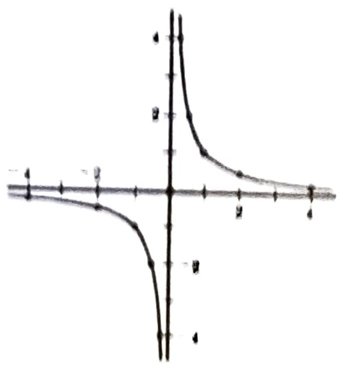
3.56 Consider the functions f and g in Problems 3.54 and 3.55 respectively. (*a*) Is $f(\mathbf{R}) = \mathbf{R}$? (*b*) Is $g(\mathbf{R}) = \mathbf{R}$?

I (*a*) Yes. As shown in Fig. 3-14, every horizontal line intersects the graph of f; hence every value of y is in the image of f. Thus $f(\mathbf{R}) = \mathbf{R}$.

(*b*) No. As shown in Fig. 3-15, some horizontal lines do not intersect the graph of g, for example, the horizontal line through $y = -20$. Thus $-20 \notin g(\mathbf{R})$, and so $g(\mathbf{R}) \neq \mathbf{R}$.

3.57 Sketch the graph of $h(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1/x & \text{if } x \neq 0 \end{cases}$.

x	$h(x)$
4	$\frac{1}{4}$
2	$\frac{1}{2}$
1	1
$\frac{1}{2}$	2
$\frac{1}{4}$	4
0	0
$-\frac{1}{4}$	-4
$-\frac{1}{2}$	-2
-1	-1
-2	$-\frac{1}{2}$
-4	$-\frac{1}{4}$



Graph of h **Fig. 3-16**

I See Fig. 3-16. (Note that this graph is only piecewise continuous. Specifically, h is continuous for $x < 0$ and for $x > 0$.)

3.58 A function $f: \mathbf{R} \to \mathbf{R}$ is a *polynomial function* if $f(x) = 0$, the zero function, or f can be expressed in the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where the a_i are real numbers and $a_n \neq 0$. Define: (*a*) the leading coefficient of f; (*b*) monic polynomial; (*c*) the degree of f, written $\deg f$.

I (*a*) The leading coefficient of f is the nonzero coefficient of the highest power of x or, in other words, a_n.

(b) A polynomial f is monic if its leading coefficient is 1, i.e., if $a_n = 1$.
(c) The degree of the zero function $f(x) = 0$ is not defined; otherwise, $\deg f = n$, the highest power of x with a nonzero coefficient.

3.59 Suppose $f(x)$ and $g(x)$ are polynomial functions such that $\deg f = m$ and $\deg g = n$. Find the degree of the product $h(x) = f(x)g(x)$.

▮ The degree of the product h is the sum of the degrees of its factors f and g; that is; $\deg h = \deg f + \deg g = m + n$.

3.60 Let $f(x) = a_n x^n + \cdots + a_1 x + a_0$ be a polynomial function of odd degree. Argue that $f(\mathbf{R}) = \mathbf{R}$.

▮ We want to show that for every $k \in \mathbf{R}$, the equation $f(x) = k$ has a solution $x \in \mathbf{R}$. We may always suppose $a_n = +1$, so that $f(x) \approx x^n$ when $|x|$ is very large. Then there must exist a (large) positive real number a such that both $f(a) > |k|$ and $f(-a) < -|k|$, which imply

$$f(-a) < k < f(a) \qquad (*)$$

Now, the graph of f is an unbroken curve connecting the points $P_1 = (-a, f(-a))$ and $P_2 = (a, f(a))$; it must therefore intersect any horizontal line included between the horizontals through P_1 and P_2. By $(*)$, $y = k$ is just such a horizontal line; in other words, $f(x) = k$ for some $-a < x < a$.

3 COMPOSITION OF FUNCTIONS

3.61 Consider functions $f: A \to B$ and $g: B \to C$; that is, where the codomain of f is the domain of g. Define the composition function of f and g.

▮ The *composition* of f and g, written $g \circ f$, is the function from A into C defined by

$$(g \circ f)(a) \equiv g(f(a))$$

That is, to find the image of a under $g \circ f$, we first find the image of a under f and then we find the image of $f(a)$ under g.

Remark: If we view f and g as relations, then the function in Problem 3.61 is the same as the composition of f and g as relations (see Section 2.4) except that here we use the functional notation $g \circ f$ for the composition of f and g instead of the notation $f \circ g$ which was used for the composition of relations.

3.62 Let the functions $f: A \to B$ and $g: B \to C$ be defined by Fig. 3-17. Find the composition function $g \circ f: A \to C$.

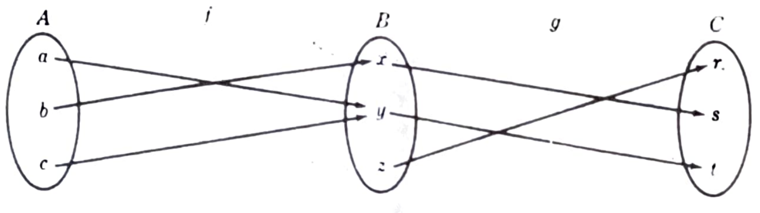


Fig. 3-17

▮ We use the definition of the composition function to compute:

$$(g \circ f)(a) = g(f(a)) = g(y) = t$$
$$(g \circ f)(b) = g(f(b)) = g(x) = s$$
$$(g \circ f)(c) = g(f(c)) = g(y) = t$$

Note that we arrive at the same answer if we "follow the arrows" in the diagram:

$$a \to y \to t, \qquad b \to x \to s, \qquad c \to y \to t$$

3.63 Give the images of the functions f and g in Fig. 3-17.

▮ The image values under the mapping f are x and y, and the image values under g are r, s and t; hence $\operatorname{Im} f = \{x, y\}$ and $\operatorname{Im} g = \{r, s, t\}$.

3.64 Figure 3-18 defines functions $f: A \to B$, $g: B \to C$, and $h: C \to D$. Find the composition function $h \circ g \circ f$.

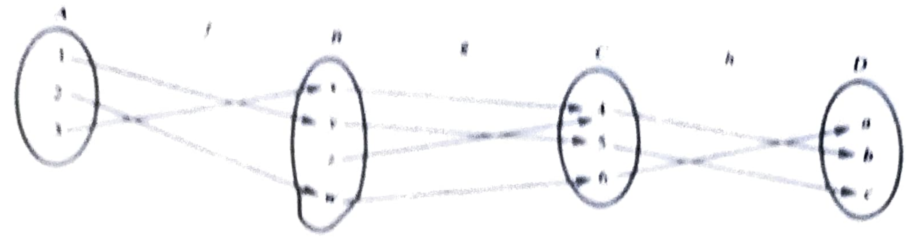


Fig. 3-18

Follow the arrows from A to B to C to D as follows:

$$1 \to y \to 5 \to c \quad \text{hence} \quad (h \circ g \circ f)(1) = c$$
$$2 \to w \to 6 \to a \quad \text{hence} \quad (h \circ g \circ f)(2) = a$$
$$3 \to x \to 4 \to b \quad \text{hence} \quad (h \circ g \circ f)(3) = b$$

3.65 Let functions f and g be defined by $f(x) = 2x + 1$ and $g(x) = x^2 - 2$ respectively. Find: **(a)** $(g \circ f)(4)$ and $(f \circ g)(4)$; **(b)** $(g \circ f)(a + 2)$; and **(c)** $(f \circ g)(a + 2)$.

(a) $f(4) = 2 \cdot 4 + 1 = 9$. Hence $(g \circ f)(4) = g(f(4)) = g(9) = 9^2 - 2 = 79$. $g(4) = 4^2 - 2 = 14$. Hence $(f \circ g)(4) = f(g(4)) = f(14) = 2 \cdot 14 + 1 = 29$. (Note that $f \circ g \neq g \circ f$ since they differ on $x = 4$.)

(b) $f(a + 2) = 2(a + 2) + 1 = 2a + 5$. Hence

$$(g \circ f)(a + 2) = g(f(a + 2)) = g(2a + 5) = (2a + 5)^2 - 2 = 4a^2 + 20a + 23$$

(c) $g(a + 2) = (a + 2)^2 - 2 = a^2 + 4a + 2$. Hence

$$(f \circ g)(a + 2) = f(g(a + 2)) = f(a^2 + 4a + 2) = 2(a^2 + 4a + 2) + 1 = 2a^2 + 8a + 5$$

3.66 Given the functions $f(x) = 2x + 1$ and $g(x) = x^2 - 2$ (Problem 3.65), find the composition functions **(a)** $g \circ f$, and **(b)** $f \circ g$.

(a) Compute the formula for $g \circ f$ as follows:

$$(g \circ f)(x) = g(f(x)) = g(2x + 1) = (2x + 1)^2 - 2 = 4x^2 + 4x - 1$$

Observe that the same answer can be found by writing $y = f(x) = 2x + 1$ and $z = g(y) = y^2 - 2$, and then eliminating y: $z = y^2 - 2 = (2x + 1)^2 - 2 = 4x^2 + 4x - 1$.

(b) $(f \circ g)(x) = f(g(x)) = f(x^2 - 2) = 2(x^2 - 2) + 1 = 2x^2 - 3$.

3.67 Given the functions $f(x) = 2x + 1$ and $g(x) = x^2 - 2$ (Problem 3.65), find the composition functions: **(a)** $f \circ f$ (sometimes denoted by f^2), and **(b)** $g \circ g$.

(a) $(f \circ f)(x) = f(f(x)) = f(2x + 1) = 2(2x + 1) + 1 = 4x + 3$.

(b) $(g \circ g)(x) = g(g(x)) = g(x^2 - 2) = (x^2 - 2)^2 - 2 = x^4 - 4x^2$.

3.68 Consider an arbitrary function $f: A \to B$. When is $f \circ f$ defined?

The composition $f \circ f$ is defined when the domain of f is equal to the codomain of f, that is, when $A = B$.

3.69 Consider any function $f: A \to B$. Show that: **(a)** $1_B \circ f = f$, **(b)** $f \circ 1_A = f$. (Here $1_B: B \to B$ and $1_A: A \to A$ are the identity functions on B and A respectively.) (See Problem 3.25.)

(a) $(1_B \circ f)(a) = 1_B(f(a)) = f(a)$, for every $a \in A$. Thus $1_B \circ f = f$.

(b) $(f \circ 1_A)(a) = f(1_A(a)) = f(a)$, for every $a \in A$. Thus $f \circ 1_B = f$.

Theorem 3.1: Consider functions $f: A \to B$, $g: B \to C$, and $h: C \to D$. Then $h \circ (g \circ f) = (h \circ g) \circ f$.

3.70 Prove Theorem 3.1 which states that composition of functions satisfies the associative law.

Consider any element $a \in A$. Then

$$(h \circ (g \circ f))(a) = h((g \circ f)(a)) = h(g(f(a))) \quad \text{and} \quad ((h \circ g) \circ f)(a) = (h \circ g)(f(a)) = h(g(f(a)))$$

Thus $(h \circ (g \circ f))(a) = ((h \circ g) \circ f)(a)$ for every $a \in A$, and so $h \circ (g \circ f) = (h \circ g) \circ f$.

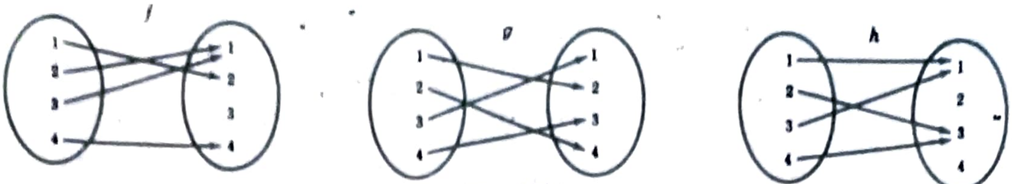


Fig. 3-19

Problems 3.71–3.76 refer to the functions f, g, and h in Fig. 3-19 where each function maps the set $A = \{1, 2, 3, 4\}$ into itself.

3.71 Find the composition function $f \circ g$.

First apply g and then f as follows:

$$(f \circ g)(1) = f(g(1)) = f(2) = 1 \qquad (f \circ g)(3) = f(g(3)) = f(1) = 2$$
$$(f \circ g)(2) = f(g(2)) = f(4) = 4 \qquad (f \circ g)(4) = f(g(4)) = f(3) = 1$$

3.72 Find the composition function $g \circ h$.

Follow the arrows using h first and then g as follows:

$$1 \to 1 \to 2, \qquad 2 \to 3 \to 1, \qquad 3 \to 1 \to 2, \qquad 4 \to 3 \to 1$$

Thus $(g \circ h)(1) = 2$, $(g \circ h)(2) = 1$, $(g \circ h)(3) = 2$, $(g \circ h)(3) = 1$.

3.73 Find the composition function $g^2 = g \circ g$.

Follow the arrows using g twice:

$$1 \to 2 \to 4, \qquad 2 \to 4 \to 3, \qquad 3 \to 1 \to 2, \qquad 4 \to 3 \to 1$$

Thus $g^2(1) = 4$, $g^2(2) = 3$, $g^2(3) = 2$, $g^2(4) = 1$.

3.74 Find the composition function $h^2 = h \circ h$.

Follow the arrows using h twice:

$$1 \to 1 \to 1, \qquad 2 \to 3 \to 1, \qquad 3 \to 1 \to 1, \qquad 4 \to 3 \to 1$$

Here h^2 is the constant function $h^2(x) = 1$.

3.75 Find the composition function $f \circ h \circ g$.

Follow the arrows using g first, then h and finally f, that is, in reverse order:

$$1 \to 2 \to 3 \to 1, \qquad 2 \to 4 \to 3 \to 1, \qquad 3 \to 1 \to 1 \to 2, \qquad 4 \to 3 \to 1 \to 2$$

Thus $f \circ h \circ g = \{(1, 1), (2, 1), (3, 2), (4, 2)\}$.

3.76 Find the composition function $f^3 = f \circ f \circ f$.

Follow the arrows using f three times as follows:

$$1 \to 2 \to 1 \to 2, \qquad 2 \to 1 \to 2 \to 1, \qquad 3 \to 1 \to 2 \to 1, \qquad 4 \to 4 \to 4 \to 4$$

Thus $f \circ f \circ f = \{(1, 2), (2, 1), (3, 1), (4, 4)\}$.

3.77 Consider the functions $f(x) = 2x - 3$ and $g(x) = x^2 + 3x + 5$. Find a formula for the composition functions (*a*) $g \circ f$ and (*b*) $f \circ g$.

(*a*) $(g \circ f)(x) = g(f(x)) = g(2x - 3) = (2x - 3)^2 + 3(2x - 3) + 5 = 4x^2 - 6x + 9 + 6x - 9 + 5 = 4x^2 + 5$.
(*b*) $(f \circ g)(x) = f(g(x)) = f(x^2 + 3x + 5) = 2(x^2 + 3x + 5) - 3 = 2x^2 + 6x + 7$.

3.78 Consider the above function $f(x) = 2x - 3$. Find a formula for the composition functions (*a*) $f^2 = f \circ f$ and (*b*) $f^3 = f \circ f \circ f$.

(*a*) $f^2(x) = f(f(x)) = f(2x - 3) = 2(2x - 3) - 3 = 4x - 9$.
(*b*) $f^3(x) = f(f^2(x)) = f(4x - 9) = 2(4x - 9) - 3 = 8x - 21$.

Diagram of Maps

3.79 Define a *diagram of maps*.

▌ A directed graph in which the vertices are sets and the edges denote maps between the sets is called a diagram of maps.

Problems 3.80–3.83 refer to maps $f: A \to B$, $g: B \to A$, $h: C \to B$, $F: B \to C$, and $G: A \to C$ which are pictured in the diagram of maps in Fig. 3-20.

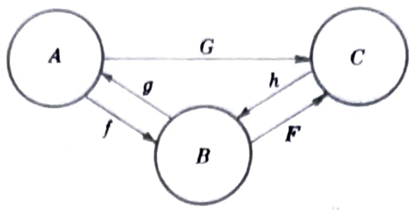


Fig. 3-20

3.80 Is $g \circ f$ defined? If so, what is its domain and codomain?

▌ Since f goes from A to B and g goes from B to A, $g \circ f$ is defined and A is its domain and codomain.

3.81 Is $h \circ f$ defined? If so, what is its domain and codomain?

▌ Note that h does not "follow" f in the diagram, i.e., the codomain B of f is not the domain of h. Hence $h \circ f$ is not defined.

3.82 Is $F \circ h \circ G$ defined? If so, what is its domain and codomain?

▌ The arrows representing G, h, and F do follow each other in the diagram and go from A to C to B to C. Thus $F \circ h \circ G$ is defined with domain A and codomain C. (We emphasize that compositions are "read" from right to left.)

3.83 Is $G \circ F \circ h$ defined? If so, what is its domain and codomain?

▌ F follows h in the diagram, but G does not follow F, i.e., the codomain C of F is not the domain of G. Hence $G \circ F \circ h$ is not defined.

3.84 Define a commutative diagram of maps.

▌ A diagram of maps is commutative if any two paths with the same initial and terminal vertices are equal.

Problems 3.85–3.90 refer to the commutative diagram of maps in Fig. 3-21.

$$
\begin{tikzcd}
A \arrow[r, "f_1"] \arrow[dr, "h_1"'] & B \arrow[r, "f_2"] \arrow[d, "h_2"] & C \arrow[d, "h_3"] \arrow[dr, "h_4"] & \\
 & X \arrow[r, "g_1"'] & Y \arrow[r, "g_2"'] & Z
\end{tikzcd}
$$

Fig. 3-21

3.85 Represent $h_2 \circ f_1$ by a single map.

▌ The composition map $h_2 \circ f_1$ goes from A to B to X. Since the diagram is commutative, $h_2 \circ f_1 = h_1$.

3.86 Represent $h_3 \circ f_2$ in as many ways as possible.

▌ The map $h_3 \circ f_2$ goes from B to C to Y. The only other path from B to Y is the map $g_1 \circ h_2$.

3.87 Represent the map $g_2 \circ h_3$ by a single map.

▌ The map $g_2 \circ h_3$ goes from C to Y to Z. The map h_4 goes from C to Z. Since the diagram is commutative, $g_2 \circ h_3 = h_4$.

Represent the map $g_1 \circ h_1$ by a single map.

The map $g_1 \circ h_1$ is not defined since the codomain Y of h_1 is not the domain of g_1.

3.89 Represent the map $g_2 \circ h_1 \circ f_2 \circ f_1$ in as many ways as possible.

The map $g_2 \circ h_1 \circ f_2 \circ f_1$ goes from A to B to C to Y to Z. There are three other paths from A to Z: (i) $g_2 \circ g_1 \circ h_1$, (ii) $g_2 \circ g_1 \circ h_2 \circ f_1$, and (iii) $h_4 \circ f_2 \circ f_1$.

3.90 Find all maps: **(a)** from A to Y, **(b)** from X to Z, **(c)** from C to X.

(a) There are three paths from A to Y which are A to B to C to Y, A to B to X to Y, and A to X to Y. Thus there are three maps from A to Y which are $h_1 \circ f_2 \circ f_1$, $g_1 \circ h_2 \circ f_1$ and $g_1 \circ h_1$.
(b) There is only one path from X to Z which is X to Y to Z. This corresponds to the map $g_2 \circ g_1$.
(c) There is no path and hence no map from C to X.

3.4 ONE-TO-ONE, ONTO, AND INVERTIBLE FUNCTIONS

3.91 Define a one-to-one (or injective) function.

A function $f: A \to B$ is said to be *one-to-one* (written 1-1) if different elements in the domain A have distinct images. Another way of saying the same thing is that f is *one-to-one* if $f(a) = f(a')$ implies $a = a'$.

3.92 Define an onto (or surjective) function.

A function $f: A \to B$ is said to be an *onto* function if each element of B is the image of some element of A. In other words, $f: A \to B$ is onto if the image of f is the entire codomain, i.e., if $f(A) = B$. In such a case we say that f is a function from A *onto* B or that f maps A *onto* B.

3.93 Define a one-to-one correspondence (or bijective function).

A function $f: A \to B$ is called a *one-to-one correspondence* or a *bijective* function between A and B if f is both one-to-one and onto. This terminology comes from the fact that each element of A will then correspond to a unique element of B and vice versa.

3.94 Define an invertible function.

A function $f: A \to B$ is said to be *invertible* if there exists a function $g: B \to A$ such that $f \circ g = 1_B$ and $g \circ f = 1_A$ (where 1_A and 1_B are the identity maps). In such a case, the function g is called the inverse of f and is denoted by f^{-1}. Alternatively, f is invertible if the inverse relation f^{-1} is a function from B to A. Also, if $b \in B$ then $f^{-1}(b) = a$ where a is the unique element of A for which $f(a) = b$. The following theorem gives a simple criterion.

Theorem 3.2: A function $f: A \to B$ is invertible if and only if f is bijective.

Problems 3.95–3.97 refer to the functions $f_1: A \to B$, $f_2: B \to C$, $f_3: C \to D$ and $f_4: D \to E$ defined in Fig. 3-22.

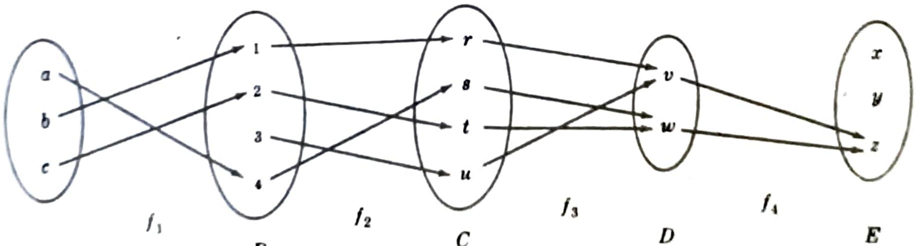


Fig. 3-22

3.95 Which of the functions in Fig. 3-22 are one-to-one?

The function f_1 is one-to-one since no element of B is the image of more than one element of A. Similarly, f_2 is one-to-one. However, neither f_3 nor f_4 is one-to-one since $f_3(r) = f_3(u)$ and $f_4(v) = f_4(w)$.

3.96 Which of the functions in Fig. 3-22 are onto functions?

▌ The functions f_2 and f_3 are both onto functions since every element of C is the image under f_2 of some element of B and every element of D is the image under f_3 of some element of C, i.e., $f_2(B) = C$ and $f_3(C) = D$. On the other hand, f_1 is not onto since $3 \in B$ is not the image under f_1 of any element of A, and f_4 is not onto since $x \in E$ is not the image under f_4 of any element of D.

3.97 Which of the functions in Fig. 3-22 are invertible.

▌ The function f_1 is one-to-one but not onto, f_3 is onto but not one-to-one and f_4 is neither one-to-one nor onto. However, f_2 is both one-to-one and onto, i.e., f_2 is a bijective function between A and B. Hence f_2 is invertible and f_2^{-1} is a function from C to B.

3.98 Let $A = \{a, b, c, d, e\}$, and let B be the set of letters in the alphabet. Let the functions f, g and h from A into B be defined as follows:

(a)	(b)	(c)
$a \xrightarrow{f} r$	$a \xrightarrow{g} z$	$a \xrightarrow{h} a$
$b \to a$	$b \to y$	$b \to c$
$c \to s$	$c \to x$	$c \to e$
$d \to r$	$d \to y$	$d \to r$
$e \to e$	$e \to z$	$e \to s$

Are any of these functions one-to-one?

▌ Recall that a function is one-to-one if it assigns distinct image values to distinct elements in the domain.
(a) No. For f assigns r to both a and d.
(b) No. For g assigns z to both a and e.
(c) Yes. For h assigns distinct images to different elements in the domain.

3.99 Determine if each function is one-to-one.
(a) To each person on the earth assign the number which corresponds to his age.
(b) To each country in the world assign the latitude and longitude of its capital.
(c) To each book written by only one author assign the author.
(d) To each country in the world which has a prime minister assign its prime minister.

▌ (a) No. Many people in the world have the same age.
(b) Yes.
(c) No. There are different books with the same author.
(d) Yes. Different countries in the world have different prime ministers.

3.100 Let the functions $f: A \to B$, $g: B \to C$, and $h: C \to D$ be defined by Fig. 3-23. Determine which of the functions are onto.

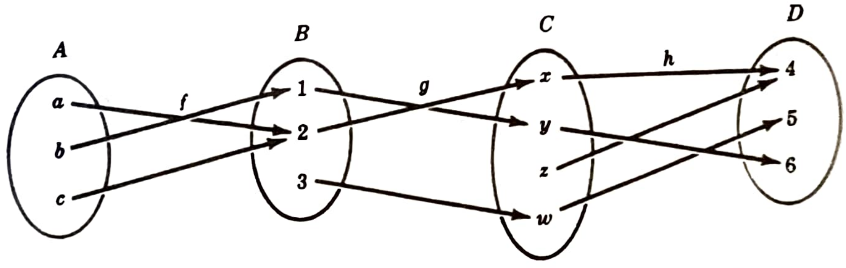


Fig. 3-23

▌ The function $f: A \to B$ is not onto since $3 \in B$ is not the image of any element in A. The function $g: B \to C$ is not onto since $z \in C$ is not the image of any element in B. The function $h: C \to D$ is onto since each element in D is the image of some element of C.

3.101 Determine which of the functions f, g, and h in Fig. 3-23 are one-to-one.

▌ The function f is not one-to-one since $f(a) = f(c) = 2$. The function h is not one-to-one since $h(x) = h(w) = 4$. The function g is one-to-one since the images of 1, 2, and 3 are distinct.

3.102 Which of the functions f, g, and h in Fig. 3-23 are invertible?

▌ The function f is neither one-to-one nor onto, g is one-to-one but not onto, and h is onto but not one-to-one. Thus none of the functions is bijective, and thus none is invertible.

3.103 Find the composition $h \circ g \circ f$ of the functions in Fig. 3-23.

▌ Now $a \to 2 \to x \to 4$, $b \to 1 \to y \to 6$, $c \to 2 \to x \to 4$. Hence $h \circ g \circ f = \{(a, 4), (b, 6), (c, 4)\}$.

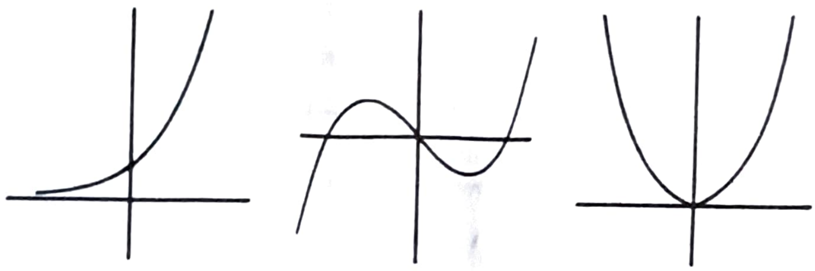
3.104 Recall that a function $f: \mathbf{R} \to \mathbf{R}$ may be identified with its graph. Give a geometrical condition which is equivalent to the property that **(a)** f is one-to-one, **(b)** f is onto, and **(c)** f is invertible.

▌ **(a)** To say that f is one-to-one means that there are no two distinct pairs (a_1, b) and (a_2, b) in the graph of f; hence each horizontal line can intersect the graph of f in at most one point.

(b) To say that f is an onto function means that for every $b \in \mathbf{R}$ there must be at least one $a \in \mathbf{R}$ such that (a, b) belongs to the graph of f; hence each horizontal line must intersect the graph of f at least once.

(c) If f is invertible, i.e., both one-to-one and onto, then each horizontal line will intersect the graph of f in exactly one point.

3.105 Consider the functions $f(x) = 2^x$, $g(x) = x^3 - x$, and $h(x) = x^2$ whose graphs appear in Fig. 3-24. Determine which of the functions are one-to-one.



$f(x) = 2^x$ $\qquad$ $g(x) = x^3 - x$ $\qquad$ $h(x) = x^2$

Fig. 3-24

▌ The function g is not one-to-one since there are horizontal lines which contain more than one point of the graph of g, e.g., $y = 0$ contains three points of g. The function h is not one-to-one since $h(2) = h(-2) = 4$, i.e., the horizontal line $y = 4$ contains two points of h. However, f is one-to-one since no horizontal line contains more than one point of f.

3.106 Determine which of the functions f, g, and h in Fig. 3-24 are onto functions.

▌ The function f is not an onto function since some horizontal lines (those below the x axis) contain no point of f. Similarly, h is not an onto function since $k = -16$ (and any other negative number) has no preimage, i.e., the horizontal line $y = -16$ contains no point of h. However, g is an onto function since every horizontal line contains at least one point of g.

3.107 Which of the functions f, g, and h in Fig. 3-24 are invertible?

▌ None of the functions f, g, and h are invertible since no function is both one-to-one and onto.

3.108 Some texts say that $f(x) = 2^x$ in Fig. 3-24 has an inverse. Why?

▌ The function $f(x) = 2^x$ is one-to-one with image $D = \{x: x > 0\}$, the positive real numbers. Suppose we redefine f to be the function $f: \mathbf{R} \to D$, that is, with D as the codomain. Then f is bijective (one-to-one and onto) and hence has an inverse function $f^{-1}: D \to \mathbf{R}$ (see Theorem 3.2).

3.109 Let $W = \{1, 2, 3, 4, 5\}$ and let $f: W \to W$, $g: W \to W$, and $h: W \to W$ be defined by the diagrams in Fig. 3-25. Determine whether each function is invertible, and, if it is, find its inverse function.

▌ In order for a function to be invertible, the function must be both one-to-one and onto. Only h is one-to-one and onto, so only h is invertible. To find h^{-1}, the inverse of h, reverse the ordered pairs which belong to h.

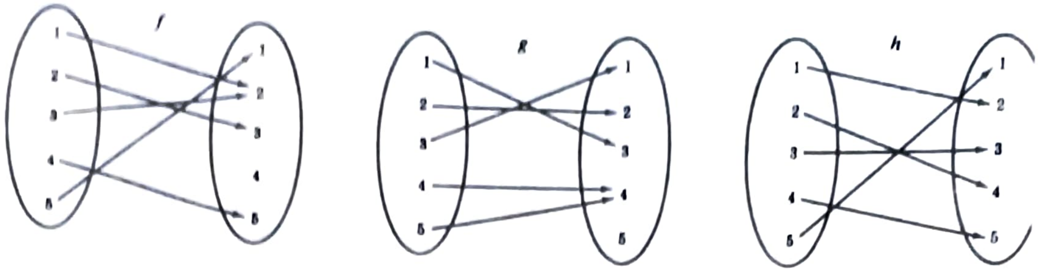


Fig. 3-25

Note

hence
$$h = \{(1, 2), (2, 4), (3, 3), (4, 5), (5, 1)\}$$
$$h^{-1} = \{(2, 1), (4, 2), (3, 3), (5, 4), (1, 5)\}$$

Observe that h^{-1} can be obtained by reversing the arrows in the diagram for h.

3.110 Let functions $f: A \to B$, $g: B \to C$, and $h: C \to D$ be defined by Fig. 3-26. Determine which of the functions ar one-to-one.

▮ The function g is not one-to-one since $g(1) = g(3) = r$. The other two functions f and h are one-to-one.

3.111 Determine which of the functions f, g, and h in Fig. 3-26 are onto functions.

▮ The function f is not an onto function since 3 in the codomain B of f has no preimage. The other two functions g and h are onto functions, that is, $g(B) = C$ and $h(C) = D$.

3.112 Determine whether each of the functions f, g, and h in Fig. 3-26 is invertible, and, if it is, find its inverse.

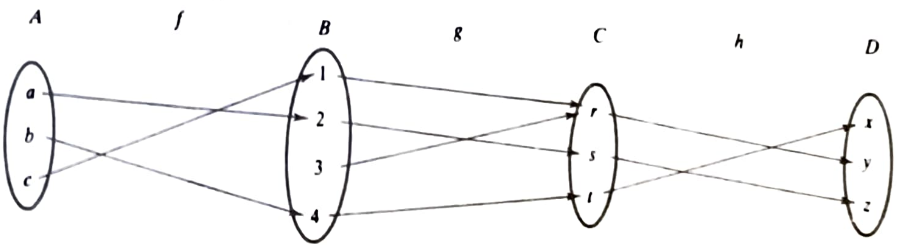


Fig. 3-

▮ Only h is both one-to-one and onto; hence only h is invertible. The inverse h^{-1} of h is obtained by reversin the ordered pairs in h. Thus

$$h = \{(r, y), (s, z), (t, x)\} \quad \text{and so} \quad h^{-1} = \{(y, r), (z, s), (x, t)\}$$

3.113 Find the composition function $h \circ g \circ f$ for the functions f, g, and h in Fig. 3-26.

▮ Follow the arrows from A to B to C to D as follows:

$$a \to 2 \to s \to z, \qquad b \to 4 \to t \to x, \qquad c \to 1 \to r \to y$$

Thus $h \circ g \circ f = \{(a, z), (b, x), (c, y)\}$.

3.114 Let $f: \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = 2x - 3$. Now f is one-to-one and onto; hence f has an inverse mapping f^{-1}. Find a formula for f^{-1}.

▮ Let y be the image of x under the mapping f, that is, set $y = 2x - 3$. Interchange x and y to obtain $x = 2y - 3$. Solve for y in terms of x to get $y = (x + 3)/2$. Thus the formula defining the inverse mapping is $f^{-1}(x) = (x + 3)/2$.

3.115 Find a formula for the inverse of $g(x) = x^2 - 1$.

▮ Set $y = x^2 - 1$. Interchange x and y to get $x = y^2 - 1$. Solve for y to get $y = \pm\sqrt{x + 1}$. The inverse of g does not exist unless the domain of g^{-1} is restricted to $x \geq -1$. In this case assume only the positive value of $\sqrt{x + 1}$

and so $g^{-1}(x) = \sqrt{x+1}$.

3.116 Find a formula for the inverse of $h(x) = \dfrac{2x-3}{5x-7}$.

▮ Set $y = h(x)$ and then interchange x and y as follows:

$$y = \frac{2x-3}{5x-7} \quad \text{and then} \quad x = \frac{2y-3}{5y-7}$$

Now solve for y in terms of x:

$$5xy - 7x = 2y - 3 \quad \text{or} \quad 5xy - 2y = 7x - 3 \quad \text{or} \quad (5x-2)y = 7x - 3$$

Thus
$$y = \frac{7x-3}{5x-2} \quad \text{and so} \quad h^{-1}(x) = \frac{7x-3}{5x-2}$$

(Here the domain of h^{-1} excludes $x = 2/5$.)

3.117 Suppose $f: A \to B$ and $g: B \to C$ are one-to-one functions. Show that $g \circ f: A \to C$ is one-to-one.

▮ Suppose $(g \circ f)(x) = (g \circ f)(y)$. Then $g(f(x)) = g(f(y))$. Since g is one-to-one, $f(x) = f(y)$. Since f is one-to-one, $x = y$. We have proven that $(g \circ f)(x) = (g \circ f)(y)$ implies $x = y$; hence $g \circ f$ is one-to-one.

3.118 Suppose $f: A \to B$ and $g: B \to C$ are onto functions. Show that $g \circ f: A \to C$ is an onto function.

▮ Suppose $c \in C$. Since g is onto, there exists $b \in B$ for which $g(b) = c$. Since f is onto, there exists $a \in A$ for which $f(a) = b$. Thus $(g \circ f)(a) = g(f(a)) = g(b) = c$; hence $g \circ f$ is onto.

3.119 Given $f: A \to B$ and $g: B \to C$. Show that if $g \circ f$ is one-to-one, then f is one-to-one.

▮ Suppose f is not one-to-one. Then there exists distinct elements $x, y \in A$ for which $f(x) = f(y)$. Thus $(g \circ f)(x) = g(f(x)) = g(f(y)) = (g \circ f)(y)$; hence $g \circ f$ is not one-to-one. Therefore, if $g \circ f$ is one-to-one, then f must be one-to-one.

3.120 Given $f: A \to B$ and $g: B \to C$. Show that if $g \circ f$ is onto, then g is onto.

▮ If $a \in A$, then $(g \circ f)(a) = g(f(a)) \in g(B)$; hence $(g \circ f)(A) \subseteq g(B)$. Suppose g is not onto. Then $g(B)$ is properly contained in C and so $(g \circ f)(A)$ is properly contained in C; thus $g \circ f$ is not onto. Accordingly, if $g \circ f$ is onto, then g must be onto.

3.121 Prove Theorem 3.2. A function $f: A \to B$ has an inverse if and only if f is bijective (one-to-one and onto).

▮ Suppose f has an inverse, i.e., there exists a function $f^{-1}: B \to A$ for which $f^{-1} \circ f = 1_A$ and $f \circ f^{-1} = 1_B$. Since 1_A is one-to-one, f is one-to-one by Problem 3.119; and since 1_B is onto, f is onto by Problem 3.120. That is, f is both one-to-one and onto.

Now suppose f is both one-to-one and onto. Then each $b \in B$ is the image of a unique element in A, say $\hat{b}$. Thus if $f(a) = b$, then $a = \hat{b}$; hence $f(\hat{b}) = b$. Now let g denote the mapping from B to A defined by $g(b) = \hat{b}$. We have:

(i) $(g \circ f)(a) = g(f(a)) = g(b) = \hat{b} = a$, for every $a \in A$; hence $g \circ f = 1_A$.
(ii) $(f \circ g)(b) = f(g(b)) = f(\hat{b}) = b$, for every $b \in B$; hence $f \circ g = 1_B$.

Accordingly, f has an inverse. Its inverse is the mapping g.

3.122 Let $P = \{A_i\}$ be a partition of a set S. **(a)** Define the natural (or canonical) map f from S to P. **(b)** Prove that the natural map $f: S \to P$ is an onto function.

▮ **(a)** Let $s \in S$. Since P is a partition of S, there is a unique index i_0 such that $s \in A_{i_0}$. Define $f: S \to P$ by $f(s) = A_{i_0}$. This is the natural map.

(b) Let $A_i \in P$. Then $A_i \neq \emptyset$. Thus there exists $s \in S$ such that $s \in A_i$ and so $f(s) = A_i$. Thus f is an onto mapping.

3.123 Let S be a subset of A and let $i: S \hookrightarrow A$ be the inclusion map (Problem 3.30). Show that the inclusion map i is one-to-one.

▮ Suppose $i(x) = i(y)$. Note $i(x) = x$ and $i(y) = y$. Hence $x = y$ and i is one-to-one.