2.1 Binary Relation

[R.G.P.V. (B.E.) Bhopal 2006; Pune (B.E.) 2005; R.G.P.V. (B.E.) Raipur 2005]

Let A and B be non-empty sets, then any subset R of the Cartesian product $A \times B$ is called a ***Relation*** from A to B and is denoted by R. Thus, R is a relation from A to $B \Rightarrow R \subseteq A \times B$. Symbolically we write

$$R = \{(x, y) : x \in A, y \in B \text{ and } x\,R\,y\}$$

where $x\,R\,y$ denotes that x is R related to y.

Example 1: *Let $A = \{1, 2, 5\}$ and $B = \{2, 4\}$ be two given sets. Now suppose a relation from the set A to B is expressed by statement 'is less than'.*

Solution: We have $A \times B = \{(1, 2), (1, 4), (2, 2), (2, 4), (5, 2), (5, 4)\}$ when $x < y$, then some ordered pairs are related and some are not. The subset $A \times B$ whose elements are related in the relation R is given by

$$R = \{(1, 2), (1, 4), (2, 4)\}$$

clearly $\quad R \subset A \times B$

Illustration: *Let A denote the set of real numbers. Define.*

$$R = \{(a, b) : 4a^2 + 25b^2 \leq 100\}$$

clearly R is a relation on A.

[P.T.U. (B.E.) Punjab 2008]

If $(a, b) \in R$, we often write ${}_aR_b$ and state, a is related to b'

If $R \subseteq A \times A$, then R is the relation from A to A and R is called relation in A

2.2 Total Number of Distinct Binary Relations from Set A to Set B

If set A has m elements and set B has n elements, then $A \times B$ will have mn elements. Therefore, power set of $A \times B$ will have 2^{mn} elements. Thus, $A \times B$ has 2^{mn} different subsets. Now every subset of $A \times B$ is a relation from A to B. Hence, the number of different relations from A to $B = 2^{mn}$

2.2.1 Domain and Range of a Relation

Let $R = \{(x, y) : x \in A, y \in B \text{ and } {}_xR_y\}$ be a relation from A to B. Then the set of first co-ordinates of every element of R is called **Domain R** and denoted by Dom (R) or $d\,(R)$ and the set of second co-ordinates of its every element is called ***Range*** of R and denoted by Ran (R) or $r(R)$. Symbolically

$$d\,(R) = \text{domain of } R = \{x : x \in A \text{ and } (x, y) \in R \text{ for some } y \in B\}$$

and $$r\,(R) = \text{range of } R = \{y : y \in B \text{ and } (x, y) \in R \text{ for some } x \in A\}$$

Example 2: *Let $A = \{a, b, c\}$, $B = \{2, 4, 6, 10\}$. A relation R from A to B is given as follows ${}_aR_2$, ${}_aR_4$, ${}_aR_6$, ${}_aR_{10}$, ${}_bR_6$, ${}_cR_{10}$, write R as set of ordered pair.*

Solution: $R = \{(a, 2), (a, 4), (a, 6), (a, 10), (b, 6), (c, 10)\}$

Illustration: *Let L be the set of all lines in a plane and R be a relation in L defined by "is perpendicular to", then*

$$R = \{(x, y) : x, y \in L \text{ and } x \perp y\}$$

is a relation in L and $R \subseteq L \times L$

Example 3: *Let $A = \{2, 4, 6\}$ and $B = \{1, 4, 5, 6\}$ then find out the relation from A to B defined by "is less than or equal to". Find out the domain and range of the relation.* **[Pune (B.E.) 2006]**

Solution: We have by Cartesian product of two sets.

$$A \times B = \{(2, 1), (2, 4), (2, 5), (2, 6), (4, 1), (4, 4), (4, 5), (4, 6), (6, 1), (6, 4), (6, 5), (6, 6)\}$$

Let R be the relation "is less than or equal to"

Since
$$2 \nless 1 \Rightarrow (2, 1) \notin \text{ i. e. } {}_2\not{R}_1$$
$$2 < 4 \Rightarrow (2, 4) \in R \text{ i.e. } {}_2R_4$$
$$2 < 5 \Rightarrow (2, 5) \in R \text{ i.e. } {}_2R_5$$
$$2 < 6 \Rightarrow (2, 6) \in R \text{ i.e. } {}_2R_6$$
$$4 \nless 1 \Rightarrow (4, 1) \notin R \text{ i.e. } {}_4\not{R}_1$$
$$4 = 4 \Rightarrow (4, 4) \in R \text{ i.e. } {}_4R_4$$
$$4 < 5 \Rightarrow (4, 5) \in R \text{ i.e. } {}_4R_5$$
$$4 < 6 \Rightarrow (4, 6) \in R \text{ i.e. } {}_4R_6$$
$$6 \nleq 1 \Rightarrow (6, 1) \notin R \text{ i.e. } {}_6\not{R}_1$$
$$6 \nleq 4 \Rightarrow (6, 4) \notin R \text{ i.e. } {}_6\not{R}_4$$
$$6 \nleq 5 \Rightarrow (6, 5) \notin R \text{ i.e. } {}_6\not{R}_5$$
$$6 = 6 \Rightarrow (6, 6) \in R \text{ i.e. } {}_6R_6$$

Hence
$$R = \{(x, y) : x \in A, y \in (B) \text{ and } x \leq y\}$$
$$= \{(2, 4), (2, 5), (2, 6), (4, 4), (4, 5), (4, 6), (6, 6)\}$$
$$d(R) = \{x : x \in A \text{ and } (x, y) \in R\}$$
$$= \{2, 4, 6\}$$
$$r(R) = \{y : y \in B \text{ and } (x, y) \in R\} = \{4, 5, 6\}$$

Example 4: *Let $A = \{a, b\}$ and A^2 is the set of all words of length 2.*

(i) *Find the elements of A^2*

(ii) *The relation R on A^2 is defind ${}_xR_y \Rightarrow$ first letter in x is same as first letter in y when $x, y \in A^2$, write R as a set of ordered pairs.* **[Delhi (B.E.) 2005, 2009]**

Solution: $A^2 = \{(a, a), (a, b), (b, a), (b, b)\}$

$$R = \{(aa, ab), (ab, aa), (ba, bb), (bb, ba)\}$$

2.3 Some Operations on Sets

1. ${}_x(S \cap T)_y = {}_xS_y \wedge {}_xT_y$
2. ${}_x(S \cup T)_y = {}_xS_y \vee {}_xT_y$
3. ${}_x(R - S)_y = {}_xR_y \wedge {}_x\not{S}_y$
4. ${}_x(R')_y = {}_x\not{R}_y$ where R' is complement of R

2.4 Operations on Relation

2.4.1 Complement of a Relation

Consider a relation R from set A to B. The complement of relation R denoted by $\bar{\mathbf{R}}$ or $\mathbf{R'}$ is a relation from A to B such that

$$\bar{R} = \{(a, b) : (a, b) \notin R\}$$

Example 5: *Let R be relation from X to Y, where $X = \{1, 2, 3\}$ and $Y = \{8, 9\}$*

and $R = \{(1, 8), (2, 8), (1, 9), (3, 9)\}$ Find the complement of relation R.

Solution: We first find $X \times Y$ *i.e.*

$$X \times Y = \{(1, 8), (1, 9), (2, 8), (2, 9), (3, 8), (3, 9)\}$$

Then complement relation $\bar{R}$ or R' w.r.t. $X \times Y$

$$\bar{R} = \{(2, 9), (3, 8)\}$$

2.4.2 Inverse Relation

Let R be relation from a set A to B. The inverse of relation R denoted by R^{-1} is a relation from B to A such that ${}_bR_a^{-1}$ iff ${}_aR_b$. Symolically

$$R^{-1} = \{(b, a) : (a, b) \in R\}$$

Thus, to find R^{-1} we write in reverse order all ordered pairs belonging to R.

Example 6: *Let $A = \{1, 2, 3\}$ and relation (R) is $\leq$ on A. Determine its inverse.* [Nagpur (B.E.) 2007]

Solution: The relation R under $\leq$ is defined as

$$R = \{(1, 2), (1, 3), (1, 1), (2, 2), (2, 3), (3, 3)\}$$

$$\Rightarrow \quad R^{-1} = \{(2, 1), (3, 1), (1, 1), (2, 2), (3, 2), (3, 3)\}$$

Example 7: *Find the R^{-1} to the relation R on A defined "$x + y$" divisible by 2. For $A = \{1, 2, 3, 4, 6\}$*

Solution: $R = \{(1, 1), (2, 2), (3, 3), (4, 4), (6, 6), (1, 3), (2, 4), (2, 6), (4, 6)$

$$\Rightarrow R^{-1} = \{(1, 1), (2, 2), (3, 3), (4, 4), (6, 6), (3, 1), (4, 2), (6, 2), (6, 4)$$

2.4.3 Intersection and Union of Relations

If R and S are the two relations then intersection of R and S denoted by $R \cap S$ and the union of R and S denoted by $R \cup S$ are two new relation that can be formed from R and S.

Thus
$$R \cup S = \{(x, y): {}_xR_y \text{ or } {}_xS_y\}$$
$$R \cap S = \{(x, y): {}_xR_y \text{ and } {}_xS_y\}$$

Illustration: *Let*
$$R_1 = \{(1, 1), (2, 2), (3, 3), (4, 4), (3, 4), (4, 3)\}$$
$$R_2 = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (2, 1)\}$$

Then
$$R_1 \cup R_2 = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (2, 1), (3, 4), (4, 3)\}$$
$$R_1 \cap R_2 = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$$

Identity relation: A relation R in a set A is said to be identity relation if $I_A = \{(x, x): x \in A\}$

2.5 Properties of Relation

[R.G.P.V. (B.E.) Raipur 2005, 2009]

A relation R on a set A satisfies certain properties. These properties are defined as.

(i) ***Reflexive Relation:*** A relation R on a set A is reflexive if ${}_aR_a \; \forall a \in A$ i.e. is $(a, a) \in R \; \forall a \in R \Rightarrow$ each element a of A is related to itself.

Illustration: Let $A = \{a, b\}$ and $R = \{(a, a), (a, b), (b, b)\}$

Then R is reflexive as ${}_aR_a, {}_bR_b \in R$

(ii) ***Irreflexive Relation:*** A relation R on set A is irreflexive if, for every $a \in A$, $(a, a) \notin R$

Illustration: *Let $A = \{1, 2\}$ and $R = \{(1, 2), (2, 1)\}$*

Then R is irreflexive, since both $(1, 1)$ and $(2, 2) \notin R$

(iii) ***Non-reflexive Relation:*** A relation R on a set A is non-reflexive if R is neither reflexive nor irreflexive.

(iv) ***Symmetric Relation:*** If R is a relation in the set A, then R is called symmetric relation if a is R related to b then b is also R-related to a.

i.e. $$(a, b) \in R \Rightarrow (b, a) \in R \text{ or } {}_aR_b \Rightarrow {}_bR_a \; \forall a, b \in A$$

Note: The relation R will be symmetric if $R = R^{-1}$

Example 8: *If* $A = \{2, 4, 5, 6\}$ *and*

$$R_1 = \{(2, 4), (4, 2), (4, 5), (5, 4), (6, 6)\}$$

and $$R_2 = \{(2, 4), (2, 6), (6, 2), (5, 4), (4, 5)\}$$

Solution: The relation R_1 is symmetric since

$$(2, 4) \in R_1 \Rightarrow (4, 2) \in R_1$$
$$(4, 5) \in R_1 \Rightarrow (5, 4) \in R_1$$
$$(6, 6) \in R_1 \Rightarrow (6, 6) \in R_1$$

Hence $(a, b) \in R \Rightarrow (b, a) \in R$ is true. But R_2 is not symmetric since $(2, 4) \in R_2 \Rightarrow (4, 2) \notin R_2$

(v) ***Antisymmetric Relation:*** A relation R is said to be antisymmetric if ${}_aR_b$ and ${}_bR_a \Rightarrow a = b$

Illustration: *In the set of natural numbers, the relation a divides b is anti-symmetric, since a divides b and b divides a is possible only when* $a = b$

i.e. $$(a, b) \in R \text{ and } (b, a) \in R \Rightarrow a = b$$

(vi) ***Asymmetric Relation:*** A relation R on set A is asymmetric if $(a, b) \in R$ then $(b, a) \notin R$ for $a \neq b$

Illustration: *Let* $A = \{1, 2, 3\}$ *and* $R = \{(1, 2), (1, 1), (2, 3), (3, 1)\}$ *is a asymmetric relation.*

(vii) ***Transitive Relation:*** The relation R on set A is called transitive relation if ${}_aR_b$ and ${}_bR_c \Rightarrow {}_aR_c$ or $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R \ \forall\ a, b, c \in A$

Illustration: *If* $A = \{1, 3, 5\}$ *and* $R = \{1, 3), (1, 5), (3, 5)\}$ *then* ${}_1R_3$ *and* ${}_3R_5 \Rightarrow {}_1R_5$

i.e. $$(1, 3) \in R \text{ and } (3, 5) \in R \Rightarrow (1, 5) \in R$$

2.6 Equality of Relation

[R.G.P.V. (B.E.) Bhopal 2003, 2007; Rohtak (M.C.A.) 2008]

Let A be non-empty set and R be a relation defined on A. Then R is said to be ***Equivalence Relation*** if it is

(i) Reflexive *i.e.* ${}_aR_a \ \forall\ a \in A$

(ii) Symmetric *i.e.* if ${}_aR_b \Rightarrow {}_bR_a \ \forall\ a, b \in A$

(iii) Transitive *i.e.* ${}_aR_b$ and ${}_bR_c \Rightarrow {}_aR_c \ \forall\ a, b, c \in A$

2.7 Composite Relation

[U.P.T.U. (M.C.A.) 2004]

Let A, B and C be three non-empty sets and R be a relation from A to B and S be a relative from B to C. Then the ***Composite Relation*** of the two relations R and S is a relation from A to C and denoted by SoR and defined as

$$SoR = \{(a, c) : \exists \text{ an element } b \in B \text{ such that } (a, b) \in R \text{ and } (b, c) \in S\} \text{ where } a \in A, c \in C$$

Hence we can say that

$$(a, b) \in R, (b, c) \in S \Rightarrow (a, c) \in SoR$$

The composition of a relation with itself is denoted with power of a relation R. Let R be a relation on the set A. Then RoR is the composition of R with itself and $RoR = R^2$. Similarly $R^3 = R^2oR = RoRoR$.

Example 9: *Let $A = \{1, 2, 3\}$, $B = \{p, q, r\}$ and $C = \{x, y, z\}$ and let $R = \{(1, p), (1, r), (2, q), (3, q)\}$ and $S = \{(p, y), (q, x), (r, z)\}$*

Compute RoS **[P.T.U. (B.E.) 2008; M.K.U. (B.E.) 2006]**

Solution:

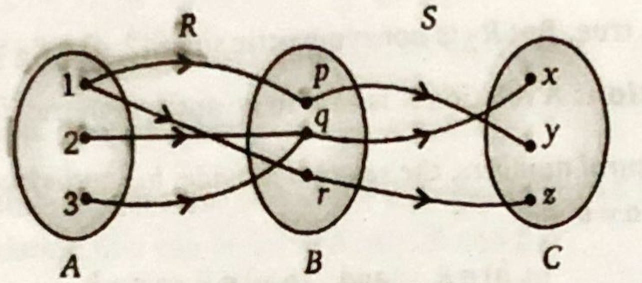

Fig. 2.1

Here, order pair $(1, p)$ in R and (p, y) in S produce the order pair $(1, y)$ in RoS. Order pair $(1, r)$ in R and (r, z) in S produce the order $(1, z)$ in RoS. Similarly $(2, q)$ in R and (q, x) in $S \Rightarrow (2, x)$ in RoS. Also $(3, q)$ in R and (q, x) in $S \Rightarrow (3, x)$ in RoS

Hence $\quad SoR = \{(1, y), (1, z), (2, x), (3, x)\}$

or

$$(1, p) \in R \text{ and } (p, y) \in S \Rightarrow (1, y) \in SoR$$
$$(1, r) \in R \text{ and } (r, z) \in S \Rightarrow (1, z) \in SoR$$
$$(2, q) \in R \text{ and } (q, x) \in S \Rightarrow (2, x) \in SoR$$
$$(3, q) \in R \text{ and } (q, x) \in S \Rightarrow (3, x) \in SoR$$

Example 10: *Let $R = \{(1, 1), (2, 1), (3, 2)\}$ and $R = \{(1, 1), (2, 1), (3, 2)\}$ compute R^2*

Solution: $R^2 = RoR$,

Now,
$$(1, 1) \in R \Rightarrow (1, 1) \in R \Rightarrow (1, 1)\, RoR$$
$$(2, 1) \in R \Rightarrow (1, 1) \in R \Rightarrow (2, 1) \in RoR$$
$$(3, 2) \in R \Rightarrow (2, 1) \in R \Rightarrow (3, 1) \in RoR$$

Hence, $\quad RoR = \{(1, 1), (2, 1), (3, 1)\}$

Theorem 1: *If R is relation from A to B, S is a relation from B to C and T is a relation from C to D. Then prove that $To\,(SoR) = (ToS)\,oR$* **[R.G.P.V. (B.E.) Bhopal 2008; R.G.P.V. Raipur (B.E.) 2006; U.P.T.U. (M.C.A.) 2001, 2003]**

Proof: Let M_R, M_S and M_T denote the matrices related to relation R, S and T respectively then,

$$M_{To\,(SoR)} = M_{SoR} \cdot M_T$$
$$= (M_R \cdot M_S) \cdot M_T$$
$$= M_R\,(M_S \cdot M_T) \quad \text{(Multiplication of matrix is associative)}$$
$$= M_R\,(M_{ToS})$$
$$= M_{(ToS)oR}$$

$\therefore \quad To\,(SoR) = (ToS)\,oR$

Theorem 2: *Let R be a relation from the set A to the set B and S be a relation from the set B to set C, then* $(SoR)^{-1} = R^{-1}\, oS^{-1}$.

[U.P.T.U. (M.C.A.) 2003; U.P.T.U. (B.E.) 2002]

Proof: Let $(c, a) \in (SoR)^{-1} \Rightarrow (a, c) \in (SoR)\ \forall\, a \in A,\, c \in C$

$\therefore$ There exist an element $b \in B$ with $(a, b) \in R$ and $(b, c) \in S$

$\therefore$ $(a, b) \in R$ and $(b, c) \in S \Rightarrow (b, a) \in R^{-1}$ and $(c, b) \in S^{-1}$

$$\Rightarrow (c, b) \in S^{-1} \text{ and } (b, a) \in R^{-1}$$

$$= (c, a) \in R^{-1} oS^{-1}$$

$\therefore$ $(c, a) \in (SoR)^{-1} \Rightarrow (c, a) \in R^{-1} oS^{-1}$

Thus, $(SoR)^{-1} = R^{-1} oS^{-1}$

Example 11: *Let* $A = \{a, b\}$

$$R = \{(a, a), (b, a), (b, b)\}$$

$$S = \{(a, b), (b, a), (b, b)\}$$

Then, verify $(SoR)^{-1} = R^{-1} oS^{-1}$ **[U.P.T.U. (M.C.A.) 2002]**

Solution: Now to find SoR

$$(a, a) \in R \text{ and } (a, b) \in S \Rightarrow (a, b) \in SoR$$

$$(b, a) \in R \text{ and } (a, b) \in S \Rightarrow (b, b) \in SoR$$

$$(b, b) \in R \text{ and } (b, a) \in S \Rightarrow (b, a) \in SoR$$

$\therefore$ $SoR = \{(a, b), (b, a), (b, b)\}$

$\Rightarrow$ $(SoR)^{-1} = \{(b, a), (a, b), (b, b)\}$

and $R^{-1} = \{(a, a), (a, b), (b, b)\}$

$$S^{-1} = \{(b, a), (a, b), (b, b)$$

$$R^{-1}\ oS^{-1} = \{(a, b), (b, a), (b, b)\}$$

$\therefore$ $(SoR)^{-1} = R^{-1} oS^{-1}$

2.8 Recursion and Recurrence Relations

Let A be given set, the successor of A is the set $A \cup \{A\}$. It is denoted by A^+

$\therefore$ $A^+ = A \cup \{A\}$

Let ϕ be the null set, then find the successor sets of ϕ, these sets are

$$\phi,\ \phi^+ = \phi \cup \{\phi\},\ \phi^{++} = \phi \cup \{\phi\} \cup \{\phi, \{\phi\}\}$$

They can be written as $\phi,\ \phi^+ = \{\phi\},\ \phi^{++} = \{\phi, \cup \{\phi\}\}$.

Renaming the ϕ as 0 (zero)

$$\phi^+ = 0^+ = \{\phi\} = 1,\ \phi^{++} = 1^+ = \{\phi, \{\phi\}\} = \{0, 1\} = 2$$

we get the set $\{0, 1, 2, 3 \ldots\}$ each element in the above set is a successor set of previous element, except 0.

Now we consider recursion in terms of successor.

Let S denote the successor, we define

(i) $x + 0 = x$ (ii) $x + S(y) = S(x + y)$

In this definition (i) is the basis and it defines addition of 0. The recursive part defines addition of the successor of y.

Illustration: $3 + 2 = 3 + S(1) = S(3 + 1) = S(S(3 + 0)) = S(S + 3) = S(4) = 5$

2.9 Order of Relation

There are two order of relation

(i) ***Partial Order Relation:*** The set A together with partially order relation R on the set A and is denoted by (A, R) is called partially ordered set or ***Poset***.

(ii) ***Total Order Relation:*** Consider the relation R on the set A.

If it is the case that for all $a, b \in A$, we have either $(a, b) \in R$ or $(b, a) \in R$ or $a = b$, then the relation R is called total order relation on set A.

A relation R on set A is called partial order relation if it satisfies the following properties

(a) Relation R is reflexive *i.e.* ${}_aR_a \ \forall\, a \in A$

(b) Relation R is antisymmetric ${}_aR_b$ and ${}_bR_a \Rightarrow a = b$

(c) Relation R is transitive ${}_aR_b$, ${}_bR_c \Rightarrow {}_aR_c$

➤ *Closure of Relation*

Let R be a relation on a set A.R may or may not have some property P, such as reflexivity, symmetry or transitivity. If there is a relation S with property P containing R such that S is a subset of every relation with P containing R, then S is called the closure of R with respect to P.

Reflexive Closure: Let R be a relation on a set A, and R is not reflexive (*i.e.* some pairs of the diagonal relation Δ are not in (R). A relation $R_1 = R \cup \Delta$ is the reflexive closure of the relation R if $R \cup \Delta$ is the smallest relation containing R which is reflexive.

Illustration: *$A = \{a, b, c\}$ and relation R is given by*

$R = \{(a, a), (a, b), (b, c)\}$ *then*

Reflexive closure $= R_1 = R \cup \Delta$ *where*

$\Delta =$ is the set of elements of the type (a, a) where $a \in A$

i.e. $\Delta = \{(a, a), (b, b), (c, c)\}$

$\therefore$ $R_1 = R \cup \Delta = \{(a, a), (a, b), (b, b), (b, c), (c, c)\}$

Symmetric Closure: Let R be relation on A which is not symmetric and R^{-1} be inverse relation of R on A then symmetric closure $R*$ is defined as

$$R* = R \cup R^{-1}$$

Example 12: *Is $R = \{(1, 2), (4, 3), (2, 2), (2, 1), (3, 1)\}$ be a relation on $S = \{1, 2, 3, 4\}$? Find the symmetric closure.*

Solution: The symmetric closure can be found by taking the union of R and R^{-1}

Now $\quad R^{-1} = \{(2, 1), (3, 4), (2, 2), (1, 2), (1, 3)\}$

Then $\quad R* = R \cup R^{-1} = \{(1, 2), (2, 1), (4, 3), (3, 4), (3, 1), (1, 3)\}$

➤ *Transitive Closure*

The relation obtained by adding the least number of ordered pairs to ensure transitivity is called the ***Transitive Closure*** of the relation. The transitive closure of R is denoted by R^+. Let a relation R be defined on A and contains m elements, one never needs more than m steps. Consequently to make a relation R transitive, one has to add all pairs of R^2. All pairs of R^3 and so on upto all pairs of R^m, unless these pairs are already in R. Thus

$$R^+ = R \cup R^2 \cup R^3 \ldots \cup R^m$$

Example 13: *Let $A = \{1, 2, 3, 4\}$ and $R = \{(1, 2), (2, 3), (3, 4)\}$ be a relation on A. Find transitive closure R^+.*

Solution: We have $R = \{(1, 2), (2, 3), (3, 4)\}$

$$R^2 = RoR = \{(1, 2), (2, 3), (3, 4)\} \, O \, \{(1, 2), (2, 3), (3, 4)\}$$
$$= \{(1, 3), (2, 4)\}$$
$$R^3 = R^2oR = \{(1, 3), (2, 4)\} \, O \, \{(1, 2), (2, 3), (3, 4)\}$$
$$= \{(1, 4)\}$$
$$R^4 = R^3oR = \{(1, 4)\} \, O \, \{(1, 2), (2, 3), (3, 4)\}$$
$$= \phi$$

Hence $\quad R^+ = R \cup R^2 \cup R^3 \cup R^4$

$$= \{(1, 2), (2, 3), (3, 4), (1, 3), (2, 4), (1, 4)\}$$

Theorem 3: *Let R be relation from A to B and let A_1 and A_2 be two subsets of A, then*

(i) $A_1 \subseteq A_2 \Rightarrow R(A_1) \subseteq R(A_2)$ $\quad$ (ii) $R(A_1 \cup A_2) = R(A_1) \cup R(A_2)$

(iii) $R(A_1 \cap A_2) \subseteq R(A_1) \cap R(A_2)$ **[P.T.U. (B.E.) Punjab 2006, 2009]**

Proof: (i) Let $y \in R(A_1) \Rightarrow {}_xR_y$ for some $x \in A_1$

$\Rightarrow x \in A_2$ $\quad [\because A_1 \subseteq A_2]$

$\therefore \quad R(A_1) \subseteq R(A_2)$

(ii) Let $y \in R(A_1 \cup A_2) \Rightarrow {}_xR_y$ for some $x \in A_1 \cup A_2$

Now $\quad x \in A_1 \cup A_2 \Rightarrow x \in A_1$ or $x \in A_2$

If $x \in A_1$, then ${}_xR_y \Rightarrow y \in R(A_1)$, similarly if $x \in A_2 \Rightarrow y \in R(A_2)$

In either case $y \in R(A_1) \cup R(A_2)$ $\therefore \quad R(A_1 \cup A_2) \subseteq R(A_1) \cup R(A_2)$ ...(1)

Again $\quad A_1 \subseteq A_1 \cup A_2 \Rightarrow R(A_1) \subseteq R(A_1 \cup A_2)$

$\quad A_2 \subseteq A_1 \cup A_2 \Rightarrow R(A_2) \subseteq R(A_1 \cup A_2)$

Therefore $\quad R(A_1) \cup R(A_2) \subseteq R(A_1 \cup A_2)$ ...(2)

From (1) and (2) $R(A_1 \cup A_2) = R(A_1) \cup R(A_2)$

(iii) Let $y \in R(A_1 \cap A_2) \Rightarrow {}_xR_y$ for some x in $A_1 \cap A_2$

Now, $\quad x \in A_1 \cap A_2 \Rightarrow x \in A_1$ and $x \in A_2$

$\Rightarrow \quad y \in R(A_1)$ and $y \in R(A_2) \Rightarrow y \in R(A_1) \cap R(A_2)$

$\therefore \quad R(A_1 \cap A_2) \subseteq R(A_1) \cap R(A_2)$

2.10 Matrix Representation of Relations

2.10.1 Matrix of Relation

Let R be the relation from the set A to B, where $\begin{cases} A = \{a_1, a_2, a_3, \ldots a_m\} \\ B = \{b_1, b_2, b_3, \ldots b_n\} \end{cases}$

be finite sets having m and n elements respectively. Then R can be represented by mn matrix and defined as $M_R = \{m_{ij}\}$

where $\quad m_{ij} = \begin{cases} 1 \text{ if } (a, b) \in R \\ 0 \text{ if } (a, b) \notin R \end{cases}$

The matrix M_R is called the matrix of R.

Note: $M_{SoT} = M_S \cdot M_T$

Example 14: *Let R be the relation from the set $A = \{1, 3, 4\}$ on itself and defined by*

$R = \{(1, 1), (1, 3), (3, 3), (4, 4)\}$ then find relation matrix.

[R.G.P.V. (B.E.) Raipur 2007; M.K.U. (B.E.) 2005, 2009; Pune (B.E.) 2008]

Solution: Let M_R denotes the matrix of R. The number of rows in M_R = number of elements in $A = 3$. Since the relation from the set A on itself, the number of columns in M_R is also 3 *i.e.* M_R is 3×3 matrix

Here $\quad a_1 = 1,\ a_2 = 3$ and $a_3 = 4$

$b_1 = 1,\ b_2 = 3$ and $b_3 = 4$

Since $\quad {}_1R_1 \Rightarrow m_{11} = 1$ as $(a_1, b_1) = (1, 1) = 1$

${}_1R_3 \Rightarrow m_{13} = 1$ as $(a_1, b_3) = (1, 3) = 1$

${}_3R_3 \Rightarrow m_{33} = 1$ as $(a_3, b_3) = (3, 3) = 1$

${}_4R_4 \Rightarrow m_{44} = 1$ as $(a_4, b_4) = (4, 4) = 1$

and all other elements of M_R are zero.

Hence, $$M_R = \begin{array}{c} \\ 1 \\ 3 \\ 4 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 1 & 3 & 4 \end{array} \\ \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \end{array}$$

$$= \begin{array}{c|ccc} & 1 & 3 & 4 \\ \hline 1 & 1 & 1 & 0 \\ 3 & 0 & 1 & 0 \\ 4 & 0 & 0 & 1 \end{array}$$

Example 15: *Let $A = \{1, 2, 3, 4, 8\}$, $B = \{1, 4, 6, 9\}$ Let $_aR_b$ iff $a|b$ i.e. a divides b. Find the relation matrix.*

[Delhi (B.E.) 2005]

Solution: We have $R = \{(1, 1), (1, 4), (1, 6), (1, 9), (2, 4), (2, 6), (3, 6), (3, 9), (4, 4)\}$

$$M_R = \begin{array}{c|cccc} & 1 & 4 & 6 & 9 \\ \hline 1 & 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 & 0 \\ 3 & 0 & 0 & 1 & 1 \\ 4 & 0 & 1 & 0 & 0 \\ 8 & 0 & 0 & 0 & 0 \end{array}$$

Example 16: *Let $A = \{a, b, c, d\}$ and let*

$$M_R = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \text{find } R$$

Solution: $R = \{(a, a), (a, b), (b, c), (b, d), (c, c), (c, d), (d, a), (d, c)\}$

Example 17: *Let $A = \{1, 2, 3, 4\}$ and let, R be a relation on A whose matrix is*

$$M_R = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Show that R is transitive **[U.P.T.U. (M.C.A.) 2003; Rohtak (M.C.A.) 2007]**

Solution: By the transitivity of R means that if $M_R^2 = M_R \cdot M_R$ and $M_R^2 + M_R = M_R$

$$\therefore \quad M_R^2 = M_R \cdot M_R = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{So,} \quad M_R^2 + M_R = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} = M_R$$

$\therefore$ The relation R is transitive

2.11 Digraphs

If A is a finite set and R is a relation on A, we can also represent R pictorially as

(i) Draw the small circle for each element of A and label the circle with corresponding element of A. These circles are called ***vertices***.

(ii) Draw an arrow, called an **edge**, from vertex a_i to $a_j \Leftrightarrow {}_{a_i}R_{a_j}$

The resulting pictorial representation of R is called a ***Directed Graph*** or ***Digraph of R***.

The directed graph representing a relation can be used to determine whether the relation has various properties.

(i) A relation is reflexive iff there is a loop at every vertex of the directed graph. So that the ordered pair of the form (a, a) occurs in the relation. If no vertex has a loop, then the relation is irreflexive.

(ii) A relation is ***Symmetric*** iff for every edge between distinct vertices in its digraph there is an edge in the opposite direction, so that (b, a) is in the relation whenever (a, b) is in the relation. A relation is anti-symmetric if no two distinct points in a digraph have an edge going between them in both direction.

(iii) A relation is transitive iff whenever there is a directed edge from a vertex a to a vertex b and from a vertex b to vertex c, then there is also a directed edge from a to c.

Example 18: *Let* $A = \{1, 2, 3, 4\}$

$$R = \{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (2, 4), (3, 4), (4, 1), (4, 4)\}$$

Construct the digraph of R **[Nagpur (B.E.) 2006]**

Solution: The digraph of R is

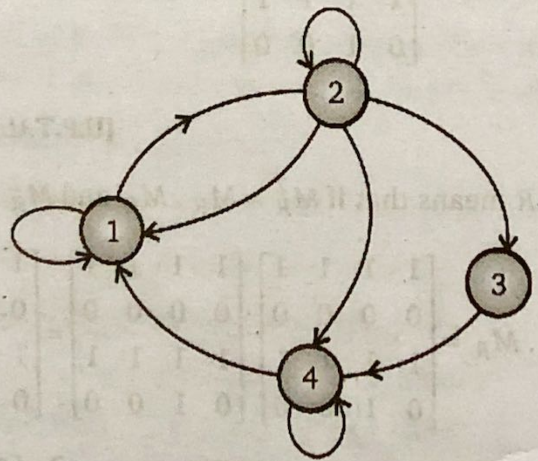

Fig. 2.2

Example 19: *Find the relation determine by*

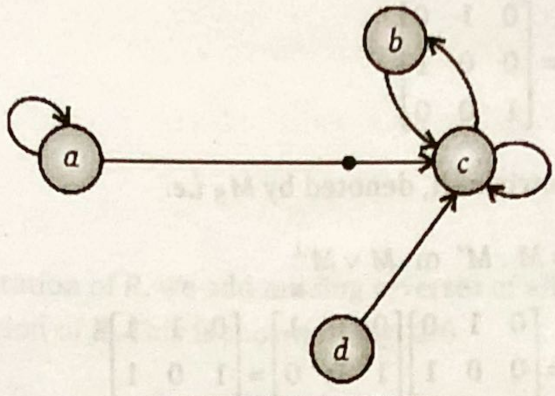

Fig. 2.3

Solution: The relation R of the digraph is

$$R = \{(a, a), (a, c), (b, c), (c, b), (c, c), (d, c)\}$$

Example 20: *Let $R = \{(1, 2), (2, 3), (3, 1)\}$ and $A = \{1, 2, 3\}$, find the reflexive, symmetric and transitive closure of R, using*

(i) *Composition of relation R*

(ii) *Composition of matrix relation R* **[R.G.P.V. (B.E.) Raipur 2005, 2009]**

(iii) *Graphical representation of R*

[P.T.U. (B.E.) Punjab 2002, 2006, 2009; M.K.U. (B.E.) 2005, 2008; Osmania (B.E.) 2003]

Solution: (i) The reflexive closure of R is denoted by R_1 and given by

$$R_1 = R \cup \Delta \text{ or } R \cup I_A$$

$$I_A = \text{identity relation}$$

$$R_1 = \{(1, 2), (2, 3), (3, 1)\} \cup \{(1, 1), (2, 2), (3, 3)\}$$

$$= \{(1, 1), (1, 2), (2, 2), (2, 3), (3, 1), (3, 3)\}$$

The symmetric closure of R is denoted by $R*$ is given by

$$R* = R \cup R^{-1}$$

$$= \{(1, 2), (2, 3), (3, 1)\} \cup \{(2, 1), (3, 2), (1, 3)\}$$

$$= \{(1, 2), (1, 3), (2, 1), (2, 3), (3, 1), (3, 2)\}$$

The transitive closure of R is denoted by R^+

Now $\quad RoR = \{(1, 2), (2, 3), (3, 1)\}\, o\, \{(1, 2), (2, 3), (3, 1)\}$

$\Rightarrow \quad R^2 = \{(1, 3), (2, 1), (3, 2)\}$

$$R^3 = R^2 oR = \{(1, 3), (2, 1), (3, 2)\}\, o\, \{(1, 2), (2, 3), (3, 1)\}$$

$$= \{(1, 1), (2, 2), (3, 3)\}$$

$$R^4 = R^3 oR = \{(1, 1), (2, 2), (3, 3)\}\, o\, \{(1, 2), (2, 3), (3, 1)\}$$

$$= \{(1, 2), (2, 3), (3, 1)\} = R$$

Thus $\quad R^+ = R \cup R^2 \cup R^3 = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$

(ii) Let M be the relation matrix of R, then

$$M = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

The symmetric closure matrix of R, denoted by M_S i.e.

$$M_S = M \,.\, M' \text{ or } M \vee M^{\perp}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$R_S^* = \{(1, 2), (1, 3), (2, 1), (2, 3), (3, 1), (3, 2)\}$$

The reflexive closure matrix of R is R_1 is given by

$$R_1 = M \vee I_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \vee \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$\Rightarrow$ $$R_1 = \{(1, 1), (1, 2), (2, 2), (2, 3), (3, 1), (3, 3)\}$$

Now $$M^2 = M \,.\, M = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$M^3 = M^2 \,.\, M = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The transitive closure relation matrix of R, denoted by M_T

i.e. $$M_T = M \vee M^2 \vee M^3 = M.M^2.M^3$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$\Rightarrow$ $$R^+ = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$$

(iii) The graphical representation of R is shown in Fig. 2.4

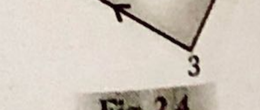

Fig. 2.4

To find out the reflexive closure representation of R we add all the arrows from points to themselves which is shown in Fig. 2.5

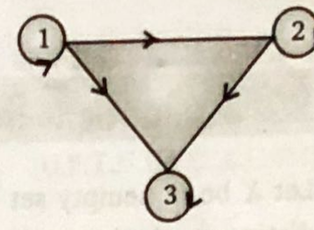

Fig. 2.5

To find symmetric closure representation of R, we add missing reverses of all the arrows in graphical representation of R. This is shown in Fig. 2.6

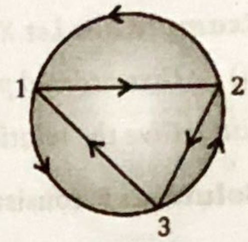

Fig. 2.6

To find transitive arrow we add 1 to 1 since $1 \to 2 \to 3 \to 1$. Similarly 2 to 2, 3 to 3. Again we add arrow 1 to 3, since $1 \to 2 \to 3$. Similarly 2 to 1 and 3 to 2. This is shown in Fig. 2.7

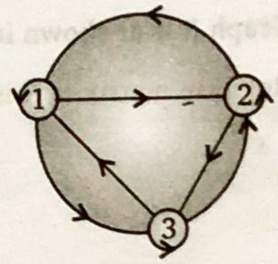

Fig. 2.7

2.12 Equivalence Classes

Consider an equivalence relation R on a set A. The equivalence class of an element $a \in A$ is the set of elements of A to which element a is related. It is denoted by $[a]$ or $\bar{a}$.

Example 21: *Let $A = \{a, b, c\}$ and let $R = \{(a, a), (b, b), (c, c), (a, b), (b, a)\}$ where R is clearly an equivalence relation. Find equivalence classes of the elements of A.*

Solution: The equivalence classes are

$$[a] = \{a, b\}, \; [b] = \{b, a\} = [a], \; [c] = \{c\}$$

and rank of R is 2.

Example 22: *Let $A = \{1, 2, 3, 4\}$ and let*

$$R = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (3, 1), (2, 3), (3, 2), (3, 3), (4, 4)\}$$

Determine the equivalence classes and find rank of R.

Solution: The equivalence class of A are

$$[1] = R\,(1) = \{1, 2, 3\} \quad [2] = R\,(2) = \{1, 2, 3\} = [1] \text{ or } R\,(1)$$

$$[3] \text{ or } R\,(3) = \{3, 1, 2\} = [1] \text{ or } R\,(1) \quad [4] \text{ or } R\,(4) = \{4\}$$

Hence, there are two distinct equivalence classes. So rank of R is 2.

2.13 Quotient Set

Let X be nonempty set and let R be an equivalence relation defined on X. The family consisting of all distinct equivalence classes, into which X is decomposed with respect to R, is called the quotient set of X with respect to equivalence R and is denoted by X / R.

Example 23: *Let $X = \{1, 2, 3, 4\}$ and $R = \{<x, y> | x > y\}$*

(i) *Give ordered pairs of R* (ii) *Draw graph of R*

(iii) *Give the relation matrix of R* **[U.P.T.U. (B.Tech.) 2005; R.G.P.V. (B.E.) Bhopal 2007]**

Solution: R consists of ordered pairs $<x, y>$ such that

$$x \in X, y \in X \text{ and } x > y. \text{ Therefore}$$

$$R = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$$

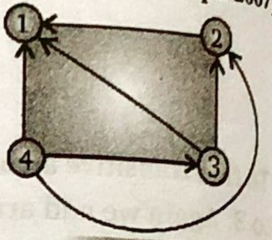

Fig. 2.8

Graph R is as shown in Fig. 2.8

Relation matrix of R is given by

$$\begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Example 24: *Let F be the collection of all function $f : (1, 2, 3) \rightarrow \{1, 2, 3\}$*

If f and g Σ F, define an equivalence relation ~ by $f \sim g$ iff $f(3) = g(3)$ **[Osmania (B.E.) 2006]**

(i) *Find the number of equivalence classes defined by ~*

(ii) *Find the number of elements in each equivalence class* **[U.P.T.U. (M.C.A.) 2004]**

Solution: F be the collection of all function $f : \{1, 2, 3\} \rightarrow \{1, 2, 3\}$

Total number function in $f = 3^3 = 27$

Again f and g are any two function in F such that $f \sim g$ iff $f(3) = g(3)$

Then, (i) Number of equivalence classes defined by the relation $\sim = 3$

(ii) Number of elements in each class = 9

Example 25: *Suppose $A = \{a, b, c, d\}$ and π_1 is the following partition $\pi_1 = \{\{a, b, c\}, \{d\}\}$*

(i) *List the ordered pairs of the equivalence relations induced by π_1.*

(ii) *Draw the graph of the above equivalence relation.* **[Osmania (M.C.A.) 2009; U.P.T.U. (M.C.A.) 2004]**

Solution: Given a set $A = \{a, b, c, d\}$ and a partition of A is given by

$$\pi_1 = \{\{a, b, c\}, \{d\}\}, \text{ then}$$

(i) The ordered pairs of the equivalence relation induced by π_1, are given by

(ii) The graph of above relation is shown in Fig. 2.9

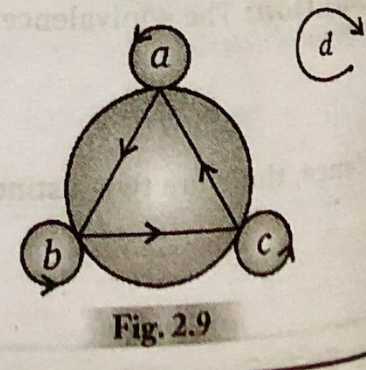

Fig. 2.9

Example 26: *Let A be the set {1, 2, 3}, define the following types of binary relation on A.*

(i) *A relation that is both symmetric and anti-symmetric*

(ii) *A relation that is neither symmetric nor anti-symmetric* **[U.P.T.U. (M.C.A.) 2004]**

Solution: We have $A = \{1, 2, 3\}$

(i) For a binary relation R on A to be symmetric, we have

$${}_aR_b \Rightarrow {}_bR_a \;\; \forall \;\; a, b \in A \;\; \text{i.e. if } (a, b) \in R \text{ then } (b, a) \in R$$

For a binary relation R on A to be anti-symmetric, we have

$${}_aR_b \text{ and } {}_bR_a \Rightarrow a = b \;\; \text{i.e. } (a, b) \in R \text{ and } (b, a) \in R \text{ only when } a = b$$

which means that if $a \neq b$ then either ${}_a\not{R}_b$ or ${}_b\not{R}_a$

The binary relation on A is symmetric as well as anti-symmetric is given by

$$R = \{(1, 1), (2, 2), (3, 3)\}$$

(ii) A relation R on A shall be

(*a*) neither symmetric *i.e.* $(a, b) \in R$ and $(b, a) \not{R}$

(*b*) nor anti-symmetric *i.e.* $(a, b) \in R$ and $(b, a) \in R$ even when $a \neq b$.

The relation is given by

$$R = \{(1, 2), (2, 1), (1, 3)\} \text{ in which } (1, 3) \in R \text{ but } (3, 1) \notin \not{R}$$

$\Rightarrow$ It is not symmetric and $(1, 2) \in \not{R}$ as well as $(2, 1) \in R$ but $1 \neq 2$, showing that R is not anti-symmetric.

Example 27: *If R is an equivalence relation on A, then prove that R^{-1} is also equivalence relation on A.* **[U.P.T.U. (M.C.A.) 2002-2003, 2005-2006]**

Solution: (i) Let $x \in A$. Since R is a reflexive relation, $(x, x) \in R$

$\Rightarrow$ $(x, x) \in R^{-1}$.

So R^{-1} is reflexive

(ii) Let $x, y \in R$, as R is symmetric relation

$(x, y) \in R \Rightarrow (y, x) \in R$

$\Rightarrow$ $(y, x) \in R^{-1}$ and $(x, y) \in R^{-1}$

So $(y, x) \in R^{-1} \Rightarrow (x, y) \in R^{-1}$

So R^{-1} is symmetric

(iii) Let $x, y, z \in A$, as R is a transitive relation

$(x, y) \in R$ and $(y, z) \in R \Rightarrow (x, z) \in R$

which means that $(y, x) \in R^{-1}$ and $(z, y) \in R^{-1} \Rightarrow (z, x) \in R^{-1}$

or $(z, y) \in R^{-1}$ and $(y, x) \in R^{-1} \Rightarrow (z, x) \in R^{-1}$

So R^{-1} is transitive

Hence, R^{-1} is an equivalence relation.

Example 28: *Let $N = \{1, 2, 3 \ldots\}$ and a relation is defined in $N \times N$ as follows: (a, b) is related to (c, d) iff $ad = bc$, then show whether R is a equivalence relation or not.* **[U.P.T.U. (B.Tech.) 2006, 2008]**

Solution: (i) *Reflexive*: We know that $a.b = b.a \Rightarrow (a, b)R(a, b)$. Hence it is reflexive.

(ii) *Symmetric*: Let $(a, b) \in R$ and $(c, d) \in R$ and $(a, b)R(c, d)$

$$ad = bc$$

$\Rightarrow$

$$cb = da$$

$\Rightarrow$

$\Rightarrow (cd)R(a, b)$. Hence, it is symmetric

(iii) *Transitive*: Let $(a, b)R(c, d)$ and $(c, d)R(e, f)$

$$ad = bc \quad \text{and} \quad cf = de$$

$\Rightarrow$

Multiplying these

$$(ad)(ef) = (bc)(de) \text{ or } af = be$$

$\Rightarrow (a, b)R(e, f)$ hence it is transitive

Thus, relation is equivalence.

Example 29: *Let $X = \{1, 2, 3, \ldots 7\}$ and $R = \{(x, y) : (x - y)$ is divisible by 3$\}$. Show that R is an equivalence relation.* **[U.P.T.U. (B.Tech.) 2007]**

Solution: Given that $X = \{1, 2, 3, 4, 5, 6, 7\}$

and $\qquad R = \{(x, y) : (x - y)$ is divisible by 3 $\}$

Then R is an equivalence relation if

(i) *Reflexive*: $\forall\, x \in X \Rightarrow (x - x)$ is divisible by 3

So, $\qquad (x, x) \in X \;\; \forall\, x \in X$

or, R is reflexive.

(ii) *Symmetric*: Let $x, y \in X$ and $(x, y) \in R$

$\Rightarrow (x - y)$ is divisible by 3

$\Rightarrow (x - y) = 3n_1$, ($n_1$ being an integer)

$\Rightarrow (y - x) = -3n_2 = 3n_2$, n_2 is also an integer

So, $y - x$ is divisible by 3 or R is symmetric

(iii) *Transitive*: Let $x, y, z \in X$ and $(x, y) \in R$, $(y, z) \in R$

Then $x - y = 3n_1$, $\; y - z = 3n_2$, $\; n_1, n_2$ being integers

$\Rightarrow x - z = 3(n_1 + n_2)$, $\; n_1 + n_2 = n_3$ be any integer

So, $(x - z)$ is also divisible by 3 or $(x, z) \in R$

So, R is transitive

Hence, R is an equivalence relation.

Example 30: *Let R be a binary relation defined as*

$$R = \{(a, b) \in R^2 : (a - b) \leq 3\}$$

determine whether R is reflexive, symmetric, anti-symmetric and transitive and how many distinct binary relations are there on the finite set?

[U.P.T.U. (B.Tech.) 2002, 2007; R.G.P.V. (B.E.) Bhopal 2009]

Solution: Let $R = \{(a, b) \in R^2 \mid (a - b) \leq 3\}$

Reflexive: $\forall\, a \in R \Rightarrow a - a = 0 \leq 3$

so R is reflexive.

Symmetric: $a, b \in R \Rightarrow a - b \leq 3$ then $b - a \geq 3$ so R is not symmetric.

Anti-symmetric: Let $(a, b) \in R \Rightarrow a - b \leq 3$

$$\Rightarrow b - a \geq 3$$

$\therefore$ $a - b \leq 3$ and $b - a \geq 3$ is possible only if $a = b$, so R is anti-symmetric.

Transitive: Let $(a, b) \in R$ and $(b, c) \in R$

$\Rightarrow$ $a - b \leq 3$ and $b - c \leq 3$

then $a - c \leq 6$

$\Rightarrow R$ is not transitive

Let n be the number of distinct elements in a finite set, then binary relation $= n^{n^2}$

Example 31: *Let* $A = \{1, 2, 3, 4\}$. *Give an example of R on A which is*

(i) *Neither symmetric nor anti-symmetric*

(ii) *Anti-symmetric and reflexive but not transitive*

(iii) *Transitive and reflexive but not anti-symmetric*

Solution: (i) $R = \{(1, 3), (1, 1), (3, 1), (1, 2), (3, 3), (4, 4)\}$ is neither symmetric nor anti-symmetric as $(1, 2) \in R$ but $(2, 1) \notin R$ and $1, 3 \in A$ such that $(1, 3) \in R$ and $(3, 1) \in R$ but $3 \neq 1$

(ii) $R = \{(1, 1), (1, 3), (2, 2), (3, 3), (3, 4), (4, 4)\}$ is reflexive as ${}_aR_a\ \forall\, a \in A$ is anti-symmetric as no pair of distinct elements x and $y \in A$ exist such that $(x, y) \in R$ and $(y, x) \in R$ is not transitive as ${}_1R_3$, and ${}_3R_4 \Rightarrow {}_1\not{R}_4$

(iii) $R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (4, 3), (4, 4)\}$ is reflexive as ${}_1R_1$, ${}_2R_2$, ${}_3R_3$, ${}_4R_4$ is transitive as ${}_aR_b$, ${}_bR_c \Rightarrow {}_aR_c$ is not anti-symmetric as ${}_1R_2$ and ${}_2R_1$ but $1 \neq 2$

Example 32: *Let R be a binary relation on the set of all strings of 0's and 1's such that*

$R = \{(a, b) \mid$ a and b are strings that have same number of 0's$\}$. Is R reflexive, symmetric, transitive or a partial order relation?

[U.P.T.U. (B.Tech.) 2003]

Solution: *Reflexive:* Since every string is related to itself because it has same number of zeros.

Symmetric: Let ${}_aR_b \Rightarrow a$ and b both have same number of zeros then b and a will have same number of zeros.

$\Rightarrow {}_bR_a$. Hence R is symmetric

Transitive: Let $_aR_b$ and $_bR_c \Rightarrow {_aR_c}$

i.e. a and b have same number of zeros, b and c have same number of zeros then a and c have same number of zeros.

Anti-symmetric: Since it is symmetric it will not be anti-symmetric. Hence R is not a partial order relation.

Example 33: *Let R be the relation on set $A = \{a, b, c, d\}$ and*

$$R = \{(a, b), (b, c), (d, c), (d, a), (a, d), (d, d)\}. \text{ Determine}$$

(i) *Reflexive closure of R* (ii) *Symmetric closure of R* (iii) *Transitive closure of R*

[U.P.T.U. (B.Tech.) 2003]

Solution: (i) *Reflexive Closure:* $= R \cup \Delta = R \cup \{(a, a), (b, b), (c, c), (d, d)\}$

$$= \{(a, b), (b, c), (d, c), (d, a), (a, d), (a, a), (b, b), (c, c), (d, d)\}$$

(ii) *Symmetric Closure:* $= R \cup R^{-1}$

$$= \{(a, b), (b, c), (d, c), (d, a), (a, d), (d, d)\} \cup \{(b, a), (c, b), (c, d), (a, d), (d, a), (d, d)\}$$

$$= \{(a, b), (b, a), (b, c), (c, b), (d, c), (c, d), (d, a), (a, d), (d, d)\}$$

(iii) *Transitive Closure:* $= R \cup R^2 \cup R^3 \ldots$

$$R^2 = RoR = \{(a, c), (d, b), (d, d), (a, a), (a, c), (d, c), (d, a)\}$$

$$R^3 = R^2oR = \{(d, c), (d, d), (d, a), (a, b), (a, d), (a, c), (a, a), (d, b)\} = R^2$$

$$R^+ = R \cup R^2 = \{(a, b), (b, c), (d, c), (d, a), (a, d), (d, d), (a, c), (d, b), (a, d), (a, a)\}$$

Example 34: *Let S be the set of all points in a plane. Let R be a relation such that for any two points a and b, $(a, b) \in R$ if b is within two centimeter from a, show that R is not an equivalence relation.*

[U.P.T.U. (B. Tech.) 2003; Kurukshetra (B.E.) 2008]

Solution: The relation R will be equivalence relation if

(i) ***Reflexive:*** $\forall\, a \in S \Rightarrow {_aR_a}$ *i.e.* every element of the plane is related to itself being within the 2 cm from itself.

(ii) ***Symmetric:*** Let $_aR_b$ *i.e.* a and b are within 2 cm distance.

$\Rightarrow b$ and a will also be within 2 cm distance.

$\Rightarrow {_bR_a}$

(iii) Let $_aR_b \Rightarrow a$ and b are within 2 cm distance *i.e.* $|a - b| < 2$

and $_bR_c \Rightarrow b$ and c are within 2 cm distance *i.e.* $|b - c| < 2$

$$\Rightarrow \qquad |a - b| + |b - c| < 4$$

$$\Rightarrow \qquad |a - b + b - c| < 4 \quad \text{or} \quad |a - c| < 4 \Rightarrow {_a\not{R}_c}.$$

Then R is not transitive.

Hence, R is not an equivalence relation.

Example 35: *Let $A = \{1, 2, 3, 4, 5, 6\}$ and let R be the relation defined by x divides y written as x / y*

(i) *Write R as a set of ordered pairs* (ii) *Draw its directed graph* (iii) *Find R^{-1}*

[Rohtak (B.E.) 2007; Delhi (B.E.) 2005]

Solution:

(i) $R = \{(1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 4), (2, 6), (3, 6), (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}$

(ii) The directed graph of R is

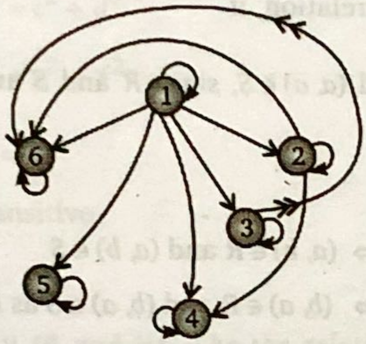

Fig. 2.10

(iii) $R^{-1} = \{(2, 1), (3, 1), (4, 1), (5, 1), (6, 1), (4, 2), (6, 2), (6, 3), (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}$

Example 36: *Let $S = \{1, 2, 3, 4, 5\}$ and $A = S \times S$. Define relation on A: $(a, b) R (a', b')$. Show that R is an equivalence relation iff $ab' = a'b$* **[U.P.T.U. (B.Tech.) 2004; Pune 2003, 2007]**

Solution: We have $A = S \times S$

$\Rightarrow$ $A = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5)\}$

$\therefore R = \{((1, 1), (5, 5)), ((1, 1), (2, 2)), ((1, 1), (3, 3)), ((1, 1), (4, 4)) \ldots\}$

Then R is an equivalence relation, if

(i) ***Reflexive:*** Let $(a, b) R (a', b') \Rightarrow ab' = a'b$

$\Rightarrow$ $(a, b) R (a, b) \Rightarrow ab = ba$

$\Rightarrow R$ is reflexive

(ii) ***Symmetric:*** Let $(a, b) R (a', b') \Rightarrow ab' = ba'$

$\Rightarrow ab' = a'b$

$\Rightarrow (a', b') R (a, b)$

$\Rightarrow R$ is symmetric

(iii) ***Transitive:*** Let $(a, b) R (a', b') \Rightarrow ab' = a'b$ ...(1)

and $(a', b') R (a'', b'') \Rightarrow a'b'' = a''b'$ ...(2)

From (1) $a' = \dfrac{ab'}{b}$

put in (2), we find $\dfrac{ab'}{b} b'' = a'' b'$

$$\Rightarrow \qquad ab'' = a'' b$$

$$\Rightarrow \qquad (a, b) R (a'', b'')$$

Hence it is transitive

Therefore, R is an equivalence relation.

Example 37: *If R and S are equivalence relations on the set A, show that the following are equivalence relation.*

(i) $R \cap S$ *(ii)* $R \cup S$ **[U.P.T.U. (B.Tech.) 2003]**

Solution: (i) $R \cap S$ is an equivalence relation, if

(a) ***Reflexive:*** $\forall a \in A, (a, a) \in R$ and $(a, a) \in S$, since R and S are equivalence relations. This implies $\forall a \in A, \ (a, a) \in R \cap S$

Hence, $R \cap S$ is reflexive

(b) ***Symmetric:*** Let $(a, b) \in R \cap S \Rightarrow (a, b) \in R$ and $(a, b) \in S$

$\Rightarrow (b, a) \in R$ and $(b, a) \in S$ as R, S is symmetric

$\Rightarrow (b, a) \in R \cap S$

(c) ***Transitive:*** Let $(a, b) \in R \cap S, (b, c) \in R \cap S$

$\Rightarrow$ $(a, b) \in R, (a, b) \in S$ and $(b, c) \in R, (b, c) \in S$

$\therefore$ $(a, b) \in R, (b, c) \in R$ and R is transitive $\Rightarrow (a, c) \in R$

and $(a, b) \in S, (b, c) \in S$ and S is transitive $\Rightarrow (a, c) \in S$

$\therefore$ $(a, c) \in R, (a, c) \in S \Rightarrow (a, c) \in R \cap S$

Hence $R \cap S$ is an equivalence relation.

(ii) The union of two equivalence relation on a set is not necessarily an equivalence relation. For example

$A = \{a, b, c\}$ and R, S be two relation on A given as

$R = \{(a, a), (b, b), (c, c), (a, b), (b, a)\}$

and $S = \{(a, a), (b, b) (c, c), (b, c), (c, b)\}$

Each R and S is an equivalence relation on A. But $R \cup S$ is not transitive, because $(a, b) \in R \cup S$ and $(b, c) \in R \cup S \Rightarrow (a, c) \notin R \cup S$

Hence $R \cup S$ is not equivalence relation.

Example 38: *Let* $A = R \times R$ *(R be the set of real numbers) and define the following relation on A.*

$$(a, b) R (c, d) \Rightarrow a^2 + b^2 = c^2 + d^2$$

(i) *Verify that (A, R) is an equivalence relation.*

(ii) *Describe geometrically what the equivalence classes are for this relation (justify).*

[U.P.T.U. (B.Tech.) 2002; R.G.P.V. (B.E.) Bhopal 2006]

Solution: *Reflexive:* Let $(a, b)R(a, b) \Rightarrow a^2 + b^2 = a^2 + b^2$ which is true.

Hence R is reflexive.

Symmetric: $(a, b)R(c, d) \Rightarrow a^2 + b^2 = c^2 + d^2$

$$\Rightarrow c^2 + d^2 = a^2 + b^2$$

$$\Rightarrow (c, d)R(a, b)$$

$$\Rightarrow R \text{ is symmetric}$$

Transitive: Let $(a, b)R(c, d) \Rightarrow a^2 + b^2 = c^2 + d^2$

and $(c, d)R(e, f) \Rightarrow c^2 + d^2 = e^2 + f^2$

$\therefore \quad a^2 + b^2 = e^2 + f^2 \Rightarrow (a, b)R(e, f)$

$$\Rightarrow R \text{ is transitive}$$

Hence, R is an equivalence relation.

Example 39: *Let $A = \{1, 2, 3, 4, 6, 7, 8, 9\}$ and let $\sim$ be the relation on $A \times A$ defined as $(a, b) \sim (c, d)$ if $a + d = b + c$. Prove that*

(*i*) $\sim$ is an equivalence relation

(*ii*) Find $[(2, 5)]$, the equivalence class of $(2, 5)$

Solution: $\sim$ is an equivalence, if

(*a*) ***Reflexive:*** $(a, b) \sim (a, b)$ *i.e.* $a + b = a + b$ which is true. Hence $\sim$ is reflexive

(*b*) ***Symmetric:*** $(a, b) \sim (c, d) \Rightarrow a + d = b + c$

$$\Rightarrow b + c = a + d \Rightarrow c + b = d + a \Rightarrow (c, d) \sim (a, b)$$

Hence relation is symmetric.

(c) ***Transitive:*** Let $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$ then $a + d = b + c$ and $c + f = d + e$ then add $a + d + c + f = b + c + d + e$ or $a + f = b + e$ or $(a, b) \sim (e, f)$ then $\sim$ is equivalence relation.

(i) $R[(2, 5)] = \{(2, 5), (1, 4), (3, 6), (4, 7), (5, 8), (6, 9)\}$

Example 40: *Let $A = \{1, 2, 3, 4, 5, 6\}$, construct description of relation R on A for the following:*

(*i*) $R = \{(j, k\} : k \text{ is multiple of } j\}$

(*ii*) $R = \{(j, k) : (J - k)^2 \in A\}$

(*iii*) $R = \{(j, k) : j \text{ divide } k\}$

(*iv*) $R = \{(j, k) : j \times k \text{ is prime}\}$ **[U.P.T.U. (B.Tech.) 2008]**

Solution: (i) We have $R = \{1, 2, 3, 4, 5, 6\}$, then

$$R = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 2), (2, 4), (2, 6), (3, 3), (3, 6), (4, 4), (5, 5), (6, 6)\}$$

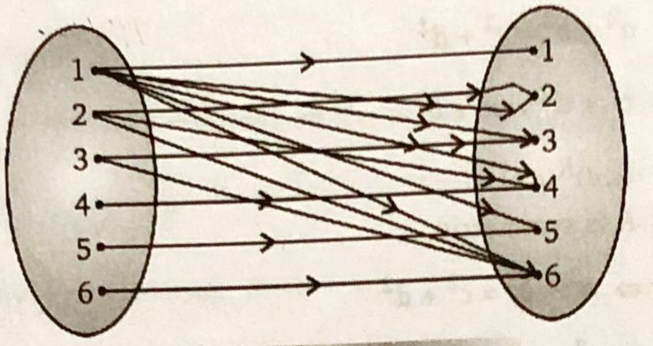

Fig. 2.11

(ii) $R = \{(1, 2), (1, 3), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (3, 4), (3, 5), (4, 2), (4, 3), (4, 5), (4, 6), (5, 3), (5, 4), (6, 5)\}$

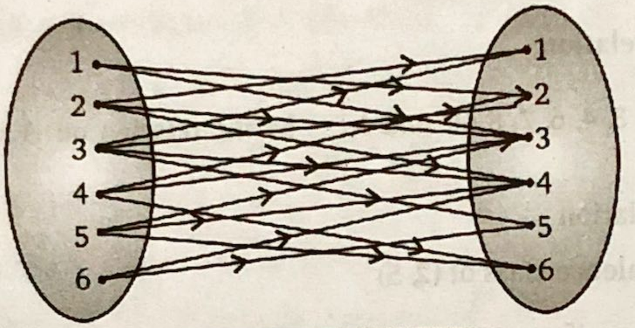

Fig. 2.12

(iii) $R = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 2), (2, 4), (2, 6)\}$

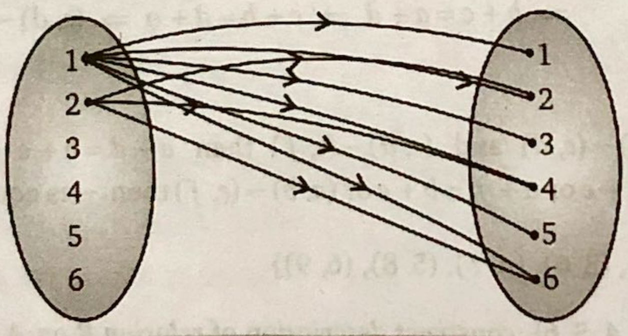

Fig. 2.13

(iv) $R = \{(1, 2), (1, 3), (1, 5), (2, 1), (3, 1), (5, 1)\}$

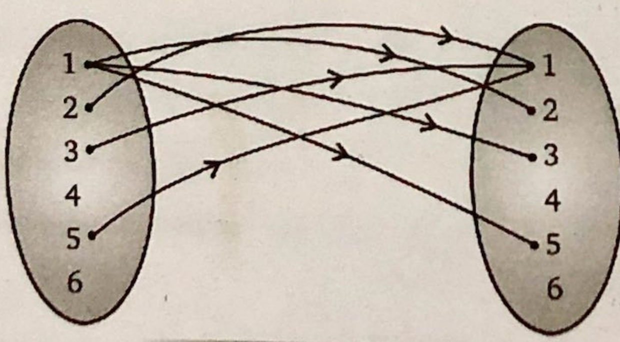

Fig. 2.14

Example 41: *How many reflexive relation and symmetric relations are there on a set with n elements?*

[Pune (B.E.) 2003, 2009; Kurukshetra (B.E.) 2007]

Solution: A relation R on a set A is a subset of $A \times A$. Thus a relation is determined by specifying whether each of the n^2 ordered pairs in $A \times A$ is in R. If R is reflexive, each of n order pair (a, a) for $a \in A$ must be in R. Each of other $n^2 - n = n(n-1)$ ordered pairs may or may not be in R. By **product rule of counting,** there are $2^{n(n-1)}$ Reflexive Relations.

If R is symmetric, **each of the n ordered pair (a, a) for $a \in A$ must be in R and** set of pairs elements of the form (a, b) and (b, a) for $a, b \in A$ be in R. There are $\frac{n(n-1)}{2}$ such pairs. By **product rule of counting,** there are $2^n . 2^{\frac{n(n-1)}{2}} = 2^{n(n+1)/2}$ symmetric relation.

Example 42: *Let A be a set with 10 distinct elements. Determine the following*

(i) *Number of distinct binary relations on A*

(ii) *Number of different symmetric binary relation A* **[U.P.T.U. (B.Tech.) 2009]**

Solution: (i) We know if a set A has n elements then number of distinct binary relation on $(A \times A) = 2^{n^2}$

But $n = 10$ in A.

Then number of distinct binary relation $A = 2^{(10)^2} = 2^{100}$

(ii) We know number of symmetric relation on set A with n elements $= 2^{n(n+1)/2}$

But $n = 10$, then number of symmetric relation on $A = 2^{10 \times (11)/2} = 2^{55}$

Example 43: *If R_1 and R_2 are two equivalence relations on a set A, then prove that $R_1 \cap R_2$ is also an equivalence relation on A.* **[Rohtak (M.C.A.) 2008]**

Solution:

(i) ***Reflexive:*** Let $x \in A$, then $(x, x) \in R_1$ as R_1 is reflexive, similarly $(x, x) \in R_2$ then $(x, x) \in R_1 \cap R_2$

Hence, $R_1 \cap R_2$ is reflexive

(ii) ***Symmetric:*** Let $x, y \in A, (x, y) \in R_1 \Rightarrow (y, x) \in R_1$ (R_1 is reflexive)

$(x, y) \in R_2 \Rightarrow (y, x) \in R_2$ (R_2 is reflexive)

$\Rightarrow$ $(x, y) \in R_1 \cap R_2$ then $(y, x) \in R_1 \cap R_2$

So, $R_1 \cap R_2$ is symmetric

(iii) ***Transitive:*** Let $x, y, z \in A$, since R_1, R_2 are transitive

$\therefore$ $(x, y), (y, z) \in R_1 \Rightarrow (x, z) \in R_1$

and $(x, y), (y, z) \in R_2 \Rightarrow (x, z) \in R_2$

Now, $(x, y), (y, z) \in R_1 \cap R_2 \Rightarrow (x, z) \in R_1 \cap R_2$

So, $R_1 \cap R_2$ is transitive

Hence, $R_1 \cap R_2$ is an equivalence relation.

Example 44: *Let A be the set of all integers and a relation R is defined as*

$R = \{(x, y) : x \equiv y \pmod m)$, m divide $(x - y)$ where m is positive integer. Prove that R is an equivalence relation. **[U.P.T.U. (M.C.A.) 2008]**

Solution: (i) Since $(x - x)$ is divisible by m, therefore

$$x \equiv x \text{ (mod m) } \textit{ i.e. } {}_xR_x.$$

$\Rightarrow$ R is reflexive

(ii) If $x, y \in A$ and $(x - y)$ is divisible by m, then $(y - x) = -(x - y)$ is also divisible by m.

$\therefore \quad x \equiv y \pmod m \Rightarrow y \equiv x \pmod y$

or $\quad {}_xR_y \Rightarrow {}_yR_x.$

So R is symmetric

(iii) If $x, y, z \in A$ and $x - y$, $y - z$ are divisible by m

$\therefore$ $x - z = (x - y) + (y - z)$ is also divisible by m

$\Rightarrow x \equiv z$ (mod m)

$\therefore \quad {}_xR_y, {}_yR_z \Rightarrow {}_xR_z.$

So R is transitive. Hence R is an equivalence relation.

Exercise

Relation

1. Give an example of a relation which is:
   (i) reflexive and transitive but not symmetric.
   (ii) symmetric and transitive but not reflexive.
   (iii) reflexive and symmetric but not transitive.
   (iv) reflexive and transitive but neither symmetric nor anti-symmetric.

2. Prove that if a relation R on set A is transitive and irreflexive, then it is symmetric

3. If R be a relation in the set of integer I defined by $R = \{(x, y) : x \in I, y \in I, (x - y) = 8k$ or $(x - y)$ is divisible by $8\}$. Prove that R is an equivalence relation.

4. List the order pairs in the relation R from $A = \{0, 1, 2, 3, 4\}$ to $B = \{0, 1, 2, 3\}$ where $(a, b) \in R$ if and only if
   (i) $a = b$ (ii) $a + b = 3$ (iii) $a \times b$
   (iv) $a \perp b$ (v) $g.c.d\,(a, b) = 1$ (vi) $\text{lcm}\,(a, b) = 2$

5. Let $A = \{1, 2, 3, 4\}$, determine whether the relation are reflexive, symmetric, anti-symmetric or transitive
   (i) $R = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$
   (ii) $R = \{(1, 3), (4, 2), (2, 4), (3, 1), (2, 2)\}$
   (iii) $R = \{(1, 2), (1, 3), (3, 1), (1, 1), (3, 3), (3, 2), (1, 4), (4, 2), (3, 4)\}$

6. Write down the relations in the square of the set {1, 2, 4, 8, 16, 32, 64}

7. Let f be a mapping of a set X onto a set B. Then if we define $(a, b) \in R$ for $a, b \in X$ provided $f(a) = f(b)$. Prove that R is an equivalence relation.

8. Let R be the relation in the natural numbers $N = \{1, 2, 3 \ldots\}$

   Define by $x + 2y = 10$ *i.e.* Let $R = \{(x, y) 1\, x, y \in N, x + 2y = 10\}$

   Find (*a*) The domain and range of R (*b*) R^{-1}

9. Show that the relation is congruent module 4 to on the set of integers {0, 1, 2, ... 10} is an equivalence relation.

10. Determine which of the following are equivalence relations and or partial ordering relations for the given sets.

    (*a*) A = {lines in the plane}: ${}_xR_y$ iff x is parallel to y

    (*b*) A = {the set of real numbers: ${}_xR_y$ iff $|x - y| \leq 7$

11. Let $A = \{1, 2, 3, 6\}$. If for $x, y \in A$

    $R = \{(x, y) : x \leq y\}$

    $S = \{(x, y) : x \text{ divides } y\}$

    write R and S as sets and find $R \cap S$

12. For a set X with n elements, find how many relations on X, which are

    (i) Symmetric [U.P.T.U. (M.C.A.) 2004]

    (ii) Anti-symmetric [P.T.U. (B.E.) Punjab 2008]

    (iii) reflexive [U.P.T.U. (B.Tech.) 2009]

    (iv) irreflexive

    (v) reflexive and symmetric [Pune (B.E.) 2007]

    (vi) reflexive and not symmetric [Rohtak (M.C.A.) 2007]

    (vii) symmetric but not reflexive [Rohtak (M.C.A.) 2007]

    (viii) neither reflexive nor irreflexive.

13. Let L be the set of lines in the euclidean plane and let R be the relation in L defined by "x is parallel to y". Is R a symmetric relation? Why? Is R a transitive relation?

14. Prove that if R is asymmetric relation then

$$R \cap R^{-1} = R$$

15. Find all partitions on

    (i) $A = \{1, 2, 3\}$ (ii) $A = \{a, b, c, d\}$