GRAPH THEORY

Graph: A graph G consists of a finite set of vertices V and a finite set of edges E. Mathematically,

$$G = (V, E)$$

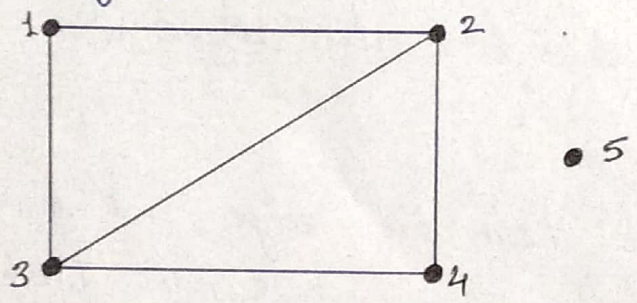
where, $E = \{(v_i, v_j) \mid v_i, v_j \in V\}$

Let us consider $V = \{1, 2, 3, 4, 5\}$

and $E = \{(1,2), (1,3), (2,3), (2,4), (3,4)\}$.

Hence the graph $G = (V, E)$ becomes

G:

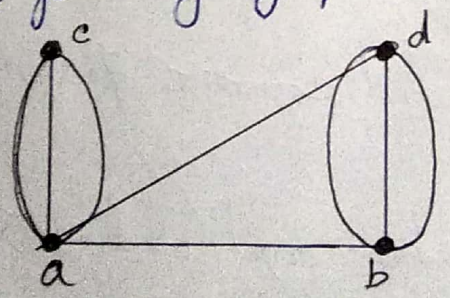


Order and Size:

The number of vertices in a graph $G(V, E)$ is called its order, and the number of edges is its size. That is the order of G is $|V|$ and its size $|E|$

Consider the following graph G

G:



The order of G i.e. $|V| = 4$

The size of G i.e. $|E| = 8$

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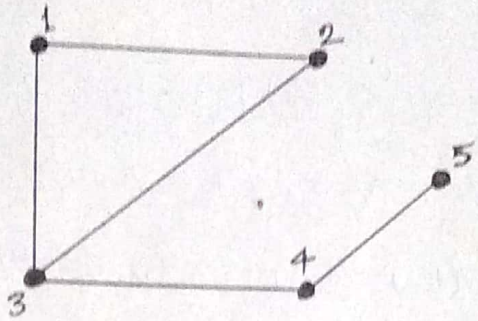
Adjacent Vertices: Two vertices v_i and v_j are said to be adjacent if there exists an edge (v_i, v_j) in the graph $G(V, E)$.

Consider the graph G as

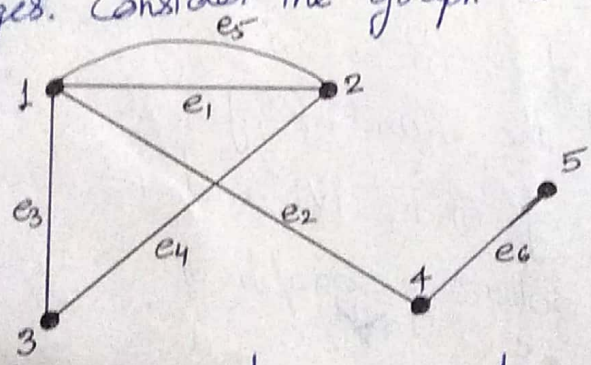
G:



Here the vertices 1 and 2 are adjacent. Similarly, the vertices 1 and 3 are also adjacent.

Parallel edges:

If there is more than one edge between the same pair of vertices, then the edges are termed as parallel edges. Consider the graph G as



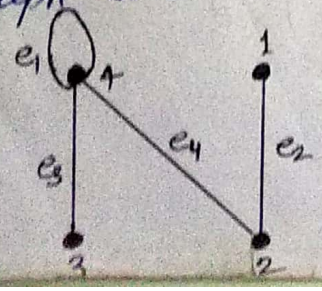
Here the edges e_1 and e_5 are parallel edges.

Loop:

An edge whose starting and ending vertex are same is known as a loop. Mathematically $e = (v_i, v_j)$.

Consider the graph G as

G:



From the graph, it is clear that the edge e_1 is a loop.

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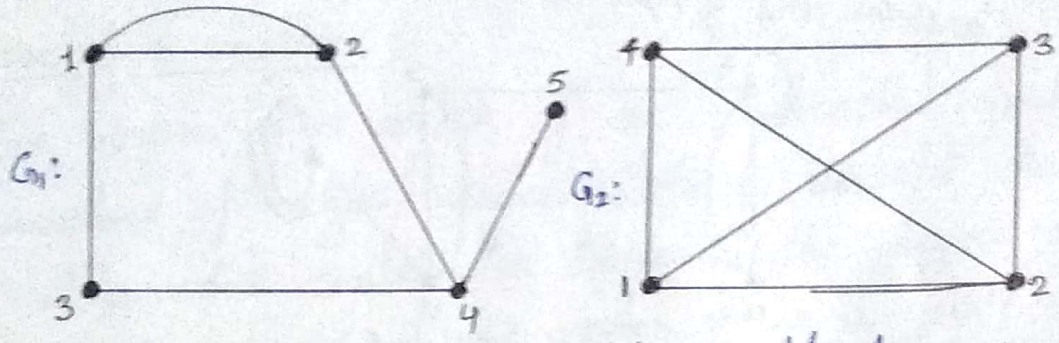
Kinds of Graph

1. Simple Graph:

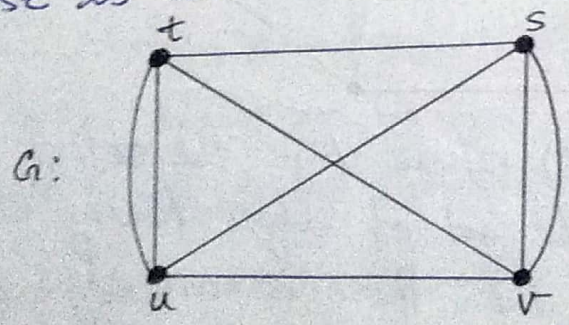
A graph $G(V, E)$ that has no self-loop or parallel edges is called a simple graph. Consider the graphs G_1 and G_2 as



The graph G_1 is not a simple graph because there exists parallel edges between the vertices 1 and 2 whereas the graph G_2 is a simple graph.

2. Multi Graph:

A graph $G(V, E)$ is known as a multi graph if it contains parallel edges, i.e. two or more edges between a pair of vertices. It is to be noted that every simple graph is a multi graph but the converse is not true. Consider the graph G as



The above graph is a multi graph because there are parallel edges between the vertices u, t and v, s.

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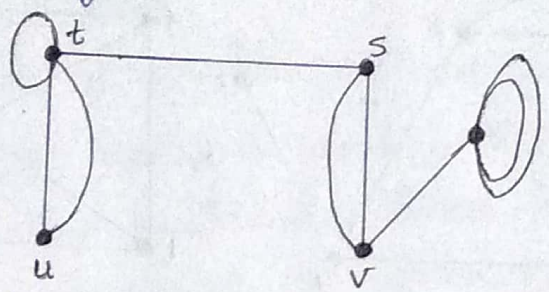
3. Pseudo Graph:

A Graph $G(V, E)$ is known as a pseudo graph if we allow both parallel edges and loops. It is to be noted that every simple graph and multi graph are pseudo graph but the converse is not true.

Consider the graph G as

G:

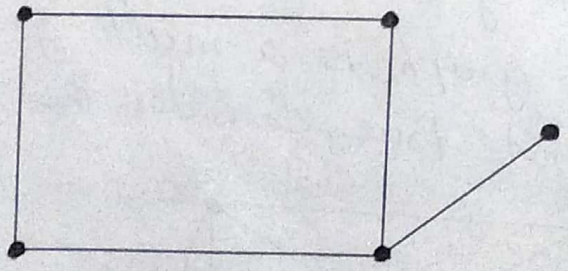


Undirected and Directed Graph

An undirected graph G consists of set V of vertices and a set E of edges such that each edge $e \in E$ is associated with an unordered pair of vertices.

Fig 1. is an example of an undirected graph we can refer to an edge joining the vertex pair i and j as either (i, j) or (j, i).

Fig 1.



A Graph $G(V, E)$ where V is the set of nodes or vertices and E is the set of edges having direction. If (v_i, v_j) is an edge, then there is an edge from the vertex v_i to the vertex v_j. A directed graph is also called a digraph. Let us consider

$V = \{1, 2, 3, 4, 5\}$ and $E = \{(1,2), (1,3), (2,4), (2,3), (3,4), (4,5)\}$

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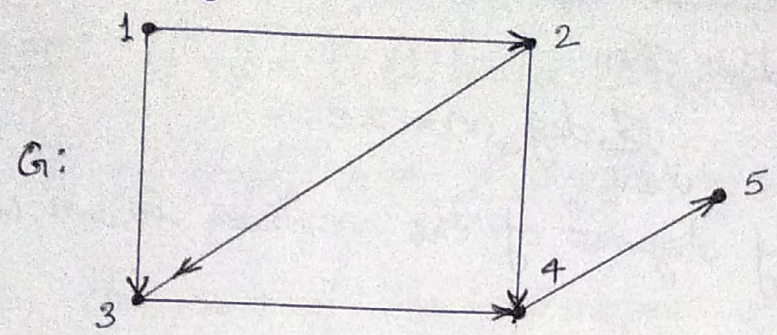
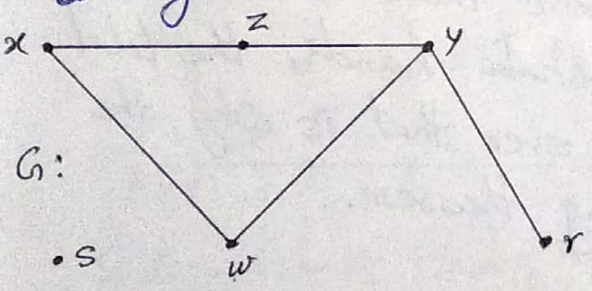
Hence, the digraph G becomes



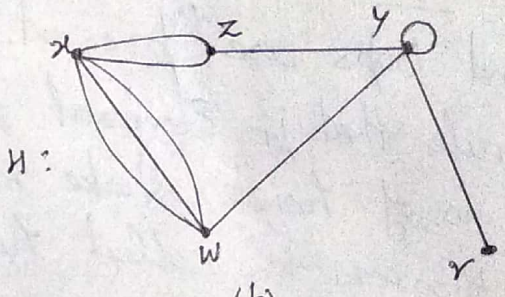
Degree of a Vertex:

The degree of a vertex of an undirected graph is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex. The degree of the vertex v in a graph G may be denoted by $deg_G(v)$.

The degree of vertices in the graph G and H are given below:



(a)



(b)

In G as shown in Fig (a)

$deg_G(x) = 2 = deg_G(z) = deg_G(w)$, $deg_G(y) = 3$, $deg_G(r) = 1$.

and in H as shown in Fig (b).

$deg_H(x) = 5$, $deg_H(z) = 3$, $deg_H(y) = 5$, $deg_H(w) = 4$, $deg_H(r) = 1$

A vertex of degree 0 is called **isolated vertex**. A vertex is **pendant** if and only if it has a degree 1. Vertex s in the graph G is isolated and vertex r is pendant.

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Handshaking Theorem: If $G=(V,E)$ be an undirected graph with e edges. Then

$$\sum_{v \in V} \deg_G(v) = 2e$$

i.e., the sum of degrees of the vertices in an undirected graph is even.

Proof. Since the degree of a vertex is the number of edges incident with that vertex, the sum of the degree counts the total number of times an edge is incident with a vertex. Since every edge is incident with exactly two vertices, each edge gets counted twice, once at each end. Thus the sum of the degree is equal twice the number of edges.

Note: This theorem applies even if multiple edges and loops are present. The above theorem holds this rule that if several people shake hands, the total no. of hands shake must be even that is why the theorem is called handshaking theorem.

In degree and Out degree:

In a directed graph G, the out degree of a vertex v of G, denoted by $\text{outdeg}_G(v)$ or $\deg_G^+(v)$, is the no. of edges begining at v and the indegree of v, denoted by $\text{indeg}_G(v)$ or $\deg_G^-(v)$, is the no. of edges ending at v. The sum of the in degree and out degree of a vertex is called the total degree of the vertex. A vertex with zero in degree is called a source and a vertex with zero out degree is called a sink. Since each edge has an initial vertex and

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terminal vertex, the immediate theorem follows:

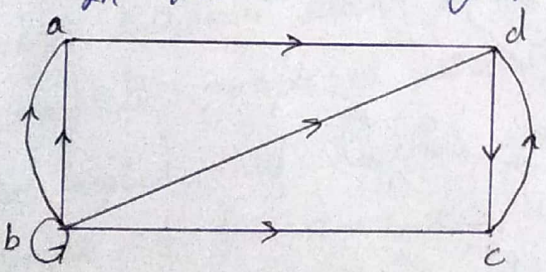
Theorem: If $G = (V, E)$ be a directed graph with e edges, then

$$\sum_{v \in V} \deg_G^+(v) = \sum_{v \in V} \deg_G^-(v) = e$$

i.e, the sum of the outdegrees of the vertices of a diagraph G equals the sum of indegrees of the vertices ~~with~~ which equals the number of edges in G.

Proof: Any directed edge (u, v) contributes 1 to the in degree of u and 1 to the out degree of v. Further, a loop at v contributes 1 to the in degree and 1 to the out degree of v. Hence the proof.

In the directed graph G in Fig. 1.

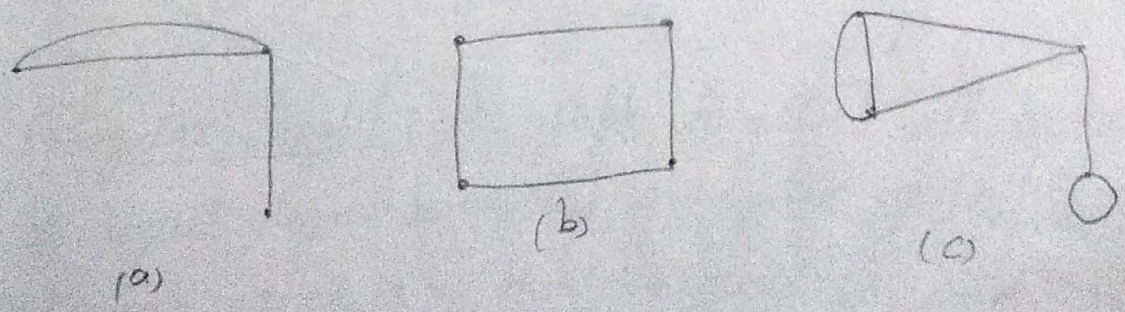


$\text{Indeg}_G(a) = 2$, $\text{Indeg}_G(b) = 1$, $\text{Indeg}_G(c) = 2$, $\text{Indeg}_G(d) = 3$

$\text{Outdeg}_G(a) = 1$, $\text{Outdeg}_G(b) = 5$, $\text{Outdeg}_G(c) = 1$, $\text{Outdeg}_G(d) = 1$.

Note that, the sum of the indegrees and the sum of the out degrees each equal to 8, the no. of edges.

Ex: 1. State which of the following graphs are simple?



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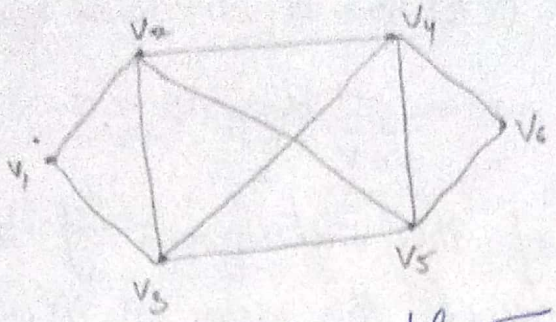
Ex:2. Find the degree of each vertex of the following graph:

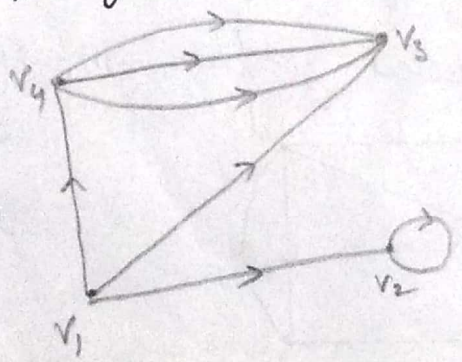


Sol. It is an undirected graph. Then

$\deg(v_1) = 2$, $\deg(v_2) = 4$,

$\deg(v_4) = 4$

Ex:3. Find the indegree out degree and of total degree of each vertex of the following graph.



Sol It is a directed graph.

in deg $(v_1) = 0$ out degree $(v_1) = 3$ total deg $(v_1) = 3$

in deg $(v_2) = 2$ out degree $(v_2) = 1$ total deg $(v_2) = 3$

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Types of Graphs:

Null Graph: A graph which contains only isolated node is called a null graph i.e, the set of edges in a null graph is empty. Null graph is denoted on n vertices by N_n: N_4 is shown in Fig. Note that each vertex of a null graph is isolated.

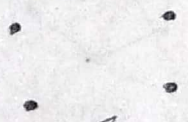
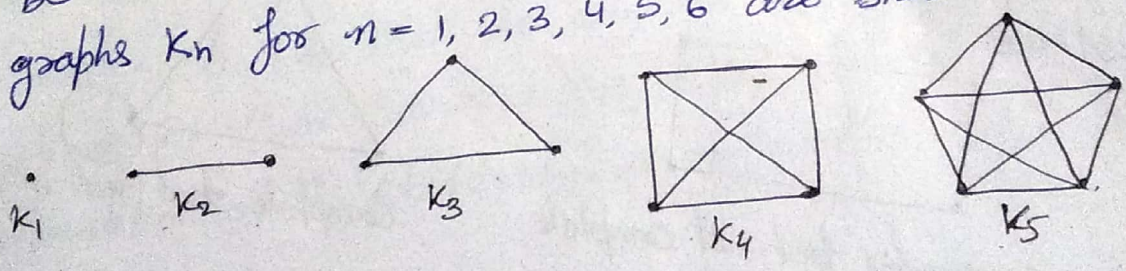
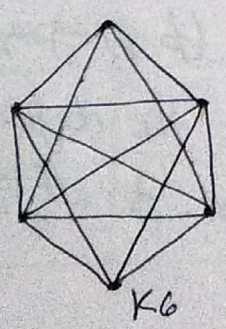


Fig.

Complete Graph: A simple graph G is said to be complete if every vertex in G is connected with every other vertex i.e., if G contains exactly one edge between each pair of distinct vertices. A complete graph is usually denoted by K_n. It should be noted that K_n has exactly $\frac{n(n-1)}{2}$ edges. The graphs K_n for $n = 1, 2, 3, 4, 5, 6$ are shown





Regular Graph: A graph in which all vertices are of equal degree is called a regular graph. If the degree of each vertex is r, then the graph is called a regular

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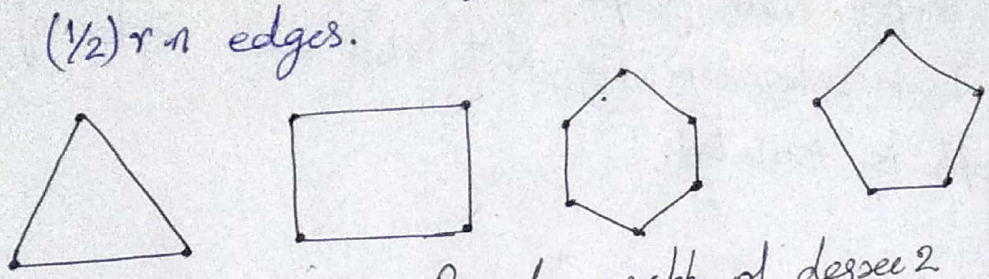
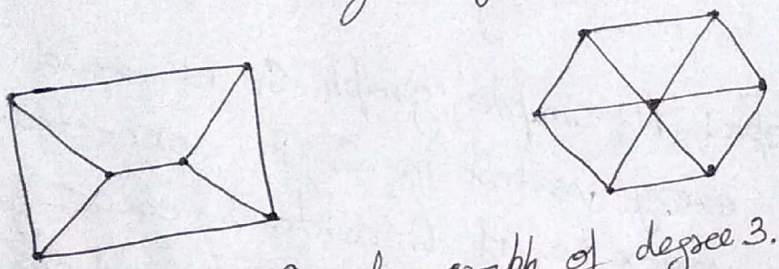
graph of degree r. Note that every null graph is regular of degree zero, and that the complete graph K_n is a regular of degree $n-1$. Also, note that if G has n vertices and is regular of degree r, then G has $(1/2)rn$ edges.

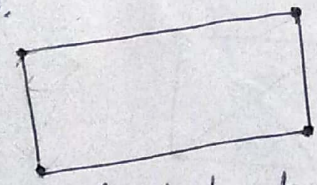


Regular graph of degree 2

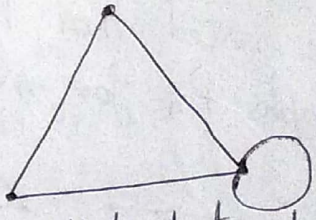


Regular graph of degree 3.

Note that complete graph need not be regular and a regular need not be complete as the following ex. illustrate.



2 regular but not complete



Complete but not regular

Ex: 1. What is the size of an r-regular (p,q)-graph?

Sol Since G is a r-regular graph, by the definition of regularity of G, we have, $\deg_G(v_i) = r$ for all $v_i \in V(G)$.

By the hand shaking theorem,

$$2q = \sum \deg_G(v_i)$$

$$2q = \sum r = p \times r \Rightarrow q = \frac{p \times r}{2}.$$

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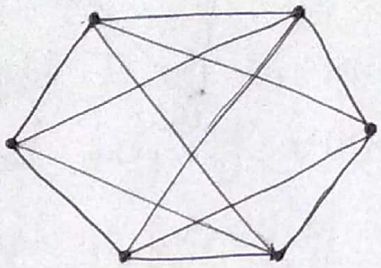
Ex:2. Does there exists a 4-regular graph on 6 vertices? If so construct a graph.

Sol.

$$q = \frac{p \times r}{2} = \frac{6 \times 4}{2} = 12.$$

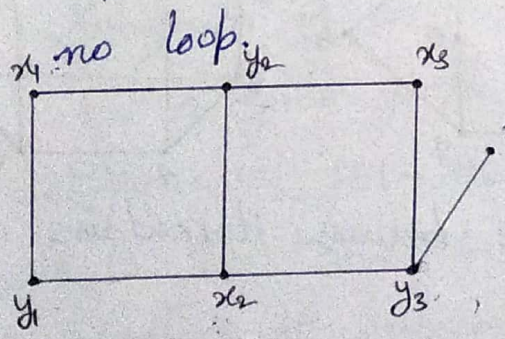
Four regular graph on 6 vertices is possible and it contains 12 edges. One of the graph is shown below.



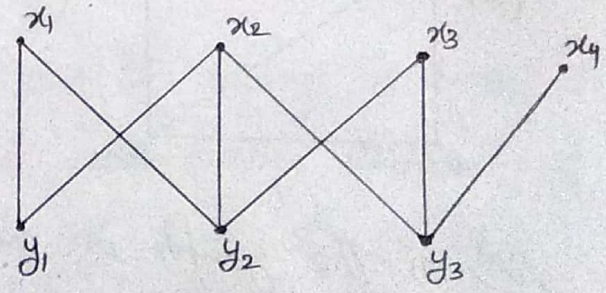
Note: Every 4-regular graph contains a 3-regular graph.

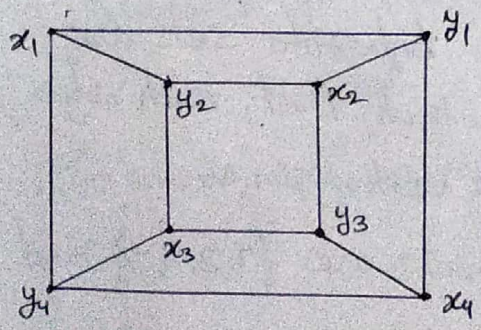
Bipartite Graph:

A graph $G = (V, E)$ is bipartite if the vertex set V can be partitioned into two subsets (disjoint) V_1 and V_2 such that every edge in E connects a vertex in V_1 and a vertex V_2 (so that no edge in G connects either two vertices in V_1 or two vertices in V_2). (V_1, V_2) is called a bipartition of G: Obviously, a bipartite graph can have no loop.

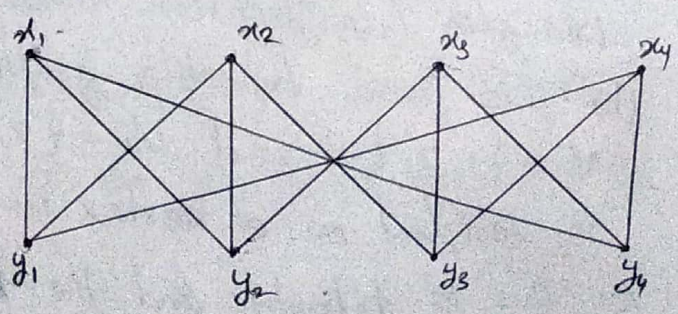


which redraw as:





which redraw as:



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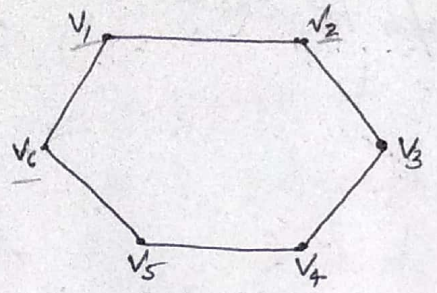
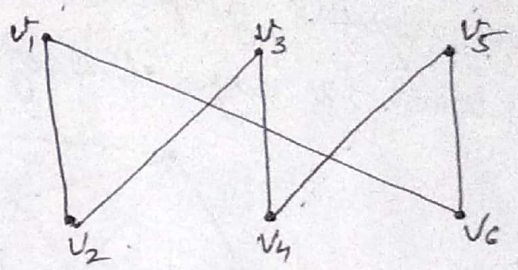
Ex.1. Show that C_6 is a bipartite graph.

Sol. C_6 is a bipartite graph since its vertex set can be partitioned into the two sets $V_1 = \{v_1, v_3, v_5\}$ and $V_2 = \{v_2, v_4, v_6\}$ and every edge of C_6 connects a vertex in V_1 and a vertex in V_2.



C_6

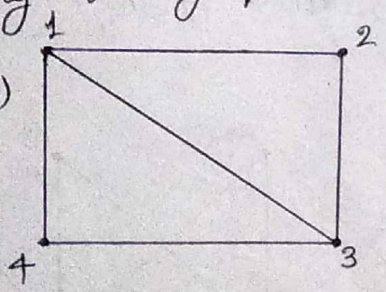


Ex:2 (i) Prove that a graph which contains a triangle cannot be bipartite.

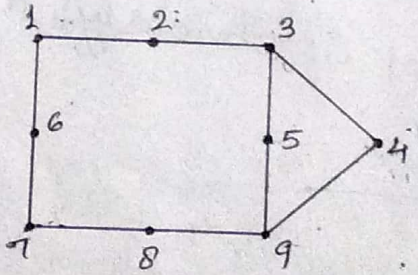
Sol. At least two of the three vertices must lie in one of the bipartite sets since these two are joined by an edge, the graph can not be bipartite.

(ii) Determine whether or not each of the graphs is bipartite. In each case, give the bipartition sets or explain why the graph is not bipartite.

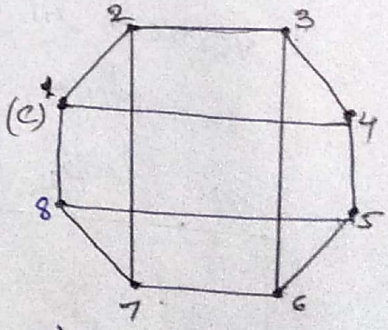
(a)



(b)



(c)



Sol. (a) The graph is not bipartite because it contains triangles (in fact two triangles)

(b) This is a bipartite graph and the bipartite sets are $V_1 = \{1, 3, 7, 9\}$ and $V_2 = \{2, 4, 5, 6, 8\}$ such that each edge is incident on a vertex in V_1 and a vertex in V_2.

(c) This is bipartite and the bipartite sets are $\{1, 3, 5, 7\}$ and $\{2, 4, 6, 8\}$.

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Subgraph:

Consider a graph $G = (V, E)$. A graph $H = (V', E')$ is called a subgraph of G if the vertices and edges of H are contained in the vertices and edges of G that is, if $V' \subseteq V$ and $E' \subseteq E$. In particular,

(i) If $V' \subset V$ and $E' \subset E$, then H is called a proper subgraph of G.

(ii) A subgraph H of G, is called a spanning subgraph of G if and only if $V(H) = V(G)$.

(iii) A subgraph $H(V', E')$ of $G(V, E)$ is called the induced subgraph of G if its edge set E' contain all edges in G whose end points belong to vertices in G.

(iv) If a subset U of V and all the edges incident on the elements of U are deleted from a graph $G(V, E)$, then the resulting subgraph is called a vertex deleted subgraph of $G(V, E)$.

(v) If a subset S of E from a graph $G(V, E)$ is deleted, then the resulting subgraph is called an edge deleted subgraph of G.

For a given graph G, there can be many subgraphs. Let $|V| = m$, $|E| = n$. The total non-empty subsets of V is $2^m - 1$ and total subsets of E is 2^n. Thus, number of subgraph is equal to $(2^m - 1) \times 2^n$.

Then no. of spanning subgraph is equal to 2^n because all vertices are to be included in a spanning subgraph.

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For eg., in Q_3 we have $|V| = 8$, $|E| = 12$, Then total no of subgraphs are

$$(2^8 - 1) \times 2^{12} = 127 - 4096 = 520192$$

and the total no of spanning subgraphs are

$$2^{12} = 4096.$$

So, if H is a subgraph of G, then

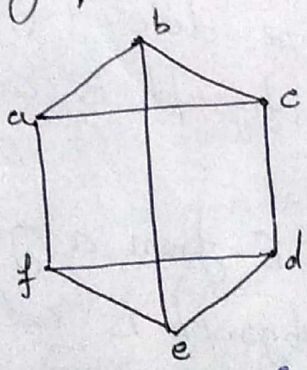
(i) All the vertices of H are in G.

(ii) All the edges of H are in G and

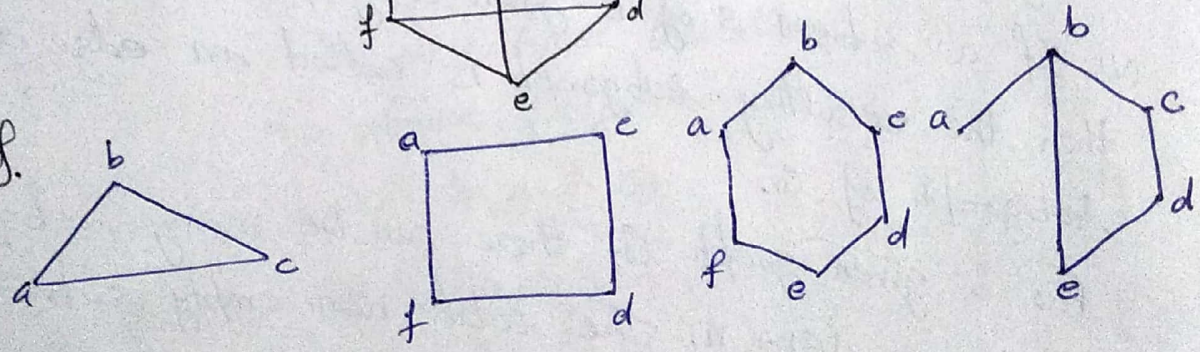
(iii) Each edge of H has the same end points in H as in G.

Any subgraph of a graph G can be obtained by removing certain vertices and edges from G.

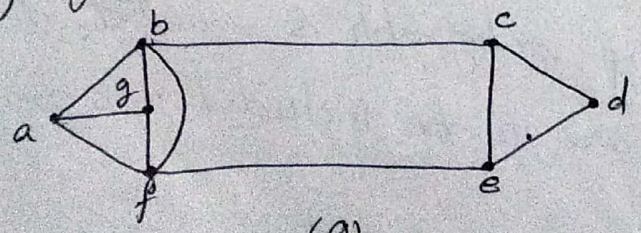
Ex: 1. Consider the graph. Find the different subgraphs of this graph.



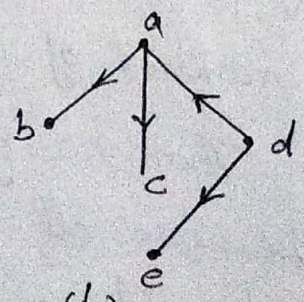
Sol.



Ex: 2. Find the two subgraphs of Fig (a) and one subgraph of Fig (b).



(a)



(b)

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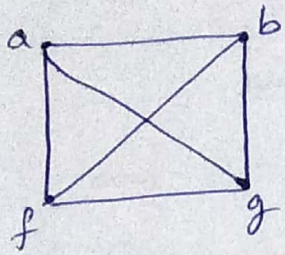
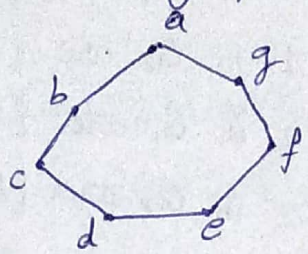
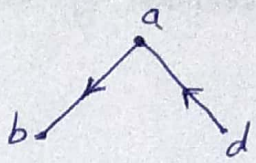
Sol. Two subgraphs of the undirected graph and one subgraph of the directed graph are







Isomorphic Graph:

Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are said to be isomorphic if there exists a functions $f: V_1 \to V_2$ such that

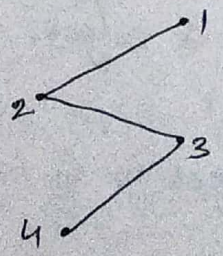
(i) f is one-one, onto i.e f is bijective.

(ii) $\{a, b\}$ is an edge in E_1, if and only if $\{f(a), f(b)\}$ is an edge in E_2 for any two elements $a, b \in V_1$.

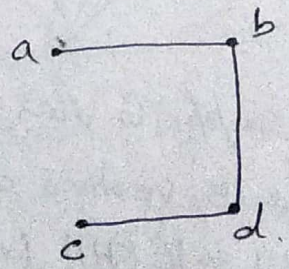
The condition (ii) says that if vertices a and b are adjacent in G_1 then $f(a)$ and $f(b)$ are adjacent in G_2. In other words the function f preserves adjacency relationship and consequently the corresponding vertices in G_1 and G_2 will have the same degree. Any function f with the above properties is called an **isomorphism** between G_1 and G_2.

Ex:1. Show that the given pair of graphs are isomorphic.

G_1:



G_2:



Sol. Here, $V(G_1) = \{1, 2, 3, 4\}$, $V(G_2) = \{a, b, c, d\}$,

$E(G_1) = \{(1,2), (2,3), (3,4)\}$ and $E(G_2) = \{(a,b), (b,d), (d,c)\}$.

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Hence $|V(G_1)| = |V(G_2)|$ and $|E(G_1)| = |E(G_2)|$

The vertices of degree 1 in G_1 are $\{1,4\}$ and

" 1 in G_2 are $\{a,c\}$

" " 2 in G_1 are $\{2,3\}$ and

" " 2 in G_2 are $\{b,d\}$

Define a function $f: V(G_1) \to V(G_2)$ as

$f(1) = a$, $f(2) = b$, $f(3) = d$ and $f(4) = c$.

f is clearly one-one and onto.

Further,

$\{1,2\} \in E(G_1)$ and $\{f(1), f(2)\} = \{a,b\} \in E(G_2)$

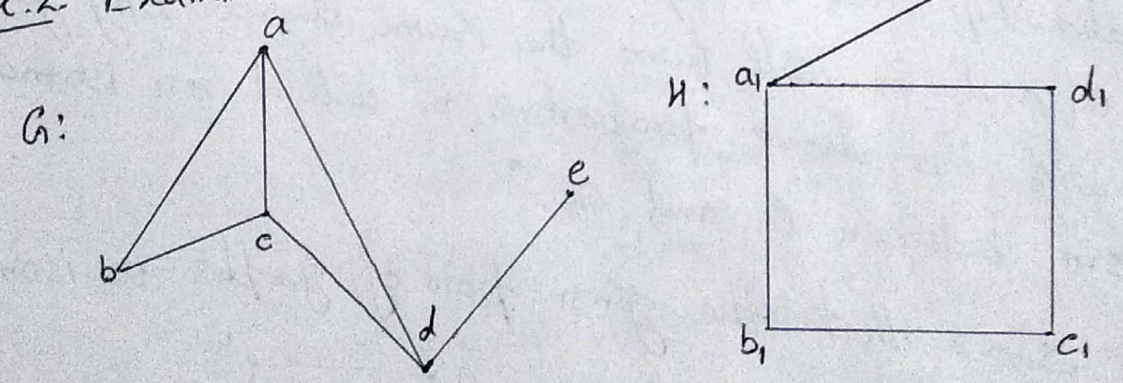
$\{2,3\} \in E(G_1)$ and $\{f(2), f(3)\} = \{b,d\} \in E(G_2)$

$\{3,4\} \in E(G_1)$ and $\{f(3), f(4)\} = \{d,c\} \in E(G_2)$

Hence f preserves adjacency of the vertices.

Therefore, G_1 and G_2 are isomorphic.

Ex:2. Examine G and H for isomorphism.



Sol. The graph G has 5 vertices and 6 edges. The graph H has 5 vertices and 5 edges.

So $|E(G)| \neq |E(H)|$ i.e. the no. of edges in G is not equal the no. of edges in H.

$\therefore$ G and H are not isomorphic.

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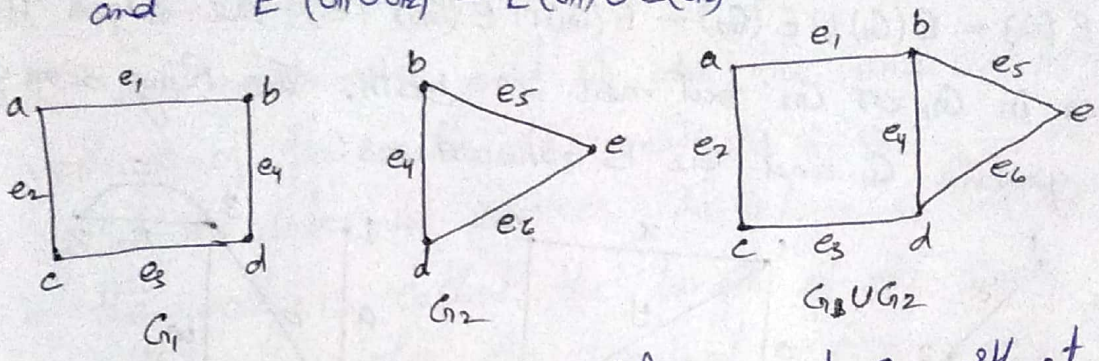
Operations of Graphs

Union: Given two graphs G_1 and G_2 their union will be a graph such that

$$V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$$

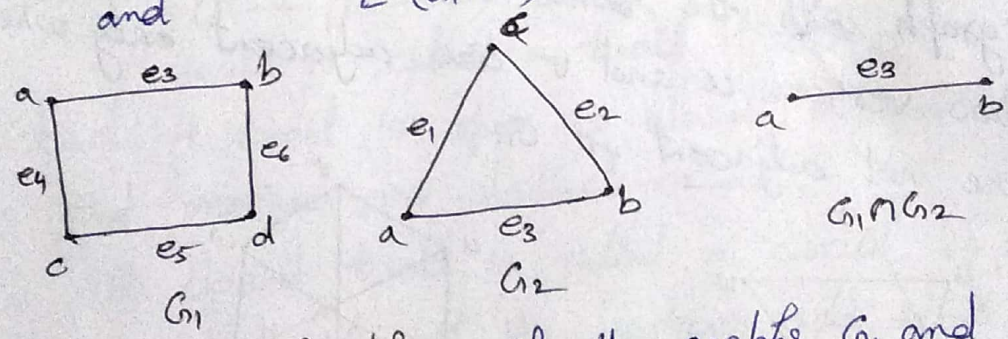
and $$E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$$



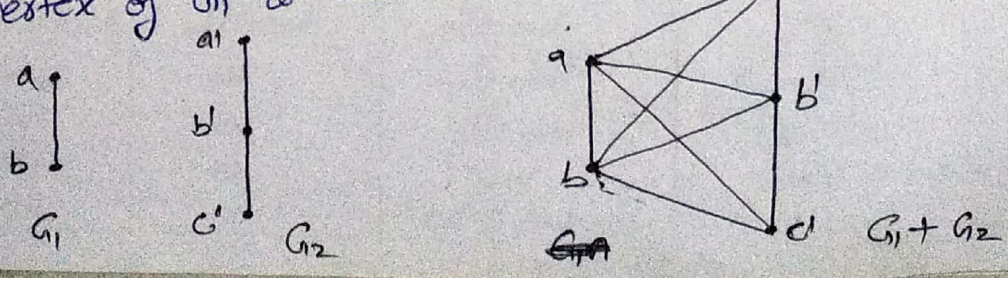
Intersection: Given two graphs G_1 and G_2 with at least one vertex in common then their intersection will be a graph such that

$$V(G_1 \cap G_2) = V(G_1) \cap V(G_2)$$

and $$E(G_1 \cap G_2) = E(G_1) \cap E(G_2)$$



Sum of Two Graphs: If the graphs G_1 and G_2 such that $V(G_1) \cap V(G_2) = \phi$; then the sum $G_1 + G_2$ is defined as the graph whose vertex set is $V(G_1) + V(G_2)$ and the edge set is consisting those edges, which are in G_1 and in G_2 and the edges obtained, by joining each vertex of G_1 to each vertex of G_2. For example



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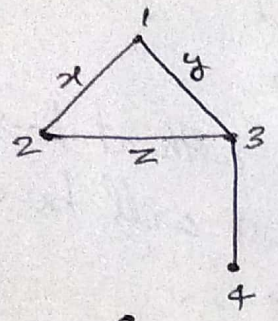
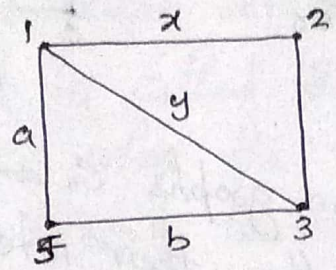
Ring Sum: Let $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ be two graphs. Then the ring sum of G_1 and G_2, denoted by $G_1 \oplus G_2$, is defined as the graph G such that

(i) $V(G) = V(G_1) \cup V(G_2)$

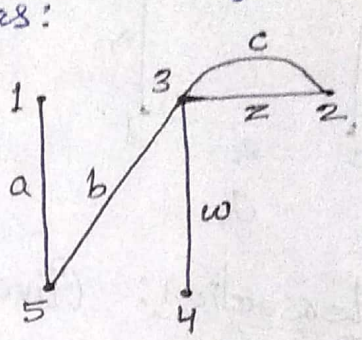
(ii) $E(G) = E(G_1) \cup E(G_2) - E(G_1) \cap E(G_2)$ i.e, the edges that either in G_1 or G_2 but not in both. The Ring sum of two graphs G_1 and G_2 is shown as:



G_1



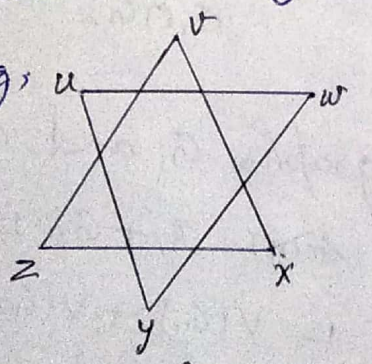
G_2



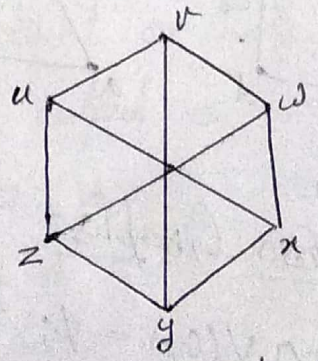
$G_1 \oplus G_2$

Complement: The complement G' of G is defined as a simple graph with the same vertex set as G and where two vertices u and v are adjacent only when they are not adjacent in G.

For eg,



G



Complement of G

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Walk

A walk in a graph G is a finite alternating sequence

$$v_0 - e_1 - v_1 - e_2 - v_2 - e_3 \cdots - e_n - v_n$$

of vertices and edges of the graph such that each edge e_i in the sequence joins vertices v_{i-1} and v_i, $1 \le i \le n$. The end vertices v_0 and v_n are the end or terminal vertices of the walk. The vertices $v_1, v_2, \ldots, v_{n-1}$ are called its internal vertices. The integer n, the no. of edges in the walk is called the length of the walk. A walk is called open when the terminal vertices are distinct. For the same end terminal vertices, it is termed as closed. Note that a walk may repeat both vertices and edges.

Special Types of Walk:

A walk is called a **trail** if all its edges are distinct. A trail is open or closed depends on whether its end vertices are distinct or not.

A closed trail is called a **circuit**. A walk is called a **path** if all its vertices and edges are distinct. Obviously, every path is a trail. A path in which only repeated vertex is the first vertex is called a **cycle** to describe such a closed path.

	Repeated Edge	Repeated Vertex	Starts and Ends at same points?
walk (open)	allowed	allowed	no
walk (closed)	allowed	allowed	yes
Trail	no	allowed	no
circuit	no	allowed	yes
path	no	no	no
cycle	no	first & last only	yes

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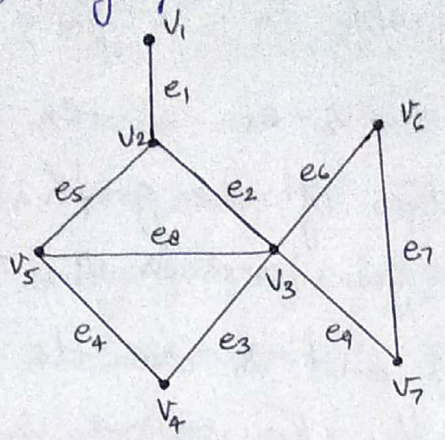
For example, in the graph



(i) The sequence $v_1 - e_1 - v_2 - e_5 - v_5 - e_8 - v_3 - e_3 - v_4 - e_4 - v_5 - e_5 - v_2 - e_2 - v_3 - e_6 - v_6$ walk of length 8. It contains repeated vertices v_2, v_3 and v_5 and repeated edge e_5.

(ii) The sequence $v_1 - e_1 - v_2 - e_5 - v_5 - e_8 - v_3 - e_3 - v_4 - e_4 - v_5$ is a trail. It contains repeated vertex v_5 but does not contain repeated edge.

(iii) The sequence $v_1 - e_1 - v_2 - e_5 - v_5 - e_8 - v_3 - e_3 - v_4$ is a path. It does not contain repeated vertex and repeated edge.

(iv) The sequence $v_2 - e_2 - v_3 - e_3 - v_4 - e_4 - v_5 - e_5 - v_2$ is a cycle. It does not contain repeated vertex and repeated edge except the first and last vertex.

Connectedness in Undirected Graphs

A graph is connected if it is possible to travel from any vertex to any other vertex along a sequence of adjacent edges of the graph. Thus a graph G is **connected** if given any vertices v and w in G, there is a path from v to w. i.e., if any pair of vertices are reachable from one another. A graph that is not connected is called **disconnected**. A disconnected graph

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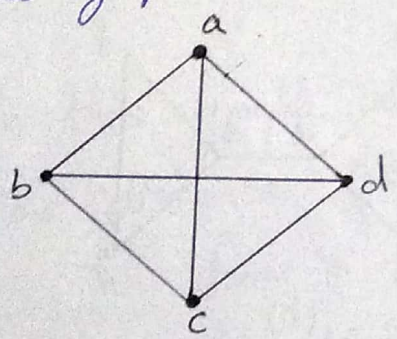
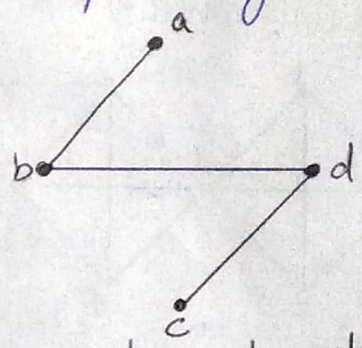
is the union of two or more connected subgraphs each pair of which has no vertex in common. These disjoint connected subgraphs are called the connected components of the graph. A connected component of a graph is a connected subgraph of largest possible size. Thus a graph H is a connected or simply component of a graph G if and only if

1. H is a subgraph of G;
2. H is connected;
3. No connected subgraph of G has H as a subgraph and contains vertices or edges that are not in H.

The graph G and a connected component of the graph is

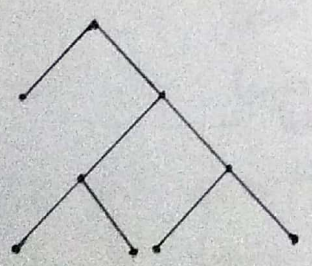


G:

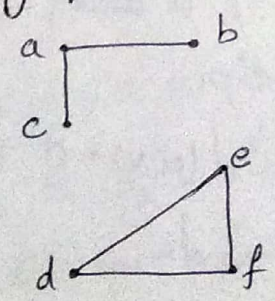


Connected component of G

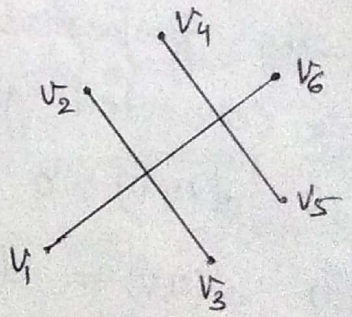
Ex:1. Which of the graphs below are connected?



(a)



(b)



(c)

Sol. (a) The graph is a connected graph since for every pair of distinct vertices there is a path between them.

(b) The graph is not connected since there is no path in the graph between vertices b and d.

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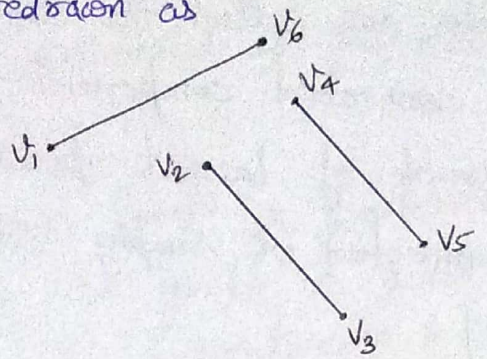
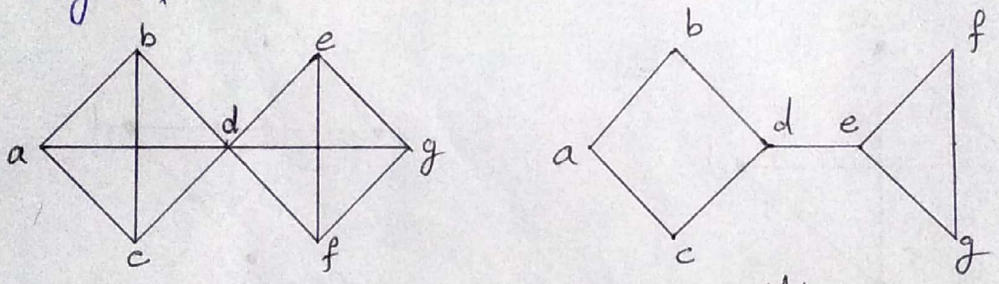
(c) The graph is not connected. In drawing a graph two edges may cross at a point which is not a vertex. The graph can be redrawn as



Distance and Diameter:

In a connected graph G, the distance between the vertices u and v, denoted by d(u,v) is the length of the shortest path. In Fig. (a), d(a,f) = 2 and in Fig (b), d(a,e) = 3



(a) (b)

The distance function as defined above has the following properties. If u, v and w are any three vertices of a connected graph then

(i) $d(u,v) \geq 0$ and $d(u,v) = 0$ iff $u = v$

(ii) $d(u,v) = d(v,u)$ and

(iii) $d(u,v) \geq d(u,w) + d(w,v)$

This shows that distance in a graph is metric.

The diameter of G, written as diam(G) is the maximum distance between any two vertices in G. In Fig (a), diam(G) = 2 and in Fig (b), diam(G) = 4.

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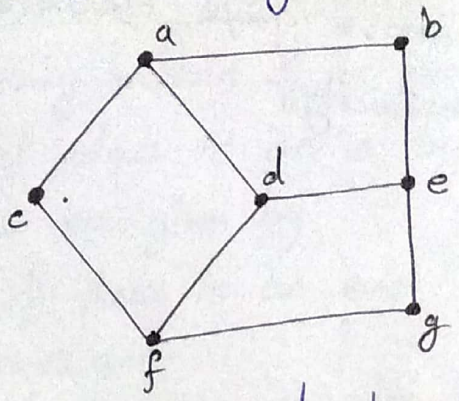
Eccentricity of a Vertex

The maximum distance between a vertex to all other vertices is considered as the eccentricity of vertex.

Notation: e(V)

The distance from a perticular vertex to all other vertices in the graph is taken and among those distances, the eccentricity is the highest of distances.

Ex:



In this graph, the ecentricity of 'a' is 3.

The distance from 'a' to 'b' is 1 ('ab')

from 'a' to 'c' is 1 ('ac'),

from 'a' to 'd' is 1 ('ad'),

from a to e is 2 (ab-be) or (ad-de),

from a to f is 2 (ac-cf) or (ad-df)

from a to g is 3 (ac-cf-fg) or (ad-df-fg).

So the eccentricity is 3, which is maximum from vertex 'a' from the distance between 'ag' which is maximum.

In other words,

e(b) = 3, e(c) = 3, e(d) = 2, e(e) = 3, e(f) = 3, e(g) = 3.

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Diameter of a Graph :

The maximum eccentricity from all the vertices is considered as the diameter of the graph G. The maximum among all the distances between a vertex to all other vertices is considered as the diameter of the Graph G.

It is written as diam (G).

Example. In the above graph, diam (G) = 3; which is the maximum eccentricity.

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Representation of Graphs

Matrix Representation: The matrix are commonly used to represent graphs for computer processing. There are no. of matrices which one can associate with any graph. We shall discuss adjacency matrix and the incidence matrix.

Adjacency Matrix.

(a) Representation of Undirected Graph:

The adjancency matrix of a graph G with n vertices and no parallel edges is an n by n matrix $A = [a_{ij}]$ whose elements are given by

$a_{ij} = 1$, if there is an edge between ith and jth vertices and

$= 0$, if there is no edge between them.

Note that for a given graph, the adjacency matrix is based on ordering chosen for the vertices. Hence, there are as many as n! different adjacency matrices for a graph with n vertices, since there are n! different ordering of n vertices. However, any two such adjacency matrices are closely related in that one can be obtained from the other by simply interchanging rows and columns.

There are a no. of observations that one can make about the adjacency matrix A of a graph G.

(i) A is symmetric i.e. $a_{ij} = a_{ji}$ for all i & j.

(ii) The entries along the principal diagonal of A all 0's if and only if the graph has no self loops. A self loop at the vertex corresponds to $a_{ii} = 1$.

(iii) If the graph is simple (no self loop, no parallel edges),

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the degree of vertex equals the no. of 1's in the corresponding row or column of A.

(iv) The (i,j) entry of A^m is the no. of paths of length (no. of occurence of edges) m from vertex v_i to vertex v_j.

(v) If G be a graph with n vertices $v_1, v_2, \ldots v_n$ and let A denote the adjacency matrix of G w.r.t. this listing of the vertices. Let B be the matrix.

$$B = A + A^2 + A^3 + \cdots + A^n \quad (n>1)$$

Then G is a connected graph iff B has no zero entries.

This result can be used to check the connectedness of a graph by using its adjacency matrix.

(b) <u>Representation</u> of <u>Directed Graph</u>:

The adjacency matrix of a diagraph D, with n vertices is the matrix $A = [a_{ij}]_{n\times n}$ in which

$$a_{ij} = 1 \text{ if arc } (v_i, v_j) \text{ is in } D$$
$$= 0 \text{ otherwise.}$$

One can make a no. of observations about the adjacency matrix of a digraph.

(i) A is not necessary symmetric, since there may not be an edge from v_i to v_j when there is an edge from v_j to v_i.

(ii) The sum of any column of j of A is equal to the no. of arcs directed towards v_j.

(iii) The sum of entries in row i is equal to the no. of arcs directed away from vertex v_i (out degree of vertex v_i)

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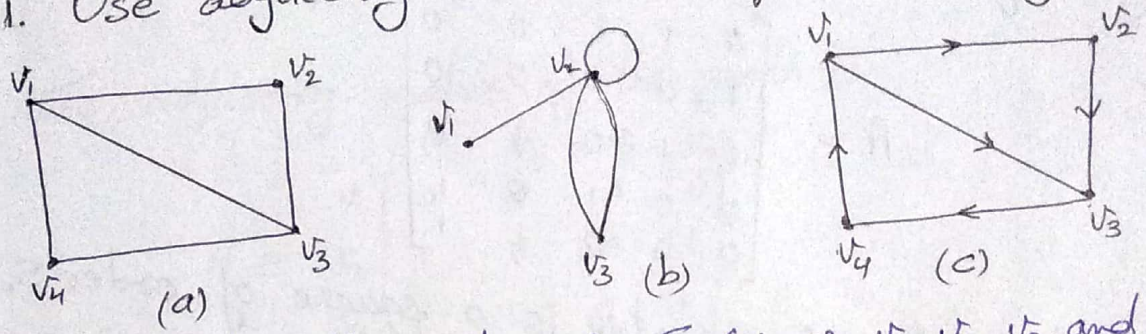
(iv) The (i,j) entry of A^m is equal to the no. of path of length m from vertex v_i to vertex v_j.

(v) The diagonal elements of $A.A^T$ show that out degree of the vertices. The diagonal entries of $A^T.A$ shows the in degree of the vertices.

Ex.1. Use adjacency matrix to represent the graphs:



Sol. We order the vertices in Fig (a) as v_1, v_2, v_3 and v_4. Since there are four vertices, the adjacency matrix representing the graph will be a square matrix of order four. The required adjacency matrix A is

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

We order the vertices in Fig (b) as v_1, v_2 and v_3. The adjacency matrix representing the graph with loop and multiple edges is given by

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 2 \\ 0 & 2 & 0 \end{bmatrix} \end{array}$$

Taking the order of the vertices in Fig (c) as v_1, v_2, v_3, v_4. The adjacency matrix representing the digraph is given by

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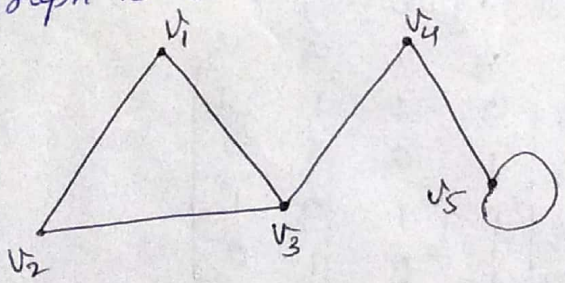
$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Ex:2. Draw the undirected graph represented by adjacency matrix A given by

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

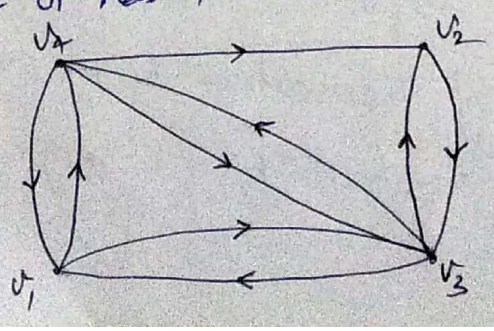
Sol Since the given matrix is a square of order 5, the graph G has five vertices v_1, v_2, v_3, v_4 & v_5. Draw an edge from v_i to v_j where $a_{ij} = 1$. The required graph is drawn as:



Ex:3. Draw the digraph G corresponding to adjacency matrix.

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Sol. Since the given matrix is square matrix of order four, the graph G has 4 vertices v_1, v_2, v_3 & v_4.



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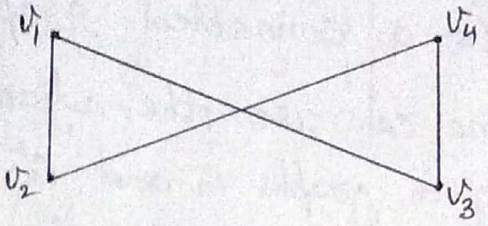
Ex. 4. Consider the graph shown here. Find the no. of walk of length three from v_2 to v_4 and also check the connectedness of the graph.



Sol. The adjacency matrix of the graph is

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Now $A^2 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix}$

Similarly, using matrix multiplication,

$$A^3 = \begin{bmatrix} 0 & 4 & 4 & 0 \\ 4 & 0 & 0 & 4 \\ 4 & 0 & 0 & 4 \\ 0 & 4 & 4 & 0 \end{bmatrix} \text{ and } A^4 = \begin{bmatrix} 8 & 0 & 0 & 8 \\ 0 & 8 & 8 & 0 \\ 0 & 8 & 8 & 0 \\ 8 & 0 & 0 & 8 \end{bmatrix}$$

Now the number of walks of length 3 from v_2 to v_4 = the element in the $(2,4)$ th entry in $A^3 = 4$.

The four different edge sequence are $v_2 - v_1 - v_3 - v_4$, $v_2 - v_1 - v_2 - v_4$, $v_2 - v_4 - v_2 - v_4$ and $v_2 - v_4 - v_3 - v_4$.

Again $B = A + A^2 + A^3 + A^4$

$$B = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 4 & 4 & 0 \\ 4 & 0 & 0 & 4 \\ 4 & 0 & 0 & 4 \\ 0 & 4 & 4 & 0 \end{bmatrix} + \begin{bmatrix} 8 & 0 & 0 & 8 \\ 0 & 8 & 8 & 0 \\ 0 & 8 & 8 & 0 \\ 8 & 0 & 0 & 8 \end{bmatrix}$$

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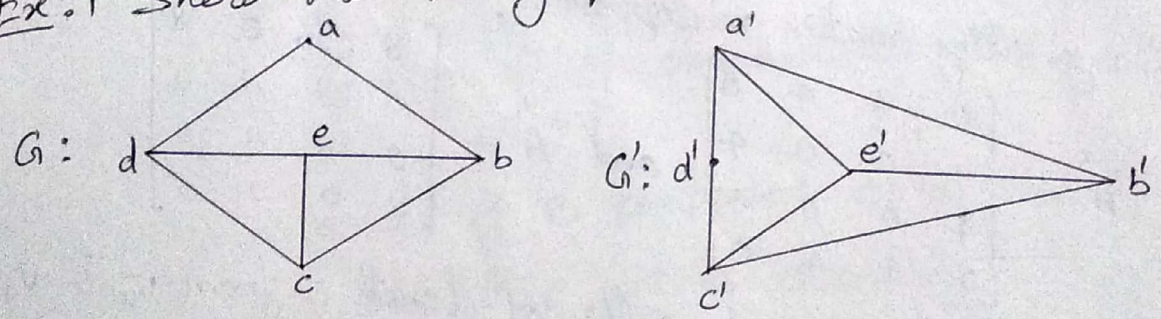
$$B = \begin{bmatrix} 10 & 5 & 5 & 10 \\ 5 & 10 & 10 & 5 \\ 5 & 10 & 10 & 5 \\ 10 & 5 & 5 & 10 \end{bmatrix}$$ which contains no zero element.

Hence the graph is a connected graph.

<u>Note</u>: Note that one can use the adjacency matrix to check whether or not the given graphs G and G' are isomorphic. Two graphs are isomorphic if and only if their vertices can be labled in such a way that the corresponding adjacency matrices are equal.

<u>Theorem</u>: Two graphs G_1 and G_2 are isomorphic if and only if the adjacency matrix of one is obtained from that of the other by interchanging rows and also columns in the same way.

<u>Ex</u>: 1 Show that the graphs G and G' are isomorphic.



<u>Sol</u>: Consider the map $f: G \to G'$, defined as $f(a) = d'$, $f(b) = a'$, $f(c) = b'$ and $f(e) = e'$. The adjacency matrix of G for the ordering a, b, c, d and e is

$$A(G) = \begin{array}{c} \\ a' \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

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The adjacency matrix of G' for the ordering d', a', b', c' & e' is

$$A(G') = \begin{array}{c} \\ d' \\ a' \\ b' \\ c' \\ e' \end{array} \begin{array}{c} \begin{array}{ccccc} d' & a' & b' & c' & e' \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

i.e. $A(G) = A(G')$

$\therefore$ G and G' are isomorphic.

Incidence Matrix:

(a) Representation of Undirected Graph

Consider a undirected graph $G = (V, E)$ which has n vertices and m edges all labelled. The incidence matrix $I(G) = [b_{ij}]$, is then $n \times m$ matrix, where

$$b_{ij} = 1, \text{ when edge } e_j \text{ is incident with } v_i$$
$$= 0, \text{ otherwise}$$

We can make a no. of observations about the incidence matrix of G.

(i) Each column of B comprises exactly two unit entries.

(ii) A row with all 0 entries corresponds to an isolated vertex.

(iii) A row with a single unit entry corresponds to a pendant vertex.

(iv) The no. of unit entries in row i of B is equal to the degree of the corresponding vertex v_i.

(v) The permutation of any two rows (any two columns) of $I(G)$ corresponds to a relabelling of the vertices (edges) of G.

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(vi) Two graphs are isomorphic if and only if their corresponding incidence matrices differ only by a permutation of rows and columns.

(vii) If G is connected with n vertices then the rank of I(G) is n-1.

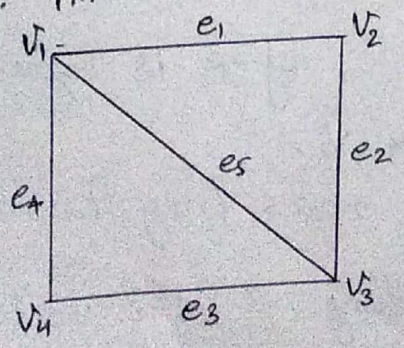
Incidence matrices can also be used to represent multiple edges and loops. Multiple edges are represented in the incidence matrix using columns with identical entries. Since these edges are incident with the same pair of vertices. Loops are represented using a column with exactly one entry equal to 1, corresponding to the vertex that is incident with this loop.

(b) Representation of Directed Graph:

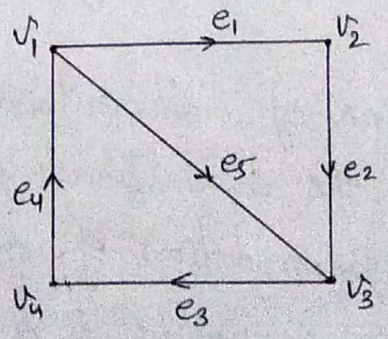
The incidence matrix $I(D) = [b_{ij}]$ of digraph D with n vertices and m edges is the $n \times m$ matrix in which

$$b_{ij} = 1 \text{ if arc } j \text{ is directed away from a vertex } v_i$$
$$= -1 \text{ if arc } j \text{ is directed towards vertex } v_i$$
$$= 0 \text{ otherwise}$$

Ex: 1. Find the incidence matrix to represent the graph



(a)



(b)

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Sol. The incidence matrix of Fig (a) is obtained by entering for row v and column e is 1 if e is incident on v and 0 otherwise. The incidence matrix is.

$$I(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

The incidence matrix of the digraph of Fig (b) is

$$I(D) = \begin{bmatrix} 1 & 0 & 0 & -1 & 1 \\ -01 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 & 0 \end{bmatrix}$$

Ex:2. Draw the graph whose incidence matrix is

$$\begin{bmatrix} 0 & 01 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Sol Since the given matrix has 5 rows and 6 columns, its corresponding graph has 5 vertices and 6 edges. We re-write the incidence matrix as follows:

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Since first column contains two 1 placed at 2nd and 3rd row, vertices v_2 and v_3 are connected by e_1. Similarly, the vertices v_1 and v_4 are connected by e_2 and so on.

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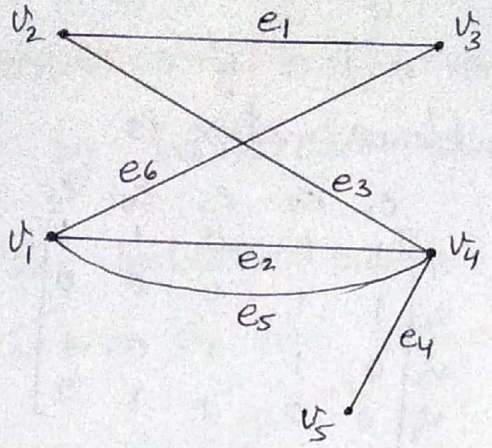
Therefore, the graph of the given incidence matrix is



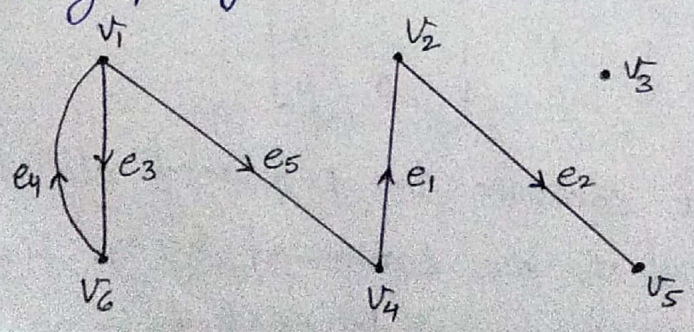
Ex: 3. Draw the graph whose incidence matrix is given below:

$$\begin{bmatrix} 0 & 0 & 1 & -1 & 1 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 01 & 0 & 0 & 0 & -01 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \end{bmatrix}$$

Sol. Since the given matrix has 6 rows and 5 columns, its corresponding graph has 6 vertices and 5 edges. Since the matrix has the element -1, it is a digraph. We rewrite the incidence matrix as follows:

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & -1 & 1 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 01 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \end{bmatrix} \end{array}$$

Hence, the graph of the incidence matrix is



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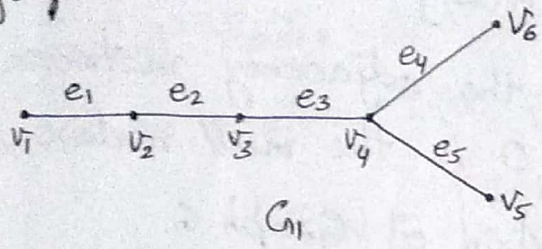
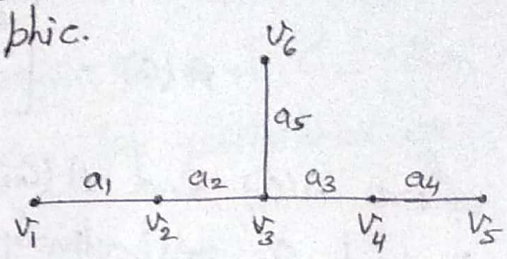
Theorem: Two graphs G_1 and G_2 are isomorphic if and only if the incidence matrix of one is obtained from that of the other by permutation of rows/columns of the matrix.

Ex: 4. Using incidence matrix find whether the two given graphs G_1 and G_2 are isomorphic.



G_1



G_2

Sol. The incidence matrices of the two graphs are

$$I(G_1) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array} \quad \text{and} \quad I(G_2) = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} \begin{array}{c} \begin{array}{ccccc} a_1 & a_2 & a_3 & a_4 & a_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Now, if $I(G_2)$ can be obtained from $I(G_1)$ by interchanging rows and columns, then G_1 and G_2 are isomorphic.

We see 1st three columns of $I(G_1)$ and $I(G_2)$ are same. To make the fourth column identical we interchange 5th row and 6th row of $I(G_1)$.

and get a matrix $B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

We see first four columns of $I(G_2)$ and B are same. The only fifth column of B is not same as that of $I(G_2)$. So, we conclude $I(G_2)$ can not be obtained from $I(G_1)$ by interchanging rows & columns. Hence G_1 & G_2 are <u>not isomorphic</u>.

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Matrix Representation of disconnected Graph and Diagraphs:

1. **Adjacency Matrix:** Let G be a disconnected graph (Digraph) having two components G_1 and G_2. Then the adjacency matrix $A(G)$ of G can be partitioned as

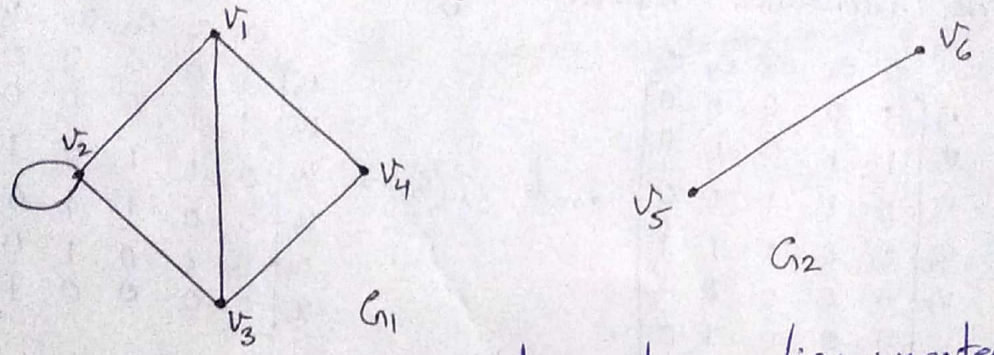
$$A(G) = \left[\begin{array}{c|c} A(G_1) & 0 \\ \hline 0 & A(G_2) \end{array}\right]$$

where $A(G_1)$ and $A(G_2)$ are the adjacency matrices of G_1 and G_2 respectively and 0 is the null matrix.

Ex: 1. Find the adjacency matrix of Graph G.

Sol.

G:



Sol. The adjancency matrix of a disconnected graph is given by

$$A(G) = \left[\begin{array}{c|c} A(G_1) & 0 \\ \hline 0 & A(G_2) \end{array}\right]$$ where $A(G_1)$ and $A(G_2)$ are the adjacency matrices of G_1 and G_2 and 0 is the null matrix.

Here, $$A(G_1) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$ and $$A(G_2) = \begin{array}{c} \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cc} v_5 & v_6 \end{array} \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array}$$

Hence the required adjacency matrix is

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$$A(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

2. <u>Incidence Matrix</u>: Let G be a disconnected graph (digraph) having two components G_1 and G_2. Then the incidence matrix $I(G)$ of G can be partitioned as

$$I(G) = \left[\begin{array}{c|c} I(G_1) & 0 \\ \hline 0 & I(G_2) \end{array}\right]$$

<u>Ex:2.</u> Draw the graph corresponding to the following incidence matrix

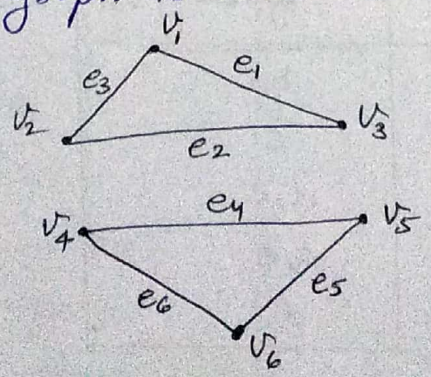
$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

<u>Sol</u> Since the given matrix has 6 rows and 6 columns, its corresponding graph has 6 vertices and 6 edges.

This matrix is of the form

$$I(G) = \left[\begin{array}{c|c} I(G_1) & 0 \\ \hline 0 & I(G_2) \end{array}\right]$$

Hence the graph is disconnected. They represent the graphs.



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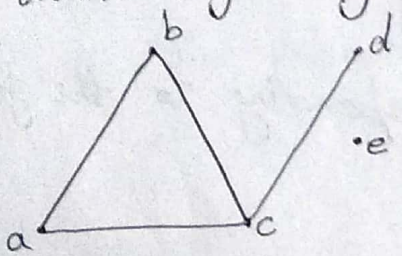
Linked Representation:

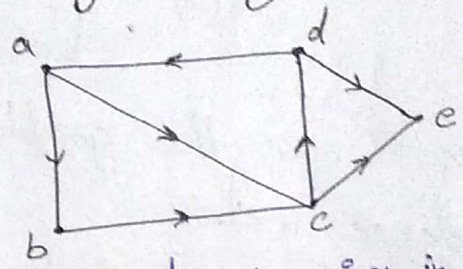
In this representation, a list of vertices adjacent to each vertex is maintained. This representation is also called adjacency structure representation. In case of a directed graph, a care has to be taken, according to the direction of an edge, while placing a vertex in the adjacent structure representation of another vertex. The linked representation is described by means of two egs.

Ex: Write adjacency structure for the graphs:

(a)



(b)



Sol. The adjacency structure represention is given in the table for Fig (a):

Vertex	Adjacency list
a	b, c
b	a, c
c	a, b, d
d	c
e	ϕ

The adjacency structure representation is given in the table for the directed graph:

Vertex	Adjacency list
a	b, c
b	c
c	d, e
d	a, e
e	ϕ

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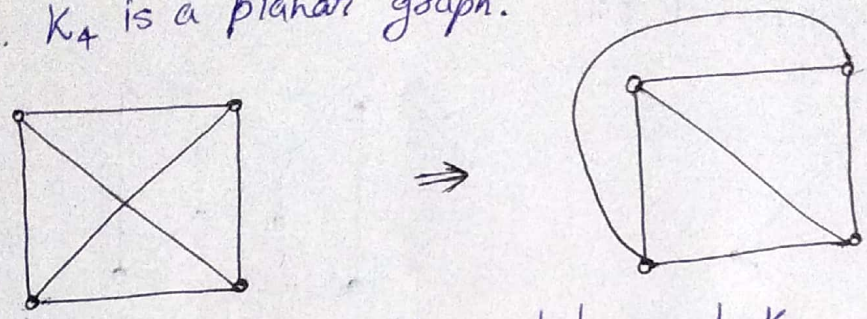
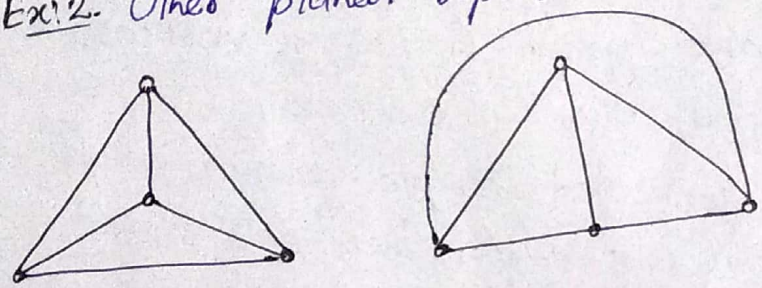
Planar Graph:

A planar graph is an undirected graph that can be drawn on a plane without any edges crossing. Such a drawing is called a planar representation of the graph in the plane.

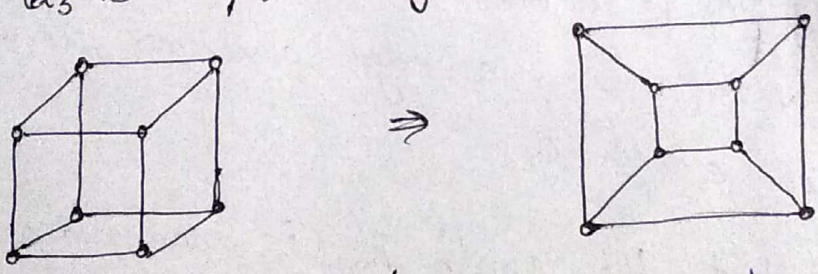
Ex:1. K_4 is a planar graph.



Ex:2. Other planar representations of K_4

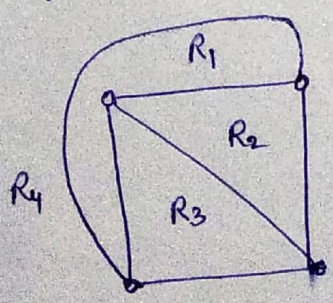


Ex:3. Q_3 is a planar graph.



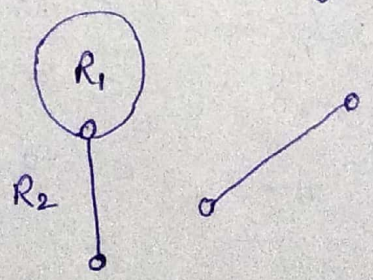
Euler's Planar Formula: A planar representation of a graph splits the plane into regions, where one of them has infinite area and is called the infinite region.

Ex:



4 regions

(R_4 = infinite region)



2 regions

(R_2 = infinite region)

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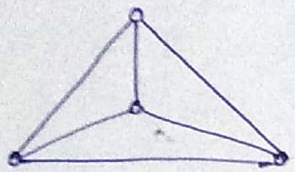
Euler's Planar Formula

Let G be a connected planar graph, and consider a planar representation of G. Let

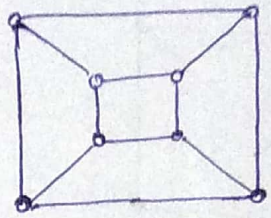
V = # vertices, E = # edges, F = # regions.

Theorem: $V + F = E + 2$.

Ex:



$V = 4,\ F = 4,\ E = 6$



$V = 8,\ F = 6,\ E = 12$

If a connected planar graph G has n vertices, e edges and r region, then $n - e + r = 2$.

Corollary: If G is connected simple planar graph with $n\ (\geq 3)$ vertices and e edges, then $e \leq 3n - 6$.

Corollary: If G is connected simple planar graph with $n\ (\geq 3)$ vertices and e edges and no circuits of length 3, then $e \leq 2n - 4$.

Ex: Show that the graph K_5 is non-planar.

Sol. Since K_5 is a simple graph, the smallest possible length for any cycle of K_5 is three. We shall suppose that the graph is planar. The graph has 5 vertices and 10 edges so that

$n = 5,\quad e = 10.$ Now $3n - 6 = 3 \times 5 - 6$

$= 9 < e.$

Thus the graph violates the inequility $e \leq 3n - 6$ and hence it is non-planar. It can be seen that K_n is planar for $n = 1, 2, 3$ & 4.

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