UNIT 2 COMBINATORICS — AN INTRODUCTION

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2.0 INTRODUCTION

Let us start with thinking about how to assess the efficiency of a computer programme. For this we would need to estimate the number of times each procedure is called during the execution of the programme. How would we do this? The theory of combinatorics helps us in this matter, as you will see while studying this unit.

Combinatorics deals with counting the number of ways in which objects can be arranged according to some pattern (listing). Mostly, it deals with a finite number of objects and a finite number of ways of arranging them. Sometimes an infinite number of objects and infinite number of ways in which they can be arranged are also considered. However, in this unit and block, we shall restrict our discussion to a finite number of objects.

We start our discussion in Sec. 2.2, with two counting principles. These principles help us in counting the number of ways in which a task can be done when it consists of several subtasks, and there are many possible ways of doing the subtasks.

In Sec. 2.3 we look at arrangements of objects in which the order matters. Such arrangements are called permutations. Here we look at various linear and circular permutations, and how to count their number in a given situation.

In Sec. 2.4, we consider arrangements of objects in which the order does not matter. Such arrangements are called combinations. We will consider situations that require us to count combinations. You will see that most of these situations require us to apply the multiplication principle also.

In the next section, Sec. 2.5, we consider binomial and multinomial coefficients. We see how they are related to the objects studied in Sec. 2.4.

Finally, in Sec. 2.6, we consider the applications of what we have presented in the rest of the unit, for finding the probability of the occurrence of an event. As you will see, this application is natural, since we use similar counting arguments for obtaining discrete probabilities. This discussion will be useful for you, for instance, in coding theory as well as in designing **reliable** computer systems.

We continue our study of combinatorics in the next unit. We also have a section of miscellaneous exercises at the end of the block of which several are based on this unit. Doing these exercises, and every exercise given in the unit, will help you achieve the following objectives of this unit.

2.1 OBJECTIVES

After going through this unit, you should be able to:

- explain the multiplication and addition principles, and apply them;
- differentiate between situations involving permutations and those involving combinations;
- perform calculations involving permutations and combinations;
- prove and use formulae involving binomial and multinomial coefficients;
- apply the concepts presented so far for calculating combinatorial probabilities.

2.2 MULTIPLICATION AND ADDITION PRINCIPLES

Let us start with considering the following situation: Suppose a shop sells six styles of pants. Each style is available in 8 lengths, six waist sizes, and four colours. How many different kinds of pants does the shop need to stock?

There are 6 possible types of pants; then for each type, there are 8 possible length sizes; for each of these, there are 6 possible waist sizes; and each of these is available in 4 different colours. So, if you sit down to count all the possibilities, you will find a huge number, and may even miss some out! However, if you apply the multiplication principle, you will have the answer in a jiffy!

So, what is the multiplication principle? There are various ways of explaining this principle. One way is the following:

Suppose that a task/procedure consists of a sequence of subtasks or steps, say, Subtask 1, Subtask 2,…, Subtask k. Furthermore, suppose that Subtask 1 can be performed in n_1, ways, Subtask 2 can be performed in n_2 ways after Subtask 1 has been performed, Subtask 3 can be performed in n_3 ways after Subtask 1 and Subtask 2 have been performed, and so on. Then **the multiplication principle** says that the number of ways in which the whole task can be performed is $n_1.n_2….n_k$.

Let us consider this principle in the context of boxes and objects filling them. Suppose there are m boxes. Suppose the first box can be filled up in k(1) ways. For every way of filling the first box, suppose there are k(2) ways of filling the second box. Then the two boxes can be filled up in k(1).k(2) ways. In general, if for every way of filling the first (r – 1) boxes, the rth box can be filled up in k(r) ways, for r = 2,3,…, m, then the total number of ways of filling all the boxes is k(1).k(2)… k(m).

So let us see how the multiplication principle can be applied to the situation above (the shop selling pants). Here k(1) = 6, k(2) = 8, k(3) = 6 and k(4) = 4. So, the different kinds of pants are $6 \times 8 \times 6 \times 4 = 1152$ in number.

Let's consider one more example.

Example 1: Suppose we want to choose two persons from a party consisting of 35 members as president and vice-president. In how many ways can this be done?

Solution: Here, Subtask 1 is 'choosing a president'. This can be done in 35 ways. Subtask 2 is 'choosing a vice-president'. For each choice of president, we can choose the vice-president in 34 ways. Therefore, the total number of ways in which Subtasks 1 and 2 can be done is $35 \times 34 = 1190$.

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There is another fundamental principle called the **addition principle**. This is applied in situations like the following one:

Suppose that a task consists of performing exactly one subtask from among a collection of disjoint (mutually exclusive) subtasks, say, Subtask 1, Subtask 2,…., Subtask k. (i.e., the task is performed if **either** Subtask 1 is performed, **or** Subtask 2,…, or Subtask k is performed.) Further, suppose that Subtask i can be performed in n_i ways, i = 1,2,…, k. Then, the number of ways in which the task can be performed is the sum $n_1+n_2+\ldots+n_k$.

Let us consider an example of its application.

Example 2: There are three political parties, P_1, P_2 and P_3. The party P_1 has 4 members, P_2 has 5 members and P_3 has 6 members in an assembly. Suppose we want to select two persons, both from the same party, to become president and vice-president. In how many ways can this be done?

Solution: From P_1, we can do the task in $4 \times 3 = 12$ ways, using the multiplication principle. From P_2, it can be done in $5 \times 4 = 20$ ways. From P_3 it can be done in $6 \times 5 = 30$ ways. So, by the addition principle, the number of ways of doing the task is $12 + 20 + 30 = 62$.

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Though both these principles seem simple, quite a number of combinatorial enumerations can be done with them. For instance, what we see from Example 2 is that the addition principle helps us to count all possible arrangements grouped into mutually exclusive and exhaustive classes.

Why don't you try a few exercises that involve the use of these principles now?

E1) Give a situation related to computing in which the addition principle is used, and one in which the multiplication principle is used.

E2) Find the number of words of length 4, meaningful or not, made with the letters a,b,…, j.

E3) If n couples are at a dance, in how many ways can the men and women be paired for a single dance?

E4) How many integers between 100 and 999 consist of distinct even digits?

E5) Consider all the numbers between 100 and 999 that have distinct digits. How many of them are odd?

Let us now consider certain arrangements of objects, in which the order in which they are arranged matters.

2.3 PERMUTATIONS

Suppose we have 15 books that we want to arrange on a shelf. How many ways are there of doing it? Using the multiplication principle, you would say –

$15 \times 14 \times 13 \times \ldots\ldots 2 \times 1 = 15!$

n! denotes '**n factorial**', which means $n \times (n-1) \times . \times 2 \times 1$ for any $n \in \mathbf{N}$.)

Each of these arrangements of the books is a permutation of the books. Let us define this term formally.

Definition: An arrangement of a set of n objects **in a given order** is called a **permutation** of the objects (taken altogether at a time).

An **ordered** arrangement of the n objects, taking r at a time, (where $r \le n$) is called **a permutation of the n objects taking r at a time**. The total number of such permutations is denoted by **P(n,r)**.

As an example, let us consider picking out books, three at a time, from the shelf of 15 books. The first book can be chosen in 15 ways, the next in 14 ways, and the third in 13 ways. So the multiplication principle tells us that the total number of permutations of the 15 books taken 3 at a time is $P(15,3) = 15 \times 14 \times 13$.

Other notations used for P(n,r) are ${}^nP_r$, P_r^n, ${}_nP_r$.

Again, consider the permutations of a,b,c,d, taken 2 at a time. These are ab, ba, ac, ca, ad, da, bc, cb, bd, db, cd, dc. (Note that ab and ba are considered different even though they consist of the same two objects.) Or, we can argue combinatorically as above: The first letter can be chosen in 4 ways, and then the next letter can be chosen. We can list out all the cases in 3 ways. So, the total number of permutations are $P(4,2) = 4 \times 3 = 12$.

Now, is there a formula for finding the value of P(n,r)? This is what the following theorem tells us.

Theorem 1: The number of permutations of n objects, taken r at a time, where $0 \le r \le n$, is given by $P(n,r) = \dfrac{n!}{(n-r)!}$

Consider r boxes arranged in a line. Choose one object out of n and place it in the first box. This can be done in n ways. Then from the remaining (n–1) objects choose one and place it in the second box. The first two boxes can be filled in n(n – 1) ways. We continue this operation till the rth box is filled. So, by the multiplication principle, the total number of ways of doing this is $n(n-1)(n-2)\ldots(n-r+1)$.

$$\begin{aligned} P(n,r) &= n(n-1)\ldots(n-r+1). \\ &= n(n-1)\ldots((n-r+1)((n-r)((n-r-1)\ldots3.2.1 \\ &= (n-r)\ldots((n-r-1)\ldots3.2.1 \\ &= n!/(n-r)! \end{aligned}$$

We define 0! = 1

Proof: In particular, Theorem 1 tells us that the number of permutation of n objects, taken all at a time, is given by

$P(n,n) = n!$

and $P(n, 0) = 1 \qquad \forall n \in \mathbf{N}$.

So, for example, by Theorem 1 we can find

$P(6,4) = 6.5.4.3 = 6!/(6-4)!$ And $P(6,0) = 1$.

Why don't you try some exercises now?

E6) If m and n are positive integers, show that $(m+n)! \ge m! + n!$.

E7) How many 3-digit numbers can be formed from the 6 digits 2,3,5,7,8,9 if repetitions are not allowed? How many of these numbers are less than 400? How many are even?

E8) How many ways are there to rank n candidates for the job of chief engineer? In how many rankings will Ms. Sheela be in the second place.

In defining the concept of permutation we assumed that the objects were distinguishable. What does this mean, and what happens if we remove this assumption? Let's see.

2.3.1 Permutation of Objects Not Necessarily Distinct

We have shown that there are P(n,r) ways to choose r objects from a set of n distinct objects and arrange them in linear order. Here we consider the same problem with the relaxed condition that some of the objects in the collection may not be distinguishable.

For example, we consider permutations of the letters of the word DISTINCT. Here there are 8 letters of which 2 are I, 2 are T, and three are 4 other different letters. To count the permutations in such a situation, we have the following result.

Theorem 2: Suppose there are n objects classified into k distinct types, with m_1 identical objects of the first type, m_2 identical objects of the second type,…, and m_k identical objects of the kth type, where $m_1+m_2+...+m_k = n$. Then the number of distinct arrangements of these n objects, denoted by $P(n; m_1, m_2,..m_k)$ is $\frac{n!}{m_1!m_2!...m_k!}$.

Proof: Let x be the number of such permutations. If the objects of Type i are considered distinct, then they can be arranged amongst themselves in $m_1!$ ways, where $i = 1,2,…, k$. Therefore, by the multiplication principle, the total number of permutations of these n distinct objects, taken all at a time, is $xm_1!m_2!…m_k!$.

But this is precisely n! when there are n distinct objects.

Hence, $xm_1!m_2!…m_k! = n!$, that is, $x = n!/m_1!m_2!…m_k!$

So for example, this result tells us that the number of distinct 8 letter words, not necessarily meaningful, that we can make from the letter of the word "DISTINCT" is $\frac{8!}{2!2!1!1!1!1!} = 14.$

Here are some related exercises.

E9) How many permutations are there of the letters, taken all at a time, of the words

(i) ASSESSES, (ii) PATTIVEERANPATTI?

E10) How many licence plates can be made if each should have 3 letters of the English alphabet with no letter repeated? What will be the answer if the letters can be repeated?

So far, we have considered permutations of objects as linear arrangements of objects; this means that we visualize arrangements of objects in a **line**. But there is a variant in which the objects are arranged along the circumference of a circle. Let us consider that now.

2.3.2 Circular Permutation

Consider an arrangement of 4 objects, a,b,c,d as in Fig. 1. We observe the objects in the clockwise direction. On the circumference there is no preferred origin, and hence the permutations abcd, bcda, cdab, dabc will look exactly alike. So, each linear

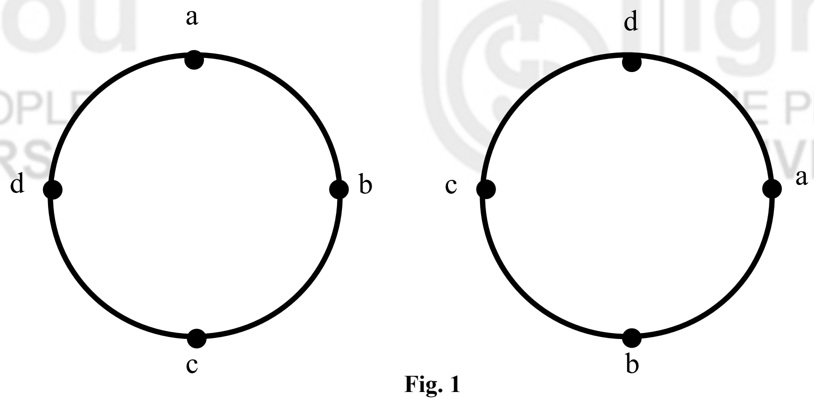

Fig. 1

permutation, when treated as a circular permutation, is repeated 4 times. Similarly, if n objects are placed in a circular arrangement, each linear arrangement is repeated n times. So, if we consider all the n! permutations of n things, each circular permutation will be indistinguishable from the (n−1) others obtained by the process of rotating the objects in the same order. So the number of distinct circular permutations will be n!/n = (n−1)!. Thus, we have shown that **the number of circular permutations of n things, taken all at a time, is (n−1)!.**

Let us consider some examples.

Example 3: In how many distinct ways is it possible to seat eight persons at a round table?

Solution: Clearly we need the number of circular permutations of 8 things. Hence the answer is 7! = 5040.

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Example 4: In the preceding question, what would be the answer if a certain pair among the eight persons

(i) must not sit in adjacent seats?

(ii) must sit in adjacent seats

Solution: To answer (i), let us first solve (ii) from 7! we have to subtract the number of cases in which the pair of persons sit together. If we consider the pair as forming one unit, then we have the circular permutations of 7 objects, which is (7−1)! (Note that this is the answer for (ii).) But even as a unit they can be arranged in two ways. Hence the required answer is 2(6!). Now to answer (i), we must subtract these possibilities from the total number of ways of seating all the people. This is 7!−2(6!) = 3600.

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Example 5: Suppose there are five married couples and they (10 people) are made to sit about a round table so that neither two men nor two women sit together. Find the number of such circular arrangements.

Solution: Five females can be made to sit about a round table in (5−1)! = 4! ways. One male can be seated in between two females. There are five positions, and hence they can be made to sit in 5! ways. By the multiplication principle, the total number of ways of such seating arrangements is 4! × 5! = 2880.

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Example 6: Consider seven people seated about a round table. How many circular arrangements are possible if at least one of them will not have the same neighbours in any two arrangements?

Solution: The two distinct arrangements in Fig. 2 show that each has the same neighbours.

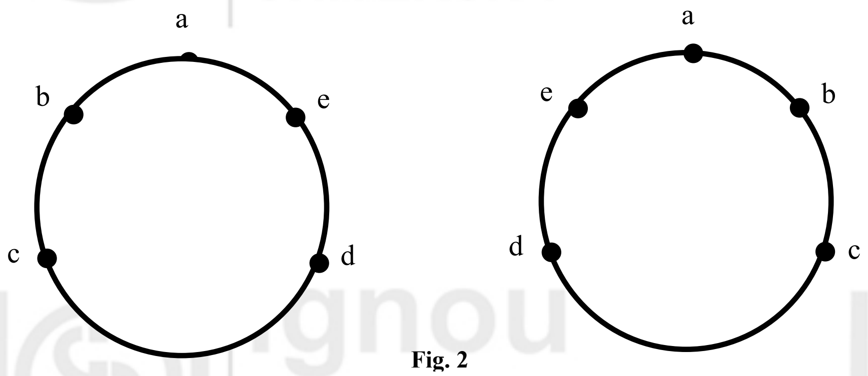

Fig. 2

Hence, the total number of circular arrangements $= (7-1)! \times \frac{1}{2} = 360$.

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You may try the following exercise.

E11) If there are 7 men and 5 women, how many circular arrangements are possible in which women do not sit adjacent to each other?

Permutations apply to ordered arrangement of objects. What happens if order does not matter? Let's see.

2.4 COMBINATIONS

Let's begin by considering a situation where we want to choose a committee of 3 faculty members from a group of seven faculty members. In how many distinct ways can this be done? Here order doesn't matter, because choosing F_1, F_2, F_3 is the same as choosing F_2, F_1, F_3, and so on. (Here F_i denotes the ith faculty member.) So, for every choice of members, to avoid repetition, we have to divide by 3!. Thus, the number would be $\frac{7 \times 6 \times 5}{3!} = \frac{7!}{3!4!}$.

More generally, suppose there are n distinct objects and we want to select r objects, where $r \leq n$, where the order of **the objects in the selection does not matter**. This is called **a combination** of n things taken r at a time. The number of ways of doing this is represented by ${}_nC_r$, ${}^nC_r$, C^n_r, $\binom{n}{r}$ or C (n, r). We will use the notation C(n, r), in conformity with the notation P(n, r) for permutations. We read C(n, r) as 'n choose r' to emphasize the fact that only **choice** is involved but **not ordering**.

In the example that we started the section with, you saw that the number of combinations was 7!/3!4!, i.e., $\frac{P(7,3)}{3!}$. In fact, this relationship between C(n, r) and P(n, r) is true in general. We have the following result.

Theorem 3: The number of combinations of n objects, taken r at a time, where $0 \leq r \leq n$ is given by

$$C(n, r) = \frac{P(n,r)}{r!} = \frac{n!}{(n-r)!r!}.$$

Proof: C(n, r) counts the number of ways of choosing r out of n distinct objects without regard to the order. Any one of these choices is simply a subset of r objects of the set of n objects we have. Such a set can be ordered in r! ways. Thus, to each combination, there corresponds r! permutations. Hence there are r! times as many permutations as there are combinations. Hence, by the multiplication principle, we get

P(n, r) = r! C(n,r)

Therefore, $C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{(n-r)!r!}$.

Using Theorem 3, we can very quickly find out, for instance, how many ways there are of choosing 2 rooms out of 20 rooms offered. This is C(20,2) = $\frac{20!}{18!2!} = 190$.

Now, to find C(20,2), I took a short cut. I cancelled 18! From the number and denominator. In practice, I only needed to calculate $\frac{20 \times 19}{2 \times 1}$. This practice is useful, in general, i.e., we use the identity C(n, r) = $\frac{n(n-1)...r \text{ factors}}{r(r-1)...r \text{ factors}}$ for calculations. In fact, sometimes r is much larger than n – r, in which case we cancel r!. This is also what the following result suggests.

Theorem 4: C(n, r) = C(n, n – r), for $0 \le r \le n$, $n \in \mathbf{N}$.

Proof 1: For every choice of r things from n things, there uniquely corresponds a choice of n – r things from those n objects, which are the unchosen objects. This one-to-one correspondence shows that these numbers must be the same. This proves the theorem.

Proof 2: C(n, r) = $\frac{n!}{(n-r)!r!} = \frac{n!}{(n-r)!(n-\overline{n-r})!} = C(n, n-r)$.

Because of these two theorems we have, for instance,

C(n, n) = C(n, 0) = P(n, 0) = 1. C(n, 1), and = C(n, n–1) = P(n, 1) = n.

The numbers C(n, r) are also called the binomial coefficients as they occur as the coefficients of x^r in the expansion of $(1+x)^n$ in ascending powers of x, as you will see in Sec. 1.5. At this stage, let us consider some examples involving C(n, r).

Example 7: Evaluate C(6, 2), C(7, 4) and C(9, 3).

Solution: C(6, 2) = $\frac{6.5}{2.1} = 15, C(7,4) = \frac{7.6.5}{3.2.1} = 35, \text{and } C(9,3) = \frac{9.8.7}{3.2.1} = 84.$

Example 8: Find the number of distinct sets of 5 cards that can be dealt from a deck of 52 cards.

Solution: The order in which the cards are dealt is not important. So, the required number is C(52, 5) = $\frac{52!}{5!47!} = \frac{52 \times 51 \times 50 \times 49 \times 48}{5 \times 4 \times 3 \times 2 \times 1} = 2{,}598{,}960.$

Example 9: Suppose a valid computer password consists of 8 characters, the first of which is the digit 1, 3 or 5. The rest of the 7 characters are either English alphabets or a digit. How many different passwords are possible?

Solution: Firstly, the initial character can be chosen in C(3, 1) ways. Now, there are 26 alphabets and 10 digits to choose the rest of the characters from, and repetition is allowed. So, the total number of possibilities for these characters is $(26+10)^7$.

Therefore, by the multiplication principle, the number of passwords possible are $C(3, 1).36^7$.

Here are some exercises now.

E12) At a certain office, a committee consisting of one male and one female worker is to be constituted from among 12 men and 15 women workers. In how many distinct ways can this be done?

E13) In how many ways can a prize winner choose any 3 CDs from the 'Ten Best' list?

E14) How many different 7-person committees can be formed, each containing 3 women and 4 men, from a set of 20 women and 30 men?

So far we have been considering combinations of distinct objects. Let us now look at combinations in which repetitions are allowed. We start with considering the following situations.

Suppose five friends stop at a sweet shop where each of them has one of the following: a samosa, a dosa, and a vada. The order of consumption does not matter. How many different purchases are possible?

Let s, t, and d represent samosa, dosa, vada, respectively. In the following table we have listed some possible ways of purchasing these. For instance, the second row represents the possibility that all 5 friends order only dosas.

s	d	**v**
x	x	xxx
xxx	xxxx	xx

These orders can also be represented by x's and $|$'s. For instance, the first row can be written as x$|$x$|$xxx. So, any order will consist of five x's and two $|$'s.

Conversely, any sequence of five x's and two $|$'s represents an order. So, there is a 1-to-1 correspondence between the orders placed and sequences of five x's and two $|$'s. But the number of such sequences is just the number of distinct ways of placing 2$|$'s in 7 possible places. This is C(7,2).

More generally, if we wish to select with repetition, r out of n distinct objects, we are considering all arrangements of r of one kind (say x's) and n – 1 of the other kind (say $|$'s) (because (n – 1) $|$'s are needed to separate n types).

The following result gives us the total number of such possibilities.

Theorem 5: Let n and r be natural numbers. Then the number of solutions in natural numbers, to the equation $x_1 + x_2 + \ldots + x_n = r$, is $C(n + r - 1,r)$. Equivalently, the number of ways to choose r objects from a collection of n objects, with repetition allowed, is C(n + r – 1,r).

Proof: Any string will consist of r objects and n – 1 bars, to denote the n different categories in which these objects can fall. So, it will be a string of length n + r – 1, containing exactly r stars and n – 1 bars. The total number of such strings is the number of ways we can position (n – 1) bars in r different places. This is C(n + r – 1,r).

Now we demonstrate how such strings correspond to solution of the equation $x_1 + \ldots + x_n = r$.

n – 1 bars in the string divide the string into n substrings of stars. The number of stars in these n substrings are the values of $x_1, x_2, \ldots, x_n$. Since there are r stars altogether, the sum is r. Therefore, is a one-to-one correspondence between the strings and the solutions, and the theorem is proved.

Let us consider examples of the use of this result.

Example 10: In how many ways can a prize winner choose three books from a list of 10 best sellers, if repeats are allowed?

Solution: Here, note that a person can choose all three books to be the same title. Applying Theorem 5, the solution is C(10 + 3 – 1, 3) = C(12, 3) = 220.

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Example 11: Determine the number of integer solutions to the equation
$x_1 + x_2 + x_3 + x_4 = 7$, where $x_i \geq 0$ for all i = 1,2,3,4.

Solution: The solution of the equation corresponds to a selection, with repetition, of size 7 from a collection of size 4. Hence, there are C(4 + 7 – 1, 7) = 120 solutions. (n = 4, r = 7 in Theorem 5.)

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So, from this sub-section, we see the equivalence of the following:

(a) The number of integer solutions of the equation $x_1 + x_2 + \ldots + x_n = r$, $x_i \geq 0$, $1 \leq i \leq n$.

(b) The number of selections, with repetition, of size r from a collection of size n.

(c) The number of ways r identical objects can be distributed among n distinct containers.

Why don't you try some exercises now?

E15) A student in a college hostel is allowed four fruits per day. There are 6 different types of fruits from which she can choose what she wants. For how many days can a student make a different selection?

E16) An urn contains 15 balls, 8 of which are red and 7 are black. In how many ways can:

i) 5 balls be chosen so that all 5 are red?

ii) 7 balls be chosen so that at least 5 are red?

In this section we have considered choosing r objects, with repetition, out of n objects, regardless of order. What happens when order comes into the picture? Let's consider an example.

Example 12: A box contains 3 red, 3 blue and 4 white socks. In how many ways can 8 socks be pulled out of the box, one at a time, if order is important?

Solution: Let us first see what happens if order isn't important. In this case we count the number of solutions of $r+b+w = 8$, $0 \le r, b \le 3$, $0 \le w \le 4$. To apply Theorem 5, we write $x = 3 - r$, $y = 3 - b$, $z = 4 - 10$.

Then we have $x+y+z = 10 - 8 = 2$, and the number of solutions this has is $C(3+2-1,2) = 6$.

These 6 solutions are (1, 0, 1) (0, 1, 1), (1, 1, 0), (2, 0, 0), (0, 2, 0), (0, 0, 2). So, the corresponding solutions for (r, b, w) are

(3, 3, 2), (2, 3, 3), (3, 2, 3), (3, 1, 4), (2, 2, 4), (1, 3, 4).

Now, we consider order. From Theorem 2 we know that the number of ways of pulling out 3 red, 3 blue and 2 white socks in some order is $\frac{8!}{3!3!2!}$. This number would be the same if you had 2 red, 3 blue and 3 white socks, etc. By this reasoning and considering all different orderings, the number of possibilities is

$$3\left(\frac{8!}{3!3!2!}\right)+2\left(\frac{8!}{3!1!4!}\right)+\frac{8!}{2!2!4!}=3220.$$

* * *

What we see, via Example 13, is that if we want to find the number of possibilities wherein order matters and repetition is allowed them:

Step 1: Find the possibilities when order doesn't matter, using Theorem 5;

Step 2: Use Theorem 2, to find the possibilities for each solution obtained in Step 1.

Why don't you try and exercise now?

E17) How many 6-letter words, not necessarily meaningful can be formed from the letters of CARACAS?

Let us now consider why C(n,r) shows up as the coefficients in the binomial expansions.

2.5 BINOMIAL COEFFICIENTS

You must be familiar with expressions like a+b, p+q, x+y, all consisting of two terms. This is why they are called binomials. You also know that a **binomial expansion** refers to the expansion of a positive integral power of such a binomial. For instance, $(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$ is a binomial expansion. Consider coefficients 1, 5, 10, 10, 5, 1 of this expansion. In particular, let us consider the coefficient 10, of a^3b^2 in this expansion. We can get this term by selecting a from 3 of the binomials and b from the remaining 2 binomials in the product (a+b) (a+b) (a+b) (a+b) (a+b). Now, a can be chosen in C(5, 3) ways, i.e., 10 ways. This is the way each coefficient arises in the expansion.

The same argument can be extended to get the coefficients of $a^r b^{n-r}$ in the expansion of $(a+b)^n$. From the n factors in $(a+b)^n$, we have to select r for a and the remaining (n – r) for b. This can be done in C(n, r) ways. Thus, the coefficient of $a^r b^{n-r}$ in the expansion of $(a+b)^n$ is C(n, r).

In view of the fact that C(n, r) = C(n, n – r), the coefficients of $a^r b^{n-r}$ and $a^{n-r} b^r$ will be the same. r can only take the values 0, 1, 2, …, n. We also see that C(n, 0) = C(n, n) = 1 are the coefficients of a^n and b^n. Hence we have established the binomial expansion.

$(a+b)^n = a^n + C(n, 1)\, a^{n-1}b + C(n, 2)a^{n-2}b^2 + \ldots + C(n, r)\, a^{n-r}b^r + \ldots + b^n.$

In analogy with 'binomial', which is a sum of two symbols, we have 'multinomial' which is a sum of two or more (though finite) distinct symbols. Multinomial expansion refers to the expansion of a positive integral power of a multinomial. Specifically we will consider the expansion of $(a_1 + a_2 + \ldots + a_m)^n$. For the expansion we can use the same technique as we use for the binomial expansion. We consider the nth power of the multinomial as the product of n factors, each of which is the same multinomial. Every term in the expansion can be obtained by picking one symbol from each factor and multiplying them. Clearly, any term will be of the form $a_1^{r_1} a_2^{r_2} \ldots a_m^{r_m}$ where $r_1, r_2, \ldots$, are non-negative integers such that $r_1+r_2+\ldots+ r_m = n$. Such a term is obtained by selecting a_1 from r_1 factors, a_2 from r_2 factors **from among the remaining ($n-r_1$) factors**, and so on. This can be done in

$C(n,r_1).\ C(n-r_1,r_2).C(n-r_1-r_2, r_3)\ldots C(n-r_1-r_2-\ldots-r_{m-1}, r_m)$ ways.

If you simplify this expression, it will reduce to $\dfrac{n!}{r_1!r_2!\ldots r_m!}$.

So, we see that the **multinomial expansion** is

$(a_1+a_2+\ldots+a_m)^n = \Sigma\ \dfrac{n!}{r_1!r_2!\ldots r_m!}\ a_1^{r_1} a_2^{r_2} \ldots a_m^{r_m}$

where the summation is over all non-negative integers $r_1, r_2, \ldots, r_m$ adding to n.

The coefficient of $a_1^{r_1} a_2^{r_2} \ldots a_m^{r_m}$ in the expansion of $(a_1+a_2+\ldots+ a_m)^n$ is $\dfrac{n!}{r_1!r_2!\ldots r_m!}$, and is called a **multinomial coefficient**, in analogy with the binomial coefficient. We represent this by $C(n; r_1, r_2, \ldots, r_m)$. This is also represented by many authors as $\left[\dfrac{n}{r_1, r_2, \ldots .r_m}\right]$.

For instance, the coefficient of $x^2y^2z^2t^2u^2$ in the expansion of $(x + y + z + t + u)^{10}$ is $C(10; 2, 2, 2, 2, 2) = 10!/(2!)^5$.

Let us see an example involving such coefficients.

Example 13: What is the sum of the coefficients of all the terms in the expansion of $(a+b+c)^7$?

Solution: The required answer is $\Sigma \dfrac{7!}{r!s!t!}$, where the summation is over all non-negative integers r, s, t adding to n. But it is also the value of $\Sigma \dfrac{7!}{r!s!t!} a^r b^s c^t$ for a = b = c = 1.

So the answer is $(1 + 1 + 1)^7 = 3^7$.

* * *

This short detour was just to give you an idea of the way in which the Cs and Ps can be extended. Let us now consider some identities involving the binomial coefficients. We first consider Pascal's formula.

Theorem 6 (Pascal's formula): For all positive integers n and all r such that $1 \le r \le n$, $C(n + 1, r) = C(n, r) + C(n, r - 1)$.

Proof 1: The left hand side of the identity represents the number of ways of choosing r things out of (n+1) distinct things. Suppose we select an object from the (n+1) things and mark it. Then the number of combinations in which the marked thing is absent is C(n, r), as we then choose r things out of the unmarked n things. The number of combinations in which the marked thing is present is C(n, r–1), as we have to choose (r – 1), things out of the unmarked things, and attach the marked thing to it to make r things. Pascal's formula now follows from the fact that the sum of the last two numbers mentioned must be equal to C(n+1, r).

Proof 2: $C(n, r) + C(n, r - 1) = \dfrac{n!}{(n-r)!r!} + \dfrac{n!}{(n-r+1)!(r-1)!}$

$$= \frac{n!}{r!(n+1-r)!}(n - r + 1 + r) = C(n+1, r).$$

Pascal's formula gives us a recursive way to calculate the binomial coefficients, since it tells us the value of C(n, r) in terms of binomial coefficients with a smaller value of n. Note that we use the fact that C(n, 0) = 1 for all $n \ge 0$ to start the recursion, since Theorem 6 only applies for $1 \le r \le n$. This recursive approach allows us to form Pascal's triangle, the display of the binomial coefficients shown in Fig.4.

The nth row of Pascal's triangle gives the binomial coefficients C(n, r) as r goes from 0 (at the left) to n (at the right); the top row is Row D. This consists of just the number 1, for the case n = 0. The left and right borders are all 1's, reflecting the fact that C(n, 0) = C(n, n) = 1 for all n. Each entry in the interior of the Pascal's triangle is the sum of the two entries immediately above it to the left and right. We call this property the **Pascal property**. For example, each 15 in Row 6 (remember that we are starting the count of rows with 0) is the sum of the 10 and the 5 immediately above it.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1
1 11 55 165 330 462 462 330 165 55 11 1

Fig. 3: Pascal's triangle

The diagonals of Pascal's triangle are also interesting. The diagonal parallel to the left edge but moved one unit to the right reads (from the top down) 1, 2, 3, 4, 5,…, reflecting the fact that C(n, 1) = n for $n \ge 1$. The next diagonal to the right, reading 1, 3, 6, 10, 15,…, reflects the fact that differences increase by 1 as we move down the diagonal.

Let us now consider some identities involving binomial coefficients.

Identity 1: $C(n, 0) + C(n, 1) + C(n, 2) + \ldots + C(n, n-1) + C(n, n) = 2^n$

By setting a = b = 1 in the binomial expansion of $(a+b)^n$, we get this identity. In the context of sets, it tells us the number of distinct subset that can be formed from a set with n elements. Note that the number of subsets containing precisely r elements is C(n, r). Hence the total number of subsets is $\sum_{r=0}^{n} C(n,r) = 2^n$, by the identity. So, this identity tells us that **the number of distinct subsets of a set with n elements is 2^n.**

Identity 2: $C(n, 0) - C(n, 1) + C(n, 2) - \ldots + (-1)^n C(n, n) = 0$.

We get this by setting a = 1, b = –1 in the expansion of $(a+b)^n$.

Now, adding the two identities, we get

$$2 \sum_{r \text{ even}} C(n, r) = 2^n, \text{ i.e., } \sum_{r \text{ even}} C(n, r) = 2^{n-1}$$

Similarly subtracting the second identity from the first leads us to the equation

$$\sum_{r \text{ odd}} C(n, r) = 2^{n-1}.$$

These two equations tell us that the number of subsets of a set of n elements with an even number of elements is equal to the number of subsets with an odd number of elements, both being 2^{n-1}.

Why don't you try to prove some identities now?

E18) Show that $C(n, m) C(m, k) = C(n, k) C(n-k, m-k)$, $1 \le k \le m \le n$.

E19) Prove that $C(k, k) + C(k + 1, k) + C(k + 2, k) + \ldots + C(n, k) = C(n+1, k+1)$ for all natural numbers $k \le n$.

Before ending this section, we just mention another extension of the definition of binomial coefficients. So far, we have defined C(n, r) for $n \ge r \ge 0$. We can extend this definition for any real number x, and any non-negative integer k, by

$$C(x, k) = \frac{x(x-1)\ldots(x-k+1)}{k!}.$$

This definition coincides with that of C(n, k), when n is a non-negative integer.

So far, in this unit, we have considered various ways of counting different kinds of arrangements. These methods are, not surprisingly, helpful in finding the probability of an event. We shall now discuss this.

2.6 COMBINATORIAL PROBABILITY

Historically, counting problems have been closely associated with probability. The probability of getting at least 6 heads on 10 flips of a fair coin, the probability of finding a defective bulb in a sample of 25 bulbs if 5 percent of the bulbs from which the sample was drawn are defective — all these probabilities are essentially counting problems. In fact, Pascal's triangle (Fig. 4) was developed by Pascal around 1650 while analysing some gambling probabilities.

Let us start by recalling some basic facts about probability. An **experiment** is a clearly defined procedure that produces one of a given set of outcomes. The set of all outcomes is called **the sample space** of the experiment.

For example, the experiment could be checking the weather to see if it is raining or not on a particular day. The sample space here would be {raining, not raining}.

Given an experiment, we can often associate more than one sample space with it. For instance, suppose the experiment is the tossing of two coins.

i) If the observer wants to record the number of tails observed as the outcomes, the sample space is {0, 1, 2}.

ii) If the outcomes are the sequence of heads and tails observed, then the sample space is {HH, HT, TH, TT}.

A subset of the sample space of an experiment is called an **event**. For example, for an experiment consisting of tossing 2 coins, with sample space {HH, HT, TH, TT}, the event that two heads do **not** show up is the subset {HT, TH, TT}.

Suppose X is a sample space of an experiment with N out comes. Then, the events are all the 2^N subsets of X. The empty set ϕ is called the **impossible event**, and the set X itself is called the **sure event**.

Now, for the purpose of this course, we will assume that all the outcomes of an experiment are **equally likely**, that is, there is nothing to prefer one case over the other. For example, in the experiment of coin tossing, we assume that the coin is unbiased. This means that 'head' and 'tail' are equally likely in a toss. The toss itself is considered a random mechanism ensuring 'equally likely' outcomes. Of course, there are coins that are 'loaded', which means that one side of the coin may be heavier than the other. But such coins are excluded from our discussion. Also, in our discussions we shall always assume that our **sample space is finite**.

Given this background, we have the following definition.

Definition: Then the **probability of the event A**, represented by **P(A)**, is $\frac{n(A)}{n(X)}$.

We represent the number of elements of a finite set A, i.e., the **cardinality** of A, by n(A) or $|A|$.

For instance, the probability that a card selected from a desk of 52 cards is a spade is $\frac{13}{52}$, because A is the set of 13 spades in the deck.

From the definition, we get the following statements:

i) As $n(\phi) = 0$, it follows that $P(\phi) = 0$.

ii) By definition, $P(X) = \frac{n(X)}{n(X)} = 1$.

iii) If A and B are two events, then $n(A \cup B) = n(A) + n(B) - n(A \cap B)$. Therefore, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

iv) (**Addition Theorem in Probability**) : If A and B are two mutually exclusive events, then the probability of their union is the sum of the probabilities of A and B. i.e., if $A \cap B = \phi$, then $\mathbf{P(A \cup B) = P(A) + P(B)}$.

[This is a consequence of (i) and (iii) above.]

v) Suppose A is an event. Then the probability of A^c (also denoted by A'), the event complementary to A, or the event 'not A' is $1 - P(A)$, i.e.,

$\mathbf{P(A^c) = 1 - P(A)}$.

The reason is that the events A and A^c are mutually exclusive and exhaustive, i.e., $A \cup A^c = X$ and $P(A) + P(A^c) = 1$.

vi) (The generalised addition theorem) : If the events $A_1, A_2, \ldots, A_m$ are pairwise disjoint (i.e., mutually exclusive), then $\mathbf{P(\underset{i}{U} A_i) = \sum_i P(A_i)}$.

Let us consider some examples from combinatorial probability.

Example 14: A die is rolled once. What are the probabilities of the following events?

(i) getting an even number,

(ii) getting at least 2,

(iii) getting at most 2,

(iv) getting at least 10.

Solution: If we call the events A, B, C and D, then we have X = {1, 2, 3, 4, 5, 6}, A = {2,4,6}, B = {2,3,4,5,6}, C = {1,2}, and D = ϕ.

Hence, P(A) = 3/6, P(B) = 5/6, P(C) = 2/6, P(D) = 0.

* * *

Example 15: A coin is tossed n times. What is the probability of getting exactly r heads?

Solution: If H and T represent head and tail, respectively, then X consists of sequences of length n that can be formed using only the letters H and T. Therefore, $n(X) = 2^n$. The event A consists of those sequences in which there are precisely r Hs. So, n(A) = C(n, r). Hence, the required probability is $C(n, r)/2^n$.

* * *

Example 16: Two dice, one red and one white, are rolled. What is the probability that the white die turns up a smaller number than the red die?

Solution: If the number on the red die is x and that on the white die is y, then X consists of the 36 pairs (x, y), where x and y can be any integer from {1, 2, 3, 4, 5, 6}.

For the event A, we need $x < y$. For x = 1, 2, 3, 4, 5, y can be x + 1, x+2,…, 6, i.e., $6 - x$ in number. Thus, by the addition principle,

$$n(A) = \sum_{x=1}^{5} (6 - x) = 5 + 4 + 3 + 2 + 1 = 15.$$

Hence, P(A) = 15/36 = 5/12.

* * *

Example 17: If a five-digit number is chosen at random, what is the probability that the product of the digits is 20?

Solution: If X is the collection of all 5-digit numbers, then $n(X) = 9.10^4 = 90000$. Now, 20 can be factored in only two ways, viz., 1.1.1.4.5 and 1.1.2.2.5, as the product of five factors. Of course, these factors can be permuted to give all possible cases for A. The numbers 5, 4, 1, 1, 1 can be permuted in 5!/1!1!3! = 20 ways, and the numbers 5, 2, 2, 1, 1 can be permuted in 5!/1!2!2! = 30 ways.

So, n(A) = 20 + 30 = 50.

Hence, P(A) = 50/90000 = 1/1800.

* * *

Example 18: Suppose A and B are mutually exclusive events such that P(A) = 0.3 and P(B) = 0.4. What is the probability that

i) A does not occur?

ii) A or B occurs?

iii) Either A or B does not occur?

Solution:

i) This is $P(A^c) = 0.7$.

ii) This is $P(A \cup B) = 0.7$.

iii) This is $P(A^c \cup B^c) = P[(A \cap B)^c] = P(\phi^c) = P(X) = 1$

* * *

Try some exercises now.

E20) A,B,C and D are four candidates for a chairperson's post. Suppose that A is twice as likely to be elected as B, B is thrice as likely as C, and C and D are equally likely to be elected. What is the probability of election of each candidate?

E21) In a ten-question true-false exam, a student must achieve six correct answers to pass. If she selects her answers randomly, what is the probability that she will pass?

There are several other methods for solving combinatorial problems. These will be taken up in the next two units. Let us now summarise what we have covered in this unit.

2.7 SUMMARY

In this unit we have discussed some counting techniques. Specifically, we have covered the following points.

1. The multiplication and addition principles for counting the number of ways in which a task can be completed.
2. What a permutation is, the derivation of the formula $P(n,r) = \frac{n!}{(n-r)!}$, and its application for solving problems.
3. The number of distinct arrangement of n objects of which m_1 are of Type 1, m_2 are of Type 2,…, m_k are of Type k, where $m_1, m_2 +\ldots, m_k = n$, is $P(n; m_1, m_2,\ldots,m_k) = \frac{n!}{m_1!m_2!\ldots m_k!}$.
4. What a circular permutation is, and that the number of such permutations of n objects, taken all at a time, is (n–1)!
5. What a combination is, the derivation of the formula

$C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{(n-r)r!}$, and its application for solving problems.

6. The proof and applications of the fact that the number of ways of choosing r objects from a collection of n objects, with repetition allowed, is C(n+r–1,r).

7. Why C(n,r) is called a binomial coefficients, and its analogue for multinomials.

8. Some identities involving C(n,r), including Pascal's formula C(n+1,r) = C(n,r) + C(n,r–1).

9. The use of counting techniques for finding some discrete probabilities.

2.8 SOLUTIONS/ ANSWERS

E1) For instance, both principles are used to find the number of ways in which 17 files are stored if there are 3 storage locations of 1000 K each and 10 files are of 100 K, 5 of 200 K 2 of 500 K.

E2) Here we apply the multiplication principle. Each letter has 10 possibilities. Therefore, the total number of words is 10^4.

E3) Suppose we number the men as 1, 2, 3,…, n. Then the first man can be paired with any of the n women, the second can be paired with any from the remaining (n – 1) women, and so on. Hence, the number of ways of pairing is n(n – 1)… 1 = n!.

E4) By the multiplication principle, the number of integers between 100 and 999 with all digits even is 4.5.5 = 100 (Note that the first digit cannot be zero, but the second and third digits can be 0.)

E5) For a number to be odd the last digit should be odd. So, the last position can be filled up in 5 ways. If the middle position is filled up by 0, then the first position can be filled up in 8 ways. Thus the number of odd numbers with 0 in the middle position and all digits distinct is 40, by the multiplication principle.

If the middle position is filled up by a digit other than 0, then this can be done in 8 ways. Then the first position can be filled up in 7 ways. So, the number of odd numbers with all digits distinct with the middle digit not zero is 5.8.7 = 280.

Thus, by the addition principle the answer is 40 + 280 = 320.

E6) $(m + n)! = (m + n)(m + n - 1) \ldots (m + 1)\, m!$.
$\Rightarrow (m + n)! - m! = \geq m^n + n! \geq m!\,[n! + m^n - 1]$
$\Rightarrow (m + n)! - m! - n! \geq n!\,(m! - 1) + m!\,(m^n - 1) \geq 0$.

E7) Without repetitions, the number is P(6, 3). For the number to be less than 400, the leftmost digit can only be 2 or 3. The rest of the digits can be filled in P(5, 2) ways. So, the total number of numbers less than 400 will be 2P(5, 2). Similarly, the total number of even numbers is 3P(5, 2).

Note: That the addition principle has been used in both cases.

E8) A ranking is an ordering of the n candidates. This can be done in P(n,n) = n! ways. The total number of rankings in which Sheela is in 2^{nd} place in P(n – 1, n – 1) = (n – 1)!

E9) In the word 'ASSESSES', we have A once, E twice, and S five times. Thus the number of permutations is 8!/1!2!5! = 168.
In the word 'PATTIVEERANPATTI', R, N and V occur once, P, E and I occur twice, A thrice and T four times. Thus the required number of permutations is 16!/1!1!1!2!2!2!3!4! = 9.10.

E10) By the multiplication principle, the answer is 26.25.24 if the letters cannot be repeated, and 26.26.26 if the letters can be repeated.

E11) The seven men can be seated first. This can be done in 6! ways. The women can sit in between two men. There are seven such places. So, the women can sit in P(7,5) ways. Hence the answer is 6! × P(7,5).

E12) This can be done in C(12, 1).C(15,1) ways, i.e., 180 ways.

E13) This can be done in C(10, 3) ways, i.e., 120 ways.

E14) The total number of possibilities is C(20,3).C(30,4) = 31,241,700.

E15) Applying Theorem 5, we get C(9, 4) = 126 days.

E16) i) Be careful! This is not an application of Theorem 5. This is only the number of ways of choosing 5 balls out of 8 balls, i.e. C(8, 5).

ii) First pick 5 red balls, in C(8,5) ways. Then pick the remaining 2 arbitrarily. These 2 can be chosen in C(2+2–1, 2) = 3 ways. So, the total number of ways is C(8, 5) × 3.

E17) We have 2Cs, 3As, 1R and 1S. If order is not a concern, we consider the solutions of

$c+a+r+s = 6, 0 \le c \le 2, \ 0 \le a \le 3, \ 0 \le r, s \le 1.$

We convert this to the equivalent problem

$x+y+z+t = 1$, where $x = 2 - c$, $y = 3 - a$, $r = 1 - z$, $s = 1 - t$,

The number of solutions of this is C(4 + 1 – 1, 1) = 4.

There are (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0) and (0, 0, 0, 1).

The corresponding solutions in (c, a, r, s) are (1, 3, 1, 1), (2, 2, 1, 1), (2, 3, 0, 1), (2, 3, 1, 0).

Now order becomes important to us. Applying Theorem 2, the required number is

$$\frac{6!}{1!3!1!1!} + \frac{6!}{2!2!1!1!} + 2\left(\frac{6!}{2!3!1!0!}\right) = 420.$$

E18) The left side counts the ways to select a group of m people chosen from a set of n people and then select a subset of k leaders, say, of this group of m. This can also be done by selecting the subset of k leaders from the set of n people first, and then selecting the remaining m – k members of the group from the remaining n – k people. The number of ways in which this can be done is given on the right hand side. Therefore, the identity.

You can also prove this algebraically.

E19) One can prove this by induction on the variable n. The base case is trivial, since if n = 0, then k = 0 as well, and the equation reduces to C(0, 0) = C(1, 1), which is true. The induction step is proved by Pascal's formula and the induction hypothesis.

E20) The relative weightages of A, B, C and D are 6.3, 1, 1, respectively. So, P(A) = $\frac{6}{11}, P(B) = \frac{3}{11}, P(C) = \frac{1}{11} = P(D)$.

E21) The answer is same as the probability of getting at least 6 heads in 10 tosses of a true coin. Hence, the answer is

$C(10, 6)/2^{10} + C(10, 7)/2^{10} + C(10, 8)/2^{10} + C(10, 9)/2^{10} + C(10, 10)/2^{10}$

$= (210 + 120 + 45 + 10 + 1)/1024 = 193/512.$