Module-3

Formation of PDE's by elimination of arbitrary constants and functions. Solution of non- homogeneous PDE by direct integration. Homogeneous PDEs involving derivative with respect to one independent variable only. Solution of Lagrange's linear PDE. Derivation of one-dimensional heat equation and wave equation.

Partial Differential equations:

Notations: If $z = f(x,\ y)$, then $p = \frac{\partial z}{\partial x} = z_x$. $q = \frac{\partial z}{\partial y} = z_y$.

$r = \frac{\partial^2 z}{\partial x^2} = z_{xx} = p_x$. $s = \frac{\partial^2 z}{\partial x \partial y} = z_{xy} = p_y = q_x$. $t = \frac{\partial^2 z}{\partial y^2} = z_{yy} = q_y$.

Formation of PDE by eliminating the arbitrary constants.

Examples:

1. Form the PDE by eliminating the arbitrary constants of the equation $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$.

Solution: $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ ……………(i)

Differentiating (i) partially with respect to x and y , we get

$2p = \frac{2x}{a^2}$ and $2q = \frac{2y}{b^2}$ $\Longrightarrow$ $\frac{x^2}{a^2} = px$ and $\frac{y^2}{b^2} = qy$

Substituting these values in (i), we get

$$\boxed{2z = px + qy}$$

2. Form the PDE by eliminating the arbitrary constants of the equation $ax^2 + by^2 + z^2 = 1$.

Solution: $ax^2 + by^2 + z^2 = 1$

$\Rightarrow$ $z^2 = 1 - ax^2 - by^2$ ……………………….. (i)

Differentiating (i) partially with respect to x and y , we get

$2zp = -2ax$ and $2zq = -2by$ $\Rightarrow$ $zpx = -ax^2$ and $zqy = -by^2$.

Substituting these values in (i) we get ,

$$\boxed{z^2 = 1 + zpx + zqy}$$

3. Find the differential equation of all planes which are at a constant distance a from the origin.

Solution: The equation of the plane in normal form is $lx + my + nz = a$ …………(i)

Differentiating (i) partially with respect to x and y , we get

$l + np = 0$ and $m + nq = 0$ $\Rightarrow$ $l = -np$ and $m = -nq$.

Substituting these values in (i) , we get $nz - npx - nqy = a$.

But $l^2 + m^2 + n^2 = 1$ $\Longrightarrow$ $n^2p^2 + n^2q^2 + n^2 = 1$ $\Longrightarrow n^2 = \frac{1}{1+p^2+q^2}$ $\Rightarrow$ $n = \frac{1}{\sqrt{1+p^2+q^2}}$

$\therefore$ $z - px - qy = \frac{a}{n}$, or $\boxed{z - px - qy = a\sqrt{1 + p^2 + q^2}}$

4. Form the PDE by eliminating the arbitrary constants of the equation $z = xy + y\sqrt{x^2 - a^2} + b$.

Solution: $z = xy + y\sqrt{x^2 - a^2} + b$ ....(i)

Differentiating (i) partially with respect to x and y , we get

$$p = y + \frac{xy}{\sqrt{x^2-a^2}} \quad and \quad q = x + \sqrt{x^2 - a^2} \quad \Longrightarrow \quad \sqrt{x^2 - a^2} = q - x$$

$$\therefore \quad p = y + \frac{xy}{q-x} \quad \Longrightarrow \quad pq - px = qy \quad or \quad \boxed{pq = px + qy}$$

Formation of PDE by eliminating the arbitrary functions:

Type 1: $z = f(ax + by) + g(cx + dy)$

$$r = a^2 f'' + c^2 g'' \quad , \quad s = abf'' + cdg'' \quad and \quad t = b^2 f'' + d^2 g'' .$$

Eliminate f'' and g'' using above three equations.

Type 2: If $u = f(v)$ or $F(u, \ v) = 0$ where u and v are functions of (x, y, z)

Then the PDE is $u_x v_y = v_x u_y$.

Because $u_x = v_x f'(v)$ and $u_y = v_y f'(v) \Rightarrow \frac{u_x}{u_y} = \frac{v_x}{v_y} \Longrightarrow u_x v_y = v_x u_y$

Similarly $F_u u_x + F_v v_x = 0$ and $F_u u_y + F_v v_y = 0 \Longrightarrow F_u u_x = -F_v v_x$ and $F_u u_y = -F_v v_y$

$$\Longrightarrow \frac{F_u u_x}{F_u u_y} = \frac{-F_v v_x}{-F_v v_y} \Rightarrow \frac{u_x}{u_y} = \frac{v_x}{v_y} \Longrightarrow u_x v_y = v_x u_y \quad .$$

Examples:

1. Form the PDE by eliminating the arbitrary functions of the equation $z = f(y + 2x) + g(y - 3x)$.

Solution: $z = f(y + 2x) + g(y - 3x)$

$$\Longrightarrow r = 4f'' + 9g'' \quad , \quad s = 2f'' - 3g'' \quad and \quad t = f'' + g''.$$

$$\Longrightarrow r + s = 6(f'' + g'') \quad or \quad \boxed{r + s = 6t}$$

2. Form the PDE by eliminating the arbitrary functions $z = f(2x + 3y) + g(x + 2y)$.

Solution: $z = f(2x + 3y) + g(x + 2y)$

$$\Longrightarrow r = 4f'' + g'' \quad , \quad s = 6f'' + 2g'' \quad and \quad t = 9f'' + 4g'' \quad .$$

$$s - 2r = -2f'' \quad , \; and \quad t - 2s = -3f'' \quad \Longrightarrow \quad 3(s - 2r) = 2(t - 2s)$$

$$\therefore \quad \boxed{2t + 6r = 7s.}$$

3. Form the PDE by eliminating the arbitrary functions $z = f(x + ct) + g(x - ct)$.

Solution: $z = f(x + ct) + g(x - ct)$

$$\Longrightarrow \quad z_{xx} = f'' + g'' \quad and \;\; z_{tt} = c^2 f'' + c^2 g'' \;\; \Longrightarrow z_{tt} = c^2 z_{xx}$$

4. Form the PDE by eliminating the arbitrary function of the equation $z = y^2 + 2f(\frac{1}{x} + \log y)$.

Solution: $z = y^2 + 2f(\frac{1}{x} + \log y) \quad \Longrightarrow \quad \frac{z-y^2}{2} = f(\frac{1}{x} + \log y)$

Let $u = \frac{z-y^2}{2}$ and $v = \frac{1}{x} + \log y$

then $u_x = \frac{p}{2}$, $v_x = -\frac{1}{x^2}$

and $\quad u_y = \frac{q-2y}{2}$, $\quad v_y = \frac{1}{y}$

$\therefore PDE \ is \ u_x v_y = v_x u_y$, $\quad ie$, $\frac{p}{2y} = -\frac{(q-2y)}{2x^2}$, $\quad \Rightarrow \quad px^2 = -qy + 2y^2$

or $\quad \boxed{px^2 + qy = 2y^2}$

5. Form the PDE by eliminating the arbitrary function of the equation $z = f\left(\frac{xy}{z}\right)$.

Solution: Let $\quad u = z \quad$ and $\quad v = \frac{xy}{z}$. Then $\quad u_x = p \quad , \quad v_x = \frac{yz - xyp}{z^2}$

and $\quad u_y = q \quad , \quad v_y = \frac{xz - xyq}{z^2}$

$\therefore PDE \ is \ u_x v_y = v_x u_y$ ie $\quad xzp - xypq = yzq - xypq \quad$ or $\quad \boxed{px = qy}$

6. Form the PDE by eliminating the arbitrary function of the equation $F(xy + z^2, \ x + y + z) = 0$.

Solution: Let $\quad u = xy + z^2 \quad$ and $\quad v = x + y + z$. Then $\quad u_x = y + 2zp \quad , \quad v_x = 1 + p$

and $\quad u_y = x + 2zq \quad , \quad v_y = 1 + q$

$\therefore PDE \ is \ u_x v_y = v_x u_y$ ie $\quad (y + 2zp)(1 + q) = (x + 2zq)(1 + p)$

$\Rightarrow \ y + 2zp + qy + 2zpq = x + 2zq + px + 2zpq$

$\Rightarrow \ \boxed{(2z - x)p + (y - 2z)q = x - y}$

7. Form the PDE by eliminating the arbitrary function of the equation $F(x + y + z, \ x^2 + y^2 + z^2) = 0$.

Solution: Let $\quad u = x + y + z \quad$ and $\quad v = x^2 + y^2 + z^2$. Then $\quad u_x = 1 + p, \quad v_x = 2x + 2zp$

and $\quad u_y = 1 + q, \quad v_y = 2y + 2zq$

$\therefore PDE \ is \ u_x v_y = v_x u_y$ ie $\quad (1 + p)2(y + zq) = (1 + q)2(x + zp)$

$\Rightarrow \ y + zq + py + zpq = x + zp + qx + zpq$

$\Rightarrow \ \boxed{(y - z)p + (z - x)q = x - y}$

8. Form the PDE by eliminating the arbitrary function of the equation $F(x^2 + y^2, \ z - xy) = 0$.

Solution: Let $\quad u = x^2 + y^2 \quad$ and $\quad v = z - xy$. Then $\quad u_x = 2x, \quad v_x = p - y$

and $\quad u_y = 2y, \quad v_y = q - x$

$\therefore PDE \ is \ u_x v_y = v_x u_y$ ie $\quad 2x(q - x) = 2y(p - y)$

$\Rightarrow \ \boxed{yp - xq = y^2 - x^2}$

9. Form the PDE of the equation $\quad z = yf(x) + xg(y)$.

Solution: $\quad z = yf(x) + xg(y) \quad$ ………………(i)

Differentiating (i) partially with respect to x and y , we get

$p = yf'(x) + g(y) \quad$ and $\quad q = f(x) + xg'(y)$

Differentiating p partially with respect to y, we get $\quad s = f'(x) + g'(y)$

$\Rightarrow \ px + qy = \overline{xyf'(x)} + \underline{xg(y) + yf(x)} + \overline{xyg'(y)}$

$= xys + z$. $\quad \therefore PDE \ is \ \boxed{px + qy = xys + z}$

Solutions of a partial differential equation.

Equation solvable by direct integration:

1. Solve $\frac{\partial^3 z}{\partial x^2 \partial y} + 18xy^2 + \sin(2x - y) = 0$.

Solution: Integrating twice with respect to x (keeping y fixed),

$$\frac{\partial^2 z}{\partial x \partial y} + 9x^2y^2 - \frac{\cos(2x-y)}{2} = f(y) \ ,$$

$$\frac{\partial z}{\partial y} + 3x^3y^2 - \frac{\sin(2x-y)}{4} = xf(y) + g(y) \ .$$

Now integrating with respect to y (keeping x fixed),

$$\boxed{z + x^3y^3 - \frac{\cos(2x-y)}{4} = xF(y) + G(y) + h(x)}$$

Where F, G, h are arbitrary functions, and $F(y) = \int f(y)dy$, $G(y) = \int g(y)dy$.

2. Solve $\frac{\partial^2 z}{\partial x \partial y} - \sin x \sin y = 0$,

for which $\frac{\partial z}{\partial y} = -2\sin y$ when $x = 0$ and $z = 0$ when y is odd multiple of $\frac{\pi}{2}$.

Solution: $\frac{\partial^2 z}{\partial x \partial y} - \sin x \sin y = 0$,

Integrating with respect to x (keeping y fixed),

$\frac{\partial z}{\partial y} + \cos x \sin y = f(y)$. Substituting $x = 0$ and $\frac{\partial z}{\partial y} = -2\sin y$ we get,

$f(y) = -2\sin y + \sin y = -\sin y$.

$\therefore \quad \frac{\partial z}{\partial y} + \cos x \sin y = -\sin y$

Now integrating with respect to y (keeping x fixed),

$z - \cos x \cos y = \cos y + g(x)$. Substituting $z = 0$ and $y = \frac{\pi}{2}$ we get,

$g(x) = 0$. $\quad \therefore \boxed{z = \cos x \cos y + \cos y}$

Solution of homogeneous PDE involving derivative with respect to one independent variable only:

3. Solve $\frac{\partial^2 z}{\partial y^2} = z$, given that when $y = 0$, $z = e^x$ and $\frac{\partial z}{\partial y} = e^{-x}$.

Solution: If z is the function of y alone, Then the equation is $(D^2 - 1)z = 0$

The solution is $z = c_1 e^{-y} + c_2 e^{y}$, where c_1 and c_2 are constants.

Since z is a function of x and y , c_1 and c_2 can be arbitrary functions of x .

Hence the solution of the given equation is $z = f(x)e^{y} + g(x)e^{-y}$

$$\therefore \quad \frac{\partial z}{\partial y} = f(x)e^{y} - g(x)e^{-y}$$

When $y = 0$, $z = e^x$ and $\frac{\partial z}{\partial y} = e^{-x}$, $f(x) + g(x) = e^x$ and $f(x) - g(x) = e^{-x}$

$\Longrightarrow f(x) = \cosh x$ and $g(x) = \sinh x$.

Hence the desired solution is $\boxed{z = e^{y}\cosh x + e^{-y}\sinh x}$

4. Solve $\frac{\partial^2 z}{\partial x^2} + 3\frac{\partial z}{\partial x} + 2z = 0$, given that when $x = 0$, $z = 0$ and $\frac{\partial z}{\partial x} = \cos y$.

Solution: If z is the function of x alone, Then the equation is $(D^2 + 3z + 2)z = 0$

The solution is $z = c_1 e^{-x} + c_2 e^{-2x}$, where c_1 and c_2 are constants.

Since z is a function of x and y , c_1 and c_2 can be arbitrary functions of y .

Hence the solution of the given equation is $z = f(y)e^{-x} + g(y)e^{-2x}$

$$\therefore \qquad \frac{\partial z}{\partial x} = -f(y)e^{-x} - 2g(y)e^{-2x}$$

When $x = 0$, $z = 0$ and $\frac{\partial z}{\partial x} = \cos y$, $\quad f(y) + g(y) = 0$ and $-f(y) - 2g(y) = \cos y$

$\Longrightarrow g(y) = -\cos y$ and $f(y) = \cos y$.

Hence the desired solution is $\boxed{z = \cos y\,(e^{-x} - e^{-2x})}$

Lagrange's linear equations of first order.

Equations of the form $Pp + Qq = R$ are called linear equations. Where P, Q and R are functions of (x, y, z).

Find the two independent solutions of subsidiary equations $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$.

Let $u(x, y, z) = a$ and $v(x, y, z) = b$ are the solutions of subsidiary equations.

Then complete solution of the linear equation is $F(u,\ v) = 0$ or $u = f(v)$.

Example:

1. Solve $(z - y)p + (x - z)q = y - x$.

Ans: $\qquad (z - y)p + (x - z)q = y - x$

Subsidiary equations are $\frac{dx}{z-y} = \frac{dy}{x-z} = \frac{dz}{y-x}$

$= \frac{dx+dy+dz}{0}$ $\qquad$ *using multipliers* 1,1 *and* 1.

$= \frac{xdx+ydy+zdz}{0}$ $\qquad$ *using multipliers* x, y *and* z.

$dx + dy + dz = 0$

Integrating, $\quad x + y + z = a$

$xdx + ydy + zdz = 0$

Integrating, $\quad \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c$ or $x^2 + y^2 + z^2 = b$.

Hence the required solution is $\boxed{F(x + y + z,\quad x^2 + y^2 + z^2) = 0}$

2. Solve $x(y^2 - z^2)p + y(z^2 - x^2)q = z(x^2 - y^2)$.

Ans: Subsidiary equations are $\frac{dx}{x(y^2-z^2)} = \frac{dy}{y(z^2-x^2)} = \frac{dz}{z(x^2-y^2)}$

$= \frac{\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}}{0}$ $\qquad$ *using multipliers* $\frac{1}{x}, \frac{1}{y}$ *and* $\frac{1}{z}$.

$= \frac{xdx+ydy+zdz}{0}$ $\qquad$ *using multipliers* x, y *and* z

$\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$

Integrating, $\quad \log x + \log y + \log z = \log a$

or $\quad xyz = a$

$xdx + ydy + zdz = 0$

Integrating, $\quad \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c$

or $\quad x^2 + y^2 + z^2 = b$.

Hence the required solution is $\boxed{xyz = F(\ x^2 + y^2 + z^2)}$

3. Solve $(x^2 - y^2 - z^2)p + 2x(qy - z) = 0$.

Ans: Given equation is $(x^2 - y^2 - z^2)p + 2xyq = 2xz$.

Subsidiary equations are $\frac{dx}{(x^2-y^2-z^2)} = \frac{dy}{2xy} = \frac{dz}{2xz}$

$= \frac{xdx+ydy+zdz}{x(x^2+y^2+z^2)}$ $\quad$ *using multipliers* x, y *and* z

$\frac{dy}{2xy} = \frac{dz}{2xz}$

$\Longrightarrow \frac{dy}{y} = \frac{dz}{z}$

on integration, $\log y = \log z + \log a$

or $\frac{y}{z} = a$

$\frac{xdx+ydy+zdz}{x(x^2+y^2+z^2)} = \frac{dz}{2xz}$

$\Longrightarrow \frac{2xdx+2ydy+2zdz}{x^2+y^2+z^2} = \frac{dz}{z}$

on integration, $\log(x^2+y^2+z^2) = \log z + \log b$

or $\frac{x^2+y^2+z^2}{z} = b$

Therefore complete solution of the given equation is $\boxed{x^2+y^2+z^2 = zf(\frac{y}{z})}$

4. Solve $(x^2-yz)p + (y^2-zx)q = z^2-xy$.

Subsidiary equations are $\frac{dx}{(x^2-yx)} = \frac{dy}{(y^2-zx)} = \frac{dz}{z^2-xy}$

$= \frac{d(x-y)}{(x-y)(x+y+z)} = \frac{d(y-z)}{(y-z)(x+y+z)}$

$= \frac{dx+dy+dz}{(x^2+y^2+z^2-yz-zx-xy)} = \frac{xdx+ydy+zdz}{(x+y+z)(x^2+y^2+z^2-yz-zx-xy)}$

$= \frac{dx+dy+dz}{1} = \frac{xdx+ydy+zdz}{(x+y+z)}$

$\frac{d(x-y)}{(x-y)} = \frac{d(y-z)}{(y-z)}$ On integration,

$\log(x-y) = \log(y-z) + \log a$

$\frac{x-y}{y-z} = a$

$(x+y+z)d(x+y+z) = xdx+ydy+zdz$

on integration $(x+y+z)^2 = x^2+y^2+z^2+2b$

$\Longrightarrow xy+yz+zx = b$

Hence the general solution is $\boxed{F\left(xy+yz+zx, \frac{x-y}{y-z}\right) = 0}$

5. Solve $(y+z)p + (x+z)q = x+y$.

Subsidiary equations are $\frac{dx}{y+z} = \frac{dy}{x+z} = \frac{dz}{x+y}$

$= \frac{d(x-y)}{-(x-y)} = \frac{d(y-z)}{-(y-z)} = \frac{dx+dy+dz}{2(x+y+z)}$

$\frac{d(x-y)}{(x-y)} = \frac{d(y-z)}{(y-z)}$

On integration,

$\log(x-y) = \log(y-z) + \log a$

$\Longrightarrow \frac{x-y}{y-z} = a$

$\frac{d(x+y+z)}{(x+y+z)} = \frac{2d(y-z)}{-(y-z)}$

On integration

$\log(x+y+z) + \log(y-z)^2 = \log b$

$\Longrightarrow (x+y+z)(y-z)^2 = b$.

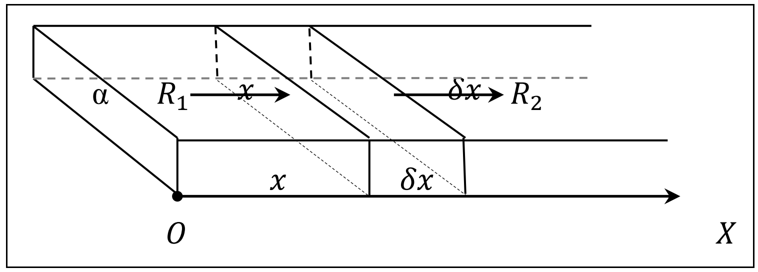
Hence the solution is $\boxed{F\left(\frac{x-y}{y-z}, \quad (x+y+z)(y-z)^2\right) = 0}$.

One-dimensional heat flow derivation:

Consider a homogeneous bar of uniform cross-section α square units. Assume that the sides are covered with heat resistant materials so that stream lines of heat flow are all parallel and perpendicular to the area α. Take one end of the bar as the origin and the direction of the heat flow as positive x-axis.

Let ρ be the density, s the specific heat and k the thermal conductivity.

Let $u(x,t)$ be the temperature at a distance x from O



If δu be the temperature change in a slab of thickness δx of the bar.

Then the quantity of heat in this slab $= s\rho\alpha\delta x\delta u$.

Hence the rate of increase of heat in this slab is $s\rho\alpha\delta x\dfrac{\partial u}{\partial t} = R_1 - R_2$. Where R_1 and R_2 are respectively rate of inflow and outflow of heat and $R_1 = -k\alpha\left(\dfrac{\partial u}{\partial x}\right)_x$, $R_2 = -k\alpha\left(\dfrac{\partial u}{\partial x}\right)_{x+\delta x}$.

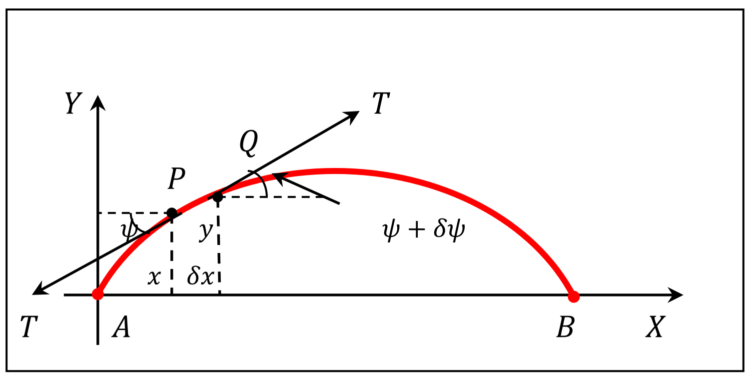
And hence $$s\rho\alpha\delta x\frac{\partial u}{\partial t} = k\alpha\left\{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right\}$$

$$\therefore \frac{\partial u}{\partial t} = \frac{k}{s\rho}\left[\frac{\left\{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right\}}{\delta x}\right].$$

Let $\dfrac{k}{s\rho} = c^2$, called diffusivity of the substance, and taking the limit as $\delta x \to 0$, we get

$$\boldsymbol{\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}}$$ This is the one-dimensional heat-flow equation.

One-dimensional wave equation derivation:



Consider a tightly stretched elastic string of length l and fixed ends A and B and subjected to constant tension. The tension T will be considered to be large as compared to the weight of the string so that the effect of gravity is negligible. Let the string be released from rest and allowed to vibrate. Assume that vibration of string is entirely in one plane. Taking the end A as origin, AB as x-axis, AY as y-axis.

So that motion takes place entirely in xy- plane. Consider the motion of the element PQ of the string between its points $P(x, y)$ and $Q(x+\delta x, y+\delta y)$, where the tangents make angles ψ and $\psi + \delta\psi$ with x- axis respectively. Clearly element is moving upwards with acceleration $\dfrac{\partial^2 y}{\partial t^2}$.

Vertical component of the force acting on this element $= T\sin(\psi+\delta\psi) - T\sin\psi$.

$= T[\tan(\psi+\delta\psi) - \tan\psi]$ $\because$ ψ is very small, $\sin\psi \approx \tan\psi \approx 0$.

$$= T\left\{\left(\frac{\partial y}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial y}{\partial x}\right)_x\right\}.$$

If m is the mass per unit length of the string, then by Newton's second law of motion, we have

$$m\delta x \frac{\partial^2 y}{\partial t^2} = T\left\{\left(\frac{\partial y}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial y}{\partial x}\right)_x\right\}$$

i.e. $\dfrac{\partial^2 y}{\partial t^2} = \dfrac{T}{m}\left[\dfrac{\left\{\left(\frac{\partial y}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial y}{\partial x}\right)_x\right\}}{\delta x}\right]$, Taking limits as $Q \to P$ i.e. $\delta x \to 0$, we have

$$\boldsymbol{\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}}, \quad \text{where } c^2 = \frac{T}{m}.$$

Self-study:

(Gaps) Method of separation of variables:

1. Using the method of separation of variables, solve $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$, where $u(x,\ 0) = 6e^{-3x}$.

Solution: Assume that $u(x,\ t) = XT$ where X is a function of x alone and T that of t alone.

Substituting in the given equation, we have $X'T = 2XT' + XT$

$$\Longrightarrow \quad \frac{X\prime}{X} = \frac{2T\prime + T}{T} = k \quad \text{(Say)}$$

Then $X' = kX$ | $2T' + T = kT \implies T' = \frac{(k-1)}{2}T$

$\Longrightarrow$ $D = k$ | $D = \frac{(k-1)}{2}$

$\Longrightarrow$ $X = c_1 e^{kx}$ | $T = c_2 e^{\frac{(k-1)t}{2}}$

Thus $u(x,\ t) = ce^{kx + \frac{(k-1)t}{2}}$ Substituting $u(x,\ 0) = 6e^{-3x}$ we get

$$6e^{-3x} = ce^{kx} \quad \Longrightarrow \quad c = 6 \quad and \quad k = -3$$

Hence the required solution is $\boxed{u(x,\ t) = 6e^{-(3x+2t)}}$

2. Solve $3\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0$, $u(x,\ 0) = 4e^{-x}$ by the method of separation of variables.

Put $z = XY$ in the given equation. Where X is a function of x alone and Y that of y alone.

Then $3X'Y + 2XY' = 0 \quad \Longrightarrow \quad \frac{X\prime}{2X} = -\frac{Y\prime}{3Y} = k$

$\therefore$ $X' = 2kX$ | $Y' = -3kY$

$\Longrightarrow$ $D = 2k$ | $D = -3k$

$\Longrightarrow$ $X = c_1 e^{2kx}$ | $Y = c_2 e^{-3ky}$

Thus $u(x,\ y) = ce^{2kx - 3ky}$ Substituting $u(x,\ 0) = 4e^{-x}$, we get

$$4e^{-x} = ce^{2kx} \quad \Longrightarrow \quad c = 4 \quad and \quad k = -\frac{1}{2}$$

Hence the required solution is $\boxed{u = 4e^{-x + \frac{3}{2}y}}$

Find all possible solutions of one dimensional heat equation $\boldsymbol{u_t = c^2 u_{xx}}$ by the method of separation of variables.

Ans: Assume that $u(x,\ t) = XT$ where X is a function of x alone and T that of t alone.

Substituting in the given equation, we have $XT' = c^2 X''T$.

$$\Longrightarrow \quad \frac{X''}{X} = \frac{T'}{c^2 T} = k$$

$$\Longrightarrow \quad \boxed{X'' = kX} \quad and \quad \boxed{T' = kc^2 T}$$

Case1: If $k > 0$, let $k = p^2$, then

$X'' = p^2 X$	$T' = p^2c^2T$
$D^2 = p^2$ or $D = \pm p$	$D = p^2c^2$
$X = c_1e^{-px} + c_2e^{px}$	$T = c_3e^{p^2c^2t}$

$\therefore$ **If** $\boldsymbol{k > 0}$, $\boldsymbol{u(x,\ t) = (c_1e^{-px} + c_2e^{px})\left(c_3e^{p^2c^2t}\right)}$.

Case2: If $k < 0$, let $k = -p^2$, then

$X'' = -p^2X$	$T' = -p^2c^2T$
$D^2 = -p^2$ or $D = \pm pi$	$D = -p^2c^2$
$X = c_4 \cos px + c_5 \sin px$	$T = c_6e^{-p^2c^2t}$

$\therefore$ **If** $\boldsymbol{k < 0}$, $\boldsymbol{u(x,\ t) = (c_4 \cos px + c_5 \sin px)\left(c_6e^{-p^2c^2t}\right)}$.

Case3: If $k = 0$, then

$X'' = 0$	$T' = 0$
$D^2 = 0$ or $D = 0,\ 0$	$D = 0$
$X = c_7 + c_8x$	$T = c_9$

$\therefore$ **If** $\boldsymbol{k = 0}$, $\boldsymbol{u(x,\ t) = (c_7 + c_8x)(c_9)}$.

Since temperature u is to decrease with increase of time t,

$\boldsymbol{u(x,\ t) = (c_1 \cos px + c_2 \sin px)\left(e^{-p^2c^2t}\right)}$ is the only solution.

Find all possible solutions of one dimensional wave equation $\boldsymbol{u_{tt} = c^2u_{xx}}$ by the method of separation of variables.

Ans: Assume that $u(x,\ t) = XT$ where X is a function of x alone and T that of t alone.

Substituting in the wave equation, we have $XT'' = c^2X''T$.

$$\Longrightarrow \quad \frac{X''}{X} = \frac{T''}{c^2T} = k$$

$$\Longrightarrow \quad \boxed{X'' = kX} \quad and \quad \boxed{T'' = kc^2T}$$

Case1: If $k > 0$, let $k = p^2$, then

$X'' = p^2X$	$T'' = p^2c^2T$
$D^2 = p^2$ or $D = \pm p$	$D^2 = p^2c^2$ or $D = \pm pc$
$X = c_1e^{-px} + c_2e^{px}$	$T = c_3e^{-pct} + c_4e^{pct}$

$\therefore$ **If** $\boldsymbol{k > 0}$, $\boldsymbol{u(x,\ t) = (c_1e^{-px} + c_2e^{px})(c_3e^{-pct} + c_4e^{pct})}$.

Case2: If $k < 0$, let $k = -p^2$, then

$X'' = -p^2X$	$T'' = -p^2c^2T$
$D^2 = -p^2$ or $D = \pm pi$	$D^2 = -p^2c^2$ or $D = \pm pci$
$X = c_5 \cos px + c_6 \sin px$	$T = c_7 \cos pct + c_8 \sin pct$

$\therefore$ **If** $\boldsymbol{k < 0}$, $\boldsymbol{u(x,\ t) = (c_5 \cos px + c_6 \sin px)(c_7 \cos pct + c_8 \sin pct)}$.

Case3: If $k = 0$, then

$X'' = 0$	$T'' = 0$
$D^2 = 0$ or $D = 0,\ 0$	$D^2 = 0$ or $D = 0,\ 0$
$X = c_9 + c_{10}x$	$T = c_{11} + c_{12}t$

$\therefore$ **If** $\boldsymbol{k = 0}$, $\boldsymbol{u(x,\ t) = (c_9 + c_{10}x)(c_{11} + c_{12}t)}$.

Since wave is periodic,

$\boldsymbol{u(x,\ t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos pct + c_4 \sin pct)}$

Is the only suitable solution of the wave equation.