Module-4: Ordinary Differential Equations of higher order

Linear Differential Equations with constant coefficients:

$f(D)y = X$ .... (1) Where $D = \frac{d}{dx}$ and X is function of x only.

Complete Solution of (1) is $y = y_c + y_p$, where y_c is complementary function, and y_p is particular integral.

To find y_c : Auxiliary equation of (1) is $f(D) = 0$. Find the roots of A.E.

Roots of A.E.	y_c
1. m_1, m_2, m_3 .....(real and different roots)	$c_1e^{m_1x} + c_2e^{m_2x} + c_3e^{m_3x} + \cdots\cdots$
2. m_1, m_1, m_2 .....(Two real and equal roots)	$(c_1 + c_2x)e^{m_1x} + c_3e^{m_2x} + \cdots\cdots$
3. m_1, m_1, m_1, m_2 .....(three real and equal roots)	$(c_1 + c_2x + c_3x^2)e^{m_1x} + c_4e^{m_2x} + \cdots\cdots$
4. $a \pm ib$, m_1 ....(a pair of imaginary roots)	$e^{ax}(c_1 \cos bx + c_2 \sin bx) + c_3e^{m_1x} + \cdots$
5. $a \pm ib$, $a \pm ib$, m_1 (2 pair of imaginary equal roots)	$e^{ax}[(c_1 + c_2x)\cos bx + (c_3 + c_4x)\sin bx] + c_4e^{m_1x}$.

Examples:

Roots of A.E.	y_c
1. 1,2,3	$\boldsymbol{y_c} = c_1e^x + c_2e^{2x} + c_3e^{3x}$.
2. $-2, -2$	$\boldsymbol{y_c} = (c_1 + c_2x)e^{-2x}$.
3. $-1, -1, -1, 1.5$	$\boldsymbol{y_c} = (c_1 + c_2x + c_3x^2)e^{-x} + c_4e^{1.5x}$.
4. $1 \pm 2i$, 3	$\boldsymbol{y_c} = e^x(c_1 \cos 2x + c_2 \sin 2x) + c_3e^{3x}$
5. $\pm 2i$	$\boldsymbol{y_c} = c_1 \cos 2x + c_2 \sin 2x$.
6. $1 \pm i$, $1 \pm i$	$\boldsymbol{y_c} = e^x[(c_1 + c_2x)\cos x + (c_3 + c_4x)\sin x]$

To find y_p : $\boldsymbol{y_p = \frac{1}{f(D)}X}$

1. $\frac{1}{D}X = \int X dx$

2. If $f(a) \neq 0$, then $\frac{1}{f(D)}e^{ax} = \frac{e^{ax}}{f(a)}$,

   If $f(a) = 0$, then $\frac{1}{f(D)}e^{ax} = x\frac{1}{f'(D)}e^{ax}$, and then put $D = a$.

   If $f(a) = 0, f'(a) = 0$, then $\frac{1}{f(D)}e^{ax} = x^2\frac{1}{f''(D)}e^{ax}$ and then put $D = a$ and so on.

3. $\frac{1}{f(D)}\sin(ax)$ or $\frac{1}{f(D)}\cos(ax)$, $put\ D^2 = -a^2$ in $f(D)$.

4. $\frac{1}{f(D)}(ax^2 + bx + c)$ is obtained by direct division.

Problems: Solve the following equations.

1. $(D^2 + D - 2)y = 0$.

   $A.E.\ is\ (D^2 + D - 2) = 0$, roots are -2 and 1.

   $\therefore\ \boldsymbol{y = c_1e^{-2x} + c_2e^x}$.

2. $(D^2 + 6D + 9)y = 0$.

$A.E.\ \ is\ \ D^2+6D+9=0$, roots are -3 and -3.

$\therefore\ \ \boldsymbol{y=(c_1+c_2x)e^{-3x}}$.

3. $(4D^3+4D^2+D)y=0$.

$A.E.\ \ is\ \ 4D^3+4D^2+D=0$, roots are $0,-\frac{1}{2}$ and $-\frac{1}{2}$.

$\therefore\ \ \boldsymbol{y=c_1+(c_2+c_3x)e^{-\frac{1}{2}x}}$.

4. $(D^4+2D^2+1)y=0$.

$A.E.\ \ is\ \ D^4+2D^2+1=0\ \ \Longrightarrow (D^2+1)^2=0$

$\Longrightarrow D^2=-1\ \ \&\ \ D^2=-1$ $\therefore$ Roots are $\pm i$ and $\pm i$.

$\therefore\ \ \boldsymbol{y=(c_1+c_2x)\cos x+(c_3+c_4x)\sin x}$.

5. $(D^4+D^3-7D^2-D+6)y=0$

$A.E.\ \ is\ \ D^4+D^3-7D^2-D+6=0$, $\Longrightarrow D=1$

Roots are $-3,-1,1\ and\ 2$

1	1	1	-7	-1	6
	0	1	2	-5	-6
	1	2	-5	-6	0

$\therefore\ \ \boldsymbol{y=c_1e^{-3x}+c_2e^{-x}+c_3e^{x}+c_4e^{2x}}$.

6. $(D^3-6D^2+11D-6)y=e^{2x}+e^{-2x}$.

$A.E.\ \ is\ \ D^3-6D^2+11D-6=0$, Roots are $1,\ 2\ and\ 3$.

$\therefore\ \ y_c=\boldsymbol{c_1e^{x}+c_2e^{2x}+c_3e^{3x}}$.

$$\boldsymbol{y_p}=\frac{\mathbf{1}}{D^3-6D^2+11D-6}(e^{2x}+e^{-2x})$$

$$=\frac{\mathbf{1}}{D^3-6D^2+11D-6}e^{2x}+\frac{\mathbf{1}}{D^3-6D^2+11D-6}e^{-2x}$$

$$=x\frac{\mathbf{1}}{3D^2-12D+11}e^{2x}-\frac{e^{-2x}}{60}=\boldsymbol{-xe^{2x}-\frac{e^{-2x}}{60}}.$$

Complete solution is $\boldsymbol{y=y_c+y_p}$.

7. $(D^4+64)y=0$.

$A.E.\ \ is\ D^4+64=0\ \ \Longrightarrow\ D^4+16D^2+64-16D^2=0$

$\Longrightarrow\ (\,D^2+8)^2-(4D)^2=0$

$\Longrightarrow\ \ (\,D^2-4D+8)(D^2+4D+8)=0$.

Roots are $2\pm 2i,\ and-2\pm 2i$.

$\therefore\ \ \boldsymbol{y=e^{2x}(c_1\cos 2x+c_2\sin 2x)+e^{-2x}(c_3\cos 2x+c_4\sin 2x)}$.

8. $(4D^4-8D^3-7D^2+11D+6)y=e^{-2x}+3$.

$A.E.\ \ is\ \ 4D^4-8D^3-7D^2+11D+6=0$

$\Longrightarrow D=-1$

-1	4	-8	-7	11	6

Roots are $-1, -\frac{1}{2}, \frac{3}{2}$ *and* 2

0	-4	12	-5	-6
4	-12	5	6	0

$\therefore \quad y_c = \boldsymbol{c_1 e^{-x} + c_2 e^{-\frac{1}{2}x} + c_3 e^{\frac{3}{2}x} + c_4 e^{2x}}$

$$\boldsymbol{y_p} = \frac{1}{4D^4-8D^3-7D^2+11D+6}\left(e^{-2x}+3e^{0x}\right) = \frac{\boldsymbol{e^{-2x}}}{\boldsymbol{84}} + \frac{\boldsymbol{1}}{\boldsymbol{2}}\ .$$

Complete solution is $\boldsymbol{y = y_c + y_p}$.

9 . $\frac{d^4x}{dt^4} = m^4 x$

$A.E.\ \ is\ \ D^4 - m^4 = 0 \Rightarrow (D^2-m^2)(D^2+m^2) = 0,$ $\therefore$roots are $\pm mi,\ \pm m$.

$\therefore \quad \boldsymbol{x = c_1 \cos mt + c_2 \sin mt + c_3 e^{-mt} + c_4 e^{mt}}$.

10. $(D^5 - D^4 - D + 1)y = 0$.

$A.E.\ \ is\ \ D^5 - D^4 - D + 1 = 0 \quad \Rightarrow (D^4-1)(D-1) = 0$.

$\Rightarrow (D^2+1)(D^2-1)(D-1) = 0$

Roots are $\pm i, -1, 1, 1$.

$\therefore \quad \boldsymbol{y = c_1 \cos x + c_2 \sin x + c_3 e^{-x} + (c_4 + c_5 x)e^{x}}$.

11. $y'' + 4y' + 4y = 0$, $y = 0, y' = -1\ at\ x = 1$.

Symbolic form is $(D^2 + 4D + 4)y = 0$.

$A.E.\ \ is\ \ D^2 + 4D + 4 = 0$, roots are $-2, -2$.

$\therefore \quad y = (c_1 + c_2 x)e^{-2x}. \quad \Rightarrow y' = -2y + c_2 e^{-2x}$.

Put $x = 1$, we get $0 = (c_1 + c_2)\, e^{-2}$ *and* $-1 = c_2 e^{-2} \Rightarrow c_2 = -e^2$ & $c_1 = e^2$.

$\therefore \quad \boldsymbol{y = (1 - x)e^{-2(x-1)}}$.

12. $y'' + 4y' + 5y = 0,\ given\ that\ y' = y''\ \ and\ y = 2\ at\ x = 0.$

Symbolic form is $(D^2 + 4D + 5)y = 0$.

$A.E.\ \ is\ \ D^2 + 4D + 5 = 0$, roots are $-2 \pm i$.

$\therefore \quad y = e^{-2x}(c_1 \cos x + c_2 \sin x). \quad \Rightarrow \quad y' = -2y + e^{-2x}(-c_1 \sin x + c_2 \cos x).$

Since $y' = y''\ and\ y = 2\ at\ x = 0,\ 5y'(0) = -10 \ \Rightarrow y(0) = 2\ \ and\ \ y'(0) = -2$.

$\therefore put\ \ x = 0,\ \ we\ get\ \ 2 = c_1. \quad \Rightarrow -2 = -4 + c_2$

$\Rightarrow \ c_1 = 2 \ \ and\ \ c_2 = 2$.

$\therefore \quad \boldsymbol{y = 2e^{-2x}(\cos x + \sin x)}.$

13. $\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 29x = 0,\ x(0) = 0, x'(0) = 15.$

Symbolic form is $(D^2 + 4D + 29)x = 0$.

$A.E.\ \ is\ \ D^2 + 4D + 29 = 0$, roots are $-2 \pm 5i$.

$x = e^{-2t}(c_1 \cos 5t + c_2 \sin 5t)$ *and* $x' = -2x + e^{-2t}(-5c_1 \sin 5t + 5c_2 \cos 5t)$.

Put $t = 0$, *we get* $c_1 = 0$ *and* $c_2 = 3$.

$\therefore \quad \boldsymbol{x = 3\, e^{-2t} \sin 5t}$.

14. $(D^2 + 2D + 3)y = \sin x$.

$A.E.$ *is* $D^2 + 2D + 3 = 0$, roots are $-1 \pm \sqrt{2}i$

$\therefore \quad y_c = \boldsymbol{e^{-x}(c_1 \cos \sqrt{2}x + c_2 \sin \sqrt{2}x)}$.

$$y_p = \frac{1}{D^2+2D+3} \sin x = \frac{1}{2D+2} \sin x$$

$$= \frac{1}{2(D+1)} \sin x = \frac{1}{2}\left[\frac{(D-1)}{D^2-1}\right] \sin x$$

$$= -\frac{1}{4}(D-1) \sin x = -\frac{1}{4}(\cos x - \sin x)$$

$$= \boldsymbol{\frac{1}{4}(\sin x - \cos x)}.$$

Complete solution is $\boldsymbol{y = y_c + y_p}$.

15. $(D^2 + 3D + 2)y = 4\cos^2 x$

$A.E.$ *is* $D^2 + 3D + 2 = 0$, roots are $-2, -1$.

$\therefore \quad y_c = \boldsymbol{c_1 e^{-2x} + c_2 e^{-x}}$.

$$y_p = \frac{1}{D^2+3D+2}(2 + 2\cos 2x) \qquad \because \cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

$$= \frac{1}{D^2+3D+2} 2\Big|_{D=0} + \frac{1}{D^2+3D+2} 2\cos 2x\Big|_{D^2=-4} = 1 + \frac{1}{3D-2} 2\cos 2x$$

$$= 1 + \left[\frac{3D+2}{9D^2-4}\right] 2\cos 2x = 1 - \frac{1}{20}(3D+2)\cos 2x$$

$$= 1 - \frac{1}{20}(-6\sin 2x + 2\cos 2x) = \boldsymbol{1 + \frac{1}{10}(3\sin 2x - \cos 2x)}$$

Complete solution is $\boldsymbol{y = y_c + y_p}$.

16. . $(D^2 - 4D + 3)y = \sin 3x \cos 2x$.

$A.E.$ *is* $D^2 - 4D + 3 = 0$, roots are $1, \ 3$. $\quad \therefore \quad y_c = \boldsymbol{c_1 e^{x} + c_2 e^{3x}}$.

$$y_p = \frac{1}{D^2-4D+3}\left[\frac{1}{2}(\sin 5x + \sin x)\right]$$

$$= \frac{1}{2}\left[\frac{1}{D^2-4D+3} \sin 5x\Big|_{D^2=-25} + \frac{1}{D^2-4D+3} \sin x\Big|_{D^2=-1}\right]$$

$$= \frac{1}{2}\left[\left(\frac{1}{-4D-22}\right)\sin 5x + \left(\frac{1}{-4D+2}\right)\sin x\right] = \frac{1}{2}\left[-\frac{1}{2}\left(\frac{1}{2D+11}\right)\sin 5x - \frac{1}{2}\left(\frac{1}{2D-1}\right)\sin x\right]$$

$$= \frac{1}{2}\left[-\frac{1}{2}\left(\frac{2D-11}{4D^2-121}\right)\sin 5x - \frac{1}{2}\left(\frac{2D+1}{4D^2-1}\right)\sin x\right]$$

$$= \left[\frac{1}{884}(2D-11)\sin 5x + \frac{1}{20}(2D+1)\sin x\right]$$

$$= \frac{1}{884}(\mathbf{10\cos 5x - 11\sin 5x}) + \frac{1}{20}(\mathbf{2\cos x + \sin x}).$$

Complete solution is $\boldsymbol{y = y_c + y_p}$.

17. $(D^3 + 2D^2 + D)y = e^{2x} + \sin 2x.$

$A.E.$ is $D^3 + 2D^2 + D = 0$, roots are $0, -1, -1$. $\therefore$ $y_c = \boldsymbol{c_1 + (c_2 + c_3 x)e^{-x}}$.

$$y_p = \frac{1}{D^3+2D^2+D}(e^{2x} + \sin 2x).$$

$$= \frac{1}{D^3+2D^2+D}(e^{2x})\Big|_{D=2} + \frac{1}{D^3+2D^2+D}(\sin 2x)\Big|_{D^2=-4 \ and \ D^3=-4D}$$

$$= \frac{e^{2x}}{18} + \frac{1}{-3D-8}\sin 2x = \frac{e^{2x}}{18} - \frac{1}{3D+8}\sin 2x$$

$$= \frac{e^{2x}}{18} - \frac{(3D-8)}{9D^2-64}\sin 2x = \frac{e^{2x}}{18} + \frac{1}{100}(3D-8)\sin 2x$$

$$= \boldsymbol{\frac{e^{2x}}{18} + \frac{1}{50}(3\cos 2x - 4\sin 2x)}\ .$$

Complete solution is $\boldsymbol{y = y_c + y_p}$.

18. $\frac{d^2y}{dx^2} - 4y = \cosh(2x-1) + 3^x$.

Symbolic form is $(D^2 - 4)y = \cosh(2x-1) + 3^x$

$A.E.$ is $D^2 - 4 = 0$, roots are $-2, +2$. $\therefore$ $y_c = \boldsymbol{c_1 e^{-2x} + c_2 e^{2x}}$.

$$y_p = \frac{1}{D^2-4}\cosh(2x-1) + \frac{1}{D^2-4}3^x$$

$$= \left[\frac{1}{D^2-4}\right]\frac{e^{(2x-1)} + e^{-(2x-1)}}{2} + \frac{1}{D^2-4}3^x$$

$$= \frac{x}{2}\left[\frac{1}{2D}\right]\left(e^{(2x-1)} + e^{-(2x-1)}\right) + \frac{1}{D^2-4}3^x$$

$$= \frac{x}{4}\left(\frac{e^{(2x-1)} - e^{-(2x-1)}}{2}\right) + \frac{1}{D^2-4}3^x$$

$$= \boldsymbol{\frac{x}{4}\sinh(2x-1) + \frac{3^x}{(\log 3)^2 - 4}} \qquad \because 3^x = e^{(\log 3)x}\ .$$

Complete solution is $\boldsymbol{y = y_c + y_p}$.

19. $(D^3 - D)y = 2x + 1 + 4\cos x + 2e^x.$

$A.E.$ is $D^3 - D = 0$, roots are $-1,\ 0, 1$. $\therefore$ $y_c = \boldsymbol{c_1 e^{-x} + c_2 + c_3 e^x}$.

$$y_p = \frac{1}{D^3-D}(2x+1) + \frac{1}{D^3-D}(4\cos x) + \frac{1}{D^3-D}(2e^x)$$

$$= \frac{1}{D(D^2-1)}(2x+1) + \frac{1}{-2D}(4\cos x) + \frac{1}{D^3-D}(2e^x)$$

$$= \int \frac{1}{D^2-1}(2x+1)dx - 2\int \cos x\, dx + 2x\frac{1}{3D^2-1}e^x$$

$$= \int \frac{1}{D^2-1}(2x+1)dx - 2\int \cos x\, dx + xe^x$$

$$= \int(-2x-1)\, dx - 2\sin x + xe^x$$

$$= \boldsymbol{-x^2 - x - 2\sin x + xe^x}$$

$$\begin{array}{r|l} & -2x-1 \\ \hline -1+D^2 & 2x+1 \\ & 2x \\ \hline & \quad 1 \\ & \quad 1 \\ \hline & \quad 0 \end{array}$$

Complete solution is $\boldsymbol{y = y_c + y_p}$.

20. $(D-2)^2 y = 8(e^{2x} + \sin 2x + x^2)$

$A.E.\ is\ \ (D-2)^2 = 0$, roots are $2, 2$. $\therefore\ y_c = \boldsymbol{(c_1 + c_2 x)e^{2x}}$.

$$y_p = \frac{1}{D^2-4D+4} 8(e^{2x} + \sin 2x + x^2)$$

$$= x\frac{1}{2D-4}8e^{2x} - \frac{1}{4D}8\sin 2x + 2x^2 + 4x + 3$$

$$= \boldsymbol{4x^2e^{2x} + \cos 2x + 2x^2 + 4x + 3}$$

$$\begin{array}{r|l} & 2x^2+4x+3 \\ \hline 4-4D+D^2 & 8x^2 \\ & 8x^2-16x+4 \\ \hline & 16x-4 \\ & 16x-16 \\ \hline & \quad 12 \\ & \quad 12 \\ \hline & \quad 0 \end{array}$$

Complete solution is $\boldsymbol{y = y_c + y_p}$.

To find y_p by method of variation of parameters:

For $f(D)y = X$, let $y_c = c_1 u + c_2 v$ then Wronskian of $u\ and\ v$ is $w = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = uv' - vu'$.

then $y_p = -u\int \frac{vX}{w}dx + v\int \frac{uX}{w}dx$.

Solve the following **by method of variation of parameters.**

1. $(D^2+1)y = \tan x$

$A.E.\ is\ \ D^2 + 1 = 0 \implies$ roots are $\pm i$.

$\therefore\ y_c = c_1 \cos x + c_2 \sin x.$

Let $u = \cos x$, $v = \sin x$, $X = \tan x$.

And $w = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1.$

$$y_p = -u\int \frac{vX}{w}dx + v\int \frac{uX}{w}dx$$

$$= -\cos x \int \frac{\sin x \tan x}{1}dx + \sin x \int \frac{\cos x \tan x}{1}dx$$

$$= -\cos x \int \frac{\sin^2 x}{\cos x}dx + \sin x \int \sin x\, dx$$

$$= -\cos x \int \frac{(1-\cos^2 x)}{\cos x}dx + \sin x \int \sin x\, dx$$

$$= -\cos x \int (\sec x - \cos x)dx + \sin x \int \sin x\, dx$$

$$= -\cos x\, (\log(\sec x + \tan x) - \sin x) + \sin x\, (-\cos x)$$

$$= -\cos x \log(\sec x + \tan x).$$

Complete solution is $\boldsymbol{y = y_c + y_p}$.

2. $(D^2+1)y = \sec x$.

$A.E.\ \ is\ \ D^2+1=0 \implies$ roots are $\pm i$.

$\therefore\ \ \mathrm{y_c} = c_1\cos x + c_2\sin x$.

Let $u=\cos x$, $v=\sin x$, $X=\sec x$.

And $w=\begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$.

$$\begin{aligned} y_p &= -u\int \frac{vX}{w}dx + v\int\frac{uX}{w}dx \\ &= -\cos x\int\frac{\sin x\sec x}{1}dx + \sin x\int\frac{\cos x\sec x}{1}dx \\ &= -\cos x\int\tan x\,dx + \sin x\int 1\,dx \\ &= -\cos x\log\sec x + x\sin x \\ &= \cos x\log\cos x + x\sin x \end{aligned}$$

Complete solution is $\boldsymbol{y = y_c + y_p}$.

3. $(D^2-2D+1)y = \frac{e^x}{x}$.

$A.E.\ \ is\ \ D^2-2D+1=0 \implies$ roots are $1, 1$.

$\therefore\ \ \mathrm{y_c} = (c_1+c_2x)e^x$.

Let $u=e^x$, $v=xe^x$, $X=\frac{e^x}{x}$.

And $w=\begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} e^x & xe^x \\ e^x & xe^x+e^x \end{vmatrix} = e^{2x}\begin{vmatrix} 1 & x \\ 1 & x+1 \end{vmatrix} = e^{2x}$.

$$\begin{aligned} y_p &= -u\int \frac{vX}{w}dx + v\int\frac{uX}{w}dx \\ &= -e^x\int\frac{xe^{2x}}{xe^{2x}}dx + xe^x\int\frac{e^{2x}}{xe^{2x}}dx \\ &= -xe^x + xe^x\log x \\ &= xe^x(\log x - 1) \end{aligned}$$

Complete solution is $\boldsymbol{y = y_c + y_p}$.

4. $(D^2-2D+1)y = e^x\log x$

$A.E.\ \ is\ \ D^2-2D+1=0 \implies$ roots are $1, 1$.

$\therefore\ \ \mathrm{y_c} = (c_1+c_2x)e^x$.

Let $u=e^x$, $v=xe^x$, $X=e^x\log x$.

And $w=\begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} e^x & xe^x \\ e^x & xe^x+e^x \end{vmatrix} = e^{2x}\begin{vmatrix} 1 & x \\ 1 & x+1 \end{vmatrix} = e^{2x}$.

$$\begin{aligned} y_p &= -u\int \frac{vX}{w}dx + v\int\frac{uX}{w}dx \\ &= -e^x\int\frac{xe^{2x}\log x}{e^{2x}}dx + xe^x\int\frac{e^{2x}\log x}{e^{2x}}dx \\ &= -e^x\int x\log x\,dx + xe^x\int\log x\,dx \\ &= -e^x\left(\frac{x^2}{2}\log x - \int\frac{1}{x}\frac{x^2}{2}dx\right) + xe^x\left(x\log x - \int x\frac{1}{x}dx\right) \\ &= -e^x\left(\frac{x^2}{2}\log x - \frac{x^2}{4}\right) + xe^x(x\log x - x) \\ &= x^2e^x\left(\frac{1}{2}\log x - \frac{3}{4}\right) \end{aligned}$$

Complete solution is $\boldsymbol{y = y_c + y_p}$.

5. $(D^2-1)y = \frac{2}{1+e^x}$

$A.E.\ \ is\ \ D^2-1=0 \implies$ roots are $-1,\ 1$.

$\therefore\ \ y_c = c_1e^{-x}+c_2e^x.$

Let $u=e^{-x}$, $v=e^x$, $X=\frac{2}{1+e^x}$.

And $w=\begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} e^{-x} & e^x \\ -e^{-x} & e^x \end{vmatrix} = 2.$

$$y_p = -e^{-x}\int \frac{e^x}{1+e^x}dx + e^x\int\frac{e^{-x}}{1+e^x}dx\ ,\qquad \text{Since } \frac{e^{-x}}{1+e^x} = \frac{1}{e^x(1+e^x)} = \frac{1}{e^x}-\frac{1}{1+e^x} = e^{-x}-1+\frac{e^x}{1+e^x}$$

$$= -e^{-x}\log(1+e^x) - e^{-x} - x + \log(1+e^x).$$

Complete solution is $\boldsymbol{y = y_c + y_p}$.

6. $(D^2-2D+2)y = e^x\tan x$.

$A.E.\ \ is\ \ D^2-2D+2=0 \implies$ roots are $1\pm i$.

$\therefore\ \ y_c = e^x(c_1\cos x + c_2\sin x).$

Let $u=e^x\cos x$, $v=e^x\sin x$, $X=e^x\tan x$.

And $w=\begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} e^x\cos x & e^x\sin x \\ e^x(\cos x-\sin x) & e^x(\sin x+\cos x) \end{vmatrix} = e^{2x}(\cos^2 x+\sin^2 x) = e^{2x}.$

$$y_p = -u\int\frac{vX}{w}dx + v\int\frac{uX}{w}dx$$
$$= -e^x\cos x\int\frac{e^{2x}\ \sin x\tan x}{e^{2x}}dx + e^x\sin x\int\frac{e^{2x}\ \cos x\tan x}{e^{2x}}dx$$
$$= -e^x\cos x\int\frac{\sin^2 x}{\cos x}dx + e^x\sin x\int\sin x\,dx$$
$$= -e^x\cos x\int\frac{(1-\cos^2 x)}{\cos x}dx + e^x\sin x\int\sin x\,dx$$
$$= -e^x\cos x\int(\sec x-\cos x)dx + e^x\sin x\int\sin x\,dx$$
$$= -e^x\cos x\,(\log(\sec x+\tan x)-\sin x) + e^x\sin x\,(-\cos x)$$
$$= -e^x\cos x\log(\sec x+\tan x).$$

Complete solution is $\boldsymbol{y = y_c + y_p}$.

7. $y''-6y'+9y = \frac{e^{3x}}{x^2}$.

$A.E.\ \ is\ \ D^2-6D+9=0 \implies$ roots are $3, 3$.

$\therefore\ \ y_c = (c_1+c_2x)e^{3x}.$

Let $u=e^{3x}$, $v=xe^{3x}$, $X=\frac{e^{3x}}{x^2}$.

And $w=\begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^x & 3xe^{3x}+e^{3x} \end{vmatrix} = e^{6x}\begin{vmatrix} 1 & x \\ 3 & 3x+1 \end{vmatrix} = e^{6x}.$

$$y_p = -u\int\frac{vX}{w}dx + v\int\frac{uX}{w}dx$$
$$= -e^{3x}\int\frac{xe^{6x}}{e^{6x}x^2}dx + xe^{3x}\int\frac{e^{6x}}{e^{6x}x^2}dx$$
$$= -e^{3x}\int\frac{1}{x}dx + xe^{3x}\int\frac{1}{x^2}dx$$
$$= -e^{3x}\log x + xe^{3x}\left(-\frac{1}{x}\right)$$
$$= -e^{3x}(\log x + 1)$$

Complete solution is $\boldsymbol{y = y_c + y_p}$.

8. $(D^2+1)y = \frac{1}{1+\sin x}$

$A.E.\ \ is\ \ D^2+1=0 \implies$ roots are $\pm i$.

$\therefore\ \ \mathrm{y_c} = c_1\cos x + c_2 \sin x.$

Let $u = \cos x$, $v = \sin x$, $X = \frac{1}{1+\sin x}$.

And $w = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1.$

$$y_p = -u\int \frac{vX}{w}dx + v\int \frac{uX}{w}dx$$
$$= -\cos x \int \frac{\sin x}{1+\sin x}dx + \sin x \int \frac{\cos x}{1+\sin x}dx$$
$$= -\cos x \int \frac{\sin x(1-\sin x)}{\cos^2 x}dx + \sin x \int \frac{\cos x}{1+\sin x}dx$$
$$= -\cos x \int (\tan x \sec x - \tan^2 x)dx + \sin x \int \frac{\cos x}{1+\sin x}dx$$
$$= -\cos x \int (\tan x \sec x - \sec^2 x + 1)dx + \sin x \int \frac{\cos x}{1+\sin x}dx$$
$$= -\cos x\,(\sec x - \tan x + x) + \sin x \log(1+\sin x)$$

Complete solution is $\boldsymbol{y = y_c + y_p}$.

9. $(D^2+1)y = \sec x \tan x$.

$A.E.\ \ is\ \ D^2+1=0 \implies$ roots are $\pm i$.

$\therefore\ \ \mathrm{y_c} = c_1\cos x + c_2 \sin x.$

Let $u = \cos x$, $v = \sin x$, $X = \sec x \tan x$.

And $w = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1.$

$$y_p = -u\int \frac{vX}{w}dx + v\int \frac{uX}{w}dx$$
$$= -\cos x \int \frac{\sin x \sec x \tan x}{1}dx + \sin x \int \frac{\cos x \sec x \tan x}{1}dx$$
$$= -\cos x \int \tan^2 x\, dx + \sin x \int \tan x\, dx$$
$$= -\cos x \int (\sec^2 x - 1)dx + \sin x \int \tan x\, dx$$
$$= -\cos x\,(\tan x - x) + \sin x \log \sec x$$

Complete solution is $\boldsymbol{y = y_c + y_p}$.

10. $(D^2+2D+2)y = e^{-x}\sec^3 x$.

$A.E.\ \ is\ \ D^2+2D+2=0 \implies$ roots are $-1 \pm i$.

$\therefore\ \ \mathrm{y_c} = e^{-x}(c_1\cos x + c_2 \sin x).$

Let $u = e^{-x}\cos x$, $v = e^{-x}\sin x$, $X = e^{-x}\sec^3 x$.

And $w = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} e^{-x}\cos x & e^{-x}\sin x \\ e^{-x}(-\cos x - \sin x) & e^{-x}(-\sin x + \cos x) \end{vmatrix} = e^{-2x}(\cos^2 x + \sin^2 x) = e^{-2x}.$

$$y_p = -u\int \frac{vX}{w}dx + v\int \frac{uX}{w}dx$$
$$= -e^{-x}\cos x \int \frac{e^{-2x}\ \sin x \sec^3 x}{e^{-2x}}dx + e^{-x}\sin x \int \frac{e^{-2x}\ \cos x \sec^3 x}{e^{-2x}}dx$$
$$= -e^{-x}\cos x \int (\tan x \sec^2 x)dx + e^{-x}\sin x \int \sec^2 x\, dx$$
$$= -e^{x}\cos x \frac{\tan^2 x}{2} + e^{x}\sin x \tan x$$

$$= -e^x \cos x \frac{\sin^2 x}{2\cos^2 x} + e^x \sin x \frac{\sin x}{\cos x}$$

$$= -e^x \frac{\sin^2 x}{2\cos x} + e^x \frac{\sin^2 x}{\cos x} = \frac{e^x}{2} \tan x \sin x$$

Complete solution is $\boldsymbol{y = y_c + y_p}$.

Legendre's linear equations: $(ax+b)^n \frac{d^n y}{dx^n} + k_1 (ax+b)^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + \cdots\cdots + k_n y = X$.

Put $ax+b = e^t$, $t = \log(ax+b)$, $D = \frac{d}{dt}$, $(ax+b)\frac{dy}{dx} = aDy$, $(ax+b)^2 \frac{d^2y}{dx^2} = a^2 D(D-1)y$

$(ax+b)^3 \frac{d^3y}{dx^3} = a^3 D(D-1)(D-2)y$ and so on.

If $a = 1,\ b = 0$ then the equation is **Cauchy's equations**. i.e.

$x^n \frac{d^n y}{dx^n} + k_1 x^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + \cdots\cdots + k_n y = X$. Put $x = e^t$, $t = \log x$, $D = \frac{d}{dt}$, $x\frac{dy}{dx} = Dy$,

$x^2 \frac{d^2y}{dx^2} = D(D-1)y$, $x^3 \frac{d^3y}{dx^3} = D(D-1)(D-2)y$ and so on.

A) Solve the following Cauchy's equations.

1. $x^2 y'' - xy' + y = \log x$

Put $x = e^t$, $t = \log x$, $D = \frac{d}{dt}$, $x\frac{dy}{dx} = Dy$, $x^2 \frac{d^2y}{dx^2} = D(D-1)y$.

We get $[D(D-1) - D + 1]y = t \implies (D^2 - 2D + 1)y = t$

$A.E.\ is\ D^2 - 2D + 1 = 0$, roots are 1 ,1.

$\therefore\ \boldsymbol{y_c = (c_1 + c_2 t)e^t}$.

$y_p = \frac{1}{D^2 - 2D + 1} t = \boldsymbol{t + 2}$.

Complete solution is $\boldsymbol{y = y_c + y_p}$, where $t = \log x$.

$$\begin{array}{r|l} & t+2 \\ \hline 1 - 2D + D^2 & t \\ & t-2 \\ \hline & \quad 2 \end{array}$$

2. $x^3 y''' + 3x^2 y'' + xy' + 8y = 65\cos(\log x)$

Put $x = e^t$, $t = \log x$, $D = \frac{d}{dt}$, $xy' = Dy$, $x^2 y'' = D(D-1)y$, $x^3 y''' = D(D-1)(D-2)y$.

We get $[D(D-1)(D-2) + 3D(D-1) + D + 8]y = 65\cos t$

$\implies (D^3 + 8)y = 65\cos t$,

$A.E.\ is\ D^3 + 8 = 0$, roots are $-2, 1 \pm \sqrt{\boldsymbol{3}}i$.

$\therefore \boldsymbol{y_c = e^t\left(c_1 \cos\sqrt{3}t + c_2 \sin\sqrt{3}t\right) + c_3 e^{-2t}}$.

$$y_p = \frac{1}{D^3+8} 65\cos t = \frac{1}{8-D} 65\cos t$$

$$= \frac{8+D}{64-D^2} 65\cos t = 8\cos t - \sin t$$

Complete solution is $\boldsymbol{y = y_c + y_p}$ **,** where $t = \log x$

3. $xy'' - \frac{2y}{x} = x + \frac{1}{x^2}$. That is $x^2 y'' - 2y = x^2 + \frac{1}{x}$.

Put $x = e^t$, $t = \log x$, $D = \frac{d}{dt}$, $xy' = Dy$, $x^2 y'' = D(D-1)y$.

We get $[D(D-1) - 2]y = e^{2t} + e^{-t} \implies (D^2 - D - 2)y = e^{2t} + e^{-t}$.

$A.E.$ is $D^2 - D - 2 = 0$, roots are $-1, 2.$ $\therefore \boldsymbol{y_c} = c_1 e^{-t} + \boldsymbol{c_2} e^{2t}.$

$$y_p = \frac{1}{D^2-D-2}[e^{2t} + e^{-t}] = t\left[\frac{1}{2D-1}\right][e^{2t} + e^{-t}] = t\left[\frac{e^{2t}}{3} - \frac{e^{-t}}{3}\right]$$

Complete solution is $\boldsymbol{y = y_c + y_p}$ **,** where $t = \log x$

4. $x^3 y''' + 2x^2 y'' + 2y = 10\left[x + \frac{1}{x}\right]$

Put $x = e^t$, $t = \log x$, $D = \frac{d}{dt}$, $xy' = Dy$, $x^2 y'' = D(D-1)y$, $x^3 y''' = D(D-1)(D-2)y$.

We get $[D(D-1)(D-2) + 2D(D-1) + 2]y = 10(\, e^t + e^{-t}\,)$ $\Longrightarrow (D^3 - D^2 - 2D + 2)y = 10(\, e^t + e^{-t}\,)$

$A.E.$ is $D^3 - D^2 + 2 = 0$, roots are $1 \pm i, -1.$ $\therefore \boldsymbol{y_c} = c_1 e^{-t} + e^t(\boldsymbol{c_2} \cos t + \boldsymbol{c_2} \sin t).$

$$y_p = \frac{1}{D^3-D^2+2} 10[e^t + e^{-t}] = 5e^t + 10t\left[\frac{1}{3D^2-2D}\right]e^{-t} = 5e^{-t} + 2te^t.$$

Complete solution is $\boldsymbol{y = y_c + y_p}$ **,** where $t = \log x.$

B) Solve the following Legendre's equations

1. $(1+x)^2 y'' + (1+x)y' + y = 4\cos(\log(1+x))$.

Clearly equation is Legendre's equation.

Put $1 + x = e^t$, $t = \log(1+x)$, $D = \frac{d}{dt}$, $(1+x)y' = Dy$, $(1+x)^2 y'' = D(D-1)y$.

We get, $[D(D-1) + Dy + 1]y = 4\cos t.$

$\Longrightarrow (D^2 + 1)y = 4\cos t.$

$A.E.$ is $D^2 + 1 = 0$, roots are $\pm i.$ $\therefore \boldsymbol{y_c} = c_1 \cos t + c_2 \sin t.$

$$y_p = \frac{1}{D^2+1} 4\cos t = t\frac{1}{2D} 4\cos t = 2t\sin t.$$

Complete solution is $\boldsymbol{y = y_c + y_p}$ **, where** $t = \log(1+x)$.

2. $(2x+3)^2 y'' - (2x+3)y' - 12y = 6x$.

Clearly equation is Legendre's equation.

Put $2x + 3 = e^t$, $t = \log(2x+3)$, $D = \frac{d}{dt}$, $(2x+3)y' = 2Dy$, $(2x+3)^2 y'' = 4D(D-1)y$,

and $2x = e^t - 3$ $\Longrightarrow 6x = 3e^t - 9$

We get, $[4D(D-1) - 2Dy - 12]y = 3e^t - 9.$

$\Longrightarrow (4D^2 - 6D - 12)y = 3e^t - 9.$

$A.E.$ is $4D^2 - 6D - 12 = 0$, roots are $\frac{3+\sqrt{57}}{4}, \frac{3-\sqrt{57}}{4}.$ $\therefore \boldsymbol{y_c} = c_1 e^{\left(\frac{3+\sqrt{57}}{4}\right)x} + c_2 e^{\left(\frac{3-\sqrt{57}}{4}\right)x}.$

$$y_p = \frac{1}{4D^2-6D-12}(3e^t - 9) = -\frac{3e^t}{14} + \frac{3}{4} .$$

Complete solution is $\boldsymbol{y = y_c + y_p}$ **,** where $t = \log(2x+3)$.

3. $(2+3x)^2 y'' + 3(2+3x)y' - 36y = 3x^2 + 4x + 1$.

Clearly equation is Legendre's equation.

Put $3x + 2 = e^t$, $t = \log(3x+2)$, $D = \frac{d}{dt}$, $(3x+2)y' = 3Dy$, $(3x+2)^2 y'' = 9D(D-1)y$,

and $3x = e^t - 2$ $\Longrightarrow x = \frac{e^t - 2}{3}$,

$$\therefore\ 3x^2 + 4x + 1 = 3\left(\frac{e^t-2}{3}\right)^2 + 4\left(\frac{e^t-2}{3}\right) + 1 = \frac{e^{2t}-1}{3}$$

We get, $[9D(D-1)+9Dy-36]y=\frac{e^{2t}-1}{3}$.

$\Rightarrow (D^2-4)y=\frac{e^{2t}-1}{27}$.

$A.E.\ is\ D^2-4=0$, roots are $-2,\ 2$. $\therefore \boldsymbol{y_c}=c_1e^{-2x}+c_2e^{2x}$.

$y_p=\frac{1}{D^2-4}\left(\frac{e^{2t}-1}{27}\right)=\frac{1}{108}(te^{2t}+1)$.

Complete solution is $y=y_c+y_p$ **,** where $t=\log(3x+2)$.

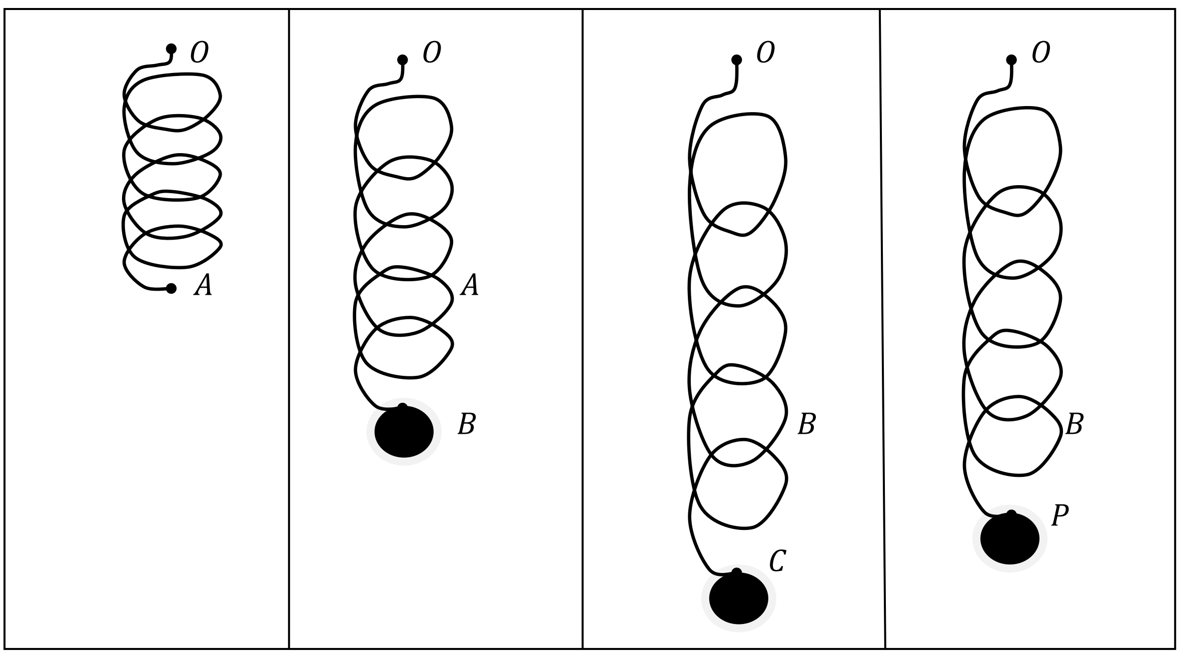
Self study: Oscillation of a spring:

1. Free oscillations: Suppose a mass m is suspended from the end A of a light spring, the other end of which is fixed at O. Let $e=AB$ be the elongation produced by the mass m hanging in equilibrium. If k be the restoring force per unit stretch of the spring due to elasticity, then for the equilibrium at B,

$$mg=T=ke.$$

The mass is pulled down to C bellow the static equilibrium position B and then released.

At any time , after the motion ensues, let the mass be at P, where $BP=x$.



Then the equation of motion of m is $m\frac{d^2x}{dt^2}=mg-k(e+x)$, or $m\frac{d^2x}{dt^2}=-kx$.

By taking $\frac{k}{m}=\mu^2$, equation becomes $\frac{d^2x}{dt^2}+\mu^2x=0$.

Its solution is $x=a\cos\mu t+b\sin\mu t$ and the velocity $\frac{dx}{dt}=-\mu a\sin\mu t+b\mu\cos\mu t$.

When $t=0,\ x=BC=x_0,\ \frac{dx}{dt}=0$. $\Rightarrow a=x_0,\ b=0$.

Therefore $\boldsymbol{x=x_0\cos\mu t}$ be the required solution. Which is simple harmonic.

Maximum velocity is $\mu\boldsymbol{x_0}$**, Period** of oscillation is $\frac{2\pi}{\mu},\ \mu=\sqrt{\frac{k}{m}}$,

At equilibrium position, $mg=T=ke$.

Example:

1. A body weighing 10kg is hung from a spring. A pull of 20kg.wt. will stretch the spring to 10 cm. The body is Pulled down to 20cm below the equilibrium position and then released. Find the displacement of the body From the equilibrium position at time t sec., the maximum velocity and the period of oscillation.

Solution: When $mg = 20kg,\ e = 0.1$ meter. Since, $mg = ke,\quad k = \frac{20}{0.1} = 200kg/m$.

When $mg = 10kg\ ,\ e = \frac{mg}{k} = \frac{10}{200} = 0.05m.\ \ m = \frac{10}{9.8}$ and $\mu = \sqrt{\frac{k}{m}} = \sqrt{\frac{200\times 9.8}{10}} = \sqrt{196} = 14.$

Now the body is pulled down to 20cm below the equilibrium position, $x_0 = 0.2m$.

Therefore $x = x_0 \cos \mu t \implies x = 0.2\cos(14t)$ be the required displacement of the body.

Velocity is $\frac{dx}{dt} = -2.8\sin(14t)$, hence maximum velocity $= 2.8\ m/sec$.

Period of oscillation is $\frac{2\pi}{\mu} = \frac{\pi}{7} \approx 0.45\ sec$.

***L* − *C* Circuit**: A circuit containing an inductance $\boldsymbol{L}$, capacitance $\boldsymbol{C}$, current $\boldsymbol{i}$ and $\boldsymbol{q}$ be the charge in the condenser plate at any time t then the voltage drop across $\boldsymbol{L}$ is $\frac{di}{dt} = L\frac{d^2q}{dt^2}$.

Voltage drop across $\boldsymbol{C}$ is $\frac{q}{C}$.

Since there is no applied e.m.f. in the circuit, by Kirchhoff's first law, $L\frac{d^2q}{dt^2} + \frac{q}{C} = 0.$

Let $\mu^2 = \frac{1}{LC}$, then equation becomes $\frac{d^2q}{dt^2} + \mu^2 q = 0$, and its solution is $q = a\cos\mu t + b\sin\mu t$.

it represents free electrical oscillations of the current having period $\frac{2\pi}{\mu} = 2\pi\sqrt{LC}$. Therefore the discharging of a condenser through an inductance L is same as the motion of the mass m at the end of a spring.

***L*-*C*-R Circuit:** The discharging of a condenser $\boldsymbol{C}$ through an inductance $\boldsymbol{L}$ and the resistance $\boldsymbol{R}$ is

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0.$$

***L* − *C* Circuit with e.m.f. $\boldsymbol{p\cos nt}$:** $L\frac{d^2q}{dt^2} + \frac{q}{C} = p\cos nt.$

***L*-*C*-R Circuit with e.m.f. $\boldsymbol{p\cos nt}$:** $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = p\cos nt.$

Example: An uncharged condenser of capacity C is charged by an e.m.f. $E\sin\left(\frac{t}{\sqrt{LC}}\right)$ thrugh leads of self- inductance L and negligible resistance. Prove that at any time t charge on one of the plate is $\frac{EC}{2}\left[\sin\left(\frac{t}{\sqrt{LC}}\right) - \frac{t}{\sqrt{LC}}\cos\left(\frac{t}{\sqrt{LC}}\right)\right]$

Solution: If q be the charge on the condenser, the differential equation of the circuit is

$$L\frac{d^2q}{dt^2} + \frac{q}{C} = E\sin\left(\frac{t}{\sqrt{LC}}\right) \quad \text{Or} \ \left(LD^2 + \frac{1}{C}\right)q = E\sin\left(\frac{t}{\sqrt{LC}}\right)$$

A.E. is $LD^2 + \frac{1}{C} = 0,$ roots are $D = \pm\frac{1}{\sqrt{LC}}i$

Therefore $C.F. = c_1\cos\left(\frac{t}{\sqrt{LC}}\right) + c_2\sin\left(\frac{t}{\sqrt{LC}}\right).$

$$P.I. = \frac{1}{LD^2 + \frac{1}{C}}\ E\sin\left(\frac{t}{\sqrt{LC}}\right)$$

$$= t\frac{1}{2LD}\ E\sin\left(\frac{t}{\sqrt{LC}}\right) = -\sqrt{\frac{C}{L}}\frac{Et}{2}\ \cos\left(\frac{t}{\sqrt{LC}}\right)$$

Hence $q = c_1 \cos\left(\frac{t}{\sqrt{LC}}\right) + c_2 \sin\left(\frac{t}{\sqrt{LC}}\right) - \sqrt{\frac{C}{L}}\frac{Et}{2}\cos\left(\frac{t}{\sqrt{LC}}\right)$

$$i = \frac{dq}{dt} = -\frac{1}{\sqrt{LC}}c_1 \sin\left(\frac{t}{\sqrt{LC}}\right) + \frac{1}{\sqrt{LC}}c_2 \cos\left(\frac{t}{\sqrt{LC}}\right) - \sqrt{\frac{C}{L}}\frac{E}{2}\left[\frac{t}{\sqrt{LC}}\sin\left(\frac{t}{\sqrt{LC}}\right) + \cos\left(\frac{t}{\sqrt{LC}}\right)\right]$$

When $t = 0,\ c = 0\ and\ i = 0 \Longrightarrow\ c_1 = 0$ and $\frac{1}{\sqrt{LC}}c_2 - \sqrt{\frac{C}{L}}\frac{E}{2} = 0 \Longrightarrow c_2 = \sqrt{\frac{C}{L}}\frac{E}{2}\sqrt{LC} = \frac{EC}{2}$

$$\therefore\ q = \frac{EC}{2}\sin\left(\frac{t}{\sqrt{LC}}\right) - \sqrt{\frac{C}{L}}\frac{Et}{2}\cos\left(\frac{t}{\sqrt{LC}}\right) = \frac{EC}{2}\left[\sin\left(\frac{t}{\sqrt{LC}}\right) - \frac{t}{\sqrt{LC}}\cos\left(\frac{t}{\sqrt{LC}}\right)\right].$$

2. A condenser of capacity C discharged through an inductance L and resistance R in series and the charge q at time t satisfies the equation $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0$. Given that $L = 0.25\ henries, R = 250\ ohms$, $C = 2 \times 10^{-6}$ farads, and when $t = 0, q = 0.002$ coulombs and the current $i = 0$

Obtain the value of q in terms of t.

Solution: $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0 \ \Longrightarrow\ 0.25\frac{d^2q}{dt^2} + 250\frac{dq}{dt} + \frac{q}{2\times10^{-6}} = 0$

$\Longrightarrow\ (D^2 + 1000D + 2000000)q = 0$, Roots of A.E. are $-500 \pm 1323i$

$\therefore\ q = e^{-500t}(c_1\cos 1323t + c_2 \sin 1323t)$

And $i = \frac{dq}{dt} = e^{-500t}1323(-c_1\sin 1323t + c_2\cos 1323t) - 500q$

Given that when $t = 0$, $q = 0.002$, $i = 0$

$c_1 = 0.002$ and $1323c_2 - 500 \times 0.002 = 0 \Longrightarrow c_2 = 0.0008$

Hence $q = e^{-500t}(0.002\cos 1323t + 0.0008 \sin 1323t)$.