Module-2: Vector Calculus
Vector Differentiation: Scalar and vector fields. Gradient, directional derivative, curl and divergence - physical interpretation, solenoidal and irrotational vector fields. Problems. **Vector Integration:** Line integrals, Surface integrals. Applications to work done by a force and flux. Statement of Green's theorem and Stoke's theorem. Problems. **Self-Study**: Volume integral and Gauss divergence theorem. **(RBT Levels: L1, L2 and L3)**

Scalar point function: $f = f(x,\ y,\ z)$

Vector point function: Functions of the type $F = f_1\boldsymbol{i} + f_2\boldsymbol{j} + f_3\boldsymbol{k}$**.**
Where f_1, f_2 and f_3 are Scalar point functions.

Vector operator: del : $\boldsymbol{\nabla} = \sum \boldsymbol{i}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial x}} = \boldsymbol{i}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial x}} + \boldsymbol{j}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial y}} + \boldsymbol{k}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial z}}$ **.**

Gradient of scalar point function: $\boldsymbol{grad(f)} = \boldsymbol{\nabla} f = \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} + \frac{\partial f}{\partial z}\boldsymbol{k}$ **.**

Note: 1. $\boldsymbol{\nabla} f$ is normal to the surface $f(x,y,z) = 0$. ∴ unit normal vector to the surface $f = c$ is $\frac{1}{|\boldsymbol{\nabla} f|}(\boldsymbol{\nabla} f)$

2. Angle between the two surfaces $f = 0$ & $g = 0$ is $\theta = \cos^{-1}\left|\frac{\boldsymbol{\nabla} f \circ \boldsymbol{\nabla} g}{|\boldsymbol{\nabla} f|\,|\boldsymbol{\nabla} g|}\right|$.

3. Directional derivative of f along $\vec{a}$ is $\frac{\boldsymbol{\nabla} f \circ \vec{a}}{|\vec{a}|}$.

Maximum Directional derivative is $|\boldsymbol{\nabla} f|$ and is along $\boldsymbol{\nabla} f$.

Divergence of a vector field: $\boldsymbol{Div(F)} = \boldsymbol{\nabla} \circ \mathrm{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$. Where $F = f_1\boldsymbol{i} + f_2\boldsymbol{j} + f_3\boldsymbol{k}$**.**

Note: F is **Solenoidal** $\Leftrightarrow Div(F) = 0$.

Curl of a vector field: $\boldsymbol{Curl(F)} = \boldsymbol{\nabla} \times \mathrm{F} = \begin{vmatrix} i & j & k \\ \boldsymbol{\frac{\partial}{\partial x}} & \boldsymbol{\frac{\partial}{\partial y}} & \boldsymbol{\frac{\partial}{\partial z}} \\ f_1 & f_2 & f_3 \end{vmatrix}$. Where $F = f_1\boldsymbol{i} + f_2\boldsymbol{j} + f_3\boldsymbol{k}$**.**

Note: F is **irrotational** $\Leftrightarrow \boldsymbol{Curl}(F) = 0$.

If F is irrotational then their exist a scalar potential ∅ such that $= \nabla\emptyset$, and

$$\emptyset = \int_{(y,z\ constant)} f_1\,dx + \int_{(z\ constant)} (\text{terms of } f_2 \text{ not containing } x)\,dy + \int (\text{terms of } f_3 \text{ not containing } x \text{ and } y)\,dz + c.$$

Theorems:

1. Prove that $curl grad\emptyset = 0$.

Proof: $L.H.S = \nabla \times (\nabla\emptyset)$

$$= \nabla \times \left[\frac{\partial\emptyset}{\partial x}i + \frac{\partial\emptyset}{\partial y}j + \frac{\partial\emptyset}{\partial z}k\right] = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\emptyset}{\partial x} & \frac{\partial\emptyset}{\partial y} & \frac{\partial\emptyset}{\partial z} \end{vmatrix}$$

$$= \sum \left[\frac{\partial^2\emptyset}{\partial y\partial z} - \frac{\partial^2\emptyset}{\partial z\partial y}\right] i = 0 = R.H.S \ .$$

2. Prove that $div curl\, F = 0$.

Proof: $L.H.S = \nabla \circ (\nabla \times \mathrm{F}) = \sum \mathrm{i}\frac{\partial}{\partial x} \circ \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$

$$= \sum \frac{\partial}{\partial x}\left[\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right] = \frac{\partial^2 f_3}{\partial x\partial y} - \frac{\partial^2 f_2}{\partial x\partial z} + \frac{\partial^2 f_1}{\partial y\partial z} - \frac{\partial^2 f_3}{\partial y\partial x} + \frac{\partial^2 f_2}{\partial z\partial x} - \frac{\partial^2 f_1}{\partial z\partial y} = 0 \ .$$

3. Prove that $div(\emptyset F) = grad\emptyset \circ F + \emptyset div F.$

Proof: $L.H.S = \nabla \circ (\emptyset F) = \sum i\frac{\partial}{\partial x} \circ (\emptyset F) = \sum i \circ \frac{\partial}{\partial x}(\emptyset F)$

$$= \sum i \circ [\frac{\partial\emptyset}{\partial x}F + \emptyset\frac{\partial F}{\partial x}] = \sum i\frac{\partial\emptyset}{\partial x} \circ F + \emptyset\sum i \circ \frac{\partial F}{\partial x}$$

$$= \nabla\emptyset \circ F + \emptyset(\nabla \circ F) = R.H.S \ .$$

4. Prove that $curl(\emptyset F) = grad\emptyset \times F + \emptyset curl F.$

Proof: $L.H.S = \nabla \times (\emptyset F) = \sum i\frac{\partial}{\partial x} \times (\emptyset F) = \sum i \times \frac{\partial}{\partial x}(\emptyset F)$

$$= \sum i \times [\frac{\partial\emptyset}{\partial x}F + \emptyset\frac{\partial F}{\partial x}] = \sum i\frac{\partial\emptyset}{\partial x} \times F + \emptyset\sum i \times \frac{\partial F}{\partial x}$$

$$= \nabla\emptyset \times F + \emptyset(\nabla \times F) = R.H.S \ .$$

Note: 1. $grad[f(r)] = \nabla f(r) = \frac{f'(r)}{r}R$ where $R = \vec{r} = xi + yj + zk, \ r = \sqrt{x^2 + y^2 + z^2}$.

2. $div(grad[f(r)]) = \nabla \circ \nabla f(r) = \nabla^2 f(r) = f''(r) + \frac{2}{r}f'(r)$.

3. $div(grad f) = \nabla^2 f = f_{xx} + f_{yy} + f_{zz}$.

Examples:

1. Find $\nabla \log(x^2 + y^2 + z^2)$ and grad $\left(\frac{1}{r}\right)$.

Solution: $\nabla \log(x^2 + y^2 + z^2) = \nabla \log(r^2)$

$$= \nabla 2\log(r) = \frac{f'(r)}{r}R = \frac{2}{r^2}R$$

$$= \frac{2}{x^2+y^2+z^2}(xi + yj + zk)$$

And grad $\left(\frac{1}{r}\right) = \frac{f'(r)}{r}R = -\frac{1}{r^3}R.$

2. Find the unit normal vector to the surface $x^3 + y^3 + 3xyz = 3$ at the point $(1,\ 2, -1)$.

Solution: Since $\boldsymbol{\nabla} f$ is normal to the surface $f(x, y, z) = 0$. Let $f = x^3 + y^3 + 3xyz - 3$.

$$\boldsymbol{\nabla} f = (3x^2 + 3yz)\boldsymbol{i} + (3y^2 + 3xz)\boldsymbol{j} + 3xy\boldsymbol{k}$$

At the point $(1,\ 2, -1)$, $\boldsymbol{\nabla} f = -3i + 9j + 6k$ **.**

Unit normal vector$= \frac{\boldsymbol{\nabla} f}{|\boldsymbol{\nabla} f|} = \frac{1}{\sqrt{14}}(-i + 3j + 2k)$.

3. Find the directional derivative of $\emptyset = x^2yz + 4xz^2$ at the point $(1, -2, -1)$ in the direction of $2i - j - 2k$.

Solution: Clearly $\nabla\emptyset = (2xyz + 4z^2)i + x^2zj + (x^2y + 8xz)k$.

At the point $(1, -2, -1)$, $\nabla\emptyset = 8i - j - 10k$. And let $\vec{a} = 2i - j - 2k$.

Directional derivative of $\emptyset$ along $\vec{a}$ is $\frac{\nabla\emptyset \circ \vec{a}}{|\vec{a}|} = \frac{16+1+20}{\sqrt{4+1+4}} = \frac{37}{3}$.

4. Calculate the angle between the normals to the surface $xy = z^2$ at the points $(4,\ 1,\ 2)$ and $(3,\ 3, -3)$.

Solution: Since ∇f is normal to the surface $f = 0$, let $f = xy - z^2$. $\nabla f = yi + xj - 2zk$.

Normal at $(4,\ 1,\ 2) = N_1 = i + 4j - 4k$.

And normal at $(3,\ 3, -3) = N_2 = 3i + 3j + 6k$.

angle between them is $\theta = \cos^{-1}\left|\frac{N_1 \circ N_2}{|N_1|\,|N_2|}\right| = \cos^{-1}\left|\frac{3+12-24}{\sqrt{33}\sqrt{54}}\right| = \cos^{-1}\frac{1}{\sqrt{22}}$.

5. Find the angle between the two surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at $(2, -1,\ 2)$.

Solution: Let $f = x^2 + y^2 + z^2$ and $g = x^2 + y^2 - z$

Then $\nabla f = 2xi + 2yj + 2zk$, and $\nabla g = 2xi + 2yj - k$,

At $(2, -1,\ 2)$, $\nabla f = 4i - 2j + 4k$, and $\nabla g = 4i - 2j - k$,

Angle between the two surfaces $f = 9$ & $g = 3$ is

$$\theta = \cos^{-1}\left|\frac{\nabla f \circ \nabla g}{|\nabla f|\,|\nabla g|}\right| = \cos^{-1}\left(\frac{16+4-4}{\sqrt{16+4+16}\sqrt{16+4+1}}\right) = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right).$$

6. Show that $\nabla^2 (r^n) = n(n+1)r^{n-2}$.

Solution: If $f(r) = r^n$ then $f'(r) = nr^{n-1}$ *and* $f''(r) = n(n-1)r^{n-2}$.

Since $\nabla^2 f(r) = f''(r) + \frac{2}{r}f'(r)$,

$$\nabla^2 (r^n) = n(n-1)r^{n-2} + \frac{2}{r}nr^{n-1} = [n(n-1) + 2n]r^{n-2} = n(n+1)r^{n-2}.$$

In particular $\nabla^2 \left(\frac{1}{r}\right) = 0.$

7. Prove that $\nabla(r^n) = nr^{n-2}R,\ R = xi + yj + zk$.

Proof: Since $\nabla f(r) = \frac{f'(r)}{r}R$, $f(r) = r^n$, $f'^{(r)} = nr^{n-1}$.

$$\nabla(r^n) = \frac{nr^{n-1}}{r}R = nr^{n-2}R.$$

8. If $\nabla(u) = 2r^4R$, find u.

Solution: $nr^{n-2}R = \nabla(r^n) \Rightarrow \; r^{n-2}R = \frac{\nabla(r^n)}{n}$

$r^4R = \frac{\nabla(r^6)}{6} \quad \Rightarrow 2r^4R = \nabla\left(\frac{r^6}{3}\right)$. Therefore $u = \frac{r^6}{3} + c$.

9. If $F = grad\,[x^3 + y^3 + z^3 - 3xyz]$, find $divF$ and $curlF$.

Solution: Given that $F = \nabla f$, where $f = x^3 + y^3 + z^3 - 3xyz$.

$$F = \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} + \frac{\partial f}{\partial z}\boldsymbol{k} = (3x^2 - 3yz)i + (3y^2 - 3xz)j + (3z^2 - 3xy)k.$$

$\therefore F = f_1\boldsymbol{i} + f_2\boldsymbol{j} + f_3\boldsymbol{k}$, where $f_1 = (3x^2 - 3yz)$, $f_2 = (3y^2 - 3xz)$, $f_3 = (3z^2 - 3xy)$.

$$Div(F) = \nabla \circ \mathrm{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = 6x + 6y + 6z.$$

And $Curl(F) = \nabla \times \mathrm{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2 - 3yz & 3y^2 - 3xz & 3z^2 - 3xy \end{vmatrix}$

$$= (-3x + 3x)i - (-3y + 3y)j + (-3z + 3z)k = \mathbf{0.}$$

Or $Curl(F) = \mathrm{Curl}(\mathrm{grad}f) = 0$.

10. Find the value of a if $F = (ax^2y + yz)i + (xy^2 - xz^2)j + (2xyz - 2x^2y^2)k$ is solenoidal.

Solution: Since F is solenoidal, $div(F) = 0$, that is $\nabla \circ \mathrm{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = 0$.

$\therefore \; 2axy + 2xy + 2xy = 0 \quad \Rightarrow a = -2.$

11. Find a, b, c, if $F = (x + by - z)i + (2x - y + cz)j + (ax + y - z)k$ is irrotational. And also find scalar potential $\emptyset$ such that $F = \nabla\emptyset$.

Solution: F is irrotational $\Rightarrow$ $Curl(F) = 0$

$$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + by - z & 2x - y + cz & ax + y - z \end{vmatrix} = 0$$

$\Rightarrow (1 - c)i - (a + 1)j + (2 - b)k = 0 \quad \Rightarrow a + 1 = 0,\; 2 - b = 0$ and $1 - c = 0$.

$\therefore \; a = -1, \; b = 2, \; c = 1.$

$$\emptyset = \underset{(y,z\ constant)}{\int f_1\,dx} + \underset{(z\ constant)}{\int} (\text{terms of } f_2 \text{ not containing } x\,)\,dy + \int (\text{terms of } f_3 \text{ not containing } x \text{and} y\,)\,dz + c.$$

$$= \int(x + 2y - z)\,dx + \int (-y + z\,)\,dy + \int (-z\,)\,dz = \frac{x^2}{2} + 2xy - xz - \frac{y^2}{2} + yz - \frac{z^2}{2} + c\,.$$

12. Show that $\frac{xi+yj}{x^2+y^2}$ is both solenoidal and irrotational.

Solution: Given that $F = \frac{x}{x^2+y^2}i + \frac{y}{x^2+y^2}j + 0k$.

$$Div(F) = \nabla \circ \mathrm{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right) = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} + \frac{x^2+y^2-2y^2}{(x^2+y^2)^2} = \frac{0}{(x^2+y^2)^2} = 0$$

Therefore F is solenoidal .

$$Curl(F) = \nabla\times \mathrm{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{x^2+y^2} & \frac{y}{x^2+y^2} & 0 \end{vmatrix} = 0i - 0j + \left[\frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right)\right]k$$

$$= \left[\frac{0-2xy}{(x^2+y^2)^2} - \frac{0-2xy}{(x^2+y^2)^2}\right]k = 0.$$ Hence F is irrotational.

13. Prove that $grad[f(r)] = \nabla f(r) = \frac{f'(r)}{r}R$ where $= \vec{r} = xi + yj + zk,\ r = \sqrt{x^2+y^2+z^2}$.

Solution: $grad[f(r)] = \nabla f(r) = \sum i \frac{\partial}{\partial x} f(r)$

$$= \sum i\, f'(r)\frac{\partial r}{\partial x} = \sum i\, f'(r)\frac{x}{r}$$

$$= \frac{f'(r)}{r}\sum xi = \frac{f'(r)}{r}R.$$

14. Prove that $div(grad[f(r)]) = \nabla\circ\nabla f(r) = \nabla^2 f(r) = f''(r) + \frac{2}{r}f'(r)$.

Solution: $div(grad[f(r)]) = \nabla\circ\nabla f(r) = \nabla^2 f(r)$

$$= \sum \frac{\partial^2}{\partial x^2} f(r) = \sum \frac{\partial}{\partial x}\left[\frac{f'(r)}{r}x\right]$$

$$= \sum \frac{\partial}{\partial x}\left[f'(r)\frac{1}{r}x\right] = \sum\left[f''(r)\frac{x^2}{r^2} - \frac{f'(r)}{r^2}\frac{x^2}{r} + \frac{f'(r)}{r}\right]$$

$$= \frac{f''(r)}{r^2}(x^2+y^2+z^2) - \frac{f'(r)}{r^3}(x^2+y^2+z^2) + 3\frac{f'(r)}{r}$$

$$= f''(r) + \frac{2}{r}f'(r) \quad .$$

Exercise :

1. In what direction from $(3,\ 1, -2)$ is the directional derivative of $\emptyset = x^2y^2z^4$ maximum? Find also the magnitude of this maximum.
2. Find the directional derivative of $\emptyset = xyz$ in the direction of the normal to the surface $x^2z + y^2x + z^2y = 3$ at $(1,\ 1,\ 1)$.
3. If $f = (x^2+y^2+z^2)^{-n}$, Find div(grad$f$) and determine n if div(grad$f$) $= 0$.
4. If $F = (x+y+1)i + j - (x+y)k$, Show that $F\circ curlF = 0$.
5. Find $curlR$ and $divR$. Where $R = xi + yj + zk$.
6. If $= e^{xyz}(i+j+k)$, find $curlF$ and $divF$.
7. Find $\nabla^2 f$ at $(1,\ 1,\ 0)$ if $f = 3x^2z - y^2z^3 + 4x^3y + 2x - 3y - 5$. $(\nabla^2 f = f_{xx} + f_{yy} + f_{zz})$
8. If $F = xy^2i + 2x^2yzj - 3yz^2k$, find $curl(F)$ and $div(F)$.
9. Show that $F = x(y-z)i + y(z-x)j + z(x-y)k$ is solenoidal.
10. Show that $F = (y+z)i + (z+x)j + (x+y)k$ is irrotational.
11. Find the constants a and b if $F = (axy + z^3)i + (3x^2 - z)j + (bxz^2 - y)k$ is irrotational, and also find the scalar potential.

12. Find the constants a, b and c if $F = (\sin y + az)i + (bx\cos y + z)j + (x + cy)k$ is irrotational.

Vector integration: Line integrals-definition and problems, surface and volume integrals-definition, Green's theorem in a plane, Stokes and Gauss-divergence theorem (without proof) and problems.

Line integrals:

The tangential line integral of vector function F along a curve C is $\int_C F\cdot dR$ or $\int_C F\cdot\frac{dR}{dt}dt$.

If $F = f_1\boldsymbol{i} + f_2\boldsymbol{j} + f_3\boldsymbol{k}$ and $dR = dx\boldsymbol{i} + dy\boldsymbol{j} + dz\boldsymbol{k}$

Then $\int_C F\cdot dR = \int_C f_1dx + f_2dy + f_3dz$.

Other types of line integrals are $\int_C F\times dR$ and $\int_C f dR$.

If F represents the velocity of a fluid particle then $\int_C F\cdot dR$ is called the circulation of F around the curve.
If the circulation of F around every closed curve in a region E is zero, then F is irrotational in E.

If F represents the force acting on a particle moving along an arc AB
Then the total work done by F during the displacement from A to B is $\int_A^B F\cdot dR$.

Examples:

1. If $F = 3xy\boldsymbol{i} - y^2\boldsymbol{j}$, Evaluate $\int_C F\cdot dR$, where C is the curve in the xy-plane $y = 2x^2$ from $(0,0)$ to $(1,2)$.

Sol: $F\cdot dR = (3xy\boldsymbol{i} - y^2\boldsymbol{j})\cdot(dx\boldsymbol{i} + dy\boldsymbol{j})$ $\because$ In $xy-$plane , $z = 0$.

$= 3xy\boldsymbol{dx} - y^2\boldsymbol{dy}$

$= (6x^3 - 16x^5)\boldsymbol{dx}$ $\because$ Along C, $y = 2x^2$, $dy = 4xdx$.

Therefore $\int_C F\cdot dR = \int_0^1(6x^3 - 16x^5)\boldsymbol{dx} = \left[6\frac{x^4}{4} - 16\frac{x^6}{6}\right]_0^1 = -\frac{7}{6}$.

2. A vector field is given by $F = \sin y\,\boldsymbol{i} + x(1 + \cos y)\boldsymbol{j}$. Evaluate the line integral over a circular path given by $x^2 + y^2 = a^2,\ z = 0$.

Sol: Along $x^2 + y^2 = a^2,\ x = a\cos t,\ y = a\sin t,$ and $dx = -a\sin t\,dt,\ dy = a\cos t\,dt,\ 0 \le t \le 2\pi$.

$F\cdot dR = (\sin y\,\boldsymbol{i} + x(1 + \cos y)\boldsymbol{j})\cdot(dx\boldsymbol{i} + dy\boldsymbol{j})$ $\because$ In $xy-$plane , $z = 0$.

$= \sin y\,\boldsymbol{dx} + x(1 + \cos y)\boldsymbol{dy}$

$= [-a\sin t\sin(a\sin t) + a^2\cos^2 t\,(1 + \cos(a\sin t))]dt$

$= a^2\cos^2 t\,dt + [-a\sin t\sin(a\sin t) + a^2\cos^2 t\cos(a\sin t)]dt$

$= a^2\cos^2 t\,dt + [-a\sin t\sin(a\sin t) + a^2\cos^2 t\cos(a\sin t)]dt$

$= a^2\cos^2 t\,dt + d[a\cos t\sin(a\sin t)]$

Therefore $\int_C F\cdot dR = \int_0^{2\pi}\{a^2\cos^2 t\,dt + d[a\cos t\sin(a\sin t)]\}$

$= \int_0^{2\pi}\{a^2\cos^2 t\,dt + d[a\cos t\sin(a\sin t)]\}$

$= 4a^2\int_0^{\frac{\pi}{2}}\cos^2 t\,dt + [a\cos t\sin(a\sin t)]_0^{2\pi}$

$= \pi a^2$.

3. Find the work done in moving a particle by the force $F = 3x^2\boldsymbol{i} + (2xz - y)\boldsymbol{j} + z\boldsymbol{k}$ **,** along

i) a straight line from $(0,\ 0,\ 0)$ to $(2,\ 1,\ 3)$

ii) the curve defined by $x^2 = 4y,\ \ 3x^3 = 8z$ from $x = 0$ to $x = 2$.

Sol: i) Equation of the straight line passing through the points $(0,\ 0,\ 0)$ and $(2,\ 1,\ 3)$ is

$\frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t,$ or $x = 2t,\quad y = t,\quad z = 3t \implies dx = 2dt,\ \ dy = dt,\ \ dz = 3dt.$

$$F \cdot dR = 3x^2\boldsymbol{dx} + (2xz - y)\boldsymbol{dy} + z\boldsymbol{dz}$$

$$= (24t^2 + 12t^2 - t + 9t)\boldsymbol{dt} = (36t^2 + 8t)\boldsymbol{dt}$$

Work done $= \int_C F \cdot dR = \int_0^1 (36t^2 + 8t)\boldsymbol{dt}$

$$= (12t^3 + 4t^2)\Big|_0^1 = 16\ .$$

ii) Since $y = \frac{x^2}{4},\ z = \frac{3x^3}{8},\qquad dy = \frac{x}{2}dx,\quad dz = \frac{9x^2}{8}dx$,

$$F \cdot dR = 3x^2\boldsymbol{dx} + (2xz - y)\boldsymbol{dy} + z\boldsymbol{dz}$$

$$= \left(3x^2 + \frac{51x^5}{64} - \frac{x^3}{8}\right)\boldsymbol{dx}$$

Work done $= \int_C F \cdot dR = \int_0^2 \left(3x^2 + \frac{51x^5}{64} - \frac{x^3}{8}\right)\boldsymbol{dx}$

$$= \left(x^3 + \frac{17x^6}{128} - \frac{x^4}{32}\right)\Big|_0^2 = 16.$$

Surface integral: The normal surface integral of F over the surface S is given by $\int_S F \cdot dS$ or $\int_S F \cdot N\, ds$. Where N is a unit outward normal vector at P to .

Other types of surface integrals are $\int_S F \times dS$ and $\int_S f dS$.

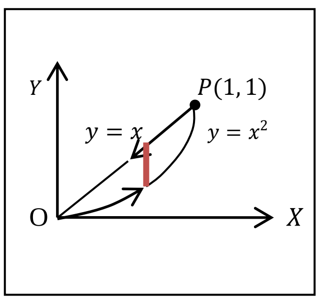
Flux across a surface: If F represents the velocity of a fluid particle then the total outward flux of F across a closed surface S is the surface integral $\int_S F \cdot dS$.

If the flux across every closed surface S in a region E is zero, then F is solenoidal vector point function in E.

Green's theorem in the plane: If $M(x, y)$, $N(x, y)$, $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ are continuous in a region E of the xy-plane bounded by a closed curve C, then $\int_C Mdx + Ndy = \iint_E \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] dxdy.$

1. Using Green's theorem evaluate $\int_C [(xy + y^2)\boldsymbol{dx} + x^2\boldsymbol{dy}]$, where C is bounded by $y = x$ and $y = x^2$.

Solution: Clearly $M = xy + y^2,\quad N = x^2$, $\frac{\partial N}{\partial x} = 2x$ and $\frac{\partial M}{\partial y} = x + 2y.$



By Green's theorem

$$\int_C Mdx + Ndy = \iint_E \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] dxdy$$

$$= \int_0^1 \int_{x^2}^{x} (x-2y)dydx = \int_0^1 [xy-y^2]_{x^2}^{x} dx$$

$$= \int_0^1 [x^4 - x^3]\, dx = \left[\frac{x^5}{5} - \frac{x^4}{4}\right]_0^1 = \frac{1}{5} - \frac{1}{4} = -\frac{1}{20}.$$

2. If C is the simple closed curve in xy-plane not enclosing the origin, show that $\int_C F \cdot dR = 0$, where $F = \frac{yi - xj}{x^2+y^2}$.

Solution: $F \cdot dR = \frac{ydx}{x^2+y^2} - \frac{xdy}{x^2+y^2}$

$$\int_C F \cdot dR = \int_C \left[\frac{ydx}{x^2+y^2} - \frac{xdy}{x^2+y^2}\right] ,$$

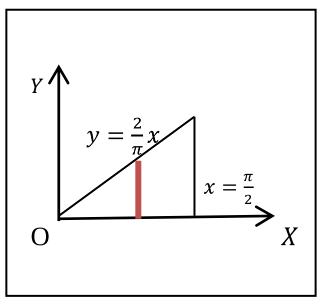
$$M = \frac{y}{x^2+y^2}, \quad N = \frac{-x}{x^2+y^2}, \qquad \frac{\partial N}{\partial x} = \frac{x^2-y^2}{(x^2+y^2)^2}, \quad \frac{\partial M}{\partial y} = \frac{x^2-y^2}{(x^2+y^2)^2}.$$

By Green's theorem. $\int_C F \cdot dR = \iint_E \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] dxdy$

$$= \iint_E \left[\frac{x^2-y^2}{(x^2+y^2)^2} - \frac{x^2-y^2}{(x^2+y^2)^2}\right] dxdy = 0.$$

3. Using the Green's theorem, evaluate $\int_C [(y - \sin x)dx + \cos x\, dy]$ where C is the plane triangle enclosed by the lines $y = 0$, $x = \frac{\pi}{2}$ and $y = \frac{2}{\pi}x$.

Sol: Clearly $M = y - \sin x$, $N = \cos x$, $\frac{\partial N}{\partial x} = -\sin x$ and $\frac{\partial M}{\partial y} = 1.$



$$\int_C [(y - \sin x)dx + \cos x\, dy] = \iint_E \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] dxdy$$

$$= -\int_0^{\frac{\pi}{2}} \int_0^{\frac{2}{\pi}x} (\sin x + 1)dydx = -\int_0^{\frac{\pi}{2}} [(\sin x + 1)y]_0^{\frac{2}{\pi}x} dx$$

$$= -\frac{2}{\pi}\int_0^{\frac{\pi}{2}} (x\sin x + x)dx = -\frac{2}{\pi}\left[-x\cos x + \sin x + \frac{x^2}{2}\right]_0^{\frac{\pi}{2}}$$

$$= -\frac{2}{\pi}\left[1 + \frac{\pi^2}{8}\right] = -\left[\frac{2}{\pi} + \frac{\pi}{4}\right].$$

4. Apply Green's theorem to evaluate $\int_C [(2x^2 - y^2)dx + (x^2 + y^2)dy]$ where C is the boundary of the area enclosed by the x-axis and the upper half of the circle $x^2 + y^2 = a^2$.

By Green's theorem,

$$\int_C [(2x^2 - y^2)dx + (x^2 + y^2)dy] = \iint_E \left[\frac{\partial}{\partial x}(x^2 + y^2) - \frac{\partial}{\partial y}(2x^2 - y^2)\right] dxdy$$

$$= 2\iint_E [x + y]\, dxdy = 2\int_0^a \int_0^{\pi} (\cos\theta + \sin\theta) r^2 d\theta dr$$

$$= 2\int_0^a r^2 d\theta dr \times \int_0^\pi (\cos\theta + \sin\theta) d\theta \ = \frac{4}{3}a^3 \ .$$

5. Apply Green's theorem to prove that the area enclosed by a plane curve C is $\frac{1}{2}\int_C (xdy - ydx)$. Hence find the area of an ellipse whose semi-major and semi-minor axes are of lengths a and b.

Solution: By the Green's theorem, $\int_C (xdy - ydx) = \iint_E \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] dxdy$

$$= \iint_E [1 - (-1)]\, dxdy$$

$$= 2\iint_E \ dxdy = 2 \times \text{Area enclosed by the plane curve } C.$$

Therefore area enclosed by a plane curve $C = \frac{1}{2}\int_C (xdy - ydx)$.

Area of an ellipse$= \frac{1}{2}\int_C (xdy - ydx)$ Put $x = a\ \cos t\ ,\ y = b\sin t$

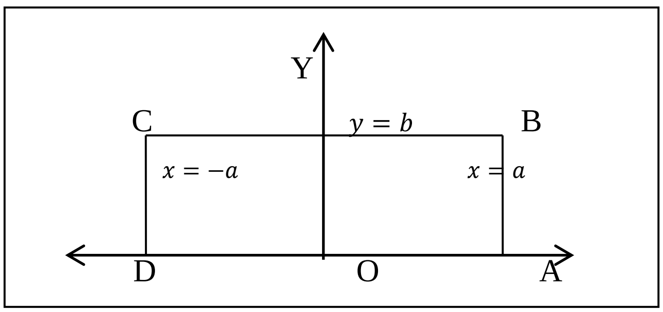
$$= \frac{1}{2}\int_0^{2\pi} (ab\cos^2 t + ab\sin^2 t)\, dt = \frac{ab}{2}\int_0^{2\pi} dt = \pi ab.$$

Stoke's theorem: If S be an open surface bounded by a closed curve C and $F = f_1\boldsymbol{i} + f_2\boldsymbol{j} + f_3\boldsymbol{k}$ be any continuously differentiable vector point function, then

$\int_C F\cdot dR = \int_S curlF \cdot N\, ds$. Where N is a unit external normal vector at any point of S.

1. Using Stoke's theorem evaluate $\int_C F\cdot dR$, where $F = (x^2+y^2)\boldsymbol{i} - 2xy\boldsymbol{j}$ taken around the rectangle bounded by the lines $x = \pm a,\ y = 0,\ y = b$.

Sol:



$$curlF = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y^2 & -2xy & 0 \end{vmatrix} = 0i - 0j + (-2y-2y)k = -4yk \ , \quad \text{and} \quad N = k.$$

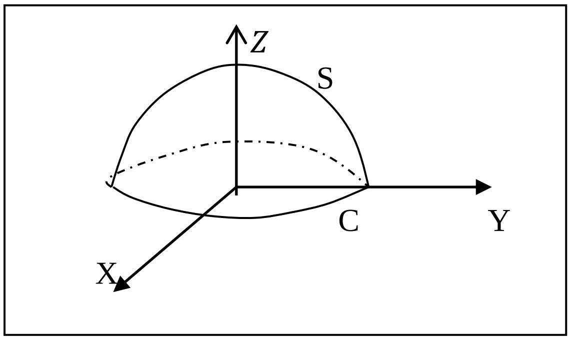
$$curlF\cdot N = -4y$$

By Stoke's theorem, $\int_C F\cdot dR = \int_S curlF\cdot N\, ds$

$$= \int_{-a}^{a}\int_0^b -4y\,dydx = \int_{-a}^{a} -2b^2\, dx = -4ab^2.$$

7. Using Stoke's theorem evaluate $\int_C F\cdot dR$, where $F = (2x-y)\boldsymbol{i} - yz^2\boldsymbol{j} - y^2z\boldsymbol{k}$ over the upper half surface of $x^2+y^2+z^2 = 1$, bounded by its projection on the xy-plane.

Sol:



The projection of the upper half surface of $x^2 + y^2 + z^2 = 1$ is the circle C: $x^2 + y^2 = 1$.

$$curlF = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x - y & -yz^2 & -y^2 z \end{vmatrix} = 0\boldsymbol{i} + 0\boldsymbol{j} + \boldsymbol{k} = \boldsymbol{k}$$, and $N = k$. $curlF \cdot N = 1$.

$\int_C F \cdot dR = \int_S curlF \cdot N\, ds = \int_A 1.dxdy = area\ of\ the\ circle\ C = \pi r^2 = \pi.$

$\because$ A is the projection of S in xy-plane .

8. Using Stoke's theorem evaluate $\int_C [(x + y)\boldsymbol{dx} + (2x - z)\boldsymbol{dy} + (y + z)\boldsymbol{dz}]$ where C is the boundary of the triangle with vertices $(2, 0, 0)$, $(0, 3, 0)$ and $(0, 0, 6)$.

Sol: Let $A \equiv (2, 0, 0)$, $B \equiv (0, 3, 0)$ and $C \equiv (0, 0, 6)$

Clearly $F = (x + y)\boldsymbol{i} + (2x - z)\boldsymbol{j} + (y + z)\boldsymbol{k}$, $\therefore curlF = 2\boldsymbol{i} + 0\boldsymbol{j} + \boldsymbol{k}$.

Equation to the triangle is $\frac{x}{2} + \frac{y}{3} + \frac{z}{6} = 1$, or $3x + 2y + z - 6 = 0$.

Normal vector is $\nabla(3x + 2y + z - 6) = 3i + 2j + k$. $\therefore N = \frac{1}{\sqrt{14}}(3i + 2j + k)$.

$curlF \cdot N = \frac{7}{\sqrt{14}}$.

By Stoke's theorem,

$$\int_C [(x + y)\boldsymbol{dx} + (2x - z)\boldsymbol{dy} + (y + z)\boldsymbol{dz}] = \int_C F \cdot dR$$

$$= \int_S curlF \cdot N\, ds = \frac{7}{\sqrt{14}}\int_S ds = \frac{7}{\sqrt{14}} \times Area\ of\ the\ \Delta ABC = \frac{7}{\sqrt{14}} \times 3\sqrt{14} = 21.$$

9. If $F = 3y\boldsymbol{i} - xz\boldsymbol{j} + yz^2\boldsymbol{k}$ and S is the surface of the paraboloid $2z = x^2 + y^2$ bounded by $z = 2$. Evaluate $\int\int_S (\nabla \times F) \cdot ds$ using Stoke's theorem.

Sol: By Stoke's theorem,

$$\int\int_S (\nabla \times F) \cdot ds = \int_C F \cdot dR \qquad \text{When } z = 2,\ x^2 + y^2 = 4\ , dz = 0$$

$$= \int_C [3y\boldsymbol{dx} - xz\boldsymbol{dy} + yz^2\boldsymbol{dz}] \qquad Put\ x = 2\cos\theta\ ,\ y = 2\sin\theta$$

$$= -\int_0^{2\pi} 20\sin^2\theta\, d\theta = -80\int_0^{\frac{\pi}{2}} \sin^2\theta\, d\theta = -20\pi.$$

10. Apply Stoke's theorem to evaluate $\int_C [y\boldsymbol{dx} + z\boldsymbol{dy} + x\boldsymbol{dz}]$, where C is the curve of intersection of

$x^2 + y^2 + z^2 = a^2 \;\; and \;\; x + z = a.$

Solution: Clearly the curve C is the circle with $(a, 0, 0)$ and $(0, 0, a)$ as end points of a diameter. Therefore radius is $\frac{a}{\sqrt{2}}$.

Since $F = y\boldsymbol{i} + z\boldsymbol{j} + x\boldsymbol{k}$

$$curlF = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = -\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k} \quad \text{and} \quad N = \frac{1}{\sqrt{2}}(\boldsymbol{i} + \boldsymbol{k}) \;\; \Longrightarrow curlF \cdot N = -\sqrt{2} \; .$$

By Stoke's theorem,

$$\int_C [y\boldsymbol{dx} + z\boldsymbol{dy} + x\boldsymbol{dz}] = \int_S curlF \cdot N\, ds$$

$$= -\sqrt{2}\int_S ds = -\sqrt{2} \times Area\ of\ the\ circle$$

$$= -\sqrt{2} \times \pi\left(\frac{a}{\sqrt{2}}\right)^2 = -\frac{\pi a^2}{\sqrt{2}}.$$

Self-study:

Volume integral: If $F = f_1\boldsymbol{i} + f_2\boldsymbol{j} + f_3\boldsymbol{k}$ and $dv = dxdydz$, then the volume integral of F over E is

$$\int_E F dv = \left(\iiint_E f_1 dxdydz\right)\boldsymbol{i} + \left(\iiint_E f_2 dxdydz\right)\boldsymbol{j} + \left(\iiint_E f_3 dxdydz\right)\boldsymbol{k}.$$

Gauss-divergence theorem: If F is a continuously differentiable vector point function in the region E bounded by the closed surface S, then

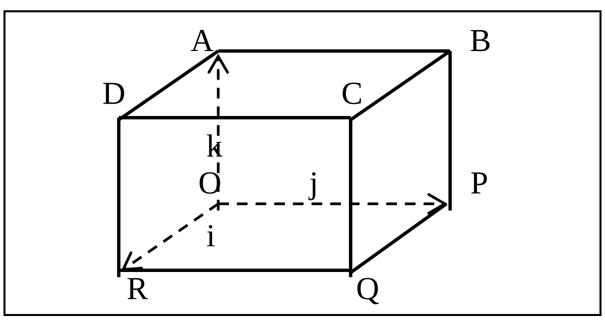
$$\int_S F \cdot N\, ds = \int_E divF dv$$. Where N is a unit external normal vector at any point of S.

1. Using Divergence theorem evaluate $\int_S F \cdot N\, ds,$ if $F = (x^2 - yz)\boldsymbol{i} + (y^2 - xz)\boldsymbol{j} + (z^2 - xy)\boldsymbol{k}$ taken over the rectangular parallelepiped $\leq x \leq a,\;\; o \leq y \leq b,\;\; o \leq z \leq c$.

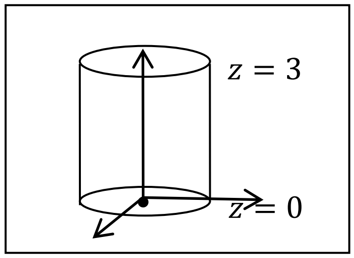
Sol: $\int_S F \cdot N\, ds = \int_E divF dv$

$$divF = \frac{\partial}{\partial x}(x^2 - yz) + \frac{\partial}{\partial y}(y^2 - xz) + \frac{\partial}{\partial z}(z^2 - xy) = 2(x + y + z)$$

$$\int_E divF dv = \int_0^a \int_0^b \int_0^c 2(x + y + z) dzdydx = abc(a + b + c).$$



2. Evaluate $\int_S F\cdot ds$ where $F = 4x\boldsymbol{i} - 2y^2\boldsymbol{j} + z^2\boldsymbol{k}$ and S is the surface bounding the region $x^2+y^2=4$, $z = 0 \ and \ z = 3$.



Solution: By the divergence theorem,

$$\int_S F\cdot ds = \int_V divF dv = \int_V (4-4y+2z)dv \quad = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_0^3 (4-4y+2z)\, dzdydx$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4z-4yz+z^2)\Big|_0^3 dydx \quad = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (21-12y)\, dydx$$

$$= 2\int_{-2}^{2}\int_0^{\sqrt{4-x^2}} (21)\, dydx = 42\int_{-2}^{2}\sqrt{4-x^2}dx$$

$$= 84\int_0^2\sqrt{4-x^2}dx \ = 84\int_0^{\frac{\pi}{2}} 4\cos^2 t\, dt = 84\pi.$$

3. Using divergence theorem, evaluate $\int_S R\cdot N\, ds$, where S is the surface of the sphere $x^2+y^2+z^2=9$.

Solution: $divR = \nabla\circ R = \nabla\circ(xi+yj+zk) = 3$.

$$\int_S R\cdot N\, ds = \int_V divRdv = 3\int_V dv = 3\times \text{Volume of the sphere} = 108\pi.$$

Since $r=3$, volume of the sphere $= \frac{4}{3}\pi r^3 = 36\pi.\ \ r=3.$