Module-5: Numerical methods -2
Numerical Solution of Ordinary Differential Equations (ODE's): Numerical solution of ordinary differential equations of first order and first degree: Taylor's series method, Modified Euler's method, Runge-Kutta method of fourth order, Milne's predictor-corrector formula (No derivations of formulae). Problems. **Self-Study:** Adam-Bashforth method. **(RBT Levels: L1, L2 and L3)**

Numerical solution of Ordinary differential equations:

1. Taylor's series method: To find $y(x)$ from $\frac{dy}{dx} = f(x,y),\ given\ y(x_0) = y_0$.

First find the values of $y'(x_0),\ y''(x_0),\ y'''(x_0),\ y'^v(x_0)$ ……

Then $y(x) = y_0 + y'(x_0)(x - x_0) + \frac{y''(x_0)}{2!}(x - x_0)^2 + \frac{y'''(x_0)}{3!}(x - x_0)^3 + \frac{y^{iv}(x_0)}{4!}(x - x_0)^4 + \cdots$

If $\boldsymbol{\frac{dy}{dx} = f(x,y),\ given\ y(0) = y_0}$ **then** $\boldsymbol{x_0 = 0}$

$$\boldsymbol{y(x) = y_0 + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y^{iv}(0)}{4!}x^4 + \cdots}$$

2. **Modified Euler's method**: To find $y(x_1) = y_1$ from $\frac{dy}{dx} = f(x,y),\ given\ y(x_0) = y_0$.

Initial approximation of y_1 is $y_1^{(I)} = y_0 + hf(x_0,\ y_0)$ where $h = x_1 - x_0$.

Better approximation of y_1 is $y_1^{(M_1)} = y_0 + \frac{h}{2}\left[f(x_0,\ y_0) + f(x_1,\ y_1^{(I)})\right]$

$$y_1^{(M_2)} = y_0 + \frac{h}{2}\left[f(x_0,\ y_0) + f(x_1,\ y_1^{(M_1)})\right]$$

We repeat this step, till two consecutive values of y agree.

Once y_1 is obtained to desired degree of accuracy, y_2 can be obtained by replacing $x_0\ by\ x_1$, x_1 by x_2 , y_0 by y_1 and y_1 by y_2 in above formulae.

3. **Fourth order Runge-Kutta method**: To find $y(x_1) = y_1$ from $\frac{dy}{dx} = f(x,y),\ given\ y(x_0) = y_0$.

Calculate successively $k_1 = h\,f(x_0,\ y_0)$, $k_2 = h\,f(x_0 + \frac{h}{2},\ y_0 + \frac{k_1}{2})$

$k_3 = h\,f(x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2})$ and $k_4 = h\,f(x_0 + h,\ y_0 + k_3)$

Finally compute $k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

Then $\boldsymbol{y_1 = y_0 + k}$

Once y_1 is obtained, y_2 can be obtained by replacing $x_0\ by\ x_1$, $y_0\ by\ y_1$ and $y_1\ by\ y_2$ in above formulae.

4. **Milne's method**: To find $y(x_4) = y_4$ from $\frac{dy}{dx} = f(x,y),\ given\ y(x_0) = y_0 , y(x_1) = y_1$, $y(x_2) = y_2$, and $y(x_3) = y_3$.

First calculate $y_1' = f(x_1,\ y_1)$, $y_2' = f(x_2,\ y_2)$, and $y_3' = f(x_3,\ y_3)$

Then the Predictor formula is $y_4^{(p)} = y_0 + \frac{4h}{3}\,[2y_1' - y_2' + 2y_3']$

Corrector formula is $y_4^{(c_1)} = y_2 + \frac{h}{3}\,[y_2' + 4y_3' + y_4'^{(p)}]$ where $y_4'^{(p)} = f(x_4,\ y_4^{(p)})$

$$y_4^{(c_2)} = y_2 + \frac{h}{3}\,[y_2' + 4y_3' + y_4'^{(c_1)}]$$

We repeat this step, till two consecutive values of y agree.

Problems:

1. Using Taylor's series method, solve $y' = x + y^2$, $given\ y(0) = 1$, at $x = 0.1,\ 0.2,$ considering upto 4th degree term.

Solution: $y(0) = 1,$

$y' = x + y^2 \implies y'(0) = 1$

$y'' = 1 + 2yy' \implies y''(0) = 3$

$y''' = 2yy'' + 2(y')^2 \implies y'''(0) = 8$

$y^{iv} = 2yy''' + 6y'y'' \implies y^{iv}(0) = 34$

If $\frac{dy}{dx} = f(x,y),\ given\ y(0) = y_0$ then $x_0 = 0$

$$y(x) = y_0 + x\,y'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \frac{x^4}{4!}y^{iv}(0) + \cdots$$

$$\therefore\ y(x) = 1 + x + \frac{3x^2}{2} + \frac{4x^3}{3} + \frac{17x^4}{12}$$

And $y(0.1) = 1.1165$, $y(0.2) = 1.2729.$

2. Find an approximate value of y when $x = 1.1$, if $\frac{dy}{dx} = 1 - x^2 y$, $given\ y(1) = 0$, using Taylor's method.

Solution: Given, $y' = 1 - x^2 y$

$y'' = -x^2 y' - 2xy$

$y''' = -x^2 y'' - 4xy' - 2y$

$y^{iv} = -x^2 y''' - 6xy'' - 6y'$

$\implies y(1) = 0,\ \ y'(1) = 1\ ,\ y''(1) = -1,\ y'''(1) = -3,\ \ y^{iv}(1) = 3$

$$y(x) = y_0 + (x - x_0)y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \frac{(x-x_0)^3}{3!}y'''(x_0) + \frac{(x-x_0)^4}{4!}y^{iv}(x_0) + \cdots$$

$$\therefore\ y(x) = (x - 1) - \frac{(x-1)^2}{2!} - \frac{(x-1)^3}{2} + \frac{(x-1)^4}{8},$$

$y(1.1) = 0.0945$.

3. Using Taylor's series method, compute the solution of $\frac{dy}{dx} = xy^2 - 1,\ given\ y(0) = 1$ at $x = 0.1$.

Solution: $y' = xy^2 - 1,\ \ y'' = 2xyy' + y^2$, $y''' = 2xyy'' + 2x(y')^2 + 4yy'$

$\implies y(0) = 1,\ \ y'(0) = -1,\ \ y''(0) = 1,\ \ y'''(0) = -4$

$$y(x) = y_0 + x\,y'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \frac{x^4}{4!}y^{iv}(0) + \cdots$$

$\therefore\ y(x) = 1 - x + \frac{x^2}{2} - \frac{2x^3}{3}$ and $y(0.1) = 0.9043$.

4. Find an approximate value of y when $x = 1.1$, if $\frac{dy}{dx} = x - y^2,\ given\ y(1) = 0,$ using Taylor's method.

Solution: $y' = x - y^2$, $y'' = 1 - 2yy'$ $y''' = -2yy'' - 2(y')^2$

$\implies y(1) = 0,\ \ y'(1) = 1\ ,\ \ y''(1) = 1\ \ y'''(1) = -2$.

$$y(x) = y_0 + (x - x_0)y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \frac{(x-x_0)^3}{3!}y'''(x_0)$$

$\therefore\ y(x) = (x - 1) + \frac{(x-1)^2}{2} - \frac{\left((x-1)\right)^3}{3}$ and $y(1.1) = 0.1047$.

5. If $\frac{dy}{dx} = 1 + y^2$, $y(0) = 0$, then find the values of $y(0.1)$ and $y(0.2)$ by Modified Euler's method. Perform two iterations in each stage.

Solution: Clearly $f(x, y) = 1 + y^2$, $x_0 = 0$, $y_0 = 0$, $x_1 = 0.1$, $x_2 = 0.2$, & $h = 0.1$.

To find y_1: $y_1^{(I)} = y_0 + hf(x_0,\ y_0) = 0.1$.

$y_1^{(M_1)} = y_0 + \frac{h}{2}\left[f(x_0,\ y_0) + f\left(x_1,\ y_1^{(I)}\right)\right] = 0.1005$. $y_1^{(M_2)} = 0.1005$.

To find y_2: $y_2^{(I)} = y_1 + hf(x_1,\ y_1) = 0.2015$.

$y_2^{(M_1)} = y_1 + \frac{h}{2}\left[f(x_1,\ y_1) + f\left(x_2,\ y_2^{(I)}\right)\right] = 0.2030$. $y_2^{(M_2)} = 0.2031$.

$y(0.1) = 0.1005$ and $y(0.2) = 0.2031$.

6. Using modified Euler's method, find an approximate value of y when $x = 0.2$, given that $\frac{dy}{dx} = x + y$, and $y = 1$, when $x = 0$ taking $h = 0.1$. Perform two iterations in each stage.

Ans: Let $f(x,\ y) = x + y$, $h = 0.1$ $x_0 = 0$, $y_0 = 1$.

Initial approximation of y_1 is $y_1^{(I)} = y_0 + hf(x_0,\ y_0) = 1.1$

Better approximation of y_1 is $y_1^{(M_1)} = y_0 + \frac{h}{2}\left[f(x_0,\ y_0) + f\left(x_1,\ y_1^{(I)}\right)\right]$

$= 1 + \frac{0.1}{2}[0 + 1 + 0.1 + 1.1] = 1.11$

$y_1^{(M_2)} = y_0 + \frac{h}{2}\left[f(x_0,\ y_0) + f\left(x_1,\ y_1^{(M_1)}\right)\right] = 1 + \frac{0.1}{2}[0 + 1 + 0.1 + 1.11] = 1.1105$

$\therefore \boldsymbol{y(0.1) = 1.1105}$.

Initial approximation of y_2 is $y_2^{(I)} = y_1 + hf(x_1,\ y_1) = 1.2316$.

$y_2^{(M_1)} = y_1 + \frac{h}{2}\left[f(x_1,\ y_1) + f\left(x_2,\ y_2^{(I)}\right)\right]$

$= 1.1105 + \frac{0.1}{2}[0.1 + 1.1105 + 0.2 + 1.2316] = 1.2426.$

$y_2^{(M_2)} = y_0 + \frac{h}{2}\left[f(x_1,\ y_1) + f\left(x_2,\ y_2^{(M_1)}\right)\right]$

$= 1.1105 + \frac{0.1}{2}[0.1 + 1.1105 + 0.2 + 1.2426] = 1.2432$

$\therefore \boldsymbol{y(0.2) = 1.2432}$.

7. Using modified Euler's method, find an approximate value of y when $x = 0.1$, given $\frac{dy}{dx} = \frac{y-x}{y+x}$, $y(0) = 1$. Perform two iterations.

Solution: Let $f(x,\ y) = \frac{y-x}{y+x}$, $h = 0.1$ $x_0 = 0$, $y_0 = 1$.

Initial approximation of y_1 is $y_1^{(I)} = y_0 + hf(x_0,\ y_0) = 1.1$

Better approximation of y_1 is $y_1^{(M_1)} = y_0 + \frac{h}{2}\left[f(x_0,\ y_0) + f\left(x_1,\ y_1^{(I)}\right)\right] = 1.0917$

$y_1^{(M_2)} = y_0 + \frac{h}{2}\left[f(x_0,\ y_0) + f\left(x_1,\ y_1^{(M_1)}\right)\right] = 1.0916$

$\therefore \boldsymbol{y(0.1)} = 1.0916$.

8. Using modified Euler's method, find an approximate value of y when $x = 1.1$, given $\frac{dy}{dx} = 2x - \frac{y}{x}$, $given\ y(1) = 1$

Let $f(x,\ y) = 2x - \frac{y}{x}$, $h = 0.1$ $x_0 = 1$, $y_0 = 1$.

Initial approximation of y_1 is $y_1^{(I)} = y_0 + hf(x_0,\ y_0) = 1.1$

Better approximation of y_1 is $y_1^{(M_1)} = y_0 + \frac{h}{2}\left[f(x_0,\ y_0) + f\left(x_1,\ y_1^{(I)}\right)\right]$

$= 1 + \frac{0.1}{2}[1 + 1.2] = 1.11$

$y_1^{(M_2)} = y_0 + \frac{h}{2}\left[f(x_0,\ y_0) + f\left(x_1,\ y_1^{(M_1)}\right)\right] = 1 + \frac{0.1}{2}[1 + 1.1909] = 1.1095$

$\therefore \boldsymbol{y(1.1) = 1.1095}$.

9. Apply fourth order Runge-Kutta method to find the solution of $\frac{dy}{dx} = x + y$, given $y(0) = 1$ at $x = 0.2$

Solution: Clearly $f(x,\ y) = x + y$. $x_0 = 0,\quad y_0 = 1,\quad h = 0.2$

$$k_1 = h\,f(x_0,\ y_0) = 0.2f(0,\ 1) = 0.2.$$

$$k_2 = h\,f\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_1}{2}\right) = 0.2f(0.1,\ 1.1) = 0.24$$

$$k_3 = h\,f\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2}\right) = 0.2f(0.1,\ 1.12) = 0.244$$

and $k_4 = h\,f(x_0 + h,\ y_0 + k_3\,) = 0.2f(0.2,\ 1.244) = 0.2888.$

$$y(0.2) = \boldsymbol{y_1} = \boldsymbol{y_0} + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 1.2428.$$

10. Using fourth order Runge-Kutta method find the solution of $10\frac{dy}{dx} = x^2 + y^2,\quad y(0) = 1$ at $x = 0.2$.

Solution: Clearly $f(x,\ y) = \frac{x^2+y^2}{10}$. $x_0 = 0,\quad y_0 = 1,\quad h = 0.2$

$$k_1 = h\,f(x_0,\ y_0) = 0.02.$$

$$k_2 = h\,f\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_1}{2}\right) = 0.0206.$$

$$k_3 = h\,f\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2}\right) = 0.0206$$

and $k_4 = h\,f(x_0 + h,\ y_0 + k_3\,) = 0.0216.$

$$y(0.2) = \boldsymbol{y_1} = \boldsymbol{y_0} + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 1.0207.$$

11. Using Runge-Kutta method of fourth order, solve $y' = \log_{10}\left[\frac{y}{1-x}\right]$ *given* $y(0) = 1$ at $x = 0.2$.

Solution: Clearly $f(x,\ y) = \log_{10}\left[\frac{y}{1-x}\right]$. $x_0 = 0,\quad y_0 = 1,\quad h = 0.2$

$$k_1 = h\,f(x_0,\ y_0) = 0.$$

$$k_2 = h\,f\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_1}{2}\right) = 0.2\,f(0.1,\ 1\,) = 0.0092.$$

$$k_3 = h\,f\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2}\right) = 0.2\,f(0.1,\ 1.0046\,) = 0.0096$$

and $k_4 = h\,f(x_0 + h,\ y_0 + k_3\,) = 0.2\,f(0.2,\ 1.0096\,) = 0.0202.$

$$y(0.2) = \boldsymbol{y_1} = \boldsymbol{y_0} + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 1.0096.$$

12. Using Runge-Kutta method of fourth order, find an approximate value of y when $x = 0.1$, given $\frac{dy}{dx} = \frac{y-x}{y+x},\quad y(0) = 1$.

Solution: Clearly $f(x,\ y) = \frac{y-x}{y+x}$. $x_0 = 0,\quad y_0 = 1,\quad h = 0.1$

$$k_1 = h\,f(x_0,\ y_0) = 0.1$$

$$k_2 = h\,f\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_1}{2}\right) = 0.1\,f(0.05,\ 1.05\,) = 0.0909.$$

$$k_3 = h\,f\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2}\right) = 0.1\,f(0.05,\ 1.0455\,) = 0.0909.$$

and $k_4 = h\,f(x_0 + h,\ y_0 + k_3\,) = 0.1\,f(0.1,\ 1.0909\,) = 0.00832.$

$$y(0.1) = \boldsymbol{y_1} = \boldsymbol{y_0} + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 1.0911.$$

13. Given $\frac{dy}{dx} = x - y^2$ and $y(0) = 0,\ y(0.2) = 0.02,\ y(0.4) = 0.0795,\ y(0.6) = 0.1762$, evaluate $y(0.8)$ by Milne's method.

Solution: $y' = x - y^2, \quad h = 0.2$.

$x_0 = 0, \quad y_0 = 0,$

$x_1 = 0.2, \quad y_1 = 0.02, \quad y_1' = \mathbf{0.1996}$.

$x_2 = 0.4, \quad y_2 = 0.0795, \quad y_2' = \mathbf{0.3937}$.

$x_3 = 0.6, \quad y_3 = 0.1762 , \quad y_3' = \mathbf{0.5690}$.

Then predictor formula is $y_4 = y_0 + \frac{4h}{3}\ [2y_1' - y_3' + 2y_3'] = 0.3049.$

$$y_4' = 0.8 - 0.3049^2 = 0.7070$$

Corrector formula is $y_4 = y_2 + \frac{h}{3}\ [y_2' + 4y_3' + y_4']$

$$= 0.0795 + \frac{0.2}{3}\ [0.3937 + 4 \times 0.5690 + 0.7070] = 0.3046.$$

14. Given $\frac{dy}{dx} = x^2(1 + y)$ and $y(1) = 1,\ y(1.1) = 1.233,\ y(1.2) = 1.548,\ y(1.3) = 1.979$, evaluate $y(1.4)$ by Milne's method.

Solution: $y' = x^2(1 + y), \quad h = 0.1$.

$x_0 = 1, \quad y_0 = 1,$.

$x_1 = 1.1, \quad y_1 = 1.233, \quad y_1' = \mathbf{2.7019}$.

$x_2 = 1.2, \quad y_2 = 1.548, \quad y_2' = \mathbf{3.6691}$.

$x_3 = 1.3, \quad y_3 = 1.979 , \quad y_3' = \mathbf{5.0345}$.

Then predictor formula is $y_4 = y_0 + \frac{4h}{3}\ [2y_1' - y_3' + 2y_3'] = 2.5738$.

$$y_4' = 1.4^2(1 + 2.5738) = 7.0047.$$

Corrector formula is $y_4 = y_2 + \frac{h}{3}\ [y_2' + 4y_3' + y_4'] = 2.5751.$

Self-study:

Adams-Bash forth method: To find $y(x_4) = y_4$ from $\frac{dy}{dx} = f(x,y),\ given\ y(x_0) = y_0$, $y(x_1) = y_1$, $y(x_2) = y_2$, and $y(x_3) = y_3$.

First calculate $f_0 = f(x_0,\ y_0)$, $f_1 = f(x_1,\ y_1)$, $f_2 = f(x_2,\ y_2)$, and $f_3 = f(x_3,\ y_3)$

Then predictor formula is $y_4 = y_3 + \frac{h}{24}\ [55f_3 - 59f_2 + 37f_1 - 9f_0]$

$$f_4 = f(x_4,\ y_4)$$

Corrector formula is $y_4 = y_3 + \frac{h}{24}\ [9f_4 + 19f_3 - 5f_2 + f_1]$

15. Given $\frac{dy}{dx} = x^2 - y$ and $y(0) = 0,\ y(0.1) = 0.90516,\ y(0.2) = 0.82127,\ y(0.3) = 0.74918$. Evaluate $y(0.4)$ by Adams-Bashforth method.

Solution: $f(x,y) = x^2 - y , \quad h = 0.1$.

$x_0 = 0, \quad y_0 = 0, \quad f_0 = \mathbf{0}$.

$x_1 = 0.1, \quad y_1 = 0.90516, \quad f_1 = \mathbf{-0.8952}$.

$x_2 = 0.2, \quad y_2 = 0.82127, \quad f_2 = \mathbf{-0.7813}$.

$x_3 = 0.3, \quad y_3 = 0.74918 , \quad f_3 = \mathbf{-0.6592}$.

Predictor value is $y_4 = y_3 + \frac{h}{24}(55f_3 - 59f_2 + 37f_1 - 9f_0) = 0.6522,$

$$f_4 = f(x_4,\ y_4) = 0.4^2 - 0.6522 = -0.4922 .$$

Corrector value is $y_4 = y_3 + \frac{h}{24}(9f_4 + 19f_3 - 5f_2 + f_1) = 0.6911.$

16. Solve by Adoms-Bhash forth method for $x = 0.8$, Given that $\frac{dy}{dx} = 2y - 2x + 1$ and

x	0	0.2	0.4	0.6
y	1	1.6918	2.6255	3.9201

Ans: Given that $f = 2y - 2x + 1, \quad h = 0.2.$

$x_0 = 0, \qquad y_0 = 1, \qquad f_0 = \mathbf{3}$.

$x_1 = 0.2, \qquad y_1 = 1.6918, \qquad f_1 = \mathbf{3.9836}$.

$x_2 = 0.4, \qquad y_2 = 2.6255, \qquad f_2 = \mathbf{5.4510}$.

$x_3 = 0.6, \qquad y_3 = 3.9201, \qquad f_3 = \mathbf{7.6402}$.

Predictor value: $y_4 = y_3 + \frac{h}{24}(55f_3 - 59f_2 + 37f_1 - 9f_0) = 5.7451.$

$$f_4 = f(x_4,\ y_4) = 2 \times 5.7451 - 2 \times 0.8 + 1 = 10.8902$$

Corrector value: $y_4 = y_3 + \frac{h}{24}(9f_4 + 19f_3 - 5f_2 + f_1) = 5.7526.$

$$\therefore y(0.8) = 5.7526 \quad .$$

17. Given $\frac{dy}{dx} = x^2(1 + y)$ and $y(1) = 1$, $y(1.1) = 1.233$, $y(1.2) = 1.548$, $y(1.3) = 1.979$, evaluate $y(1.4)$ by Adams-Bashforth method.

Solution: $f(x, y) = x^2(1 + y), \quad h = 0.1$.

$x_0 = 1, \qquad y_0 = 1, \qquad f_0 = \mathbf{2}$.

$x_1 = 1.1, \qquad y_1 = 1.233, \qquad f_1 = \mathbf{2.7019}$.

$x_2 = 1.2, \qquad y_2 = 1.548, \qquad f_2 = \mathbf{3.6691}$.

$x_3 = 1.3, \qquad y_3 = 1.979 , \qquad f_3 = \mathbf{5.0345}$.

Predictor value: $y_4 = y_3 + \frac{h}{24}(55f_3 - 59f_2 + 37f_1 - 9f_0) = 2.5723$.

$$f_4 = f(x_4,\ y_4) = 1.4^2(1 + 2.5723) = 7.0017.$$

Corrector value: $y_4 = y_3 + \frac{h}{24}(9f_4 + 19f_3 - 5f_2 + f_1) = 2.5749.$