Note:
- $\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$
- $\cos A \sin B = \frac{1}{2}[\sin(A+B) - \sin(A-B)]$
- $\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$
- $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

- $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$, $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$.
- $\sin^3\theta = \frac{1}{4}(3\sin\theta - \sin 3\theta)$, $\cos^3\theta = \frac{1}{4}(3\cos\theta + \cos 3\theta)$.

1. $\psi = \theta + \phi$. **2** $\tan\boldsymbol{\phi} = \boldsymbol{r}\frac{d\theta}{dr}$. **3.** $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$.

Note: 1. Angle between the two polar curves is $|\boldsymbol{\phi}_1 - \boldsymbol{\phi}_2|$

Find $\tan\phi_1 = \frac{r}{r_1}$ for the first curve and $\tan\phi_2 = \frac{r}{r_1}$ for the second curve

And if $\tan\phi_1 . \tan\phi_2 = -1$. Then angle of intersection is $\frac{\pi}{2}$.

2. Equation involving only $\boldsymbol{p}$ *and* $\boldsymbol{r}$ is called **pedal equation.**

To find the pedal equation, find $\frac{r_1}{r}$ and use it in $\frac{1}{p^2} = \frac{1}{r^2}\left[1 + \left(\frac{r_1}{r}\right)^2\right]$ and then eliminate θ .

Curvature $K = \frac{d\psi}{ds}$, **Radius of curvature** $\rho = \frac{ds}{d\psi}$.

Radius of curvature in Cartesian form: $\rho = \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2}$

Radius of curvature in Polar form: $\rho = \frac{(r^2+r_1^2)^{\frac{3}{2}}}{r^2+2r_1^2-rr_2}$.

Module2:

Taylor's series: Expansion of $f(x)$ about $x = a$ (or in powers of $(x-a)$) is

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \frac{(x-a)^4}{4!}f'^{v}(a) + \cdots$$

Or $y = y(a) + y_1(a)(x-a) + \frac{y_2(a)}{2!}(x-a)^2 + \frac{y_3(a)}{3!}(x-a)^3 + \frac{y_4(a)}{4!}(x-a)^4 + \cdots$

If $a = 0$ then series is called **Maclaurin's series** i.e.

Expansion of $f(x)$ about $x = 0$ (or in powers of x) is

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f'^{v}(0) + \cdots$$

Or $y = y(0) + y_1(0)x + \frac{y_2(0)}{2!}x^2 + \frac{y_3(0)}{3!}x^3 + \frac{y_4(0)}{4!}x^4 + \cdots$

Indeterminate forms:

$\left(\frac{0}{0}\right)$ form: If $f(a) = 0 = g(a)$ then $\lim_{x \to a} \frac{f(x)}{g(x)}$ is called $\frac{0}{0}$ form.

L'Hospital's rule: If $f(a) = 0 = g(a)$ then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$.

Note: Forms 1^{∞}, $\infty^0, 0^0$ can be reducible to form $\left(\frac{0}{0}\right)$ or form $\frac{\infty}{\infty}$ by taking log.

Partial derivatives –

Total derivatives:

1. If $u = f(x, y)$ $and\ x = g(t),\ y = h(t)\ then$ $\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$.
2. If $f(x, y) = constant$, then $\frac{dy}{dx} = -\frac{f_x}{f_y}$.
3. If $u = f(x, y)$ subject to $\varphi(x, y) = c$. Then $\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}$, where $\frac{dy}{dx} = -\frac{\varphi_x}{\varphi_y}$.
4. If $u = f(r, s, t)$ where $r, s\ and\ t\ are\ functions\ of\ (x,\ y,\ z)$, then by Chain rule

   $u_x = u_r r_x + u_s s_x + u_t t_x$, $u_y = u_r r_y + u_s s_y + u_t t_y$ and $u_z = u_r r_z + u_s s_z + u_t t_z$.

Jacobian:

$$\boldsymbol{J} = \frac{\partial(\boldsymbol{u},\ \boldsymbol{v},\ \boldsymbol{w})}{\partial(\boldsymbol{x},\ \boldsymbol{y},\ \boldsymbol{z})} = \begin{vmatrix} \boldsymbol{u_x} & \boldsymbol{u_y} & \boldsymbol{u_z} \\ \boldsymbol{v_x} & \boldsymbol{v_y} & \boldsymbol{v_z} \\ \boldsymbol{w_x} & \boldsymbol{w_y} & \boldsymbol{w_z} \end{vmatrix}.$$

Note: 1. If $u = f(r)\ and\ r = \sqrt{x^2+y^2}$ then $u_{xx} + u_{yy} = f''(r) + \frac{1}{r}f'(r)$.

2. If $u = f(r)\ and\ r = \sqrt{x^2+y^2+z^2}$ then $u_{xx} + u_{yy} + u_{zz} = f''(r) + \frac{2}{r}f'(r)$.

Maxima and minima of functions of two variables:

1. $f(x,\ y)$ is stationary at $(a,\ b)$ i.e. $f(a,\ b)$ is the stationary value of f if $f_x = 0 = f_y$ at $(a,\ b)$.
2. $f(x,\ y)$ is maximum at $(a,\ b)$ i.e. $f(a,\ b)$ is the maximum value of f

   If at $(a,\ b)$ i) $f_x = 0 = f_y$ ii) $f_{xx}f_{yy} - f_{xy}^2 > 0$ iii) $f_{xx} < 0$.
3. $f(x,\ y)$ is minimum at $(a,\ b)$ i.e. $f(a,\ b)$ is the minimum value of f

   If at $(a,\ b)$ i) $f_x = 0 = f_y$ ii) $f_{xx}f_{yy} - f_{xy}^2 > 0$ iii) $f_{xx} > 0$.
4. $(a,\ b)$ is said to be saddle point of $f(x,\ y)$ if i) $f_x = 0 = f_y$ ii) $f_{xx}f_{yy} - f_{xy}^2 < 0$ at $(a,\ b)$.
5. If $f_x = 0 = f_y$ and $f_{xx}f_{yy} - f_{xy}^2 = 0$ at $(a,\ b)$, then by discussion find maxima and minima.

Module3:

Differential Equations:

Solution of first order and first degree differential equations –

Exact equation:

The necessary and sufficient condition for the differential equation $Mdx + Ndy = 0$ to be exact is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.\ (\text{ i.e. } M_y = N_x\ )$$

Solution of exact equation is $\underset{(y\ constant)}{\int M\,dx} + \int$ (terms of N not containing x) $dy = c$.

Reducible to exact:

In the non-exact equation $Mdx + Ndy = 0$,

i) if $\frac{1}{N}\left[\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right]=f(x)$ i.e. the function of x only, then $\text{I.F.}=e^{\int f(x)dx}$.

ii) if $\frac{1}{M}\left[\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right]=g(y)$ i.e. the function of y only, then $\text{I.F.}=e^{\int g(y)dy}$.

Bernoulli's differential equations: Reducible to linear form

Equation of the type $\frac{dy}{dx}+Py=Qy^n$ where P and Q are functions of x, is called **Bernoulli's equation.**

And Dividing by y^n and substituting $z=\frac{1}{y^{n-1}}$ equation reduces to linear form.

To find the O.T. of Cartesian curves:

If $f(x,\ y,\ c)=0$ be the given family of curves with c is the arbitrary constant.

Step1: Find the differential equation of the given family by eliminating c. Let it be $F\left(x,\ y,\ \frac{dy}{dx}\right)=0.$

Step2: Find the differential equation of the orthogonal trajectory

by replacing $\frac{dy}{dx}=-\frac{dx}{dy}$ i.e. $F\left(x,\ y,-\frac{dx}{dy}\right)=0.$

Step3: Solve $F\left(x,\ y,-\frac{dx}{dy}\right)=0$ to get orthogonal trajectory.

To find the O.T. of Polar curves:

If $f(r,\ \theta,\ c)=0$ be the given family of curves with c is the arbitrary constant.

Step1: Find the differential equation of the given family by eliminating c. Let it be $F\left(r,\ \theta,\ \frac{dr}{d\theta}\right)=0.$

Step2: Find the differential equation of the orthogonal trajectory

by replacing $\frac{dr}{d\theta}=-r^2\frac{d\theta}{dr}$ i.e. $F\left(r,\ \theta,-r^2\frac{d\theta}{dr}\right)=0.$

Step3: Solve $F\left(r,\ \theta,-r^2\frac{d\theta}{dr}\right)=0$ to get orthogonal trajectory.

Newton's law of cooling: Suppose that a body whose temperature is initially $T_1^{\circ}C$ is allowed to cool in air which is maintained at a constant temperature $T_2^{\circ}C$. Let $T^{\circ}C$ is the temperature of the body at time t. Then,

$$\boxed{T=T_2+(T_1-T_2)e^{-kt}}$$

1. **Equations solvable for p** : n^{th}degree, first order equations of the form
   $p^n+P_1p^{n-1}+P_2p^{n-2}+\cdots P_n=0$, (where $P_1,\ P_2\ ,\cdots P_n$ are functions of $x,\ y$)
   are solvable for p.

   Splitting up the equation in to n linear factors we get , $[p-f_1(x,\ y)][p-f_2(x,\ y)]\cdots[p-f_n(x,\ y)]=0$.
   Equating each of the factors to zero and solving we get ,
   $$F_1(x,y,c)=0,\quad F_2(x,y,c)=0\ ,\quad \cdots\cdots,\quad F_n(x,y,c)=0.$$
   These n solutions constitute the general solution of the equation.
   Or we can write the general solution as $F_1(x,y,c)F_2(x,y,c)\ \cdots\cdots\ F_n(x,y,c)=0.$

2. **Clairaut's equations:** An equation of the form $y=px+f(p)$ is called Clairaut's equation.
   General solution is $y=cx+f(c)$.
   To find the singular solution eliminate c from $y=cx+f(c)$ using $x=-f'(c)$.

Module4:

Linear Differential Equations with constant coefficients:

$f(D)y = X$ .... (1) Where $D = \frac{d}{dx}$ and X is function of x only.

Complete Solution of (1) is $y = y_c + y_p$, where y_c is complementary function, and y_p is particular integral.

To find y_c : Auxiliary equation of (1) is $f(D) = 0$. Find the roots of A.E.

Roots of A.E.	y_c
1. m_1, m_2, m_3 .....(real and different roots)	$c_1e^{m_1x} + c_2e^{m_2x} + c_3e^{m_3x} + \cdots\cdots$
2. m_1, m_1, m_2 .....(Two real and equal roots)	$(c_1 + c_2x)e^{m_1x} + c_3e^{m_2x} + \cdots\cdots$
3. m_1, m_1, m_1, m_2 .....(three real and equal roots)	$(c_1 + c_2x + c_3x^2)e^{m_1x} + c_4e^{m_2x} + \cdots\cdots$
4. $a \pm ib$, m_1 ....(a pair of imaginary roots)	$e^{ax}(c_1 \cos bx + c_2 \sin bx) + c_3e^{m_1x} + \cdots$

Examples:

Roots of A.E.	y_c
1. 1,2,3	$\boldsymbol{y_c} = c_1e^x + c_2e^{2x} + c_3e^{3x}$.
2. $-2, -2$	$\boldsymbol{y_c} = (c_1 + c_2x)e^{-2x}$.
3. $-1, -1, -1$, 1.5	$\boldsymbol{y_c} = (c_1 + c_2x + c_3x^2)e^{-x} + c_4e^{1.5x}$.
4. $1 \pm 2i$, 3	$\boldsymbol{y_c} = e^x(c_1 \cos 2x + c_2 \sin 2x) + c_3e^{3x}$
5. $\pm 2i$	$\boldsymbol{y_c} = c_1 \cos 2x + c_2 \sin 2x$.

To find y_p : $\boldsymbol{y_p = \frac{1}{f(D)}X}$

1. $\frac{1}{D}X = \int X dx$

2. If $f(a) \neq 0$, then $\frac{1}{f(D)}e^{ax} = \frac{e^{ax}}{f(a)}$,

   If $f(a) = 0$, then $\frac{1}{f(D)}e^{ax} = x\frac{1}{f'(D)}e^{ax}$, and then put $D = a$.

   If $f(a) = 0, f'(a) = 0$, then $\frac{1}{f(D)}e^{ax} = x^2\frac{1}{f''(D)}e^{ax}$ and then put $D = a$ and so on.

3. $\frac{1}{f(D)}\sin(ax)$ or $\frac{1}{f(D)}\cos(ax)$, *put* $D^2 = -a^2$ in $f(D)$.

4. $\frac{1}{f(D)}(ax^2 + bx + c)$ is obtained by direct division.

Module5: **If a matrix A is equivalent to an echelon matrix E, then $\rho(A) =$ Number of nonzero rows in E .**

Consistency of Homogeneous linear equations, $AX = 0$:

$X = 0$ is the trivial solution. Thus the homogeneous system is always consistent.

Note: 1. If $\rho(A) =$ number of unknowns, then the system has only trivial solution.

2. If $\rho(A) <$ number of unknowns, then the system has an infinite number of solutions.

Consistency of non-homogeneous linear equations, $AX = B$:

1. If $\rho(A) = \rho(A|B) =$ number of unknowns, then the system has unique solution.
2. If $\rho(A) = \rho(A|B) <$ number of unknowns, then the system has an infinite number of solutions.
3. If $(A) \neq \rho(A|B)$, then system has no solution.

Gauss-Seidel iteration method: Consider the equations $a_1x + b_1y + c_1z = d_1,\ a_2x + b_2y + c_2z = d_2,$ $a_3x + b_3y + c_3z = d_3$, If $a_1,\ b_2,\ c_3$ are numerically large as compared to other coefficients in their respective equations. Then iterative formula for $x, y\ and\ z$ are given by

$$x_{n+1} = \frac{1}{a_1}(d_1 - c_1 z_n - b_1 y_n\,), \qquad y_{n+1} = \frac{1}{b_2}(d_2 - a_2 x_{n+1} - c_2 z_n\,) \text{ and}$$

$$z_{n+1} = \frac{1}{c_3}(d_3 - b_3 y_{n+1} - a_3 x_{n+1}\,)$$

Characteristic equation: $|A - \lambda I| = 0$ is the characteristic equation of the square matrix A. Roots are called **Characteristic roots** or **Eigen values** or **latent roots** of A.

Any vector X satisfying $[A - \lambda I]X = 0$ is called **Eigen vector** corresponding to the Eigen value.

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then Characteristic equation is $\lambda^2 - (a + d)\lambda + (ad - cb) = 0$.

if $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, then Characteristic equation is

$$\lambda^3 - (a_1 + b_2 + c_3)\lambda^2 + (sum\ of\ the\ minors\ of\ a_1, b_2 \& c_3)\lambda - |A| = 0\,.$$

<u>Determination of largest Eigen value by Rayleigh's power method:</u>

Let A be the given square matrix and a column vector X_0 be the initial Eigen vector. Evaluate $AX_0 = \lambda_1 X_1$ where λ_1 is the first approximation of the Eigen value and X_1 is the corresponding Eigen vector.
$AX_1 = \lambda_2 X_2$. Where λ_2 is the 2^{nd} approximation of the Eigen value and X_2 is the corresponding Eigen vector.
$AX_2 = \lambda_3 X_3$. Where λ_3 is the 3^{rd} approximation of the Eigen value and X_3 is the corresponding Eigen vector.
Repeat this process till $X_n - X_{n-1}$ becomes negligible.