Module-1: Differential Calculus - 1
Polar curves, angle between the radius vector and the tangent, angle between two curves. Pedal equations. Curvature and Radius of curvature - Cartesian, Parametric, Polar and Pedal forms. Problems.   **Self-study:** Center and circle of curvature, evolutes and involutes.   **(RBT Levels: L1, L2 and L3)**

Note:
- $\sin A\cos B=\frac{1}{2}[\sin(A+B)+\sin(A-B)]$
- $\cos A\sin B=\frac{1}{2}[\sin(A+B)-\sin(A-B)]$
- $\cos A\cos B=\frac{1}{2}[\cos(A+B)+\cos(A-B)]$
- $\sin A\sin B=\frac{1}{2}[\cos(A-B)-\cos(A+B)]$
- $\sin^2\theta=\frac{1}{2}(1-\cos 2\theta)$, $\cos^2\theta=\frac{1}{2}(1+\cos 2\theta)$.
- $\sin^3\theta=\frac{1}{4}(3\sin\theta-\sin 3\theta)$, $\cos^3\theta=\frac{1}{4}(3\cos\theta+\cos 3\theta)$.

n^{th} derivatives:

1. $[e^{ax}]_n=a^n e^{ax}$ **2.** $[a^{bx}]_n=(b\log a)^n a^{bx}$ **3.** $[\sin(ax+b)]_n=a^n\sin(ax+b+n\frac{\pi}{2})$

4. $[\cos(ax+b)]_n=a^n\cos(ax+b+n\frac{\pi}{2})$

5. $[e^{ax}\sin(bx+c)]_n=(a^2+b^2)^{\frac{n}{2}}e^{ax}\sin(bx+c+n\tan^{-1}\frac{b}{a})$

6. $[e^{ax}\cos(bx+c)]_n=(a^2+b^2)^{\frac{n}{2}}e^{ax}\cos(bx+c+n\tan^{-1}\frac{b}{a})$

7. $[(ax+b)^m]_n=a^n.m.(m-1).(m-2)\cdots\cdots(m-n+1)(ax+b)^{m-n}$

8. $\left[\frac{1}{ax+b}\right]_n=\frac{(-1)^n n!\,a^n}{(ax+b)^{n+1}}$ **9.** $[\log(ax+b)]_n=\frac{(-1)^{n-1}(n-1)!\,a^n}{(ax+b)^n}$.

Leibnitz's rule: $[\boldsymbol{uv}]_n=\ {}^n\boldsymbol{C_0 u v_n}+\ {}^n\boldsymbol{C_1 u_1 v_{n-1}}+\ {}^n\boldsymbol{C_2 u_2 v_{n-2}}+\cdots\cdots\cdots+\ {}^n\boldsymbol{C_n u_n v}$

$$=\boldsymbol{uv_n+nu_1v_{n-1}+\frac{n(n-1)}{2!}u_2v_{n-2}+\cdots\cdots\cdots+u_nv.}$$

Problems: Find the n^{th} derivative of the following.

1. $\cos x\cos 2x\cos 3x$

Let $y=\cos x\cos 2x\cos 3x=(\cos 3x\cos 2x)\cos x=\frac{1}{2}[\cos 5x+\cos x]\cos x$

$$=\frac{1}{4}[\cos 6x+\cos 4x+\cos 2x+1]$$

$$\therefore\ y_n=\frac{1}{4}\left[6^n\cos\left(6x+n\frac{\pi}{2}\right)+4^n\cos\left(4x+n\frac{\pi}{2}\right)+2^n\cos\left(2x+n\frac{\pi}{2}\right)\right].$$

2. $e^x\cos^2 2x$

Let $y=e^x\cos^2 2x=\frac{1}{2}e^x[1+\cos 4x]=\frac{1}{2}[e^x+e^x\cos 4x]$

$$\therefore\ y_n = \frac{1}{2}\left[e^x + 17^{\frac{n}{2}} e^x \cos(4x + n\tan^{-1}4)\right]$$

3. $\dfrac{x}{1+3x+2x^2}$

Let $y = \dfrac{x}{1+3x+2x^2} = \dfrac{x}{(x+1)(2x+1)} = \dfrac{1}{(x+1)} - \dfrac{1}{(2x+1)}$

$$\therefore\ y_n = \frac{(-1)^n n!}{(x+1)^{n+1}} - \frac{(-1)^n n! 2^n}{(2x+1)^{n+1}}$$

4. If $y = \sin(m\sin^{-1}x)$, Prove that $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0$.

Proof: $y = \sin(m\sin^{-1}x) \implies \sin^{-1}y = m\sin^{-1}x$

Differentiating with respect to x we get, $\dfrac{y_1}{\sqrt{1-y^2}} = \dfrac{m}{\sqrt{1-x^2}}$.

Squaring and rearranging, $(1-x^2)y_1^2 = m^2(1-y^2)$.

Differentiating once again with respect to x, $(1-x^2)2y_1y_2 - 2xy_1^2 = m^2(-2yy_1)$

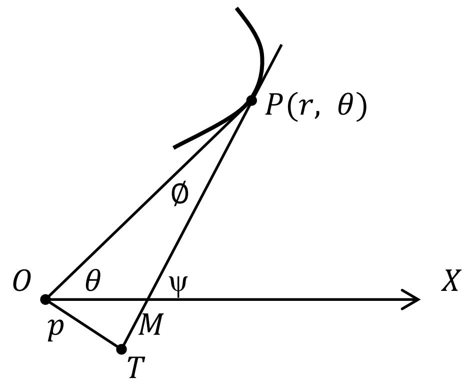
Dividing by $2y_1$, $(1-x^2)y_2 - xy_1 + m^2y = 0$.

Differentiating n times using **Leibnitz's** rule

$$(1-x^2)y_{n+2} + n(-2x)y_{n+1} + \frac{n(n-1)}{2!}(-2)y_n - [xy_{n+1} + n(1)y_n] + m^2y_n = 0$$

$$\implies (1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0\ .$$

Polar curves:



O is the pole, OX is the initial line, OP the radius vector , PT is the tangent to the curve at P .

And $OT = p$.

In ΔOPM , $\psi = \theta + \phi$.

1. Prove that $\boldsymbol{\tan\phi = r\frac{d\theta}{dr}}$.

Proof: Since $x = r\cos\theta$, $y = r\sin\theta$

Slope of the tangent $= \tan\psi = \frac{dy}{dx} = \frac{dy/_{dr}}{dx/_{dr}}$

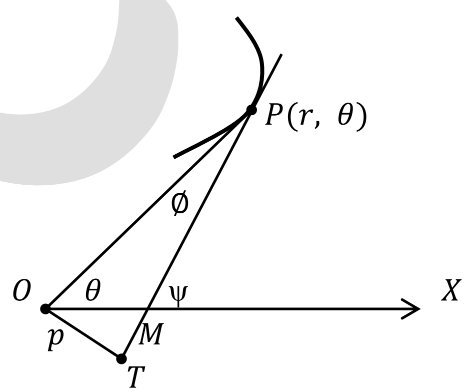
$$= \frac{\sin\theta + r\cos\theta\frac{d\theta}{dr}}{\cos\theta - r\sin\theta\frac{d\theta}{dr}} = \frac{\tan\theta + \boldsymbol{r\frac{d\theta}{dr}}}{1 - \tan\theta\ \boldsymbol{r\frac{d\theta}{dr}}} \qquad \cdots\cdots\cdots\cdots(1)$$

But $\psi = \theta + \phi \implies \tan\psi = \tan(\theta + \phi) = \frac{\tan\theta + \boldsymbol{\tan\emptyset}}{1 - \tan\theta\ \boldsymbol{\tan\emptyset}}$ $\cdots\cdots\cdots\cdots(2)$

By (1) and (2) $\boldsymbol{\tan\phi = r\frac{d\theta}{dr}}$.

2. Prove that $\boldsymbol{\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2}$.

Proof:



In ΔOPT $\frac{OT}{OP} = \sin\phi \implies \frac{p}{r} = \sin\phi$ *or* $p = r\sin\phi$.

$$p = r\sin\phi \implies \frac{1}{p^2} = \frac{1}{r^2}\ \operatorname{cosec}^2\phi$$

$$= \frac{1}{r^2}\ (1+\cot^2\phi) \qquad (\tan\phi = r\frac{d\theta}{dr} \Longrightarrow \cot\phi = \frac{1}{r}\frac{dr}{d\theta})$$

$$= \frac{1}{r^2}[1+\frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2] = \boldsymbol{\frac{1}{r^2}+\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2}.$$

Note: 1. Angle between the two polar curves is $|\boldsymbol{\phi}_1 - \boldsymbol{\phi}_2|$

Find $\tan\phi_1 = \frac{r}{r_1}$ for the first curve and $\tan\phi_2 = \frac{r}{r_1}$ for the second curve

And if $\tan\phi_1 . \tan\phi_2 = -1$. Then angle of intersection is $\frac{\pi}{2}$.

2. Equation involving only $\boldsymbol{p}$ *and* $\boldsymbol{r}$ is called **pedal equation.**

To find the pedal equation, find $\boldsymbol{\frac{r_1}{r}}$ and use it in $\boldsymbol{\frac{1}{p^2}} = \boldsymbol{\frac{1}{r^2}}\ [\mathbf{1}+\left(\boldsymbol{\frac{r_1}{r}}\right)^{\mathbf{2}}]$ and then eliminate θ .

Problems:

1. Find the angle between the following two curves.

a) $r = a(1-\sin\theta)$, $r = b(1+\sin\theta)$

$r = a(1-\sin\theta)$

Diff. w.r.to θ we get, $r_1 = a(-\cos\theta)$

$\therefore \tan\phi_1 = \frac{r}{r_1} = -\frac{(1-\sin\theta)}{\cos\theta}$

$r = b(1+\sin\theta)$

Diff. w.r.to θ we get, $r_1 = b(\cos\theta)$

$\therefore \tan\phi_2 = \frac{r}{r_1} = \frac{(1+\sin\theta)}{\cos\theta}$

$$\Longrightarrow \tan\phi_1 . \tan\phi_2 = -\frac{(1-\sin^2\theta)}{\cos^2\theta} = -1$$

Hence angle between them is $\frac{\pi}{2}$.

b) $r^n = a^n\cos n\theta$, $r^n = b^n\sin n\theta$.

$r^n = a^n\cos n\theta$

Diff. w.r.to θ we get,

$nr^{n-1}r_1 = -na^n\sin n\theta$

Or $r^n\,\boldsymbol{\frac{r_1}{r}} = -a^n\sin n\theta$

$\therefore \tan\phi_1 = \frac{r}{r_1} = -\cot n\theta$

$r^n = b^n\sin n\theta$

Diff. w.r.to θ we get,

$nr^{n-1}r_1 = nb^n\cos n\theta$

Or $r^n\,\boldsymbol{\frac{r_1}{r}} = b^n\cos n\theta$

$\therefore \tan\phi_2 = \frac{r}{r_1} = \tan n\theta$

$$\Longrightarrow \tan\phi_1 . \tan\phi_2 = -\cot n\theta\tan n\theta = -1$$

Hence the angle of intersection is $\frac{\pi}{2}$.

c) $r = \frac{2a}{(1-\cos\theta)}$, $r = \frac{2b}{(1+\cos\theta)}$

$r(1-\cos\theta) = 2a$

Diff. w.r.to θ we get,

$r(1+\cos\theta) = 2b$

Diff. w.r.to θ we get,

$r_1(1-\cos\theta)+r\sin\theta=0$

Or $r\sin\theta=-r_1(1-\cos\theta)$

$\therefore \tan\phi_1=\frac{r}{r_1}=-\frac{(1-\cos\theta)}{\sin\theta}$

$r_1(1+\cos\theta)-r\sin\theta=0$

Or $r\sin\theta=r_1(1+\cos\theta)$

$\therefore \tan\phi_2=\frac{r}{r_1}=\frac{(1+\cos\theta)}{\sin\theta}$

$\Longrightarrow \tan\phi_1 . \tan\phi_2=-\frac{(1-\cos^2\theta)}{\sin^2\theta}=-1.$ Hence the angle of intersection is $\frac{\pi}{2}$.

d) $r=\sin\theta+\cos\theta$, $r=2\sin\theta$

$r=\sin\theta+\cos\theta$

Diff. w.r.to θ we get,

$r_1=\cos\theta-\sin\theta$

$\therefore \tan\phi_1=\frac{r}{r_1}=\frac{\sin\theta+\cos\theta}{\cos\theta-\sin\theta}=\frac{1+\tan\theta}{1-\tan\theta}$

$=\tan\left(\theta+\frac{\pi}{4}\right)$

$\Longrightarrow \quad \phi_1=\theta+\frac{\pi}{4}$

$r=2\sin\theta$

Diff. w.r.to θ we get,

$r_1=2\cos\theta$

$\therefore \tan\phi_2=\frac{r}{r_1}=\frac{\sin\theta}{\cos\theta}=\tan\theta$

$\Longrightarrow \quad \phi_2=\theta$

Hence the angle of intersection $=|\phi_1-\phi_2|=\frac{\pi}{4}$.

2 . Find the pedal equation of the following curves.

a) $r=a(1-\sin\theta)$

Diff. w.r.to θ we get, $r_1=a(-\cos\theta)$

$\therefore \frac{\boldsymbol{r_1}}{\boldsymbol{r}}=-\frac{\cos\theta}{(1-\sin\theta)}$

$\frac{1}{p^2}=\frac{1}{r^2}\left[1+\left(\frac{\cos\theta}{1-\sin\theta}\right)^2\right]=\frac{1}{r^2}\left[\frac{2(1-\sin\theta)}{(1-\sin\theta)^2}\right]=\frac{\boldsymbol{2a}}{r^3}$

Hence Pedal equation is $\boxed{r^3=2ap^2}$.

b) $r^n=a^n\cos n\theta$

Diff. w.r.to θ we get, $nr^{n-1}r_1=-na^n\sin n\theta$

Or $r^n\,\frac{\boldsymbol{r_1}}{\boldsymbol{r}}=-a^n\sin n\theta$

$\therefore \frac{r_1}{r}=-\tan n\theta \quad \Longrightarrow \quad \frac{1}{p^2}=\frac{1}{r^2}\,[1+\tan^2 n\theta\,]=\frac{1}{r^2\cos^2 n\theta}$

$\Longrightarrow p=r\cos n\theta \quad \Longrightarrow$ Pedal equation is $\boxed{pa^n=r^{n+1}}$

c) $r(1-\cos\theta)=2a$

Diff. w.r.to θ we get, $r_1(1-\cos\theta)+r\sin\theta=0$

Or $r\sin\theta=-r_1(1-\cos\theta) \quad \Longrightarrow \frac{r_1}{r}=-\frac{\sin\theta}{(1-\cos\theta)}$

$\frac{1}{p^2}=\frac{1}{r^2}\left[1+\left(\frac{\sin\theta}{1-\cos\theta}\right)^2\right]=\frac{1}{r^2}\left[\frac{2(1-\cos\theta)}{(1-\cos\theta)^2}\right]=\frac{\boldsymbol{1}}{ar}$

Hence Pedal equation is $\boxed{p^2 = ar}$.

d) $r = m\theta$

Diff. w.r.to θ we get, $r_1 = m \implies \frac{1}{p^2} = \frac{1}{r^2}\left[1 + \frac{m^2}{r^2}\right]$

Hence Pedal equation is $\boxed{r^4 = [r^2 + m^2]p^2}$

Derivative of arc length: $\frac{ds}{dx} = \sqrt{1 + y_1^2} = \sec\psi$, $\frac{ds}{dy} = \sqrt{1 + \frac{1}{y_1^2}} = \operatorname{cosec}\psi$.

Parametric: $\frac{ds}{dt} = \sqrt{\dot{x}^2 + \dot{y}^2}$ where $\dot{x} = \frac{dx}{dt}$, $\dot{y} = \frac{dy}{dt}$

Polar form: $\frac{ds}{d\theta} = \sqrt{r^2 + r_1^2} = r\operatorname{cosec}\emptyset$, $\frac{ds}{dr} = \sqrt{\left(\frac{r}{r_1}\right)^2 + 1} = \sec\emptyset$,

Therefore $\sin\emptyset = r\frac{d\theta}{ds}$ and $\cos\emptyset = \frac{dr}{ds}$.

Curvature $\boldsymbol{K} = \frac{d\psi}{ds}$, **Radius of curvature** $\boldsymbol{\rho} = \frac{ds}{d\psi}$.

Radius of curvature in Cartesian form: $\rho = \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2}$

Proof: We know that $\tan\psi = y_1$ or $\psi = \tan^{-1} y_1$.

Differentiating both sides w.r.t. x,

$$\frac{d\psi}{dx} = \frac{1}{1+y_1^2} \cdot \frac{dy_1}{dx} = \frac{y_2}{1+y_1^2} .$$

$$\rho = \frac{ds}{d\psi} = \frac{ds}{dx} \cdot \frac{dx}{d\psi} = \sqrt{1 + y_1^2} \cdot \frac{1+y_1^2}{y_2} = \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2} .$$

Radius of curvature in Parametric form:

$$\rho = \frac{(\dot{x}^2+\dot{y}^2)^{\frac{3}{2}}}{\dot{x}\ddot{y}-\dot{y}\ddot{x}} \quad \text{Where } \dot{x} = \frac{dx}{dt},\ \dot{y} = \frac{dy}{dt},\ \ddot{x} = \frac{d^2x}{dt^2},\ \ddot{y} = \frac{d^2y}{dt^2} .$$

Proof: $y_1 = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\dot{y}}{\dot{x}}$ and

$$y_2 = \frac{dy_1}{dx} = \frac{dy_1/dt}{dx/dt} = \frac{\dot{x}\ddot{y}-\dot{y}\ddot{x}}{\dot{x}^2} \cdot \frac{1}{\dot{x}}$$

$$\therefore \rho = \frac{\left(1+y_1^2\right)^{\frac{3}{2}}}{y_2} = \frac{\left(1+\left(\frac{\dot{y}}{\dot{x}}\right)^2\right)^{\frac{3}{2}}}{\frac{\dot{x}\ddot{y}-\dot{y}\ddot{x}}{\dot{x}^3}} = \frac{\left(\dot{x}^2+\dot{y}^2\right)^{\frac{3}{2}}}{\dot{x}\ddot{y}-\dot{y}\ddot{x}}.$$

Radius of curvature in Polar form: $\rho = \dfrac{\left(r^2+r_1^2\right)^{\frac{3}{2}}}{r^2+2r_1^2-rr_2}$.

Proof: We know that $\tan\phi = \frac{r}{r_1}$, diff. w. r. t. θ we get $\sec^2 \emptyset \frac{d\phi}{d\theta} = \frac{r_1^2-rr_2}{r_1^2}$

$$\frac{d\phi}{d\theta} = \frac{r_1^2-rr_2}{r_1^2\left[\left(\frac{r}{r_1}\right)^2+1\right]} = \frac{r_1^2-rr_2}{r^2+r_1^2}.$$

But $\psi = \theta + \phi$,

$$\therefore \frac{d\psi}{d\theta} = 1 + \frac{d\phi}{d\theta} = 1 + \frac{r_1^2-rr_2}{r^2+r_1^2}$$

$$= \frac{r^2+2r_1^2-rr_2}{r^2+r_1^2}.$$

Finally $\rho = \frac{ds}{d\psi} = \frac{ds}{d\theta}\cdot\frac{d\theta}{d\psi}$

$$= \sqrt{r^2+r_1^2}\cdot\frac{r^2+r_1^2}{r^2+2r_1^2-rr_2} = \frac{\left(r^2+r_1^2\right)^{\frac{3}{2}}}{r^2+2r_1^2-rr_2}.$$

Radius of curvature in Pedal form: $\rho = r\dfrac{dr}{dp}$.

Proof: We know that $p = r\sin\emptyset$,

diff. w. r. t. r we get, $\frac{dp}{dr} = r\frac{d\emptyset}{dr}\cdot\cos\emptyset + \sin\emptyset$

$$= r\frac{d\emptyset}{dr}\frac{dr}{ds} + r\frac{d\theta}{ds} = r\frac{d}{dr}(\phi+\theta)$$

$$= r\frac{d\psi}{dr} = \frac{r}{\rho}.$$

Therefore, $\frac{\rho}{r} = \frac{dr}{dp}$ or $\rho = r\frac{dr}{dp}$.

Note: i) If x-axis is a tangent to a curve at $(0,\ 0)$, then ρ at $(0,\ 0) = \lim\limits_{x\to 0}\frac{x^2}{2y}$.

ii) If y-axis is a tangent to a curve at the origin, then ρ at $(0,\ 0) = \lim\limits_{x\to 0}\frac{y^2}{2x}$.

Problems: 1. Find the radius of curvature at $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ of the Folium $x^3 + y^3 = 3axy$.

Differentiating with respect to x, we get

$$3x^2 + 3y^2 y_1 = 3a(xy_1 + y) \implies y_1 = -\frac{x^2 - ay}{y^2 - ax} \quad \ldots\ldots..(i)$$

$$\therefore y_1 \text{ at } \left(\frac{3a}{2}, \frac{3a}{2}\right) = -1 \ .$$

Differentiating (i) , $y_2 = -\frac{(y^2-ax)(2x-ay_1)-(x^2-ay)(2yy_1-a)}{(y^2-ax)^2}$

$$\therefore y_2 \text{ at } \left(\frac{3a}{2}, \frac{3a}{2}\right) = -\frac{32}{3a} \ .$$

Hence ρ at $\left(\frac{3a}{2}, \frac{3a}{2}\right) = \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2} = \frac{3a}{8\sqrt{2}}$.

2. Find the radius of curvature at any point of the cycloid

$x = a(\theta + \sin\theta),\ y = a(1 - \cos\theta)$.

$$x = a(\theta + \sin\theta) \ , \qquad y = a(1 - \cos\theta)$$

$$\implies \quad \dot{x} = a(1 + \cos\theta) \ , \qquad \dot{y} = a\sin\theta$$

And $\quad \ddot{x} = -a\sin\theta \quad , \qquad \ddot{y} = a\cos\theta$

$$\therefore \quad \rho = \frac{(\dot{x}^2+\dot{y}^2)^{\frac{3}{2}}}{\dot{x}\ddot{y}-\dot{y}\ddot{x}} = \frac{(a^2(1+\cos\theta)^2+a^2\sin^2\theta)^{\frac{3}{2}}}{a^2(1+\cos\theta)\cos\theta+a^2\sin^2\theta}$$

$$= a2^{\frac{3}{2}}\sqrt{(1+\cos\theta)} = 4a\cos\frac{\theta}{2}.$$

3. For the cardioid $r = a(1 + \cos\theta)$, show that ρ^2 is proportional to r .

$$r = a(1 + \cos\theta) \implies r_1 = -a\sin\theta \quad \text{and} \quad r_2 = -a\cos\theta$$

$$\therefore \ \rho = \frac{(r^2+r_1^2)^{\frac{3}{2}}}{r^2+2r_1^2-rr_2} = \frac{(a^2(1+\cos\theta)^2+a^2\sin^2\theta)^{\frac{3}{2}}}{a^2(1+\cos\theta)^2+2a^2\sin^2\theta+a^2(1+\cos\theta)\cos\theta}$$

$$= \frac{a[2(1+\cos\theta)]^{\frac{3}{2}}}{3(1+\cos\theta)}$$

$$\implies \quad \rho^2 = \frac{8a^2(1+\cos\theta)^3}{9(1+\cos\theta)^2} = \frac{8a}{9}r, \qquad \text{and hence} \qquad \rho^2 \propto r \ .$$

4. Find the radius of curvature for $p^2 = ar$.

$$p^2 = ar \implies r = \frac{p^2}{a}$$

$$\therefore \frac{dr}{dp} = \frac{2p}{a} = 2\sqrt{\frac{r}{a}} \qquad \implies \rho = r\frac{dr}{dp} = \frac{2r\sqrt{r}}{\sqrt{a}} \ .$$

5. Find the radius of curvature of the curve $r^n = a^n \cos n\theta$.

Solution: Given curve is $r^n = a^n \cos n\theta$

Diff. w.r.to θ we get,

$$nr^{n-1}r_1 = -na^n \sin n\theta$$

Or $r^n \frac{r_1}{r} = -a^n \sin n\theta$

$$\therefore \frac{r_1}{r} = -\tan n\theta \implies \frac{1}{p^2} = \frac{1}{r^2}\left[1 + \tan^2 n\theta\right] = \frac{1}{r^2 \cos^2 n\theta}$$

$\implies p = r\cos n\theta \implies$ Pedal equation is $\boxed{pa^n = r^{n+1}}$

Diff. w.r.to r we get , $\frac{dp}{dr}a^n = (n+1)r^n$

Therefore $\rho = r\frac{dr}{dp} = \frac{a^n}{(n+1)r^{n-1}}$.

Or Given curve is $r^n = a^n \cos n\theta$

Diff. w.r.to θ we get,

$$nr^{n-1}r_1 = -na^n \sin n\theta$$

$$\therefore \quad r_1 = -r\tan n\theta$$

$$r_2 = -r_1 \tan n\theta - nr\sec^2 n\theta$$

$$= r\tan^2 n\theta - nr\sec^2 n\theta$$

$$\rho = \frac{(r^2+r_1^2)^{\frac{3}{2}}}{r^2+2r_1^2-rr_2} = \frac{r^3(1+\tan^2 n\theta)^{\frac{3}{2}}}{r^2+2r^2\tan^2 n\theta - r(r\tan^2 n\theta - nr\sec^2 n\theta)}$$

$$= \frac{r\sec^3 n\theta}{1+\tan^2 n\theta + n\sec^2 n\theta} = \frac{r\sec n\theta}{n+1}$$

$$= \frac{ra^n}{(n+1)r^n} = \frac{a^n}{(n+1)r^{n-1}} \ .$$

6. Find the radius of curvature of the curve $x^4 + y^4 = 2$ at the point $(1,\ 1)$.

Solution: $x^4 + y^4 = 2$

Differentiating with respect to x, we get

$$4x^3 + 4y^3y_1 = 0 \implies y_1 = -\frac{x^3}{y^3} \quad \text{........}(i) \quad \therefore y_1 \text{ at } (1,\ 1) = -1 \ .$$

Differentiating (i) with respect to x, we get,

$$y_2 = -\frac{y^3 3x^2 - x^3 3y^2 y_1}{y^6} \qquad \therefore y_2 \text{ at } (1,\ 1) = -6 \ .$$

Hence ρ at $(1, 1) = \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2} = \frac{2\sqrt{2}}{6} = \frac{\sqrt{2}}{3}$.

7. Find the radius of curvature of the curve $x^{2/3} + y^{2/3} = a^{2/3}$ at any point (x, y).

Solution: $y_1 = -\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}} \quad \Rightarrow y_1^3 = -\frac{y}{x} \quad \Rightarrow xy_1^3 = -y \quad \Rightarrow 3xy_1^2y_2 + y_1^3 = -y_1$

$$\Rightarrow 3xy_1y_2 + y_1^2 = -1 \quad \Rightarrow y_2 = -\left(\frac{1+y_1^2}{3xy_1}\right)$$

$$\mathrm{P} = \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2} = \frac{(1+y_1^2)^{\frac{3}{2}}}{-\left(\frac{1+y_1^2}{3xy_1}\right)} = -3xy_1(1+y_1^2)^{\frac{1}{2}}$$

$$= 3x\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}\left(\frac{x^{\frac{2}{3}}+y^{\frac{2}{3}}}{x^{\frac{2}{3}}}\right)^{\frac{1}{2}} = 3a^{\frac{1}{3}}x^{\frac{1}{3}}y^{\frac{1}{3}} = 3\sqrt[3]{axy} .$$

Or Parametric equation is $x = a\cos^3 t \quad and \quad y = a\sin^3 t$

$\Rightarrow \quad \dot{x} = -3a\cos^2 t \sin t$, $\quad \dot{y} = 3a\sin^2 t\cos t$

$\ddot{x} = -3a(-2\cos t\sin^2 t + \cos^3 t) \quad , \quad \ddot{y} = 3a(2\sin t\cos^2 t - \sin^3 t)$

$$\therefore \quad \rho = \frac{(\dot{x}^2+\dot{y}^2)^{\frac{3}{2}}}{\dot{x}\ddot{y}-\dot{y}\ddot{x}} = \frac{(3a)^3\cos^3 t \sin^3 t}{-9a^2\cos^2 t \sin t(2\sin t\cos^2 t-\sin^3 t)+9a^2\sin^2 t \cos t(2\sin t\cos^2 t-\sin^3 t)}$$

$$= \frac{3a\cos^3 t \sin^3 t}{-\cos^2 t \sin^2 t} = 3a\cos t\sin t = 3a\left(\frac{x}{a}\right)^{\frac{1}{3}}\left(\frac{y}{a}\right)^{\frac{1}{3}} = 3a^{\frac{1}{3}}x^{\frac{1}{3}}y^{\frac{1}{3}} = 3\sqrt[3]{axy}$$

8. Find the radius of curvature of the curve $r^n = a^n \sin n\theta$.

Solution: $r^n = a^n \sin n\theta$

Diff. w.r.to θ we get, $nr^{n-1}r_1 = na^n\cos n\theta$

Or $r^n\,\frac{\boldsymbol{r_1}}{\boldsymbol{r}} = a^n\cos n\theta$

$\therefore \frac{r_1}{r} = \cot n\theta \quad \Rightarrow \quad \frac{1}{p^2} = \frac{1}{r^2}\,[1+\cot^2 n\theta\,] = \frac{1}{r^2\sin^2 n\theta}$ (3)

$\Rightarrow\; p = r\sin n\theta \quad \Rightarrow$ Pedal equation is $\boxed{pa^n = r^{n+1}} \Rightarrow \frac{dp}{dr} = \frac{(n+1)r^n}{a^n}$

$\therefore \rho = r\frac{dr}{dp} = \frac{a^n}{(n+1)r^{n-1}}$.

9. Find the radius of curvature of the curve $y^2 = \frac{a^2(a-x)}{x}$ at the point $(a, 0)$.

$$y^2 = \frac{a^2(a-x)}{x}$$

$$y^2x = a^3 - a^2x$$

Differentiating w.r.t x

$2xyy_1 + y^2 = -a^2 \quad \Rightarrow y_1 = -\frac{a^2+y^2}{2xy}$.

Since $y_1 = \infty$ at $(a, 0)$, therefore $\rho = \frac{1}{x_2}$.

$x_1 = -\frac{2xy}{y^2+a^2}$ then $x_1 = 0\ at\ (a, 0)$

$(y^2 + a^2)x_1 = -2xy$.

Again differentiating w.r.t y

$(y^2 + a^2)x_2 + 2yx_1 = -2x - 2yx_1$

$\Longrightarrow\ (y^2 + a^2)x_2 = -2x - 2yx_1 - 2yx_1$

Then $x_2 = -\frac{2}{a}$ at $(a, 0)$

$\therefore \rho = \frac{1}{x_2} = -\frac{a}{2}$

$\therefore$ The radius of curvature of the given curve is $\frac{a}{2}$.

10. Find the radius of curvature of the curve $x = a\log(\sec t + \tan t)$, $y = a\sec t$.

Given that, $x = a\log(\sec t + \tan t)$, $y = a\sec t$.

$$\Longrightarrow \quad \dot{x} = a\frac{\sec t\ \tan t + \sec^2 t}{\sec t + \tan t} = a\sec t\ , \qquad \dot{y} = a\sec t\ \tan t$$

$$\ddot{x} = a\sec t\ \tan t\ , \qquad \ddot{y} = a(\sec^3 t + \sec t\tan^2 t)$$

$$\therefore \quad \rho = \frac{(\dot{x}^2+\dot{y}^2)^{\frac{3}{2}}}{\dot{x}\ddot{y}-\dot{y}\ddot{x}} = \frac{a^3(\sec^2 t+\sec^2 t\tan^2 t)^{\frac{3}{2}}}{a^2\sec^2 t\,(\sec^2 t+\tan^2 t)-a^2\sec^2 t\tan^2 t}$$

$$= \frac{a\sec^6 t}{\sec^4 t} = a\ \sec^2 t.$$

Exercise:

a) Find the angle between the following pair of curves.

1. $r^2 = a^2 \operatorname{cosc} 2\theta$ and $r^2 = b^2\sec 2\theta$, 2. $r = ae^{\theta}$ and $re^{\theta} = b$.

3. $r^n = a^n\sec(n\theta + \propto)$ and $r^n = b^n\sec(n\theta + \beta)$, 4. $r = \frac{a\theta}{1+\theta}$ and $r = \frac{a}{1+\theta^2}$.

c) Find the pedal equation of the following curves.

1. $r^2 = a^2\sec 2\theta$ 2. $r^m = a^m(\cos m\theta + \sin m\theta)$ 3. $r^n = a^n \operatorname{sech} n\theta$ 4. $r = ae^{\theta\cot\propto}$

5. $r^n = a^n\sin n\theta + b^n\cos n\theta$.

d) 1. Find the radius of curvature of the curve $\sqrt{x} + \sqrt{y} = 4$ at the point where it cuts the line $y = x$.

2. Find the radius of curvature of the curve $\theta = \frac{\sqrt{r^2-a^2}}{a} - \cos^{-1}\left(\frac{a}{r}\right)$ at any point on it.

e) Find ρ for the following curves.

1. $xy^3 = a^4$ at the point $(a,\ a)$ 2. $y = 4\sin x - \sin 2x$ at $(\frac{\pi}{2},\ 4)$

3. $r^2 = a^2\sec 2\theta$ 4. $x = a\log\sec\theta,\quad y = a(\tan\theta - \theta)$

f) 1. If ρ_1 and ρ_2 be the radii of curvature at the extremities of any chord of the cardioid $a(1 + \cos\theta)$ which passes through the pole, show that $\rho_1^2 + \rho_2^2 = \frac{16a^2}{9}$.

2. Find the radius of curvature of the curve $\theta = \frac{\sqrt{r^2-a^2}}{a} - \cos^{-1}\left(\frac{a}{r}\right)$ at any point on it.
3. Prove that $\rho = p + \frac{d^2p}{d\psi^2}$ with usual notations.

Self-study:

Centre of curvature: A point $\boldsymbol{C}$ on the normal to any point $\boldsymbol{P}$ of a curve at a distance $\boldsymbol{\rho}$ from it, is called center of curvature.

Circle of curvature: A circle with center $\boldsymbol{C}$ (center of curvature at $\boldsymbol{P}$) and radius $\boldsymbol{\rho}$ is called circle of curvature Or osculating circle at $\boldsymbol{P}$.

Centre of curvature at any point $P(x,\ y)$ on the curve $y = f(x)$ is given by

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2} \quad , \qquad \bar{y} = y + \frac{1+y_1^2}{y_2} \quad .$$

Evolute: The locus of the center of the curvature for a curve is called its **evolute** and the curve is called an **Involute** of its evolute.

Problems:

1. If the center of curvature of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at one end of the minor axis lies at the other end, then show that the eccentricity of the ellipse is $\frac{1}{\sqrt{2}}$.

Solution: Given that the center of curvature at $(0,\ b)$ is $(0, -b)$. Therefore $\bar{y} = -b$.

For the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $y_1 = -\frac{b^2x}{a^2y}$, and $y_2 = -\frac{b^2}{a^2}\left(\frac{y-xy_1}{y^2}\right)$.

At the point $(0,\ b)$, $y_1 = 0$, and $y_2 = -\frac{b}{a^2}$.

$$\bar{y} = y + \frac{1+y_1^2}{y_2} \Longrightarrow \ -b = b - \frac{a^2}{b} \ \ \Longrightarrow 2b^2 = a^2 \ \text{ or } \ \frac{b^2}{a^2} = \frac{1}{2} \ .$$

Hence eccentricity $e = \sqrt{1 - \frac{b^2}{a^2}} = \frac{1}{\sqrt{2}}$.

2. Find the coordinates of the center of curvature at any point of the parabola $y^2 = 4ax$. Hence find its evolute.

Solution: Differentiating with respect to *x*, we get

$$2yy_1 = 4a \ \Longrightarrow \ \ y_1 = \frac{2a}{y} \quad \text{........} (i)$$

Differentiating (i), $y_2 = -\frac{2ay_1}{y^2} \ \ \Longrightarrow y_2 = -\frac{4a^2}{y^3}$.

Centre of curvature at any point is $\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2} = x + \frac{\frac{2a}{y}\left(1+\frac{4a^2}{y^2}\right)}{\frac{4a^2}{y^3}} = x + \frac{y^2+4a^2}{2a} = 3x + 2a$,

$$\bar{y} = y + \frac{1+y_1^2}{y_2} = y - \frac{\left(1+\frac{4a^2}{y^2}\right)}{\frac{4a^2}{y^3}} = y - \frac{y(y^2+4a^2)}{4a^2} = -\frac{y^3}{4a^2} \ .$$

To find the evolute, $(\bar{y})^2 = \frac{y^6}{16a^4} = \frac{(4ax)^3}{16a^4} = \frac{4x^3}{a} = \frac{4}{a}\left(\frac{\bar{x}-2a}{3}\right)^3$ or $(\bar{y})^2 = \frac{4}{a}\left(\frac{\bar{x}-2a}{3}\right)^3$.

Therefore the locus of $(\bar{x},\ \bar{y})$ i.e., evolute, is $27ay^2 = 4(x-2a)^3$.

Assignment questions:

g) Find the coordinates of the center of curvature at $(2,\ 1)$ on the parabola $x^2 = 4y$.

h) Find the coordinates of the center of curvature at any point of the parabola $x^2 = 4ay$. Hence find its evolute.

i) Show that the equation of the evolute of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}$.