Module-5: Linear Algebra

Elementary transformation of a matrix:

1. The interchange of any two rows (columns)
2. The multiplication of any row (column) by a non-zero number.
3. The addition of a constant multiple of the elements of any row (column) to the corresponding elements of any other row (column)

Two matrices A and B are said to be **equivalent** if one can be obtained from the other by a sequence of Elementary transformation. Equivalent matrices are denoted by $A \sim B$.

A matrix is obtained from the unit matrix by any one of the elementary transformations is called **Elementary matrix.**

Rank: A matrix is said to be of rank $\boldsymbol{r}$ **,** if it has at least one nonzero minor of order r and every minor of order higher then r vanishes. Rank of A is denoted by $\rho(A)$.

Note: 1. If a matrix has nonzero minor of order r , then its rank is $\geq r$.
2. If all the minors of order $r+1$ are zero, then its rank is $\leq r$.
3. Elementary transformations do not change the rank of a matrix.

Echelon Form: A rectangular matrix is in echelon form if,
1. All nonzero rows are above any zero rows.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zero.

Row reduced Echelon Form: An echelon form is said to be row reduced if, the leading entry in each nonzero row is 1 and each leading 1 is the only nonzero entry in its column.

If a matrix $\boldsymbol{A}$ is equivalent to an echelon matrix $\boldsymbol{E}$, then $\boldsymbol{\rho(A)} =$ Number of nonzero rows in $\boldsymbol{E}$.

Examples: a) Find the rank of the following matrix.

1. $$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix} \xrightarrow[R_3 = R_3 - 2R_1]{R_2 = R_2 - R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix} \xrightarrow{R_3 = R_3 - R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Clearly reduced matrix is in echelon form with 2 nonzero rows. $\therefore\ \boldsymbol{\rho(A) = 2}$ **.**

2. $$\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix} \xrightarrow{R_4 = R_4 - (R_1 + R_2 + R_3)} \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow[R_3 = 2R_3 - 3R_1]{R_2 = -(2R_2 - R_1)} \begin{bmatrix} 2 & 3 & -1 & -1 \\ 0 & 5 & 3 & 7 \\ 0 & -7 & 9 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_3 = 5R_3 + 7R_2} \begin{bmatrix} 2 & 3 & -1 & -1 \\ 0 & 5 & 3 & 7 \\ 0 & 0 & 66 & 44 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Clearly reduced matrix is in echelon form with 3 nonzero rows. $\therefore\ \boldsymbol{\rho(A) = 3}$ **.**

3. $$\begin{bmatrix} 90 & 91 & 92 & 93 & 94 \\ 91 & 92 & 93 & 94 & 95 \\ 92 & 93 & 94 & 95 & 96 \\ 93 & 94 & 95 & 96 & 97 \\ 94 & 95 & 96 & 97 & 98 \end{bmatrix} \xrightarrow[R_3 = R_3 - R_2,\ R_2 = R_2 - R_1]{R_5 = R_5 - R_4,\ R_4 = R_4 - R_3} \begin{bmatrix} 90 & 91 & 92 & 93 & 94 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow[R_3 = R_3 - R_2]{R_5 = R_5 - R_2,\ R_4 = R_4 - R_2} \begin{bmatrix} 90 & 91 & 92 & 93 & 94 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad R_2 = -(90R_2 - R_1) \begin{bmatrix} 90 & 91 & 92 & 93 & 94 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Clearly reduced matrix is in echelon form with 2 nonzero rows. $\therefore$ $\boldsymbol{\rho(A) = 2}$ **.**

4. $$\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_4} \begin{bmatrix} 1 & 1 & -2 & 0 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 0 & 1 & -3 & -1 \end{bmatrix}$$

$$\xrightarrow[R_3 = -\frac{1}{2}(R_3 - 3R_1)]{R_2 = R_2 - R_1} \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & -1 & 3 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix} \xrightarrow{R_4 = R_4 + R_2,\ R_3 = R_3 + R_2} \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & -1 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Clearly reduced matrix is in echelon form with 2 nonzero rows. $\therefore$ $\boldsymbol{\rho(A) = 2}$ **.**

Consistency of Homogeneous linear equations, $\boldsymbol{AX = 0}$:

$X = 0$ is the trivial solution. Thus the homogeneous system is always consistent.

Note: 1. If $\rho(A) =$ number of unknowns, then the system has only trivial solution.

2. If $\rho(A) <$ number of unknowns, then the system has an infinite number of solutions.

Consistency of non-homogeneous linear equations, $\boldsymbol{AX = B}$:

1. If $\rho(A) = \rho(A|B) =$ number of unknowns, then the system has unique solution.
2. If $\rho(A) = \rho(A|B) <$ number of unknowns, then the system has an infinite number of solutions.
3. If $(A) \neq \rho(A|B)$, then system has no solution.

Examples:

1. Test for consistency and solve the system $x + 4 + 3z = 0,\quad x - y + z = 0,\ 2x - y + 3z = 0$.

Solution: Augmented matrix $[A|B]$ is

$$\left[\begin{array}{ccc|c} 1 & 4 & 3 & 0 \\ 1 & -1 & 1 & 0 \\ 2 & -1 & 3 & 0 \end{array}\right]$$

$$\xrightarrow[R_3 = -(R_3 - 2R_1)]{R_2 = -(R_2 - R_1)} \left[\begin{array}{ccc|c} 1 & 4 & 3 & 0 \\ 0 & 5 & 2 & 0 \\ 0 & 9 & 3 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 = 5R_3 - 9R_2} \left[\begin{array}{ccc|c} 1 & 4 & 3 & 0 \\ 0 & 5 & 2 & 0 \\ 0 & 0 & -3 & 0 \end{array}\right]$$

Clearly $\rho(A) = \rho(A|B) = 3 =$ number of unknowns, the system has unique solution that is trivial. $x = y = z = 0$.

2. For what values of λ and μ do the system of equations: $x + y + z = 6, \quad x + 2y + 3z = 10,$ $x + 2y + \lambda z = \mu$ have (i) no solution (ii) unique solution (iii) infinite solutions.

Solution: Augmented matrix $[A|B]$ is

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right]$$

$$\xrightarrow[R_3 = R_3 - R_1]{R_2 = R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda - 1 & \mu - 6 \end{array}\right]$$

$$\xrightarrow{R_3 = R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda - 3 & \mu - 10 \end{array}\right]$$

(i) If $(A) \neq \rho(A|B)$, then the system has no solution.

If $\lambda - 3 = 0$ and $\mu - 10 \neq 0$ then $\rho(A) = 2 \neq \rho(A|B) = 3$.

Therefore, if $\lambda = 3$ and $\mu \neq 10$ then the system has no solution.

(ii) If $\rho(A) = \rho(A|B) =$ number of unknowns, then the system has unique solution.

If $\lambda - 3 \neq 0$ and for any value of μ, $\rho(A) = \rho(A|B) = 3 =$ number of unknowns.

Hence for $\lambda \neq 3$, the system has unique solution.

(iii) If $\rho(A) = \rho(A|B) <$ number of unknowns, then the system has an infinite number of solutions.

If $\lambda - 3 = 0$ and $\mu - 10 = 0$ then $\rho(A) = 2 = \rho(A|B) < 3$.

Therefore if $\lambda = 3$ and $\mu = 10$ then the system has an infinite number of solutions.

3. Show that if $\lambda \neq -5$, the system $3x - y + 4z = 3, \quad x + 2y - 3z = -2, \; 6x + 5y + \lambda z = -3$ have a unique solution. Find the solution if $\lambda = -5$.

Solution: Augmented matrix $[A|B]$ is

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & -2 \\ 3 & -1 & 4 & 3 \\ 6 & 5 & \lambda & -3 \end{array}\right]$$

$$\xrightarrow[R_3 = R_3 - 6R_1]{R_2 = R_2 - 3R_1} \left[\begin{array}{ccc|c} 1 & 2 & -3 & -2 \\ 0 & -7 & 13 & 9 \\ 0 & -7 & \lambda + 18 & 9 \end{array}\right]$$

$$\xrightarrow{R_3 = R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 2 & -3 & -2 \\ 0 & -7 & 13 & 9 \\ 0 & 0 & \lambda+5 & 0 \end{array}\right]$$

Clearly if $\lambda + 5 \neq 0$, $\rho(A) = \rho(A|B) = 3 =$ number of unknowns.

Therefore if $\lambda \neq -5$ then the system has unique solution.

if $\lambda = -5$ then $\rho(A) = 2 = \rho(A|B) < 3$, the system has an infinite number of solutions.

$x + 2y - 3z = -2$ and $-7y + 13z = 9 \Rightarrow y = \frac{13z-9}{7}$, $x = -2 - 2\left(\frac{13z-9}{7}\right) + 3z = \frac{4-5z}{7}$

Therefore solutions are $\begin{pmatrix} \frac{4-5z}{7} \\ \frac{13z-9}{7} \\ z \end{pmatrix}$ for any value of z.

Exercise:

1. Test for consistency and solve the system $x + y + z = 3$, $2x - y + 3z = 10$, $4x + y + 5z = 16$.
2. For what values of λ and μ do the system of equations: $2x + 3y + 5z = 9$, $7x + 3y - 2z = 8$, $2x + 3y + \lambda z = \mu$ have (i) no solution (ii) unique solution (iii) infinite solutions.
3. Test for consistency of the system $x + y + z = 3$, $2x + y + 3z = 5$, $x + 2y = 3$.

Solution of linear simultaneous equations:

1. **Gauss elimination method**:

Consider the equations $a_1x + b_1y + c_1z = d_1$, $a_2x + b_2y + c_2z = d_2$, $a_3x + b_3y + c_3z = d_3$

Reduce augmented matrix into an upper triangular matrix as below

$$\left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right]$$

$$\xrightarrow[R_3 = a_1R_3 - a_3R_1]{R_2 = a_1R_2 - a_2R_1} \left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ 0 & b'_2 & c'_2 & d'_2 \\ 0 & b'_3 & c'_3 & d'_3 \end{array}\right]$$

$$\xrightarrow{R_3 = b'_2R_3 - b'_3R_2} \left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ 0 & b'_2 & c'_2 & d'_2 \\ 0 & 0 & c''_3 & d''_3 \end{array}\right]$$

Then $z = \frac{d''_3}{c''_3}$, $y = \frac{d'_2 - zc'_2}{b'_2}$, $x = \frac{d_1 - yb_1 - zc_1}{a_1}$.

Example:

1. Solve by Gauss elimination method, $2x - 3y + z = -1$, $x + 4y + 5z = 25$, $3x - 4y + z = 2$.

   Solution: Augmented matrix is

$$\left[\begin{array}{ccc|c} 2 & -3 & 1 & -1 \\ 1 & 4 & 5 & 25 \\ 3 & -4 & 1 & 2 \end{array}\right]$$

$$\xrightarrow[R_3 = 2R_3 - 3R_1]{R_2 = 2R_2 - R_1} \left[\begin{array}{ccc|c} 2 & -3 & 1 & -1 \\ 0 & 11 & 9 & 51 \\ 0 & 1 & -1 & 7 \end{array}\right]$$

$$\xrightarrow{R_3 = 11R_3 - R_2} \left[\begin{array}{ccc|c} 2 & -3 & 1 & -1 \\ 0 & 11 & 9 & 51 \\ 0 & 0 & -20 & 26 \end{array}\right]$$

$\therefore \quad z = -\frac{26}{20} = -1.3, \qquad y = \frac{51-9\times(-1.3)}{11} = 5.7 \quad and \quad z = \frac{-1+3\times 5.7+1.3}{2} = 8.7$.

2. Solve by Gauss elimination method, $2x + 3y + z = -1, \quad x - y + z = 6, \quad 3x + 2y - z = -4.$

Solution: Augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 6 \\ 2 & 3 & 1 & -1 \\ 3 & 2 & -1 & -4 \end{array}\right]$$

$$\xrightarrow[R_3 = R_3 - 3R_1]{R_2 = R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 6 \\ 0 & 5 & -1 & -13 \\ 0 & 5 & -4 & -22 \end{array}\right]$$

$$\xrightarrow{R_3 = R_3 - R_2} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 6 \\ 0 & 5 & -1 & -13 \\ 0 & 0 & -3 & -9 \end{array}\right]$$

$\therefore \quad -3z = -9, \qquad 5y - z = -13, \quad x - y + z = 6 \implies \qquad z = 3, \quad y = -2, \quad and \quad x = 1$.

2. Gauss Jordan method:

Reduce augmented matrix into a diagonal matrix as below

$$\left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right]$$

$$\xrightarrow[R_3 = a_1R_3 - a_3R_1]{R_2 = a_1R_2 - a_2R_1} \left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ 0 & b'_2 & c'_2 & d'_2 \\ 0 & b'_3 & c'_3 & d'_3 \end{array}\right]$$

$$\xrightarrow[R_3 = b'_2R_3 - b'_3R_2]{R_1 = b'_2R_1 - b_1R_2} \left[\begin{array}{ccc|c} a'_1 & 0 & c'_1 & d'_1 \\ 0 & b'_2 & c'_2 & d'_2 \\ 0 & 0 & c''_3 & d''_3 \end{array}\right]$$

$$\xrightarrow[R_2 = c''_3R_2 - c'_2R_3]{R_1 = c''_3R_1 - c'_1R_3} \left[\begin{array}{ccc|c} a''_1 & 0 & 0 & d''_1 \\ 0 & b''_2 & 0 & d''_2 \\ 0 & 0 & c''_3 & d''_3 \end{array}\right]$$

Then $\quad x = \frac{d''_1}{a''_1}, \qquad y = \frac{d''_2}{b''_2} \qquad$ and $\qquad z = \frac{d''_3}{c''_3}$.

Examples:

1. Solve by Gauss Jordan method, $2x - y + 3z = 1, \quad -3x + 4y - 5z = 0, \quad x + 3y - 6z = 0.$

Solution: Augmented matrix is

$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & 1 \\ -3 & 4 & -5 & 0 \\ 1 & 3 & -6 & 0 \end{array}\right]$$

$$\xrightarrow[R_3 = 2R_3 - R_1]{R_2 = 2R_2 + 3R_1} \left[\begin{array}{ccc|c} 2 & -1 & 3 & 1 \\ 0 & 5 & -1 & 3 \\ 0 & 7 & -15 & -1 \end{array}\right]$$

$$\xrightarrow[R_3 = 5R_3 - 7R_2]{R_1 = 5R_1 + R_2} \left[\begin{array}{ccc|c} 10 & 0 & 14 & 8 \\ 0 & 5 & -1 & 3 \\ 0 & 0 & -68 & -26 \end{array}\right] \to \left[\begin{array}{ccc|c} 5 & 0 & 7 & 4 \\ 0 & 5 & -1 & 3 \\ 0 & 0 & 34 & 13 \end{array}\right]$$

$$\xrightarrow[R_2 = 34R_2 + R_3]{R_1 = 34R_1 - 7R_3} \left[\begin{array}{ccc|c} 170 & 0 & 0 & 45 \\ 0 & 170 & 0 & 115 \\ 0 & 0 & 34 & 13 \end{array}\right]$$

$\therefore \quad x = \frac{45}{170} = \frac{9}{34} = 0.2647$, $\quad y = \frac{115}{170} = \frac{23}{34} = 0.6765$ and $\quad z = \frac{13}{34} = 0.3824$.

2. Solve by Gauss Jordan method, $2x + y + z = 10$, $\quad 3x + 2y + 3z = 18$, $\quad x + 4y + 9z = 16$.

Solution: Augmented matrix is

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 10 \\ 3 & 2 & 3 & 18 \\ 1 & 4 & 9 & 16 \end{array}\right]$$

$$\xrightarrow[R_3 = 2R_3 - R_1]{R_2 = 2R_2 - 3R_1} \left[\begin{array}{ccc|c} 2 & 1 & 1 & 10 \\ 0 & 1 & 3 & 6 \\ 0 & 7 & 17 & 22 \end{array}\right]$$

$$\xrightarrow[R_3 = -\frac{1}{4}(R_3 - 7R_2)]{R_1 = \frac{1}{2}(R_1 - R_2)} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 3 & 6 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

$$\xrightarrow[R_2 = R_2 - 3R_3]{R_1 = R_1 + R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -9 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

$\therefore \quad x = 7$, $\quad y = -9$ and $\quad z = 5$.

Exercise:

1. Applying Gauss Jordan method solve,
   i) $2x + 3y - z = 5$, $\quad 4x + 4y - 3z = 3$, $\; 2x - 3y + 2z = 2$.
   ii) $x + y + z = 6$, $\quad x - 2y + 3z = 8$, $\; 2x + y - z = 3$.
2. Applying Gauss elimination method solve the above system of equations.

3. Gauss-Seidel iteration method: Consider the equations $a_1x + b_1y + c_1z = d_1$, $a_2x + b_2y + c_2z = d_2$, $a_3x + b_3y + c_3z = d_3$, If a_1, b_2, c_3 are numerically large as compared to other coefficients in their respective equations. Then iterative formula for x, y *and* z are given by

$x_{n+1} = \frac{1}{a_1}(d_1 - c_1z_n - b_1y_n)$, $\qquad y_{n+1} = \frac{1}{b_2}(d_2 - a_2x_{n+1} - c_2z_n)$ and

$z_{n+1} = \frac{1}{c_3}(d_3 - b_3y_{n+1} - a_3x_{n+1})$

Start with initial approximations $x_0 = 0$, $y_0 = 0$, $z_0 = 0$ for x, y, z respectively .

Note: **Gauss-Seidel method converges if in each equation, the absolute value of the largest coefficient is Greater than the sum of the absolute values of the remaining coefficients.**

Example:

1. Solve $54x + y + z = 110$, $2x + 15y + 6z = 72$, $-x + 6y + 21z = 85$ by Gauss-Seidel iteration method.

$x_{n+1} = \frac{1}{54}(110 - z_n - y_n)$	$y_{n+1} = \frac{1}{15}(72 - 2x_{n+1} - 6z_n)$	$z_{n+1} = \frac{1}{21}(85 - 6y_{n+1} + x_{n+1})$
Let $x_0 = 0$	$y_0 = 0$	$z_0 = 0$
$x_1 = \frac{1}{54}(110 - z_0 - y_0)$ $= 2.037$	$y_1 = \frac{1}{15}(72 - 2x_1 - 6z_0)$ $= 4.528$	$z_1 = \frac{1}{21}(85 - 6y_1 + x_1)$ $= 2.851$
$x_2 = \frac{1}{54}(110 - z_1 - y_1)$ $= 1.900$	$y_2 = \frac{1}{15}(72 - 2x_2 - 6z_1)$ $= 3.406$	$z_2 = \frac{1}{21}(85 - 6y_2 + x_2)$ $= 3.165$
$x_3 = \frac{1}{54}(110 - z_2 - y_2)$ $=1.915$	$y_3 = \frac{1}{15}(72 - 2x_3 - 6z_2)$ $=3.279$	$z_3 = \frac{1}{21}(85 - 6y_3 + x_3)$ $=3.202$
$x_4 = \frac{1}{54}(110 - z_3 - y_3)$ $=1.917$	$y_4 = \frac{1}{15}(72 - 2x_4 - 6z_3)$ $=3.264$	$z_4 = \frac{1}{21}(85 - 6y_4 + x_4)$ $=3.206$
$x_5 = \frac{1}{54}(110 - z_4 - y_4)$ $=1.917$	$y_5 = \frac{1}{15}(72 - 2x_5 - 6z_4)$ $=3.262$	$z_5 = \frac{1}{21}(85 - 6y_5 + x_5)$ $=3.207$

$\therefore \quad x = 1.917 \ , \quad y = 3.262 \quad and \quad z = 3.207$.

2. Use Gauss-Seidel method to solve $20x + y - 2z = 17$, $3x + 20y - z = 18$, $2x - 3y + 20z = 25$. Carry out 2 iterations with $x_0 = 0$, $y_0 = 0$, $z_0 = 0$.

$x_{n+1} = \frac{1}{20}(17 + 2z_n - y_n)$	$y_{n+1} = \frac{1}{20}(18 - 3x_{n+1} + z_n)$	$z_{n+1} = \frac{1}{20}(25 + 3y_{n+1} - 2x_{n+1})$
Let $x_0 = 0$	$y_0 = 0$	$z_0 = 0$
$x_1 = \frac{1}{20}(17 + 2z_0 - y_0)$ $= 0.85$	$y_1 = \frac{1}{20}(18 - 3x_1 + z_0)$ $= 0.7725$	$z_1 = \frac{1}{20}(25 + 3y_1 - 2x_1)$ $= 1.2809$
$x_2 = \frac{1}{20}(17 + 2z_1 - y_1)$ $= 0.9395$	$y_2 = \frac{1}{20}(18 - 3x_2 + z_1)$ $= 0.8231$	$z_2 = \frac{1}{20}(25 + 3y_2 - 2x_2)$ $= 1.2795$

$\therefore \quad x = 0.9395 \ , \quad y = 0.8231 \quad and \quad z = 1.2795$.

Exercise:

Solve by Gauss-Seidel method.

1. $2x + y + 6z = 9$, $8x + 3y + 2z = 13$, $x + 5y + z = 7$.
2. $83x + 11y - 4z = 95$, $7x + 52y + 13z = 104$, $3x + 8y + 29z = 71$.

Characteristic equation: $|A - \lambda I| = 0$ is the characteristic equation of the square matrix A. Roots are called **Characteristic roots** or **Eigen values** or **latent roots** of A.

Any vector X satisfying $[A - \lambda I]X = 0$ is called **Eigen vector** corresponding to the Eigen value.

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then Characteristic equation is $\lambda^2 - (a + d)\lambda + (ad - cb) = 0$.

if $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, then Characteristic equation is

$$\lambda^3 - (a_1 + b_2 + c_3)\lambda^2 + (sum\ of\ the\ minors\ of\ a_1, b_2 \& c_3)\lambda - |A| = 0\ .$$

Properties of Eigen values:

1) The sum of the Eigen values of a matrix is the sum of the principal diagonal elements.
2) The product of the Eigen values of a matrix is equal to its determinant.
3) If λ is the Eigen value of A , then $1/\lambda$ is Eigen value of A^{-1}.
4) If λ is the Eigen value of an orthogonal matrix, then $1/\lambda$ is also its Eigen value .
5) If λ is the Eigen value of A, then λ^n is the Eigen value of A^n . But Eigen vectors are same.

Examples:

1) Find the Eigen values and Eigen vectors of the following matrices.

i) Let $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

Characteristic equation is $|A - \lambda I| = 0$.

$$\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$$

$\sum D = 1 + 5 + 1 = 7.$

$\sum M\ D = 4 - 8 + 4 = 0$

$|A| = -36$

$\Longrightarrow\ \lambda^3 - (\ 7\ )\lambda^2 + (\ \ 0\ )\lambda - (\,-36) = 0$

$\Longrightarrow\ \lambda^3 - 7\lambda^2 + 0\lambda + 36 = 0$.

Roots are $-2,\ 3,\ 6$

$\lambda_1 = -2$	$\lambda_2 = 3$	$\lambda_3 = 6$
$3x + y + 3z = 0$	$-2x + y + 3z = 0$	$-5x + y + 3z = 0$
$x + 7y + z = 0$	$x + 2y + z = 0$	$x - y + z = 0$
$\Rightarrow 20y = 0$, and $z = -x$	$\Longrightarrow\ y = -z$	$\Longrightarrow\ z = x$
$\therefore\ X_1 = [1,\ \ 0,\ \ -1]'$	$X_2 = [1,\ \ -1,\ \ 1]'$	$X_3 = [1,\ \ 2,\ \ 1]'$

Eigen values are $-2,\ 3$ and 6 , the corresponding Eigen vectors are
$[1,\ \ 0,\ \ -1]'$, $[1,\ \ -1,\ \ 1]'$ and $[1,\ \ 2,\ \ 1]'$ respectively.

ii) Let $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

Characteristic equation is $|A - \lambda I| = 0$.

$$\Longrightarrow \begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$\sum D = 8 + 7 + 3 = 18.$

$\sum M\ D = 5 + 20 + 20 = 45$

$|A| = 0.$

$\Longrightarrow\ \lambda^3 - (\ 18\ )\lambda^2 + (\ \ 45\ )\lambda - (\,0) = 0$

$\Longrightarrow\ \lambda^3 - 18\lambda^2 + 45\lambda + 0 = 0$

Roots are $0,\ 3,\ 15$

$\lambda_1 = 0$	$\lambda_2 = 3$	$\lambda_3 = 15$
$8x - 6y + 2z = 0$	$5x - 6y + 2z = 0$	$-7x - 6y + 2z = 0$

$-6x+7y-4z=0$	$-6x+4y-4z=0$	$-6x-8y-4z=0$
$\Rightarrow 10x-5y=0$,	$\Rightarrow\ 4x-8y=0$	$\Rightarrow -20x-20y=0$
$\Rightarrow\ y=2x$	$\Rightarrow\ x=2y$	$\Rightarrow\ y=-x$
$\therefore\ X_1=[1,\quad 2,\quad 2]'$	$X_2=[2,\quad 1,\quad -2]'$	$X_3=\left[1,\quad -1,\quad \frac{1}{2}\right]'$

Eigen values are $0,\ 3$ and 15 , the corresponding Eigen vectors are
$[1,\quad 2,\quad 2]'$, $[2,\quad 1,\quad -2]'$ and $[2,\quad -2,\quad 1]'$ respectively.

iii) Let $A=\begin{bmatrix}6 & -2 & 2\\ -2 & 3 & -1\\ 2 & -1 & 3\end{bmatrix}$

Characteristic equation is $|A-\lambda I|=0$.

$$\Rightarrow \begin{vmatrix}6-\lambda & -2 & 2\\ -2 & 3-\lambda & -1\\ 2 & -1 & 3-\lambda\end{vmatrix}=0$$

$\sum D=6+3+3=12.$

$\sum M\ D=8+14+14=36$

$|A|=32$

$\Rightarrow\ \lambda^3-(\ 12\ )\lambda^2+(\ 36\ )\lambda-(\ 32)=0$
$\Rightarrow\ \lambda^3-12\lambda^2+36\lambda-32=0$

Roots are $2,\ 2,\ 8$

(The sum of the Eigen values of a matrix is the sum of the principal diagonal elements.

$\lambda_1=2$	$\lambda_2=2$	$\lambda_3=8$
$4x-2y+2z=0$	$4x-2y+2z=0$	$-2x-2y+2z=0$
Or $2x-y+z=0$	Or $2x-y+z=0$	$-2x-5y-z=0$
Let $y=0$, and $z=-2x$	Let $z=0$, and $y=2x$	$\Rightarrow 3y+3z=0$
$\therefore\ X_1=[1,\quad 0,\quad -2]'$	$X_2=[1,\quad 2,\quad 0]'$	$X_3=[2,\quad -1,\quad 1]'$

Eigen values are $2,\ 2$ and 8 , the corresponding Eigen vectors are
$[1,\quad 0\quad -2]'$, $[1,\quad 2,\quad 0]'$ and $[2\quad -1\quad 1]'$ respectively.

Exercise:

1. Find the Eigen values and Eigen vectors of the following matrices.

i) $\begin{bmatrix}3 & 1 & 1\\ 1 & 3 & -1\\ 1 & -1 & 3\end{bmatrix}$ ii) $\begin{bmatrix}0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{bmatrix}$ iii) $\begin{bmatrix}3 & -1 & 1\\ -1 & 5 & -1\\ 1 & -1 & 3\end{bmatrix}$

<u>Determination of largest Eigen value by Rayleigh's power method:</u>

Let A be the given square matrix and a column vector X_0 be the initial Eigen vector. Evaluate $AX_0=\lambda_1X_1$ where λ_1 is the first approximation of the Eigen value and X_1 is the corresponding Eigen vector.
$AX_1=\lambda_2X_2$. Where λ_2 is the 2^{nd} approximation of the Eigen value and X_2 is the corresponding Eigen vector.
$AX_2=\lambda_3X_3$. Where λ_3 is the 3^{rd} approximation of the Eigen value and X_3 is the corresponding Eigen vector.
Repeat this process till X_n-X_{n-1} becomes negligible.

Example: 1. Find the largest Eigen value and corresponding Eigen vector of the matrix by power method.

$$A = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \text{ taking initial eigen vector } X_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Solution:

$$AX_0 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \lambda_1 X_1, \qquad AX_1 = \begin{bmatrix} 7 \\ 3 \\ 0 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 0.43 \\ 0 \end{bmatrix} = \lambda_2 X_2,$$

$$AX_2 = \begin{bmatrix} 3.57 \\ 1.86 \\ 0 \end{bmatrix} = 3.57 \begin{bmatrix} 1 \\ 0.52 \\ 0 \end{bmatrix} = \lambda_3 X_3, \qquad AX_3 = \begin{bmatrix} 4.12 \\ 2.04 \\ 0 \end{bmatrix} = 4.12 \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = \lambda_4 X_4,$$

$$AX_4 = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = \lambda_5 X_5.$$

Since X_4 and X_5 are same, the largest Eigen value is 4 and the corresponding Eigen vector is $[1,\ 0.5,\ 0]'$

2. Find the largest Eigen value and corresponding Eigen vector of the matrix by power method.

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \text{ taking initial Eigen vector } X_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \text{ Carry out 4 iterations.}$$

Solution:

$$AX_0 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ 0.5 \end{bmatrix} = \lambda_1 X_1, \qquad AX_1 = \begin{bmatrix} 2.5 \\ 0 \\ 2 \end{bmatrix} = 2.5 \begin{bmatrix} 1 \\ 0 \\ 0.8 \end{bmatrix} = \lambda_2 X_2,$$

$$AX_2 = \begin{bmatrix} 2.8 \\ 0 \\ 2.6 \end{bmatrix} = 2.8 \begin{bmatrix} 1 \\ 0 \\ 0.93 \end{bmatrix} = \lambda_3 X_3, \qquad AX_3 = \begin{bmatrix} 2.93 \\ 0 \\ 2.86 \end{bmatrix} = 2.93 \begin{bmatrix} 1 \\ 0 \\ 0.98 \end{bmatrix} = \lambda_4 X_4.$$

The largest Eigen value is 2.93 and the corresponding Eigen vector is $[1,\ 0,\ 0.98]'$.

Exercise:

Find the largest Eigen value and corresponding Eigen vector of the following matrix by power method.

1. $\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$, $X_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.
2. $\begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}$, $X_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.
3. $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$, $X_0 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.
4. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $X_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Self-Study: Gauss-Jacobi's iteration method:

Consider the equations $a_1 x + b_1 y + c_1 z = d_1$; $a_2 x + b_2 y + c_2 z = d_2$; $a_3 x + b_3 y + c_3 z = d_3$,

If $a_1,\ b_2,\ c_3$ are numerically large as compared to other coefficients in their respective equations.

Then iterative formula for x, y and z are given by

$$x = \frac{1}{a_1}(d_1 - b_1 y - c_1 z), \qquad y = \frac{1}{b_2}(d_2 - a_2 x - c_2 x) \text{ and } z = \frac{1}{c_3}(d_3 - a_3 x - b_3 y) \cdots\cdots (1)$$

If not given assume that initial value of $(x,\ y,\ z) \equiv (0,\ 0,\ 0)$. Substitute these values in (1) and find

$x_1 = \frac{1}{a_1}(d_1)$, $y_1 = \frac{1}{b_2}(d_2)$ and $z_1 = \frac{1}{c_3}(d_3)$

Then find, $x_2 = \frac{1}{a_1}(d_1 - b_1y_1 - c_1z_1)$, $y_2 = \frac{1}{b_2}(d_2 - a_2x_1 - c_2z_1)$ and $z_2 = \frac{1}{c_3}(d_3 - a_3x_1 - b_3y_1)$

Continuing like this using

$x_{n+1} = \frac{1}{a_1}(d_1 - b_1y_n - c_1z_n)$, $y_{n+1} = \frac{1}{b_2}(d_2 - a_2x_n - c_2z_n)$ and $z_{n+1} = \frac{1}{c_3}(d_3 - a_3x_n - b_3y_n)$

until two set of values coincides.

Example:

1. Solve $54x + y + z = 110$, $2x + 15y + 6z = 72$, $-x + 6y + 21z = 85$ by **Gauss-Jacobi's** iteration method by taking initial values $(x,\ y,\ z) \equiv (2,\ 3,\ 4)$.

$x_{n+1} = \frac{1}{54}(110 - z_n - y_n)$	$y_{n+1} = \frac{1}{15}(72 - 2x_n - 6z_n)$	$z_{n+1} = \frac{1}{21}(85 - 6y_n + x_n)$
Let $x_0 = 2$	$y_0 = 3$	$z_0 = 4$
$x_1 = \frac{1}{54}(110 - z_0 - y_0)$ $= 1.907$	$y_1 = \frac{1}{15}(72 - 2x_0 - 6z_0)$ $= 2.933$	$z_1 = \frac{1}{21}(85 - 6y_0 + x_0)$ $= 3.286$
$x_2 = \frac{1}{54}(110 - z_1 - y_1)$ $= 1.922$	$y_2 = \frac{1}{15}(72 - 2x_1 - 6z_1)$ $= 3.231$	$z_2 = \frac{1}{21}(85 - 6y_1 + x_1)$ $= 3.300$
$x_3 = \frac{1}{54}(110 - z_2 - y_2)$ $=1.916$	$y_3 = \frac{1}{15}(72 - 2x_2 - 6z_2)$ $=3.224$	$z_3 = \frac{1}{21}(85 - 6y_2 + x_2)$ $=3.216$
$x_4 = \frac{1}{54}(110 - z_3 - y_3)$ $=1.918$	$y_4 = \frac{1}{15}(72 - 2x_3 - 6z_3)$ $=3.258$	$z_4 = \frac{1}{21}(85 - 6y_3 + x_3)$ $=3.218$
$x_5 = \frac{1}{54}(110 - z_4 - y_4)$ $=1.917$	$y_5 = \frac{1}{15}(72 - 2x_4 - 6z_4)$ $=3.257$	$z_5 = \frac{1}{21}(85 - 6y_4 + x_4)$ $=3.208$
$x_6 = \frac{1}{54}(110 - z_5 - y_5)$ $=1.917$	$y_6 = \frac{1}{15}(72 - 2x_5 - 6z_5)$ $=3.261$	$z_6 = \frac{1}{21}(85 - 6y_5 + x_5)$ $=3.208$

$\therefore \quad x = 1.917$, $y = 3.261$ *and* $z = 3.208$.

2. Use Gauss-Seidel method to solve $20x + y - 2z = 17$, $3x + 20y - z = 18$, $2x - 3y + 20z = 25$. with $x_0 = 0$, $y_0 = 0$, $z_0 = 1$.

$x_{n+1} = \frac{1}{20}(17 + 2z_n - y_n)$	$y_{n+1} = \frac{1}{20}(18 - 3x_n + z_n)$	$z_{n+1} = \frac{1}{20}(25 + 3y_n - 2x_{n+1})$
Let $x_0 = 0$	$y_0 = 0$	$z_0 = 1$
$x_1 = \frac{1}{20}(17 + 2z_0 - y_0)$ $= 0.95$	$y_1 = \frac{1}{20}(18 - 3x_0 + z_0)$ $= 0.95$	$z_1 = \frac{1}{20}(25 + 3y_0 - 2x_0)$ $= 1.25$
$x_2 = \frac{1}{20}(17 + 2z_1 - y_1)$ $= 0.928$	$y_2 = \frac{1}{20}(18 - 3x_1 + z_1)$ $= 0.820$	$z_2 = \frac{1}{20}(25 + 3y_1 - 2x_1)$ $= 1.298$
$x_3 = \frac{1}{20}(17 + 2z_2 - y_2)$ $= 0.939$	$y_3 = \frac{1}{20}(18 - 3x_2 + z_2)$ $= 0.826$	$z_3 = \frac{1}{20}(25 + 3y_2 - 2x_2)$ $= 1.28$
$x_4 = \frac{1}{20}(17 + 2z_3 - y_3)$ $= 0.937$	$y_4 = \frac{1}{20}(18 - 3x_3 + z_3)$ $= 0.823$	$z_4 = \frac{1}{20}(25 + 3y_3 - 2x_3)$ $= 1.28$

$\therefore \quad x = 0.937 \ , \quad y = 0.823 \quad and \quad z = 1.28 \ .$

Cayley-Hamilton theorem: Every square matrix satisfies its characteristic equation.

Example: 1. Find the inverse of the matrix $A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$, by Cayley-Hamilton theorem.

Sol: Characteristic equation is $|A - \lambda I| = 0$.

$$\Rightarrow \begin{vmatrix} 3-\lambda & 1 \\ 2 & 4-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 - 7\lambda + 10 = 0 \ .$$

By Cayley-Hamilton theorem $A^2 - 7A + 10I = 0$

$$\Rightarrow 10A^{-1} = 7I - A$$

$$\therefore A^{-1} = \frac{1}{10}\left\{\begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}\right\} = \frac{1}{10}\begin{bmatrix} 4 & -1 \\ -2 & 3 \end{bmatrix}.$$