Module-1: Integral Calculus
Multiple Integrals: Evaluation of double and triple integrals, evaluation of double integrals by change of order of integration, changing into polar coordinates. Applications to find: Area and Volume by double integral. Problems. **Beta and Gamma functions:** Definitions, properties, relation between Beta and Gamma functions. Problems. **Self-Study:** Center of gravity. **(RBT Levels: L1, L2 and L3)**

Reference: https://youtu.be/ZFYhQVC1RFI

Reduction formulae:

1. $\int \sin^n x\,dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\,dx.$

2. $\int \cos^n x\,dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\,dx.$

3. $\int \sin^m x \cos^n x\,dx = -\frac{\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n}\int \sin^{m-2} x \cos^n x\,dx.$

4. $\int \sin^m x \cos^n x\,dx = \frac{\sin^{m+1} x \cos^{n-1} x}{m+n} + \frac{n-1}{m+n}\int \sin^m x \cos^{n-2} x\,dx.$

If $I_n = \int_0^{\frac{\pi}{2}} \sin^n x\,dx = \int_0^{\frac{\pi}{2}} \cos^n x\,dx = \frac{(n-1)(n-3)\cdots\cdots}{n(n-2)(n-4)\cdots\cdots}\,k.$ where $k = \begin{cases} \frac{\pi}{2}, & \text{if } n \text{ is even} \\ 1, & \text{if } n \text{ is odd} \end{cases}$,

and if $I_{m,n} = \int_0^{\frac{\pi}{2}} \sin^m x \cos^n x\,dx = \frac{(m-1)(m-3)\cdots\cdots(n-1)(n-3)\cdots\cdots}{(m+n)(m+n-2)(m+n-4)\cdots\cdots}\,k.$

Where $k = \begin{cases} \frac{\pi}{2}, & \text{if both } m \text{ and } n \text{ are even} \\ 1, & \text{otherwise} \end{cases}$.

Double integrals: Integral of $f(x,\ y)$ a function of two independent variables x and y in the region R bounded by $x = a,\ x = b,\ y = g_1(x)$ and $y = g_2(x)$.

[or $y = c,\ y = d,\ \ x = h_1(y)$ and $x = h_2(y)$]

is called double integral over the region R, and is denoted by $\iint_R f(x,y)dxdy$.

Double integral is evaluated by evaluating successive single integrals as follows (when we integrate with respect to y treat x as constant , similarly when we integrate with respect to x treat y as constant.

$$\iint_R f(x,y)dxdy = \int_{x=a}^{x=b}\left\{\int_{y=g_1(x)}^{y=g_2(x)} f(x,y)dy\right\}dx = \int_{y=c}^{y=d}\left\{\int_{x=h_1(y)}^{x=h_2(y)} f(x,y)dx\right\}dy$$

Double integrals in Polar co-ordinates:

$$\iint_{R(x,y)} f(x,y)dxdy = \iint_{R(r,\theta)} f(r\cos\theta, r\sin\theta)rdrd\theta$$

Examples:

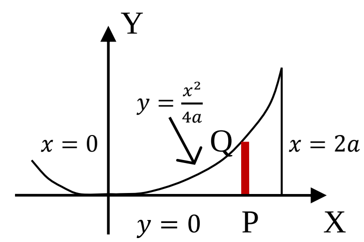
1) $\int_0^1 \int_0^3 x^3y^3dxdy = \int_0^1 \frac{x^4}{4}\Big|_0^3 y^3dy = \frac{81}{4}\int_0^1 y^3dy = \frac{81}{4}.\frac{y^4}{4}\Big|_0^1 = \frac{81}{16}.$

2) $\int_0^1 \int_1^{x^2} x(x^2+y^2)dxdy = \int_0^1 \int_1^{x^2}(x^3+xy^2)dydx = \int_0^1 (x^3y + x\frac{y^3}{3})\Big|_{y=1}^{y=x^2} dx$

$= \int_0^1 (x^5 - x^3 + \frac{x^7}{3} - \frac{x}{3})dx = \frac{x^6}{6} - \frac{x^4}{4} + \frac{x^8}{24} - \frac{x^2}{6}\Big|_{x=0}^{x=1} = -\frac{5}{24}.$

3) $\int_0^a \int_0^{\sqrt{a^2-x^2}} x^2y\, dy\, dx = \int_0^a x^2\frac{y^2}{2}\Big|_{y=0}^{y=\sqrt{a^2-x^2}} dx = \frac{1}{2}\int_0^a (a^2x^2 - x^4)dx = \frac{1}{2}\left(a^2\frac{x^3}{3} - \frac{x^5}{5}\right)_{x=0}^{x=a} = \frac{a^5}{15}.$

4) Evaluate $\iint_R xydxdy$, where R is the domain bounded by x-axis, ordinate $x = 2a$ and the curve $x^2 = 4ay$.



As x varies from 0 to 2a ,

For each x , y varies from 0 to $\frac{x^2}{4a}$.

$\iint_R xydxdy = \int_0^{2a}(\int_0^{\frac{x^2}{4a}} xydy)dx$

$= \int_0^{2a} x\frac{y^2}{2}\Big|_{y=0}^{y=\frac{x^2}{4a}} dx = \int_0^{2a} \frac{x^5}{32a^2}dx$

$= \frac{x^6}{32\times 6a^2}\Big|_{x=0}^{x=2a} = \frac{a^4}{3}.$

Reference: https://youtu.be/EXYW8QwQH4M

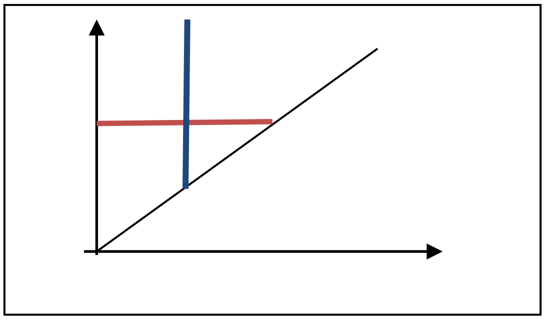
5. Evaluate by changing the order of integration $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx$

Solution: $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx$

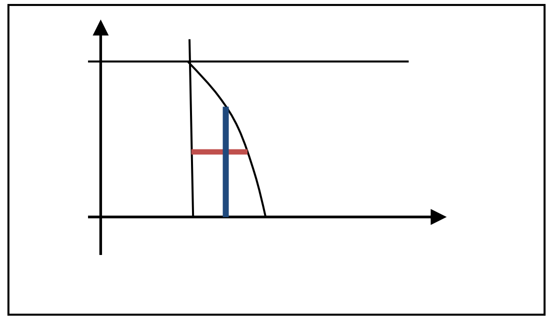
$= \int_0^\infty \int_0^y \frac{e^{-y}}{y}\, dx\, dy$

$= \int_0^\infty \frac{xe^{-y}}{y}\Big|_0^y\, dx = \int_0^\infty e^{-y}\, dx$

$= \int_0^\infty e^{-y}\, dx = e^{-y}\Big|_0^\infty = 1$



6. Evaluate by changing the order of integration $\int_0^3 \int_1^{\sqrt{4-y}} (x+y)\ dxdy$



Ans: $\int_0^3 \int_1^{\sqrt{4-y}} (x+y)\ dxdy$

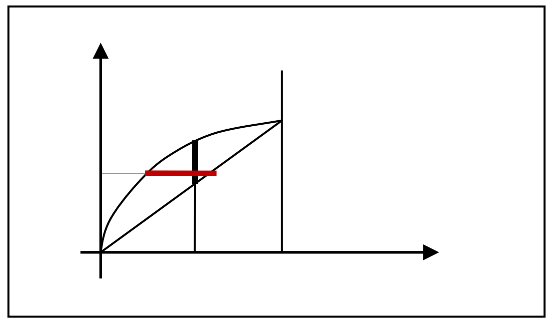
$$= \int_1^2 \int_0^{4-x^2} (x+y)\, dy\, dx$$

$$= \int_1^2 \left[xy + \frac{xy^2}{2}\right]\Bigg|_0^{4-x^2} dx$$

$$= \int_1^2 \left[8 + 4x - 4x^2 - x^3 + \frac{x^4}{2}\right] dx$$

$$= \left[8x + 2x^2 - \frac{4x^3}{3} - \frac{x^4}{4} + \frac{x^5}{10}\right]\Bigg|_1^2 = \frac{241}{60}.$$

7. Evaluate $\int_0^1 \int_x^{\sqrt{x}} xy\, dy\, dx$ by changing the order of integration.



Ans: $\int_0^1 \int_x^{\sqrt{x}} xy\, dy\, dx$

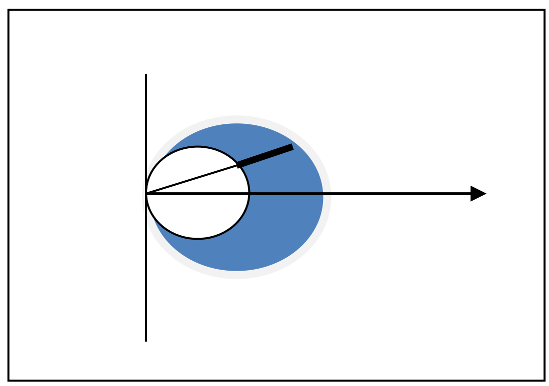
$$= \int_0^1 \int_{y^2}^{y} xy\, dx\, dy$$

$$= \int_0^1 \left[\frac{yx^2}{2}\right]_{y^2}^{y} dy$$

$$= \int_0^1 \left[\frac{y^3}{2} - \frac{y^5}{2}\right] dy = \frac{y^4}{8} - \frac{y^6}{12}\Bigg|_0^1 = \frac{1}{24}.$$

Reference: https://youtu.be/hEf2Q41iP28

8. Calculate $\iint r^3\, dr d\theta$ over the area included between the circles $r = 2\cos\theta$, $and\ r = 4\cos\theta$.



$$\iint r^3\, dr d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{2\cos\theta}^{4\cos\theta} r^3 dr d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{r^4}{4}\Bigg|_{2\cos\theta}^{4\cos\theta} d\theta$$

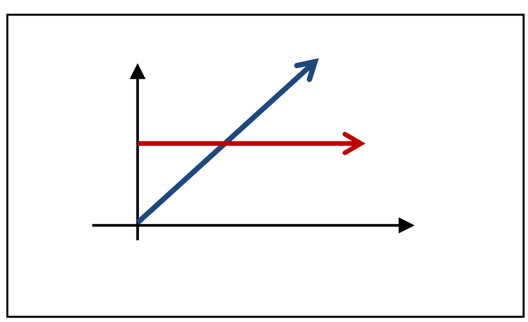
$$= 60 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4\theta\ d\theta$$

$$= 120 \int_0^{\frac{\pi}{2}} \cos^4\theta\, d\theta$$

$$= 120 \times \frac{3\times 1\times \pi}{4\times 2\times 2} = \frac{45}{2}\pi$$

9. Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\, dy$ by changing to polar coordinates.

Ans: $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\, dy = \int_0^{\frac{\pi}{2}} \int_0^\infty re^{-r^2} dr\, d\theta$

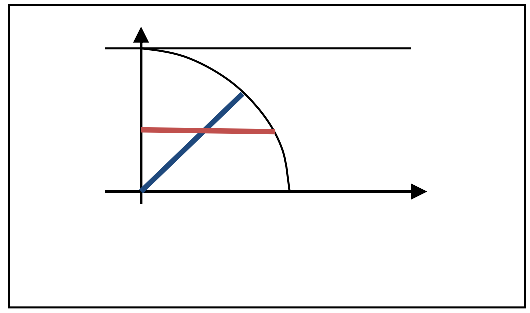


$$= \int_0^{\frac{\pi}{2}} \int_0^\infty re^{-r^2} dr\, d\theta$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} d\theta = \frac{\pi}{4}$$

10. Evaluate by changing to polar coordinates, $\int_0^1 \int_0^{\sqrt{1-y^2}} (x^2+y^2)dx\,dy$.

$$\int_0^1 \int_0^{\sqrt{1-y^2}} (x^2+y^2)dx\,dy$$
$$= \int_0^{\frac{\pi}{2}} \int_0^1 r^3\,drd\theta$$
$$= \int_0^{\frac{\pi}{2}} \frac{r^4}{4}\Big|_0^1 d\theta = \frac{1}{4}\int_0^{\frac{\pi}{2}} d\theta = \frac{\pi}{8}$$



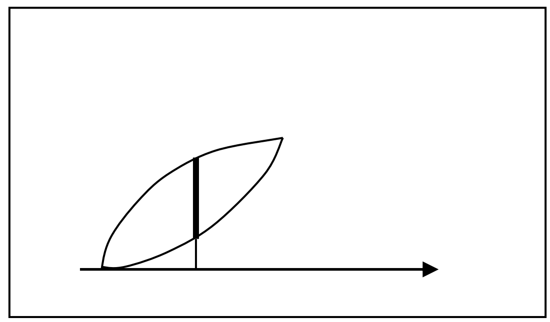
Reference: https://youtu.be/RYqV_OuYFpU

$Area = \iint_A 1dxdy$, $\quad In\ polar\ form\ Area = \iint_A r dr d\theta$.

11. Find the area between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$ by double integration.

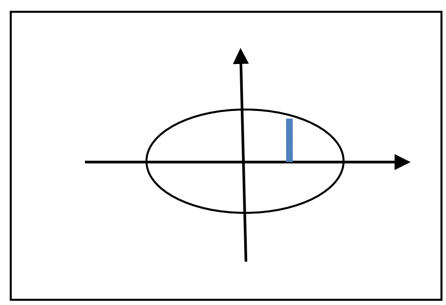
Ans: $A = \int_0^{4a} \int_{\frac{x^2}{4a}}^{2\sqrt{ax}} 1\,dy\,dx$

$$= \int_0^{4a} y\Big|_{\frac{x^2}{4a}}^{2\sqrt{ax}} dx$$
$$= \int_0^{\infty} \left[2\sqrt{a}x^{\frac{1}{2}} - \frac{x^2}{4a}\right] dx$$
$$= \frac{4\sqrt{a}\,x\sqrt{x}}{3} - \frac{x^3}{12a}\Big|_0^{4a} = \frac{32a^2}{3} - \frac{16a^2}{3} = \frac{16a^2}{3}.$$



12. Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ by double integration.

$$A = 4\int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} 1dydx$$
$$= \frac{4b}{a}\int_0^a \sqrt{a^2-x^2}\,dx$$
$$= \frac{4b}{a}\int_0^{\frac{\pi}{2}} a^2\cos^2\theta\,d\theta = 4ab\frac{\pi}{4} = \pi ab.$$

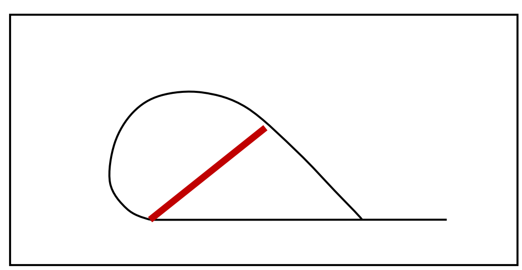


13. Find the area enclosed by the curve $r = a(1+\cos\theta)$ above the initial line.

Ans:

Required area is

$$A = \int_0^{\pi} \int_0^{a(1+\cos\theta)} r dr d\theta$$
$$= \int_0^{\pi} \frac{r^2}{2}\Big|_0^{a(1+\cos\theta)} d\theta$$



$$= \int_0^{\pi} \frac{a^2(1+2\cos\theta+\cos^2\theta)}{2}\, d\theta = \frac{a^2}{2}\left[\int_0^{\pi} 1 d\theta + 2\int_0^{\frac{\pi}{2}} \cos^2\theta\, d\theta\right] = \frac{a^2}{2}\left[\pi + \frac{\pi}{2}\right] = \frac{3a^2\pi}{4}.$$

Exercise:

A. Evaluate the following integrals.

1. $\int_0^1 \int_0^y e^{\frac{x}{y}}\, dx\, dy.$ 2. $\int_0^1 \int_x^{\sqrt{x}} (x^2+y^2)\, dy\, dx.$
3. $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dydx}{1+x^2+y^2}$.
4. $\iint xy(x+y)\, dxdy$ over the area between $y = x^2$ and $y = x$.

B. Evaluate the following integrals by changing the order of integration.

1. $\int_0^a \int_y^a \frac{xdxdy}{x^2+y^2}$. 2. $\int_0^3 \int_1^{\sqrt{4-y}} (x+y)\, dxdy$. 3. $\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{xdydx}{\sqrt{x^2+y^2}}$. 4. $\int_0^1 \int_x^{\sqrt{x}} xy\, dy\, dx$.
5. $\int_0^{\infty} \int_x^{\infty} \frac{e^{-y}}{y}\, dy\, dx$. 6. $\int_0^1 \int_{x^2}^{2-x} xy\, dy\, dx.$

C

1. Calculate $\iint r^3\, dr d\theta$ over the area included between the circles $r = 2\cos\theta$, $and\; r = 4\cos\theta$.
2. Evaluate $\iint r^2 \sin\theta\, dr d\theta$ over the semi-circle $r = 2a\cos\theta$ above the initial line.
3. Evaluate $\iint \frac{rdrd\theta}{\sqrt{a^2+r^2}} drd\theta$ over one loop of lemniscate $r^2 = a^2\cos 2\theta$.

D. Evaluate by changing to polar coordinates,

1. $\int_0^1 \int_0^{\sqrt{1-y^2}} (x^2+y^2) dx\, dy.$ 2. $\int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{xdydx}{x^2+y^2}$. 3. $\int_0^{4a} \int_{\frac{y^2}{4a}}^{y} \frac{x^2-y^2}{x^2+y^2} dx\, dy.$

E.

1. Find the area between the parabola $y = 4x - x^2$ and the line $y = x$.
2. Find the area lying between the circle $x^2+y^2 = a^2$ and the line $x + y = a$ in the first quadrant.
3. Find the area lying inside the cardioid $r = a(1+\cos\theta)$ and outside the circle $r = a$.
4. Find the area lying inside the circle $r = a\sin\theta$ and outside the cardioid $r = a(1-\cos\theta)$.

Reference: https://youtu.be/mvcktis9bc0

Triple integrals:

1) $$\int_0^1\int_0^2\int_0^3 (x+y+z)dzdydx = \int_0^1\int_0^2 \left(xz+yz+\frac{z^2}{2}\right)\Big|_{z=0}^{z=3} dydx = \int_0^1\int_0^2 \left(3x+3y+\frac{9}{2}\right) dydx$$
$$= \int_0^1 \left(3xy + 3\frac{y^2}{2} + \frac{9y}{2}\right)\Big|_{y=0}^{y=2} dx = \int_0^1 (6x+15)dx = \left(6\frac{x^2}{2}+15x\right)\Big|_{x=0}^{x=1} = 18.$$

2) $$\int_{-c}^{c}\int_{-b}^{b}\int_{-a}^{a} (x^2+y^2+z^2)dzdydx = \int_{-c}^{c}\int_{-b}^{b}\left(x^2z+y^2z+\frac{z^3}{3}\right)\Big|_{z=-a}^{z=a} dydx$$
$$= \int_{-c}^{c}\int_{-b}^{b}\left(2ax^2+2ay^2+\frac{2a^3}{3}\right) dydx$$
$$= \int_{-c}^{c}\left(2ax^2y+2a\frac{y^3}{3}+\frac{2a^3y}{3}\right)\Big|_{y=-b}^{y=b} dx$$
$$= \int_{-c}^{c}\left(4abx^2+\frac{4ab^3}{3}+\frac{4a^3b}{3}\right) dx$$
$$= \left(4ab\frac{x^3}{3}+\frac{4ab^3x}{3}+\frac{4a^3bx}{3}\right)\Big|_{x=-c}^{x=c} = \frac{8abc^3}{3}+\frac{8ab^3c}{3}+\frac{8abc^3}{3} = \frac{8abc(a^2+b^2+c^2)}{3}$$

3) $\int_0^1\int_0^2\int_1^2 x^2yz\,dxdydz = \int_0^1 zdz \times \int_0^2 ydy \times \int_1^2 x^2dx = \frac{z^2}{2}\Big|_0^1 \times \frac{y^2}{2}\Big|_0^2 \times \frac{x^3}{3}\Big|_1^2 = \frac{1}{2}\cdot 2\cdot\frac{7}{3} = \frac{7}{3}$.

4) $\int_0^1\int_0^1\int_0^1 e^{x+y+z}dxdydz = \int_0^1\int_0^1\int_0^1 e^xe^ye^z dxdydz = \int_0^1 e^zdz \times \int_0^1 e^ydy \times \int_0^1 e^xdx = (e-1)^3$.

5) Evaluate $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} xyz\,dzdydx$

$$\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} xyz\,dzdydx$$

$$= \int_0^1\int_0^{\sqrt{1-x^2}} \frac{xyz^2}{2}\Big|_0^{\sqrt{1-x^2-y^2}} dydx$$

$$= \int_0^1\int_0^{\sqrt{1-x^2}} \left[\frac{xy}{2} - \frac{x^3y}{2} - \frac{xy^3}{2}\right] dydx$$

$$= \int_0^1 \left[\frac{xy^2}{4} - \frac{x^3y^2}{4} - \frac{xy^4}{8}\right]\Bigg|_0^{\sqrt{1-x^2}} dx$$

$$= \int_0^1 \left[\frac{x}{8} - \frac{x^3}{4} + \frac{x^5}{8}\right] dx = \left[\frac{x^2}{16} - \frac{x^4}{16} + \frac{x^6}{48}\right]\Bigg|_0^1 = \frac{1}{16} - \frac{1}{16} + \frac{1}{48} = \frac{1}{48} \ .$$

6) Evaluate $\int_{-1}^1\int_0^z\int_{x-z}^{x+z}(x+y+z)dxdydz$

$$\int_{-1}^1\int_0^z\int_{x-z}^{x+z}(x+y+z)dxdydz = \int_{-1}^1\int_0^z\int_{x-z}^{x+z}(x+y+z)dydxdz$$

$$= \int_{-1}^1\int_0^z \left[xy + \frac{y^2}{2} + zy\right]\Bigg|_{x-z}^{x+z} dxdz = \int_{-1}^1\int_0^z [4xz + 2z^2]\ dxdz$$

$$= \int_{-1}^1 [2zx^2 + 2z^2x]_0^z\ dz = \int_{-1}^1 4z^3\ dz = z^4\Big|_{-1}^1 = 0$$

Evaluate the following triple integrals.

1. $\int_0^1\int_{y^2}^1\int_0^{1-x} x\,dzdxdy$.
2. $\int_0^a\int_0^x\int_0^{x+y} e^{x+y+z}dxdydz$.
3. $\int_0^{\frac{\pi}{2}}\int_0^{a\sin\theta}\int_0^{\frac{a^2-r^2}{a}} r\,dzdrd\theta$.
4. $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} \frac{dzdydx}{\sqrt{1-x^2-y^2-z^2}}$.

Reference: https://youtu.be/Ky4onfGuXHA

Volume as double integral: $\boldsymbol{Volume = \iint_A zdxdy = \iint_A f(x,y)dxdy.}$

1. A pyramid is bounded by three coordinate planes and the plane $x + 2y + 3z = 6$. Compute the volume by double integration.

$$Volume = \iint_A zdxdy = \int_0^6\int_0^{\frac{1}{2}(6-x)} \frac{1}{3}(6 - x - 2y)dydx$$

$$= \frac{1}{3}\int_0^6 (6y - xy - y^2)\Big|_0^{\frac{1}{2}(6-x)} dx = \frac{1}{3}\int_0^6 \left(3(6-x) - \frac{1}{2}x(6-x) - \frac{1}{4}(6-x)^2\right)dx$$

$$= \frac{1}{3}\int_0^6 \left(9 - 3x + \frac{x^2}{4}\right)dx = \frac{1}{3}\left(9x - \frac{3x^2}{2} + \frac{x^3}{12}\right)_0^6 = 6.$$

2. Find the volume of the tetrahedron bounded by the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ and the coordinate planes, using

double integration.

$$Volume = \iint_A zdxdy = c\int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} \left(1-\frac{x}{a}-\frac{y}{b}\right) dydx$$

$$= c\int_0^a \left(y - \frac{xy}{a} - \frac{y^2}{2b}\right)_0^{b\left(1-\frac{x}{a}\right)} dx$$

$$= \frac{bc}{2}\int_0^a \left(1-\frac{x}{a}\right)^2 dx = -\frac{abc}{6}\left(1-\frac{x}{a}\right)^3\Big|_0^a = \frac{abc}{6}\ .$$

3. Calculate the volume of the solid bounded by the planes $x = 0,\ y = 0,\ x + y + z = 1,\ and\ z = 0$.
   Required volume is

$$V = \int_0^1 \int_0^{1-x} (1-x-y)dy\, dx$$

$$= \int_0^1 \left[(1-x)y - \frac{y^2}{2}\right]_0^{1-x} dx$$

$$= \int_0^1 \frac{(1-x)^2}{2} dx = -\left[\frac{(1-x)^3}{6}\right]_0^1 = \frac{1}{6}.$$

4. Find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.
   Required volume is 8 times of volume in the first octant

$$\therefore V = 8c\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\, dy\, dx$$

$$= \frac{8c}{b}\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \sqrt{b^2\left(1-\frac{x^2}{a^2}\right) - y^2}\, dy\, dx$$

$$= 2bc\pi \int_0^a \left(1-\frac{x^2}{a^2}\right) dx$$

$$= 2bc\pi \left(x - \frac{x^3}{3a^2}\right)_0^a$$

$$= \frac{4\pi abc}{3}.$$

$$\int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \sqrt{b^2\left(1-\frac{x^2}{a^2}\right) - y^2}\, dy$$

$$= \int_0^{\frac{\pi}{2}} A^2 \cos^2\theta\ d\theta$$

$$= \frac{\pi}{4}A^2 = \frac{\pi}{4}b^2\left(1-\frac{x^2}{a^2}\right)$$

Exercise:

1. Find the volume bounded by the cylinder $x^2 + y^2 = 4$ and the planes $x + y = 4,\ z = 0$.
2. Find the volume bounded by the xy-plane, the cylinder $x^2 + y^2 = 1$ and the plane $x + y + z = 3$.

Reference: https://youtu.be/hEsejhABdpQ

Beta, Gamma functions:

$$\beta(m\,,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$$
$$= 2\int_0^{\frac{\pi}{2}} \sin^{2m-1} x \cos^{2n-1} x \, dx = \beta(n\,,m)$$

$$\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}dx$$
$$= 2\int_0^\infty e^{-x^2}x^{2n-1}dx.$$

$$\Longrightarrow \quad \int_0^{\frac{\pi}{2}} \sin^p x \cos^q x \, dx = \frac{1}{2}\beta\left(\frac{p+1}{2}\,,\frac{q+1}{2}\right) \quad , \quad \int_0^\infty e^{-x^2}x^p dx = \frac{1}{2}\Gamma\left(\frac{p+1}{2}\right) ,$$

and $\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}$

Theorems:

1. $\beta(m\,,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$ 2. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ 3. $\Gamma(n+1) = n\Gamma(n)$

Proofs:

1. Since $\Gamma(n) = 2\int_0^\infty e^{-x^2}x^{2n-1}dx$ ....(1)

$\Gamma(m) = 2\int_0^\infty e^{-y^2}y^{2m-1}dy$ ....(2)

$\Gamma(m+n) = 2\int_0^\infty e^{-r^2}r^{2m+2n-1}dr$ ....(3)

$\beta(m\,,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$ ....(4)

(1)× (2) $\Longrightarrow$ $\Gamma(m)\Gamma(n) = 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}x^{2n-1}y^{2m-1}dxdy$ (by changing into polar co-ordinates)

$$= 2\int_0^\infty e^{-r^2}r^{2m+2n-1}dr \;.\, 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\cos^{2n-1}\theta \, d\theta$$
$$= \Gamma(m+n).\,\beta(m\,,n)$$
$$\therefore\ \beta(m\,,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \ .$$

2. $\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-x^2}x^{\frac{2\times 1}{2}-1}dx = 2\int_0^\infty e^{-x^2}dx$ or $\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-y^2}dy$

$\Longrightarrow$ $[\Gamma\left(\frac{1}{2}\right)]^2 = 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}dxdy$ (by changing into polar co-ordinates)

$$= 4\int_0^{\frac{\pi}{2}}\int_0^\infty e^{-r^2}rdrd\theta$$
$$= 4\int_0^{\frac{\pi}{2}} d\theta \times \int_0^\infty e^{-r^2}rdr = 4\times\frac{\pi}{2}\times\frac{1}{2} = \pi.$$

$\therefore$ $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

Or Since $\beta(m\,,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta$ and $\beta(m\,,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

$$\Longrightarrow \quad \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta$$

Put $m = n = \frac{1}{2}$, we get $\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(1)} = 2\int_0^{\frac{\pi}{2}} \sin^0\theta\cos^0\theta\,d\theta$

Or $\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = 2\int_0^{\frac{\pi}{2}} d\theta = 2\frac{\pi}{2} = \pi$ $\therefore$ $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

3. $\Gamma(n+1)=\int_0^\infty e^{-x}x^n dx = x^n(-e^{-x})\Big|_0^\infty + n\int_0^\infty e^{-x}x^{n-1}dx = 0 + n\int_0^\infty e^{-x}x^{n-1}dx = n\Gamma(n).$

4) Evaluate $\int_0^\infty e^{-4x}x^2 dx$ using gamma function.

Put $4x = y \implies x = \frac{y}{4}$ $and\ dx = \frac{1}{4}dy$.

$\int_0^\infty e^{-y}y^n dx = \Gamma(n+1) = n!$.

$$\int_0^\infty e^{-4x}x^2 dx = \frac{1}{64}\int_0^\infty e^{-y}y^2 dx$$
$$= \frac{1}{64}\,\Gamma(2+1) = \frac{\Gamma(3)}{64} = \frac{1}{32} \ .$$

5) $\beta\left(\frac{5}{2}\,,\,\frac{3}{2}\right) = \frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{5}{2}+\frac{3}{2}\right)} = \frac{\frac{3}{2}\frac{1}{2}\sqrt{\pi}\frac{1}{2}\sqrt{\pi}}{\Gamma(4)} = \frac{\pi}{16}.$

6) Evaluate $\int_0^\infty e^{-4x}x^{3/2}dx$ using gamma function

$$\int_0^\infty e^{-4x}x^{3/2}dx = \frac{1}{32}\int_0^\infty e^{-y}y^{3/2}dy$$

Put $4x = y \implies x = \frac{y}{4}\,,\ x^{3/2} = \frac{y^{3/2}}{8}$ and $dx = \frac{1}{4}dy$

$$= \frac{1}{32}\Gamma\left(3/2+1\right) = \frac{1}{32}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi} = \frac{3}{128}\sqrt{\pi}$$

7) Prove that $\int_0^{\frac{\pi}{2}}\sqrt{\sin\theta}\,d\theta \times \int_0^{\frac{\pi}{2}}\frac{1}{\sqrt{\sin\theta}}d\theta = \pi$

$$\int_0^{\frac{\pi}{2}}\sqrt{\sin\theta}\,d\theta \times \int_0^{\frac{\pi}{2}}\frac{1}{\sqrt{\sin\theta}}d\theta = \int_0^{\frac{\pi}{2}}\sin^{\frac{1}{2}}\theta\cos^0\theta\,d\theta \times \int_0^{\frac{\pi}{2}}\sin^{-\frac{1}{2}}\theta\cos^0\theta\,d\theta$$
$$= \frac{1}{2}\beta\left(\frac{3}{4}\,,\frac{1}{2}\right)\times\frac{1}{2}\beta\left(\frac{1}{4}\,,\frac{1}{2}\right)$$

$\because \int_0^{\frac{\pi}{2}}\sin^p x\cos^q x\,dx = \frac{1}{2}\beta\left(\frac{p+1}{2}\,,\frac{q+1}{2}\right)$

$$= \frac{1}{4}\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{5}{4}\right)}\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}\right)} = \frac{\frac{1}{4}\,\Gamma\left(\frac{1}{4}\right)\ \pi}{\Gamma\left(\frac{5}{4}\right)} = \pi \ .$$

8) Show that $\Gamma(n) = \int_0^1(\log\frac{1}{x})^{n-1}\,dx$

$$\Gamma(n) = \int_0^\infty e^{-y}y^{n-1}dy$$

Put $x = e^{-y} \implies \log x = -y\,,\ or\ \ y = -\log x = \log\frac{1}{x}$ $\quad dy = -\frac{1}{x}dx$

$$= -\int_1^0 x(\log\tfrac{1}{x})^{n-1}\frac{1}{x}dx$$

$y\to\infty \implies x\to 0,\quad and\ \ y\to 0 \implies x\to 1$

$$= \int_0^1(\log\tfrac{1}{x})^{n-1}\,dx \ .$$

9) $\int_0^{\frac{\pi}{2}}\sqrt{\tan\theta}\,d\theta = \int_0^{\frac{\pi}{2}}\sin^{\frac{1}{2}}\theta\cos^{-\frac{1}{2}}\theta\,d\theta = \frac{1}{2}\beta\left(\frac{3}{4}\,,\frac{1}{4}\right) = \frac{1}{2}\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma(1\,)} = \frac{\sqrt{2}\pi}{2} = \frac{\pi}{\sqrt{2}}.$ $\quad\because \Gamma(n)\Gamma(n-1) = \frac{\pi}{\sin n\pi}$.

10) $\int_0^{\frac{\pi}{2}}\sqrt{\cot\theta}\,d\theta = \int_0^{\frac{\pi}{2}}\sin^{-\frac{1}{2}}\theta\cos^{\frac{1}{2}}\theta\,d\theta = \frac{1}{2}\beta\left(\frac{1}{4}\,,\frac{3}{4}\right) = \frac{1}{2}\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}{\Gamma(1\,)} = \frac{\sqrt{2}\pi}{2} = \frac{\pi}{\sqrt{2}}.$ $\quad\because \Gamma(n)\Gamma(n-1) = \frac{\pi}{\sin n\pi}$.

11. $\int_0^1 x^3(1-\sqrt{x})^5dx$

Put $\sqrt{x} = y\ \ or\ \ x = y^2\ \ and\ \ dx = 2ydy$

$$= 2\int_0^1 y^7(1-y)^5dy$$

$\int_0^1 y^m(1-y)^n dy = \beta(m+1\,,n+1)$

$$= 2\beta(8\,,6)$$
$$= 2\frac{\Gamma(8)\Gamma(6)}{\Gamma(14)} = \frac{2\times 7!\times 5!}{13!} = \frac{1}{5148}.$$

12. $\int_0^1 x^5(1-x^3)^{10}dx$ Put $x^3 = y$ or $x = y^{\frac{1}{3}}$ and $dx = \frac{1}{3}y^{-\frac{2}{3}}\, dy$

$= \frac{1}{3}\int_0^1 y\ \ (1-y)^{10}dy$ $\int_0^1 y^m(1-y)^n dy = \beta(m+1\,, n+1)$

$= \frac{1}{3}\beta(2\,,11)$

$= \frac{1}{3}\frac{\Gamma(2)\Gamma(11)}{\Gamma(13)} = \frac{1!\times 10!}{3\times 12!} = \frac{1}{396}\,.$

Self-study: Centre of Gravity: C.G. $(\bar{x}\,,\bar{y})$ of a surface S is

$$\bar{x} = \frac{\iint x\rho dxdy}{\iint \rho dxdy}\ ,\ \bar{y} = \frac{\iint y\rho dxdy}{\iint \rho dxdy}\ \text{ integrals over S.}$$

Using polar coordinates $\bar{x} = \frac{\iint r^2\rho\cos\theta drd\theta}{\iint r\rho drd\theta}\ ,\ \bar{y} = \frac{\iint r^2\rho\sin\theta drd\theta}{\iint \rho rdrd\theta}$.

1. Find by double integration, the centre of gravity of the area of the cardioid $r = a(1+\cos\theta)$, if density is constant.

   Since cardioid is symmetric about initial line, C.G. $(\bar{x}\,,\bar{y})$ lies on the initial line. Therefore $\bar{y} = 0$.

$$\iint r^2\cos\theta\, drd\theta = \int_{-\pi}^{\pi}\int_0^{a(1+\cos\theta)} r^2\cos\theta\, drd\theta$$
$$= \frac{a^3}{3}\int_{-\pi}^{\pi}\cos\theta\,(1+\cos\theta)^3\, d\theta$$
$$= \frac{4a^3}{3}\int_0^{\frac{\pi}{2}}[3\cos^2\theta + \cos^4\theta]\, d\theta = \frac{4a^3}{3}\left[\frac{3\pi}{4} + \frac{3\pi}{16}\right] = \frac{5a^3}{4}\pi$$
$$\iint rdrd\theta = \int_{-\pi}^{\pi}\int_0^{a(1+\cos\theta)} rdrd\theta = \frac{a^2}{2}\int_{-\pi}^{\pi}(1+\cos\theta)^2\, d\theta$$
$$= 2a^2\int_0^{\frac{\pi}{2}}(1+\cos^2\theta)\, d\theta$$
$$= 2a^2\left[\frac{\pi}{2} + \frac{\pi}{4}\right] = \frac{3a^2}{2}\pi.$$
$$\therefore \bar{x} = \frac{\iint r^2\cos\theta drd\theta}{\iint rdrd\theta} = \frac{5a}{6}\ . \qquad \because \int_{-\pi}^{\pi}\cos^n\theta\, d\theta = \begin{cases} 0 & if\ n\ is\ odd \\ 4\int_0^{\frac{\pi}{2}}\cos^n\theta\, d\theta & if\ n\ is\ even \end{cases}.$$

2. Find by double integration, the centre of gravity of the area of first quadrant of the curve $\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1$, density being $\rho = kxy$, where k is constant.

$$\bar{x} = \frac{\iint x\rho dxdy}{\iint \rho dxdy} = \frac{\int_0^a\int_0^{b\left[1-\left(\frac{x}{a}\right)^{\frac{2}{3}}\right]^{\frac{3}{2}}} x^2ydydx}{\int_0^a\int_0^{b\left[1-\left(\frac{x}{a}\right)^{\frac{2}{3}}\right]^{\frac{3}{2}}} xydydx}$$

$$= \frac{\int_0^a x^2b^2\left[1-\left(\frac{x}{a}\right)^{\frac{2}{3}}\right]^3 dx}{\int_0^a xb^2\left[1-\left(\frac{x}{a}\right)^{\frac{2}{3}}\right]^3 dx}$$

Put $x = a\cos^3 t$ $\quad dx = -3a\cos^2 t\ \sin t\, dt$

$$= \frac{-3a^3b^2 \int_{\frac{\pi}{2}}^{0} \sin^7 t \cos^8 t \, dt}{-3a^2b^2 \int_{\frac{\pi}{2}}^{0} \sin^7 t \cos^5 t \, dt} \qquad t \text{ varies from } \frac{\pi}{2} \text{ to } 0.$$

$$= a \frac{\int_0^{\frac{\pi}{2}} \sin^7 t \cos^8 t \, dt}{\int_{\frac{\pi}{2}}^{0} \sin^7 t \cos^5 t \, dt} = a \frac{\frac{6\times4\times2}{15\times13\times11\times9}}{\frac{4\times2}{12\times10\times8}} = \frac{128}{429} a$$

$$\bar{y} = \frac{\iint y\rho dxdy}{\iint \rho dxdy} = \frac{\int_0^a \int_0^{b\left[1-\left(\frac{x}{a}\right)^{\frac{2}{3}}\right]^{\frac{3}{2}}} xy^2 dydx}{\int_0^a \int_0^{b\left[1-\left(\frac{x}{a}\right)^{\frac{2}{3}}\right]^{\frac{3}{2}}} xydydx} = \frac{2}{3} \frac{\int_0^a xb^3 \left[1-\left(\frac{x}{a}\right)^{\frac{2}{3}}\right]^{\frac{9}{2}} dx}{\int_0^a xb^2 \left[1-\left(\frac{x}{a}\right)^{\frac{2}{3}}\right]^{3} dx} \qquad \text{Put } x = a\cos^3 t, \; dx = -3a\cos^2 t \sin t \, dt$$

$$= \frac{2}{3} \times \frac{-3a^2b^3 \int_{\frac{\pi}{2}}^{0} \sin^{10} t \cos^5 t \, dt}{-3a^2b^2 \int_{\frac{\pi}{2}}^{0} \sin^7 t \cos^5 t \, dt} \qquad t \text{ varies from } \frac{\pi}{2} \text{ to } 0.$$

$$= \frac{2}{3} b \times \frac{\int_0^{\frac{\pi}{2}} \sin^{10} t \cos^5 t \, dt}{\int_{\frac{\pi}{2}}^{0} \sin^7 t \cos^5 t \, dt} = \frac{2}{3} b \frac{\frac{4\times2}{15\times13\times11}}{\frac{4\times2}{12\times10\times8}} = \frac{128}{429} b \quad .$$

Hence the required C.G. is $\left(\frac{128}{429}a, \; \frac{128}{429}b\right)$.

Exercise:

1. Find the centroid of the area enclosed by the parabola $y^2 = 4ax$, x-axis and its latus-rectum.
2. In a semi-circular disc bounded by a diameter OA, the density at any point varies as the distance from O, find the position of the centre of gravity.
3. The density at any point (x, y) of a lamina is $\frac{\sigma}{a}(x + y)$ where σ and a are constants. The lamina is bounded by the lines $x = 0$, $y = 0$, $x = a$, $y = b$. Find the position of its centre of gravity.