Module-4: Numerical methods -1
Solution of polynomial and transcendental equations: Regula-Falsi and Newton-Raphson methods (only formulae). Problems. Finite differences, Interpolation using Newton's forward and backward difference formulae, Newton's divided difference formula and Lagrange's interpolation formula (All formulae without proof). Problems. N**umerical integration**: Simpson's $(1/3)^{rd}$ and $(3/8)^{th}$ rules(without proof). Problems.

Solution of algebraic and transcendental equations:

1. Method of false position (Regula-falsi method):

To find the root of the equation $f(x) = 0$ by Regula-falsi method,

Step1: Find *a* and *b* such $that$ $f(a).f(b) < 0$, then root lies between *a* and *b*.

Step2: $x_1 = \frac{af(b)-b(fa)}{f(b)-f(a)}$.

Step3: If $f(a).f(x_1) < 0$, then root lies between *a* and x_1.

Set $a = a$ and $b = x_1$ go to step 2.

If $f(b).f(x_1) < 0$, then root lies between x_1 and b .

Set $a = x_1$ and $b = b$ go to step 2.

This procedure is repeated till the root is found to desired accuracy.

Examples: 1. Find a real root of the equation $x \log_{10} x = 1.2$ by regula-falsi method correct to four decimal places.

Solution : Let $f(x) = x \log_{10} x - 1.2$.

Since $f(1) = -1.2$, $f(2) = -0.5979$, $f(3) = 0.2314$, root lies between 2 and 3.

a $f(a)$	b $f(b)$	$x = \dfrac{af(b) - bf(a)}{f(b) - f(a)}$	$f(x)$
$a = 2$ $f(a) = -0.5979$	$b = 3$ $f(b) = 0.2314$	$x_1 = \dfrac{2 \times 0.2314 + 3 \times 0.5979}{0.2314 + 0.5979}$ $= 2.7210$	$f(x_1) = -0.0171$
Since $f(b).f(x_1) < 0$, root lies between 2.7210 and 3.			
$a = 2.7210$ $f(a) = -0.0171$	$b = 3$ $f(b) = 0.2314$	$x_2 = \dfrac{2.7210 \times 0.2314 + 3 \times 0.0171}{0.2314 + 0.0171}$ $= 2.7402$	$f(x_2) = -0.0004$
Since $f(b).f(x_2) < 0$, root lies between 2.7402 and 3.			
$a = 2.7402$ $f(a) = -0.0004$	$b = 3$ $f(b) = 0.2314$	$x_3 = \dfrac{2.7402 \times 0.2314 + 3 \times 0.0004}{0.2314 + 0.0004}$ $= 2.7406$	$f(x_3) = 0.0000$

Hence the root is 2.7406 correct to four decimal places.

2) Solve $xe^x - 2 = 0$ using Regula-false method.

Solution: Let $f(x) = xe^x - 2$.

Since $f(0.8) = -0.2196$, $f(0.9) = 0.2136$, root lies between 0.8 and 0.9 .

a $f(a)$	b $f(b)$	$x = \dfrac{af(b) - bf(a)}{f(b) - f(a)}$	$f(x)$
$a = 0.8$ $f(a) = -0.2196$	$b = 0.9$ $f(b) = 0.2136$	$x_1 = \dfrac{0.8 \times 0.2136 + 0.9 \times 0.2196}{0.2136 + 0.2196}$ $= 0.8507$	$f(x_1) = -0.0083$
Since $f(b).f(x_1) < 0$, root lies between 0.8507 and 0.9.			
$a = 0.8507$ $f(a) = -0.0083$	$b = 0.9$ $f(b) = 0.2136$	$x_2 = \dfrac{0.8507 \times 0.2136 + 0.9 \times 0.0083}{0.2136 + 0.0083}$ $= 0.8525$	$f(x_2) = -0.0005$
Since $f(b).f(x_2) < 0$, root lies between 0.8525 and 0.9.			
$a = 0.8525$ $f(a) = -0.0005$	$b = 0.9$ $f(b) = 0.2136$	$x_3 = \dfrac{0.8525 \times 0.2136 + 0.9 \times 0.0005}{0.2136 + 0.0005}$ $= 0.8526$	$f(x_3) = 0.0000$

Hence the root is 0.8526 correct to four decimal places.

Exercise: Using Regula-falsi method compute the real root of the following equations correct to four decimal places. 1. $x^3 - 2x - 5 = 0$ 2. $xe^x - \sin x = 0$ 3. $2x - \log x = 6$.

2. Newton-Raphson method: To find the root of the equation $f(x) = 0$ by N-R method,
Find $f'(x)$ and x_0 such that $f(x_0)$ is nearer to 0.

A closer approximation of the root is $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

Similarly, starting with x_1, better approximation of the root is $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

In general, $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ (this is Newton-Raphson formula or Newton's iteration formula)

Examples: 1. Find by Newton's method, the real root of the equation $3x = \cos x + 1$.

Solution: Let $f(x) = 3x - \cos x - 1$ then $f'(x) = 3 + \sin x$.

Since $f(0) = -2$, $f(1) = 1.4597$, $f(0.5) = -0.3776$.

Let us take $x_0 = 0.5$.

Newton-Raphson formula is $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

$$= x_n - \frac{3x_n - \cos x_n - 1}{3 + \sin x_n} .$$

$$x_1 = 0.5 - \frac{3\times0.5 - \cos 0.5 - 1}{3 + \sin 0.5} = 0.6085 .$$

$$x_2 = 0.6085 - \frac{3\times0.6085 - \cos 0.6085 - 1}{3 + \sin 0.6085} = 0.6071.$$

$$x_3 = 0.6071 - \frac{3\times0.6071 - \cos 0.6071 - 1}{3 + \sin 0.6071} = 0.6071$$

Hence the root is 0.6071 correct to four decimal places.

2. Use Newton-Raphson method to find a root of the equation $\tan x - x = 0$ near $x = 4.5$.
Carry out two iterations. (05 Marks)

Solution: Let $f(x) = \tan x - x$ then $f'(x) = \sec^2 x - 1 = \tan^2 x$.

$x_0 = 4.5$.

Newton-Raphson formula is $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

$$= x_n - \frac{\tan x_n - x_n}{\tan^2 x_n} .$$

$$x_1 = x_0 - \frac{\tan x_0 - x_0}{\tan^2 x_0} = 4.4936 .$$

$$x_2 = x_1 - \frac{\tan x_1 - x_1}{\tan^2 x_1} = 4.4934.$$

Therefore root is 4.4934 .

Exercise:

Using Newton-Raphson method compute the real root of the following equations correct to four decimal places.

1. $\sin x + \cos x = 0$, near $x = \pi$. 2. $\tan x + \tanh x = 1$ 3. $e^x = x^3 + \cos 25x$ near $x = 4.5$

Forward and backward differences:

For the given set of values x_0 , x_1 , x_2 , $\cdots\cdots x_n$ coresponding y values are y_0 , y_1 , y_2 , $\cdots\cdots y_n$

First forward differences are $\Delta y_0 = y_1 - y_0$, $\Delta y_1 = y_2 - y_1$, $\Delta y_2 = y_3 - y_2$ and so on .

Second forward differences are $\Delta^2 y_0 = \Delta y_1 - \Delta y_0$, $\Delta^2 y_1 = \Delta y_2 - \Delta y_1$, $\Delta^2 y_2 = \Delta y_3 - \Delta y_2$ and so on .

r^{th} forward differences are $\Delta^r y_0 = \Delta^{r-1} y_1 - \Delta^{r-1} y_0$, $\Delta^{r-1} y_1 = \Delta^{r-1} y_2 - \Delta^{r-1} y_1$, and so on .

First backward differences are $\nabla y_1 = y_1 - y_0$, $\nabla y_2 = y_2 - y_1$, $\nabla y_3 = y_3 - y_2$ and so on .

Second backward differences are $\nabla^2 y_1 = \nabla y_1 - \nabla y_0$, $\nabla^2 y_2 = \nabla y_2 - \nabla y_1$, $\nabla^2 y_3 = \nabla y_3 - \nabla y_2$ and so on .

r^{th} backward differences are $\nabla^r y_1 = \nabla^{r-1} y_1 - \nabla^{r-1} y_0$, $\nabla^{r-1} y_2 = \nabla^{r-1} y_2 - \nabla^{r-1} y_1$, and so on .

Newton's forward interpolation formula: To find $y(x)$ near x_0 from the given set of values

$x_0\ ,\ x_1\ ,\ x_2\ ,\ \cdots\cdots x_n$ coresponding y values are $y_0\ ,\ y_1\ ,\ y_2\ ,\ \cdots\cdots y_n$

With $x_1 - x_0 = x_2 - x_1 = x_3 - x_2 = \cdots = h$

Then, $y(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots\cdots$ where $p = \frac{x-x_0}{h}$.

Newton's backward interpolation formula : To find $y(x)$ near x_n from the given set of values

$x_0\ ,\ x_1\ ,\ x_2\ ,\ \cdots\cdots x_n$ coresponding y values are $y_0\ ,\ y_1\ ,\ y_2\ ,\ \cdots\cdots y_n$

with $x_1 - x_0 = x_2 - x_1 = x_3 - x_2 = \cdots = h.$

Then, $y(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \cdots\cdots$ where $p = \frac{x-x_n}{h}$.

Example:Using suitable interpolation formulae, find $y(38)$ and $y(85)$ for the following data:

x	40	50	60	70	80	90
y	184	204	226	250	276	304

Ans: The difference table is

x	y	$\Delta\ or\ \nabla$	$\Delta^2\ or\ \nabla^2$
40	184		
		20	
50	204		2
		22	
60	226		2
		24	
70	250		2
		26	
80	276		2
		28	
90	304		

(i) Since $x_0 = 40,\ h = 10\quad p = \frac{x-x_0}{h} = \frac{38-40}{10} = -0.2.$

$y(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0$

$\therefore\ y(38) = 184 - 0.2 \times 20 - \frac{0.2\times -1.2\times 2}{2} = 180.24$.

(ii) Since $x_n = 90,\ h = 10\quad p = \frac{x-x_n}{h} = \frac{85-90}{10} = -0.5.$

$y(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n$

$\therefore\ y(85) = 304 - 0.5 \times 28 - \frac{0.5\times 0.5\times 2}{2} = 289.75.$

Lagrange's interpolation formula: If $y = f(x)$ or $x = g(y)$ then

$$y = f(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)y_0}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)} + \frac{(x-x_0)(x-x_2)\cdots(x-x_n)y_1}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)} + \cdots\cdots \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})y_n}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})}$$

Example: If $y(0) = -12,\ y(1) = 0,\ y(3) = 6,\ and\ y(4) = 12$, Find the Lagrange's interpolation polynomial and estimate $y(2)$.

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)y_0}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} + \frac{(x-x_0)(x-x_2)(x-x_3)y_1}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} + \frac{(x-x_0)(x-x_1)(x-x_3)y_2}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} + \frac{(x-x_0)(x-x_1)(x-x_2)y_3}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}$$

$$= \frac{(x-1)(x-3)(x-4)(-12)}{(-1)(-3)(-4)} + 0 + \frac{x(x-1)(x-4)6}{(3)(2)(-1)} + \frac{x(x-1)(x-3)12}{(4)(3)(1)} = x^3 - 9x^2 + 18x - 12.$$

$\therefore\ y(2) = -4$.

Newton's divided difference formula:

$$y = f(x) = y_0 + (x - x_0) \times 1^{st} D.D. + (x - x_0)(x - x_1) \times 2^{nd} D.D.$$
$$+ (x - x_0)(x - x_1)(x - x_2) \times 3^{nd} D.D. + \cdots\cdots\cdots .$$

Where $1^{st} D.D. = [x_0,\ x_1] = \frac{y_1 - y_0}{x_1 - x_0}$

$$2^{st} D.D. = [x_0,\ x_1,\ x_2] = \frac{[x_1, x_2] - [x_0, x_1]}{x_2 - x_0} \quad , \quad 3^{rd} D.D. = [x_0, x_1, x_2, x_3] = \frac{[x_1, x_2, x_3] - [x_0, x_1,\ x_2]}{x_3 - x_0} . \cdots\cdots$$

Example

1) Find the interpolating polynomial $f(x)$ by using Newton's divided difference interpolation formula from the data

x	0	1	2	3	4	5
$f(x)$	3	2	7	24	59	118

Sol: The divided difference table is

x	y	$1st\ D.D$	$2nd\ D.D$	$3rd\ D.D$
0	3			
		-1		
1	2		3	
		5		1
2	7		6	
		17		1
3	24		9	
		35		1
4	59		12	
		59		
5	118			

$$f(x) = y_0 + (x - x_0) \times 1^{st} D.D. + (x - x_0)(x - x_1) \times 2^{nd} D.D.\ + (x - x_0)(x - x_1)(x - x_2) \times 3^{nd} D.D.$$
$$= 3 + x \times -1 + x(x-1) \times 3\ + x(x-1)(x-2)$$
$$= x^3 - 2x + 3 \qquad .$$

Problems:

1. From the following table, estimate the number of students who have obtained the marks between 40 and 45.

Marks	30-40	40-50	50-60	60-70	70-80
Number of students	31	42	51	35	31

Sol: Let y be the number of students with marks less than or equal to x marks.

Then the number of students who have obtained the marks between 40 and 45 $= y(45) - y(40)$

x	40	50	60	70	80
y	31	73	124	159	190

The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
40	31				
		42			
50	73		9		
		51		-25	
60	124		-16		37
		35		12	
70	159		-4		
		31			
80	190				

Since $x_0 = 40,\ \ h = 10\ \ \ p = \frac{x-x_0}{h} = \frac{45-40}{10} = 0.5.$

$$y(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0$$

$$\therefore\ y(45) = 31 + 0.5 \times 32 + \frac{0.5(-0.5)(9)}{2} + \frac{0.5(-0.5)(-1.5)(-25)}{6} + \frac{0.5(-0.5)(-1.5)(-2.5)}{24}(37)$$
$$= 47.8672 \approx 48 \text{ students.}$$

Therefore the number of students with marks between 40 and 45 $= y(45) - y(40) = 48 - 31 = 17$.

2. The following data is on melting point of an alloy of lead and zine where t is temperature in celsius and P is percentage of lead alloy. Find the melting point of the alloy containing 86% of lead.

P	40	50	60	70	80	90
t	180	204	226	250	276	304

Sol: The difference table is

$P = x$	$t = y$	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
40	180					
		24				
50	204		-2			
		22		4		
60	226		2		-4	
		24		0		4
70	250		2		0	
		26		0		
80	276		2			
		28				
90	304					

Since $x_n = 90,\ \ h = 10\ \ \ p = \frac{x-x_n}{h} = \frac{86-90}{10} = -0.4.$

$$y(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \cdots\cdots + \frac{p(p+1)(p+2)(p+3)(p+4)}{5!}\nabla^5 y_n + \cdots\cdots$$

$$\therefore\ y(86) = 304 - 0.4 \times 28 - \frac{0.4(0.6)(2)}{2} + 0 + 0 - \frac{0.4(0.6)(1.6)(2.6)(3.6)(4)}{120}$$

$= 292.4402$ celsius.

Therefore melting point of the alloy containing 86% of lead $\cong 292$ celsius.

3. Given

x	1	2	3	4	5	6	7	8
$f(x)$	1	8	27	64	125	216	343	512

Estimate $f(7.5)$ using Newton-Gregory Backward difference interpolation formula.

Sol: The difference table is

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
1	1			
		7		
2	8		12	
		19		6
3	27		18	
		37		6
4	64		24	
		61		6
5	125		30	
		91		6
6	216		36	
		127		6
7	343		42	
		169		
8	512			

Since $x_n = 8,\ h = 1 \quad p = \frac{x - x_n}{h} = \frac{7.5-8}{1} = -0.5.$

$$\therefore\ f(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \cdots\cdots$$

$$\therefore\ f(7.5) = 512 - 0.5 \times 169 - \frac{0.5(0.5)(42)}{2} - \frac{0.5(0.5)(1.5)(6)}{6} = 421.8750.$$

4. Using the Lagrange's formula find an interpolation polynomial and estimate $y(3)$.

x	0	1	2	5
y	2	3	12	147

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)y_0}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} + \frac{(x-x_0)(x-x_2)(x-x_3)y_1}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} + \frac{(x-x_0)(x-x_1)(x-x_3)y_2}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} + \frac{(x-x_0)(x-x_1)(x-x_2)y_3}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}$$

$$= \frac{(x-1)(x-2)(x-5)(2)}{(-1)(-2)(-5)} + \frac{(x)(x-2)(x-5)(3)}{(1)(-1)(-4)} + \frac{x(x-1)(x-5)12}{(2)(1)(-3)} + \frac{x(x-1)(x-2)147}{(5)(4)(3)}$$

$$= -\frac{1}{5}(x^3 - 8x^2 + 17x - 10) + \frac{3}{4}(x^3 - 7x^2 + 10x) - 2(x^3 - 6x^2 + 5x) + \frac{49}{20}(x^3 - 3x^2 + 2x)$$
$$= x^3 + x^2 - x + 2\,.$$

$\therefore\ y(3) = 35\,.$

5. The following table gives the premium payable at ages in years completed. Interpolate the premium payable at age 35 completed using Lagrange's formula. (5 Marks)

$x =$ Age completed	25	30	40	60
$y =$ Premium in Rs	50	55	70	95

Solution:

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)y_0}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} + \frac{(x-x_0)(x-x_2)(x-x_3)y_1}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} + \frac{(x-x_0)(x-x_1)(x-x_3)y_2}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} + \frac{(x-x_0)(x-x_1)(x-x_2)y_3}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}$$

$$= \frac{5\times -5\times -25\times 50}{-5\times -15\times -35} + \frac{10\times -5\times -25\times 55}{5\times -10\times -30} + \frac{10\times 5\times -25\times 70}{15\times 10\times -20} + \frac{10\times 5\times -5\times 95}{35\times 30\times 20} = 61.9643 \approx 62.$$

5. Fit an interpolating polynomial for $f(x)$ using divided difference formula and hence evaluate $f(5)$, given $f(0) = -5,\ f(1) = -14,\ f(4) = -125,\ f(8) = -21,\ f(10) = 355\,.$

Sol: The divided difference table is

x	y	$1st\ D.D$	$2nd\ D.D$	$3rd\ D.D$
0	-5			
		-9		
1	-14		-7	
		-37		2
4	-125		9	
		26		2
8	-21		27	
		188		
10	355			

$$f(x) = y_0 + (x - x_0) \times 1^{st} D.D. + (x - x_0)(x - x_1) \times 2^{nd} D.D.\ + (x - x_0)(x - x_1)(x - x_2) \times 3^{nd} D.D.$$
$$= -5 + x \times -9 + x(x-1) \times -7\ + x(x-1)(x-4) \times 2$$
$$= 2x^3 - 17x^2 + 6x - 5 \quad .$$

$\therefore f(5) = -150\,.$

6. Construct an interpolating polynomial for the data given below using Newton's divided difference formula.

x	2	4	5	6	8	10
$f(x)$	10	96	196	350	868	1746

Sol: The divided difference table is

x	y	$1st\ D.D$	$2nd\ D.D$	$3rd\ D.D$
2	10			
		43		
4	96		19	
		100		2
5	196		27	
		154		2
6	350		35	
		259		2
8	868		45	
		439		
10	1746			

$$f(x) = y_0 + (x - x_0) \times 1^{st} D.D. + (x - x_0)(x - x_1) \times 2^{nd} D.D. \ + (x - x_0)(x - x_1)(x - x_2) \times 3^{nd} D.D.$$

$$= 10 + (x - 2) \times 43 + (x - 2)(x - 4) \times 19 \ + (x - 2)(x - 4)(x - 5) \times 2$$

$$= 2x^3 - 3x^2 + 5x - 4 \quad .$$

7. Given the values

x	5	7	11	13	17
$f(x)$	150	392	1452	2366	5202

Evaluate $f(9)$, using Newton's divided difference formula.

Sol: The divided difference table is

x	y	$1st\ D.D$	$2nd\ D.D$	$3rd\ D.D$
5	150			
		121		
7	392		24	
		265		1
11	1452		32	
		457		1
13	2366		42	
		709		
17	5202			

$$f(x) = y_0 + (x - x_0) \times 1^{st} D.D. + (x - x_0)(x - x_1) \times 2^{nd} D.D. \ + (x - x_0)(x - x_1)(x - x_2) \times 3^{nd} D.D.$$

$$\therefore f(9) = 150 + 4 \times 121 + 4 \times 2 \times 24 \ + 4 \times 2 \times -2 \times 1$$

$$= 810 \quad .$$

Numerical-integration:

Simpson's one-third rule: $\int_a^b f(x)dx = (\frac{h}{3})[1 \quad 4 \quad 1]$

Simpson's three-eighth rule: $\int_a^b f(x)\,dx = (\frac{3h}{8})\,[1 \quad 3 \quad 3 \quad 1]$

Where $h = \frac{b-a}{n}$

$\mathbf{n}$ *is number of intervals*

or $n = number\ of\ ordinates - 1.$

1. Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using i. Simpson's one-third rule by taking 10 sub intervals.

   ii. Simpson's $\frac{3}{8}th$ rule by taking 10 ordinates.

i. Simpson's one-third rule by taking 10 sub intervals.

$n = 10, \quad h = \frac{b-a}{n} = \frac{1}{10} = 0.1 \quad \text{and } y = \frac{1}{1+x^2}.$

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
y	1	0.9901	0.9615	0.9174	0.8621	0.8	0.7353	0.6711	0.6098	0.5525	0.5
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}
	1	4	1	4	1	4	1	4	1	4	1
			1		1		1		1		

$$\int_0^1 \frac{dx}{1+x^2} = \frac{h}{3}\left[y_0 + y_{10} + 2\left(y_2 + y_4 + y_6 + y_8\right) + 4\left(y_1 + y_3 + y_5 + y_7 + y_9\right)\right] = 0.7854\,.$$

ii. Simpson's $\frac{3}{8}th$ rule by taking 10 ordinates.

$n = 9, \quad h = \frac{b-a}{n} = \frac{1}{9} \quad \text{and } y = \frac{1}{1+x^2}.$

x	0	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{3}{9}$	$\frac{4}{9}$	$\frac{5}{9}$	$\frac{6}{9}$	$\frac{7}{9}$	$\frac{8}{9}$	1
y	1	0.9878	0.9529	0.9	0.8351	0.7642	0.6923	0.6231	0.5586	0.5
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9
	1	3	3	1	3	3	1	3	3	1
				1			1			

$$\int_0^1 \frac{dx}{1+x^2} = \frac{3h}{8}\left[y_0 + y_9 + 2\left(y_3 + y_6\right) + 3\left(y_1 + y_2 + y_4 + y_5 + y_7 + y_8\right)\right] = 0.7854\,.$$

2. Find an approximate value of $\log_e 5$, by Simpson's 1/3 rule, from $\int_0^5 \frac{dx}{4x+5}$, dividing the range into 10 equal parts.

   Since $n = 10, \quad h = \frac{b-a}{n} = \frac{5}{10} = 0.5 \quad \text{and } y = \frac{1}{4x+5}.$

x	0	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5
y	$\frac{1}{5}$	$\frac{1}{7}$	$\frac{1}{9}$	$\frac{1}{11}$	$\frac{1}{13}$	$\frac{1}{15}$	$\frac{1}{17}$	$\frac{1}{19}$	$\frac{1}{21}$	$\frac{1}{23}$	$\frac{1}{25}$
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}
	1	4	1	4	1	4	1	4	1	4	1
			1		1		1		1		

$\int_0^5 \frac{\boldsymbol{dx}}{\boldsymbol{4x+5}} = \frac{h}{3}\left[y_0 + y_{10} + 2\left(y_2 + y_4 + y_6 + y_8\right) + 4\left(y_1 + y_3 + y_5 + y_7 + y_9\right)\right] = 0.4025$.

But $\int_0^5 \frac{\boldsymbol{dx}}{\boldsymbol{4x+5}} = \frac{1}{4}\log(4x+5)\Big|_0^5 = \frac{1}{4}[\log 25 - \log 5] = \frac{\log 5}{4} \cong 0.4025$.

And hence $\log 5 \cong 4 \times 0.4025 = 1.61$.

3. Evaluate $\int_0^{\frac{\pi}{2}} \sqrt{\cos\theta}\, d\theta$ by Simpson's $\frac{3}{8}th$ rule by taking 10 ordinates.

$n = 9, \quad h = \frac{b-a}{n} = \frac{\pi}{18}$ and $y = \sqrt{\cos\theta}$.

θ	0	$\frac{\pi}{18}$	$\frac{2\pi}{18}$	$\frac{3\pi}{18}$	$\frac{4\pi}{18}$	$\frac{5\pi}{18}$	$\frac{6\pi}{18}$	$\frac{7\pi}{18}$	$\frac{8\pi}{18}$	$\frac{\pi}{2}$
y	1	0.9924	0.9694	0.9306	0.8752	0.8017	0.7071	0.5848	0.4167	0

y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9
1	3	3	1	3	3	1	3	3	1
			1			1			

$\int_0^{\frac{\pi}{2}} \sqrt{\cos\theta}\, d\theta = \frac{3h}{8}\left[y_0 + y_9 + 2\left(y_3 + y_6\right) + 3\left(y_1 + y_2 + y_4 + y_5 + y_7 + y_8\right)\right]$

$= \frac{\pi}{48}[1 + 0 + 2(0.9306 + 0.7071) + 3(0.9924 + 0.9694 + 0.8752 + 0.8017 + 0.5848 + 0.4167)]$

$= 1.1909$

Self-study:

Bisection Method: To find the root of the equation $f(x) = 0$ by Bisection method

Step1: Find a and b such that $f(a).f(b) < 0$, then root lies between a and .

Step2: Find $x_1 = \frac{a+b}{2}$

Step3: If $f(a).f(x_1) < 0$, then root lies between a and x_1 .

Set $a = a$ and $b = x$ go to step 2.

If $f(b).f(x) < 0$, then root lies between x_1 *and* b .

Set $a = x$ and $b = b$ go to step 2.

This procedure is repeated till the root is found to desired accuracy.

Example: Solve $xe^x - 2 = 0$ using Bisection method.

Let $(x) = xe^x - 2$. Since $f(0.8) = -0.2196$, $f(0.9) = 0.2136$, root lies between 0.8 and 0.9 .

$x = \frac{a+b}{2} = 0.85$. $f(x) = -0.0113$.

Since $f(b).f(x) < 0$, root lies between 0.85 and 0.9 .

$a = 0.85, \; b = 0.9$

$x = \frac{a+b}{2} = 0.875$, $f(x) = 0.0990$.

Since $f(a).f(x) < 0$, root lies between 0.85 and 0.875 .

$a = 0.85, \ \ b = 0.875$

$x = \frac{a+b}{2} = 0.8625$, $f(x) = 0.0433$.

Since $f(a).f(x) < 0$, root lies between 0.85 and 0.8625 .

$a = 0.85, \ \ b = 0.8625$

$x = \frac{a+b}{2} = 0.8563$, $f(x) = 0.0159$.

Since $f(a).f(x) < 0$, root lies between 0.85 and 0.8563 .

$a = 0.85, \ \ b = 0.8563$

$x = \frac{a+b}{2} = 0.8532$, $f(x) = 0.0024$.

Since $f(a).f(x) < 0$, root lies between 0.85 and 0.8532 .

$a = 0.85, \ \ b = 0.8532$

$x = \frac{a+b}{2} = 0.8516$, $f(x) = -0.0044$.

Lagrange's interpolation inverse formula: If $y = f(x)$ or $x = g(y)$ then

$$x = g(y) = \frac{(y-y_1)(y-y_2)\cdots(y-y_n)x_0}{(y_0-y_1)(y_0-y_2)\cdots(y_0-y_n)} + \frac{(y-y_0)(y-y_2)\cdots(y-y_n)x_1}{(y_1-y_0)(y_1-y_2)\cdots(y_1-y_n)} + \cdots\cdots \frac{(y-y_0)(y-y_1)\cdots(y-y_{n-1})x_n}{(y_n-y_0)(y_n-y_1)\cdots(y_n-y_{n-1})} .$$

Example

1. Find the root of the equation $f(x) = 0$, using Lagrange interpolation formula given,

x	5	8	18	21
$f(x)$	-3	2	4	7

Solution: Let $y = f(x)$,

Then, $x = \frac{(y-y_1)(y-y_2)(y-y_3)x_0}{(y_0-y_1)(y_0-y_2)(y_0-y_3)} + \frac{(y-y_0)(y-y_2)(y-y_3)x_1}{(y_1-y_0)(y_1-y_2)(y_1-y_3)} + \frac{(y-y_0)(y-y_1)(y-y_3)x_3}{(y_2-y_0)(y_2-y_1)(y_2-y_3)} + \frac{(y-y_0)(y-y_1)(y-y_2)x_3}{(y_3-y_0)(y_3-y_1)(y_3-y_2)}$.

$$x(0) = \frac{(-2)(-4)(-7)5}{(-5)(-7)(-10)} + \frac{(3)(-4)(-7)8}{(5)(-2)(-5)} + \frac{(3)(-2)(-7)18}{(7)(2)(-3)} + \frac{(3)(-2)(-4)21}{(10)(5)(3)}$$

$= -0.4$.

Numerical-integration:

Weddle's rule: $\int_a^b f(x)\,dx = (\frac{3h}{10})[1 \ \ 5 \ \ 1 \ \ \ 6 \ \ 1 \ \ \ 5 \ \ \ 1]$

Where $h = \frac{b-a}{n}$, $\mathbf{n}$ *is number of intervals* **,** or $n = number\ of\ ordinates - 1$.

1. Evaluate $\int_0^{\frac{\pi}{2}} e^{\sin x} dx$ by Weddle's rule.

Sol: $n = 6$, $h = \frac{b-a}{n} = \frac{\pi}{12}$ and $y = e^{\sin x}$.

x	0	$\frac{\pi}{12}$	$\frac{2\pi}{12}$	$\frac{3\pi}{12}$	$\frac{4\pi}{12}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$
y	1	1.2954	1.6487	2.0281	2.3774	2.6272	2.7183
	y_0	y_1	y_2	y_3	y_4	y_5	y_6
	1	5	1	6	1	5	1

$\int_0^{\frac{\pi}{2}} e^{\sin x} dx = \frac{3h}{10}\left[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6\right] = 3.1044$.

2. Evaluate $\int_4^{5.2} \log_e x\, dx$ by Weddle's rule by taking $n = 6$.

$n = 6, \quad h = \frac{b-a}{n} = 0.2 \quad \text{and } y = \log_e x$.

x	4	4.2	4.4	4.6	4.8	5	5.2
y	$\log_e 4$	$\log_e 4.2$	$\log_e 4.4$	$\log_e 4.6$	$\log_e 4.8$	$\log_e 5$	$\log_e 5.2$
	y_0	y_1	y_2	y_3	y_4	y_5	y_6
	1	5	1	6	1	5	1

$\int_4^{5.2} \log_e x\, dx = \frac{3h}{10}\left[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6\right]$.

$= \left(\frac{3\times0.2}{10}\right)[\log_e 4 + 5\log_e 4.2 + \log_e 4.4 + 6\log_e 4.6 + \log_e 4.8 + 5\log_e 5 + \log_e 5.2]$

$= 1.8278.$

3. Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using Weddle's rule by taking $n = 6$.

$n = 6, \quad h = \frac{b-a}{n} = \frac{1}{6} \quad \text{and } y = \frac{1}{1+x^2}$.

x	0	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	1
y	1	0.9730	0.9	0.8	0.6923	0.5902	0.5
	y_0	y_1	y_2	y_3	y_4	y_5	y_6
	1	5	1	6	1	5	1

$\int_0^1 \frac{dx}{1+x^2} = \frac{3h}{10}\left[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6\right] = 0.7854$.