Module-2: Differential Calculus - 2
Taylor's and Maclaurin's series expansion for one variable (Statement only) – problems. Indeterminate forms-L'Hospital's rule. Partial differentiation, total derivative-differentiation of composite functions. Jacobian and problems. Maxima and minima for a function of two variables. Problems. **Self-study:** Euler's Theorem and problems. Method of Lagrange undetermined multipliers with single constraint. **(RBT Levels: L1, L2 and L3)**

Taylor's theorem: If i) $f(x)$ and its first $(n-1)$ derivatives be continuous in the interval $[a,\ a+h]$, and

ii) n^{th} derivative of $f(x)$ exists for every values of x in $(a,\ a+h)$, then there is at least one number θ in $(0,\ 1)$ such that,

$$f(a+h) = f(a) + hf\,'(a) + \frac{h^2}{2!}f\,''(a) + \cdots\cdots\cdots\cdots + \frac{h^n}{n!}f^n\,(a+\theta h)$$

If $a = 0$ then the Taylor's theorem is called Maclaurin's theorem.

Taylor's series: Expansion of $f(x)$ about $x = a$ (or in powers of $(x-a)$) is

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \frac{(x-a)^4}{4!}f'^{v}(a) + \cdots$$

Or $$y = y(a) + y_1(a)(x-a) + \frac{y_2(a)}{2!}(x-a)^2 + \frac{y_3(a)}{3!}(x-a)^3 + \frac{y_4(a)}{4!}(x-a)^4 + \cdots$$

If $a = 0$ then series is called **Maclaurin's series** i.e.

Expansion of $f(x)$ about $x = 0$ (or in powers of x) is

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f'^{v}(0) + \cdots$$

Or $$y = y(0) + y_1(0)x + \frac{y_2(0)}{2!}x^2 + \frac{y_3(0)}{3!}x^3 + \frac{y_4(0)}{4!}x^4 + \cdots$$

Examples:

- Expand $y = \sin x$ in powers of $(x - \frac{\pi}{2})$.

Clearly $a = \frac{\pi}{2}$ and $y = \sin x$, $y_1 = \cos x$, $y_2 = -\sin x$, $y_3 = -\cos x$, $y_4 = \sin x$ and so on.

$$\therefore\ y\left(\frac{\pi}{2}\right) = 1,\quad y_1\left(\frac{\pi}{2}\right) = 0,\quad y_2\left(\frac{\pi}{2}\right) = -1,\quad y_3\left(\frac{\pi}{2}\right) = 0,\quad y_4\left(\frac{\pi}{2}\right) = 1 \quad \ldots\ldots\ldots$$

Substituting in the Taylor's formula

$$y = y(a) + y_1(a)(x-a) + \frac{y_2(a)}{2!}(x-a)^2 + \frac{y_3(a)}{3!}(x-a)^3 + \frac{y_4(a)}{4!}(x-a)^4 + \cdots$$

$$\sin x = 1 - \frac{\left(x-\frac{\pi}{2}\right)^2}{2!} + \frac{\left(x-\frac{\pi}{2}\right)^4}{4!} - \cdots$$

- Find Maclaurin's series of a) e^x b) $\cos x$ c) $\sin x$ d) $\tan x$ e) $\cosh x$ f) $\sinh x$ g) $\log(1+x)$

  h) $\log \sec x$ i) $e^{\sin x}$ j) $\tan^{-1} x$ k) $\sqrt{(1+\sin 2x)}$.

a) $y = e^x = y_1 = y_2 = y_3 = \cdots$ And hence $y(0) = 1 = y_1(0) = y_2(0) = y_3(0) = y_4(0) =$ ....

Maclaurin's series is

$$y = y(0) + y_1(0)x + \frac{y_2(0)}{2!}x^2 + \frac{y_3(0)}{3!}x^3 + \frac{y_4(0)}{4!}x^4 + \cdots$$

$$\Rightarrow \quad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

b) $y = \cos x$, $y_1 = -\sin x$, $y_2 = -\cos x$, $y_3 = \sin x$, $y_4 = \cos x$ $\cdots\cdots\cdots$

$y(0) = 1,\quad y_1(0) = 0,\quad y_2(0) = -1,\quad y_3(0) = 0,\quad y_4(0) = 1,$ ....

$$\Rightarrow \quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

c) $y = \sin x \Rightarrow y_1 = \cos x$, $y_2 = -\sin x$, $y_3 = -\cos x$, $y_4 = \sin x$ $\cdots\cdots\cdots$

$y(0) = 0,\quad y_1(0) = 1,\quad y_2(0) = 0,\quad y_3(0) = -1,\quad y_4(0) = 0,$ ....

$$\Rightarrow \quad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

d) $y = \tan x \Rightarrow y_1 = \sec^2 x = 1 + y^2$, $\quad \Rightarrow \quad y(0) = 0,\quad y_1(0) = 1$

$$y_2 = 2yy_1\ , \qquad \Rightarrow \quad y_2(0) = 0$$

$$y_3 = 2yy_2 + 2y_1^2\ , \qquad \Rightarrow \quad y_3(0) = 2$$

$$y_4 = 2yy_3 + 6y_1y_2, \qquad \Rightarrow \quad y_4(0) = 0$$

$$y_5 = 2yy_4 + 8y_1y_3 + 6y_2^2, \Rightarrow \quad y_5(0) = 16. \qquad \cdots\cdots$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots$$

e) $y = \cosh x$ f) $\sinh x$

Since $\quad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

$$\Rightarrow \quad e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\cosh x = \frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots \quad \text{and} \quad \sinh x = \frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$$

g) $y = \log(1+x) \quad \Rightarrow y_n = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$

$\therefore\ y(0) = 0,\ y_1(0) = 1\ ,\quad y_2(0) = -1,\quad y_3(0) = 2,\quad y_4(0) = -6,\quad \cdots\cdots$

$$\Rightarrow \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} \cdots\cdots$$

h) $y = \log \sec x \quad \Rightarrow \quad y_1 = \tan x \qquad \Rightarrow \quad y(0) = 0,\quad y_1(0) = 0$

$$y_2 = \sec^2 x = 1 + y_1^2\ , \qquad \Rightarrow \quad y_2(0) = 1$$

$$y_3 = 2y_1y_2\ , \qquad \Rightarrow \quad y_3(0) = 0$$

$$y_4 = 2y_1y_3 + 2y_2^2\ , \qquad \Rightarrow \quad y_4(0) = 2$$

$$y_5 = 2y_1y_4 + 6y_2y_3, \qquad \Rightarrow \quad y_5(0) = 0$$

$$y_6 = 2y_1y_5 + 8y_2y_4 + 6y_3^2, \qquad \Rightarrow \quad y_6(0) = 16. \qquad \cdots\cdots$$

$$\therefore \log\sec x = \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45}\cdots$$

i) $\quad y = e^{\sin x} \;\Rightarrow y_1 = y\cos x \qquad \Rightarrow \; y(0) = 1, \quad y_1(0) = 1$

$$y_2 = y_1\cos x - y\sin x, \qquad \Rightarrow \; y_2(0) = 1$$

$$y_3 = y_2\cos x - 2y_1\sin x - y_1 \qquad \Rightarrow \; y_3(0) = 0$$

$$y_4 = y_3\cos x - 3y_2\sin x - 2y_1\cos x - y_2\,, \qquad \Rightarrow \; y_4(0) = -3$$

$$\Rightarrow e^{\sin x} = 1 + x + \frac{x^2}{2} - \frac{x^4}{8}\cdots\cdots$$

j) $\quad y = \tan^{-1}x \;\Rightarrow y_1 = \frac{1}{1+x^2}$ or $(1+x^2)y_1 = 1 \qquad \Rightarrow \; y(0) = 0, \quad y_1(0) = 1$

$$(1+x^2)y_2 + 2xy_1 = 0 \qquad \Rightarrow y_2(0) = 0$$

$$(1+x^2)y_3 + 4xy_2 + 2y_1 = 0 \qquad \Rightarrow y_3(0) = -2$$

$$(1+x^2)y_4 + 6xy_3 + 6y_2 = 0 \qquad \Rightarrow y_4(0) = 0$$

$$(1+x^2)y_5 + 8xy_4 + 12y_3 = 0 \qquad \Rightarrow y_5(0) = 24$$

$$\Rightarrow \tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5}\cdots\cdots$$

k) $\quad y = \sqrt{(1+\sin 2x)} = \sqrt{(\sin x + \cos x)^2} = \sin x + \cos x$

$y_1 = \cos x - \sin x \quad, y_2 = -\sin x - \cos x\,, \quad y_3 = -\cos x + \sin x, \qquad y_4 = \sin x + \cos x.$

$$\Rightarrow y(0) = 1,\; y_1(0) = 1, \quad y_2(0) = -1, \quad y_3(0) = -1, \quad y_4(0) = -1$$

$$\therefore \sqrt{(1+\sin 2x)} = 1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} - \cdots$$

Exercise:

Using Maclaurin's series expand the following functions

1. $\log\sqrt{\frac{1+x}{1-x}}$ 2. $\frac{x}{\sin x}$ 3. $\sec x$ 4. $\log(1+\sin x)$ 5. $\log(1+e^x)$

6. $e^x\cos x$ 7. $e^{x\sin x}$ 8. $\frac{e^x}{e^x+1}$ 9. $\sin x\cosh x$

10. Find the Maclaurin's series of a) $\sin^{-1}(3x-4x^3)$ b) $\log\left(\frac{\sin x}{x}\right)$.

Indeterminate forms:

$\left(\frac{0}{0}\right)$ form: If $f(a) = 0 = g(a)$ then $\lim_{x\to a}\frac{f(x)}{g(x)}$ is called $\frac{0}{0}$ form.

L'Hospital's rule: If $f(a) = 0 = g(a)$ then $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$.

Note: Forms 1^∞, $\infty^0, 0^0$ can be reducible to form $\left(\frac{0}{0}\right)$ or form $\frac{\infty}{\infty}$ by taking log.

Examples: Evaluate the following limits.

1. $\underset{x \to \frac{\pi}{2}}{Lt} (\sin x)^{\tan x} \quad \cdots\cdots\cdots (1^\infty \ form)$

Let $k = \underset{x \to \frac{\pi}{2}}{Lt} (\sin x)^{\tan x}$

Taking log on both sides,

$$\log k = \underset{x \to \frac{\pi}{2}}{Lt} \frac{\log \sin x}{\cot x} \quad \cdots\cdots\cdots \left(\frac{0}{0} \ form\right)$$

$$= \underset{x \to \frac{\pi}{2}}{Lt} \frac{\frac{\cos x}{\sin x}}{-\operatorname{cosec}^2 x} = 0$$

And hence $k = e^0 = 1.$

2. $\underset{x \to 0}{Lt} \left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}} \quad \cdots\cdots\cdots (1^\infty \ form)$

Let $k = \underset{x \to 0}{Lt} \left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}$

Taking log on both sides,

$$\log k = \underset{x \to 0}{Lt} \frac{\log \tan x - \log x}{x^2} \quad \cdots\cdots\cdots \left(\frac{0}{0} \ form\right)$$

$$= \underset{x \to 0}{Lt} \frac{\frac{\sec^2 x}{\tan x} - \frac{1}{x}}{2x} = \underset{x \to 0}{Lt} \frac{\frac{\sec^2 x}{\tan x} - \frac{1}{x}}{2x}$$

$$= \underset{x \to 0}{Lt} \frac{1}{2x}\left[\frac{1}{\sin x \cos x} - \frac{1}{x}\right] = \underset{x \to 0}{Lt} \frac{1}{2x}\left[\frac{2}{\sin 2x} - \frac{1}{x}\right]$$

$$= \underset{x \to 0}{Lt} \frac{2x - \sin 2x}{2x^2 \sin 2x} = \underset{x \to 0}{Lt} \frac{2x - \sin 2x}{2x^2 2x \frac{\sin 2x}{2x}} = \underset{x \to 0}{Lt} \frac{2x - \sin 2x}{4x^3} \quad \cdots\cdots\cdots \left(\frac{0}{0} \ form\right)$$

$$= \underset{x \to 0}{Lt} \frac{2 - 2\cos 2x}{12x^2} = \underset{x \to 0}{Lt} \frac{1 - \cos 2x}{6x^2}$$

$$= \underset{x \to 0}{Lt} \frac{2\sin^2 x}{6x^2} = \frac{1}{3}$$

And hence $k = e^{\frac{1}{3}}$.

Or $\underset{x \to 0}{Lt} \left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}} = \underset{x \to 0}{Lt} \left(\frac{x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots\cdots}{x}\right)^{\frac{1}{x^2}}$

$$= \underset{x \to 0}{Lt} (1 + tx^2)^{\frac{1}{x^2}}, \quad \text{where } t = \frac{1}{3} + \frac{2x^2}{15} + \cdots\cdots$$

$$= \underset{x \to 0}{Lt} e^t = e^{\frac{1}{3}} . \qquad \because \underset{z \to 0}{Lt} (1 + tz)^{\frac{1}{z}} = e^z$$

3. $\underset{x \to 0}{Lt} \left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}} \quad \cdots\cdots\cdots (1^\infty \ form)$

Let $k = \underset{x \to 0}{Lt} \left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}}$

Taking log on both sides,

$$\log k = \underset{x \to 0}{Lt} \frac{\log \sin x - \log x}{x^2} \quad \cdots\cdots\cdots \left(\frac{0}{0}\ form\right)$$

$$= \underset{x \to 0}{Lt} \frac{\frac{\cos x}{\sin x} - \frac{1}{x}}{2x}$$

$$= \underset{x \to 0}{Lt} \frac{1}{2x}\left[\frac{x\cos x - \sin x}{x \sin x}\right] = \underset{x \to 0}{Lt} \frac{x\cos x - \sin x}{2x^3} \quad \cdots\cdots\cdots \left(\frac{0}{0}\ form\right)$$

$$= \underset{x \to 0}{Lt} \frac{\cos x - x\sin x - \cos x}{6x^2} = \underset{x \to 0}{Lt} \frac{-\sin x}{6x} = -\frac{1}{6}.$$

And hence $k = e^{-\frac{1}{6}}$.

Or $$\underset{x \to 0}{Lt} \left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}} = \underset{x \to 0}{Lt} \left(\frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\cdots}{x}\right)^{\frac{1}{x^2}}$$

$$= \underset{x \to 0}{Lt} (1 + tx^2)^{\frac{1}{x^2}}, \quad \text{where } t = -\frac{1}{3!} + \frac{x^2}{5!} - \cdots\cdots$$

$$= \underset{x \to 0}{Lt} e^t = e^{-\frac{1}{6}}. \qquad \because \underset{z \to 0}{Lt} (1+tz)^{\frac{1}{z}} = e^z$$

4. $\underset{x \to a}{Lt} \left(2 - \frac{x}{a}\right)^{\tan\left(\frac{\pi x}{2a}\right)} \quad \cdots\cdots\cdots (1^\infty\ form)$

Let $k = \underset{x \to a}{Lt} \left(2 - \frac{x}{a}\right)^{\tan\left(\frac{\pi x}{2a}\right)}$

Taking log on both sides,

$$\log k = \underset{x \to a}{Lt} \frac{\log\left(2 - \frac{x}{a}\right)}{\cot\left(\frac{\pi x}{2a}\right)} \quad \cdots\cdots\cdots \left(\frac{0}{0}\ form\right)$$

$$= \underset{x \to a}{Lt} \frac{\frac{1}{2 - \frac{x}{a}} \times \left(-\frac{1}{a}\right)}{-\left(\frac{\pi}{2a}\right)\operatorname{cosec}^2\left(\frac{\pi x}{2a}\right)} = \frac{2}{\pi}.$$

And hence $k = e^{\frac{2}{\pi}}$.

5. $\underset{x \to 0}{Lt} \left(\frac{a^x + b^x + c^x}{3}\right)^{\frac{1}{x}} \quad \cdots\cdots\cdots (1^\infty\ form)$

Let $k = \underset{x \to 0}{Lt} \left(\frac{a^x + b^x + c^x}{3}\right)^{\frac{1}{x}}$

Taking log on both sides,

$$\log k = \underset{x \to 0}{Lt} \frac{\log(a^x + b^x + c^x) - \log 3}{x} \quad \cdots\cdots\cdots \left(\frac{0}{0}\ form\right)$$

$$= \underset{x \to 0}{Lt} \frac{\frac{1}{a^x+b^x+c^x}(a^x \log a + b^x \log b + c^x \log c)}{1} = \frac{1}{3}\log(abc) = \log(\sqrt[3]{abc}).$$

And hence $k = \sqrt[3]{abc}$.

6. $\underset{x \to 0}{Lt} (a^x + x)^{\frac{1}{x}}$ ……… $(1^\infty \ form)$

Let $k = \underset{x \to 0}{Lt} (a^x + x)^{\frac{1}{x}}$

Taking log on both sides,

$$\log k = \underset{x \to 0}{Lt} \frac{\log(a^x+x)}{x} \quad \dots\dots\dots \left(\frac{0}{0} \ form\right)$$

$$= \underset{x \to 0}{Lt} \frac{\frac{1}{a^x+x}(a^x \log a + 1)}{1} = \log a + 1 = \log a + \log e = \log(ea).$$

Hence $k = ea$.

7. $\underset{x \to 0}{Lt} (1 + \sin x)^{\cot x}$ ……… $(1^\infty \ form)$

Let $k = \underset{x \to 0}{Lt} (1 + \sin x)^{\cot x}$

Taking log on both sides,

$$\log k = \underset{x \to 0}{Lt} \frac{\log(1+\sin x)}{\tan x} \quad \dots\dots\dots \left(\frac{0}{0} \ form\right)$$

$$= \underset{x \to 0}{Lt} \frac{\frac{\cos x}{1+\sin x}}{\sec^2 x} = 1.$$

And hence $k = e^1 = e$.

8. $\underset{x \to \frac{\pi}{2}}{Lt} (\sec x)^{\cot x}$ ……… $(\infty^0 \ form)$

Let $k = \underset{x \to \frac{\pi}{2}}{Lt} (\sec x)^{\cot x}$

Taking log on both sides,

$$\log k = \underset{x \to \frac{\pi}{2}}{Lt} \frac{\log(\sec x)}{\tan x} \quad \dots\dots\dots \left(\frac{\infty}{\infty} \ form\right)$$

$$= \underset{x \to \frac{\pi}{2}}{Lt} \frac{\frac{\sec x \tan x}{\sec x}}{\sec^2 x} = \underset{x \to \frac{\pi}{2}}{Lt} \frac{\tan x}{\sec^2 x}.$$

$$= \underset{x \to \frac{\pi}{2}}{Lt} \sin x \cos x = 0.$$

And hence $k = e^0 = 1$.

9. $\underset{x \to 0}{Lt} (\cot x)^{\frac{1}{\log x}}$ ……… $(\infty^0 \ form)$

Let $k = \underset{x \to 0}{Lt} (\cot x)^{\frac{1}{\log x}}$

Taking log on both sides,

$$\log k = \underset{x \to 0}{Lt} \frac{\log(\cot x)}{\log x} \quad \cdots\cdots\cdots \left(\frac{\infty}{\infty} \; form\right)$$

$$= \underset{x \to 0}{Lt} \frac{-\frac{\operatorname{cosec}^2 x}{\cot x}}{\frac{1}{x}} = \underset{x \to 0}{Lt} - \frac{x}{\sin x \cos x}.$$

$$= \underset{x \to 0}{Lt} - \frac{1}{\frac{\sin x}{x}\cos x} = -1.$$

And hence $k = e^{-1} = \frac{1}{e}$.

10. $\underset{x \to \frac{\pi}{2}}{Lt} (\tan x)^{\tan 2x} \quad \cdots\cdots\cdots \; (\infty^0 \; form)$

Let $k = \underset{x \to \frac{\pi}{2}}{Lt} (\tan x)^{\tan 2x}$

Taking log on both sides,

$$\log k = \underset{x \to \frac{\pi}{2}}{Lt} \frac{\log(\tan x)}{\cot 2x} \quad \cdots\cdots\cdots \left(\frac{\infty}{\infty} \; form\right)$$

$$= \underset{x \to \frac{\pi}{2}}{Lt} \frac{\frac{\sec^2 x}{\tan x}}{-2\operatorname{cosec}^2 2x} = \underset{x \to \frac{\pi}{2}}{Lt} - \frac{\sin^2 2x}{2\sin x \cos x}.$$

$$= \underset{x \to \frac{\pi}{2}}{Lt} - \frac{\sin^2 2x}{\sin 2x} = \underset{x \to \frac{\pi}{2}}{Lt} - \sin 2x = 0.$$

Hence $k = e^0 = 1$.

11. $\underset{x \to \frac{\pi}{2}}{Lt} (\cos x)^{\frac{\pi}{2}-x} \quad \cdots\cdots\cdots \; (0^0 \; form)$

Let $k = \underset{x \to \frac{\pi}{2}}{Lt} (\cos x)^{\frac{\pi}{2}-x}$

Taking log on both sides,

$$\log k = \underset{x \to \frac{\pi}{2}}{Lt} \frac{\log(\cos x)}{\frac{1}{\frac{\pi}{2}-x}} \quad \cdots\cdots\cdots \left(\frac{\infty}{\infty} \; form\right)$$

$$= \underset{x \to \frac{\pi}{2}}{Lt} \frac{-\tan x}{\frac{1}{\left(\frac{\pi}{2}-x\right)^2}} = \underset{x \to \frac{\pi}{2}}{Lt} - \frac{\left(\frac{\pi}{2}-x\right)^2}{\cot x}.$$

$$= \underset{x \to \frac{\pi}{2}}{Lt} - \frac{\left(\frac{\pi}{2}-x\right)^2}{\tan\left(\frac{\pi}{2}-x\right)} = \underset{x \to \frac{\pi}{2}}{Lt} - \left(\frac{\pi}{2} - x\right) = 0.$$

Hence $k = e^0 = 1$.

12. $\underset{x \to 1}{Lt} (1 - x^2)^{\frac{1}{\log(1-x)}} \quad \cdots\cdots\cdots \; (0^0 \; form)$

Let $k = \underset{x \to 1}{Lt} (1 - x^2)^{\frac{1}{\log(1-x)}}$

Taking log on both sides,

$$\log k = \underset{x \to 1}{Lt} \frac{\log(1-x^2)}{\log(1-x)} \quad \cdots\cdots\cdots \left(\frac{\infty}{\infty}\ form\right)$$

$$= \underset{x \to 1}{Lt} \frac{\frac{2x}{1-x^2}}{\frac{1}{(1-x)}} = \underset{x \to 1}{Lt} \frac{2x}{1+x}.$$

$$= 1$$

Hence $k = e^1 = e.$

Partial derivatives –

- Let $z = f(x,y)$ be a function of two variables in x and y.

The first order partial derivative of z w.r.t. x, denoted by $\frac{\partial z}{\partial x}$ or z_x (i.e. Derivative of z w.r.to x keeping 'y' fixed.). Similarly $\frac{\partial z}{\partial y}$ or z_y is the derivative of $\boldsymbol{z}$ w.r.to y keeping 'x' fixed.

$$\frac{\partial z}{\partial x} = \underset{h \to 0}{Lt} \frac{f(x+h,\ y)-f(x,y)}{h} \quad . \text{ And } \frac{\partial z}{\partial y} = \underset{h \to 0}{Lt} \frac{f(x,\ y+h)-f(x,y)}{h}$$

Higher order partial derivatives also obtained in the same way.

In all ordinary cases, it can be verified that $z_{xy} = z_{yx}$.

Total derivatives:

1. If $u = f(x,y)$ *and* $x = g(t),\ y = h(t)$ *then* $\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$.
2. If $f(x,y) = constant$, then $\frac{dy}{dx} = -\frac{f_x}{f_y}$.
3. If $u = f(x,y)$ subject to $\varphi(x,y) = c$. Then $\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}$, where $\frac{dy}{dx} = -\frac{\varphi_x}{\varphi_y}$.
4. If $u = f(r,s,t)$ where r, s *and* t *are functions of* $(x,\ y,\ z)$, then by Chain rule

   $u_x = u_r r_x + u_s s_x + u_t t_x$, $u_y = u_r r_y + u_s s_y + u_t t_y$ and $u_z = u_r r_z + u_s s_z + u_t t_z$.

Jacobian:

$$J = \frac{\partial(u,\ v,\ w)}{\partial(x,\ y,\ z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}.$$

Note: 1. If $u = f(r)$ *and* $r = \sqrt{x^2+y^2}$ then $u_{xx} + u_{yy} = f''(r) + \frac{1}{r}f'(r)$.

2. If $u = f(r)$ *and* $r = \sqrt{x^2+y^2+z^2}$ then $u_{xx} + u_{yy} + u_{zz} = f''(r) + \frac{2}{r}f'(r)$.

Problems:

1. If $u = \sin\left(\frac{x}{y}\right)$, $x = e^t$ and $y = t^2$ find $\frac{du}{dt}$ *as a function of* t.

Solution: $\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} = \frac{1}{y}\cos\left(\frac{x}{y}\right)e^t - \frac{x}{y^2}\cos\left(\frac{x}{y}\right)2t$

$$= \frac{1}{t^2}\cos\left(\frac{e^t}{t^2}\right)e^t - \frac{e^t}{t^4}\cos\left(\frac{e^t}{t^2}\right)2t$$

$$= e^t \cos\left(\frac{e^t}{t^2}\right)\left[\frac{1}{t^2} - \frac{2}{t^3}\right]$$

2. If x increases at the rate of 2 cm/sec at the instant when $x = 3\ cm.$ and $y = 1\ cm.$, at what rate must y changing in order that the function $2xy - 3x^2y$ shell be neither increasing nor decreasing?

Solution: Let $u = 2xy - 3x^2y$, given that $\frac{dx}{dt} = 2$, $\frac{du}{dt} = 0$, $x = 3$ and $y = 1$.

So that

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} = (2y - 6xy)\frac{dx}{dt} + (2x - 3x^2)\frac{dy}{dt}$$

$$\Longrightarrow \quad 0 = 2(2 - 18) + (6 - 27)\frac{dy}{dt} \quad \Longrightarrow \frac{dy}{dt} = -\frac{32}{21}\text{ cm/sec.}$$

Thus y is decreasing at the rate of $\frac{32}{21}$ cm/sec.

3. If $u = x\log xy$ where $x^3 + y^3 + 3xy = 1$ find $\frac{du}{dx}$.

Solution: If $u = f(x,y)$ subject to $\varphi(x,y) = c$. Then $\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}$, where $\frac{dy}{dx} = -\frac{\varphi_x}{\varphi_y}$.

Given that $u = x\log xy$, $\varphi(x,y) = x^3 + y^3 + 3xy$

Clearly $\frac{dy}{dx} = -\frac{\varphi_x}{\varphi_y} = -\frac{3x^2+3y}{3y^2+3x} = -\frac{x^2+y}{y^2+x}$, $\quad \frac{\partial u}{\partial x} = \log xy + 1$, $\quad \frac{\partial u}{\partial y} = \frac{x}{y}$.

Hence $\frac{du}{dx} = \log xy + 1 - \frac{x(x^2+y)}{y(y^2+x)}$

4. If $u = \sqrt{x^2 + y^2}$ and $x^3 + y^3 + 3axy = 5a^2$, find $\frac{du}{dx}$ when $x = y = a$.

Solution: If $u = f(x,y)$ subject to $\varphi(x,y) = c$. Then $\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}$, where $\frac{dy}{dx} = -\frac{\varphi_x}{\varphi_y}$.

Hear $u = \sqrt{x^2 + y^2}$ and $\varphi = x^3 + y^3 + 3axy = 5a^2$

$$\Longrightarrow \frac{dy}{dx} = -\frac{\varphi_x}{\varphi_y} = -\frac{3x^2+3ay}{3y^2+3ax} = -\frac{x^2+ay}{y^2+ax} = -1 \text{ at } x = y = a$$

Then $\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} = \frac{x}{\sqrt{x^2+y^2}} + \frac{y}{\sqrt{x^2+y^2}}\left(-\frac{x^2+ay}{y^2+ax}\right)$

$$= \frac{a}{\sqrt{2a^2}} - \frac{a}{\sqrt{2a^2}} = 0\ , \text{ at } x = y = a.$$

5. If $u = f(y - z, z - x, x - y)$, then prove that $u_x + u_y + u_z = 0$.

Proof: Let $r = y - z$, $\quad s = z - x$, $\quad t = x - y$

Then $r_x = 0$, $r_y = 1$, $r_z = -1$, $s_x = -1$, $s_y = 0$, $s_z = 1$, $t_x = 1$, $t_y = -1$, $t_z = 0$.

If $u = f(r,s,t)$ where r, s and t are functions of (x, y, z), then by Chain rule

$u_x = u_r r_x + u_s s_x + u_t t_x$, $\quad u_y = u_r r_y + u_s s_y + u_t t_y$ and $\quad u_z = u_r r_z + u_s s_z + u_t t_z$.

$\Longrightarrow u_x = 0 - u_s + u_t$, $\quad u_y = u_r + 0 - u_t$ and $\quad u_z = -u_r + u_s + 0$.

$$\therefore u_x + u_y + u_z = -u_s + u_t + u_r - u_t - u_r + u_s = 0.$$

6. If $u = f(2x - 3y, 3y - 4z, 4z - 2x)$, *then find the value of* $\frac{1}{2}u_x + \frac{1}{3}u_y + \frac{1}{4}u_z$.

Solution: Let $r = 2x - 3y, \quad s = 3y - 4z, \quad t = 4z - 2x$

Then $r_x = 2, \quad r_y = -3, \quad r_z = 0, \quad s_x = 0, \quad s_y = 4, \quad s_z = -4, \quad t_x = -2, \quad t_y = 0, \quad t_z = 4.$

If $u = f(r, s, t)$ where r, s *and* t *are functions of* (x, y, z) , then by Chain rule

$$u_x = u_r r_x + u_s s_x + u_t t_x \ , \quad u_y = u_r r_y + u_s s_y + u_t t_y \quad \text{and} \quad u_z = u_r r_z + u_s s_z + u_t t_z \ .$$

$$\Rightarrow u_x = 2u_r + 0 - 2u_t \ , \quad u_y = -3u_r + 3u_s + 0 \quad \text{and} \quad u_z = 0 - 4u_s + 4u_t \ .$$

$$\therefore \frac{1}{2}u_x + \frac{1}{3}u_y + \frac{1}{4}u_z = u_r - u_t - u_r + u_s - u_s + u_t = 0.$$

7. If $u = f\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$, then show that $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0.$

Solution: Let $r = \frac{y-x}{xy} = \frac{1}{x} - \frac{1}{y}, \quad s = \frac{z-x}{xz} = \frac{1}{x} - \frac{1}{z}$.

Then $r_x = -\frac{1}{x^2}, \quad r_y = \frac{1}{y^2}, \quad r_z = 0, \quad s_x = -\frac{1}{x^2}, \quad s_y = 0, \quad s_z = \frac{1}{z^2},$

If $u = f(r, s)$ where r, *and* s *are functions of* (x, y, z) , then by Chain rule

$$u_x = u_r r_x + u_s s_x \ , \quad u_y = u_r r_y + u_s s_y \quad \text{and} \quad u_z = u_r r_z + u_s s_z$$

$$\Rightarrow u_x = -\frac{1}{x^2}u_r - \frac{1}{x^2}u_s \ , \quad u_y = \frac{1}{y^2}u_r + 0 \quad \text{and} \quad u_z = 0 + \frac{1}{z^2}u_s$$

$$\Rightarrow x^2\frac{\partial u}{\partial x} = -u_r - u_s, \quad y^2\frac{\partial u}{\partial y} = u_r \quad \text{and} \quad z^2\frac{\partial u}{\partial z} = u_s.$$

Therefore $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0.$

8. If $x = r\cos\theta\,, \ y = r\sin\theta$, then verify that $JJ' = 1$.

Solution: $x = r\cos\theta\,, \ y = r\sin\theta$

$$J = \frac{\partial(x,\ y)}{\partial(r,\ \theta)} = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r\,.$$

$x = r\cos\theta\,, \ y = r\sin\theta$

$$\Rightarrow \quad r = \sqrt{x^2 + y^2} \quad \text{and} \quad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

And $$J' = \frac{\partial(r,\ \theta)}{\partial(x,\ y)} = \begin{vmatrix} r_x & r_y \\ \theta_x & \theta_y \end{vmatrix} = \begin{vmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right) & \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right) \end{vmatrix}$$

$$= \begin{vmatrix} \frac{x}{r} & \frac{y}{r} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix} = \frac{x^2}{r(x^2+y^2)} + \frac{y^2}{r(x^2+y^2)} = \frac{1}{r}\,.$$

$\therefore JJ' = 1.$

7. If $x = r\cos\varphi, \ y = r\sin\varphi, \ z = z$, then find $\boldsymbol{J} = \frac{\partial(x\ \ y,\ z)}{\partial(r,\ \varphi,\ z)}$

Solution: $J = \frac{\partial(x\ \ y,\ z)}{\partial(r,\ \varphi,\ z)} = \begin{vmatrix} x_r & x_\varphi & x_z \\ y_r & y_\varphi & y_z \\ z_r & z_\varphi & z_z \end{vmatrix} = \begin{vmatrix} \cos\varphi & -r\sin\varphi & 0 \\ \sin\varphi & r\cos\varphi & 0 \\ 0 & 0 & 1 \end{vmatrix} = r\cos^2\varphi + r\sin^2\varphi = r$.

8. If $x = u(1+v),\ \ y = v(1+u), show\ that\ \frac{\partial(\boldsymbol{x},\ \boldsymbol{y})}{\partial(\boldsymbol{u},\ \boldsymbol{v})} = 1 + \mathrm{u} + \mathrm{v}$.

Solution: Given that $x = u(1+v), y = v(1+u)$

$$\frac{\partial(x,\ y)}{\partial(u,\ v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} (1+v) & u \\ v & (1+u) \end{vmatrix} = (1+v)(1+u) - uv = 1 + u + v + uv - uv = 1 + u + v.$$

9. If $u = x + y + z,\ v = x^2 + y^2 + z^2\ and\ w = xy + yz + zx, then\ find\ \frac{\partial(\boldsymbol{u},\ \boldsymbol{v},\ \boldsymbol{w})}{\partial(\boldsymbol{x},\ \boldsymbol{y},\ \boldsymbol{z})}$.

Solution: Given that, $u = x + y + z,\ v = x^2 + y^2 + z^2\ and\ w = xy + yz + zx$

$$\frac{\partial(u,\ v,\ w)}{\partial(x,\ y,\ z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y+z & x+z & x+y \end{vmatrix} = 2\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ y+z & x+z & x+y \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x+y+z & x+y+z & x+y+z \end{vmatrix}$$

$$= 2(x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

10. If $u = x^2 - 2y^2$, $v = 2x^2 - y^2$ and $x = r\cos\theta$, $y = r\sin\theta$, then show that $\frac{\partial(u,\ v)}{\partial(r,\ \theta)} = 6r^3\sin 2\theta$.

Solution: Since $\frac{\partial(u,\ v)}{\partial(r,\ \theta)} = \frac{\partial(u,\ v)}{\partial(x,\ y)} \times \frac{\partial(x,\ y)}{\partial(r,\ \theta)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \times \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix}$

$$= \begin{vmatrix} 2x & -4y \\ 4x & -2y \end{vmatrix} \times \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = 12xyr = 12r^3\sin\theta\cos\theta = 6r^3\sin 2\theta.$$

10. If $z = f(x,y)\ and\ \ x = r\cos\theta$, $y = r\sin\theta$ then show that, $(z_x)^2 + (z_y)^2 = (z_r)^2 + \frac{1}{r^2}(z_\theta)^2$.

Solution: $z_r = z_x x_r + z_y y_r = z_x\cos\theta + z_y\sin\theta$ ……… (1)

And $z_\theta = z_x x_\theta + z_y y_\theta = -rz_x\sin\theta + \mathrm{r}z_y\cos\theta$

$\Longrightarrow\ \frac{1}{r}z_\theta = -z_x\sin\theta + z_y\cos\theta$ ……… (2)

$(1)^2 + (2)^2 \quad \Longrightarrow (z_r)^2 + \frac{1}{r^2}(z_\theta)^2 = (z_x)^2\cos^2\theta + (z_y)^2\sin^2\theta + 2z_xz_y\cos\theta\sin\theta$

$+(z_x)^2\sin^2\theta + (z_y)^2\cos^2\theta - 2z_xz_y\cos\theta\sin\theta$

$= (z_x)^2 + (z_y)^2.$

Exercise:

1. If $u = \sin(x^2 + y^2)$ where $a^2x^2 + b^2y^2 = c^2$ find $\frac{du}{dx}$.
2. If $u = \tan^{-1}\left(\frac{y}{x}\right)$, $x = e^t - e^{-t}$ and $y = e^t + e^{-t}$ find $\frac{du}{dt}$ $as\, a function\, of\, t$.
3. If $\phi(cx - az,\ cy - bz) = 0$, then show that $a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} = c$.
4. If $f(x,\ y) = 0,\ \phi(y,\ z) = 0$, show that $\frac{\partial f}{\partial y} . \frac{\partial \phi}{\partial z} . \frac{dz}{dx} = \frac{\partial f}{\partial x} . \frac{\partial \phi}{\partial y}$
5. If $z = f(x,\ y)$ and $x = e^u + e^{-v}, y = e^{-u} - e^v$, prove that $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$.
6. If $\phi\left(\frac{z}{x^3},\ \frac{y}{x}\right) = 0$, then show that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 3z$.
7. If $J = \frac{\partial(u,\ v)}{\partial(x,\ y)}$ and $J' = \frac{\partial(x,\ y)}{\partial(u,\ v)}$ then show that $JJ' = 1$.
8. Prove that $\frac{\partial(u,\ v)}{\partial(x,\ y)} \times \frac{\partial(x,\ y)}{\partial(r,\ \theta)} = \frac{\partial(u,\ v)}{\partial(r,\ \theta)}$.
9. If $u = x^2 - y^2$, $v = 2xy$ and $x = r\cos\theta$, $y = r\sin\theta$ find $\frac{\partial(u,\ v)}{\partial(r,\ \theta)}$.
10. If $ux = yz,\ vy = zx,\ wz = xy$, then show that $\frac{\partial(u,\ v,\ w)}{\partial(x,\ y,\ z)} = 4$.
11. If $u = x + y + z,\ uv = y + z\ and\ uvw = z, then\ find$ $\frac{\partial(x,\ y,\ z)}{\partial(u,\ v,\ w)}$.
12. If $x = r\sin\theta\cos\emptyset$, $y = r\sin\theta\sin\emptyset$ and $z = r\cos\theta$, then find $J = \frac{\partial(x\ \ y,\ z)}{\partial(r,\ \theta,\ \emptyset)}$.
13. At a given instant the sides of a rectangle are 4ft and 3 ft, and they are increasing at the rate of 1.5ft/sec and 0.5ft/sec respectively. Find the rate at which the area is increasing at that instant.
14. If $u^3 + v^3 = x + y$ and $u^2 + v^2 = x^3 + y^3$, show that $\frac{\partial(u,\ v)}{\partial(x,\ y)} = \frac{1}{2}\frac{y^2 - x^2}{uv(u-v)}$.
15. If the kinetic energy T is given by $T = \frac{1}{2}mv^2$, find approximately the change in T , as the mass m changes from 49 to 49.5 and velocity v changes from 1600 to 1590.

Maxima and minima of functions of two variables:

1. $f(x,\ y)$ is stationary at $(a,\ b)$ i.e. $f(a,\ b)$ is the stationary value of f if $f_x = 0 = f_y$ at $(a,\ b)$.
2. $f(x,\ y)$ is maximum at $(a,\ b)$ i.e. $f(a,\ b)$ is the maximum value of f

   If at $(a,\ b)$ i) $f_x = 0 = f_y$ ii) $f_{xx}f_{yy} - f_{xy}^2 > 0$ iii) $f_{xx} < 0$.
3. $f(x,\ y)$ is minimum at $(a,\ b)$ i.e. $f(a,\ b)$ is the minimum value of f

   If at $(a,\ b)$ i) $f_x = 0 = f_y$ ii) $f_{xx}f_{yy} - f_{xy}^2 > 0$ iii) $f_{xx} > 0$.
4. $(a,\ b)$ is said to be saddle point of $f(x,\ y)$ if i) $f_x = 0 = f_y$ ii) $f_{xx}f_{yy} - f_{xy}^2 < 0$ at $(a,\ b)$.
5. If $f_x = 0 = f_y$ and $f_{xx}f_{yy} - f_{xy}^2 = 0$ at $(a,\ b)$, then by discussion find maxima and minima.

Examples:

1. Examine the function $f(x,\ y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$ for extreme values.

Solution: $f(x,\ y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$.

Differentiating partially we get, $f_x = 4x^3 - 4x + 4y,$ $f_y = 4y^3 + 4x - 4y,$

$f_{xx} = 12x^2 - 4$, $f_{yy} = 12y^2 - 4$ and $f_{xy} = 4$.

Now for extreme values $f_x = 0$, $f_y = 0$

$\Longrightarrow 4x^3 - 4x + 4y = 0$ and $4y^3 + 4x - 4y = 0$.

Adding these, we get $4(x^3 + y^3) = 0$ or $y = -x$.

Put $y = -x$ in $x^3 - x + y = 0$, we get $x^3 - 2x = 0$

$\Longrightarrow$ $x = 0, \sqrt{2}, -\sqrt{2}$ and corresponding values of y are $0, -\sqrt{2}, \sqrt{2}$.

Point	f_{xx}	f_{yy}	f_{xy}	$f_{xx}f_{yy} - f_{xy}^2$	Conclusion
$(\sqrt{2}, -\sqrt{2})$	$20 > 0$	20	4	$384 > 0$	$f(\sqrt{2}, -\sqrt{2}) = -8$ is minimum
$(-\sqrt{2}, \sqrt{2})$	$20 > 0$	20	4	$384 > 0$	$f(-\sqrt{2}, \sqrt{2}) = -8$ is minimum
$(0, 0)$	$-4 < 0$	-4	4	0	Since $f_{xx}f_{yy} - f_{xy}^2 = 0$ Further investigation is needed.

Clearly $f(0,\ 0) = 0$, $f(0.1,\ 0) = -0.0199$, $f(0.1,\ 0.1) = 0.0002$.

Thus, in the neighborhood of $(0,\ 0)$, $f > f(0,\ 0)$ at some points and $f < f(0,\ 0)$ at some points.

Hence $f(0,\ 0)$ is not an extreme value. The point $(0,\ 0)$ is saddle point.

2. Discuss the maxima and minima of $f(x,\ y) = x^3y^2(1 - x - y)$.

Solution: $f(x,\ y) = x^3y^2 - x^4y^2 - x^3y^3$.

Differentiating partially we get, $f_x = 3x^2y^2 - 4x^3y^2 - 3x^2y^3$, $f_y = 2x^3y - 2x^4y - 3x^3y^2$,

$f_{xx} = 6xy^2 - 12x^2y^2 - 6xy^3$, $f_{yy} = 2x^3 - 2x^4 - 6x^3y$ and $f_{xy} = 6x^2y - 8x^3y - 9x^2y^2$.

Now for extreme values $f_x = 0$, $f_y = 0$

$\Longrightarrow 3x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0$ and $2x^3y - 2x^4y - 3x^3y^2 = 0$.

$\Longrightarrow 3 - 4x - 3y = 0$ and $2 - 2x - 3y = 0$.

Therefore stationary points are $\left(\frac{1}{2},\ \frac{1}{3}\right)$ and $(0,\ 0)$.

Point	f_{xx}	f_{yy}	f_{xy}	$f_{xx}f_{yy} - f_{xy}^2$	Conclusion
$\left(\frac{1}{2}, \frac{1}{3}\right)$	$-\frac{1}{9} < 0$	$-\frac{1}{8}$	$-\frac{1}{12}$	$\frac{1}{144} > 0$	$f\left(\frac{1}{2}, \frac{1}{3}\right) = \frac{1}{432}$ is maximum
$(0, 0)$	0	0	0	0	Since $f_{xx}f_{yy} - f_{xy}^2 = 0$ Further investigation is needed.

Clearly $f(0,\ 0) = 0$, $f(0.1,\ 0.1) > 0$, $f(-0.1, -0.1) < 0$.

Thus in the neighborhood of $(0,\ 0)$, $f > f(0,\ 0)$ at some points and $f < f(0,\ 0)$ at some points.

Hence $f(0,\ 0)$ is not an extreme value. The point $(0,\ 0)$ is saddle point.

Exercise:

Find the maximum and minimum values of

i) $x^3+y^3-2x^2-3axy$. ii) $x^3+3xy^2-15x^2-15y^2+72x$. iii) $\sin x\sin y\sin(x+y)$.

Self-study: Euler's Theorem and problems. Method of Lagrange undetermined multipliers with single constraint.

Homogeneous Function:

If a function can be expressed in the form of $x^n\varphi\left(\frac{y}{x}\right)$ is called homogeneous of degree n.

Euler's theorem:- If u is a homogenous function of x and y with degree n, then $xu_x+yu_y=nu$.

Proof: - Since u is a homogenous function of degree n, we can put u is the form $u=x^n\varphi\left(\frac{y}{x}\right)$ $\cdots\cdots\cdots 1$

Differentiating 1 w.r.to x partially, $u_x=x^n\left(-\frac{y}{x^2}\right)\varphi'\left(\frac{y}{x}\right)+nx^{n-1}\varphi\left(\frac{y}{x}\right)$

$$\Rightarrow xu_x=-x^{n-1}y\varphi'\left(\frac{y}{x}\right)+nu \cdots\cdots 2.$$

Differentiating 1 w.r.to y partially, $u_y=x^n\left(\frac{1}{x}\right)\varphi'\left(\frac{y}{x}\right)$

$$\Rightarrow yu_y=x^{n-1}y\varphi'\left(\frac{y}{x}\right) \cdots\cdots 3.$$

Adding 2 and 3 we get, $xu_x+yu_y=nu$.

2. If u is a homogenous function of x and y with degree n, then $x^2u_{xx}+2xyu_{xy}+y^2u_{yy}=n(n-1)u$.

Proof: By Euler's theorem, $xu_x+yu_y=nu \cdots\cdots\cdots (1)$

Differentiating 1 w.r.to x partially, $xu_{xx}+u_x+yu_{xy}=nu_x \implies xu_{xx}+yu_{xy}=(n-1)u_x$.

$\therefore x^2u_{xx}+xyu_{xy}=(n-1)xu_x \cdots\cdots (2)$. Similarly $y^2u_{yy}+xyu_{xy}=(n-1)yu_y \cdots\cdots (3)$.

Adding 2 and 3 we get, $x^2u_{xx}+2xyu_{xy}+y^2u_{yy}=(n-1)\left[xu_x+yu_y\right]=n(n-1)u$.

3. If $u=\sin^{-1}\left(\frac{x^2y^2}{x+y}\right)$, then prove that $xu_x+yu_y=3\tan u$.

$u=\sin^{-1}\left(\frac{x^2y^2}{x+y}\right)\Rightarrow \sin u=\frac{x^2y^2}{x+y}=\frac{x^4(y^2/x^2)}{x(1+y/x)}=x^3\varphi\left(\frac{y}{x}\right)$.

$\therefore \sin u$ is homogenous function of degree 3,

Then by Euler's theorem, $x\frac{\partial \sin u}{\partial x}+y\frac{\partial \sin u}{\partial y}=3\sin u \quad\Rightarrow \cos u\left(xu_x+yu_y\right)=3\sin u$

Or $xu_x+yu_y=3\tan u$.

4. If $u=\frac{x^3y^3}{x^3+y^3}$, then prove that $xu_x+yu_y=3u$.

$u=\frac{x^3y^3}{x^3+y^3}=\frac{x^6(y^3/x^3)}{x^3(1+y^3/x^3)}=x^3\varphi\left(\frac{y}{x}\right)$.

$\therefore u$ is homogenous function of degree 3,

Then by Euler's theorem, $xu_x+yu_y=3u$.

5. If $u=\frac{x^2y^2}{x+y}$, then find the value of $x^2u_{xx}+2xyu_{xy}+y^2u_{yy}$.

If $u=\frac{x^2y^2}{x+y}=\frac{x^4(y^2/x^2)}{x(1+y/x)}=x^3\varphi\left(\frac{y}{x}\right)$.

$\therefore u$ is homogenous function of degree 3, Then by Euler's theorem, $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = n(n-1)u = 6u$.

Exercise:

1. If $u = \tan^{-1}\frac{x^3+y^3}{x-y}$, then prove that $xu_x + yu_y = \sin 2u$ and $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = 2\cos 3u \sin u$.
2. If $z = x\,\varphi\left(\frac{y}{x}\right) + \emptyset\left(\frac{y}{x}\right)$, prove that $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = 0$.
3. If $u = \frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y}$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$.

Extremum by Lagrange's multiplier method:

To find extremum of $f(x,y,z)$ subject to $\emptyset(x,y,z) = c$, first we write $F = f(x,y,z) + \lambda\emptyset(x,y,z)$

Next we obtain the equations $F_x = 0,\ F_y = 0,\ F_z = 0.$

Then solve the above equations with $\emptyset(x,y,z) = c$.

The valve of x, y, z so obtained will give the stationary value of $f(x,y,z)$.

Example:

1. A rectangular box open at top is to have volume 32 cubic ft. Find the dimension of the box requiring least material for its construction.

Solution: Clearly $f(x,y,z) = xy + 2yz + 2zx$ (open at top) and $\emptyset(x,y,z) = xyz = 32.$

Let $F = xy + 2yz + 2zx + \lambda xyz$

$$F_x = 0,\ F_y = 0,\ F_z = 0 \Longrightarrow \quad y + 2z + \lambda yz = 0, \quad \cdots \text{(i)}$$

$$x + 2z + \lambda xz = 0\,, \quad \cdots \text{(ii)}$$

$$2y + 2x + \lambda xy = 0\,. \quad \cdots \text{(iii)}$$

$\text{(i)}x - \text{(ii)}y \ \Longrightarrow 2z(x-y) = 0 \quad \Longrightarrow x = y,$

$\text{(ii)}y - \text{(iii)}z \ \Longrightarrow x(y-2z) = 0 \quad \Longrightarrow y = 2z.$

$xyz = 32 \ \Longrightarrow 4z^3 = 32 \Longrightarrow \ z = 2,\ x = 4,\ y = 4.$

Therefore $x = 4ft,\ y = 4ft,\ z = 2ft.$

2. In a plane triangle find the maximum value of $\cos A\ \cos B\ \cos C$.

Solution: Let $x = A,\ y = B,\ z = C.$

Then the question is to find the maximum value of $f = \cos x\ \cos y\ \cos z$ subject to $x + y + z = \pi$

Let $F = \cos x\ \cos y\ \cos z + \lambda(x + y + z)$

$$F_x = 0,\ F_y = 0,\ F_z = 0 \Longrightarrow \quad -\sin x\ \cos y\ \cos z + \lambda = 0, \quad \cdots \text{(i)}$$

$$-\cos x\ \sin y\ \cos z + \lambda = 0, \quad \cdots \text{(ii)}$$

$$-\cos x\ \cos y\ \sin z + \lambda = 0\ . \quad \cdots \text{(iii)}$$

$$\Longrightarrow \lambda = \sin x\ \cos y\ \cos z = \cos x\ \sin y\ \cos z = \cos x\ \cos y\ \sin z$$

$$\Longrightarrow \sin x\ \cos y\ = \cos x\ \sin y \quad \text{and} \quad \sin y\ \cos z = \ \cos y\ \sin z$$

$$\Longrightarrow \sin(x-y) = 0 \quad \text{and} \quad \sin(y-z) = 0\,.$$

Therefore $x = y = z$ and $x + y + z = \pi \implies x = y = z = \frac{\pi}{3}$.

Hence the maximum value of $\cos A\ \cos B\ \cos C = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$.

3. Find the maximum and minimum distances of the point $(3, 4, 12)$ from the sphere $x^2 + y^2 + z^2 = 4$.

Solution: Let $P(x, y, z)$ be any point on the sphere, and $A \equiv (3, 4, 12)$.

Then the distance $AP = \sqrt{(x-3)^2 + (y-4)^2 + (z-12)^2}$

Without loss of generality consider $f = (x-3)^2 + (y-4)^2 + (z-12)^2$

and $\emptyset = x^2 + y^2 + z^2 = 4$.

Let $F = (x-3)^2 + (y-4)^2 + (z-12)^2 + \lambda(x^2 + y^2 + z^2)$.

$$F_x = 0,\ F_y = 0,\ F_z = 0 \implies \quad (x-3) + x\lambda = 0, \quad \cdots \text{(i)}$$

$$(y-4) + y\lambda = 0, \quad \cdots \text{(ii)}$$

$$(z-12) + z\lambda = 0\ . \quad \cdots \text{(iii)}$$

$\text{(i)}y - \text{(ii)}x \implies 4x - 3y = 0$ and $\text{(ii)}z - \text{(iii)}y \implies 12y - 4z = 0$

$$\implies z = 3y \quad \& \quad x = \frac{3}{4}y$$

$$x^2 + y^2 + z^2 = 4 \implies \left(\frac{9}{16} + 1 + 9\right)y^2 = 4 \implies \left(\frac{169}{16}\right)y^2 = 4$$

$$\implies y^2 = \frac{64}{169} \implies y = \pm\frac{8}{13}\ .$$

When $y = \frac{8}{13}$, $x = \frac{6}{13}$, $z = \frac{24}{13}$ and when $y = -\frac{8}{13}$, $x = -\frac{6}{13}$, $z = -\frac{24}{13}$.

$$AP = \sqrt{\left(\frac{6}{13} - 3\right)^2 + \left(\frac{8}{13} - 4\right)^2 + \left(\frac{24}{13} - 12\right)^2} = 11.$$

And $$AP = \sqrt{\left(-\frac{6}{13} - 3\right)^2 + \left(-\frac{8}{13} - 4\right)^2 + \left(-\frac{24}{13} - 12\right)^2} = 15.$$

Hence maximum distance is 15 and minimum distance is 11.

4. The temperature T at any point (x, y, z)in space is $T = 400xyz^2$. Find the highest temperature on the surface of the unit sphere $x^2 + y^2 + z^2 = 1$.

Solution: Let $F = xyz^2 + \lambda(x^2 + y^2 + z^2)$.

$$F_x = 0,\ F_y = 0,\ F_z = 0 \implies \quad yz^2 + 2x\lambda = 0, \quad \cdots \text{(i)}$$

$$xz^2 + 2y\lambda = 0, \quad \cdots \text{(ii)}$$

$$2xyz + 2z\lambda = 0\ . \quad \cdots \text{(iii)}$$

$\text{(i)}y - \text{(ii)}x \implies y^2z^2 - x^2z^2 = 0$ and $\text{(ii)}z - \text{(iii)}y \implies xz^3 - 2xy^2z = 0$

$$\implies y^2 = x^2 \quad \& \ z^2 = 2y^2$$

$$x^2 + y^2 + z^2 = 1 \implies \quad y^2 + y^2 + 2y^2 = 1 \implies y^2 = \frac{1}{4}.$$

$$\implies x = y = \frac{1}{2} \text{ and } z^2 = \frac{1}{2}.$$

Therefore highest temperature is $T = 400xyz^2 = \frac{400}{8} = 50$ units.

Exercise:

1. Find the minimum value of $x^2 + y^2 + z^2$ subject to the condition, i) $xyz = a^3$ ii) $xy + yz + zx = 3a^2$.
2. Find the dimensions of the rectangular box, open at top, of maximum capacity whose surface is $432\ cm^2$.

Assignment:

1. If $u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$.
2. If $u = \cos^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$, then prove that $xu_x + yu_y = -\frac{1}{2}\cot u$.
3. Find the minimum value of $x^2 + y^2 + z^2$ subject to the condition $ax + by + cz = p$.
4. Find the maximum and minimum distances from the origin to the surface $5x^2 + 6xy + 5y^2 = 8$.