H.8

a) $$\phi(x) = \frac{\delta}{\delta j(x)} E[j] = \frac{\delta}{\delta j(x)} \frac{i}{1!} \int d^4x_1\, G_c(x_1)\, j(x_1)$$

$\leftarrow j=0$

$$= i\, G_1(x)$$

$$= i \langle 0 | T\, \phi(x_1) | 0 \rangle$$

$$= i \langle 0 | \phi(x_1) | 0 \rangle$$

in Heisenberg picture.
Isn't it zero?

b) $$\frac{\delta}{\delta \phi(x)} \Gamma[\phi] = \frac{\delta}{\delta \phi(x)} \left(E - \int d^4y\, j(y)\phi(y) \right)$$

$$= \frac{\delta E[j[\phi]]}{\delta \phi(x)} - \int d^4y \frac{\delta j[\phi(y)]}{\delta \phi(x)} \phi(y) - j(x)$$

chain rule of functional deri. involves integral to get rid of variable. It's the continuum generalization of chain rule for multivariant derivative.

$$= \int d^4y \left\{ \underbrace{\frac{\delta E[j[\phi]]}{\delta j[\phi(y)]}}_{=\phi(y)} \frac{\delta j[\phi(y)]}{\delta \phi(x)} - \frac{\delta j[\phi(y)]}{\delta \phi(x)} \phi(y) \right\} - j(x)$$

$$= -j(x)$$

c) I) $$\frac{\delta \phi(x)}{\delta \phi(y)} = \delta^{(4)}(x-y)$$

II) $$\frac{\delta}{\delta \phi(y)} \frac{\delta}{\delta j(x)} E[j] = \int d^4z\, \frac{\delta^2 E[j]}{\delta j(x)\, \delta j(z)} \frac{\delta j(z)}{\delta \phi(y)}$$

$$= \int d^4z\, i G_c(x-z) \left[- \frac{\delta^2 \Gamma[\phi]}{\delta \phi(y) \delta \phi(z)} \right]$$

$$= -i \int d^4z\, G_c(x-z)\, \Gamma(y-z) = \delta^{(4)}(x-y)$$

Fourier trafo:

$$\cancel{\frac{1}{(2\pi)^4}} \int d^4x\, e^{i(x-y)p}\, \delta^{(4)}(x-y) = -i \int d^4z \int \frac{d^4x}{\cancel{(2\pi)^4}} e^{i(x-y)p} G_c(x-z)\Gamma(y-z)$$

$$1 = -i\int d^4z \underbrace{\int d^4x\, e^{i(x-z)p} G_c(x-z)}_{\tilde{G}_c(p)} e^{-i(y-z)p} \Gamma(y-z)$$

But there is z in this too ? Hmm

$$= -i\, \tilde{G}_c(p)\, \tilde{\Gamma}(p)$$

d) $$\frac{\delta^2 \phi(x)}{\delta\phi(z)\delta\phi(y)} = 0$$

$$= \frac{\delta}{\delta\phi(z)} \frac{\delta}{\delta\phi(y)} \frac{\delta}{\delta j(x)} E[j]$$

$$= \frac{\delta}{\delta\phi(z)} \int d^4x_1 \frac{\delta^2 E[j]}{\delta j(x)\delta j(x_1)} \frac{\delta j(x_1)}{\delta\phi(y)}$$

$$= \int d^4x_1 \int d^4x_2 \frac{\delta^3 E[j]}{\delta j(x)\delta j(x_2)\delta j(x_1)} \frac{\delta j(x_1)}{\delta\phi(y)} \frac{\delta j(x_2)}{\delta\phi(z)}$$

$$+ \int d^4x_1 \frac{\delta^2 E[j]}{\delta j(x)\delta j(x_1)} \frac{\delta^2 j(x_1)}{\delta\phi(y)\delta\phi(z)}$$

$$= \int d^4x_1 \int d^4x_2 (-1) G(x, x_2, x_1)\, \Gamma(x_1, y)\, \Gamma(x_2, z)$$

$$+ \int d^4x_1 (-1) G(x, x_1)\, \Gamma(x_1, y, z)$$

$$0 = -\int d^4x_1 \int d^4x_2\, G(x, x_1, x_2)\, \Gamma(x_1, y)\, \Gamma(x_2, z) - \int d^4x_1\, G(x, x_1)\, \Gamma(x_1, y, z)$$

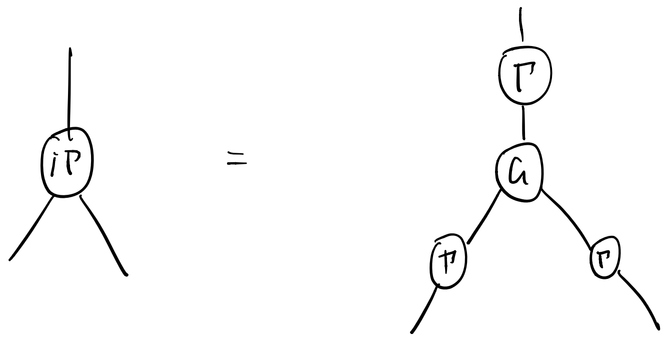
Multiply with $\int d^4x\, \Gamma(x, w)$

$$0 = -\int d^4x_1 \int d^4x_2 \int d^4x\, G(x, x_1, x_2)\Gamma(x_1, y)\Gamma(x_2, z)\Gamma(x, w)$$

$$- \underbrace{\int d^4x \int d^4x_1\, G(x, x_1)\, \Gamma(x_1, y, z)\, \Gamma(x, w)}_{= i\delta^{(4)}(w - x_1)}$$

$$\Rightarrow \quad i\,\Gamma(w, y, z) = \int d^4x_1 \int d^4x_2 \int d^4x\, G(x, x_1, x_2)\, \Gamma(x_1, y)\, \Gamma(x_2, z)\, \Gamma(x, w)$$

$$\downarrow$$



$\Gamma \sim G^{-1}$