H.6

a) $\theta_i^2 = 0, \ \forall i, \quad \theta_i \in G$

$\Rightarrow$ Terms with Grassmann variable with power higher than 2 is automatically zero. Thus any function can be decomposed into monomial in θ_i

Here:

$$F(\breve{\xi}) = F(\xi^*, \xi)$$
$$= F^{(0)} + \sum_i F_i^{(1)} \breve{\xi}_i + \dots + \sum_{i,\dots,k} F^{(2n)}_{i,\dots,k} \breve{\xi}_i \dots \breve{\xi}_j \dots \breve{\xi}_k$$

$$F(\breve{\xi}(\breve{\eta})) = F^{(0)} + \sum_i F_i^{(1)} M_{i\mu} \breve{\eta}_\mu + \dots + \sum_{i,\dots,k} F^{(2n)}_{i,\dots,k} M_{i\mu} \breve{\xi}_\mu \dots M_{j\nu} \breve{\xi}_\nu \dots M_{k\lambda} \breve{\xi}_k$$

In the integral only the terms with $(2n)$ (unique) Grassmann variables are contributing. All other terms vanish because of $\int d\theta_i = 0$.

$$\int d\breve{\xi}_1 \dots d\breve{\xi}_{2n} F(\breve{\xi}) = \int d\breve{\xi}_1 \dots d\breve{\xi}_{2n} \sum_{i,\dots,j,\dots,k}^{2n} F^{(2n)}_{ijk} \breve{\xi}_i \dots \breve{\xi}_k$$

$$= \sum_{\alpha,\dots,\beta} \int M_{1\alpha} \, d\breve{\eta}_\alpha \dots M_{2n,\beta} \, d\breve{\eta}_\beta \times \underbrace{\sum_{i\dots k}^{2n} F^{(2n)}_{i\dots k} \sum_{\mu,\dots,\nu} M_{i\mu} \breve{\eta}_\mu \dots M_{k\nu} \breve{\eta}_\nu}_{\sim F(\breve{\xi}(\breve{\eta}))}$$

✓

Need to take care of order of $\alpha \dots \beta$.

$$= \sum_{\alpha\dots\beta} M_{1\alpha} \dots M_{2n,\beta} \, \epsilon_{\alpha\dots\beta} \int d\breve{\eta}_1 \dots d\breve{\eta}_{2n} \, F(\breve{\xi}(\breve{\eta}))$$

$$= \underbrace{(\det M)^{-1}}_{\left| \frac{\partial(\eta, \eta^*)}{\partial(\xi, \xi^*)} \right|} \int d\breve{\eta}_1 \dots d\breve{\eta}_{2n} \, F(\breve{\xi}(\breve{\eta}))$$

✓

b) $\int \prod_{i=1}^{n} d\eta_i^* d\eta_i \exp(\underbrace{-\eta_k^* H_{kl} \eta_l + \xi_k^* \eta_k + \eta_k^* \xi_k}$)

$$= -\eta^\dagger H \eta + \xi^\dagger \eta + \eta^\dagger \xi$$
$$= -\omega^\dagger \Delta \omega + J^\dagger H^{-1} J$$

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with $\Delta = U \ H U^{-1}$ $\quad \Delta = \mathrm{diag}(\lambda_{11}, \dots, \lambda_{nn})$

$\omega = U(\eta - H^{-1}\xi)$, $\quad \det(U) = 1$, $\quad U^\dagger U = \mathbb{1}_n$

probably no need to diagonalize

$$-\underbrace{(\eta - H^{-1}\xi)^\dagger} \underbrace{U^\dagger \Delta U}_{= H} (\eta - H^{-1}\xi)$$

$(= \eta^\dagger - \xi^\dagger H$

$$= -(\eta^\dagger - \xi^\dagger H) H (\eta - H^{-1}\xi)$$
$$= -\eta^\dagger H \eta - \xi^\dagger H H H^{-1} \xi + \xi^\dagger H H \eta + \eta^\dagger H H^{-1} \xi$$

$$= \left|\frac{\partial(\omega^*, \omega)}{\partial(\eta^*, \eta)}\right| \int \prod_{i=1}^{n} d\omega_i^* d\omega_i \exp(-\omega^\dagger \Delta \omega + \xi^\dagger H^{-1} \xi)$$

$\left(= |\det U| \right.$ $\left. = 1 \right)$

$$= \exp(\xi^\dagger H^{-1} \xi) \int \prod_{i=1}^{n} d\omega_i^* d\omega_i \exp(\underbrace{-\omega^\dagger \Delta \omega}_{= -\omega_1^* \lambda_{11} \omega_1 - \dots - \omega_n^* \lambda_{nn} \omega_n})$$

diagonal

$$\left(\int d\theta^* d\theta \underbrace{\exp(-\theta^* \theta)}_{= 1 - \theta^* \theta} = -\int d\theta^* d\theta \, \theta^* \theta = +1 \right.$$

$$= \exp(\xi^\dagger H^{-1} \xi) \int \prod d\omega_i^* d\omega_i (1 - \omega_1^* \lambda_{11} \omega_1 - \dots - \omega_n^* \lambda_{nn} \omega_n)$$

$$= \exp(\xi^\dagger H^{-1} \xi) \underbrace{\lambda_{11} \cdot \dots \cdot \lambda_{nn}}_{= \det \Delta = \det H}$$

$$I = \exp(\xi^* H^{-1} \xi) \int d\theta_1^* d\theta_1 \dots d\theta_n^* d\theta_n$$

$$\exp\Big(-\underbrace{(\eta^* - \xi^* H^{-1})}_{\theta^*} H \underbrace{(\eta - H^{-1}\xi)}_{\theta} + \xi^* H^{-1} \xi\Big)$$

$$= \exp(\dots) \int d\theta_1^* d\theta_1 \dots d\theta_n^* d\theta_n \,(1 - \theta_{1k}^* H_{kl} \theta_{l1}) \dots (1 - \theta_n^* H \theta_n)$$

Only terms with all θ_n, θ_n^* are non-zero.

$$= \exp(\dots) \det(H)$$

H.7
$$\mathcal{L} = \frac{1}{2}\left[(\partial_\mu \phi)(\partial^\mu \phi) - m^2 \phi^2\right] + j\phi$$

a)
$$W[j] = \frac{\int D\phi \exp(i \int d^4x\, \mathcal{L}_0 + j\phi)}{\int D\phi \exp(i \int d^4x\, \mathcal{L}_0)}$$

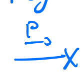
$$= \exp\left(-\frac{1}{2} \int d^4x\, d^4y\; j(x) D_F(x-y) j(y)\right)$$

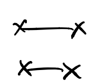
$$= \exp\left(-\frac{1}{2} \int d^4x\, d^4y \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip(x-y)} j(x) j(y)\right)$$

$$= \exp\left(-\frac{1}{2} \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} \tilde{j}(p) \tilde{j}(-p)\right)$$

b)
$$= 1 - \frac{1}{2} \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} \overbrace{\tilde{j}(p) \underbrace{\tilde{j}(-p)}_{= \tilde{j}(p)^*}}^{|\tilde{j}(p)|} + \mathcal{O}(\tilde{j}^2)$$

Feynman rules

 $\sim i\tilde{j}(p)$

$1 - \frac{1}{2}$  $\underbrace{-\frac{1}{4}}_{+\frac{1}{8}}$  $+ \dots$

$+\frac{1}{8} \leftarrow \frac{1}{2!}$ from Taylor

$$W[j] = \exp(-\tfrac{1}{2}\lambda)$$

$$P(0 \to 0) = |W[j]|^2 = \exp(-\lambda)$$

$$\langle 0 | S | 0 \rangle = W[j]$$

c)
$$S = \exp\left(-i \int d^4x\, \phi_I^-(x) j(x)\right) \exp\left[-i \int d^4x\, \phi_I^+(x) j(x)\right] W[j]$$

$$\langle k | S | 0 \rangle$$

$$= \langle 0 | a_{\vec{k}} \sqrt{2E_p}\, e^{-i\int d^4x\, \phi_I^-(x) j(x)} e^{-i\int d^4x\, \phi_I^+(x) j(x)} e^{-\frac{1}{2}\lambda} | 0 \rangle$$

$$= \sqrt{2E_{\vec{k}}}\, e^{-\frac{1}{2}\lambda} \quad \langle 0|\, a_{\vec{k}} \exp\left(-i\int d^4x \int \frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_p}} e^{+ip\cdot x} a^\dagger_{\vec{p}}\, j(x)\right) \exp\left(-i\int d^4x \int \frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_p}} e^{-ipx} a_{\vec{p}}\, j(x)\right)|0\rangle$$

$$\langle 0|\exp(\cdots)\ -i\int d^4x \int \frac{d^3\vec{p}}{\cancel{(2\pi)^3}\sqrt{2E_p}} e^{+ipx} \cdot \cancel{(2\pi)^3}\, \delta^{(3)}(\vec{p}-\vec{k})\, j(x)\, |0\rangle$$

$$\langle 0|\ -i\int d^4x \frac{1}{\sqrt{2E_k}} e^{ikx}\, j(x)\ |0\rangle$$

$$= e^{-\frac{1}{2}\lambda} \cdot -i\int d^4x\, e^{ikx}\, j(x)$$

$$\Rightarrow \langle n|S|0\rangle = e^{-\frac{1}{2}\lambda} \frac{1}{n!} (\tilde{j}(k))^n$$

$$\langle k_1 \cdots k_n | S | 0 \rangle$$

$$= \langle k_1 \cdots k_2 |\exp(-i\int d^4x\, \phi_I^-(x)\, j(x))\,|0\rangle \exp(-\tfrac{1}{2}\lambda)$$

$$= \frac{(-i)^n}{n!} \int d^4x_1 \ldots d^4x_n \langle k_1 \ldots k_n |\, \phi_I^-(x_1)\, j(x_1) \ldots \phi_I^-(x_n)\, j(x_n)\, |0\rangle\, W[j]$$

$$= (-i)^n \left(\int d^4x_1\, e^{ik_1x_1} j(x_1)\right) \ldots \left(\int d^4x_n\, e^{ik_nx_n} j(x_n)\right) W[j]$$

$$= (-i)^n\ \tilde{j}(k_1) \cdots \tilde{j}(k_n)\ W[j]$$

$$P(0 \to n) = \frac{1}{n!} \int \frac{d^3k_1}{(2\pi)^3 2E_1} \cdots \int \frac{d^3k_n}{(2\pi)^n 2E_n} |\langle k_1 \ldots k_n | S | 0 \rangle|^2$$

$$= \frac{\lambda^n e^{-\lambda}}{n!} \longrightarrow \text{Poisson distribution}$$

$$\sum_{n=0}^{\infty} P(0 \to n) = e^{-\lambda} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} = 1$$

$$\sum_{n=0}^{\infty} n\, P(0 \to n) = e^{-\lambda} \sum_{n=0}^{\infty} \frac{\lambda^n}{(n-1)!} = \lambda$$