Advanced Quantum Field Theory

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see Peskin

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2.1 Reminder: Path integrals in Quantum Mechanics

Transition amplitude is given by

$$\langle x_b|\mathrm{e}^{-iH(t_b-t_a)}|x_a\rangle_S = \langle x_b, t_b|x_a, t_a\rangle_H \tag{2.1.1}$$

Here we denotes the Schrödinger picture states by $_S$ and Heisenberg picture states by $_H$.

$$\begin{aligned} |x_a, t_a\rangle &= \mathrm{e}^{iHt_a}\,|x_a\rangle & (2.1.2)\\ \hat{H}_a(t_a) &= \mathrm{e}^{iHt_a}\,\hat{x}_S\,\mathrm{e}^{-iHt_a} & (2.1.3)\\ \hat{x}_H(t_a)\,|x_a, t_a\rangle &= \mathrm{e}^{iHt_a}\,\hat{x}_s\,|x_a\rangle = \mathrm{e}^{iHt_a}\,x_a\,|x_a\rangle & \\ &= x_a\mathrm{e}^{iHt_a}\,|x_a\rangle = x_a\,|x_a, t_a\rangle & (2.1.4) \end{aligned}$$

We are looking at time evolution in position space.

It can be calculated directly for free particle with Hamiltonian $H = H_0 = \frac{\hat{p}^2}{2m}$

$$\langle x_b|\mathrm{e}^{-i\frac{\hat{p}^2}{2m}(t_b-t_a)}|x_a\rangle = \sqrt{\frac{m}{2\pi i(t_b - t_a)}}\mathrm{e}^{i(x_b-x_a)^2\frac{m}{2(t_b-t_a)}} \tag{2.1.5}$$

We are going to insert $1 = \int \mathrm{d}^3p\,|p\rangle\,\langle p|$ and use $\langle x|p\rangle$ is the plane wave

For general Hamiltonian $H = H_0 + V$ and $[H_0, V] \neq 0$ the procedure is as following

- divide t into N small intervals $t = N \cdot \epsilon$
- use Lie-Kato-Trotter product formula

$$\mathrm{e}^{A+B} = \lim_{N\to 0}\left(\mathrm{e}^{A/N}\mathrm{e}^{B/N}\right)^N \qquad A, B \in GL(n, \mathbb{C}) \tag{2.1.6}$$

Then we get a functional for path $x(t')$

$$\langle x_b|\mathrm{e}^{-iH(t_b-t_a)}|x_a\rangle = \int \mathcal{D}x\mathrm{e}^{iS[x]/\hbar} \tag{2.1.7}$$

with $S[x] = \int_{t_a}^{t_b}\mathrm{d}t'\left[\frac{m}{2}\dot{x}(t') - V(x(t'))\right]$

[†]see also in Peskin and Schroeder Ch 9.1, Ryder Ch 5.1, L.S.Brown Ch1 1-3

Definition integration measure

$$\mathcal{D}x = D\left[x(t)\right] = \lim_{N\to\infty}\left(\frac{mN}{2\pi i\Delta t}\right)^{N/2} \mathrm{d}x(t_1)\ldots\mathrm{d}x(t_{N-1}) \tag{2.1.8}$$

with $\Delta t = (t_b - t_a)/N$

Pictorially we sum over all paths (i.e. amplitudes). Remember the superposition principle in quantum mechanics!

Classical path comes from Hamilton principle $\delta S = 0$

$$\left.\frac{\delta S\left[x\right]}{\delta x(t)}\right|_{x=x_{\text{cl}}} = 0 \tag{2.1.9}$$

Classical path dominates the transition probability in the limit $\hbar \to 0$. It is the contribution with fewest oscillations in the path integral. Others interfere destructively (averaged out). This is essentially stationary phase approximation.

Example harmonic oscillation

$$L = \frac{m}{2}\left(\dot{x}^2 - \omega^2 x^2\right) \tag{2.1.10}$$

Then the classical path obeys the equation of motion

$$\ddot{x}_{\text{cl}}(t) + \omega^2 x_{\text{cl}}(t) = 0 \tag{2.1.11}$$

Split a general path into classical and fluctuations $x(t) = x_{\text{cl}}(t) + y(t)$. The action turns into

$$S\left[x\right] = S\left[x_{\text{cl}}\right] + \underbrace{\int \mathrm{d}t\, \frac{\delta s}{\delta x(t)}|_{x=x_{\text{cl}}} y(t)}_{=0} + \frac{1}{2}\int \mathrm{d}t \int \mathrm{d}t'\, \frac{\delta^2 S}{\delta x(t)\delta x(t')}|_{x=x_{\text{cl}}} y(t)y(t') + \ldots$$

Then we can factor out the classical path contribution in transition probability

$$\langle x_b|\mathrm{e}^{-iHT}|x_a\rangle = \int \mathcal{D}x \mathrm{e}^{\frac{i}{\hbar}S[x]} = \mathrm{e}^{\frac{i}{\hbar}S[x_a]}\int \mathcal{D}x \mathrm{e}^{\frac{i}{\hbar}S[y]}$$

The integral is to sum over fluctuations around the classical path. Ideally suited to treat fluctuations (quantum and thermal). The explicit calculation for harmonics oscillator can be found in AQT course.

Physical Interpretation the transition probability is the propagator

$$\langle x_b|\mathrm{e}^{-iH(t_b-t_a)}|x_a\rangle = U(x_b t_b; x_a t_a) \tag{2.1.12}$$

Superposition principle takes the form

$$\begin{aligned}
\psi(x_b, t_b) &= \langle x_b|\psi(t_b)\rangle = \langle x_b|\mathrm{e}^{-iHt_b}|\psi\rangle \\
&= \int \mathrm{d}x_a \langle x_b|\mathrm{e}^{-iH(t_b-t_a)}|x_a\rangle \langle x_a|\mathrm{e}^{-iHt_a}|\psi\rangle \\
&= \int \mathrm{d}x_a\, U(x_b t_b; x_a t_a) \underbrace{\langle x_a|\psi(t_a)\rangle}_{\psi(x_a,t_a)}
\end{aligned}$$

2.2 Quantum Mechanical Path Integrals and External Forces

Definition Time evolution operator in path integral representation

$$\begin{aligned} U(x_b, t_b; x_a, t_a) &= \langle x_b, t_b | x_a, t_a \rangle \\ &= \int \mathcal{D}x(t) \mathrm{e}^{iS[x]} \\ &= \int \mathcal{D}x(t) \mathrm{e}^{i \int_{t_a}^{t_b} \mathrm{d}t L(x, \dot{x})} \end{aligned} \tag{2.2.1}$$

Add coupling to an external force (source) $f(t)$

$$L = L_0 + f(t)x(t) \tag{2.2.2}$$

Definition functional derivative with respect to $f(t)$

$$\frac{\delta}{\delta f(t)} \int \mathrm{d}t' \, f(t')g(t') = g(t) \tag{2.2.3}$$

For a general functional of external forces

$$F[f] = \int \mathrm{d}t_1 \, K_1(f_1)f(t_1) + \frac{1}{2!} \int \mathrm{d}t_1 \, \mathrm{d}t_2 \, K_2(t_1, t_2)f(t_1)f(t_2) + \dots \tag{2.2.4}$$

with the $K_n(t_1, \dots, t_n)$ totally symmetric in the arguments $t_1, \dots, t_n$, since antisymmetric contributions drop automatically upon integration. The functional derivatives is then

$$\frac{\delta F}{\delta f(t)} = K_1(t) + \int \mathrm{d}t_2 \, K_2(t, t_2)f(t_2) + \frac{1}{2!} \int \mathrm{d}t_2 \, \mathrm{d}t_1 \, K_3(t, t_2, t_3)f(t_2)f(t_3) + \dots \tag{2.2.5}$$

Consider functional derivative of time evolution operator

$$\begin{aligned} \frac{1}{i} \frac{\delta}{\delta f(t)} \langle x_b, t_b | x_a, t_a \rangle^f &= \int \mathcal{D}x \exp\left(i \int_{t_a}^{t_b} \mathrm{d}t' \, L_0 \right) \frac{1}{i} \frac{\delta}{\delta f(t)} \exp\left(i \int_{t_a}^{t_b} \mathrm{d}t' \, f(t')x(t') \right) \\ &= \int \mathcal{D}x \, x(t) \exp\left(i \int_{t_a}^{t_b} \mathrm{d}t' \, [L_0 + f(t')x(t')] \right) \end{aligned}$$

To split the path integral into two parts, time before and after t (superposition principle). M steps before t and $N - M - 1$ steps after t. The integration over $x(t)$ is to sum over all possible positions at time t.

$$\int_{t_a}^{t_b} \mathcal{D}x = \int \mathrm{d}x(t) \int_t^{t_b} \mathcal{D}x \int_{t_a}^{t} \mathcal{D}x$$

Then

$$\begin{aligned} \frac{1}{i} \frac{\delta}{\delta f(t)} \langle x_b, t_b | x_a, t_a \rangle^f &= \int \mathrm{d}x(t) \underbrace{\int \mathcal{D}x \exp\left(i \int_t^{t_b} \mathrm{d}t' \, (L_0 + fx) \right)}_{N-M-1 \text{ factor}} x(t) \underbrace{\int \mathcal{D}x \exp\left(i \int_{t_a}^{t_b} \mathrm{d}t' \, (L_0 + fx) \right)}_{M \text{ factor}} \\ &= \int \mathrm{d}x(t) \, \langle x_b, t_b | x(t), t \rangle^f \, x(t) \, \langle x(t), t | x_a, t_a \rangle^f \end{aligned}$$

Here $x(t)$ is an eigenvalue, not an operator, so we write $x(t) = \bar{x}$ with

$$\int \mathrm{d}\bar{x}\, |\bar{x}, t\rangle\, \bar{x}\, \langle \bar{x}, t| = \int \mathrm{d}\bar{x}\, \bar{x}\, \langle \bar{x}, t|\bar{x}, t\rangle = x(t)$$

the Heisenberg operator.

We get

$$\frac{1}{i}\frac{\delta}{\delta f(t)}\langle x_b t_b | x_a t_a\rangle^f = \langle x_b, t_b | x(t) | x_a, t_a\rangle \tag{2.2.6}$$

The functional derivative with respect to the external force $f(t)$ which couples to $x(t)$, to "insert" the operator $x(t)$ into the matrix element.

Now consider *two* functional derivatives with $t_b \geq t, t' \geq t_a$

$$\frac{1}{i}\frac{\delta}{\delta f(t)}\frac{1}{i}\frac{\delta}{\delta f(t')}\langle x_b, t_b | x_a, t_a\rangle^f = \int \mathcal{D}x\, x(t) x(t') \mathrm{e}^{i\int_{t_a}^{t_b} \mathrm{d}t' [L_0 + f\cdot x]} \tag{2.2.7}$$

In general

$$\begin{aligned}
\frac{1}{i}\frac{\delta}{\delta f(t)}\frac{1}{i}\frac{\delta}{\delta f(t')}\langle x_b, t_b | x_a, t_a\rangle^f &= \frac{1}{i}\frac{\delta}{\delta f(t)}\langle x_b, t_b | x(t') | x_a, t_a\rangle^f \\
&= \frac{1}{i}\frac{\delta}{\delta f(t)}\int \mathrm{d}\bar{x}'\, \langle x_b, t_b | \bar{x}', t\rangle^f\, \bar{x}'\, \langle \bar{x}', t' | x_a, t_a\rangle^f \\
&= \int \mathrm{d}\bar{x}' \left(\frac{1}{i}\frac{\delta}{\delta f(t)}\langle x_b, t_b | \bar{x}', t\rangle^f\right) \bar{x}'\, \langle \bar{x}', t' | x_a, t_a\rangle^f \\
&\quad + \int \mathrm{d}\bar{x}'\, \langle x_b, t_b | \bar{x}', t\rangle^f\, \bar{x}' \left(\frac{1}{i}\frac{\delta}{\delta f(t)}\langle \bar{x}', t' | x_a, t_a\rangle^f\right)
\end{aligned}$$

Then transition amplitudes only depend on the time interval, where the external forces actually act

$$\frac{1}{i}\frac{\delta}{\delta f(t)}\langle x_b, t_b | \bar{x}', t'\rangle^f = \begin{cases} \langle x_b, t_b | x(t) | \bar{x}', t'\rangle^f & t > t' \\ 0 & t < t' \end{cases}$$

$$\frac{1}{i}\frac{\delta}{\delta f(t)}\langle \bar{x}', t' | x_b, t_b\rangle^f = \begin{cases} 0 & t > t' \\ \langle \bar{x}', t' | x(t) | x_b, t_b\rangle^f & t < t' \end{cases}$$

Eliminate $\bar{x}'$ integration as before

$$\frac{1}{i}\frac{\delta}{\delta f(t)}\frac{1}{i}\frac{\delta}{\delta f(t')}\langle x_b, t_b | x_a, t_a\rangle^f = \langle x_b, t_b | T\left[x(t), x(t')\right] | x_a, t_a\rangle^f \tag{2.2.8}$$

This can be easily generalised

$$\frac{1}{i}\frac{\delta}{\delta f(t')}\frac{1}{i}\frac{\delta}{\delta f(t'')}\ldots\langle x_b, t_b | x_a, t_a\rangle^f = \langle x_b, t_b | T\left[x(t')x(t'')\ldots\right] | x_a, t_a\rangle^f \tag{2.2.9}$$

$$= \int \mathcal{D}x\, x(t')x(t'')\ldots \exp\left(i\int_{t_a}^{t_b} \mathrm{d}t\, (L_0(x,\dot{x}) + f(t)x(t))\right) \tag{2.2.10}$$

Interpretation the addition of external force to the Lagrangian of a path integral produces a "generating functional" for a matrix element which contain time-ordered products of arbitrary many position operators. The functional derivative is just a trick to generate the matrix element in the propagator. This is called Schwinger source theory.

Now we can set $f = 0$

$$\langle x_b, t_b | T\left[x(t')x(t'')\dots\right] | x_a, t_a\rangle^{f=0} = \int \mathcal{D}x x(t')x(t'')\dots \exp\left(i\int_{t_a}^{t_b} L_0(x,\dot{x})\right) \tag{2.2.11}$$

or in case of an arbitrary generating functional $F[x]$

$$\langle x_b, t_b | T\left\{F[x]\right\} | x_a, t_a\rangle^{f=0} = \int \mathcal{D}x F[x] \exp\left(i\int_{t_a}^{t_b} L_0(x,\dot{x})\right) \tag{2.2.12}$$

for example

$$\langle x_b, t_b | x_a, t_a\rangle^{f} = \langle x_b, t_b | T \mathrm{e}^{i\int_{t_a}^{t_b} \mathrm{d}t' q(t') f(t')} | x_a, t_a\rangle^{f=0}$$

2.3 Scalar Field Theories and Feynman Rules

We are going to generalise the concept of path integral to field theories. Simplest example is a neutral (real) scalar field $\phi(x)$ coupled to an external classical "current"/source $j(x)$

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi\right)^2 - \frac{1}{2}m^2\phi^2 + \phi j(x) = \mathcal{L}_0 + \phi(x) j(x) \tag{2.3.1}$$

Proceed along the lines of quantum mechanical path integral with external forces

- construct a generating functional
- using the functional-integral-representation derive expressions for the correlation functions $\widetilde{=}$ Feynman rules

Sufficient to consider vacuum-to-vacuum amplitudes in the presence of $j(x)$. Consider $t_a = -\infty(1 - i\epsilon)$, $t_b = +\infty(1 - i\epsilon)$ and $j(x) = 0$ for $t \mapsto \pm\infty$

$$\langle 0|0\rangle^j = \int \mathcal{D}x\, \phi(x) \exp\left(i\int \mathrm{d}^4x\, \mathcal{L}(\phi, \partial_\mu\phi)\right)$$

where $\mathcal{D}\phi(x)$ in the generalization $\mathcal{D}x \mapsto \mathcal{D}(\text{field})$

Compute $\langle 0|0\rangle^j$ (exact for a free field theory). First to solve with classical action

$$\delta\int \mathrm{d}^4x\left[\frac{1}{2}\left(\partial_\mu\phi_{\mathrm{cl}}\right)^2 - \frac{1}{2}m^2\phi_{\mathrm{cl}}^2 + \phi_{\mathrm{cl}} j\right] = 0$$

$$\left(\partial^2 + m^2\right)\phi_{\mathrm{cl}}(x) = j(x)$$

Solution

$$\phi_{\mathrm{cl}}(x) = i\int \mathrm{d}^4y\, D_F(x-y) j(y) \tag{2.3.2}$$

since Feynman-propagator is the Green's function of the KG operator.

$$\left(\partial^2 + m^2\right) D_F(x-y) = -i\delta^{(4)}(x-y) \tag{2.3.3}$$

To define the "fluctuation" field $\phi'(x)$ via $\phi(x) = \phi_{\text{cl}}(x) + \phi'(x)$. Then the Lagrangian is

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\left(\partial_\mu\phi_{\text{cl}} + \partial_\mu\phi'\right)^2 - \frac{m^2}{2}\left(\phi_{\text{cl}} + \phi'\right)^2 + \left(\phi_{\text{cl}} + \phi'\right)\cdot j(x) \\ &= \mathcal{L}_{\text{cl}} + \mathcal{L}' + \left[(\partial_\mu\phi_{\text{cl}})(\partial^\mu\phi') - m^2\phi_{\text{cl}}\phi' + j\phi\right]\end{aligned}$$

after integration by parts and using equation of motion the last part vanishes. Then ϕ' (per construction) is a free field. Thus

$$\langle 0|0\rangle^j = \int \mathcal{D}\phi' \exp\left(i\int \mathrm{d}^4x\,(\mathcal{L}_{\text{cl}} + \mathcal{L}')\right) = \mathrm{e}^{iS_{\text{cl}}}\,\langle 0|0\rangle^{j=0} \tag{2.3.4}$$

On the other hand, iS_{cl} can be rewritten as

$$\begin{aligned}iS_{\text{cl}} &= i\int \mathrm{d}^4x\left[\frac{1}{2} - \frac{m}{2}\phi_{\text{cl}}^2 + \phi_{\text{cl}}j\right] \\ &= i\int \mathrm{d}^4x\Big[-\frac{1}{2}\phi_{\text{cl}}\underbrace{\left(\partial^2 + m^2\right)\phi_{\text{cl}}}_{=j\text{ from e.o.m.}} + \phi_{\text{cl}}j\Big] \\ &= \frac{i}{2}\int \mathrm{d}^4x\,\phi_{\text{cl}}(x)j(x) \\ &= -\frac{1}{2}\int \mathrm{d}^4x\,\mathrm{d}^4y\; j(x)D_F(x-y)j(y)\end{aligned}$$

Definition generating functional in the free scalar field theory

$$\begin{aligned}W_0[j] &= \frac{Z[j]}{Z[j=0]} = \frac{\langle 0|0\rangle^j}{\langle 0|0\rangle^{j=0}} \\ &= \exp\left(-\frac{1}{2}\int \mathrm{d}^4x\,\mathrm{d}^4y\; j(x)D_F(x-y)j(y)\right)\end{aligned} \tag{2.3.5}$$

Connection to the S-matrix

$$\begin{aligned}S &= U(-\infty, \infty) \\ &= \lim_{t_i\mapsto-\infty(1-i\epsilon)}\;\lim_{t_f\mapsto+\infty(1-i\epsilon)} T\exp\left(-i\int_{t_i}^{t_f}\mathrm{d}t\,\mathcal{H}_{\text{int}}(t)\right) \\ &= T\exp\left(-i\int \mathrm{d}^4x\,\mathcal{H}_{\text{int}}(x)\right) \\ &= T\exp\left(i\int \mathrm{d}^4x\,\phi(x)j(x)\right) \\ &= T\sum_{n=0}^{\infty}\frac{i^n}{n!}\int \mathrm{d}^4x_1\ldots\mathrm{d}^4x_n\; j(x_1)\ldots j(x_n)\phi(x_1)\ldots\phi(x_n)\end{aligned}$$

$$\langle 0|S|0\rangle = \sum_{n=0}^{\infty}\frac{i^n}{n!}\int \mathrm{d}^4x_1\ldots\mathrm{d}^4x_n\; j(x_1)\ldots j(x_n)G_n^0(x_1,\ldots,x_n) \tag{2.3.6}$$

where $G_n^0(x_1,\ldots,x_n) = \langle 0|T[\phi(x_1)\ldots\phi(x_n)]|0\rangle$ the n-point-Green's function of the free scalar field theory.

We can calculate the Green's function as for the quantum mechanical path integral with external forces via functional derivatives of the generating functional

$$W_0[j] = \frac{\int \mathcal{D}\phi \exp\Big(i \int \mathrm{d}^4x(\mathcal{L}_0(\phi, \partial_\mu\phi) + \phi j)\Big)}{\int \mathcal{D}\phi \exp\Big(i \int \mathrm{d}^4x(\mathcal{L}_0(\phi, \partial_\mu\phi))\Big)} \tag{2.3.7}$$

$$\begin{aligned} G_n^0(x_1, \dots, x_n) &= \frac{1}{i}\frac{\delta}{\delta j(x_1)} \dots \frac{1}{i}\frac{\delta}{\delta j(x_n)} W_0[j]|_{j=0} \\ &= \frac{\int \mathcal{D}\phi \exp\Big(i \int \mathrm{d}^4x(\mathcal{L}_0(\phi, \partial_\mu\phi))\Big)\phi(x_1)\dots\phi(x_n)}{\int \mathcal{D}\phi \exp\Big(i \int \mathrm{d}^4x(\mathcal{L}_0(\phi, \partial_\mu\phi))\Big)} \\ &= \langle 0|T\phi(x_1)\dots\phi(x_n)|0\rangle \end{aligned} \tag{2.3.8}$$

The central result here is that these three things are closely related: S-matrix ↔ Green's function ↔ Path integral

Special case, two-point function

$$\begin{aligned} G_2^0(x_1, x_2) &= \langle 0|T\phi(x_1)\phi(x_2)|0\rangle \\ &= \frac{1}{i}\frac{\delta}{\delta j(x_1)}\frac{1}{i}\frac{\delta}{\delta j(x_2)} \exp\left[-\frac{1}{2}\int \mathrm{d}^4x\,\mathrm{d}^4y\, j(x)D_F(x-y)j(y)\right]\Bigg|_{j=0} \end{aligned}$$

The exponential is the generating functional.

$$\begin{aligned} &= -\frac{1}{\delta j(x_1)}\left[-\frac{1}{2}\int \mathrm{d}^4y\, D_F(x_2-y)j(y) - \frac{1}{2}\int \mathrm{d}^4x\, j(x)D_F(x-x_2)\right] W_0[j]\Bigg|_{j=0} \\ &= D_F(x_1 - x_2) \end{aligned}$$

Four-point function. Use abbreviations $\phi_i = \phi(x_i)$, $j_x = j(x)$, $D_{x_i} = D_F(x - x_i)$ and integration over the repeated index is implied.

$$\begin{aligned} G_4^0(x_1, x_2, x_3, x_4) &= \langle 0|T\phi_1\phi_2\phi_3\phi_4|0\rangle \\ &= \left(\frac{1}{i}\right)^4 \frac{\delta}{\delta j_1}\frac{\delta}{\delta j_2}\frac{\delta}{\delta j_3}\frac{\delta}{\delta j_4}\ \mathrm{e}^{-\frac{1}{2}j_xD_{xy}j_y}\Big|_{j=0} \end{aligned}$$

Since $D(x-y) = D(y-x)$, one can combine two integrals after substituition into one.

$$\begin{aligned} &= \frac{\delta}{\delta j_1}\frac{\delta}{\delta j_2}\frac{\delta}{\delta j_3}\left[-j_{\tilde{x}}D_{\tilde{x}4}\right] \mathrm{e}^{-\frac{1}{2}j_xD_{xy}j_y}\Big|_{j=0} \\ &= \frac{\delta}{\delta j_1}\frac{\delta}{\delta j_2}\left[-D_{34} + j_{\tilde{x}}D_{\tilde{x}4}j_{\tilde{y}}D_{\tilde{y}3}\right]\mathrm{e}^{-\frac{1}{2}j_xD_{xy}j_y}|_{j=0} \\ &= \frac{\delta}{\delta j_1}\left[D_{34}j_{\tilde{x}}D_{\tilde{x}2} + D_{24}j_{\tilde{y}}D_{\tilde{y}3} + j_{\tilde{x}}D_{\tilde{x}4}D_{23} + \dots\right]\mathrm{e}^{-\frac{1}{2}j_xD_{xy}j_y}|_{j=0} \end{aligned}$$

Dots are the terms contains j.

$$= D_{34}D_{12} + D_{24}D_{13} + D_{14}D_{23}$$

$$\begin{aligned} G_4^0(x_1, x_2, x_3, x_4) =& D_F(x_3 - x_4)D_F(x_1 - x_2) \\ &+ D_F(x_2 - x_4)D_F(x_1 - x_3) + D_F(x_1 - x_4)D_F(x_2 - x_3) \end{aligned} \tag{2.3.9}$$

So we recovered Wick's theorem in path integral representation.

$$\frac{1}{N}\int \mathcal{D}\phi e^{i\int d^4x \mathcal{L}_0[\phi]}\phi(x_1)\phi(x_2) = \wick{\c1\phi(x_1)\c1\phi(x_2)}$$

with $N = \int \mathcal{D}\phi e^{i\int d^4x \mathcal{L}_0[\phi]}$

$$\frac{1}{N}\int \mathcal{D}\phi e^{i\int d^4x \mathcal{L}_0[\phi]}\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) = \wick{\c1\phi(x_1)\c1\phi(x_2)\c2\phi(x_3)\c2\phi(x_4)} + \wick{\c1\phi(x_1)\c2\phi(x_2)\c1\phi(x_3)\c2\phi(x_4)} + \wick{\c1\phi(x_1)\c2\phi(x_2)\c2\phi(x_3)\c1\phi(x_4)}$$

Interacting field Consider now an interacting field theory, e.g. ϕ^4

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 + \phi j \\ &= \mathcal{L}_0 - \frac{\lambda}{4!}\phi^4 + \phi j\end{aligned}$$

The generating functional or the normalized vacuum-to-vacuum transition amplitude is given

$$W[j] = \frac{\langle 0|0\rangle^j}{\langle 0|0\rangle^0} = \sum_{n=0}^{\infty}\frac{i^n}{n!}\int d^4x_1 \dots d^4x_n\, j(x_1)\dots j(x_n) G_n(x_1,\dots,x_n) \tag{2.3.10}$$

$$\langle 0|0\rangle^j = \int \mathcal{D}\phi \exp\left\{i\int d^4x\left[\frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 + \phi j\right]\right\} \tag{2.3.11}$$

thus

$$G_n(x_1,\dots,x_n) = \frac{\int \mathcal{D}\phi\phi(x_1)\dots\phi(x_n)\exp\left\{i\int d^4x\, \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4\right\}}{\int \mathcal{D}\phi \exp\left\{i\int d^4x\, \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4\right\}} \tag{2.3.12}$$

Our aim is to get perturbative expansion of G_n and thus for the S-matrix. For this purpose, expanse $\lambda\phi^4$ term!

$$\sum_{n=0}^{\infty}\int \mathcal{D}\phi\phi(x_1)\dots\phi(x_n)\frac{1}{n!}\left(\int d^4y\, \frac{-i\lambda}{4!}\phi^4(y)\right)^n e^{i\int d^4x\mathcal{L}_0}$$

This expansion is equivalent to the Dyson-Wick expansion of the S-matrix in powers of $\mathcal{H}_{\text{int}}$.

Two-point Green's function

Note that denominator cancels the vacuum diagrams, so we only have perturbation theory for connected graphs.

- $\mathcal{O}(\lambda^0)$

$$G_2^0 = \wick{\c1\phi(x_1)\c1\phi(x_2)} = D_F(x_1 - x_2) \tag{2.3.13}$$

- $\mathcal{O}(\lambda^1)$

$$\begin{aligned}G(x_1,x_2) &= \frac{\int \mathcal{D}\phi\phi(x_1)\phi(x_2)e^{i\int d^4x(\mathcal{L}_0(x) - \frac{\lambda}{4!}\phi^4(x))}}{\int \mathcal{D}\phi e^{i\int d^4x(\mathcal{L}_0(x) - \frac{\lambda}{4!}\phi^4(x))})} \\ &= \frac{\int \mathcal{D}\phi\phi(x_1)\phi(x_2)\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{i\lambda}{4!}\int d^4y\,\phi^4(y)\right)^n e^{i\int d^4x\mathcal{L}_0(x)}}{\int \mathcal{D}\phi\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{i\lambda}{4!}\int d^4y\,\phi^4(y)\right)^n e^{i\int d^4x\mathcal{L}_0(x)}}\end{aligned}$$

$$
\begin{aligned}
G_2^1(x_1,x_2) &= \frac{1}{N}\int \mathcal{D}\phi\phi(x_1)\phi(x_2)\left(-\frac{i\lambda}{4!}\int \mathrm{d}^4y\,\phi^4(y)\right)\mathrm{e}^{i\int \mathrm{d}^4x\mathcal{L}_0(x)}\\
&= \frac{1}{N}\left(\frac{-i\lambda}{4!}\right)\int \mathcal{D}\phi\phi(x_1)\phi(x_2)\phi(y)\phi(y)\phi(y)\phi(y)\mathrm{e}^{i\int \mathrm{d}^4x\mathcal{L}_0(x)}\\
&= \frac{1}{N}\left(\frac{-i\lambda}{4!}\right)\int \mathrm{d}^4y\Big[\phi(x_1)\phi(x_2)\Big(\phi(y)\phi(y)\phi(y)\phi(y)+\phi(y)\phi(y)\phi(y)\phi(y)+\phi(y)\phi(y)\phi(y)\phi(y)\Big)\\
&\quad+\phi(x_1)\phi(x_2)\phi(y)\phi(y)\phi(y)\phi(y)+11\text{ more terms}\Big]\\
&= \frac{1}{N}\left\{-\frac{i\lambda}{8}\int \mathrm{d}^4y\,\phi(x_1)\phi(x_2)\Big(\phi(y)\phi(y)\Big)^2-\frac{i\lambda}{2}\phi(x_1)\phi(y)\phi(x_2)\phi(y)\phi(y)\phi(y)\right\}
\end{aligned}
$$

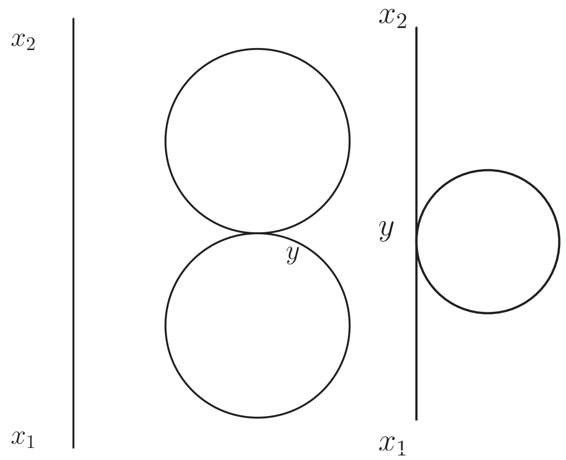


Figure 2.1: Feynman diagrams

General case: common (position space) Feynman rules for interacting Green's functions $G_n(x_1,\dots,x_n)$

2.4 Photon Propagator in Path Integrals*

How can we derive the Feynman rule for photon propagator?

$$\frac{-ig^{\mu\nu}}{k^2+i\epsilon} \tag{2.4.1}$$

What is the problem? The functional integral $\int \mathcal{D}A_\mu \mathrm{e}^{iS[A]}$ incorporated the action

$$
\begin{aligned}
S &= \int \mathrm{d}^4x\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right)\\
&= \frac{1}{2}\int \mathrm{d}^4x\,A_\mu(x)\left(\partial^2g^{\mu\nu}-\partial^\mu\partial^\nu\right)A_\nu(x)\\
&= \frac{1}{2}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\tilde{A}_\mu(k)(-k^2g^{\mu\nu}+k^\mu k^\nu)\tilde{A}_\nu(-k)
\end{aligned}
\tag{2.4.2}
$$

This expression has a great deal of (interconnected) problems

*see also Ryder, Chap 7.1-2; P & S, Chap 9.4

1. Assume the photon propagator $D_{\mu\nu}(x-y)$ to be solution of
$$\left(\partial^2 g^{\mu\nu} - \partial^\mu\partial^\nu\right) D_{\nu\lambda}(x-y) = i g^\mu_\lambda \delta^{(4)}(x-y)$$
multiply with partial derivative
$$\left(\partial^\nu\partial^2 - \partial^2\partial^\nu\right) D_{\nu\lambda} = 0 \cdot \partial^\nu D_{\nu\lambda} \neq i\partial^\nu\delta^{(4)}(x-y)$$
$D_{\nu\lambda}$ has no inverse (formally singular). The same holds in momentum space.
2. We need this inverse for the derivation of the generating function with external currents.
3. $\frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\tilde{A}_\mu(k)(-k^2 g^{\mu\nu} + k^\mu k^\nu)\tilde{A}_\nu(-k)$ vanished for all $\tilde{A}_\nu(k) = k_\mu\alpha(k)$ all these field configurations have the same weight 1 in $\int \mathcal{D}A_\mu e^{iS[A]}$. It will terribly diverge.
4. These configurations correspond to gauge transformation $\mathcal{L}$ (also S) invariant under $A_\mu(x) \mapsto A_\mu(x) + \frac{1}{e}\partial_\mu\alpha(x)$. Thus redundant (path-integral) integration over gauge equivalent configurations.

Schematically to split each A_μ into some fixed $\bar{A}_\mu$ and a gauge transformation α
$$A_\mu(x) \mapsto \bar{A}_\mu(x), \alpha(x)$$
then the path integral is given by
$$Z = \int \mathcal{D}A_\mu e^{iS} \sim \int \mathcal{D}\bar{A}_\mu e^{iS} \int \mathcal{D}\alpha$$
since S is gauge invariant, i.e. independent of α.

The divergence stems from $\int \mathcal{D}\alpha$ which cancels in the ratio $W[j] = \frac{Z[j]}{Z[0]}$

The aim is to factorize of the gauge part in the path integral via a "smart integration" of the gauge-fixing as a constraint. This is called Faddeev-Popov method.

2.4.1 Factorization of Constraints*

Externally simple example
$$I = \int dx\, dy\, e^{-(x^2+y^2)}$$
it is rotation invariant, use polar coordinates
$$= \int d\theta\, dr\, r e^{-r^2}$$
$\int d\theta = 2\pi$ corresponds to $\int \mathcal{D}\alpha$ in the path integral.

A more general expansion for this separation
$$I = \int d\theta' \int dr \int d\theta\, r e^{-r^2} \delta(\theta)$$
The delta function reduced the integration path to one along the x-axis ($\theta = 0$).

*Ryder, Chap 7.2

A more general path

$$f(\theta) = y\cos\theta - x\sin\theta = 0 \tag{2.4.3}$$

i.e. $\theta \neq 0$.

How can we include this constraint in path integral?

$$\delta(f(\theta)) = \sum_i \left|\frac{\partial f(\theta_i)}{\partial\theta}\right|^{-1} \delta(\theta - \theta_i) \tag{2.4.4}$$

with θ_i the roots of $f(\theta)$.

$$\theta_1 = \arctan\left(\frac{y}{x}\right) \quad \theta_2 = \pi + \arctan\left(\frac{y}{x}\right)$$
$$\left|\frac{\partial f}{\partial\theta}\right| = y\sin\theta + x\cos\theta = r = \left|\frac{\partial f}{\partial\theta}\right|_{\theta_1,\theta_2}$$

thus

$$\delta(f(\theta)) = \frac{1}{r}\left(\delta(\theta - \theta_1) + \delta(\theta - \theta_2)\right)$$
$$\int \delta(f(\theta))\mathrm{d}\theta = \frac{2}{r} = \frac{2}{\sqrt{x^2+y^2}}$$

rewrite this as $\Delta(r)\int \delta(f(\theta))\,\mathrm{d}\theta = 1$, i.e.

$$\Delta(r) = \frac{r}{2} = \frac{\sqrt{x^2+y^2}}{2} \tag{2.4.5}$$

Note that $f(\theta)$ can simply be obtained by a rotation from y-axis

$$y' = y\cos\theta - x\sin\theta$$
$$x' = x\cos\theta + y\sin\theta$$
$$x^2 + y^2 = x'^2 + y'^2$$

$$\Delta(r)\int \mathrm{d}\theta\,\delta(f(\theta)) = 1$$
$$\Delta\left(\sqrt{x'^2+y'^2}\right)\int \mathrm{d}\theta\,\delta(y') = 1$$

remember $y' = f(\theta) = y\cos\theta - x\sin\theta$.

Insert this unity into $I = \int \mathrm{d}x\,\mathrm{d}y\,\mathrm{e}^{-(x^2+y^2)}$

$$I = \int \mathrm{d}\theta \int \mathrm{d}x'\,\mathrm{d}y'\,\mathrm{e}^{-(x'^2+y'^2)}\Delta\left(\sqrt{x'^2+y'^2}\right)\delta(y')$$

It exhibits separation of variables made possible by the rotation invariance of the integral. The integral $\int \mathrm{d}x'\,\mathrm{d}y' \ldots$ is independent of θ, so $\int \mathrm{d}\theta$ is simply an overall multiplication factor in integral.

Finally $\Delta(r)$ can also be rewritten as

$$\begin{aligned}\Delta(r)^{-1} &= \int \mathrm{d}\theta\,\delta(f(\theta)) \\ &= \int \delta(f(\theta))\det\left|\frac{\mathrm{d}\theta}{\mathrm{d}f}\right|\mathrm{d}f \\ &= \det\left|\frac{\mathrm{d}\theta}{\mathrm{d}f}\right|_{f=0}\end{aligned} \tag{2.4.6}$$

then we have the functional determinant

$$\Delta(r) = \det \left|\frac{\mathrm{d}f}{\mathrm{d}\theta}\right|_{f=0} \tag{2.4.7}$$

2.4.2 Gauge Fixing in a Path Integral

Consider a gauge transformation

$$A_\mu(x) \mapsto A_\mu^\alpha(x) = A_\mu(x) + \frac{1}{e}\partial_\mu \alpha(x) \tag{2.4.8}$$

and a gauge-fixing condition

$$F[A_\mu] = 0 \tag{2.4.9}$$

In analogy to the condition 2.4.6

$$\Delta_F^{-1}[A_\mu] = \int \mathcal{D}\alpha \delta\left(F[A_\mu^\alpha]\right) \tag{2.4.10}$$

where δ is a δ-functional.

$\Delta_F^{-1}[A_\mu]$ is gauge invariant

$$\Delta_F^{-1}\left[A_\mu^{\alpha'}\right] = \int \mathcal{D}\alpha\delta\left(F[A_\mu^{\alpha+\alpha'}]\right)$$

group invariant measure $\alpha'' = \alpha + \alpha'$

$$= \int \mathcal{D}\alpha''\delta\left(F[A_\mu^{\alpha''}]\right) = \Delta_F^{-1}[A_\mu]$$

Insert $1 = \Delta_F[A_\mu] \int \mathcal{D}\alpha\delta(F[A_\mu^\alpha])$ into the path integral Z

$$\begin{aligned} Z &= \int \mathcal{D}A_\mu \mathrm{e}^{iS[A_\mu]} \\ &= \int \mathcal{D}A_\mu \Delta_F[A_\mu] \int \mathcal{D}\alpha\delta\left(F[A_\mu^\alpha]\right)\mathrm{e}^{iS[A_\mu]} \end{aligned}$$

using the gauge transformation $A_\mu \mapsto A_\mu^\alpha$

$$= \int \mathcal{D}A_\mu^\alpha \Delta[A_\mu^\alpha] \int \mathcal{D}\alpha\delta\left(F[A_\mu^\alpha]\right)\mathrm{e}^{iS[A_\mu^\alpha]}$$

rename $A_\mu^\alpha = A_\mu$

$$= \int \mathcal{D}\alpha \int \mathcal{D}A_\mu \Delta_F[A_\mu]\delta\left(F[A_\mu]\right)\mathrm{e}^{iS[A_\mu]} \tag{2.4.11}$$

S and $\Delta_F(\dots)$ are gauge invariant and $\mathcal{D}A_\mu^\alpha = \mathcal{D}A_\mu$, since

$$\begin{aligned} Z &= \int \mathcal{D}A_\mu \mathrm{e}^{iS[A_\mu]} \\ &= \int \mathcal{D}A_\mu^{\tilde{\alpha}} \mathrm{e}^{iS[A_\mu^{\tilde{\alpha}}]} \\ &= \int \mathcal{D}A_\mu^{\tilde{\alpha}} \mathrm{e}^{iS[A_\mu]} \end{aligned}$$

or $A_\mu \mapsto A_\mu^{\tilde{\alpha}}$ is just a shift of integration. Integrand of equation 2.4.11 independent of gauge α. Thus $\int \mathcal{D}\alpha$ can be moved in front and is therefore already separated!

Now use

$$\Delta_F[A_\mu] = \det\left|\frac{\delta F}{\delta\alpha}\right|_{F=0} \tag{2.4.12}$$

We will apply gauge-fixing conditions of the form (generalization of Lorenz gauge)

$$F[A_\mu] = \partial^\mu A_\mu + C(x) = 0 \tag{2.4.13}$$

with $c(x)$ any scalar function. Thus

$$F[A_\mu^\alpha] = F[A_\mu] + \frac{1}{e}\partial^2\alpha(x) \tag{2.4.14}$$

$$\Delta_F[A_\mu] = \det\left|\frac{1}{e}\partial^2\right| \tag{2.4.15}$$

independent of A_μ. $\Delta_F[A_\mu]$ can be moved in front of the path integral. (Not valid in the non-abelian case.)

Since Z and in the end, the physics does not depend on value of $C(x)$, we are free to have a linear combination of different $C(x)$. Now multiply equation (2.4.11) with a weight $\int \mathcal{D}C \exp\left(\frac{-i}{2\xi}C^2\,\mathrm{d}x\right)$. After integrating $\int \mathcal{D}C$) and using $\delta\left(F[A_\mu]\right) = \delta\left(\partial^\mu A_\mu - C(x)\right)$

$$Z = N\int \mathcal{D}A_\mu \exp\left\{i\int \mathrm{d}^4x\left(\mathcal{L} - \frac{1}{2\xi}\left(\partial_\mu A^\mu\right)^2\right)\right\} \tag{2.4.16}$$

$$\mathcal{L}_{\text{eff}} = \mathcal{L} + \mathcal{L}_{\text{GF}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 \tag{2.4.17}$$

with the gauge-fixing term in Lagrangian.

Now consider an n-point Green's function a la

$$\langle 0|T\left(O[A_\mu]\right)|0\rangle$$

Coming back to our starting problem, to find photon propagator

$$\left(-k^2 g_{\mu\nu} + \left(1 - \frac{1}{\xi}\right)k_\mu k_\nu\right)\tilde{D}_F^{\nu\lambda}(k) = i\delta_\mu^\lambda$$

possess the solution

$$\tilde{D}_F^{\mu\nu}(k) = \frac{-i}{k^2 + i\epsilon}\left(g^\mu\nu - (1-\xi)\frac{k^\mu k^\nu}{k^2}\right) \tag{2.4.18}$$

Most commonly used gauges Feynman gauge $\xi = 1$ and Landau gauge $\xi = 0$

2.5 Path Integral Quantization of Fermion Fields*

The fermionic fields anti-commute, therefore the integration over (complexed-valued) fermion fields is non-trivial.

*see also Ryder Chap 6,7, P& S, Chap 9.5

2.5.1 Grassmann Algebra*

Consider an algebra $\mathcal{G}_n$ generated by n anticommuting generators $\theta_1, \dots, \theta_n$.

$$\theta_i\theta_j + \theta_j\theta_i = \left\{\theta_i, \theta_j\right\} = 0 \tag{2.5.1}$$

They are variables, not field operators (yet)!

Implicitly $\theta_i^2 = 0$ for all i Thus a basis is given monomials (polynomial of first order of one term) $1; \theta_1, \dots, \theta_n; \theta_1\theta_2, \dots \theta_{n-1}\theta_n; \dots; \theta_1 \dots \theta_n$

Each element $F(\theta) \in \mathcal{G}_n$ can be expressed as a linear combination of these monomials.

$$F(\theta) = F^{(0)} + \sum_i F_i^{(1)}\theta_i + \cdots + \sum_{i \dots k} F^{(n)}_{i,\dots,k}\theta_i \dots \theta_j \dots \theta_k \tag{2.5.2}$$

All coefficient are totally antisymmetric under exchange of the indices. We call elements with even (odd) monomials even (odd) algebra. Every $F(\theta)$ can be uniquely decomposed into a sum of even and odd monomials. Even elements commute with each other and odd elements anti-commute with each other.

In $\mathcal{G}_n$ we can define sums $F(\theta)+G(\theta)$ products $F(\theta)\cdot G(\theta)$ and functions $\mathrm{e}^{F(\theta)} = 1+F(\theta)+\frac{1}{2!}(F(\theta))^2+\dots$. All these therms can easily be expressed as linear combinations of moments.

Differentiation

$$\frac{\partial}{\partial\theta_j}\theta_i = \delta_{ij} \tag{2.5.3}$$

$$\frac{\partial}{\partial\theta_i}c = 0, \quad c \in \mathbb{C} \tag{2.5.4}$$

from anti-commutation relation and sign convention and derivative always acts on variable directly following

$$\frac{\partial}{\partial\theta_i}(\theta_1 \dots \theta_n) = \delta_{i1}\theta_2 \dots \theta_n - \delta_{i2}\theta_1\theta_3 \dots \theta_n + \cdots + (-1)^{n-1}\theta_1 \dots \theta_{n-1} \tag{2.5.5}$$

it has the consequence

$$\left\{\frac{\partial}{\partial\theta_i}, \frac{\partial}{\partial\theta_j}\right\} = 0 \tag{2.5.6}$$

$$\left\{\frac{\partial}{\partial\theta_i}, \theta_j\right\} = \delta ij \tag{2.5.7}$$

We can then read θ_i and $\frac{\partial}{\partial\theta_j}$ as a representation of fermion creation and annihilation operators.

Integration The goal is to find generalization of functional integrals, so we only need the analogue of $\int_{-\infty}^{\infty} \mathrm{d}x$, no need of finite integrals. First consider one single Grassmann variable θ

$$\int \mathrm{d}\theta\, f(\theta) = \int \mathrm{d}\theta\, (A + B\theta)$$

it should be a linear function of A and B because of linearity of integration. To enable variable shift $\theta \mapsto \theta + \eta$

$$\int \mathrm{d}\theta\, (A + B\theta) = \int \mathrm{d}\theta\, ((A + b\eta) + B\theta)$$

$$\Rightarrow \int 1 \cdot \mathrm{d}\theta = 0 \tag{2.5.8}$$

*see also F.A.Berezin, the method of second quantization, 1966 in addition we define

$$\int d\theta\, \theta = 1 \tag{2.5.9}$$

in general

$$\int d\theta_i = 0 \tag{2.5.10}$$

$$\int d\theta_i\, \theta_j = \delta_{ij} \tag{2.5.11}$$

$$\left\{d\theta_i\, , d\theta_j\right\} = 0 = \left\{\theta_i, d\theta_j\right\} \tag{2.5.12}$$

multiple integrals

$$\int d\theta_n \ldots d\theta_1\, F(\theta) = \int d\theta_n \ldots d\theta_1 \left(\sum_{i,\ldots,k}^{n} F^{(n)}_{i\ldots k} \theta_i \ldots \theta_k \right) = n! F^{(n)}_{12\ldots n} \tag{2.5.13}$$

All terms with $k < n$ vanish due to $\int d\theta_i = 0^n$ Note that differentiation and integration with respect to Grassmann variables yield same result.

Gaussian integrals for <u>even</u> numbers of generators and A skew-symmetric matrix.

$$\int d\theta_1 \ldots d\theta_n\, e^{-\frac{1}{2}\theta_i A_{ij} \theta_j} = \sqrt{\det\{A\}} \tag{2.5.14}$$

Here consider only example for $n = 2$, i.e. $A_{11} = A_{22} = 0$ and $A_{12} = -A_{21}$

$$\begin{aligned} e^{-\frac{1}{2}\theta_i A_{ij} \theta_j} &= e^{-\frac{1}{2}(\theta_1\theta_2 A_{12} + \theta_2\theta_1 A_{21})} \\ &= e^{-A_{12}\theta_1\theta_2} \\ &= 1 - A_{12}\theta_1\theta_2 \end{aligned}$$

hence

$$\begin{aligned} \int d\theta_1\, d\theta_2\, e^{-\frac{1}{2}\theta_i A_{ij} \theta_j} &= \int d\theta_1\, d\theta_2\, (1 - A_{12}\theta_1\theta_2) = A_{12} \\ &= \sqrt{\det\{A\}} \end{aligned}$$

There is a subtlety in the equation (2.5.14). Unlike two dimensional case where we only consider the terms up to linear term, we need to take care of higher order terms in higher dimension, otherwise the integral vanishes! There is a subtlety in the equation. Unlike two dimensional case where we only consider the terms up to linear term, we need to take care of higher order terms in higher dimension, otherwise the integral vanishes!

For each skew-symmetric matrix of even rank, the determinant is a perfect square while for each skew-symmetric matrix of odd rank, $\det\{A\} = 0$. $\sqrt{\det\{A\}}$ = Pfaffian form

$$\begin{aligned} n &= 2 \qquad & P &= A_{12} = \frac{1}{2}\epsilon A_{ij} \\ n &= 4 \qquad & P &= A_{12}A_{34} - A_{13}A_{24} + A_{14}A_{23} = \frac{1}{8}\epsilon_{ijkl}A_{ij}A_{kl} \end{aligned}$$

2.5.2 Fermion Fields

Definition of Grassmann fields as functions of space-time, whose values are anti-commuting numbers, .e.g.

$$\psi(x) = \sum_i \psi_i \phi_i(x) \tag{2.5.15}$$

with $\psi_i \in \mathcal{G}, \phi_i \in \mathbb{C}$. For Dirac fields, ϕ_i are 4-component spinors.

As in the scalar case, to add external sources $\eta, \bar{\eta}$ to the Dirac Lagrangian

$$\mathcal{L} = \bar{\psi}\,(i\partial\!\!\!/ - m)\,\psi + \bar{\psi}\eta + \bar{\eta}\psi \tag{2.5.16}$$

Obviously the sources must be Grassmann valued $\{\eta, \eta\} = \{\eta, \bar{\eta}\} = \{\bar{\eta}, \bar{\eta}\} = \{\psi, \eta\} = \{\bar{\psi}, \eta\} = \{\psi, \bar{\eta}\} = \{\bar{\psi}, \bar{\eta}\} = 0$.

Vacuum to vacuum transition amplitude in presence of external sources

$$\langle 0|0\rangle^{\eta\bar{\eta}} = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left\{ i \int \mathrm{d}^4x\,[\bar{\psi}(i\partial\!\!\!/ - m)\psi + \bar{\eta}\eta + \bar{\psi}\eta] \right\} \tag{2.5.17}$$

Determine classical solution from the least-action principle

$$\psi_{\mathrm{cl}}(x) = i \int \mathrm{d}^4y\, S_F(x-y)\eta(y) \tag{2.5.18}$$

$$\bar{\psi}_{\mathrm{cl}}(x) = i \int \mathrm{d}^4y\, \bar{\eta}(y) S_F(x-y) \tag{2.5.19}$$

with already known Dirac propagator

$$S_F(x-y) = \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{i(k\!\!\!/ + m)}{k^2 - m^2 + i\epsilon} \mathrm{e}^{-ik(x-y)} \tag{2.5.20}$$

as the Green's function of the Dirac operator $(i\partial\!\!\!/ - m)$.

By expansion of $\psi(x)$ around the classical solution $\psi_{\mathrm{cl}}(x)$ we find, like in the scalar case,

$$\langle 0|0\rangle^{\eta,\bar{\eta}} = \mathrm{e}^{iS_{\mathrm{cl}}}\,\langle 0|0\rangle^{\eta=\bar{\eta}=0} \tag{2.5.21}$$

and classical action can be rewritten as

$$iS_{\mathrm{cl}} = i \int \mathrm{d}^4x\,[\bar{\psi}_{\mathrm{cl}}\,(i\partial\!\!\!/ - m)\,\psi_{\mathrm{cl}} + \bar{\psi}_{\mathrm{cl}}\eta + \bar{\eta}\psi_{\mathrm{cl}}] \tag{2.5.22}$$

$$= -\int \mathrm{d}^4x\,\mathrm{d}^4y\, \bar{\eta}(x) S_F(x-y)\eta(y) \tag{2.5.23}$$

hence the generating functional for the free Dirac field is given as

$$W_0[\eta, \bar{\eta}] = \frac{\langle 0|0\rangle^{\eta,\bar{\eta}}}{\langle 0|0\rangle^{\eta=\bar{\eta}=0}} = \exp\left\{ -i \int \mathrm{d}^4x\,\mathrm{d}^4y\, \bar{\eta}(x) S_F(x-y)\eta(y) \right\} \tag{2.5.24}$$

Derive n-point functions from generating functional

$$\langle 0|T\psi(x)\bar{\psi}(y)|0\rangle = \frac{1}{i}\frac{\delta}{\delta\eta(y)}\frac{1}{i}\frac{\delta}{\delta\bar{\eta}(x)}\, W_0[\eta, \bar{\eta}]\bigg|_{\eta=\bar{\eta}=0} \tag{2.5.25}$$

2.5.3 QED

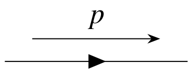
Lagrangian given as

$$\begin{aligned}\mathcal{L}_{\text{QED}} &= \bar{\psi}\left(i\not{D} - m\right)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2\xi}\left(\partial_\mu A^\mu\right)^2 \\ &= \bar{\psi}\left(i\not{\partial} - m\right)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2\xi}\left(\partial_\mu A^\mu\right)^2 - e\bar{\psi}\gamma^\mu\psi A_\mu \\ &= \mathcal{L}_0 + \mathcal{L}_{\text{GF}} - e\bar{\psi}\gamma^\mu\psi A_\mu\end{aligned} \tag{2.5.26}$$

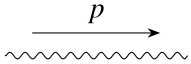
Expand the exponential of the interaction term.

$$\exp\left\{i\int \mathrm{d}^4x\,\mathcal{L}_{\text{QED}}\right\} = \exp\left\{i\int \mathrm{d}^4x\,(\mathcal{L}_0 + \mathcal{L}_{\text{GF}})\right\}\cdot\left[1 - ie\int \mathrm{d}^4x\,\bar{\psi}\gamma^\mu\psi A_\mu + \ldots\right]$$

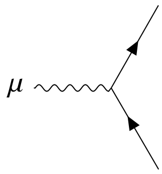
Feynman rules (position space)



$$= \int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon}\mathrm{e}^{-ip(x-y)} \tag{2.5.27}$$



$$= \int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{-i}{q^2 + i\epsilon}\left(g^{\mu\nu} - (1-\xi)\frac{q^\mu q^\nu}{q^2}\right)\mathrm{e}^{-ip(x-y)} \tag{2.5.28}$$



$$= -ie\gamma^\mu\int \mathrm{d}^4x \tag{2.5.29}$$

Generating functional for QED

$$Z[j_\mu, \eta, \bar{\eta}] = \int \mathcal{D}A_\mu\mathcal{D}\bar{\psi}\mathcal{D}\psi\,\exp\left\{i\int \mathrm{d}^4x\left(\mathcal{L}_0 + \mathcal{L}_{\text{GF}} + \mathcal{L}_{\text{int}} + j_\mu A^\mu + \bar{\psi}\eta + \bar{\eta}\psi\right)\right\} \tag{2.5.30}$$

$$Z[j_\mu, \eta, \bar{\eta}] = \frac{Z[j_\mu, \eta, \bar{\eta}]}{Z[j_\mu = 0, \eta = \bar{\eta} = 0]} \tag{2.5.31}$$

2.6 Generating Functional for Fully Connected Green's Functions

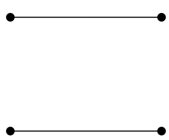
Return to scalar theory for simplicity. In section 2.2 we calculated 4-point function to λ^0

$$\langle 0|T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|0\rangle = D_F(x_3 - x_4)D_F(x_1 - x_2) + D_F(x_2 - x_4)D_F(x_1 - x_3) \tag{2.6.1}$$

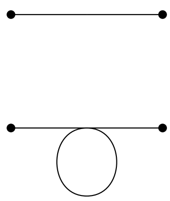
$$+ D_F(x_1 - x_4)D_F(x_2 - x_3) \tag{2.6.2}$$

To next order in perturbation theory $O(\lambda^1)$

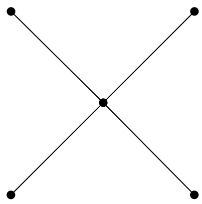
$$\langle 0|T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|0\rangle = 3$$



$$- 3i\lambda$$



$$- i\lambda$$



$$+ O\left(\lambda^2\right) \tag{2.6.3}$$

only the last term is fully connected and contributes to T-matrix!

There exists a generating functional $E[j]$ that only generates the fully connected diagrams

$$iE[j] = \ln(Z[j]) \tag{2.6.4}$$

Note that often in the literature, $E[j]$ is often called $W[j]$!

To show that $E[j]$ in $\lambda\phi^4$ generates no disconnected contributions in the 2- and 4-point functions!

2-point function

$$\frac{\delta^2 iE[j]}{i\delta j(x_1) i\delta j(x_2)} = \frac{1}{Z}\frac{\delta^2 Z}{i\delta j(x_1) i\delta j(x_2)} - \frac{1}{Z^2}\frac{\delta Z}{i\delta j(x_1)}\frac{\delta Z}{i\delta j(x_2)}$$

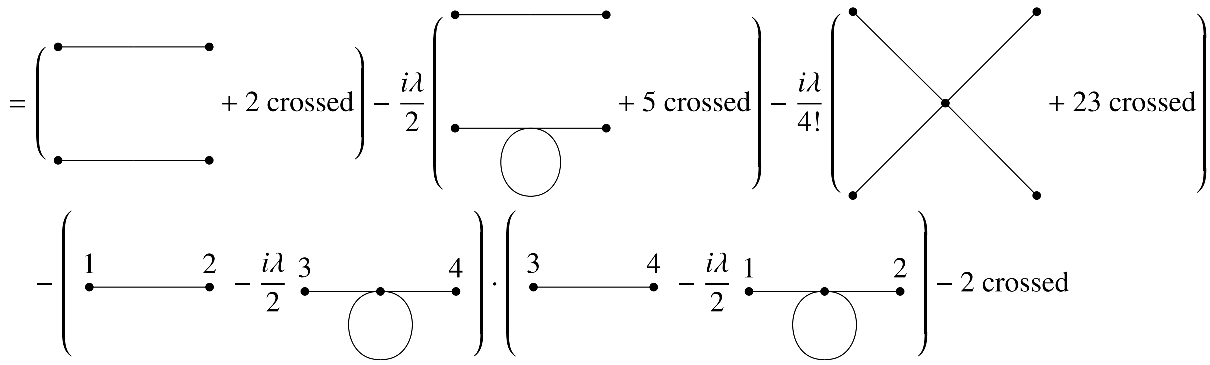
since Z is quadratic in j, for $j = 0$ term with one derivative drops put. Hence

$$\begin{aligned}\left.\frac{\delta^2 iE[j]}{i\delta j(x_1) i\delta j(x_2)}\right|_{j=0} &= \left.\frac{1}{Z}\frac{\delta^2 Z}{i\delta j(x_1) i\delta j(x_2)}\right|_{j=0} \\ &= D_F(x-y)\end{aligned}$$

there is (to arbitrary order in λ) the propagator. It doesn't have disconnected poles.

4-point function

$$\begin{aligned}&\left.\frac{\delta^4 iE[j]}{i\delta j(x_1) i\delta(x_2) i\delta j(x_3) i\delta(x_4)}\right|_{j=0} \\ &= \left.\frac{1}{Z}\,\frac{\delta^4 Z[j]}{\delta j(x_1)\delta(x_2)\delta j(x_3)\delta(x_4)}\right|_{j=0} - \left.\frac{1}{Z^2}\frac{\delta^2 Z}{\delta j(x_1) j(x_2)}\frac{\delta^2 Z}{\delta j(x_3)\delta j(x_4)}\right|_{j=0} - \left.\frac{1}{Z^2}\frac{\delta^2 Z}{\delta j(x_1) j(x_3)}\frac{\delta^2 Z}{\delta j(x_2)\delta j(x_4)}\right|_{j=0} \\ &\quad - \left.\frac{1}{Z^2}\frac{\delta^2 Z}{\delta j(x_1) j(x_4)}\frac{\delta^2 Z}{\delta j(x_2)\delta j(x_3)}\right|_{j=0} \\ &= \langle 0|T\phi_1\phi_2\phi_3\phi_4|0\rangle - \langle 0|T\phi_1\phi_2|0\rangle\langle 0|T\phi_3\phi_4|0\rangle - \langle 0|T\phi_1\phi_3|0\rangle\langle 0|T\phi_2\phi_4|0\rangle - \langle 0|T\phi_1\phi_4|0\rangle\langle 0|T\phi_2\phi_3|0\rangle\end{aligned}$$



Indeed only the fully connected term survive!

So define in general the "connected" or "irreducible" n-point function by

$$\langle 0|T\phi(x_1)\ldots\phi(x_n)|0\rangle_{\mathrm{c}} = \frac{1}{i}\frac{\delta}{\delta j(x_1)}\cdots\frac{1}{i}\frac{\delta}{\delta j(x_n)} iE[j] \tag{2.6.5}$$

We have shown that

$$\langle 0|T\phi_1\phi_2\phi_3\phi_4|0\rangle = \langle 0|T\phi_1\phi_2\phi_3\phi_4|0\rangle_c + \sum_P \langle 0|T\phi_{i_1}\phi_{i_2}\rangle_{\mathrm{c}}\langle 0|T\phi_{i_3}\phi_{i_4}|0\rangle_{\mathrm{c}}$$

Example for 6-point function

2.7 Effective action and Legendre Transform

$iE[j]$ is generating functional for irreducible Green's functions. Formally

$$iE[j] = \sum_n \frac{i^n}{n!} \mathrm{d}^4x_1 \ldots \mathrm{d}^4x_n \, G_\mathrm{c}(x_1, \ldots, x_n) j(x_1) \ldots j(x_n) \tag{2.7.1}$$

Remember the LSZ reduction formula. It is the relation between S-matrix and Green's function

$$_{\text{out}}\langle k_1 \ldots k_m | k_{m+1} \ldots k_n \rangle_{\text{in}} = \text{disconnected terms} + \prod_{j=1}^{n} \left(\frac{iZ}{k_j^2 - m^2 + i\epsilon} \right)^{-1} \sqrt{Z}^n G(k_1, \ldots, k_n) \tag{2.7.2}$$

Conclusion is that n-point Green's function contain poles in all external legs. S-matrix elements are amputated Green's functions. In the following to derive generating functional for amputated, fully connected (one-particle-irreducible) Green's functions. It leads to "effective action".

Define the classical field

$$\phi(x) = \frac{\delta}{\delta j(x)} E[j] \tag{2.7.3}$$

For $j \neq 0$, $\phi = \phi[j]$ can in principle be inverted to $j = j[\phi]$.

Effective action is Legendre transform of $E[j]$

$$\Gamma[\phi] = E - \int \mathrm{d}^4x \, j(x)\phi(x) \Big|_{j=j[\phi]} \tag{2.7.4}$$

$j(x)$ can be recovered from Γ by functional derivation with respect to ϕ

$$\frac{\delta}{\delta\phi(x)} \Gamma[\phi] = -j(x) \tag{2.7.5}$$

Note that $E = E[j[\phi]]$, so this calculation is not quite as trivial as it seems.

Define $\Gamma(x_1, \ldots, x_n)$ through the formal expansion

$$\Gamma[\phi] = -\sum \frac{(-i)^n}{n!} \int \mathrm{d}x_1 \ldots \mathrm{d}x_n \, \Gamma(x_1, \ldots, x_n) \phi(x_1) \ldots \phi(x_n) \tag{2.7.6}$$

$\Gamma(x_1, \ldots, x_n)$ can be obtained from $\Gamma[\phi]$ by repeated function derivative with respect to $\phi(x_i)$.

Calculate first

$$\begin{aligned} \frac{\delta}{\delta\phi(y)} \phi(y) &= \delta(x - y) \\ &= \frac{\delta}{\delta\phi(y)} \left(\frac{\delta}{\delta j(x)} E[j] \right) \end{aligned}$$

use the chain rule

$$\begin{aligned} &= \int \mathrm{d}^4z \, \frac{\delta^2 E}{\delta j(x) \delta j(z)} \frac{\delta}{\delta\phi(y)} j(z) \\ &= -i \int \mathrm{d}^4z \, G_\mathrm{c}(x - z) \Gamma(y - z) \end{aligned}$$

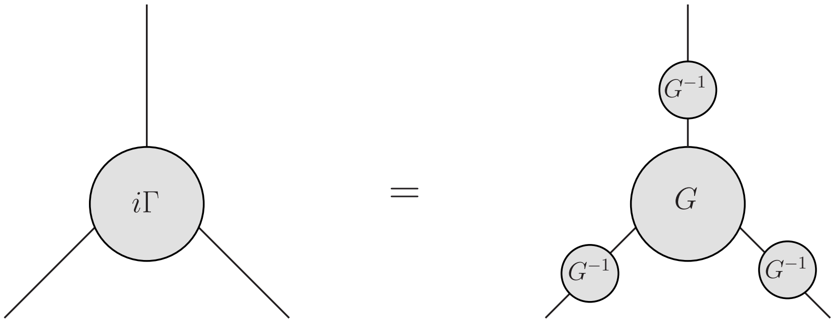
Fourier transform the result we have $i = \tilde{G}(p)\tilde{\Gamma}(p)$, or $\tilde{\Gamma}(p) = Z^{-1}(p^2 - m^2)$. Differentiate this once again

$$\begin{aligned}
&\frac{\delta^2\phi(x)}{\delta\phi(x)\delta(y)} = 0 \\
&\qquad = \int \mathrm{d}^4z\,\mathrm{d}^4[v]\left[\frac{\delta^3 E}{\delta j(x)\delta j(z)\delta(v)}\frac{\delta j(z)}{\delta\phi(y)}\frac{\delta j(v)}{\delta\phi(n)}\right] + \int \mathrm{d}^4z\,\frac{\delta^2 E}{\delta j(x)\delta j(z)}\frac{\delta^2 j(z)}{\delta\phi(y)\delta\phi(x)} \\
&\qquad = -\int \mathrm{d}^4z\,\mathrm{d}^4v\,G(x,z,v)\Gamma(z,y)\Gamma(v,u) - \int \mathrm{d}^4z\,G(x,z)\Gamma(z,y,x) = 0
\end{aligned}$$

multiply with $\int \mathrm{d}^4x\,\Gamma(x,\omega)$, use $\int \mathrm{d}^4x\,\Gamma(x-\omega)G(x-z) = i\delta(\omega - z)$

$$i\Gamma(\omega,y,u) = \int \mathrm{d}^4z\,\mathrm{d}^4v\,\mathrm{d}^4x\,G(x,z,v)\Gamma(z,y)\Gamma(x,\omega)\Gamma(u,v)$$

graphically



In word, $\Gamma(\omega,y,u)$, third derivative of $\Gamma[\phi]$, is the 1-particle irreducible (amputated and fully connected) version of $G(x,z,v)$! (Only two-point function has non-1PI part?)

2.8 Ward-Takahashi Identity for QED

Ward Takahashi identities are relations between one-particle irreducible vertex functions and propagators that hold to all orders in perturbation theory. It is in fact consequence of gauge invariance. It also plays key role in the proof of renormalizability of QED.

Generating functional of QED

$$Z[j_\mu,\eta,\bar{\eta}] = N\int \mathcal{D}A_\mu\mathcal{D}\phi\mathcal{D}\bar{\psi}\,\exp\left\{i\int \mathrm{d}^4x\,\mathcal{L}_{\text{eff}}\right\} \tag{2.8.1}$$

$$\begin{aligned}
\mathcal{L}_{\text{eff}} &= \underbrace{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}\left[i\left(\partial\!\!\!/ + ieA\!\!\!/\right) - m\right]\psi}_{\mathcal{L}} \\
&\quad \underbrace{-\frac{1}{2\xi}(\partial_\mu A^\mu)^2 + j_\mu A^\mu + \bar{\psi}\eta + \bar{\eta}\psi}_{\mathcal{L}_1}
\end{aligned} \tag{2.8.2}$$

Two observations are seemingly contradicting to each other

- $\mathcal{L}_{\text{eff}}$ is obviously <u>not</u> gauge invariant, since we have introduced a gauge fixing term.
- On the other hand, physics as expressed through Green's functions <u>must</u> be independent of gauge.

This non-trivial connection leads to differential equation for Z!

Consider infinitesimal gauge transformation

$$\begin{aligned} A_\mu &\mapsto A_\mu + \partial_\mu \Lambda \\ \psi &\mapsto \psi - ie\Lambda\psi \\ \bar{\psi} &\mapsto \bar{\psi} + ie\Lambda\bar{\psi} \end{aligned}$$

In the decomposition $\mathcal{L} + \mathcal{L}_1$, all the changes are induced via $\mathcal{L}_1$ ($\mathcal{L}$ is gauge invariant)

$$\delta \int \mathrm{d}^4x\, \mathcal{L}_{\text{eff}} = \delta \int \mathrm{d}^4x\, \mathcal{L}_1 = \int \mathrm{d}^4x \left[-\frac{1}{\xi}(\partial_\mu A^\mu)\partial^2\Lambda + j_\mu \partial^\mu \Lambda - ie\Lambda(\bar{\eta}\psi - \bar{\psi}\eta) \right]$$

Hence the change in $Z[j, \eta, \bar{\eta}]$ is

$$\begin{aligned} \delta Z[j, \eta, \bar{\eta}] = N \int \mathcal{D}A_\mu \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp\left\{ i \int \mathrm{d}^4x\, \mathcal{L}_{\text{eff}} \right\} \\ \times i \int \mathrm{d}^4x \left[-\frac{1}{\xi}\partial^2(\partial_\mu A^\mu) - \partial_\mu j^\mu - ie(\bar{\eta}\psi - \bar{\psi}\eta) \right] \Lambda(x) \end{aligned}$$

As $\Lambda(x)$ is arbitrary, the term is bracket needs to vanish. Take this bracket in front of the functional Z using

$$\begin{aligned} \psi &\mapsto \frac{1}{i}\frac{\delta}{\delta\bar{\eta}} \\ \bar{\psi} &\mapsto -\frac{1}{i}\frac{\delta}{\delta\eta} \\ A_\mu &\mapsto \frac{1}{i}\frac{\delta}{\delta j^\mu} \end{aligned}$$

$$\left[\frac{i}{\xi}\partial^2(\partial_\mu \frac{\delta}{\delta j_\mu}) - \partial_\mu j^\mu - e\left(\bar{\eta}\frac{\delta}{\delta\bar{\eta}} - \eta\frac{\delta}{\delta\eta} \right) \right] Z[j, \eta, \bar{\eta}] = 0$$

Transform this into PDE for generating functional of irreducible Green's functions $Z = \mathrm{e}^{iE}$

$$-\frac{1}{\xi}\partial^2 \left(\partial_\mu \frac{\delta E}{\delta j_\mu} \right) - \partial_\mu j^\mu - ie\left(\bar{\eta}\frac{\delta E}{\delta\bar{\eta}} - \eta\frac{\delta E}{\delta\eta} \right) = 0 \tag{2.8.3}$$

Finally use the effective action to derive relations for irreducible amputated vertex functions

$$\Gamma[A_\mu, \psi, \bar{\psi}] = E[j_\mu, \eta, \bar{\eta}] - \int \mathrm{d}^4x \left(j_\mu A^\mu + \bar{\eta}\psi + \bar{\psi}\eta \right) \tag{2.8.4}$$

$$\begin{aligned} \frac{\delta\Gamma}{\delta A_\mu} &= -j^\mu & \frac{\delta E}{\delta j^\mu} &= A_\mu \\ \frac{\delta\Gamma}{\delta\psi} &= +\bar{\eta} & \frac{\delta E}{\delta\bar{\eta}} &= \psi \\ \frac{\delta\Gamma}{\delta\bar{\psi}} &= -\eta & \frac{\delta E}{\delta\eta} &= -\bar{\psi} \end{aligned}$$

After the replacement, equation 2.8.3 becomes

$$\left[\frac{1}{\xi}\partial^2\partial_\mu A^\mu - \partial_\mu\frac{\delta\Gamma}{\delta A_\mu} + ie\left(\bar{\psi}\frac{\delta\Gamma}{\delta\bar{\psi}} + \frac{\delta\Gamma}{\delta\psi}\psi\right)\right] = 0 \tag{2.8.5}$$

Take functional derivative $\frac{\delta}{\delta\bar{\psi}}\frac{\delta}{\delta\psi}$ and subsequently put $\bar{\psi} = \psi = A_\mu = 0$

$$-\partial_x^\mu\frac{\delta^3\Gamma}{\delta\bar{\psi}(x_1)\delta\psi(y_1)\delta A^\mu(x)} = -ie\delta^{(4)}(x-x_1)\frac{\delta^2\Gamma}{\delta\bar{\psi}(x_1)\delta\psi(y_1)} + ie\delta^{(4)}(x-y_1)\frac{\delta^2\Gamma}{\delta\bar{\psi}(x_1)\delta\psi(y_1)} \tag{2.8.6}$$

This becomes more intuitive in momentum space. First define

$$\int \mathrm{d}^4x\,\mathrm{d}^4x_1\,\mathrm{d}^4y_1\exp[i(p'x_1 - py_1 - qx)]\frac{\delta^3\Gamma}{\delta\bar{\psi}(x_1)\delta\psi(y_1)\delta A^\mu(x)} = e(2\pi)^4\delta^{(4)}(p'-p-q)\Gamma(p,q,p') \tag{2.8.7}$$

Know already the one-particle irreducible two-point function

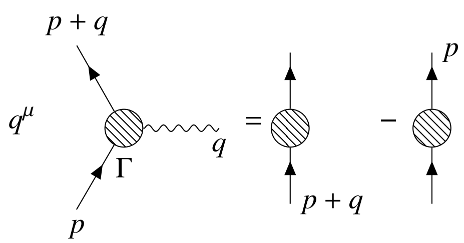
$$\int \mathrm{d}^4x_1\,\mathrm{d}^4y_1\exp[i(p'x_1 - py_1)]\frac{\delta^2\Gamma}{\delta\bar{\psi}(x_1)\delta\psi(y_1)} = -(2\pi)^4\delta^{(4)}(p'-p)iS_F^{-1}(p) \tag{2.8.8}$$

Multiply 2.8.6 with $\exp[i(p'x_1 - py_1 - qx)]$ and integrate over x, x_1 and y_1

$$q_\mu\Gamma^\mu(p,q,p+q) = iS_F^{-1}(p+q) - iS_F^{-1}(p) \tag{2.8.9}$$

This is Ward Takahashi identity.

Graphically



At lowest order in QED

$$\begin{aligned} S_F(p) &= \frac{i}{\not{p} - m} \\ \Gamma^\mu(p,q,p+q) &= \gamma^\mu \\ \not{q} &= (\not{p} - \not{q} - m) - (\not{p} - m) \end{aligned}$$

In the limit $q^\mu \to 0$, we obtain Ward identity

$$\Gamma^\mu(p,0,p) = \frac{\partial iS_F^{-1}}{\partial p_\mu} \tag{2.8.10}$$

There are more Ward identities that can be derived using different functional derivatives. Start again with 2.8.3 and differentiate with respect to $j_\nu(y)$, then put $\eta = \bar{\eta} = j = 0$

$$-\frac{1}{\xi}\partial^2\partial_x^\mu\frac{\delta^2 E}{\delta j^\mu(x)\delta j^\nu(y)} = \partial_x^\mu g_{\mu\nu}\delta^{(4)}(x-y)$$

Remember photon propagtor is given by

$$\frac{i\delta^2 E}{i\delta j^\mu(x) i\delta j^\nu(y)} = \langle 0|TA_\mu(x)A_\nu(y)|0\rangle = G_{\mu\nu}(x-y)$$

$$\frac{1}{\xi}\partial^2\partial_x^\mu G_{\mu\nu}(x-y) = i\partial_x^\mu g_{\mu\nu}\delta^{(4)}(x-y) \tag{2.8.11}$$

After Fourier transform

$$\frac{i}{\xi}k^2 k^\mu \tilde{G}_{\mu\nu}(k) = k_\nu \tag{2.8.12}$$

Again it is true to all orders in perturbation theory. To say that the longitudinal component of $G_{\mu\nu}$ is fixed and not modified by interactions

$$\tilde{G}_{\mu\nu}(k) = \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)G_T(k^2) + \frac{k_\mu k_\nu}{k^2}G_L(k^2) \tag{2.8.13}$$

with Ward identity

$$\frac{i}{\xi}k^2 G_L(k^2)k_\nu = k_\nu$$

$$G_L(k^2) = \frac{-i\xi}{k^2}$$

Propagator at leading order

$$\hat{G}_{\mu\nu}(k) = \frac{-i}{k^2}\left(g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right)$$

$$G_T(k^2) = \frac{-i}{k^2}$$

$$G_L(k^2) = -\frac{i\xi}{k^2}$$

We will make heavy use of the fact Ward Takahashi identities hold to all orders in the following sections on the renormalization of QED.

2.9 Renormalization of QED I: Divergences and Dimensional Analysis

It is known from QFT course last term that loop diagrams are often (UV-)divergent. To obtain a sensible theory, need to regularise these divergences and remove or absorb them by renormalization.

Analyse divergences structure of QED by dimensional analysis. Superficial degree of divergences D of a Feynman diagram with

- d space dimension
- L number of loops
- P_γ number of photon propagators
- P_e number of electron propagators
- N_γ number of external photons
- N_e number of external electrons

- V number of vertices

An arbitrary diagram contains the integral like

$$\int \frac{\mathrm{d}^d k_1 \ldots \mathrm{d}^d k_L}{(\not{k}_{i_1} - m)\ldots(\not{k}_{i_{P_e}}) k_{j_1}^2 \ldots k_{j_{P_\gamma}}^2} \sim k^D$$

thus

$$D = dL - 2P_\gamma - P_e \tag{2.9.1}$$

We want to eliminate L, P_γ and P_e in favour of V, N_γ and N_e

- L is the number of undetermined momenta

$$L = P - V + 1 = P_\gamma + P_e - V + 1 \tag{2.9.2}$$

- Each vertex is connected to 2 electron and 1 photon line. External lines are attached to 1 vertex, internal to 2 vertices.

$$V = 2P_\gamma + N_\gamma = \frac{1}{2}(2P_e + N_e) \tag{2.9.3}$$

Put together

$$D = d + V\left(\frac{d-4}{2}\right) - N_e\left(\frac{d-1}{2}\right) - N_\gamma\left(\frac{d-2}{2}\right) \tag{2.9.4}$$

for $d = 4$

$$D = 4 - \frac{3}{2}N_e - N_\gamma \tag{2.9.5}$$

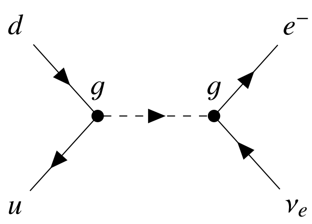
In four dimension, D is independent of number of vertices, only dependent on N_e and N_γ. $D \geq 0$ only for certain, finite "small" N_e and N_γ.

There are three different categories

- $d < 4$ super-renormalizable $\Leftrightarrow [e] > 0$
- $d = 4$ renormalizable $\Leftrightarrow [e] = 0$
- $d > 4$ non-renormalizable$\Leftrightarrow [e] < 0$

Mass dimension of coupling constant $[\psi] = (d-1)/2$ and $[A_\mu] = (d-2)/2$. Interaction $eA_\mu\bar{\psi}\psi$ leads to $[e] = 2 - d/2$

Fermi theory of weak interaction contains the interaction term $G_F\left(\bar{\psi}\gamma_\mu(1-\gamma_5)\psi\right)(\bar{\psi}\gamma^\mu(1-\gamma_5)\psi)$. Coupling constant has negative mass dimension $[G_F] = -2$, non-renormalizable.

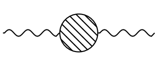


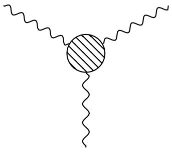
$$\sim \frac{g^2}{q^2 - M_W^2} \approx \frac{g^2}{M_W^2} = G_F$$

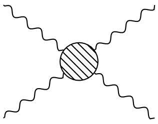
Back to QED in $d = 4$; divergent amplitudes

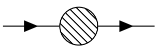
(a)  $D = 4$

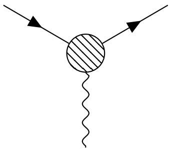
(b)  $D = 3$

(c)  $D = 2$

(d)  $D = 1$

(e)  $D = 0$

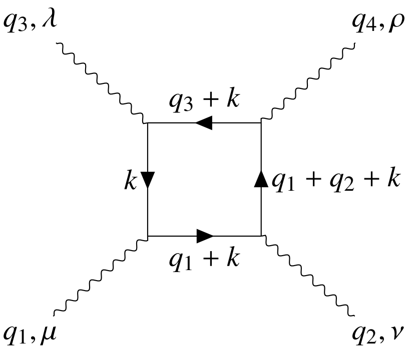
(f)  $D = 1$

(g)  $D = 0$

Need to show that all divergences can be absorbed into renormalization of the parameters of the theory ($e_0 \to e$, $m_0 \to m$) and by adjusting the field strength $\psi \mapsto Z_2^{-1/2}\psi$ and $A_\mu \mapsto Z_3^{-1/2}A_\mu A$

To ignore (a). QED is C-invariant, $A_\mu \mapsto -A_\mu$, correlation functions of odd numbers of photons vanish. (Furry's theorem) Then ignore ((b)) ((d)).

((e)) could be potentially dangerous. Need counter-terms like $(F_{\mu\nu}F^{\mu\nu})^2$, $(F_{\mu\nu}\tilde{F}^{\mu\nu})^2$, but they have dimension 8, i.e. need -4 mass dimension coupling constant. Then the theory becomes non-renormalizable! Gauge invariance saves us.



$$= \frac{e^4}{i}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\gamma_\mu(\not{k}+m)\gamma_\lambda(\not{q}_3+\not{k}+m)\gamma_\rho(\not{q}_1+\not{q}_2+\not{k}+m)\gamma_\nu(\not{q}_1+\not{k}+m)}{(k^2-m^2)((q_3+k)^2-m^2)((q_1+q_2+k)^2-m^2)((q_1+k)^2-m^2)}$$

We can evaluate the divergent part by putting the external momenta to zero, since the part dependent on momenta is finite. This divergence is UV-divergence, so we can also put mass to zero in the limit of large momentum.

$$= \frac{e^4}{i}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\gamma_\mu\not{k}\gamma_\lambda\not{k}\gamma_\rho\not{k}\gamma_\nu\not{k}}{(k^2)^4} + \text{finite}$$

$$= e^4 I(g_{\mu\lambda}g_{\nu\rho} + g_{\mu\rho}g_{\nu\lambda} + g_{\mu\nu}g_{\rho\lambda}) + \text{finite}$$

From gauge invariance or another Ward identity, one can show $q_1^\mu(\dots) = 0$

$$e^4 I(q_{1\lambda}g_{\rho\nu} + q_{1\rho}g_{\lambda\nu} + q_{1\nu}g_{\lambda\rho}) + \text{finite} = 0$$

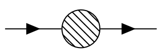
Thus I has to be finite! Symmetries can renders amplitudes more convergent than they appear superficially!

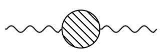
Conclusion: primitively divergent amplitudes are ((c)) photon self energy, ((f)) electron self energy and ((g)) the vertex graph.

Renormalization of QED, schematically Original Lagrangian including gauge fixing

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi_0}(\partial_\mu A^\mu)^2 + \bar{\psi}(i\partial\!\!\!/ - m_0)\psi - e_0\bar{\psi}\gamma^\mu\psi A_\mu \tag{2.9.6}$$

Calculation of self-energy graphs leads to expressions

 $= \dfrac{iZ_2}{p\!\!\!/ - m} + \ldots \qquad \langle 0|T\psi\bar{\psi}|0\rangle \sim Z_2$

 $= \dfrac{-iZ_3 g_{\mu\nu}}{q^2} + \cdots \qquad \langle 0|TA_\mu A_\nu|0\rangle \sim Z_3$

To reinstate residues of 1, define renormalized field strengths

$$\psi = Z_2^{1/2}\psi_r \tag{2.9.7}$$

$$A^\mu = Z_3^{1/2}A_r^\mu \tag{2.9.8}$$

then

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4}Z_3 F_{r\mu\nu}F_r^{\mu\nu} + Z_2\bar{\psi}_r(i\partial\!\!\!/ - m_0)\psi_r - e_0 Z_2 Z_3^{1/2}\bar{\psi}_r\gamma^\mu\psi A_{r\mu} - \frac{Z_3}{2\xi_0}(\partial_\mu A_r^\mu)^2 \tag{2.9.9}$$

Define the physical electric coupling by

$$e_0 Z_2 Z_3^{1/2} = eZ_1 \tag{2.9.10}$$

In addition,

$$\xi_0 = Z_\xi \xi \tag{2.9.11}$$

which means

$$\frac{Z_3}{\xi_0} = \frac{Z_3}{Z_\xi \xi} = \frac{1+\delta_3}{(1-\delta_\xi)\xi} = \frac{1}{\xi}(1 + \underbrace{\delta_3 - \delta_\xi}_{=0}) + O\big(\xi^2\big)$$

$\delta_3 = \delta_\xi$ is a consequence of Ward Identity for G_L.

Define

$$\delta_i = Z_i - 1 \tag{2.9.12}$$

for $i = 1, 2, 3, \xi$

$$\delta_m = Z_2 m_0 - m \tag{2.9.13}$$

Therefore the Lagrangian with counter-terms is

$$\begin{aligned}\mathcal{L}_{\text{QED}} = &-\frac{1}{4}Z_3 F_{r\mu\nu}F_r^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A_r^\mu)^2 + \bar{\psi}_r(i\partial\!\!\!/ - m)\psi_r - e\bar{\psi}_r\gamma^\mu\psi_r A_{r\mu} \\ &- \frac{1}{4}\delta_3 F_{r\mu\nu}F_r^{\mu\nu} - \frac{1}{2\xi}(\delta_3 - \delta_\xi)(\partial_\mu A_r^\mu)^2 + \bar{\psi}_r(i\delta_2\partial\!\!\!/ - \delta_m)\psi_r - e\delta_1\bar{\psi}_r\gamma_\mu\psi_r A_r^\mu\end{aligned} \tag{2.9.14}$$

Introduced four counter-terms in the Lagrangian. They are to be determined such that observables are finite.

2.10 Renormalization of QED II: One Loop*

Use dimensional regularisation $\mathrm{d}^4k \mapsto \mathrm{d}^dk$. In order to give the Lagrangian density in d dimensions a unique mass dimension, introduce (arbitrary) mass parameter μ.

$$\mathcal{L}_{\mathrm{QED}} = -\frac{1}{4}(F_{\mu\nu})^2 + \bar{\psi}(i\partial\!\!\!/ - m)\psi - e\mu^{2-d/2}\bar{\psi}\gamma_\mu\psi A^\mu \tag{2.10.1}$$

with $[A] = (d-2)/2$, $[\psi] = (d-1)/2$ and $[e] = 0$.

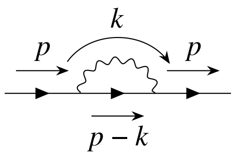
Repeat the two important formula. Feynman parameter

$$\frac{1}{AB} = \int_0^1 \frac{\mathrm{d}z}{[zA + (1-z)B]^2} \tag{2.10.2}$$

Dimensional regularisation formula

$$\frac{1}{i}\int \frac{\mathrm{d}^dk}{(2\pi)^d}\frac{1}{[k^2-\Delta^2]^n} = \frac{(-1)^n}{(4\pi)^{d/2}}\frac{\Gamma(n-d/2)}{\Gamma(n)}\Delta^{d/2-n} \tag{2.10.3}$$

Electron self-energy



$$= -i\Sigma(p) = \mu^{4-d}\int \frac{\mathrm{d}^dk}{(2\pi)^d} ie\gamma_\mu \frac{i(p\!\!\!/ - k\!\!\!/ + m)}{(p-k)^2 - m^2} ie\gamma_\nu \frac{-ig^{\mu\nu}}{k^2}$$

$$\Sigma(p) = e^2\mu^{4-d}\frac{1}{i}\int \frac{\mathrm{d}^dk}{(2\pi)^d}\frac{\gamma_\mu(p\!\!\!/ - k\!\!\!/ + m)\gamma^\mu}{((p-k)^2-m^2)k^2}$$

Introduce Feynman parameter z

$$= e^2\mu^{4-d}\int_0^1 \mathrm{d}z\,\frac{1}{i}\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{\gamma_\mu(p\!\!\!/ - k\!\!\!/ + m)\gamma^\mu}{\left[(p-k)^2z - m^2z + k^2(1-z)\right]^2}$$

Make substituition $k' = k - zp$

$$= e^2\mu^{4-d}\int_0^1 \mathrm{d}z\,\frac{i}{i}\int\frac{\mathrm{d}^dk'}{(2\pi)^d}\frac{\gamma_\mu(p\!\!\!/(1-z) - k\!\!\!/' + m)\gamma^\mu}{\left[k'^2 - m^2z + z(1-z)p^2\right]^2}$$

Because of parity of integral $k' \leftrightarrow -k'$, the term linear in k' vanishes.

$$= e^2\mu^{4-d}\int_0^1 \mathrm{d}z\,\gamma_\mu(p\!\!\!/(1-z)+m)\gamma^\mu\frac{1}{i}\int\frac{\mathrm{d}^dk'}{(2\pi)^d}\frac{1}{\left[k'^2 - m^2z + z(1-z)p^2\right]^2}$$

Use dimensional regularization formula

$$= \mu^{4-d}e^2\frac{\Gamma(2-d/2)}{(4\pi)^{d/2}}\int_0^1 \mathrm{d}z\,\gamma_\mu(p\!\!\!/(1-z)+m)\gamma^\mu\left[m^2z - p^2z(1-z)\right]^{d/2-2}$$

set $\epsilon = 4-d$, use identities $\gamma_\mu\gamma^\mu = d$, $\gamma_\mu p\!\!\!/\gamma^\mu = (2-d)p\!\!\!/$

$$= \frac{e^2}{(4\pi)^2}\Gamma\left(\frac{\epsilon}{2}\right)\int_0^1 \mathrm{d}z\left\{2p\!\!\!/(1-z) - 4m - \epsilon\left[p\!\!\!/(1-z) - m\right]\right\}\left(\frac{m^2z - p^2z(1-z)}{4\pi\mu^2}\right)^{-\epsilon/2}$$

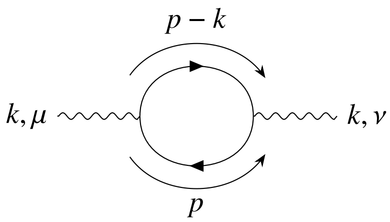
*see also Ryder Ch. 9.6

use $\Gamma(\epsilon/2) = \frac{2}{\epsilon} - \gamma_E + O(\epsilon)$

$$= \frac{e^2}{8\pi^2\epsilon}(-\not{p} + 4m) + \frac{e^2}{16\pi^2}\left[\not{p}(1+\gamma_E) - 2m(1+2\gamma_E) + 2\int_0^1 \mathrm{d}z\,(\not{p}(1-z) - 2m)\ln\!\left(\frac{m^2 z - p^2 z(1-z)}{4\pi\mu^2}\right) + O(\epsilon)\right]$$

$$= \frac{e^2}{8\pi^2\epsilon}(-\not{p} + 4m) + \text{finite}$$

Photon self-energy



$$= i\Pi_{\mu\nu}(k)$$

$$= -\mu^{4-d}e^2 \int \frac{\mathrm{d}^d p}{(2\pi)^d}\,\mathrm{tr}\!\left[\gamma_\mu \frac{1}{\not{p} - m}\gamma_\nu \frac{1}{\not{p} - \not{k} - m}\right]$$

$$= -\mu^{4-d}e^2 \int \frac{\mathrm{d}^d p}{(2\pi)^d}\frac{\mathrm{tr}\!\left[\gamma_\mu(\not{p} + m)\gamma_\nu(\not{p} - \not{k} + m)\right]}{(p^2 - m^2)((p-k)^2 - m^2)}$$

same Feynman parameter trick $p' = p - kz$

$$= -\mu^{4-d}e^2 \int_0^1 \mathrm{d}z\,\frac{1}{i}\int \frac{\mathrm{d}^d p'}{(2\pi)^d}\frac{\mathrm{tr}\!\left[\gamma_\mu(\not{p}' + \not{k}z + m)\gamma_\nu(\not{p}' - \not{k}(1+z) + m)\right]}{[p'^2 - m^2 + z(1-z)k^2]^2}$$

Linear terms in p' drop out. Focus on the numerator

$$\begin{aligned} N &= \left[p'^\kappa p'^\lambda - k^\kappa k^\lambda z(1-z)\right]\mathrm{tr}\!\left(\gamma_\mu\gamma_\kappa\gamma_\nu\gamma_\lambda\right) + m^2\,\mathrm{tr}\!\left(\gamma_\mu\gamma_\nu\right) \\ &= 4\left[2p'_\mu p'_\nu - 2z(1-z)(k_\mu k_\nu - k^2 g_{\mu\nu}) - g_{\mu\nu}(p'^2 - m^2 + z(1-z)k^2)\right] \end{aligned}$$

Therefore

$$\Pi_{\mu\nu}(k) = -\mu^{4-d}e^2 4\int_0^1 \mathrm{d}z\,\frac{1}{i}\int\frac{\mathrm{d}^d p}{(2\pi)^d}\left\{\frac{2p_\mu p_\nu}{[p^2 - m^2 + z(1-z)k^2]^2} - \frac{2z(1-z)(k_\mu k_\nu - k^2 g_{\mu\nu})}{[p^2 - m^2 + z(1-z)k^2]^2} - \frac{g_{\mu\nu}}{p^2 - m^2 + z(1-z)k^2}\right\}$$

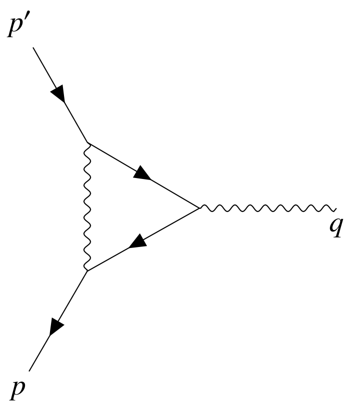
Dimensional regularisation

$$\frac{1}{i}\int\frac{\mathrm{d}^d p}{(2\pi)^d}\frac{p_\mu p_\nu}{(p^2 - \Delta^2)^n} = \frac{g_{\mu\nu}}{2(n-1)}\frac{1}{i}\int\frac{\mathrm{d}^d p}{(2\pi)^d}\frac{1}{(p^2 - \Delta)^{n-1}}$$

so the first and third term cancel!

$$\begin{aligned} \Pi_{\mu\nu}(k) &= \frac{e^2}{2\pi^2}(k_\mu k_\nu - g_{\mu\nu}k^2)\left[\frac{1}{3\epsilon} - \frac{\gamma_E}{6} - \int_0^1 \mathrm{d}z\, z(1-z)\log\!\left(\frac{m^2 - z(1-z)k^2}{4\pi\mu^2}\right)\right] \\ &= \frac{e^2}{6\pi^2}(k_\mu k_\nu - k^2 g_{\mu\nu})\left(\frac{1}{\epsilon} + \text{finite const.} + \frac{k^2}{10m^2} + \ldots\right) \end{aligned}$$

Vertex graph



$$= ie\mu^{2-d/2}\Lambda_\mu(p,q,p')$$

$$\Lambda(p,q,p') = \frac{e^2}{8\pi^2\epsilon}\gamma_\mu + \text{finite}$$

Remember "finite" include a new Dirac structure anomalous magnetic moment

$$\frac{\alpha}{2\pi}\frac{i\sigma_{\mu\nu}q^\nu}{2m}$$

It has to be finite!

Summary of all divergences

$$\Sigma(p) = \frac{e^2}{8\pi^2\epsilon}(-\not{p} + 4m) + \text{finite} \tag{2.10.4}$$

$$\Pi_{\mu\nu}(p) = \frac{e^2}{6\pi^2\epsilon}(k_\mu k_\nu - g_{\mu\nu}k^2) + \text{finite} \tag{2.10.5}$$

$$\Lambda_\mu(p,q,p') = \frac{e^2}{8\pi^2\epsilon}\gamma_\mu + \text{finite} \tag{2.10.6}$$

Electron propagator combines tree, loop and counter-terms.

$$\begin{aligned}
\Gamma^{(2)}(p) = iS_F^{-1}(p) = \not{p} - m - \frac{e^2}{8\pi^2\epsilon}(-\not{p} + 4m) + (\delta_2\not{p} - \delta_m) \\
= \not{p}\left(1 + \frac{e^2}{8\pi^2\epsilon} + \delta_2\right) - m\left(1 + \frac{e^2}{2\pi^2\epsilon}\right) - \delta_m \\
\overset{!}{=} \text{finite}
\end{aligned}$$

Comparing the coefficients leads to

$$\delta_2 = -\frac{e^2}{8\pi^2\epsilon} \tag{2.10.7}$$

$$\delta_m = -\frac{me^2}{2\pi^2\epsilon} \tag{2.10.8}$$

Or

$$\psi = \sqrt{Z_2}\psi_r \tag{2.10.9}$$

$$Z_2 = 1 + \delta_2 = 1 - \frac{e^2}{8\pi^2\epsilon} \tag{2.10.10}$$

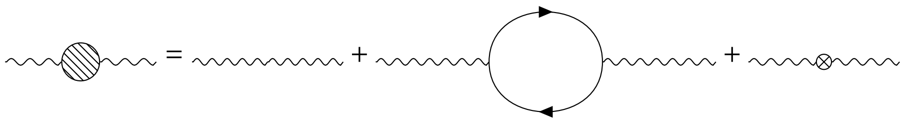
$$m_0 = Z_2^{-1}(m + \delta_m) = m\left(1 - \frac{3e^2}{8\pi^2\epsilon}\right) \tag{2.10.11}$$

Note for m_0 approximation is implicit applied. It doesn't matter in the end, since no physics is dependent on the this factor.

This kind of renormalization, to cancel only the infinite terms $\propto 1/(d-4) \propto 1(\epsilon)$ by the counter-terms, is called minimal subtraction (MS). Alternative is modified minimal subtraction ($\overline{\text{MS}}$). It subtracts terms proportional to

$$\frac{1}{\epsilon} - \frac{1}{2}\left(\gamma_E - \log(4\pi)\right)$$

Photon propagator The modified photon propagator is (using Feynman gauge!)



$$\begin{aligned} D'_{\mu\nu}(k) &= D_{\mu\nu}(k)(1-\delta_3) - D_{\mu\alpha}(k)\Pi^{\alpha\beta}(k)D_{\beta\nu}(k) - \frac{\delta_\xi}{k^2}\frac{k_\mu k_\nu}{k^2} \\ &= -\frac{g_{\mu\nu}}{k^2}(1-\delta_3) - \frac{g_{\mu\alpha}}{k^2}\frac{e^2}{6\pi^2}\left[(k^\alpha k^\beta - k^2 g^{\alpha\beta})\left(\frac{1}{\epsilon} + \frac{k^2}{10m^2} + \dots\right)\right]\frac{g_{\beta\nu}}{k^2} - \frac{\delta_\xi}{k^2}\frac{k_\mu k_\nu}{k^2} \\ &= -\frac{g_{\mu\nu}}{k^2}\left(1 - \delta_3 - \frac{e^2}{6\pi^2\epsilon} - \frac{e^2}{60\pi^2}\frac{k^2}{m^2}\right) - \left(\delta_\xi + \frac{e^2}{6\pi^2\epsilon}\right)\frac{1}{k^2}\frac{k_\mu k_\nu}{k^2} + \dots \end{aligned}$$

resulting propagator not automatically in Feynman gauge, which it should be, since we use Feynman gauge along the way. Need gauge counter-term

$$\delta_\xi = -\frac{e^2}{6\pi^2\epsilon} = \delta_3 \tag{2.10.12}$$

Wave function renormalization of the photon

$$Z_3 = 1 + \delta_3 = 1 - \frac{e^2}{6\pi^2\epsilon} \tag{2.10.13}$$

Note that renormalization does not generate a photon mass term!

Renormalization does not eliminate the finite effect.

$$D'_{\mu\nu} = -g_{\mu\nu}\left(\frac{1}{k^2} - \frac{e^2}{60\pi^2 m^2} + O\left(k^2\right)\right) \tag{2.10.14}$$

Fourier transform yields the potential between two charges

$$V(r) = \frac{e^2}{4\pi r} + \frac{e^4}{60\pi^2 m^2}\delta^{(3)}(\boldsymbol{r}) \tag{2.10.15}$$

The second shifts the S-levels in hydrogen, Lamb shift!

Vertex function

$$\begin{aligned} \Gamma_\mu(p,q,p') &= \gamma_\mu(1+\delta_1) + \Lambda_\mu(p,q,p') \\ &= \gamma_\mu(1 + \delta_1 + \frac{e^2}{8\pi^2\epsilon}) + \text{finite} \end{aligned} \tag{2.10.16}$$

$$Z_1 = 1 + \delta_1 = 1 - \frac{e^2}{8\pi^2\epsilon} = Z_2 \tag{2.10.17}$$

The resulting charge renormalization is

$$e_0 = \mu^{\epsilon/2}\frac{Z_1}{Z_2}Z_3^{-1/2}e = \mu^{\epsilon/2}Z_3^{-1/2}e$$
$$e_0 = \mu^{\epsilon/2}\left(1 + \frac{e^2}{12\pi^2\epsilon}e\right) \tag{2.10.18}$$

Charge renormalization and fermionic field renormalization are related $Z_1 = Z_2$. It is no coincidence, but a result of the Ward identity

$$\Gamma_\mu(p, 0, p) = \frac{\partial i S_F^{-1}}{\partial p^\mu}$$

By renormalization $\Gamma_\mu \mapsto Z_1\gamma_\mu$ and $S_F \mapsto Z_2^{-1}S_F$. Thus $Z_1 = Z_2$

Conclusion: charge renormalization only depends on the photon vacuum polarisation $\sim Z_3$. Essential many particles have the same charge e, electron, muon, proton and so on. If e depends, via Z_1 Z_2, on the properties (masses, type of photon coupling) of the particles conserved, this would be a huge coincidence! It is a consequence of Ward identity, i.e. gauge invariance.

Summary We have shown that the Lagrangian density

$$\begin{aligned}\mathcal{L}_{\text{QED}} = &-\frac{1}{4}(F_r^{\mu\nu})^2 - \frac{1}{2\xi}(\partial_\mu A_r^\mu)^2 - \bar{\psi}_r(i\partial\!\!\!/ - m)\psi_r - e\bar{\psi}_r\gamma_\mu\psi_r A_r^\mu \\ &- \frac{\delta_3}{4}(F_r^{\mu\nu})^2 - \frac{\delta_2 - \delta_\xi}{2\xi}(\partial_\mu A_r^\mu)^2 + \bar{\psi}_r(i\delta_2\partial\!\!\!/ - \delta_m)\psi_r - e\delta_1\bar{\psi}_r\gamma_\mu\psi_r A_r^\mu\end{aligned}$$

- leads to finite physical observable
- can be written as

$$-\frac{1}{4}(F_{\text{bare}}^{\mu\nu})^2 - \frac{1}{2\xi_0}(\partial_\mu A_{\text{bare}}^\mu)^2 + \bar{\psi}_{\text{bare}}(i\partial\!\!\!/ - m_0)\psi_{\text{bare}} - e_0\bar{\psi}_{\text{bare}}A_{\text{bare}}^\mu \tag{2.10.19}$$

  which is form-identical to the original. QED up to one loop is renormalizable!

Predictions of one-loop QED beyond pure renormalization

- anomalous magnetic moment
- Lamb shift
- asymptotic behaviour of QED for large energies. Detailed discussion in renormalization group chapter.

2.11 Renormalization of QED to All Orders*

Here is just a sketch of proof using ward identity.

*see also in Ryder Ch. 9.7, P& S Ch. 10.4, Weinberg Ch. 12.3, Schwatz Ch 21.1.3 or Jauch & Rohrlich Ch. 10

Dressed propagator S'_F contains all possible numbers of self-energy diagrams. It can be written as geometric series

$$\begin{aligned} iS'_F &= iS_F + iS_F(-i\Sigma)iS_F + \dots \\ &= \frac{S_F}{1-\Sigma S_F} \\ S'^{-1}_F &= S^{-1}_F - \Sigma \end{aligned}$$

Consider the following relations for inverse electron propagator, inverse photon propagators and vertex function including Ward identity. The divergent parts have been separated from the free parts.

$$\begin{aligned} S'^{-1}_F(p) &= S^{-1}_F(p) - \Sigma(p) \\ D'^{-1}_F(k) &= D^{-1}_F(k) - \Pi(k) \\ \Gamma_\mu(p,q,p') &= \gamma_\mu + \Lambda_\mu(p,q,p') \\ -\frac{\partial\Sigma(p)}{\partial p^\mu} &= \Lambda_\mu(p,0,p) \end{aligned} \tag{2.11.1}$$

where metric tensor is taken out off the (dressed and free) photon propagator without indices

$$\begin{aligned} D^{(1)}_{\mu\nu} &= g_{\mu\nu}D^{(1)} \\ \Pi_{\mu\nu}(k) &= -g_{\mu\nu}\Pi(k) \end{aligned}$$

Σ , Π and Λ_μ are all divergent as it was shown in the one-loop case.

To show that all divergences can be removed by multiplicative renormalization, i.e. we can define finite propagators and vertex functions like

$$\begin{aligned} \tilde{S}_F &= Z_2^{-1}S'_F \\ \tilde{D} &= Z_3^{-1}D'_F \\ \tilde{\Gamma}_\mu &= Z_1\Gamma_\mu \end{aligned} \tag{2.11.2}$$

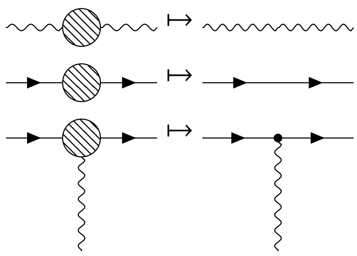
and all divergences are included in Z_1 , Z_2, Z_3 and mass renormalization.

Steps to proceed are

1. Isolated divergences
   a) in irreducible diagrams
   b) in reducible diagrams
2. Define finite propagators and vertex function and show that they satisfy (2.11.1)

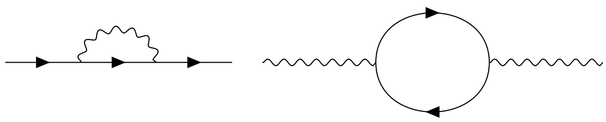
Define irreducible diagrams as a subclass of the one-particle-irreducible ones.

The skeleton F_S of a Feynman diagram F arises from F, if we replace all subgraphs according to the rules.



Graphs are irreducible if $F_S = F$.

Example The sole irreducible self-energy graphs are one-loop electron and photon self-energy graphs, which we have analysed in one-loop case.



There are numerous irreducible vertex graphs, e.g.

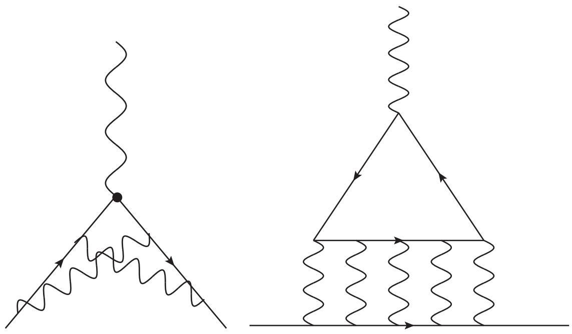


Figure 2.2: Example of irreducible graphs

In (1a) (for the irreducible case), we only need to show this step for the vertex graphs. Graphs related to $\Lambda_\mu(p, q, p')$ have all a *superficial* degree of divergence 0 (logarithmically divergent). To expand in momenta, only one infinite constant appears.

$$\Lambda_\mu = \underbrace{L}_{\text{infinite}} \gamma_\mu + \underbrace{\Lambda_\mu^{(\mathrm{f})}}_{\text{finite}} \tag{2.11.3}$$

Although the superficial degree of divergence is $D = 0$, there are still sub-divergences. For formal treatment, see Jauch and Rohrlich. The separation is defined by

$$\bar{u}(p)\Lambda_\mu^{(\mathrm{f})}(p, 0, p)u(p) = 0$$

Separating finite term from infinite quantity is not unique. This condition make sure it is not ambiguous.

In step (1b) we focus on the reducible vertex graphs. They are obtainable from the skeleton by the replacements

$$S_F \mapsto S'_F$$
$$D_F \mapsto D'_F$$
$$\gamma \mapsto \Gamma$$

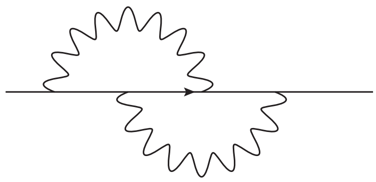
In vertex function, to replace an arbitrary graph by its skeleton connected by dressed propagators

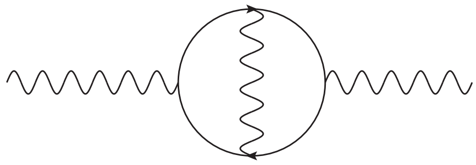
$$\Lambda^\mu(p, p'; S_F, D_F, \gamma, e) \mapsto \Lambda_s^\mu(p, p'; S'_F, D'_F, \Gamma, e)$$

Hence

$$\Gamma^\mu(p, p') = \gamma^\mu + \Lambda_s^\mu(p, p', S'_F, D'_F, \Gamma, e) \tag{2.11.4}$$

Reducible self-energy diagrams have the problem of overlapping divergences, since it is not included in vertex function.





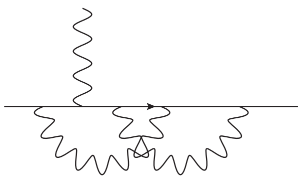
The overlapping divergences are the main difficulty in isolating divergences.

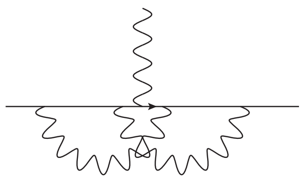
Solution here is to apply the ward identity in order to reduce them to the (eventually will be solved) vertex problem.

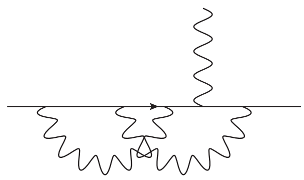
$$S'_F(p) - S'^{-1}_F(p_0) = (p - p_0)^\mu \Gamma_\mu(p, p_0) \tag{2.11.5}$$

The ward identity in (2.11.1) is simply in the limit of $p \to p_0$.

2.11.4 and 2.11.5 form coupled equations for the electron self energy and propagators. For the first overlapping divergent graph, three diagrams are obtained by differentiating with respect to external momentum p.





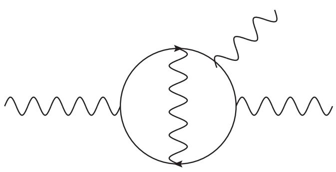


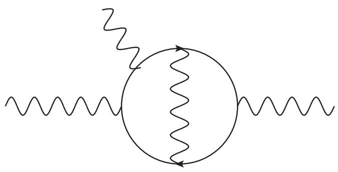
The second overlapping divergent diagram is for photon self-energy. Analogously for the photon propagator. Define operator Δ_μ (analogous to Λ_μ) as

$$\Delta_\mu(k) = -\frac{\partial \Pi(k)}{\partial k^\mu} \tag{2.11.6}$$

(analogous to $\Lambda_\mu = -\mathrm{d}\Sigma(p)/\mathrm{d}p^\mu$).

Similar as before, we get three-photon diagrams differentiating with respect to external momentum k





Furry's theorem only states that the sum of diagrams with odd number of photons must vanish. Not that a single diagram does so!

Analogously to $\Gamma_\mu = \gamma_\mu + \Lambda_\mu$ here we have

$$W_\mu(k) = -2k_\mu + \Delta_\mu(k) \tag{2.11.7}$$

Note $\frac{\partial}{\partial k^\mu}\left(-k^2\right) = -2k_\mu$. The total photon propagator

$$\begin{aligned} D'_F &= D_F + D_F \Pi D_F + \ldots \\ &= \frac{D_F}{1 - \Pi D_F} \\ (D'_F)^{-1} &= D_F^{-1} - \Pi \end{aligned}$$

Thus

$$\begin{aligned}\frac{\partial(D'_F)^{-1}}{\partial k^\mu} &= \frac{\partial}{\partial k^\mu}\left(-k^2+\Pi\right)\\ &= -2k_\mu + \Delta_\mu(k)\\ &= W_\mu(k)\end{aligned} \tag{2.11.8}$$

Analogous to 2.11.4, the 3-photon vertex function Δ_μ satisfy $\Delta_\mu[k;S_F,D_F,k,e] = \Delta_{S\,\mu}[k;S'_F,D'_F,W,e]$

$$W_\mu(k) = -2k_\mu + \Delta_{s\mu}[k;S'_F,D'_F,W,e] \tag{2.11.9}$$

Go to step 2. We have four coupled equation: 2.11.4, 2.11.5, 2.11.8 and 2.11.9 for S'_F, D'_F, Γ_μ and W_μ. All these quantities are divergent. Task is to find the corresponding set of equations for finite quantities.

Vertex function Λ_μ, Δ_μ are logarithmically divergent. Finite (denoted by tilde) after a single subtraction

$$\begin{aligned}\tilde{\Lambda}_p(p,p') &= \Lambda_{s\mu}(p,p') - \Lambda_{s\mu}(p_0,p_0)\Big|_{\not p_0=m}\\ \tilde{\Delta}_\mu(k^2) &= \Delta_{s\mu}(k^2) - \Delta_{s\mu}(\mu^2)\end{aligned} \tag{2.11.10}$$

with μ invariant photon mass and p_0 on-shell photon mass.

From $\Lambda_\mu = L\gamma_\mu + \Lambda_\mu^{(f)}$ in equation (2.11.3), we get $\Lambda_{s\mu}(p_0,p_0)\Big|_{\not p_0=m} = L\gamma_\mu$ with L infinite.

Use this result to define vertex functions and finite propagators

$$\begin{aligned}\tilde{\Gamma}_\mu(p,p') &= \gamma_\mu + \tilde{\Lambda}_{s\mu}[p,p',\tilde{S}_F,\tilde{D}_F,\tilde{\Gamma},e_r]\\ \tilde{S}_F^{-1}(p) - \tilde{S}_F^{-1}(p_0) &= (p-p_0)^\mu\tilde{\Gamma}_\mu(p,p_0)\\ \tilde{W}_\mu(k) &= -2k_\mu + \tilde{\Delta}_{s\mu}[k;\tilde{S}_F,\tilde{D}_F,\tilde{W},e_r]\\ \frac{\partial\tilde{D}_F^{-1}(k)}{\partial k^\mu} &= \tilde{W}_\mu(k)\end{aligned} \tag{2.11.11}$$

Normalization of the electron and photon propagators

$$\begin{aligned}\tilde{S}_F^{-1}(p_0) &= \not p_0 - m\\ k^2\tilde{D}_F(k^2)\Big|_{k^2=\mu^2} &= 1\end{aligned}$$

It is left to show that $\tilde{\Gamma}_\mu$, $\tilde{S}_F$, $\tilde{D}_F$ and $\tilde{W}_\mu$ are connected by multiplicative renormalization factor with Γ_μ, S'_F, D'_F and W_μ, if the charge is renormalized as $e_r^2 = Z_3 e^2$.

Consider Γ_μ or Λ_μ at order e^{2n} (apart from bare vertex)

$$\begin{aligned}V &= 2n+1\\ P_e &= \frac{1}{2}(2V-N_e) = 2n\\ P_\gamma &= \frac{1}{2}(V-N_\gamma) = n\end{aligned}$$

There are $(2n)$ electron propagator S_F, (n) photon propagator D_F, $(2n+1)$ factors of γ_μ with the transformation

$$\begin{aligned}S'_F &\mapsto Z_2^{-1}S'_F = \tilde{S}_F\\ D'_F &\mapsto Z_3^{-1}D'_F = \tilde{D}_F\\ \Gamma_\mu &\mapsto Z_1\Gamma_\mu = \tilde{\Gamma}_\mu\\ e^2 &\mapsto Z_3e^2 = e_r^2\end{aligned} \tag{2.11.12}$$

Then

$$\Lambda_{s\mu} \mapsto Z_2^{-2n} Z_3^{-n} Z_1^{2n+1} Z_3^n \Lambda_{s\mu} = Z_1 \Lambda_{s\mu}$$

or

$$\Lambda_{s\mu}[Z_2^{-1} S'_F, Z_3^{-1} D'_F, Z_1 \Gamma_\nu, Z_3 e^2] = Z_1 \Lambda_{s\mu}[S'_F, D'_F, \Gamma_\nu, e^2] \tag{2.11.13}$$

Also using 2.11.11 and 2.11.10 with 2.11.3

$$\begin{aligned}\tilde{\Gamma}_\mu &= \gamma_\mu + \tilde{\Lambda}_{s\mu} = \gamma_\mu \Lambda_{s\mu} - L\gamma_\mu \\ &= \underbrace{(1-L)}_{Z_1}\left(\gamma_\mu + \frac{1}{1-L}\Lambda_{s\mu}\right) \\ &= Z_1\left\{\gamma_\mu + \frac{1}{Z_1}\Lambda_{s\mu}[\tilde{S}_F, \tilde{D}_F, \tilde{\Gamma}_\nu, e_r^2]\right\}\end{aligned}$$

using 2.11.13

$$\begin{aligned}&= Z_1\left\{\gamma_\mu + \Lambda_{s\mu}[S'_F, D'_F, \Gamma_\nu, e_r^2]\right\} \\ &= Z_1 \Gamma_\mu\end{aligned}$$

So we see that multiplicative renormalization works and it is equivalent to the subtraction 2.11.10. Similarly it can be shown that $\tilde{W}_\mu = -2k_\mu + \tilde{\Delta}_s[k; \tilde{S}_F, \tilde{D}_F, \tilde{W}, e_r]$ works, if

$$\tilde{W}_\mu(k) = Z_3 W_\mu \tag{2.11.14}$$

where $Z_3 = 1 + \frac{1}{2}\Delta_s(\mu^2)$ and $\Delta_{s\mu} = k_\mu \Delta_s$. Furthermore, the second subtraction in 2.11.10 can be rewritten as multiplicative renormalization. Everything is consistent.

Explicitly a three-point function with three external photons contains $[2(2n+1)-0]/2 = 2n+1$ fermion propagator and $[2n+1-3]/2 = n-1$ photon propagators. Thus

$$\Delta_{s\mu} \mapsto Z_2^{-(2n+1)} Z_3^{-(n-1)} Z_1^{2n+1} Z_3^n \Delta_{s\mu} = Z_3 \Delta_{s\mu}$$

$$\Delta_{s\mu}[Z_2^{-1} S_F^{-1}, Z_3^{-1} D'_F, Z_1 \Gamma, Z_3 e^2] = Z_3 \Delta_{s\mu}[S'_F, D'_F, \Gamma, e^2] \tag{2.11.15}$$

$$\begin{aligned}\tilde{W}(k) &= -2k_\mu + \Delta_{s\mu}(k^2) - \Delta_{s\mu}(\mu^2) \\ &= -2k_\mu\left[1 + \frac{1}{2}\Delta_s(\mu^2)\right] + \Delta_{s\mu}(k^2) \\ &= Z_3\left[-2k_\mu + \frac{1}{Z_3}\Delta_{s\mu}(k^2)\right]\end{aligned}$$

use 2.11.15

$$= Z_3 W_\mu(k)$$

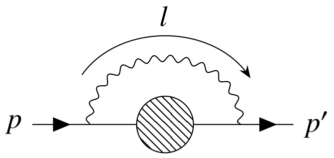
with $Z_3 = 1 + \Delta_s(\mu^2)/2$ and $\Delta_{s\mu} = k_\mu \Delta_s$.

Complete version of proof in J.M.Jauch and F.Rohrlich *The theory of photons and electrons*.

2.12 Infrared divergences

All the divergences we have discussed so far (in the context of the renormalization program) are UV divergences. They all stem from large loop momenta. There are also other, infrared (IR) divergences coming from very soft loop momenta in theories involving massless particles (photons, gluons; Goldstone bosons, chiral fermions, etc.)

Example in QED, photon joining two external electron lines

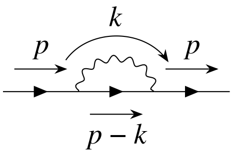


$$\sim \int \frac{d^4 l}{(2\pi)^4} \frac{1}{[(p+l)^2 - m^2][(p'-l)^2 - m^2] l^2}$$
$$= \int \frac{d^4 l}{(2\pi)^4} \frac{1}{[2p \cdot l + l^2][-2p' \cdot l + l^2] l^2}$$
$$\sim \int \frac{d^4 l}{l^4}$$

It is logarithmically divergent. There are two essential ingredients here.

- massless particles (photon), otherwise $1/(l^2 - m^2)$ regulator.
- on-shell relation $p'^2 = p^2 = m^2$. No such divergences in for example photon self-energy.

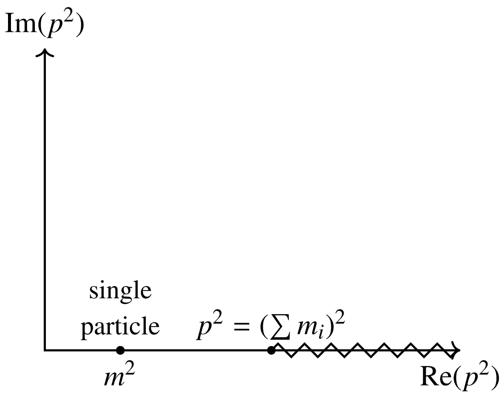
Another important, although slightly less obvious, IR-divergent QED effects exist in electron self-energy diagram



$$= -i\Sigma(p) = \int \frac{d^4 k}{(2\pi)^4} \frac{1}{[(p-k)^2 - m^2]} \frac{1}{k^2}$$

The diagram itself is IR-finite. What is divergent is precisely $\Sigma'(m^2) = \left.\frac{d\Sigma(p^2)}{dp^2}\right|_{p^2=m^2}$.

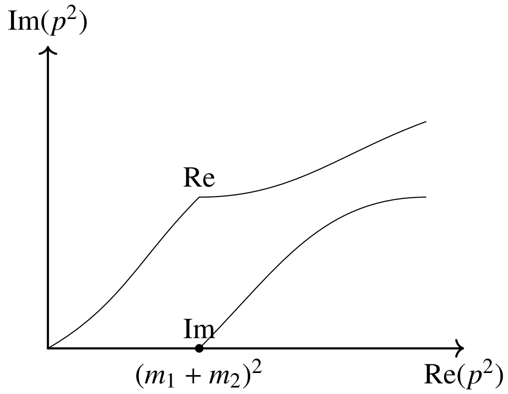
To check the plausibility: remember analytic structure of the spectral function



Multi-particle threshold starts at $p^2 = (\sum_i m_i)^2$.

$$\mathrm{Im}\, \Sigma(p^2) \sim \sqrt{p^2 - (m_1 + m_2)^2}\,\Theta(p^2 - (m_1 + m_2)^2) \tag{2.12.1}$$

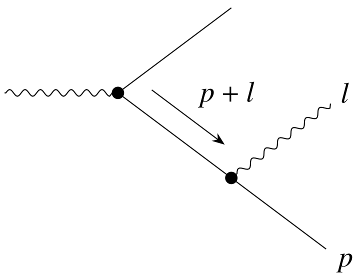
$$\mathrm{Re}\, \Sigma(p^2) \sim -\sqrt{(m_1 + m_2)^2 - p^2} \text{ below threshold} \tag{2.12.2}$$



Derivative is singular at threshold $\left.\frac{\mathrm{d}\Sigma}{\mathrm{d}p^2}\right|_{p^2=(m_1+m_2)^2}$.

In the case above, electron and photon threshold coincides with electron pole position $(m_e + m_\gamma)^2 = m_e^2$.

There is another class of process in QED that contain IR divergences: bremsstrahlung, i.e. radiation of soft photons off asymptotic charged particles.



$$\sim \frac{1}{(p+l)^2 - m^2} = \frac{1}{p \cdot l}$$

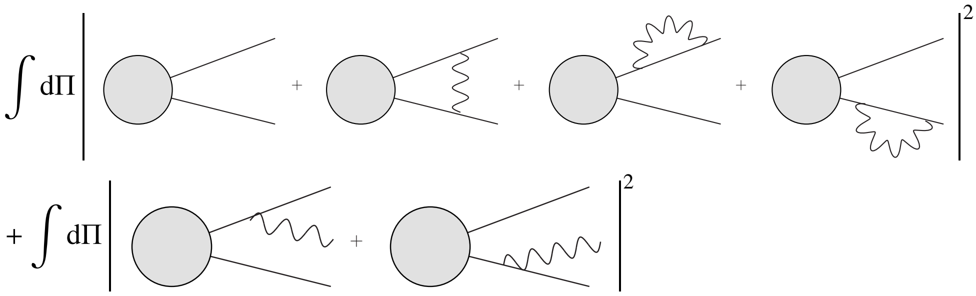
In phase space integral (when calculating cross section)

$$\sim \int \frac{\mathrm{d}^3 l}{(2\pi)^3 2 l^0} \frac{1}{(2p \cdot l)^2} \sim \int \frac{\mathrm{d}^3 l}{l^3}$$

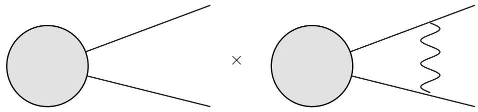
Also IR divergent!

Physical arguments to counter the divergences: we cannot measure arbitrary soft (low-energy) particles. Photons with energies below a certain detector resolution $E_{\max}$ necessarily will pass undetected. So instead of measuring $\sigma(A \to B)$, one in fact measures $\sigma(A \to B) + \sigma\,(A \to B\gamma)\big|_{E_\gamma < E_{\max}} + \ldots$. These quantities turn out to be IR-finite at each order in α_{QED}.

For example at $\mathcal{O}(\alpha)$

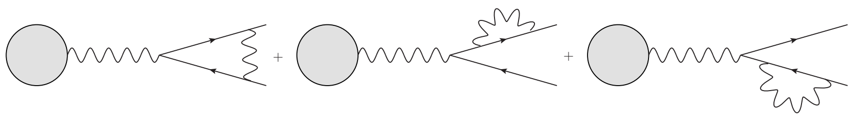


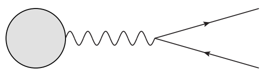
Note: cancellation on the cross section, not on the amplitude level! The first term contain interference term at $O(\alpha)$ ($\sim \alpha$ when calculating cross section)



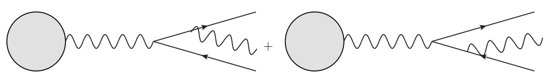
Example regulate with finite photon mass m_γ. Virtual (loop) corrections $\sim \ln\left(\frac{m_\gamma}{m_{(e)}}\right)$. Bremsstrahlung $\sim \int_{m_\gamma}^{E_{\max}} \frac{\mathrm{d}^2 l}{l^3} \ln\left(\frac{E_{\max}}{m_\gamma}\right)$. In total $\sim \ln(E_{\max}/m)$. Its IR-finite!

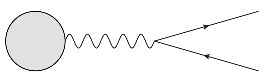
Concrete Example



$=$  $\cdot \frac{\alpha}{\pi}\left(\frac{1+\sigma^2}{2\sigma}\ln\frac{1+\sigma}{1-\sigma} - 1\right)\ln\frac{m_\gamma}{m} + \dots$

with $\sigma = \sqrt{1 - \frac{4m^2}{s}} < 1$.

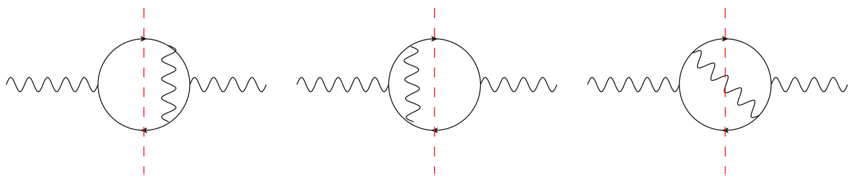
$\int \frac{\mathrm{d}^3 l}{2l}\Big|$  $\Big|^2$

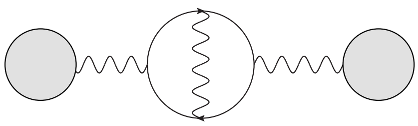
$= \Big|$  $\Big| \cdot \frac{2\alpha}{\pi}\left(\frac{1+\sigma^2}{2\sigma}\ln\frac{1+\sigma}{1-\sigma} - 1\right)\ln\frac{E_{\max}}{m_\gamma} + \dots$

Cancellation mechanism is identified diagram-wise.

Another way to see this problem using optical theorem.

$$\int \mathrm{dps}\, |\mathcal{M}|^2 + \int \mathrm{dps}^\gamma\, |\mathcal{M}_\gamma|^2$$

$\sim$ 

$\sim$ 

with dps phase space integral and red lines as cuts.

This IR cancellation mechanism can be shown to work at all orders (see e.g. Weinberg Ch13, P& S Ch.6). Re-summation of all $\ln(E_{\mathrm{mas}}/m)$ terms possible.

Advanced calculation in QED regulate also IR divergences dimensionally (instead of using m_γ). It also leads to poles in $1/(d-4)$. Formally, UV convergences only for $d < 4$, Ir convergence for $d > 4$.

There are further divergences due to small masses in many process, $E \gg m_e$, so effectively $m_e \approx 0$. Consider Bremsstrahlung pole or propagator again

$$\begin{aligned} \frac{1}{p \cdot l} &= \frac{1}{(p^0 - |\boldsymbol{p}|z)|\boldsymbol{l}|} \approx \frac{1}{|\boldsymbol{p}||\boldsymbol{l}|(1-z)} \\ p^0 &= \sqrt{m_e^2 + |\boldsymbol{p}|^2} \approx |\boldsymbol{p}| \end{aligned}$$

with $z = \cos\left(\theta_{e\gamma}\right)$. It divergences for $z = 1$. Collinear singularity $\sim \ln\left(\frac{m_e}{E}\right)$ enhancement.

Kinoshita-Lee-Navenberg theorem no such mass singularityes can survice in total/inclusive transition probabilities.

3 The Renormalization Group

3.1 The Wilsonian Renormalization Group†

We will study influence of UV fluctuations more explicitly using UV cutoff Λ. It is difficult for gauge theories, but more intuitive for ϕ^4.

Consider path integral with field vanishes in momentum space,

$$\begin{aligned} Z[J] &= \int [\mathcal{D}\phi] \exp\left\{ i \int \mathrm{d}^4 x \, (\mathcal{L} + \phi J) \right\} \\ &= \prod_k \int \mathrm{d}\phi(k) \exp\left\{ i \int \mathrm{d}^4 x \, (\mathcal{L} + \phi J) \right\} \end{aligned}$$

We would like to separate out integration over modes with $|k| \leq \Lambda$. It is difficult in Minkowski space, since Minkowski "scalar product" is not positive semi-definite. So first perform Wick rotation to Euclidean space, where momentum cutoff Λ is well defined. Euclidean path integral

$$Z_E[J]\bigg|_{J=0} = \int [\mathcal{D}\phi]_\Lambda \exp\left\{ - \int \mathrm{d}^d x_E \left(\frac{1}{2}(\partial\phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!}\phi^4 \right) \right\} \tag{3.1.1}$$

Drop the subscript E from now on. m and λ are bare parameters, there are no counter-terms yet. Dimension d to keep discussion general.

Idea is to lower the cutoff Λ somewhat, from $\Lambda \to b\Lambda$, with b a small positive number $0 < b < 1$.

Define <u>low- and high-momentum modes</u>

$$\tilde{\phi}(k) = \begin{cases} \phi(k) & |k| \leq b\Lambda \\ 0 & |k| > b\Lambda \end{cases} \tag{3.1.2}$$

$$\hat{\phi}(k) = \begin{cases} 0 & |k| \leq b\Lambda \\ \phi(k) & b\Lambda < |k| \leq \Lambda \end{cases} \tag{3.1.3}$$

so that field can be decomposed into low-momentum modes and high-momentum modes.

$$\phi(k) = \tilde{\phi}(k) + \hat{\phi}(k) \tag{3.1.4}$$

Rename low-momentum mode $\tilde{\phi}(k) = \phi(k)$.

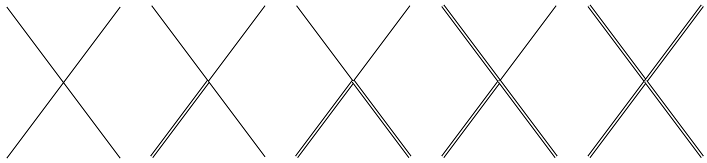
In the path integral $[\mathcal{D}\phi]_\Lambda = [\mathcal{D}\phi]_{b\Lambda}[\mathcal{D}\hat{\phi}]$ and substitute $\phi \mapsto \phi + \hat{\phi}$ in the Lagrangian.

$$\begin{aligned} Z &= \int [\mathcal{D}\phi]_{b\Lambda} \int [\mathcal{D}\hat{\phi}] \exp\left\{ - \int \mathrm{d}^d x \left[\frac{1}{2}(\partial\phi + \partial\hat{\phi})^2 + \frac{m^2}{2}(\phi + \hat{\phi})^2 + \frac{\lambda}{4!}(\phi + \hat{\phi})^4 \right] \right\} \\ &= \int [\mathcal{D}\phi]_{b\Lambda} \exp\left\{ - \int \mathrm{d}^d x \, \mathcal{L}[\phi] \right\} \int [\mathcal{D}\hat{\phi}] \exp\left\{ - \int \mathrm{d}^d x \left[\frac{1}{2}(\partial\hat{\phi})^2 + \frac{m^2}{2}\hat{\phi}^2 + \lambda \left(\frac{1}{6}\phi^3\hat{\phi} + \frac{1}{4}\phi^2\hat{\phi}^2 + \frac{1}{6}\phi\hat{\phi}^3 + \frac{1}{4!}\hat{\phi}^4 \right) \right] \right\} \end{aligned}$$

†P & S, Ch 12.1

Note that terms of order $\phi\hat{\phi}$ vanish! They would contribute to propagator-type terms, not have disjoint momentum support (different Fourier components orthogonal!)

Interaction terms of the form (double line for high momentum modes, single line for low momentum modes)



After $\int \mathcal{D}\hat{\phi}$ path integral is carried out, the generating function should look like

$$Z \stackrel{!}{=} \int [\mathcal{D}\phi]_{b\Lambda} \mathrm{e}^{-\int \mathrm{d}^d x \mathcal{L}_{\text{eff}}(\phi)}$$

Now $\mathcal{L}_{\text{eff}}$ only involves Fourier components with $|k| \leq b\Lambda$.

How does $\mathcal{L}_{\text{eff}}$ look like?

$$\mathcal{L}_{\text{eff}} = \mathcal{L}(\phi) + \text{corrections} \tag{3.1.5}$$

The corrections are in order of λ. The correction terms compensate for the removal of high-momentum Fourier components/fluctuations in $\hat{\phi}$.

We are interested in large-ish cutoffs $\Lambda^2 \gg m^2$. Treat m^2 and λ terms in the $\mathcal{D}\hat{\phi}$ path integral as perturbations. Leading propagator comes from

$$\begin{aligned}\int \mathrm{d}^d x\, \mathcal{L}_0 &= \int \mathrm{d}^d x\, \frac{1}{2}(\partial\hat{\phi})^2 \\ &= \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{2} \hat{\phi}^*(k) k^2 \hat{\phi}(k)\end{aligned}$$

The contraction is similar to a normal propagator

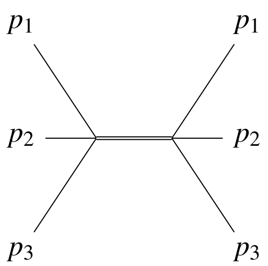
$$\begin{aligned}\hat{\phi}(k)\hat{\phi}(p) &= \frac{\int \mathcal{D}\hat{\phi}\hat{\phi}(k)\hat{\phi}(p)\mathrm{e}^{-\int \mathrm{d}^d x \mathcal{L}}}{\int \mathcal{D}\hat{\phi}\mathrm{e}^{-\int \mathrm{d}^d x \mathcal{L}_0}} \\ &= \frac{1}{k^2}(2\pi)^d \delta^{(d)}(p+k)\hat{\theta}(k)\end{aligned} \tag{3.1.6}$$

where

$$\hat{\theta}(k) = \begin{cases} 1 & b\Lambda < |k| \leq \Lambda \\ 0 & \text{otherwise} \end{cases}$$

Perturbations in m^2 and λ are calculated expanding the exponential, using Wick's theorem with propagator from above. What corrections in $\mathcal{L}_{\text{eff}}$ will the $\hat{\phi}$ field generate?

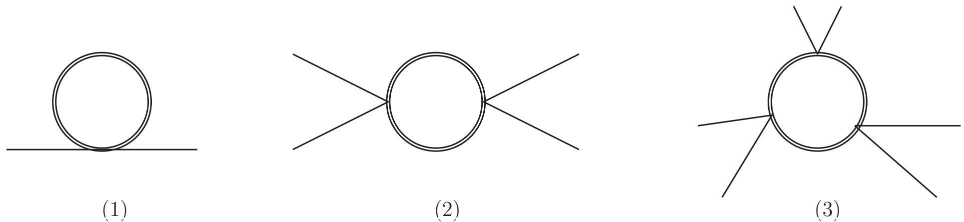
Tree level diagrams Diagram with $\phi^3\hat{\phi}\phi^3\hat{\phi}$



$$\sim \frac{\lambda^2}{(p_1+p_2+p_3)^2}\hat{\theta}(p_1+p_2+p_3)$$

does not contribute for $p_1, p_2, p_3 \ll \Lambda^2$. Similarly other tree-level diagrams won't contribute.
Consider $p_i = 0$ (external) for now!

Single $\hat{\phi}$ loop



(1) (2) (3)

Calculate (1) explicitly using equation (3.1.6)

$$(1) = -\frac{\lambda}{4}\int \mathrm{d}^d x\, \phi^2 \underset{\sqcup}{\hat{\phi}\hat{\phi}} =: -\frac{1}{2}\int \frac{\mathrm{d}^d k_1}{(2\pi)^d}\Delta m^2 \phi(k_1)\phi(-k_1)$$

in which

$$\begin{aligned}\Delta m^2 &= \frac{\lambda}{2}\int_{b\Lambda<|k|\leq\Lambda}\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{k^2}\\ &= \frac{\lambda}{2}\frac{\Omega_d}{(2\pi)^d}\int_{b\Lambda}^{\Lambda}\mathrm{d}k\, k^{d-3}\\ &= \frac{\lambda}{(4\pi)^{d/2}}\frac{1}{\Gamma(d/2)}\frac{\Lambda^{d-2}}{d-2}(1-b^{d-2})\end{aligned}$$

with n dimensional solid angle $\Omega_d = 2\pi^{d/2}/\Gamma(d/2)$. In $d = 4$

$$\Delta m^2 = \frac{\lambda}{16\pi^2}\frac{\Lambda^2}{2}(1-b^2)$$

Remember $b < 1$ so $\Delta m^2 > 1$.
Second diagram give $-\frac{\Delta\lambda}{4!}\phi^4$ in effective Lagrangian. Setting external momenta to zero

$$\begin{aligned}\Delta\lambda &= -4!\frac{2}{2!}\left(\frac{\lambda}{4}\right)^2\int\frac{\mathrm{d}^d k}{(2\pi)^d}\left(\frac{1}{k^2}\right)^2\\ &= -\frac{3}{2}\lambda^2\frac{2}{(4\pi)^{d/2}\Gamma(d/2)}\int_{b\Lambda}^{\Lambda}\mathrm{d}k\, k^{d-5}\\ &= \frac{-3\lambda^2}{(4\pi)^{d/2}\Gamma(d/2)}\frac{\Lambda^{d-4}}{d-4}(1-b^{d-4})\\ &= -\frac{3\lambda^2}{16\pi^2}\ln\left(\frac{1}{b}\right)\end{aligned}$$

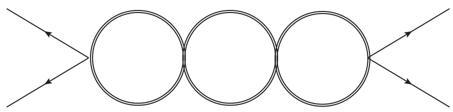
It's negative since $0 < b < 1$!
If we set external momenta $p_i \neq 0$: Taylor expand in p_i, it will generate interactions terms $(\partial\phi)^2\phi^2$, $(\partial\phi)^4, \ldots$.
Third diagram generates term $\sim \lambda^3\phi^6$ (in $d = 4$, $\propto \frac{\lambda^3\phi^6}{\Lambda^2}$). Higher dimensional, non-renormalizable interactions are generated! We will see soon why it is not a real problem.

Comments

- Everything is finite, although being cutoff-dependent
- Is loop expansion $\frac{\lambda+\Delta\lambda}{4!}\phi^4$ valid?

$$\lambda + \Delta\lambda = \lambda\Big[1 - \underbrace{\frac{3\lambda}{16\pi^2}\ln(1/b)}_{\ll 1}\Big]$$

  Higher order (N loops) will scale like $\lambda(\frac{\lambda}{16\pi^2}\ln(1/b))^N$.



  It is even smaller corrections.

More careful comparison of $\mathcal{L}$ and $\mathcal{L}_{\text{eff}}$

$$Z = \int [\mathcal{D}\phi]_{b\Lambda} e^{-\int d^d x \mathcal{L}_{\text{eff}}}$$

$$\mathcal{L}_{\text{eff}} = \frac{1}{2}(1+\Delta Z)(\partial\phi)^2 + \frac{1}{2}(m^2+\Delta m^2)\phi^2 + \frac{1}{4!}(\lambda+\Delta\lambda)\phi^4 + \Delta C((\partial\phi)^2)^2 + \Delta\tilde{C}\phi^2(\partial\phi)^2 + \Delta D\phi^6 + \dots$$

Now rescale distances and momenta $k' = k/b$ or $x' = xb$. k' in integrated up to Λ (original cutoff).

$$\int d^d x\, \mathcal{L}_{\text{eff}} = \int d^d x'\, b^{-d}\Big[\frac{1}{2}(1+\Delta z)b^2(\partial'\phi)^2 + \frac{1}{2}(m^2+\Delta m^2)\phi^2$$
$$+ \frac{1}{4!}(\lambda+\Delta\lambda)\phi^4 + \Delta C b^4(\partial\phi)^4 + \Delta\tilde{C} b^2(\partial'\phi)^2\phi^2 + \Delta D\phi^6 + \dots\Big]$$

Now rescale the fields

$$\phi' = \sqrt{b^{2-d}(1+\Delta Z)}\phi$$

to obtain canonical kinetic term

$$\int d^d x\, \mathcal{L}_{\text{eff}} = \int d^d x' \left[\frac{1}{2}(\partial'\phi')^2 + \frac{1}{2}m'^2\phi'^2 + \frac{1}{4!}\lambda'\phi'^4 + C'(\partial'\phi')^4 + \tilde{C}'(\partial'\phi')^2\phi'^2 + D'\phi'^6\right]$$

with the scaled variables

$$m'^2 = \frac{m^2+\Delta m^2}{b^2(1+\Delta Z)}$$
$$\lambda' = \frac{\lambda+\Delta\lambda}{b^{4-d}(1+\Delta Z)^2}$$
$$C' = b^d\frac{C+\Delta C}{(1+\Delta Z)^2}$$
$$\tilde{C}' = b^{d-2}\frac{\tilde{C}+\Delta\tilde{C}}{(1+\Delta Z)^2}$$
$$D' = b^{2d-6}\frac{D+\Delta D}{(1+\Delta Z)^3}$$

Even if we had $C = \tilde{C} = D = 0$ initially, it would apply as well.

So combination of integrating out degree of freedom and rescaling leads to transformation of $\mathcal{L}$ (with identical *Llamba*). $\mathcal{L}$ characterized by set of coupling constants

$$(m^2, \lambda, C, \tilde{C}, D, \dots) \mapsto (m'^2, \lambda', C', \tilde{C}', D', \dots)$$

This operation can be repeated, make it infinitesimal $b \mapsto 1 - db$, so that it's continuous. Transformation in space of all possible Lagrangians.

Study trajectory or flows leads to Renormalization Group. It is not really a group, rather a semi-group, as transformation of "integrating-out" high-momentum degree of freedom is not invertible.

Two possible ways to perform calculations of correlation functions for $|p_i| \ll \Lambda$

- use original $\mathcal{L}$, high-momentum fluctuations in loops
- use $\mathcal{L}_{\text{eff}}$ high-momentum fluctuations have been absorbed in new coupling constants. Already at tree-level. Essentially it is effective field theory.

Renormalization Group (RG) in Detail Consider $\mathcal{L}$ near the free theory $m^2 = \lambda = C = \tilde{C} = D = \dots = 0$

$$\mathcal{L}_0 = \frac{1}{2}(\partial\phi)^2$$

$\mathcal{L}_0$ is unchanged under the RG flow. It is fixed point.

Near $\mathcal{L}_0$, only consider terms <u>linear</u> in perturbation. Neglect $\Delta m^2 (\propto \lambda), \Delta\lambda(\propto \lambda^2), \Delta Z(\propto \lambda^2), \Delta C, \Delta\tilde{C}(\propto \lambda^2), \Delta D(\propto \lambda^3)$. Then simply have

$$\begin{aligned}
m'^2 &= m^2 b^{-2} \\
\lambda' &= \lambda b^{d-4} \\
\tilde{C}' &= \tilde{C} b^{d-2} \\
C' &= C b^{d} \\
D' &= D b^{2d-6}
\end{aligned}$$

Since $0 < b < 1$, behaviours are classified as following

- <u>relevant</u> term grows with RG,
- <u>marginal</u> term in $d = 4$ unchanged (higher orders unimportant)
- <u>irrelevant</u> term diminished in RG flow

This can actually be seen directly from dimensional analysis. Operator with N fields ϕ and M derivatives scales as

$$\begin{aligned}
C'_{N,M} &= b^{N(d/2-1)+M-d} C_{N,M} \\
&= b^{d_{N,M}-d} C_{N,M}
\end{aligned}$$

with $d_{N,M}$ mass dimension of the operator.

Note: relevant/marginal/irrelevant terms correspond to super-renormalizable/renormalizable/non-renormalizable interactions.

One can understand evolution of couplings near a fixed point from dimensional analysis! Near fixed point, arbitrary complicated $\mathcal{L}$ reduces to a finite number of renormalizable term! (Only near fixed point!)

Illustrate RG flow for ϕ^4 in 3 cases

- $\underline{d > 4}$ only mass term is relevant, everything else irrelevant. Only m^2 grows, since $m'^2 = m^2 b^{-2n}$ after n iterations. Ultimately $m'^2 \sim \Lambda^2$. Integrate complete momentum range between Λ and the effective mass m'.

- $\underline{d = 4}$ marginal λ? Go back to the full transformation including non-linear terms.

$$\lambda' = \frac{\lambda + \Delta\lambda}{b^{4-d}(1 + \Delta Z)^2} = \lambda - \frac{3\lambda^2}{16\pi^2} \ln(1/b) + O\big(\lambda^3\big)$$

  λ slowly decreases as high-momentum modes are integrated out. Coupling goes to zero ϕ^4 becomes non-interacting in $d = 4$.lambda

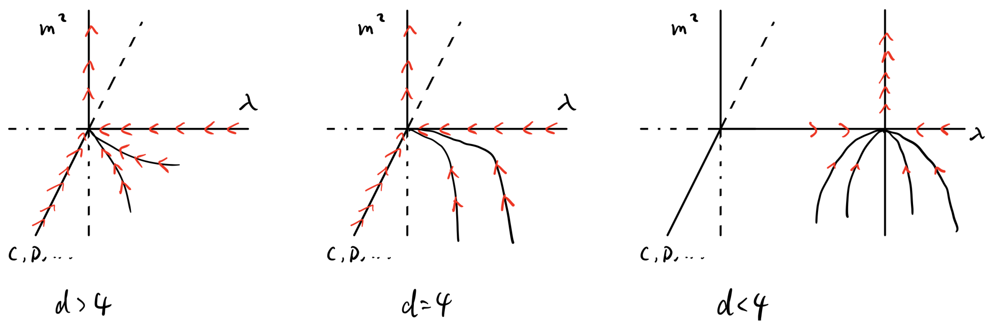
- $\underline{d < 4}$ λ is relevant! Coupling grows. Non-linear effects are important.

$$\lambda' = b^{d-4}\left[\lambda = \frac{3\lambda^2}{(4\pi)^{d/2}} \frac{1}{\Gamma(d/2)} \frac{b^{d-4} - 1}{4 - d} \Lambda^{d-4}\right]$$

  There is a second fixed point, besides a trivial one $\lambda = 0$.

$$\lambda = \frac{4 - d}{3} (4\pi)^{d/2} \Gamma(d/2) (\Lambda b)^{4-d} > 0$$

  where the non-linear effects compensate the rescaling!



Remarks

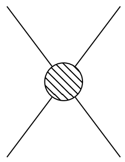
- for $d < 4$ but "close", the new fixed point will be "close" to the free fixed point. Then perturbation theory still make sense. Could have strongly-coupled theories as a fixed point. More difficult, study exactly solvable models.

- $m^2 d^2$ in ϕ^4 always relevant, diverges quickly, naturally $m \mapsto \Lambda$.Problems for theories with elementary scales (Higgs in Standard Model)

3.2 Callan-Symanzik Equation

We want to avoid technically complicated calculations with cutoffs (violate [gauge] symmetries!). Looking for a method to extract RG information from <u>renormalized</u> Green's functions (limit $\Lambda \to \infty$ already taken). Cannot extract RG from Λ-dependence any more use dependence on renormalization scale μ instead!

ϕ^4-theory near the critical point $m = 0$ Renormalization conditions so far, on-shell, $s = t = u = \frac{4}{3}m^2$ for 4-point function, are problematic in the massless case. Use arbitrary (unphysical) space-like reference point $-\mu^2$. Then the two- and four-point functions are

1. $\Sigma_{1\text{PI}}(p^2 = -\mu^2) = 0$,
2. $\frac{\mathrm{d}}{\mathrm{d}p^2}\, \Sigma_{1\text{PI}}\big|_{p^2=-\mu^2} = 0$,
3.  $= -i\lambda$ at $s = (p_1 + p_2)^2 = -\mu^2$, $t = -\mu^2$, $u = -\mu^2$.

Call μ renormalization scale.

Consequence of first two

$$\langle 0|\phi(p)\phi(-p)|0\rangle = \frac{i}{p^2}\bigg|_{p^2=-\mu^2} \tag{3.2.1}$$

in terms of the bare field $\phi_0 = \sqrt{Z}\phi$

$$\langle 0|\phi_0(p)\phi_0(-p)|0\rangle = \frac{iZ}{p^2}\bigg|_{p^2=-\mu^2} \tag{3.2.2}$$

Do renormalized perturbation theory as before, adjusting the counterterms to the conditions above.

Note that renormalization scale μ is arbitrary; one could have chosen different value for μ!

Green's functions for bare and renormalized fields

$$\begin{aligned} G^{(n)}_{\text{base}} &= \langle 0|T\phi_0(x_1)\ldots\phi_0(x_n)|0\rangle \\ G^{(n)}_{\text{ren}} &= \langle 0|T\phi(x_1)\ldots\phi(x_n)|0\rangle \\ \phi &= Z^{-1/2}\phi_0 \end{aligned}$$

Then we have the relation

$$G^{(n)}_{\text{ren}} = Z^{-n/2}G^{(n)}_{\text{bare}} \tag{3.2.3}$$

The 1PI vertex functions have n external legs amputated.

$$\Gamma^{(n)}_{\text{ren}} = Z^{n/2}\Gamma^{(n)}_{\text{bare}} \tag{3.2.4}$$

Now shift the renormalization scale $\mu \mapsto \mu + \mathrm{d}\mu$. Shift couplings to obtain the same physics

$$\begin{aligned} \lambda &\mapsto \lambda + \mathrm{d}\lambda \\ Z &\mapsto Z + \mathrm{d}Z \\ [m &\mapsto m + \mathrm{d}m] \end{aligned}$$

while the bare quantities $\lambda_0, [m_0]$ remain unchanged.

$$\Gamma^{(n)}_{\text{ren}}(x_1, \ldots, x_n, \lambda + \mathrm{d}\lambda\,, [m + \mathrm{d}m], \mu + \mathrm{d}\mu) = (Z + \mathrm{d}Z)^{n/2}\Gamma^{(n)}_{\text{bare}}(x_1, \ldots, x_n, \lambda_0, [m_0,](\Lambda)) \tag{3.2.5}$$

The infinitesimal changes are

$$\begin{aligned}
\mathrm{d}\Gamma^{(n)}_{\text{ren}} &= \frac{\partial \Gamma^{(n)}_{\text{ren}}}{\partial \mu}\,\mathrm{d}\mu + \frac{\partial \Gamma^{(n)}_{\text{ren}}}{\partial \lambda}\,\mathrm{d}\lambda \left(+ \frac{\partial \Gamma^{(n)}_{\text{ren}}}{\partial m}\,\mathrm{d}m \right) \\
&= \frac{n}{2} Z^{n/2-1}\,\mathrm{d}Z\,(Z + \mathrm{d}Z)^{n/2} \Gamma^{(n)}_{\text{bare}}(x_1, \ldots, x_n, \lambda_0, [m_0,](\Lambda))
\end{aligned}$$

Thus

$$\mu \frac{\partial \Gamma^{(n)}_{\text{bare}}}{\partial \mu} = 0$$

$$\mu \frac{\partial \Gamma^{(n)}_{\text{ren}}}{\partial \mu} + \underbrace{\mu \frac{\partial \lambda}{\partial \mu}}_{\beta} \frac{\partial \Gamma^{(n)}_{\text{ren}}}{\partial \lambda} \left[+ \underbrace{\mu \frac{\partial m}{\partial \mu}}_{m\gamma_m} \frac{\partial \Gamma^{(n)}_{\text{ren}}}{\partial m} \right] = \underbrace{\frac{n}{2} \frac{\mu}{Z} \frac{\partial Z}{\partial \mu}}_{n\gamma} \underbrace{Z^{n/2} \Gamma^{(n)}_{\text{bare}}}_{\Gamma^{(n)}_{\text{ren}}}$$

Callan-Symanzik Equation

$$\left[\mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial \lambda} \left(+ m\gamma_m \frac{\partial}{\partial m} \right) - n\gamma \right] \Gamma^{(n)}_{\text{ren}} = 0 \tag{3.2.6}$$

for Green's function $G^{(n)}_{\text{ren}}$ there is a sign change $+n\gamma$.

Per definition β and γ are independent of n. Since $\Gamma^{(n)}_{\text{ren}}$ is used, they are independent of cutoff Λ. For dimensional reasons, also independent of μ (in the massless case $m = 0$!). β and γ are only functions of the dimensionless coupling constant λ!

Compute β and λ in loop expansion (dimensional regularization and $\overline{\text{MS}}$) in ϕ^4; 2- and 4-functions must satisfy CS-equation.

Tree level

$$\begin{aligned}
i\Gamma^{(2)}_{\text{tree}} &= ip^2 \\
i\Gamma^{(4)}_{\text{tree}} &= -i\lambda
\end{aligned}$$

There is no μ-dependence, $\beta = \gamma = 0$ at tree level

One-loop

$$\begin{aligned}
i\Gamma^{(2)}_{\text{one-loop}} &= i(p^2 - \Sigma(p^2)) \\
&= \ldots \\
&= ip^2 - \frac{i\lambda\mu^{4-d}}{2} \underbrace{\int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{i}{q^2}}_{=0} - \delta^{(1)}_m + ip^2 \delta^{(1)}_Z
\end{aligned}$$

If it is massive theory, the loop integral is $\sim m^2$. So the renormalization condition is fulfilled for $\delta^{(1)}_m = 0$, $\delta^{(1)}_Z = 0$, then

$$i\Gamma^{(2)}_{\text{one-loop}} = i\Gamma^{(2)}_{\text{tree}} = ip^2 \tag{3.2.7}$$

See result from last term

$$
\begin{aligned}
i\,\Gamma^{(4)}_{\text{one-loop}}\Big|_{s=t=u=-\mu^2} &= -i\lambda - \frac{3i\lambda^2\mu^{4-d}}{16\pi^2}\left[\frac{1}{d-4} + \frac{1}{2}\left(\gamma_E - \ln(4\pi) + \underbrace{\int_0^1 \ln(x(1-x))}_{=-2}\right)\right] - i\delta_\lambda \\
&\stackrel{!}{=} -i\lambda
\end{aligned}
$$

$$
\delta_\lambda = -\frac{3\lambda^2}{16\pi^2}\mu^{4-d}\left[\frac{1}{d-4} + \frac{1}{2}\left(\gamma_E - \ln(4\pi)\right) - 1\right] \tag{3.2.8}
$$

Then the fully renormalized result is

$$
\begin{aligned}
i\Gamma^{(4)}_{\text{one-loop}}(s,t,u) &= -i\lambda - \frac{i\lambda^2}{32\pi^2}\mu^{4-d}\sum_{q^2=s,t,u}\left[\int \mathrm{d}x \ln\left(\frac{-q^2x(1-x)}{\mu^2}\right) + 2\right] \\
&\stackrel{d=4}{=} -i\lambda - \frac{i\lambda^2}{32\pi^2}\sum_{q^2=s,t,u}\left[\int \mathrm{d}x \ln\left(\frac{-q^2x(1-x)}{\mu^2}\right) + 2\right]
\end{aligned} \tag{3.2.9}
$$

Note on the convergence of the perturbative series: corrections can be large if $\lambda \ll 1$, but $\ln\left(\frac{q^2}{\mu^2}\right) \gg 1$. Choose μ^2 of the order of the typical momentum scale of the process. So large logarithm.

Conversion to different process with different momenta $q'^2 \neq q^2$ need CS-equation.

Now evaluate CS equations

$$
\begin{aligned}
i\Gamma^{(2)}_{\text{one-loop}} &= ip^2 \\
\left[\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial\lambda} - n\gamma\right]p^2 &= 0
\end{aligned}
$$

Thus

$$
\gamma = 0 \tag{3.2.10}
$$

at one-loop level.

$$
\begin{aligned}
\left[\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial\lambda} - n\gamma\right]\Gamma^{(n)}_{\text{one-loop}} &= 0 \\
\left[\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial\lambda}\right]\left[-\lambda - \frac{\lambda^2}{32\pi^2}\left(6 + \sum_{q^2=s,t,u}\int_0^1 \mathrm{d}x \ln\frac{-q^2x(1-x)}{\mu^2}\right)\right] &= 0 \\
\mu\left[0 - \frac{\lambda^2}{32\pi^2}\left(-\frac{6}{\mu}\right)\right] + \beta(-1 + O(\lambda)) = \frac{3\lambda^2}{16\pi^2} - \beta(1 + O(\lambda)) &= 0
\end{aligned}
$$

Thus

$$
\beta = \frac{3\lambda^2}{16\pi^2} + O\left(\lambda^3\right) \tag{3.2.11}
$$

Can we obtain this (expression for β) in an easier way? Yes! β is determined by the μ-dependence of the counterterm:

$$\mu \frac{\partial}{\partial \mu} \delta_\lambda^{(1)} = \mu \frac{\partial}{\partial \mu} \left(-\frac{3\lambda^2}{16\pi^2} \mu^{4-d} \left[\frac{1}{d-4} + \dots \right] \right) \tag{3.2.12}$$

$\mu^{4-d} = 1 + (4-d)\ln(\mu)$

$$= \frac{3\lambda^2}{16\pi^2} = \beta \tag{3.2.13}$$

It only works at one-loop, scheme dependence at higher loop orders.

The meaning of β and γ

$$\beta = \mu \frac{\partial}{\partial \mu} \lambda$$
$$\mathrm{d}\lambda = \beta(\lambda) \frac{\mathrm{d}\mu}{\mu}$$

It characterises rate of change in renormalized coupling for fixed bare coupling λ_0. Associate $\lambda(\mu)$ with coupling λ' of the Wilsonian RG: β is rate of RG flow for λ

- $\beta > 0$ coupling grows at large momenta and shrinks at small momenta
- $\beta < 0$ the opposite

$$\gamma(\lambda) = \frac{1}{2} \frac{\mu}{Z} \frac{\partial Z}{\partial \mu} = \frac{1}{2} \mu \frac{\partial}{\partial \mu} \ln(Z) \tag{3.2.14}$$

is the rate of change in field strength rescaling.

CS Equation for QED with $m_e = 0$

$$\left[\mu \frac{\partial}{\partial \mu} + \beta(e) \frac{\partial}{\partial e} - n\gamma_2(e) - m\gamma_3(e) \right] \Gamma_{\text{ren}}^{(n,m)} = 0$$

with n number of electron fields and m number of photon fields.

$$\beta(e) = \mu \frac{\partial}{\partial \mu} e = \frac{e^3}{12\pi^2} > 0$$
$$\gamma_2(e) = \frac{\mu}{2} \frac{\partial}{\partial \mu} \ln(Z_2) = \frac{e^2}{16\pi^2}$$
$$\gamma_3(e) = \frac{\mu}{2} \frac{\partial}{\partial \mu} \ln(Z_3) = \frac{e^2}{12\pi^2}$$

3.3 RG, Scaling of Γ and the Running Coupling

We will study the behaviour under rescaling of external momenta

$$\Gamma_{\text{ren}}^{(n)} (\{p_i\}; m, \lambda, \mu) \mapsto \Gamma_{\text{ren}}^{(n)} (s\{p_i\}; m, \lambda, \mu) \tag{3.3.1}$$

First rescale all dimensionfull quantities

$$\Gamma^{(n)}_{\text{ren}}\left(s\left\{p_i\right\};m,\lambda,\mu\right) = s^{4-n}\Gamma^{(n)}_{\text{ren}}\left(\left\{p_i\right\};m,\lambda,\mu\right) \tag{3.3.2}$$

according to mass dimensional of $\Gamma^{(n)}$ from dimensional analysis. For example $\Gamma^{(2)} = p^2 - m^2$, $\Gamma^{(4)} = -\lambda$

Expand around $s = 1$: $s = 1 + \mathrm{d}s$, so

$$\begin{aligned} p_i' &= (1+\mathrm{d}s)p_i \\ m' &= (1+\mathrm{d}s)m \\ \mu' &= (1+\mathrm{d}s)\mu \end{aligned}$$

$$\Gamma^{(n)}_{\text{ren}}\left(\left\{p_i'\right\};m',\lambda,\mu'\right) = \left[1 + (4-n)\,\mathrm{d}s\right]\Gamma^{(n)}_{\text{ren}}\left(\left\{p_i\right\};m,\lambda,\mu\right) \tag{3.3.3}$$

as a consequence

$$\left\{\sum_i p_i\frac{\partial}{\partial p_i} + m\frac{\partial}{\partial m} + \mu\frac{\partial}{\partial\mu} - (4-n)\right\}\Gamma^{(n)}_{\text{ren}}\left(\left\{p_i\right\};m,\lambda,\mu\right) = 0$$

or, slightly more elegantly

$$\left\{s\frac{\partial}{\partial s} + m\frac{\partial}{\partial m} + \mu\frac{\partial}{\partial\mu} - (4-n)\right\}\Gamma^{(n)}_{\text{ren}}\left(s\left\{p_i\right\};m,\lambda,\mu\right) = 0 \tag{3.3.4}$$

This equation expresses homogeneity properties of Γ, because Γ has mass dimension.

We also have the CS equation

$$\left\{\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial\lambda} + m\gamma_m\frac{\partial}{\partial m} - n\gamma\right\}\Gamma^{(n)}_{\text{ren}}(s\left\{p_i\right\};m,\lambda,\mu) = 0 \tag{3.3.5}$$

Combine these two equations together to eliminate $\mu\frac{\partial}{\partial\mu}$

$$\left\{-s\frac{\partial}{\partial s} + \beta\frac{\partial}{\partial\lambda} + m(\gamma_m - 1)\frac{\partial}{\partial m} - n\gamma + (4-n)\right\}\Gamma^{(n)}_{\text{ren}}(s\left\{p_i\right\};m,\lambda,\mu) = 0 \tag{3.3.6}$$

Solution

$$\Gamma^{(n)}_{\text{ren}}\left(s\left\{p_i\right\};m,\lambda,\mu\right) = s^{4-n}\Gamma^{(n)}_{\text{ren}}\left(\left\{p_i\right\};\overline{m}(s),\overline{\lambda}(s),\mu\right)\exp\left[-n\int_1^s\frac{\mathrm{d}s'}{s'}\gamma(\overline{\lambda}(s'))\right] \tag{3.3.7}$$

with s^{4-n} naive scaling according to engineering dimension. The exponential is anomalous scaling dimension, from rescaling of Z-factors.

- $\overline{\lambda}$ running coupling, $s\frac{\partial\overline{\lambda}}{\partial s} = \beta(\overline{\lambda}(s))$ and $\overline{\lambda}(1) = \lambda$
- $\overline{m}$ running mass, $s\frac{\partial\overline{\lambda}}{\partial s} = (\gamma_m - 1)\overline{m}(s)$ and $\overline{m}(1) = m$

Note: this is only solvable because β, γ_m and γ depend only on λ, but not on m! Consequence of a mass-independent renormalization scheme (not true in general).

Example massless ϕ^4 at one-loop, $\beta = \frac{3\lambda^2}{16\pi^2}$ and $\gamma = 0$

$$\left\{-s\frac{\partial}{\partial s} + \beta\frac{\partial}{\partial\lambda} + (4-n)\right\}\Gamma^{(n)}_{\text{ren}}\left(s\{p_i\};\lambda,\mu\right) = 0$$
$$\Gamma^{(n)}_{\text{ren}}\left(s\{p_i\};\lambda,\mu\right) = s^{4-n}\Gamma^{(n)}_{\text{ren}}\left(\{p_i\};\overline{\lambda}(s),\mu\right)\cdot 1$$

HOW?

$$s\frac{\partial}{\partial s}\overline{\lambda} = \frac{3\overline{\lambda}^2}{16\pi^2}$$
$$\frac{\partial}{\partial t}\overline{\lambda} = \frac{3\overline{\lambda}^2}{16\pi^2}$$

$t = \ln(s)$ RG equation for coupling

Integrate

$$\int_\lambda^{\overline{\lambda}} \frac{\mathrm{d}\overline{\lambda}'}{\overline{\lambda}'^2} = \int_0^t \mathrm{d}t'\,\frac{3}{16\pi^2}$$
$$-\frac{1}{\overline{\lambda}} + \frac{1}{\lambda} = \frac{3t}{16\pi^2}$$
$$\overline{\lambda}(t) = \frac{1}{1/\lambda - 3t/16\pi^2}$$
$$\overline{\lambda}(s) = \frac{\lambda}{1 - \frac{3\lambda}{16\pi^2}\ln s}$$

at $\ln s_0 = t_0 = \frac{16\pi^2}{3\lambda}$ or $s_0 = \exp\left(\frac{16\pi^2}{3\lambda}\right)$ the denominator vanishes. It is called *Landau pole*, where perturbation theory breaks down.

Practical application Want to predict observables for p^2 very different from μ^2. Large logs $\ln\frac{p^2}{\mu^2}$ slows down perturbation theory.

Use running coupling with $s = \sqrt{p^2/m^2} = p/\mu$, then

$$\overline{\lambda}(p) = \frac{\lambda}{1 - \frac{3\lambda}{16\pi^2\ln\left(\frac{p}{\mu}\right)}}$$
$$= \lambda\left[1 + \frac{3\lambda}{16\pi^2}\ln\frac{p}{\mu} + \dots\right]$$

λ is fixed at $p^2 = \mu^2$. Naive breakdown of perturbation theory can be deferred by re-summing "leading logarithms" (But no help if λ is large).

3.4 Running Coupling: Sign of the β-function

RG equation

$$\frac{\partial}{\partial\ln\frac{p}{\mu}}\bar{\lambda} = \beta(\bar{\lambda})$$

There are three qualitatively different behaviour

1. $\beta(\lambda) > 0$ at one-loop order

$$\phi^4 \qquad \beta(\lambda) = \frac{3\lambda^2}{16\pi^2} > 0 \tag{3.4.1}$$

$$\text{QED} \qquad \beta(e) = \frac{e^3}{12\pi^2} \tag{3.4.2}$$

In QED the running coupling would be

$$\int_e^{\bar{e}} \frac{\mathrm{d}e'}{e'^3} = \frac{1}{12\pi^2} \int_0^{\ln p/\mu} \mathrm{d}s\,,$$
$$-\frac{1}{2}\left(\frac{1}{\bar{e}^2} - \frac{1}{e^2}\right) = \frac{\ln(p/\mu)}{12\pi^2},$$
$$\bar{e}^2(p) = \frac{e^2}{1 - \frac{e^2}{6\pi^2}\ln(p/\mu)}.$$

Coupling is often written in terms of $\alpha = e^2/4\pi$

$$\bar{\alpha}(p) = \frac{\alpha}{1 - \frac{2\alpha}{3\pi}\ln(p/\mu)}. \tag{3.4.3}$$

It also has Landau pole. Coupling goes to zero in the IR limit, it becomes large in the UV limit. Long distance behaviour calculable in perturbation theory, but short-distance is not.

2. $\beta(\lambda) = 0$ and coupling does not flow. $\bar{\lambda}$ is independent of p, no UV divergences in relations between couplings, only divergences associated with z-factors, which are cancelled in the S-matrix elements. It leads to *finite QFTs*. Not very physical cases; gauge theories with extended supersymmetry.

3. $\beta(\lambda) < 0$ with QCD as an example. The β-function is

$$\beta(g) = -\frac{\beta_0 g^3}{16\pi^2}, \tag{3.4.4}$$
$$\beta_0 = 11 - \frac{2}{3}n_F,$$

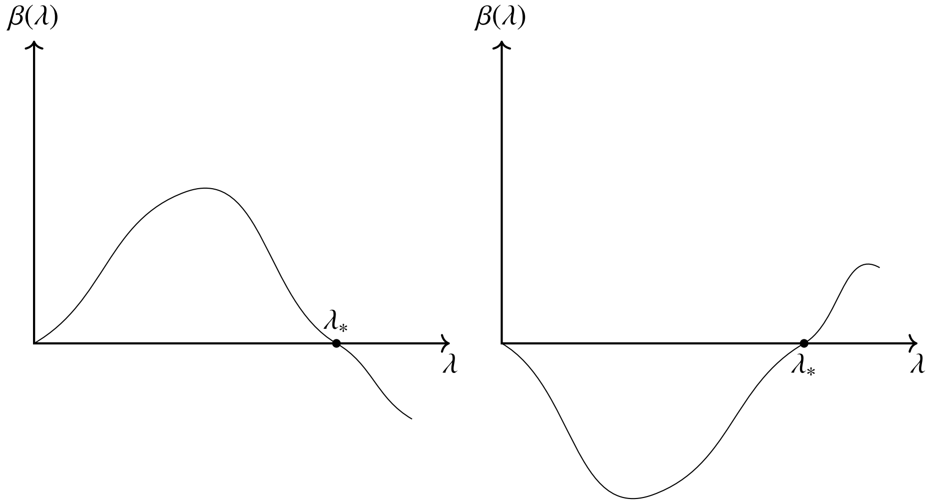
with n_F number of quark flavours. For $n_F = 6, \beta_0 > 0$. The running coupling is

$$\bar{\alpha}_s(p) = \frac{\bar{g}^2(p)}{4\pi} \tag{3.4.5}$$

$$\bar{\alpha}_s(p) = \frac{\alpha_s}{1 + \frac{\alpha_s}{2\pi}\beta_0 \ln(p/\mu)} \tag{3.4.6}$$

Coupling decreases as momentum scale increases. It is called asymptotic freedom. (Noble Prize 2004). It means that short-distance or high-energy behaviour perturbatively calculable, but not the long-distance one (confinement).

Non-trivial Fixed Points Possible forms of $\beta(\lambda)$



Near fixed point, one can approximate

$$\beta(\lambda) = -B(\lambda - \lambda_*) + O\left(\lambda^2\right) \quad \text{with } \begin{cases} B > 0 \\ B < 0 \end{cases} \tag{3.4.7}$$

So the RG for λ becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\ln(p/\mu)}\bar{\lambda} &\approx -B(\bar{\lambda} - \lambda_*) \\ \frac{\mathrm{d}\bar{\lambda}}{\bar{\lambda} - \lambda_*} &= -B\,\mathrm{d}\ln(p/\mu) \\ \ln\left(\bar{\lambda} - \lambda_*\right) &= C' - B\ln(p/\mu) \\ \bar{\lambda} &= \lambda_* + C\left(\frac{p}{\mu}\right)^{-B} \end{aligned}$$

So with $B > 0$,

$$\lim_{p\to\infty} \bar{\lambda}(p) = \lambda_*.$$

It is a UV-stable fixed point.

So with $B < 0$,

$$\lim_{p\to 0} \bar{\lambda}(p) = \lambda_*.$$

It is a IR-stable fixed point.

Note also that behaviour of the Green's functions

$$\Gamma^{(n)}_{\text{ren}}(s\{p_i\}\,;\lambda,\mu) = s^{4-n}\exp\left[-n\int_1^s \frac{\mathrm{d}s'}{s'}\gamma(\bar{\lambda}(s'))\right]\Gamma^{(n)}_{\text{ren}}(\{p_i\}\,;\bar{\lambda}(s),\mu) \tag{3.4.8}$$

for $\bar{\lambda}$ near a fixed point, $\int_1^s \frac{\mathrm{d}s'}{s'}\gamma(\bar{\lambda}(s')) \approx \gamma(\lambda_*)\ln s$

$$\Gamma^{(n)}_{\text{ren}}(s\{p_i\}\,;\lambda,\mu) = s^{4-n-n\gamma(\lambda_*)}\Gamma^{(n)}_{\text{ren}}(\{p_i\}\,;\lambda_*,\mu) \tag{3.4.9}$$

$$\Gamma^{(4)}_{\text{ren}}(p) \propto \left(\frac{p}{\mu}\right)^{4-n-n\gamma(\lambda_*)} \tag{3.4.10}$$

Simple power law, but with a different dimension from "engineering dimension" $4-n$. Hence $\gamma(\lambda_*)$ is called anomalous dimension.

4 Non-abelian Gauge Theories

4.1 Reminder: Gauge invariance

Demand invariance of Dirac theory under local phase transformation

$$\psi(x) \mapsto \mathrm{e}^{i\alpha(x)}\psi(x).$$

To have invariant Lagrangian density, mass term $-m\bar{\psi}(x)\psi(x)$ causes no problem. Definition of directional derivative by

$$n^\mu \partial_\mu \psi = \lim_{\epsilon\to 0} \frac{1}{\epsilon} \left[\psi(x+\epsilon n) + \psi(x)\right]. \tag{4.1.1}$$

$\psi(x+\epsilon n)$ and $\psi(x)$ have different behaviour under local phase (gauge) transformation.

Compensate this by introducing an operator

$$U(y,x) \mapsto \mathrm{e}^{i\alpha(y)} U(y,x) \mathrm{e}^{-i\alpha(x)}, \tag{4.1.2}$$

with $U(x,x) = 1$ and $U(y,x) = \mathrm{e}^{i\phi(x,y)}$.

Then define covariant derivative

$$n^\mu D_\mu \psi = \lim_{\epsilon\to 0} \frac{1}{\epsilon} \left[\psi(x+\epsilon n) - U(x+\epsilon n, x)\psi(x)\right]. \tag{4.1.3}$$

Infinitesimally,

$$U(x+\epsilon n, x) = 1 + ie\epsilon n^\mu A_\mu(x) + O\!\left(\epsilon^2\right). \tag{4.1.4}$$

It defines vector field A_μ. The covariant derivative is

$$D_\mu \psi(x) = \partial_\mu \psi(x) - ieA_\mu(x)\psi(x). \tag{4.1.5}$$

Combine (4.1.2) and (4.1.4),

$$\begin{aligned} 1 + ie\epsilon n \cdot A(x) &\to (1 + i\alpha(x+\epsilon n))(1 + ie\epsilon n \cdot A(x))(1 - i\alpha(x)), \\ &= 1 + ie\epsilon n \cdot \left(A(x) + \frac{1}{e}\partial\alpha(x)\right) + O\!\left(\epsilon^2\right). \end{aligned}$$

Hence the vector field transforms according to

$$A_\mu(x) \mapsto A_\mu(x) + \frac{1}{e}\partial_\mu \alpha(x) \tag{4.1.6}$$

Covariant derivative is indeed covariant

$$\begin{aligned} D_\mu\psi(x) \mapsto &\left[\partial_\mu - ie\left(A_\mu + \frac{1}{e}\partial_\mu\alpha(x)\right)\right] \mathrm{e}^{i\alpha(x)}\psi(x), \\ &= \mathrm{e}^{i\alpha(x)}\left(\partial_\mu - ieA_\mu\right)\psi(x), \\ &= \mathrm{e}^{i\alpha(x)} D_\mu\psi(x). \end{aligned}$$

In this way, we can construct derivative terms invariant under local phase transformation $i\bar{\psi}\not{D}\psi$ and potentially higher derivative if we don't care about renormalizability.

The field $A_\mu(x)$ also need kinetic term(s). Also second covariant derivatives are covariant, in particular,

$$\begin{aligned}\left[D_\mu, D_\nu\right]\psi &\mapsto \mathrm{e}^{i\alpha(x)}\left[D_\mu, D_\nu\right]\psi(x),\\ &= \left[\partial_\mu, \partial_\nu, \psi\right] - ie\left(\left[\partial_\mu, A_\nu\right] - \left[\partial_\nu, A_\mu\right]\right)\psi - e^2\left[A_\mu, A_\nu\right]\psi,\end{aligned}$$

We are dealing with classical theory at the moment. The commutator of the fields is zero.

$$\begin{aligned}&= -ie\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\psi,\\ &= -ieF_{\mu\nu}.\end{aligned}$$

Conclude $F_{\mu\nu}$ is invariant under local phase transformation.

All operators up to dimension 4

$$\mathcal{L}_4 = i\bar{\psi}\not{D}\psi - m\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - e\epsilon_{\mu\nu\alpha\beta}F^{\mu\nu}F^{\alpha\beta} \tag{4.1.7}$$

4.2 Yang-Mills Fields

It is the simplest example for a non-abelian gauge theory and was originally gauge theory for isospin.

Consider ψ with spinor in Minkowski space and "isospinor" in isospin space

$$\psi(x) = \begin{pmatrix}\psi_1(x)\\ \psi_2(x)\end{pmatrix} \tag{4.2.1}$$

Promote standard isospin invariant to a local transformation

$$\psi(x) \mapsto V(x)\psi(x), \tag{4.2.2}$$

$$V(x) = \exp\left(i\alpha^i(x)\sigma^i/2\right), \tag{4.2.3}$$

with σ^i Pauli matrices and $V(x) \in \mathbf{SU}(2)$. It is non-abelian, because different elements of $\mathbf{SU}(2)$ in general don't commute.

Repeat the construction from the previous section here. The transformation of an unitary matrix

$$U(y,x) \mapsto V(y)U(y,x)V(x)^\dagger, \tag{4.2.4}$$

with $U(x,x) = \mathbb{1}$. It is used for the construction of a covariant derivative. Infinitesimally,

$$U(x+\epsilon n, x) = \mathbb{1} + ig\epsilon n^\mu A^i_\mu\sigma^i/2 + O\left(\epsilon^2\right). \tag{4.2.5}$$

There are three vector fields A^i_μ with $i = 1,2,3$.

Covariant derivative

$$D_\mu = \partial_\mu - igA^i_\mu\sigma^i/2 \tag{4.2.6}$$

The transformation of A^i_μ is

$$1 + ig\epsilon n^\mu A^i_\mu\sigma^i/2 \mapsto V(x+\epsilon n)(1 + ig\epsilon n^\mu A^i_\mu\sigma^i/2)V(x)^\dagger.$$

Expand this to linear order in ϵ,

$$\begin{aligned} V(x+\epsilon n)V(x)^\dagger &= \Big[(1+\epsilon n^\mu \partial_\mu)V(x)\Big]V(x)^\dagger + O\big(\epsilon^2\big) \\ &= \mathbb{1} + \epsilon n^\mu(\partial_\mu V(x))V(x)^\dagger + O\big(\epsilon^2\big) \\ &= \mathbb{1} - \epsilon n^\mu V(x)(\partial_\mu V(x)^\dagger) + O\big(\epsilon^2\big) \end{aligned}$$

Hence

$$\frac{1}{2}A^i_\mu \sigma^i \mapsto V(x)\left[\frac{1}{2}A^i_\mu\sigma^i + \frac{i}{g}\partial_\mu\right]V(x)^\dagger. \tag{4.2.7}$$

For infinitesimal transformation $V(x) = \mathbb{1} + \frac{i}{2}\alpha^i(x)\sigma^i + O\big(\alpha^2\big)$. We find

$$\frac{1}{2}A^i_\mu\sigma^i \mapsto \frac{1}{2}A^i_\mu\sigma^i + \frac{1}{2g}(\partial_\mu\alpha^i)\sigma^i + \frac{i}{4}\alpha^i A^j_\mu\big[\sigma^i,\sigma^j\big], \tag{4.2.8}$$

$$A^i_\mu \mapsto A^i_\mu + \frac{1}{g}\partial_\mu\alpha^i - \epsilon^{ijk}\alpha^j A^k_\mu. \tag{4.2.9}$$

Third terms acts like a gauge field and third like an isovector. The isovector term is new compared to the abelian theory. Consequence of the non-commuting local transformation.

Introducing notation $\tilde{X} = \frac{1}{2}X^i\sigma^i$. Covariant derivative is now

$$\begin{aligned} D_\mu\psi &\mapsto \Big(\partial_\mu - ig\tilde{A}_\mu - i\partial_\mu\tilde{\alpha} + g\big[\tilde{\alpha},\tilde{A}\big]\Big)(1+i\tilde{\alpha})\psi(x), \\ &= \Big(\partial_\mu + i\tilde{\alpha}\partial_\mu - ig\tilde{A}_\mu + g\tilde{A}_\mu\tilde{\alpha} + g\big[\tilde{\alpha},\tilde{A}\big] + O\big(\alpha^2\big)\Big)\psi, \\ &= (1+i\tilde{\alpha})(\partial_\mu - ig\tilde{A}_\mu)\psi + O\big(\alpha^2\big), \\ &= (1+i\tilde{\alpha})D_\mu\psi + O\big(\alpha^2\big). \end{aligned} \tag{4.2.10}$$

Work out the details!

Introduce field strength through commutator of two covariant derivatives

$$\big[D_\mu, D_\nu\big] = -ig\tilde{F}_{\mu\nu}, \tag{4.2.11}$$

with

$$\tilde{F}_{\mu\nu} = \partial_\mu\tilde{A}_\nu - \partial_\nu\tilde{A}_\mu - ig\big[\tilde{A}_\mu,\tilde{A}_\nu\big], \tag{4.2.12}$$

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + g\epsilon^{ijk}A^j_\mu A^k_\nu. \tag{4.2.13}$$

Transformation behaviour of $\tilde{F}_{\mu\nu}$ from $\psi \mapsto V\psi$, $\big[D_\mu, D_\nu\big]\psi \mapsto V\big[D_\mu, D_\nu\big]\psi$

$$\tilde{F}_{\mu\nu}(x) \mapsto V(x)\tilde{F}_{\mu\nu}(x)V(x)^\dagger.$$

$\tilde{F}_{\mu\nu}$ is not gauge invariant any more.

Gauge invariant Lagrangian terms are however easily constructed

$$\mathcal{L} = -\frac{1}{2}\,\mathrm{tr}\Big(\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}\Big) = -\frac{1}{4}F^i_{\mu\nu}F^{i\,\mu\nu}. \tag{4.2.14}$$

Trace is taken in isospin space: $\mathrm{tr}\big(\sigma^i\sigma^j\big) = 2\delta^{ij}$. Note that in contrast to the abelian case, $\mathrm{tr}\big(\tilde{F}\tilde{F}\big)$ contains cubic and quartic terms in A^i_μ, hence already describes an <u>interacting</u> theory, not a free one.

Complete YM Lagrangian density

$$\mathcal{L} = -\frac{1}{2}\,\mathrm{tr}\Big(\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}\Big) + \bar{\psi}(i\not{D} - m)\psi. \tag{4.2.15}$$

Field equations are

$$\begin{aligned}(i\not{D} - m)\psi &= 0,\\ \Big[D^{\mu}, \tilde{F}_{\mu\nu}\Big] &= -g\tilde{j}_{\nu},\end{aligned}$$

with $j^i_\nu = \frac{1}{2}\bar{\psi}\gamma_\nu\sigma^i\psi$. These are highly non-linear set of equations.

Now

$$\begin{aligned}\Big[D_\mu, \Big[D_\nu, \tilde{F}^{\mu\nu}\Big]\Big] &= \frac{1}{2}\Big\{\Big[D_\mu, \Big[D_\nu, \tilde{F}^{\mu\nu}\Big]\Big] - \Big[D_\nu, \Big[D_\mu, \tilde{F}^{\mu\nu}\Big]\Big]\Big\}\\ &= \frac{1}{2}\Big[\Big[D_\mu, D_n u\Big], \tilde{F}^{\mu\nu}\Big]\\ &= 0\end{aligned}$$

Hence, from the second equation of motion

$$\Big[D^{\mu}, \tilde{j}_{\mu}\Big] = \partial^{\mu}\tilde{j}_{\mu} - ig\Big[\tilde{A}^{\mu}, \tilde{j}_{\mu}\Big] = 0. \tag{4.2.16}$$

This is the analogue of current conservation in Yang-Mills theory.

All this can be generalized to other (than $\mathbf{SU}(2)$) local symmetry group with

$$\sigma^i/2 \mapsto t^a,$$

a new set of hermitian generators and the general structure constants f^{abc}

$$\Big[t^a, t^b\Big] = if^{abc}t^c, \tag{4.2.17}$$

instead of ϵ^{ijk}.

4.3 Feynman Rules for Non-abelian Gauge theories

$$\mathcal{L} = \bar{\psi}\,(i\not{D} - m)\,\psi - \frac{1}{4}F^a_{\mu\nu}F^{a\,\mu\nu} \tag{4.3.1}$$

with

$$\begin{aligned}F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^v_\mu A^c_\nu\\ D_\mu &= \partial_\mu - igA^a_\mu t^a\end{aligned}$$

Fermion propagator is as before but with an internal quantum number

$$\langle 0|T\psi_A(x)\bar{\psi}_B(y)|0\rangle = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{i\delta_{AB}}{\not{k} - m}\mathrm{e}^{-ik(x-y)} \tag{4.3.2}$$

Fermion fields are (in) fundamental representation for $\mathbf{SU}(n)$ with $A, B = 1, \ldots, n$.

Suspect that there is an analogous gauge field propagator in Feynman gauge

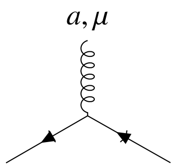
$$\langle 0|TA^a_\mu(x)A^b_\nu(y)|0\rangle = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{-ig_{\mu\nu}\delta^{ab}}{k^2}\mathrm{e}^{-ik(x-y)} \tag{4.3.3}$$

Fields $A_\mu(x)$ are in adjoint representation for $\mathbf{SU}(n)$ with $a, b = 1, \ldots, n^2 - 1$.

Interaction terms are

$$\mathcal{L} = \mathcal{L}_0 + gA^a_\lambda\bar{\psi}\gamma^\lambda t_a\psi - gf^{abc}(\partial_\kappa A^a_\lambda)A^{\kappa\, b}A^{\lambda\, c} - \frac{1}{4}g^2\left(f^{eab}A^a_\kappa A^b_\lambda\right)\left(f^{ecd}A^{\kappa\, c}A^{\lambda\, d}\right) \tag{4.3.4}$$

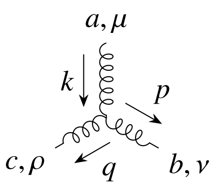
Feynman rules are as follows:



$$= ig\gamma^\mu t_a \tag{4.3.5}$$

It acts in Dirac space and on gauge group indices.

Functional derivatives or contractions generate $3! = 6$ terms and derivative turns into momentum. With all permutations



$$= gf^{abc}\left[g^{\mu\nu}(k-p)^\rho + g^{\nu\rho}(p-q)^\mu + g^{\rho\mu}(q-k)^\nu\right] \tag{4.3.6}$$

Altogether $4! = 24$ permutations for the last term. Only 4 each generate same contributions. Missing the diagram

$$= -ig^2\left\{f^{abe}f^{cde}\left(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\rho}g^{\nu\rho}\right) + f^{ace}f^{bde}\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho}\right) + f^{ade}f^{bce}\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma}\right)\right\} \tag{4.3.7}$$

By construction, as a consequence of gauge invariance, the same coupling g appears in three types of vertices.

4.4 Faddeev-Popov Quantization

We saw already for abelian gauge theory that invariance of the Lagrangian density makes the action under

$$A^a_\mu \mapsto A^a_\mu + \frac{1}{g}\partial_\mu\alpha^a - f^{abc}\alpha^b A^c_\mu, \tag{4.4.1}$$

$$= A^a_\mu + \frac{1}{g}D^{ab}_\mu\alpha^b, \tag{4.4.2}$$

leads to terribly divergent path integral for the generating functional

$$Z = \int \mathcal{D}A^a_\mu \mathrm{e}^{iS}.$$

Solution is to separate the gauge group volume by gauge fixing.

1. Gauge fixing of the form $F[A_\mu^a] = 0$, e.g. generalised Lorenz gauge
$$F[A_\mu^a] = \partial^\mu A_\mu^a(x) + C^a(x) = 0. \tag{4.4.3}$$

2. Define
$$\Delta_F^{-1}[A_\mu^a] = \int \mathcal{D}\alpha\delta(F[A_\mu^a]). \tag{4.4.4}$$
and insert $1 = \Delta_F[A_\mu^a]\int \mathcal{D}\alpha\delta(F[A_\mu^a])$ into the path integral. $\Delta_F[A_\mu^a]$ is gauge invariant. Exchange order of integration $\mathcal{D}A_\mu^a\mathcal{D}\alpha$
$$Z = \int \mathcal{D}\alpha \int \mathcal{D}A_\mu^a \Delta_F[A_\mu^a]\delta(F[A_\mu^a])\mathrm{e}^{iS}. \tag{4.4.5}$$

3. $\Delta_F[A_\mu^a]$ can be written as a functional determinant $\Delta_F[A_\mu^a] = \det\left|\frac{\delta F}{\delta\alpha}\right|_{F=0} =: \det(iM)$. From the transformation of gauge field and the generalised Lorenz gauge, we find
$$\frac{\delta F}{\delta\alpha} = \frac{1}{g}\partial^\mu D_\mu. \tag{4.4.6}$$
In the abelian case: $\mathcal{D}_\mu \mapsto \partial_\mu$ independent of A_μ, but not true here any more! $\det(iM)$ cannot simply be pulled out of the path integral!

4. Faddeev and Popov wrote $\det(iM)$ as a functional integral over (anti-commuting) Grassmann fields $\eta^a, \bar\eta^a$
$$\det(iM) = \int \mathcal{D}\bar\eta\mathcal{D}\eta \exp\left(-i\int \mathrm{d}^4x\,\bar\eta^a M_{ab}\eta^b\right). \tag{4.4.7}$$

5. Multiply Z with a constant $\int \mathcal{D}C \exp\left(-\frac{i}{\xi}\int \mathrm{d}^4x\, C^2(x)\right)$ and evaluate the delta functional. Result (in Lorenz gauge)
$$\begin{aligned} Z &= N\int \mathcal{D}A_\mu^a\mathcal{D}\bar\eta\mathcal{D}\eta \exp\left\{i\int \mathrm{d}^4x\left[\mathcal{L} - \frac{1}{2\xi}(\partial^\mu A_\mu^a)^2 - \eta^a M_{ab}\eta^b\right]\right\}, \\ &= N\int \mathcal{D}A_\mu^a\mathcal{D}\bar\eta\mathcal{D}\eta \exp\left(i\int \mathrm{d}^4x\,\mathcal{L}_{\mathrm{eff}}\right), \end{aligned} \tag{4.4.8}$$
where
$$\begin{aligned} \mathcal{L}_{\mathrm{eff}} &= \mathcal{L} + \mathcal{L}_{\mathrm{GF}} + \mathcal{L}_{\mathrm{FPG}}, \\ &= \mathcal{L} - \frac{1}{2\xi}(\partial^\mu A_\mu^a)^2 - \bar\eta^a\partial^\mu D_\mu^{ab}\eta^b. \end{aligned} \tag{4.4.9}$$
A factor $\sqrt{g}$ is absorbed in normalization of η and $\bar\eta$.

Interpretation Gauge fixing $\mathcal{L}_{\mathrm{GF}}$ is the same as in QED and it leads to gauge field propagator
$$\langle 0|TA_\mu^a(x)A_\nu^b(y)|0\rangle = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{-i}{k^2+i\epsilon}\left(g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right)\delta^{ab}\mathrm{e}^{-ik(x-y)}.$$

We also introduced Faddeev-Popov ghost fields. We find that they are anti-commuting (Grassmann-valued) but a scalar under Lorentz transformation. (It is scalar by construction!) Anything forbids us to adding Spinor structure in ghost fields? It violates the spin-statistics theorem. It cannot appear as external, asymptotic fields, but only in closed loops. (It should be later clear why they can only exist in loops.) It provides additional Feynman rules from $\mathcal{L}_{\text{FPG}}$

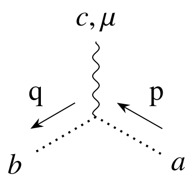
$$\mathcal{L}_{\text{FPG}} = -\bar{\eta}^a \partial^\mu D_\mu^{ab} \eta^b, \tag{4.4.10}$$

$$= -\bar{\eta}^a \left(\delta^{ab} \partial^2 - g f^{abc} (\partial^\mu A_\mu^c) - g f^{abc} A_\mu^c \partial^\mu \right) \eta^b. \tag{4.4.11}$$

First term leads to ghost propagator

$$\langle 0 | T \eta^a(x) \bar{\eta}^b(y) | 0 \rangle = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{i}{k^2 + i\epsilon} \delta^{ab} \mathrm{e}^{-ik(x-y)}. \tag{4.4.12}$$

Second and third terms introduce ghost-gauge-field interaction



$\sim -g f^{abc} p_\mu$

Note that due to anticommuting properties, there is an overall minus sign for any closed ghost loop.

Remark In the abelian case, we just ignored $\det(iM)$. We would have found only the kinetic terms the ghosts, no interaction. Ghost fields decouple from other firelds in QED.

Is there a gauge with which we might achieve this decoupling also in the non-abelian case? It leads to the axial gauge. Gauge condition $r^\mu A_\mu^a = 0$ with r^μ a space-like vector. Then the gauge fixing condition would be $F[A_\mu^a] = r^\mu A_\mu^a$. Under gauge transformation

$$A_\mu^a \mapsto A_\mu^a + \frac{1}{g} \partial_\mu \alpha^a - f^{abc} \alpha^b A_\mu^c, \tag{4.4.13}$$

we have

$$\left. \frac{\delta F^a}{\delta \alpha^b} \right|_{F=0} = \frac{1}{g} r^\mu \partial_\mu \delta^{ab}. \tag{4.4.14}$$

Independent of A_μ^a, ghosts decouple and it can be integrated out of the path integral again!

Disadvantage: complication gauge field propagator

$$\mathcal{L} + \mathcal{L}_{\text{GF}} = -\frac{1}{4} F_{\mu\nu}^a F^{a\,\mu\nu} - \frac{1}{2\xi} (r^\mu A_\mu^a)^2 \tag{4.4.15}$$

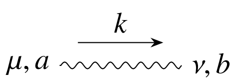
Quadratic part of the action is

$$\frac{1}{2} \int \mathrm{d}^4 x \, A_\mu^a \left(g^{\mu\nu} \partial^2 - \partial^\mu \partial^\nu - \frac{1}{\xi} r^\mu r^\nu \right) A_\nu^b$$

in momentum space

$$\ldots \left(-g^{\mu\nu} k^2 + k^\mu k^\nu - \frac{1}{\xi} r^\mu r^\nu \right) \ldots$$

Its inverse is the propagator



$$\sim -\frac{i}{k^2} \left(g^{\mu\nu} + \frac{(r^2 + \xi k^2) k^\mu k^\nu}{(k \cdot r)^2} - \frac{k^\mu r^\nu + r^\mu k^\nu}{k \cdot r} \right). \tag{4.4.16}$$

Complicated, not manifestly Lorentz-invariance.

4.5 BRST Symmetry

Rewrite the complete non-abelian Lagrangian (inclusive $\mathcal{L}_{\mathrm{GF}} + \mathcal{L}_{\mathrm{FPG}}$) with the help of the *auxiliary field* B^a

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \bar{\psi}(i\not{D} - m)\psi + \frac{\xi}{2}(B^a)^2 + B^a\partial^\mu A^a_\mu - \bar{\eta}^a\partial^\mu D^{ab}_\mu \eta^b. \tag{4.5.1}$$

No kinetic term for B^a, so it doesn't propagate but can be eliminated through the equation of motion

$$B^a = -\frac{1}{\xi}\partial^\mu A^a_\mu. \tag{4.5.2}$$

It leads back to original form of $\mathcal{L}$.

We know that $\mathcal{L}$ in equation 4.5.1 is not invariant under (infinitesimal) gauge transformations

$$\delta A^a_\mu = \frac{1}{g}D^{ab}_\mu \alpha^b, \quad \delta\psi = i\tilde{\alpha}\psi. \tag{4.5.3}$$

In QED, we used this to derive Ward-Takahashi identities.

Do something different here. In equation 4.5.3, set

$$\alpha^a = g\lambda\eta^a \Rightarrow \delta A^a_\mu = \lambda D^{ab}_\mu \eta^b.$$

η^a is the Grassmann-valued ghost field. Since $\alpha \in \mathbb{R}$, λ is an (infinitesimal) Grassmann number.

Show the invariance under the generalized transformation

$$\begin{aligned} \delta\eta^a &= -\frac{g}{2}f^{abc}\lambda\eta^b\eta^c \\ \delta\bar{\eta}^a &= \lambda B^a \\ \delta B^a &= 0 \end{aligned} \tag{4.5.4}$$

In principle, this is a supersymmetric transformation, as it links commuting and anti-commuting fields.

How the invariance of equation under 4.5.3 and 4.5.4

- $-\frac{1}{4}F^{a\mu\nu}F^a_{\mu\nu} + \bar{\psi}(i\not{D} - m)\psi$ are invariant, as we are just using re-parametrized gauge transformations.
- $+\frac{\xi}{2}(B^a)^2$ is trivially invariant as $\delta B^a = 0$.

What remains is

$$\begin{aligned} \delta\mathcal{L} &= B^a\delta^\mu\delta A^a_\mu - (\delta\bar{\eta}^a)\partial^\mu D^{ab}_\mu\eta^b - \bar{\eta}^a\partial^\mu\delta(D^{ab}_\mu\eta^b) \\ &= \underbrace{B^a\partial^\mu D^{ab}_\mu\lambda\eta^b - \lambda B^a\partial^\mu D^{ab}_\mu\eta^b}_{=0} - \bar{\eta}^a\delta^\mu\delta(D^{ab}_\mu\eta b) \end{aligned} \tag{4.5.5}$$

$$\begin{aligned} \delta D^{ab}_\mu\eta^b &= \partial(\delta\eta^a) + gf^{abc}\left[(\delta A^b_\mu)\eta^c + A^b_\mu(\delta\eta^c)\right], \\ &= -\frac{g}{2}\lambda\partial_\mu(f^{abc}\eta^b\eta^c) + gf^{abc}\lambda(\partial_\mu\eta^b)\eta^c + g^2\lambda f^{abc}f^{bde}A^d_\mu\eta^e\eta^c - \frac{1}{2}\lambda f^{abc}f^{cde}A^b_\mu\eta^d\eta^e. \end{aligned} \tag{4.5.6}$$

In $O(g)$, $\partial_\mu(\eta^b\eta^c) = (\partial_\mu\eta^b)\eta^c - (\partial_\mu\eta^c)\eta^b$ and use antisymmetry of structure constant to cancel. Terms in $O\left(g^2\right)$ can be rewritten as $-\frac{1}{2}g^2\lambda f^{abc}f^{cde}\left(A^b_\mu\eta^d\eta^e + A^d_\mu\eta^e\eta^b + A^e_\mu\eta^b\eta^d\right)$, which vanished accounting for the Jacobi identity $f^{ade}f^{bcd} + f^{bde}f^{cad} + f^{cde}f^{abd} = 0$.

BRST transformation is a global symmetry of the gauge-fixed Lagrangian (for arbitrary ξ). The corresponding conserved Noether current, charge Q is the generator of BRST symmetry transformation (with ϕ an arbitrary field)

$$\delta\phi = \lambda Q\phi,$$

e.g. $QA^a_\mu = D^{ab}_\mu \eta^b$.

What we have shown in equation 4.5.6 amounts to $Q^2 A^a_\mu = 0$. In addition $Q^2\bar{\eta}^a = 0$ as $QB^a = 0$. Further $Q^2\eta^a = 0$ because of

$$\begin{aligned}
\delta Q\eta^a &= -\frac{g^2}{2} f^{abc}\left[(\delta\eta^b)\eta^c + \eta^b(\delta\eta^c)\right],\\
&= \frac{g^2}{4} f^{abc}\left[\lambda f^{bde}\eta^d\eta^e\eta^c + \eta^b\lambda f^{cde}\eta^d\eta^e\right],\\
&= \frac{g^2}{2}\lambda f^{abc} f^{nde}\eta^c\eta^d\eta^e + \text{Jacobi identity}.
\end{aligned}$$

Finally $Q^2\psi = 0$ because of

$$\begin{aligned}
\delta(Q\psi) &= i(\delta\eta^a)t^a\psi i\eta^a t^a(\delta\psi),\\
&= -\frac{i}{2} g\lambda f^{abc}\eta^b\eta^c t^a\psi + g\lambda\eta^a t^a\eta^b t^b\psi,\\
&= -\frac{i}{2} g\lambda f^{abc}\eta^b\eta^c t^a\psi + g\lambda\frac{1}{2}\eta^a\eta^b\left[t^a, t^b\right] = \frac{i}{2}\eta^a\eta^b f^{abc} t^c,\\
&= 0.
\end{aligned}$$

Altogether we have proven the operator identity

$$Q^2 = 0 \tag{4.5.7}$$

The BRST charge operator is nilpotent and commutes with the Hamiltonian.

Q divides the space of eigenstates of H into three subspaces

$$\begin{aligned}
\mathscr{H}_1 &= \{|\psi_1\rangle \mid Q\,|\psi_1\rangle \neq 0\}\\
\mathscr{H}_2 &= Q\mathscr{H}_1\\
\mathscr{H}_0 &= \mathscr{H} - \mathscr{H}_1 - \mathscr{H}_2
\end{aligned}$$

Note $|\psi_{2(a/b)}\rangle \in \mathscr{H}_2$, $|\psi_0\rangle \in \mathscr{H}_0$

$$\begin{aligned}
\langle\psi_{1a}|Q|\psi_{2b}\rangle &= 0\\
\langle\psi_2|\psi_0\rangle = \langle\psi_1|Q|\psi_0\rangle &= 0
\end{aligned}$$

States in $\mathscr{H}_2$ have zero overlap with each other and with states in $\mathscr{H}_0$. Investigate which subpspaces the different polarization states of gauge bosons belong to: gauge bosons of momentum $k^\mu = (k^0, \boldsymbol{k})$ and $k^2 = 0$.

There two transverse polarization states, $\epsilon^T_{i\mu}$ and $\epsilon^T_{i\mu}k^\mu = 0$ with $i = 1, 2$. And two longitudinal ones $\epsilon^{+\mu} \propto k^\mu$, $\epsilon^+\mu k^\mu = 0$, forward polarization. $\epsilon^{-\mu} \propto (k_0, -\boldsymbol{k})$ and $\epsilon^-_\mu k^\mu = 0$ backward polarization. They form a complete basis.

Consider equation 4.5.3 and 4.5.4 for $g = 0$:

$$QA^a_\mu = \partial_\mu\eta^a \propto k_\mu \qquad \text{forward polarization}$$

$$Q\eta^a = 0,\ Q\bar{\eta}^a = B^a = -\frac{1}{\xi}\partial^\mu A^a_\mu \propto \epsilon_\mu k^\mu \qquad \text{backward polarization}$$

Interpretation Q transforms forward-polarization gauge bosons to ghost, ghosts to zero, anti-ghosts into backward-polarization gauge bosons.

$$\begin{aligned}\bar{\eta}, A^+_\mu &\in \mathscr{H}_1\\ \eta, A^-_\mu &\in \mathscr{H}_2\\ A^T_\mu &\in \mathscr{H}_0\end{aligned}$$

This holds in general. States with (anti-)ghosts and unphysical gauge-boson polarization states belong to $\mathscr{H}_{1/2}$; asymptotic states in $\mathscr{H}_0$ only contain physical states (transverse gauge bosons and fermions).

Consequence for the unitarity relation of the S-matrix. Let $A^T, B^T \in \mathscr{H}_0$ be asymptotic, physical states, then

$$\langle A^T|\mathbb{1}|B^T\rangle = \sum_{C^T\in\mathscr{H}_0} \langle A^T|S^\dagger|C^T\rangle\langle C^T|S|B^T\rangle$$

i.e. only physical intermediate states contribute, while unphysical states (longitudinal ghosts $\in \mathscr{H}_{1/2}$) drop out (exercise!).

Proof: as $A^T, B^T \in \mathscr{H}_0$, we know $Q|A^T\rangle = 0$. Since $[Q, H] = 0$ and hence $[Q, S] = 0$. $QS\,|A^T\rangle = 0$, therefore $S\,|A^T\rangle \in \mathscr{H}_0$ or $\mathscr{H}_2$. But states in $\mathscr{H}_2$ have zero overlap with $\mathscr{H}_0$. Thus only $\mathscr{H}_0$-states contribute!