H.10

a) $$2p_\mu I_2 = \int \frac{d^dk}{(2\pi)^d} \frac{-2i\,k^\mu p_\mu}{\ } = \frac{1}{16\pi^2} p^\mu p_\mu B_0$$

$$\frac{1}{2}\left[[k^2-m^2] - [(k-p)^2 - m^2] + p^2 \right]$$

$$= -i \int \frac{d^dk}{(2\pi)^d} \left\{ \underbrace{\frac{1}{(k-p)^2-m^2} - \frac{1}{k^2-m^2}}_{\substack{=0 \\ \text{after substitution}}} + \frac{p^2}{(\ )(\ )} \right\}$$

$$\Rightarrow 2p_\mu I_2 = \int \frac{d^dk}{(2\pi)^d} \frac{-i}{(k^2-m^2)((p-k)^2-m^2)} = I_1$$

b) $$I_1 = -i \int \frac{d^dk}{(2\pi)^d} \int_0^1 dx \left\{ \underbrace{(1-x)(k^2-m^2) + x[(p-k)^2-m^2]}_{} \right\}^{-2}$$

$$= k^2 - 2xpk + m^2 + xp^2$$

$$= (k-xp)^2 - x^2p^2 + xp^2 + m^2$$

$$= (k-xp)^2 - m^2 + x(1-x)p^2$$

$$=: q^2 - \Delta$$

with $q = k - xp$, $\Delta = m^2 + x(x-1)p^2$

Do a Wick rotation: $ik_E^0 = k^0$

$$I_1 = -i \cdot i \int \frac{d^dq_E}{(2\pi)^d} \int_0^1 dx \frac{1}{[q_E^2 + \Delta]^2}$$

$$= \int_0^1 dx\, (4\pi)^{-d/2} \frac{\Gamma(2-d/2)}{\Gamma(2)} \Delta^{d/2-2}$$

$\mu = 4 - 2\epsilon$, $\epsilon \to 0$

$$= (4\pi)^{-\frac{(4-2\epsilon)}{2}} \frac{\Gamma(2-\frac{4-\epsilon}{2})}{\Gamma(2)} \int_0^1 dx \quad \Delta^{\frac{4-2\epsilon}{2}-2}$$

$$= (4\pi)^{-2} (4\pi)^{+\epsilon} \Gamma(\epsilon) \int_0^1 dx \quad [\underbrace{m^2 + x(x-1)p^2}_{\Delta}]^{-\epsilon}$$

$$\left(a^{\epsilon} = 1 + \ln a \cdot \epsilon + \mathcal{O}(\epsilon^2) \right.$$

$$= (4\pi)^{-2} \left[1 + \ln(4\pi)\epsilon + \mathcal{O}(\epsilon^2) \right] \left(\frac{1}{\epsilon} - \gamma_E + \mathcal{O}(\epsilon) \right) \int_0^1 dx \left[1 - \ln\Delta \cdot \epsilon + \mathcal{O}(\epsilon^2) \right]$$

$$= \frac{1}{(4\pi)^2} \int_0^1 dx \left[\frac{1}{\epsilon} - \gamma_E + \ln(4\pi) - \ln\Delta + \mathcal{O}(\epsilon^2) \right]$$

c) $\int_0^1 dx \ \ln(m^2 + x(x-1)p^2)$ $\qquad y = \frac{4m^2}{p^2} < 0$, not physical!

$$= \int_0^1 dx \ln \frac{m^2 + x(x-1)p^2}{m^2} + \ln m^2$$

$$= \int_0^1 dx \left[\ln\left(1 + \tfrac{1}{4} x(x-1)/y\right) + \ln m^2 \right]$$

$$= \int_0^1 dx \left\{ \ln \underbrace{\left(\frac{\sigma+1}{2} - x \right)\left(x + \frac{\sigma-1}{2} \right)} - \ln \underbrace{\frac{\sigma^2-1}{4}} + \ln m^2 \right\} \quad , \quad \sigma = \sqrt{1-y}$$

$$= \frac{\sigma^2-1}{4} - x^2 + \frac{\sigma+1-\sigma+1}{2} x \qquad = \frac{1-y-1}{4} = -\frac{y}{4}$$

$$= \frac{1-y-1}{4} - x^2 + x$$

$$= -\frac{y}{4} + x(-x+1)$$

$$= -\ln\frac{\sigma^2-1}{4} + \ln m^2 + \underbrace{\int_0^1 dx \ln\left(\frac{\sigma+1}{2} - x\right)} + \underbrace{\int_0^1 dx \ \ln\left(\frac{\sigma-1}{2} + x\right)}$$

$$= \underline{-\frac{\sigma-1}{2} \ln\frac{\sigma-1}{2}} + \frac{\sigma-1}{2} \qquad = \underline{\frac{\sigma+1}{2} \ln\frac{\sigma+1}{2}} - \frac{\sigma+1}{2}$$

$$+ \underline{\frac{\sigma+1}{2} \ln\frac{\sigma+1}{2}} - \frac{\sigma+1}{2} \qquad \underline{- \frac{\sigma-1}{2} \ln\frac{\sigma-1}{2}} + \frac{\sigma-1}{2}$$

$$= -\ln\frac{\sigma^2-1}{4} + \ln m^2 + \ln\frac{\sigma-1}{2}\cdot(1-\sigma) + \ln\frac{\sigma+1}{2}\cdot(\sigma+1) - 2$$

$$= \ln m^2 - 2 \quad - \ln\left(\frac{\sigma+1}{2}\right) - \ln\left(\frac{\sigma-1}{2}\right) + (1-\sigma)\ln\frac{\sigma-1}{2} + (1+\sigma)\ln\frac{\sigma+1}{2}$$

$$= \ln m^2 - 2 + \sigma\ln\frac{\sigma+1}{\sigma-1}$$

if $y > 1$, $\quad \sigma = \sqrt{1-y} \in \mathbb{C}$

$$= i\sqrt{y-1}$$

$$\Rightarrow B_0(m^2, p_0, m^2) - B_0(m^2, 0, m^2)$$

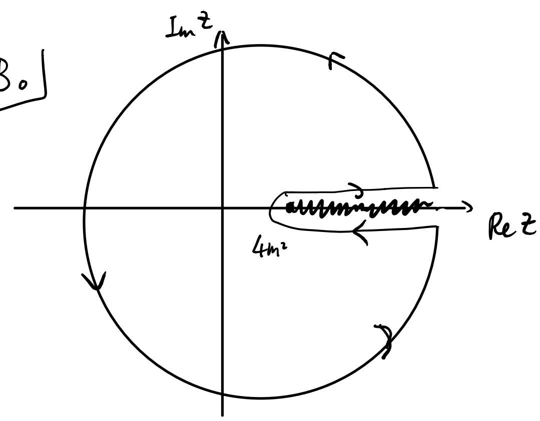
$$= i\sqrt{y-1}\,\ln\frac{i\sqrt{y-1}+1}{i\sqrt{y-1}-1}$$

$$\left(\arctan z = \frac{i}{2}\ln\frac{i+z}{i-z}\right)$$

$$= i\sqrt{y-1}\,\ln\frac{i+\frac{1}{\sqrt{y-1}}}{i-\frac{1}{\sqrt{y-1}}}$$

$$= -2\sqrt{y-1}\,\arctan\left(\frac{1}{\sqrt{y-1}}\right)$$

d) $\underline{B_0}$



$$B_0(p^2, m, m) = \frac{1}{2\pi i}\oint_\gamma dz\,\frac{B_0(z, m, m)}{z - p^2}$$

$$= \frac{1}{2\pi i}\int_{4m^2}^{\infty} dz\,\frac{B_0(z+i\varepsilon, m, m)}{z-p^2} + \frac{1}{2\pi i}\int_{\infty}^{4m} dz\,\frac{B_0(z-i\varepsilon, m, m)}{z-p^2}$$

$$+\frac{1}{2\pi i}\underbrace{\int_{z^2=\Lambda^2} dz \frac{B_0(z, m, m)}{z-p^2}}_{\to 0}$$

$$= \frac{1}{\pi}\int_{4m^2}^{\infty} dz \frac{\mathrm{Im}\, B_0(z, m, m)}{z-p^2}$$

$$\Rightarrow B_0(p^2, m, m) - B_0(0, m, m) = \frac{1}{\pi}\int_{4m^2}^{\infty} dz\, \mathrm{Im}\, B_0(z, m, m)\underbrace{\left(\frac{1}{z-p^2} - \frac{1}{z}\right)}_{= \frac{+p^2}{z(z-p^2)}}$$

$$= \frac{p^2}{\pi}\int_{4m^2}^{\infty} dz \frac{\mathrm{Im}\, B_0(z, m, m)}{z(z-p^2)}$$

$$\mathrm{Im}\, B_0(z, m, m) = (-i)^3\int \frac{d^4k}{(2\pi)^4}\, 2\pi\delta(k^2-m^2)\, 2\pi\delta((k-p)^2-m^2)$$

$$= \frac{i}{(2\pi)^2}\int d^4k\, \underbrace{\delta(k^2-m^2)\,\delta((k-p)^2-m^2)}$$

$$= \delta(k^2-m^2)\,\delta(k^2-m^2+p^2-2k\cdot p)$$

$$= \delta(p^2-2k\cdot p)$$

$$= \pi\, \Theta(p^2-4m^2)\underbrace{\sqrt{1-\frac{4m^2}{p^2}}}_{=\sqrt{1-y} = \sigma}$$

$p^2 < 0, \quad y < 0 \quad \Rightarrow \sigma \in \mathbb{R}$

$$\Rightarrow B_0(\cdot\cdot) - B_0(\cdot\cdot) = \frac{p^2}{\pi^2}\int_{4m^2}^{\infty} dz \frac{1}{z(z-p^2)}\,\pi\,\underbrace{\Theta(z-4m^2)}_{=1}\sqrt{1-\frac{4m^2}{z}}$$

$$= \frac{p^2}{\pi}\int_{4m^2}^{\infty} dz \frac{\sqrt{1-4m^2/z}}{z(z-p^2)}$$

Define $\sigma = \sqrt{1-\frac{4m^2}{p^2}}$, $\rho = \sqrt{1-\frac{4m^2}{z}}$

$$\frac{1}{z(z-p^2)} = \frac{1}{z}\cdot\frac{4m^2}{\sigma^2-\rho^2} \quad \leftarrow \quad \sigma^2-\rho^2 = 4m^2(-p^2+z)$$

$$\frac{d\rho}{dz} = \frac{1}{\sqrt{1-\frac{4m^2}{z}}}\cdot\frac{4m^2}{z^2} \quad \Rightarrow \quad dz = \rho\,\frac{z^2}{4m^2}\,d\rho$$

$$z = 4m^2 \Leftrightarrow \rho = 0$$

$$z = \infty \Leftrightarrow \rho = 1$$

$$\Rightarrow B_0(\dots) - B_0(\dots) = \frac{p^2}{\pi}\int_0^1 d\rho \;\; \rho^2\,\frac{\cancel{z^2}}{\cancel{4m^2}}\;\frac{1}{\cancel{z}}\;\frac{\cancel{4m^2}}{\sigma^2-\rho^2}$$

$$= \frac{p^2}{\pi}\int_0^1 d\rho \;\; \frac{\rho^2}{\sigma^2-\rho^2}$$

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$$= \frac{p^2}{\pi}\left(-\rho + \rho\cdot\arctan\frac{\rho}{\sigma}\right)\Big|_0^1$$

$$= \frac{p^2}{\pi}\left(-1 + \arctan\frac{1}{\sigma}\right)$$