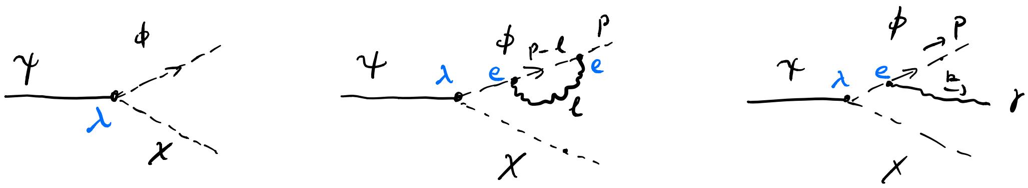
H.11

a) $\psi(p) \longrightarrow \phi(p)\,\chi(q)\,[\gamma(k)]$

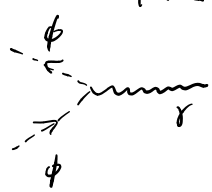


virtual correction

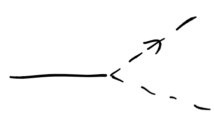
Feynman rules:

Propagator (ψ, ϕ, χ) : $\dfrac{i}{p^2 - m_i^2 + i\varepsilon}$

photon propagator : $\dfrac{i}{p^2 + i\varepsilon}$



$\sim -ie$



$\sim -i\lambda$

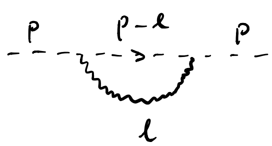
$\rightarrow \quad i\mathcal{M}_a = +i\lambda$

$$i\mathcal{M}_b = +i\lambda\,(-ie)^2 \int \frac{d^d\ell}{(2\pi)^d}\,\frac{i}{(p-\ell)^2 - m^2}\,\frac{i}{\ell^2}\cdot\frac{1}{p^2 - m^2}$$

$$i\mathcal{M}_c = +i\lambda\,(-ie)\,\frac{i}{\underbrace{(p-k)^2}_{=2p\cdot k} - m^2} = -i\mathcal{M}_a\,\frac{e}{2p\cdot k}$$

on-shell !

b)



$$-i\Sigma(p^2) = (-ie)^2 \int \frac{d^d\ell}{(2\pi)^d}\,\frac{i}{(p-\ell)^2 - m^2}\,\frac{i}{\ell^2 - \mu^2}$$

$$= e^2 \int \frac{d^d\ell}{(2\pi)^d}\,\underbrace{\frac{1}{(p-\ell)^2 - m^2}\,\frac{1}{\ell^2 - \mu^2}}$$

$$= \int_0^1 dx \; \{x[(p-\ell)^2 - m^2] + (1-x)(\ell^2 - \mu^2)\}^{-2}$$

$$= \int_0^1 dx \; \{ xp^2 - \underbrace{2xp\ell} + \cancel{x\ell^2} - xm^2 + \underline{\ell^2} - \cancel{x\ell^2} - \mu^2 + x\mu^2 \}^{-2}$$

$$= \int_0^1 dx \; \{(\ell - xp)^2 - x^2p^2 + xp^2 - xm^2 + (x-1)\mu^2\}^{-2}$$

$$= \int_0^1 dx \; \{\underbrace{(\ell - xp)^2}_{=: q^2} + \underbrace{x(-x+1)p^2 - xm^2 + (x-1)\mu^2}_{=: -\Delta(x)}\}^{-2}$$

$$= +e^2 \int \frac{d^dq}{(2\pi)^d} \int_0^1 dx \; \frac{1}{(q^2 - \Delta(x))^2}$$

Wick rotation: $q^0 \to i q^0_E$

$$= ie^2 \int \frac{d^dq_E}{(2\pi)^d} \int_0^1 dx \; \frac{1}{(q_E^2 + \Delta(x))^2}$$

$\leftarrow$ $\ell \to 0$, $q \to -xp$, $p^2 = m^2$

$\sim \int_0^1 dx \; \frac{1}{[(x-1)\mu^2]^2}$

is IR-finite!

$$= \frac{+ie^2}{(4\pi)^2} \left(\frac{1}{\epsilon} - \gamma_E + \ln 4\pi + \int_0^1 dx \; \ln\Delta(x) + \mathcal{O}(\epsilon) \right)$$

However, $$\Sigma'(m^2) = \frac{d\Sigma(p^2)}{dp^2}\Big|_{p^2=m^2}$$

$$= ie \int \frac{d^dq_E}{(2\pi)^d} \int_0^1 dx \; \underbrace{\frac{d}{dp^2} \frac{1}{(q_E^2 + \Delta)^2}}\Big|_{p^2=m^2}$$

$$= -2 \frac{1}{(q_E^2 + \Delta)^3} \frac{d\Delta}{dp^2}$$

$$\Delta(x) = x(-x+1)p^2 - xm^2 + (x-1)\mu^2$$

$$\simeq xp^2 - xm^2 - \mu^2$$

$$\to \frac{d\Delta}{dp^2} = x \; .$$

$$= ie \int \frac{d^d q_E}{(2\pi)^d} \int_0^1 dx \frac{-2}{(q_E^2+\Delta)^3}\, x \Big|_{p^2=m^2}$$

$$\frac{d}{dp^2}\int_0^1 dx \ln\Delta(x) = \int_0^1 dx \frac{1}{\Delta(x)}\cdot x(-x+1) = \int_0^1 dx \frac{x(-x+1)}{\underbrace{x(1-x)p^2 - xm^2}_{m^2=p^2 \rightarrow\, = x^2m^2} + (x-1)\mu^2}\Bigg\}$$

$$\simeq \frac{x}{x^2m^2+\mu^2} + IR-\text{finite}.$$

$$= \int_0^1 dx \frac{x(-x+1)}{(x-1)(-xp^2+\mu^2)-xm^2}$$

$1-x \simeq 1$
$$\simeq \int_0^1 dx \frac{x}{+xp^2-\mu^2-xm^2}$$

$$= \int_0^1 dx \frac{x}{x(p^2-m^2)-\mu^2}$$

if $\mu \to 0$ $\quad \sim \int_0^1 dx \frac{1}{p^2-m^2}$.

$$\rightarrow \quad \Sigma'(m^2) \sim \infty$$

$$-i\Sigma'(m^2) = e^2\int_0^1 dx \int \frac{d^dq}{(2\pi)^d}\frac{-2x(1-x)}{[q^2-\Delta]^3}$$

$$= ie^2\int_0^1 dx \underbrace{\frac{1}{(4\pi)^{d/2}}\frac{2x(1-x)}{\Delta^{3-d/2}}\frac{\Gamma(3-\frac{d}{2})}{\Gamma(3)}}_{\lim\ d\to 4\ \text{finite!}}$$

$$= \frac{ie^2}{16\pi^2}\int_0^1 \frac{x\overbrace{(1-x)}^{\simeq 1}}{x^2-m^2+\underbrace{(1-x)}_{\sim 1}\mu^2}dx$$

comes from $(1-x)\simeq 1$

The approximation separate the finite and infinite parts

$$\Rightarrow \quad \Sigma'(m^2) = \frac{-e^2}{16\pi^2}\int_0^1 \frac{x}{x^2m^2+\mu^2}dx + IR-\text{finite}$$

$$= -\frac{e^2}{16\pi^2}\frac{1}{2m^2}\underbrace{\int_{\mu^2}^{m^2+\mu^2}\frac{dy}{y}}_{=\log\left(\frac{m^2+\mu^2}{\mu^2}\right)} + IR-\text{finite}$$

$$= \frac{e^2}{16\pi^2 m^2} \log\left(\frac{\mu}{m}\right) + IR\text{-finite}$$

$$= \mathcal{M}_{\text{tree}} \times \sqrt{Z_\phi}$$

$$\frac{i}{p^2 - m_0^2 + \Sigma(p^2)} = \frac{i Z_\phi}{p^2 - m^2}$$

$$m^2 = m_0^2 + \Sigma(m^2)$$

Expand $\quad p^2 - m_0^2 - \Sigma(p^2) = \underbrace{m^2 - m_0^2 - \Sigma(m^2)}_{=0} + (p^2 - m^2)\left[1 - \Sigma'(m^2)\right] + \cdots$

$$= (p^2 - m^2)\left[1 - \Sigma'(m^2)\right] + \cdots$$

$$\frac{i Z_\phi}{p^2 - m^2} = \frac{i\left(1 - \Sigma'(m^2)\right)^{-1}}{p^2 - m^2} \longrightarrow Z_\phi = \frac{1}{1 - \Sigma'(m^2)} \simeq 1 + \Sigma'(m^2)$$

$$= 1 + \frac{e^2}{16\pi^2 m^2} \log\frac{\mu}{m} + IR\text{-finite}$$

$$|\mathcal{M}_b|^2 = |\mathcal{M}_a|^2 \left(1 + \frac{e^2}{16\pi^2 m^2} \log\frac{\mu}{m} + IR\text{-finite}\right)$$

c) $$d\Gamma_A = \frac{1}{2M} \frac{d^3 p}{(2\pi)^3 2E_p} \frac{d^3 q}{(2\pi)^3 2E_q} |\mathcal{M}_A|^2 (2\pi)^4 \delta^{(4)}(\underbrace{P - q - q}_{P - p - q})$$

$$d\Gamma_B = \frac{1}{2M} \frac{d^3 p}{(2\pi)^3 2E_p} \frac{d^3 q}{(2\pi)^3 2E_q} \frac{d^3 k}{(2\pi)^3 2E_q} |\mathcal{M}_B|^2 (2\pi)^4 \delta^{(4)}(\underbrace{P - q - q}_{P - p - q - \underbrace{k}_{\sim 0}})$$

photo extreme soft

$\longrightarrow$ equation (3)

d) $$|\tilde{\mathcal{M}}_B|^2 = \int \frac{d^3k}{(2\pi)^3 2E_k} |\mathcal{M}_B|^2$$

$$= \int \frac{d^3k}{(2\pi)^3 2E_k} \lambda^2 e^2 \left(\frac{1}{(p-k)^2 - m^2}\right)^2$$

$$= \frac{4\pi}{8\pi^3} \lambda^2 e^2 \int_{E_{min}}^{\infty} \frac{dk\, k^2}{2\sqrt{k^2+\mu^2}} [\underbrace{(p-k)^2 - m^2}_{2p\cdot k}]^{-2}$$

$$|\tilde{\mathcal{M}}_B|^2 = \int \frac{d^3k}{(2\pi)^3 2E_k} |\mathcal{M}_B|^2$$

$$= |\mathcal{M}_{tree}|^2 \int \frac{d^3k}{(2\pi)^3 2E_k} \frac{e^2}{4(p\cdot k)^2}$$

$E_k^2 = |\vec{k}|^2 + \mu^2$

switch to $d^3k = 2\pi \underbrace{dz}_{d\cos\theta} dk\, k^2 = 2\pi\, dz\, E_k^2\, dE_k$, $p = (m, 0, 0, 0)^t$, $k = (E_k, \vec{k})^t$

$$= |\mathcal{M}_{tree}|^2 \frac{e^2}{8(2\pi)^2} \int_{-1}^{1} dz \int_{\mu}^{E_{max}} \frac{dE_k}{E_k} \frac{E_k^2}{(mE_k)^2}$$

$$= |\mathcal{M}_{tree}|^2 \frac{e^2}{16\pi^2 m^2} \log\left(\frac{E_{max}}{\mu}\right)$$ denominator cancels the divergence

$$|\mathcal{M}_A|^2 + |\tilde{\mathcal{M}}_B|^2 = |\mathcal{M}_{tree}|^2 \left(1 + \frac{e^2}{16\pi^2 m^2} \log\frac{E_{max}}{m} + \text{IR-finite}\right)$$

$\rightarrow$ finite!