H.12 $$\mathcal{L} = -\frac{1}{2} H(\partial^2 + M^2) H + \bar{\Psi}(i\not{\partial} - m)\Psi - g\bar{\Psi}\Psi H$$

a)

Introduce sources j and η of H and Ψ

$$\mathcal{L} \to \mathcal{L} + Hj + \bar{\Psi}\eta + \bar{\eta}\Psi ,$$

$$\mathcal{L}_0 = -\frac{1}{2} H(\partial^2 + M^2) H + \bar{\Psi}(i\not{\partial} - m)\Psi + Hj + \bar{\Psi}\eta + \bar{\eta}\Psi$$

the free Lagrangian

$$W[J] = \frac{\langle 0|0\rangle_{j,\eta,\bar{\eta}}}{\langle 0|0\rangle_{j=\eta=\bar{\eta}=0}}$$

$$\langle 0|0\rangle_{j,\eta,\bar{\eta}} = \int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi\mathcal{D}H \exp(i\int d^4x\ \mathcal{L}[\bar{\Psi},\Psi,H,\bar{\eta},\eta,j])$$

$$= \int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi\mathcal{D}H \exp(i\int d^4x \mathcal{L}_0)\exp(-i\int d^4x\ gH\bar{\Psi}\Psi)$$

$$\langle 0|0\rangle_{j=\eta=\bar{\eta}=0} = \int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi\mathcal{D}H \exp(i\int d^4x\ \mathcal{L}[\bar{\Psi},\Psi,H,\bar{\eta}=\eta=j=0])$$

$$= \int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi\mathcal{D}H \exp(i\int d^4x \mathcal{L}_0|_{j=\eta=\bar{\eta}=0})\exp(-i\int d^4x\ gH\bar{\Psi}\Psi)$$

$$H \to \frac{1}{i}\frac{\delta}{\delta j}, \quad \Psi \to \frac{1}{i}\frac{\delta}{\delta\bar{\eta}}, \quad \bar{\Psi} \to -\frac{1}{i}\frac{\delta}{\delta\eta}$$

$$\exp(-i\int d^4x\, gH\bar{\Psi}\Psi) \to \exp\left[-ig\int d^4x \frac{1}{i}\frac{\delta}{\delta j(x)}\frac{1}{i}\frac{\delta}{\delta\eta(x)}\frac{1}{i}\frac{\delta}{\delta\bar{\eta}(x)}\right]$$

b) $F = -g\bar{\Psi}\Psi$, no external sources

$$\langle 0|0\rangle_{j,\bar{\eta},\eta} = \int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi\mathcal{D}H \exp(i\int d^4x(-\frac{1}{2}H\hat{D}_H H + \bar{\Psi}\hat{D}_\Psi\Psi + FH))$$

Shift $$H \to H' = H - i\int d^4y\ D_F(x-y)F(y)$$

$$F(y) = -g\,\bar{\Psi}(y)\,\Psi(y)\,H(y)$$

$$S_H = \int d^4x \left(-\tfrac{1}{2} H \hat{D}_H H + F H\right)$$

$$\rightarrow S_{H'} = \int d^4x \left(-\tfrac{1}{2} H' \hat{D}_H H' + F H'\right)$$

$$= \int d^4x \Big\{ -\tfrac{1}{2} H' \hat{D}_H H' + FH$$

$$- i \int d^4y \, F(x) D_F(x-y) F(y) \Big\}$$

$$\Rightarrow \langle 0|0\rangle_{j,\eta,\bar\eta} = \int \mathcal{D}H' \exp\left(-\tfrac{i}{2}\int d^4x \, H' \hat{D}_H H'\right)$$

$$\times \int \mathcal{D}\bar\Psi \mathcal{D}\Psi \exp\left(i \int d^4x \left(\bar\Psi \hat{D}_\Psi \Psi - i \int d^4y \, F(x) D_F(x-y) F(y)\right)\right)$$

$$\left(\begin{array}{l} \text{First factor is Gaussian integral} \\ \int \mathcal{D}H' \exp\left(-\tfrac{i}{2}\int d^4x \, H' \hat{D}_H H'\right) = (\det \hat{D}_H)^{-\frac{1}{2}} \\ \text{It gets cancelled out in } W \end{array}\right) \leftarrow \text{Wick rotation}$$

$$= \tilde{N} \int \mathcal{D}\bar\Psi \mathcal{D}\Psi \exp\left(i \int d^4x \left(\bar\Psi \hat{D}_\Psi \Psi - i \int d^4y \, F(x) D_F(x-y) F(y)\right)\right)$$

$$\rightarrow W = \frac{1}{\langle 0|0\rangle_{j=\bar\eta=\eta=0}} \int \mathcal{D}\bar\Psi \mathcal{D}\Psi \exp\left(i \int d^4x \left(\bar\Psi \hat{D}_\Psi \Psi - i \int d^4y \, F(x) D_F(x-y) F(y)\right)\right)$$

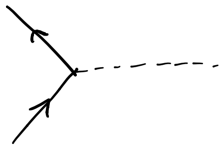
$$\mathcal{L}_{eff} = \bar\Psi \hat{D}_\Psi \Psi - i \int d^4y \, F(x) D_F(x-y) F(y)$$

c) $p^2 \ll M^2$,

$$D_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - M^2 + i\varepsilon} e^{-ip(x-y)}$$

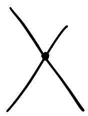
$$= \int_{|p|^2 < \Lambda^2} \frac{d^4p}{(2\pi)^4} \underbrace{\frac{i}{p^2/M^2 - 1}}_{= i(-1 + \mathcal{O}(p^2/M^2))} \cdot \frac{1}{M^2} e^{-ip(x-y)}$$

$$\rightarrow \mathcal{L}_{eff} \simeq \bar\Psi \hat{D}_\Psi \Psi + \underbrace{i \int d^4y \, F(x) \frac{i}{M^2} \delta^{(4)}(x-y) F(y)}_{\begin{array}{l} = -F^2(x) \frac{1}{M^2} \\ = -\frac{g^2}{M^2} (\bar\Psi\Psi)^2 \end{array}}$$

d) i)  $\sim -ig$

ii) 

$\sim -ig^2 \int d^4y \, D_F(x-y)$.

iii)  $\sim i\frac{g^2}{M^2}$

e) We went from small scale physics to large scale by integrating field in generating function out.