Advanced Quantum Field Theory

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see Peskin

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2.1 Reminder: Path integrals in Quantum Mechanics

Transition amplitude is given by

$$\langle x_b|\mathrm{e}^{-iH(t_b-t_a)}|x_a\rangle_S = \langle x_b, t_b|x_a, t_a\rangle_H \tag{2.1.1}$$

Here we denotes the Schrödinger picture states by $_S$ and Heisenberg picture states by $_H$.

$$\begin{aligned} |x_a, t_a\rangle &= \mathrm{e}^{iHt_a}\,|x_a\rangle && (2.1.2)\\ \hat{H}_a(t_a) &= \mathrm{e}^{iHt_a}\,\hat{x}_S\,\mathrm{e}^{-iHt_a} && (2.1.3)\\ \hat{x}_H(t_a)\,|x_a, t_a\rangle &= \mathrm{e}^{iHt_a}\,\hat{x}_s\,|x_a\rangle = \mathrm{e}^{iHt_a}\,x_a\,|x_a\rangle \\ &= x_a\mathrm{e}^{iHt_a}\,|x_a\rangle = x_a\,|x_a, t_a\rangle && (2.1.4) \end{aligned}$$

We are looking at time evolution in position space.

It can be calculated directly for free particle with Hamiltonian $H = H_0 = \frac{\hat{p}^2}{2m}$

$$\langle x_b|\mathrm{e}^{-i\frac{\hat{p}^2}{2m}(t_b-t_a)}|x_a\rangle = \sqrt{\frac{m}{2\pi i(t_b-t_a)}}\,\mathrm{e}^{i(x_b-x_a)^2\frac{m}{2(t_b-t_a)}} \tag{2.1.5}$$

We are going to insert $1 = \int \mathrm{d}^3 p\,|p\rangle\,\langle p|$ and use $\langle x|p\rangle$ is the plane wave

For general Hamiltonian $H = H_0 + V$ and $[H_0, V] \neq 0$ the procedure is as following

- divide t into N small intervals $t = N \cdot \epsilon$
- use Lie-Kato-Trotter product formula

$$\mathrm{e}^{A+B} = \lim_{N\to 0}\left(\mathrm{e}^{A/N}\mathrm{e}^{B/N}\right)^N \qquad A, B \in GL(n, \mathbb{C}) \tag{2.1.6}$$

Then we get a functional for path $x(t')$

$$\langle x_b|\mathrm{e}^{-iH(t_b-t_a)}|x_a\rangle = \int \mathcal{D}x\mathrm{e}^{iS[x]/\hbar} \tag{2.1.7}$$

with $S[x] = \int_{t_a}^{t_b} \mathrm{d}t'\left[\frac{m}{2}\dot{x}(t') - V(x(t'))\right]$

† see also in Peskin and Schroeder Ch 9.1, Ryder Ch 5.1, L.S.Brown Ch1 1-3

Definition integration measure

$$\mathcal{D}x = D[x(t)] = \lim_{N\to\infty}\left(\frac{mN}{2\pi i\Delta t}\right)^{N/2} \mathrm{d}x(t_1)\dots\mathrm{d}x(t_{N-1}) \tag{2.1.8}$$

with $\Delta t = (t_b - t_a)/N$

Pictorially we sum over all paths (i.e. amplitudes). Remember the superposition principle in quantum mechanics!

Classical path comes from Hamilton principle $\delta S = 0$

$$\left.\frac{\delta S[x]}{\delta x(t)}\right|_{x=x_{cl}} = 0 \tag{2.1.9}$$

Classical path dominates the transition probability in the limit $\hbar \to 0$. It is the contribution with fewest oscillations in the path integral. Others interfere destructively (averaged out). This is essentially stationary phase approximation.

Example harmonic oscillation

$$L = \frac{m}{2}\left(\dot{x}^2 - \omega^2 x^2\right) \tag{2.1.10}$$

Then the classical path obeys the equation of motion

$$\ddot{x}_{cl}(t) + \omega^2 x_{cl}(t) = 0 \tag{2.1.11}$$

Split a general path into classical and fluctuations $x(t) = x_{cl}(t) + y(t)$. The action turns into

$$S[x] = S[x_{cl}] + \underbrace{\int \mathrm{d}t\, \frac{\delta s}{\delta x(t)}|_{x=x_{cl}} y(t)}_{=0} + \frac{1}{2}\int \mathrm{d}t \int \mathrm{d}t'\, \frac{\delta^2 S}{\delta x(t)\delta x(t')}|_{x=x_{cl}} y(t)y(t') + \dots$$

Then we can factor out the classical path contribution in transition probability

$$\langle x_b|\mathrm{e}^{-iHT}|x_a\rangle = \int \mathcal{D}x \mathrm{e}^{\frac{i}{\hbar}S[x]} = \mathrm{e}^{\frac{i}{\hbar}S[x_a]}\int \mathcal{D}x \mathrm{e}^{\frac{i}{\hbar}S[y]}$$

The integral is to sum over fluctuations around the classical path. Ideally suited to treat fluctuations (quantum and thermal). The explicit calculation for harmonics oscillator can be found in AQT course.

Physical Interpretation the transition probability is the propagator

$$\langle x_b|\mathrm{e}^{-iH(t_b-t_a)}|x_a\rangle = U(x_b t_b; x_a t_a) \tag{2.1.12}$$

Superposition principle takes the form

$$\begin{aligned}\psi(x_b, t_b) &= \langle x_b|\psi(t_b)\rangle = \langle x_b|\mathrm{e}^{-iHt_b}|\psi\rangle \\ &= \int \mathrm{d}x_a \,\langle x_b|\mathrm{e}^{-iH(t_b-t_a)}|x_a\rangle\,\langle x_a|\mathrm{e}^{-iHt_a}|\psi\rangle \\ &= \int \mathrm{d}x_a\, U(x_b t_b; x_a t_a)\,\underbrace{\langle x_a|\psi(t_a)\rangle}_{\psi(x_a,t_a)}\end{aligned}$$

2.2 Quantum Mechanical Path Integrals and External Forces

Definition Time evolution operator in path integral representation

$$\begin{aligned}
U(x_b, t_b; x_a, t_a) &= \langle x_b, t_b | x_a, t_a \rangle \qquad (2.2.1)\\
&= \int \mathcal{D}x(t) \mathrm{e}^{iS[x]}\\
&= \int \mathcal{D}x(t) \mathrm{e}^{i\int_{t_a}^{t_b} \mathrm{d}t L(x,\dot{x})}
\end{aligned}$$

Add coupling to an external force (source) $f(t)$

$$L = L_0 + f(t)x(t) \qquad (2.2.2)$$

Definition functional derivatives with respect to $if(t)$

$$\frac{\delta}{\delta f(t)} \int \mathrm{d}t'\, f(t')g(t') = g(t) \qquad (2.2.3)$$

For a general functional of external forces

$$F[f] = \int \mathrm{d}t_1\, K_1(f_1)f(t_1) + \frac{1}{2!}\int \mathrm{d}t_1\, \mathrm{d}t_2\, K_2(t_1, t_2)f(t_1)f(t_2) + \dots \qquad (2.2.4)$$

with the $K_n(t_1, \dots, t_n)$ totally symmetric in the arguments $t_1, \dots, t_n$, since antisymmetric contributions drop automatically upon integration. The functional derivatives is then

$$\frac{\delta F}{\delta f(t)} = K_1(t) + \int \mathrm{d}t_2\, K_2(t, t_2)f(t_2) + \frac{1}{2!}\int \mathrm{d}t_2\, \mathrm{d}t_1\, K_3(t, t_2, t_3)f(t_2)f(t_3) + \dots \qquad (2.2.5)$$

Consider functional derivative of time evolution operator

$$\begin{aligned}
\frac{1}{i}\frac{\delta}{\delta f(t)} \langle x_b, t_b | x_a, t_a \rangle^f &= \int \mathcal{D}x \exp\left(i\int_{t_a}^{t_b} \mathrm{d}t'\, L_0\right)\frac{1}{i}\frac{\delta}{\delta f(t)} \exp\left(i\int_{t_a}^{t_b} \mathrm{d}t'\, f(t')x(t')\right)\\
&= \int \mathcal{D}x\, x(t) \exp\left(i\int_{t_a}^{t_b} \mathrm{d}t'\, [L_0 + f(t')x(t')]\right)
\end{aligned}$$

To split the path integral into two parts, time before and after t (superposition principle). M steps before t and $N - M - 1$ steps after t. The integration over $x(t)$ is to sum over all possible positions at time t.

$$\int_{t_a}^{t_b} \mathcal{D}x = \int \mathrm{d}x(t) \int_t^{t_b} \mathcal{D}x \int_{t_a}^{t} \mathcal{D}x$$

Then

$$\begin{aligned}
\frac{1}{i}\frac{\delta}{\delta f(t)} \langle x_b, t_b | x_a, t_a \rangle^f &= \int \mathrm{d}x(t) \underbrace{\int \mathcal{D}x \exp\left(i\int_{t}^{t_b} \mathrm{d}t'\, (L_0 + fx)\right)}_{N-M-1 \text{ factor}} x(t) \underbrace{\int \mathcal{D}x \exp\left(i\int_{t_a}^{t_b} \mathrm{d}t'\, (L_0 + fx)\right)}_{M \text{ factor}}\\
&= \int \mathrm{d}x(t)\, \langle x_b, t_b | x(t), t \rangle^f\, x(t)\, \langle x(t), t | x_a, t_a \rangle^f
\end{aligned}$$

Here $x(t)$ is an eigenvalue, not an operator, so we write $x(t) = \bar{x}$ with

$$\int \mathrm{d}\bar{x}\, |\bar{x}, t\rangle\, \bar{x}\, \langle \bar{x}, t| = \int \mathrm{d}\bar{x}\, \bar{x}\, \langle \bar{x}, t|\bar{x}, t\rangle = x(t)$$

the Heisenberg operator.

We get

$$\frac{1}{i}\frac{\delta}{\delta f(t)}\, \langle x_b t_b | x_a t_a\rangle^f = \langle x_b, t_b | x(t) | x_a, t_a\rangle \tag{2.2.6}$$

The functional derivative with respect to the external force $f(t)$ which couples to $x(t)$, to "insert" the operator $x(t)$ into the matrix element.

Now consider *two* functional derivatives with $t_b \geq t, t' \geq t_a$

$$\frac{1}{i}\frac{\delta}{\delta f(t)}\frac{1}{i}\frac{\delta}{\delta f(t')}\, \langle x_b, t_b | x_a, t_a\rangle^f = \int \mathcal{D}x\, x(t) x(t') \mathrm{e}^{i\int_{t_a}^{t_b}\mathrm{d}t'[L_0 + f\cdot x]} \tag{2.2.7}$$

In general

$$\begin{aligned}
\frac{1}{i}\frac{\delta}{\delta f(t)}\frac{1}{i}\frac{\delta}{\delta f(t')}\, \langle x_b, t_b | x_a, t_a\rangle^f &= \frac{1}{i}\frac{\delta}{\delta f(t)}\, \langle x_b, t_b | x(t') | x_a, t_a\rangle^f \\
&= \frac{1}{i}\frac{\delta}{\delta f(t)} \int \mathrm{d}\bar{x}'\, \langle x_b, t_b | \bar{x}', t\rangle^f\, \bar{x}'\, \langle \bar{x}', t' | x_a, t_a\rangle^f \\
&= \int \mathrm{d}\bar{x}' \left(\frac{1}{i}\frac{\delta}{\delta f(t)}\, \langle x_b, t_b | \bar{x}', t\rangle^f\right) \bar{x}'\, \langle \bar{x}', t' | x_a, t_a\rangle^f \\
&\quad + \int \mathrm{d}\bar{x}'\, \langle x_b, t_b | \bar{x}', t\rangle^f\, \bar{x}' \left(\frac{1}{i}\frac{\delta}{\delta f(t)}\, \langle \bar{x}', t' | x_a, t_a\rangle^f\right)
\end{aligned}$$

Then transition amplitudes only depend on the time interval, where the external forces actually act

$$\begin{aligned}
\frac{1}{i}\frac{\delta}{\delta f(t)}\, \langle x_b, t_b | \bar{x}', t'\rangle^f &= \begin{cases} \langle x_b, t_b | x(t) | \bar{x}', t'\rangle^f & t > t' \\ 0 & t < t' \end{cases} \\
\frac{1}{i}\frac{\delta}{\delta f(t)}\, \langle \bar{x}', t' | x_b, t_b\rangle^f &= \begin{cases} 0 & t > t' \\ \langle \bar{x}', t' | x(t) | x_b, t_b\rangle^f & t < t' \end{cases}
\end{aligned}$$

Eliminate $\bar{x}'$ integration as before

$$\frac{1}{i}\frac{\delta}{\delta f(t)}\frac{1}{i}\frac{\delta}{\delta f(t')}\, \langle x_b, t_b | x_a, t_a\rangle^f = \langle x_b, t_b | T\left[x(t), x(t')\right] | x_a, t_a\rangle^f \tag{2.2.8}$$

This can be easily generalised

$$\frac{1}{i}\frac{\delta}{\delta f(t')}\frac{1}{i}\frac{\delta}{\delta f(t'')} \dots \langle x_b, t_b | x_a, t_a\rangle^f = \langle x_b, t_b | T\left[x(t')x(t'')\dots\right] | x_a, t_a\rangle^f \tag{2.2.9}$$

$$= \int \mathcal{D}x\, x(t')x(t'')\dots \exp\left(i\int_{t_a}^{t_b} \mathrm{d}t\, (L_0(x, \dot{x}) + f(t)x(t))\right) \tag{2.2.10}$$

Interpretation the addition of external force to the Lagrangian of a path integral produces a "generating functional" for a matrix element which contain time-ordered products of arbitrary many position operators. The functional derivative is just a trick to generate the matrix element in the propagator. This is called Schwinger source theory.

Now we can set $f = 0$

$$\langle x_b, t_b | T\,[x(t')x(t'')\dots]\,|x_a, t_a\rangle^{f=0} = \int \mathcal{D}x x(t')x(t'')\dots \exp\left(i\int_{t_a}^{t_b} L_0(x,\dot{x})\right) \tag{2.2.11}$$

or in case of an arbitrary generating functional $F[x]$

$$\langle x_b, t_b | T\,\{F[x]\}\,|x_a, t_a\rangle^{f=0} = \int \mathcal{D}x F[x] \exp\left(i\int_{t_a}^{t_b} L_0(x,\dot{x})\right) \tag{2.2.12}$$

for example

$$\langle x_b, t_b | x_a, t_a\rangle^{f} = \langle x_b, t_b | T e^{i\int_{t_a}^{t_b} \mathrm{d}t' q(t')f(t')} | x_a, t_a\rangle^{f=0}$$

2.3 Scalar Field Theories and Feynman Rules

We are going to generalise the concept of path integral to field theories. Simplest example is a neutral (real) scalar field $\phi(x)$ coupled to an external classical "current"/source $j(x)$

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi\right)^2 - \frac{1}{2}m^2\phi^2 + \phi j(x) = \mathcal{L}_0 + \phi(x)j(x) \tag{2.3.1}$$

Proceed along the lines of quantum mechanical path integral with external forces

- construct a generating functional
- using the functional-integral-representation derive expressions for the correlation functions $\tilde{=}$ Feynman rules

Sufficient to consider vacuum-to-vacuum amplitudes in the presence of $j(x)$. Consider $t_a = -\infty(1-i\epsilon)$, $t_b = +\infty(1-i\epsilon)$ and $j(x) = 0$ for $t \mapsto \pm\infty$

$$\langle 0|0\rangle^j = \int \mathcal{D}x\,\phi(x) \exp\left(i\int \mathrm{d}^4x\,\mathcal{L}(\phi, \partial_\mu\phi)\right)$$

where $\mathcal{D}\phi(x)$ in the generalization $\mathcal{D}x \mapsto \mathcal{D}(\text{field})$

Compute $\langle 0|0\rangle^j$ (exact for a free field theory). First to solve with classical action

$$\delta\int \mathrm{d}^4x\left[\frac{1}{2}\left(\partial_\mu\phi_{cl}\right)^2 - \frac{1}{2}m^2\phi_{cl}^2 + \phi_{cl}j\right] = 0$$
$$\left(\partial^2 + m^2\right)\phi_{cl}(x) = j(x)$$

Solution

$$\phi_{cl}(x) = i\int \mathrm{d}^4y\, D_F(x-y)j(y) \tag{2.3.2}$$

since Feynman-propagator is the Green's function of the KG operator.

$$\left(\partial^2 + m^2\right)D_F(x-y) = -i\delta^{(4)}(x-y) \tag{2.3.3}$$

To define the "fluctuation" field $\phi'(x)$ via $\phi(x) = \phi_{cl}(x) + \phi'(x)$. Then the Lagrangian is

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\left(\partial_\mu\phi_{cl} + \partial_\mu\phi'\right)^2 - \frac{m^2}{2}\left(\phi_{cl} + \phi'\right)^2 + \left(\phi_{cl} + \phi'\right)\cdot j(x) \\ &= \mathcal{L}_{cl} + \mathcal{L}' + \left[(\partial_\mu\phi_{cl})(\partial^\mu\phi') - m^2\phi_{cl}\phi' + j\phi\right]\end{aligned}$$

after integration by parts and using equation of motion the last part vanishes. Then ϕ' (per construction) is a free field. Thus

$$\langle 0|0\rangle^j = \int \mathcal{D}\phi' \exp\left(i\int \mathrm{d}^4x\,(\mathcal{L}_{cl} + \mathcal{L}')\right) = \mathrm{e}^{iS_{cl}}\,\langle 0|0\rangle^{j=0} \tag{2.3.4}$$

On the other hand, iS_{cl} can be rewritten as

$$\begin{aligned}iS_{cl} &= i\int \mathrm{d}^4x\left[\frac{1}{2} - \frac{m}{2}\phi_{cl}^2 + \phi_{cl}j\right] \\ &= i\int \mathrm{d}^4x\Big[-\frac{1}{2}\phi_{cl}\underbrace{\left(\partial^2 + m^2\right)\phi_{cl}}_{=j\text{ from e.o.m.}} + \phi_{cl}j\Big] \\ &= \frac{i}{2}\int \mathrm{d}^4x\,\phi_{cl}(x)j(x) \\ &= -\frac{1}{2}\int \mathrm{d}^4x\,\mathrm{d}^4y\; j(x)D_F(x-y)j(y)\end{aligned}$$

Definition generating functional in the free scalar field theory

$$\begin{aligned}W_0[j] &= \frac{Z[j]}{Z[j=0]} = \frac{\langle 0|0\rangle^j}{\langle 0|0\rangle^{j=0}} \\ &= \exp\left(-\frac{1}{2}\int \mathrm{d}^4x\,\mathrm{d}^4y\; j(x)D_F(x-y)j(y)\right)\end{aligned} \tag{2.3.5}$$

Connection to the S-matrix

$$\begin{aligned}S &= U(-\infty,\infty) \\ &= \lim_{t_i\mapsto-\infty(1-i\epsilon)}\;\lim_{t_f\mapsto+\infty(1-i\epsilon)} T\exp\left(-i\int_{t_i}^{t_f}\mathrm{d}t\,\mathcal{H}_{int}(t)\right) \\ &= T\exp\left(-i\int \mathrm{d}^4x\,\mathcal{H}_{int}(x)\right) \\ &= T\exp\left(i\int \mathrm{d}^4x\,\phi(x)j(x)\right) \\ &= T\sum_{n=0}^{\infty}\frac{i^n}{n!}\int \mathrm{d}^4x_1\ldots\mathrm{d}^4x_n\; j(x_1)\ldots j(x_n)\phi(x_1)\ldots\phi(x_n)\end{aligned}$$

$$\langle 0|S|0\rangle = \sum_{n=0}^{\infty}\frac{i^n}{n!}\int \mathrm{d}^4x_1\ldots\mathrm{d}^4x_n\; j(x_1)\ldots j(x_n)G_n^0(x_1,\ldots,x_n) \tag{2.3.6}$$

where $G_n^0(x_1,\ldots,x_n) = \langle 0|T[\phi(x_1)\ldots\phi(x_n)]|0\rangle$ the n-point-Green's function of the free scalar field theory.

We can calculate the Green's function as for the quantum mechanical path integral with external forces via functional derivatives of the generating functional

$$W_0[j] = \frac{\int \mathcal{D}\phi \exp\Big(i \int \mathrm{d}^4 x(\mathcal{L}_0(\phi, \partial_\mu \phi) + \phi j)\Big)}{\int \mathcal{D}\phi \exp\Big(i \int \mathrm{d}^4 x(\mathcal{L}_0(\phi, \partial_\mu \phi))\Big)} \tag{2.3.7}$$

$$\begin{aligned} G_n^0(x_1, \ldots, x_n) &= \frac{1}{i}\frac{\delta}{\delta j(x_1)} \ldots \frac{1}{i}\frac{\delta}{\delta j(x_n)} W_0[j]|_{j=0} \\ &= \frac{\int \mathcal{D}\phi \exp\Big(i \int \mathrm{d}^4 x(\mathcal{L}_0(\phi, \partial_\mu \phi))\Big)\phi(x_1)\ldots\phi(x_n)}{\int \mathcal{D}\phi \exp\Big(i \int \mathrm{d}^4 x(\mathcal{L}_0(\phi, \partial_\mu \phi))\Big)} \\ &= \langle 0|T\phi(x_1)\ldots\phi(x_n)|0\rangle \end{aligned} \tag{2.3.8}$$

The central result here is that these three things are closely related: S-matrix $\leftrightarrow$ Green's function $\leftrightarrow$ Path integral