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H.4

a) A symmetric, positive definite $\rightarrow \exists A^{-1}$

$$(\vec{x} - A^{-1}\vec{J})^T A (\vec{x} - A^{-1}\vec{J}) \leftarrow \text{note it is just a "translation"}$$

$$= \vec{x}^T A \vec{x} + (A^{-1}\vec{J})^T \underbrace{A A^{-1}}_{1} \vec{J} \underbrace{- (A^{-1}\vec{J})^T A \vec{x} - \vec{x}^T A A^{-1}\vec{J}}$$

$$= -2\vec{x}^T A A^{-1}\vec{J}$$

$$= -2\vec{x}^T\vec{J}$$

$$= -2\vec{J}\cdot\vec{x}$$

$$= \vec{x}^T A \vec{x} + \vec{J}^T A^{-1}\vec{J} - 2\vec{J}\cdot\vec{x} \leftarrow A^T = A$$

Thus we can rewrite the integral as

$$\frac{1}{(2\pi)^{n/2}} \int \prod_{i=1}^{n} dx_i \exp\left(-\frac{1}{2}\vec{x}^T A \vec{x} + \vec{J}\cdot\vec{x}\right)$$

$$= \frac{1}{(2\pi)^{n/2}} \int \prod_{i=1}^{n} dx_i \left(-\frac{1}{2}(\vec{x} - A^{-1}\vec{J})^T A (\vec{x} - A^{-1}\vec{J}) + \frac{1}{2}\vec{J}^T A^{-1}\vec{J}\right)$$

$$= \frac{1}{(2\pi)^{n/2}} \exp\left(\frac{1}{2}\vec{J}^T A^{-1}\vec{J}\right) \underbrace{\int \prod_{i=1}^{n} dx_i \exp\left(-\frac{1}{2}(\vec{x} - A^{-1}\vec{J})^T A (\vec{x} - A^{-1}\vec{J})\right)}$$

$$= \int \prod_{i=1}^{n} dx_i \exp\left(-\frac{1}{2}\vec{x}^T A \vec{x}\right)$$

since $A^{-1}\vec{J}$ is just a constant

Introduce the similarity transformation

$$u^{-1} A u = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}.$$

det (u)

it bring A to its diagonal form and $\det(\mathrm{Jac}) = 1$

$$= \int \prod_{i=1}^{n} dx_i \exp\left(-\frac{1}{2}\vec{x}^T u u^{-1} A u u^{-1}\vec{x}\right)$$

$\vec{y} = u^{-1}\vec{x}$

$$= \int \prod_{i=1}^{n} dy_i \exp\left(-\frac{1}{2}\vec{y}^T (u A u^{-1})\vec{y}\right)$$

$$= \int \prod_{i=1}^{n} dy_i \exp\left(-\tfrac{1}{2}\left(y_1^2\lambda_1 + \cdots y_n^2\lambda_n\right)\right)$$

$$= \int dy_1 \exp\left(-\tfrac{1}{2}\lambda_1 y_1^2\right) \cdot \cdots \cdot \int dy_n \exp\left(-\tfrac{1}{2}\lambda_n y_n^2\right)$$

$$= (2\pi)^{n/2} \prod_{n=1}^{n} \lambda_i^{-1/2}$$

$$= (2\pi)^{n/2} \sqrt{\det(U^{-1}AU)}^{-1}$$

$$= (2\pi)^{n/2} \left(\det(A)\right)^{-1/2}$$

$$\Rightarrow \frac{1}{(2\pi)^{n/2}} \int \prod_{i=1}^{n} dx_i \exp\left(-\tfrac{1}{2}\vec{x}^T A \vec{x} + \vec{J}\cdot\vec{x}\right)$$

$$= \frac{1}{\sqrt{\det A}} \exp\left(\tfrac{1}{2} J^T A^{-1} J\right)$$

b) Complex gauß integral with scalar

$$\int \frac{dz\,dz^*}{2\pi i} \exp(-z^* a z) = \int \frac{dx\,dy}{\pi} e^{-a(x^2+y^2)}$$

$z = x+iy$
$z^* = x-iy$

$x = r\cos\varphi$
$y = r\sin\varphi$

$$= \frac{1}{\pi}\int_0^\infty \int_0^{2\pi} d\varphi\, dr\, r\, e^{-ar^2}$$

$$= 2\int_0^\infty dr\, r\, e^{-ar^2}$$

$$= \frac{1}{a}$$

$$\frac{1}{(2\pi i)^n} \int \prod_{i=1}^{n} dz_i^* dz_i \exp\left(-\vec{z}^\dagger H \vec{z} + \vec{J}^*\cdot\vec{z} + \vec{J}\cdot\vec{z}^*\right)$$

$$= \frac{1}{\pi^n} \int \prod_{i=1}^{n} dx_i\, dy_i \exp\left(-\vec{x}^T H \vec{x} - \vec{y}^T H \vec{y} + \vec{J}^*\cdot(\vec{x}+i\vec{y}) + \vec{J}\cdot(\vec{x}-i\vec{y})\right)$$

$$= \frac{1}{\pi^n} \int \prod_{i=1}^{n} dx_i\, dy_i \exp\left(-\vec{x}^T H \vec{x} + (\vec{J}^* + \vec{J})\cdot\vec{x}\right) \exp\left(-\vec{y}^T H \vec{y} + i(\vec{J}^* - \vec{J})\vec{y}\right)$$

$$= \frac{1}{\pi^n} \frac{(2\pi)^{n/2}}{\sqrt{\det(2H)}} \exp\left(\tfrac{1}{2}(\vec{J}^*+\vec{J})^T \cdot (2H)^{-1} \cdot (\vec{J}^*+\vec{J})\right) \cdot \frac{(2\pi)^{n/2}}{\sqrt{\det(2H)}} \exp\left(-\tfrac{1}{2}(\vec{J}^*-\vec{J})^T (2H)^{-1} (\vec{J}^*-\vec{J})\right)$$

$$\det(2H) = \det(H) \cdot 2^n$$

$$= \frac{1}{\pi^n} (2\pi)^n \cdot \frac{1}{2^n} \quad \frac{1}{\det H} \exp\Big(\tfrac{1}{4}\underbrace{\big[(\vec{J}^* + \vec{J})^T H (\vec{J}^* + \vec{J}) - (\vec{J}^* - \vec{J})^T H (\vec{J}^* - \vec{J})\big]}_{= \vec{J}^T H \vec{J}^* + \vec{J}^* H \vec{J} + \vec{J}^{*T} H \vec{J} + \vec{J}^T H \vec{J}^*}\Big)$$

$$= \frac{1}{\det H} \exp(\vec{J}^\dagger H^{-1} \vec{J}) \quad \checkmark$$

$$\frac{1}{(2\pi i)^n} \int \prod_{i=1}^n dz_i^* dz_i \exp(-z_k^* H_{kl} z_l + J_k^* z_k + J_k z_k^*)$$

$$z^\dagger H z + J^\dagger z + J z^\dagger = -w^\dagger \Delta w + J^\dagger H^{-1} J$$

$$U H U^{-1} = \Delta, \quad U^\dagger = U^{-1}, \quad \det U = 1$$

$$w = U(z - H^{-1} J)$$

$$dz^* dz = dw^* dw \underbrace{\det U}_{=1}$$

$$= 2i \, dx \, dy$$

$$\Rightarrow \quad \int \prod_{i=1}^n dx_i dy_i \exp(-x^T \Delta x - y^T \Delta y)$$

use (a)

H.5

a) At $\mathcal{O}(\lambda^0)$. $W_0[\lambda] = \exp(-\frac{i}{2}\int d^4x d^4y\ j(x) D(x-y) j(y))$

At $\mathcal{O}(\lambda^1)$:

$$\exp(-i\frac{\lambda}{4!}\int d^4z \ (\frac{1}{i}\frac{\delta}{\delta j(z)})^4)\exp(-\frac{i}{2}\int d^4x d^4y\ j(x) D_F(x-y) j(y))$$

$$= [1 - i\frac{\lambda}{4!}\int d^4z(\frac{1}{i}\frac{\delta}{\delta j(z)})^4 + \mathcal{O}(\lambda^2)]\exp(-\frac{i}{2}\int d^4x d^4y\ j(x) D_F(x-y) j(y))$$

$$\left(\frac{\delta}{\delta j(z)}\right)^4 \exp(-\frac{i}{2}\int d^4x\, d^4y\ j(x) D_F(x-y) j(y))$$

$$= \left(\frac{\delta}{\delta j(z)}\right)^3\left(-i\int d^4y\ D_F(z-y) j(y)\right)\exp(\cdots)$$

$$= \left(\frac{\delta}{\delta j(z)}\right)^2\left(-iD_F(z-z) + (i)^2\underbrace{\int d^4y\, d^4x\ D_F(z-y) j(y)\ D(z-x) j(x)}_{[\int d^4x\ D_F(z-x) j(x)]^2}\right)\exp(\cdots)$$

$$= \left(\frac{\delta}{\delta j(z)}\right)\left(\underbrace{-iD_F(0)\cdot\left(-i\int d^4x D_F(z-x) j(x)\right) - 2D_F(0)\int d^4x\ j(z) D(z-x) j(x)}_{-3D_F(0)\int d^4x\, D_F(z-x) j(x)}\right.$$

$$\left. + i\underbrace{\int d^4y\int d^4x\, D_F(z-y) j(y) D(z-x) j(x)\int d^4\tilde{x}\ D_F(z-\tilde{x}) j(\tilde{x})}_{=[\int dx\ D(x-z) j(x)]^3}\right)\exp(\cdots)$$

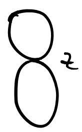
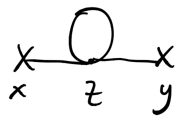
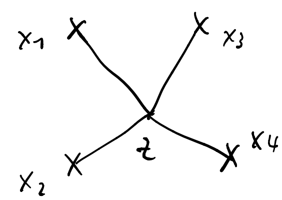
$$= \left(-3D_F^2(0) + \overset{6}{3}i D_F(0)\int d^4x\, D_F(z-x) j(x)\int d^4y\, D_F(z-y) j(y)\right.$$

$$\left. + \underbrace{\prod_{i=1}^{4}\int d^4x_i\, D_F(z-x_i) j(x_i)}_{=[\int d^4x\ D_F(z-x) j(x)]^4}\right)\exp(\cdots)$$

→ 3 come from $-3D_F(0)\int d^4x D_F(z-x) j(x)\ \frac{\delta}{\delta j(z)}\exp(\cdots)$

another 3 come from $\frac{\delta}{\delta j(z)}\left(\int d^4x\, D_F(z-x) j(x)\right)^3$

$$= \Big[-3 D_F^2(0) + \overset{6}{3} i D_F(0) \int d^4y \int d^4x \, D_F(z-y) j(y) D(z-x) j(x) + \prod_{i=1}^{4} \int d^4x_i \, D_F(z-x_i) j(x_i) \Big] \exp(\cdots)$$

-3  , $+6i$  , 

$$\Rightarrow W[j] = \frac{1}{N_0}\Big[1 - i\frac{\lambda}{4!}\int d^4z \Big(\cancel{-3D_F^2(0)} - 3i D_F(0) \int d^4y\, d^4x \, D_F(z-y) j(y) D(z-x) j(x) + \prod_{i=1}^{4} \int d^4x_i \, D_F(z-x_i) j(x_i) \Big)\Big] e^{-\frac{i}{2}\int d^4x d^4y \, j(x) D_{xy} j(y)}$$

$$N_0 = \Big(1 - i\frac{\lambda}{4!}\int d^4z \big(\cancel{-3D_F^2(0)}\big)\Big)$$

$$\frac{1}{1-x} = 1 + x + \mathcal{O}(x^2) \Rightarrow \text{no vacuum bubbles}$$

b) $$\frac{\delta^4}{\delta j(x_1)\delta j(x_2)\delta j(x_3)\delta j(x_4)} e^{-\frac{i}{2}\int d^4x d^4y \, j(x) D_{xy} j(y)}$$

$$= \frac{\delta^3}{\delta_1\delta_2\delta_3}(-i D_{4x} j_x) e^{\cdots},$$ Integration over repeated indices

$$= \frac{\delta^2}{\delta_1\delta_2}(-iD_{43} - i D_{4x} j_x D_{3y} j_y) e^{\cdots}$$

$$= \frac{\delta}{\delta_1}(-iD_{43} D_{2x} j_x - i D_{42} D_{3y} j_y - i D_{4x} j_x D_{32} - i D_{4x} j_x D_{3y} j_y D_{2z} j_z) e^{\cdots}$$

$$= (-iD_{43}D_{21} - iD_{42}D_{31} - iD_{41}D_{32} + \underbrace{\cdots}_{\text{No effect after setting } j=0}) e^{\cdots}$$

$$\frac{\delta^2}{\delta j(a)\delta j(b)} D_{zy} j_y D_{zx} j_x$$

$$= \frac{\delta}{\delta j(a)}(D_{zb} D_{zx} j_x + D_{zy} j_y D_{zb})$$

$$= 2 \frac{\delta}{\delta j(a)} D_{zb} D_{zx} j_x$$

$$= 2 D_{zb} D_{za}$$

In the 4-point function a, b can be any two of x_1, x_2, x_3, x_4. The two differentiations left are applied to the exponential.

$$\langle 0 | T \phi(x_1) \phi(x_2) \phi(x_3) \phi(x_4) | 0 \rangle = \frac{\delta^4 W[j]}{\delta j(x_1) \delta j(x_2) \delta j(x_3) \delta j(x_4)} \Big|_{j=0}$$

-3 

$$= \cancel{\frac{1}{N_0}} \Big[\big(\cancel{1 - i\frac{\lambda}{4!} \int d^4 z} - 3iD_F^2(0) \big) \cdot \big(-iD_{12} D_{34} - iD_{13} D_{24} - iD_{14} D_{23} \big)$$

$-3i\lambda$ 

$$- i \frac{\lambda}{4!} (-3i) D_F(0) \int d^4 z \big(2D_{z1} D_{z2} D_{34} + 2D_{z1} D_{z3} D_{24} + 2 D_{z1} D_{z4} D_{34} + 2 D_{z2} D_{z3} D_{14} + 2 D_{z2} D_{z4} D_{13} + 2 D_{z3} D_{z4} D_{12} \big) \Big]$$

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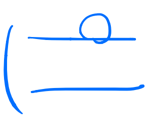
$$- i \frac{\lambda}{4!} 4! \int d^4 z \underbrace{D_{z1} D_{z2} D_{z3} D_{z4}}$$

permutation

$-i\lambda$ 

$= -3$ ( ) $-3i\lambda$ () $- i\lambda$ ()

→ Ryder ch 6.5