H19. $Z_3 = \frac{1}{1+\Pi(0)} \simeq 1 - \Pi(0)$, even though $\Pi(0)$ contains infinities ...

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a) $$\Pi(q^2) = \frac{e_0^2}{2\pi^2} \frac{\Gamma(\epsilon/2)}{(4\pi)^{-\epsilon/2}} \mu^\epsilon \int_0^1 dx \; x(1-x)[m^2 - x(1-x)q^2]^{-\epsilon/2}$$

$$= \frac{e_0^2}{2\pi^2}\left[\frac{2}{\epsilon} - \gamma_E + \mathcal{O}(\epsilon)\right]\left[1 + \frac{\epsilon}{2}\cdot \ln(4\pi)\right]\left[1 + \epsilon \underline{\ln\mu}\right]$$

$$\times \int_0^1 dx \; x(1-x)\left(1 - \frac{\epsilon}{2}\underline{\ln\Delta}(x)\right)$$

Better to write unitless logs $= \ln(\Delta/\mu^2)$

(With $\Delta(x) = m^2 - x(1-x)q^2$

$$= \frac{e_0^2}{2\pi^2}\Big[\frac{2}{\epsilon}\cdot 1 \cdot \int_0^1 dx \; x(1-x) - \gamma_E \cdot \frac{1}{6}$$

$$+ \frac{2}{\cancel{\epsilon}}\frac{\cancel{\epsilon}}{2}\ln(4\pi)\cdot 1 \cdot \int_0^1 dx \; x(1-x) + \frac{2}{\epsilon}\cdot\epsilon\ln(\mu)\cdot\int_0^1 dx \; x(1-x)$$

$$- \frac{2}{\cancel{\epsilon}}\cdot 1 \int_0^1 dx \; x(1-x)\cdot\frac{\cancel{\epsilon}}{\cancel{2}}\ln\Delta(x) \quad + \mathcal{O}(\epsilon)\Big]$$

✓

$$\left(\int_0^1 dx \; x(1-x) = \int_0^1 dx(x - x^2) = \left[\tfrac{1}{2}x^2\right]_0^1 - \left[\tfrac{1}{3}x^3\right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}\right.$$

$$= \frac{e_0^2}{2\pi^2}\left[\frac{1}{3}\cdot\frac{1}{\epsilon} + \frac{\ln(4\pi)}{6} - \frac{\gamma_E}{6} + \underbrace{\frac{1}{3}\ln\mu}_{\frac{1}{6}\ln\mu^2} - \int_0^1 dx \; x(1-x)\ln\Delta(x) + \mathcal{O}(\epsilon)\right]$$

a) $$\Pi(q^2) = \frac{e_0^2}{2\pi^2} \frac{\Gamma(\epsilon/2)}{(4\pi)^{-\epsilon/2}} \mu^\epsilon \int_0^1 dx \; x(1-x)[m^2 - x(1-x)q^2]^{-\epsilon/2}$$

so that $\ln$ is unitless

$$= \frac{e_0^2}{2\pi^2}\frac{1}{(4\pi)^{-\epsilon/2}}\left(\frac{2}{\epsilon} - \gamma_E + \mathcal{O}(\epsilon)\right)\int_0^1 dx \; x(1-x)\left[\frac{m^2 - x(1-x)q^2}{\mu^2}\right]^{-\epsilon/2}$$

$$= \frac{e_0^2}{\pi^2}\left(\frac{1}{\epsilon} - \frac{1}{2}(\gamma_E - \ln 4\pi)\right)\int_0 dx \; x(1-x)$$

$$- \frac{e_0^2}{2\pi^2}\int_0^1 dx \; x(1-x)\ln\left(\frac{m^2 - x(1-x)q^2}{\mu^2}\right)$$

b) $\tilde{\Pi}(q^2) = \Pi(q^2) - \Pi(0)$,

$$\Pi(0) = \frac{e_0^2}{2\pi^2}\left[\frac{1}{3}\cdot\frac{1}{\epsilon} + \frac{\ln(4\pi)}{6} - \frac{\gamma_E}{6} + \frac{1}{3}\ln\mu - \underbrace{\int_0^1 dx \; x(1-x)\ln(m^2)}_{=\frac{\ln(m^2)}{6}} + \mathcal{O}(\epsilon)\right]$$

$$= \frac{e_0^2}{2\pi^2}\left[\frac{1}{3}\cdot\frac{1}{\epsilon} + \frac{1}{6}\ln(4\pi) - \frac{\gamma_E}{6} + \frac{1}{3}\ln\mu - \frac{1}{6}\ln(m^2) + \mathcal{O}(\epsilon)\right]$$

$$\Rightarrow \tilde{\Pi}(q^2) = \Pi(q^2) - \Pi(0)$$

$$= \frac{e_0^2}{2\pi^2}\cdot\left[-\underbrace{\int_0^1 dx\, x(1-x)\ln\Delta(x)}_{=:I} + \frac{1}{6}\ln(m^2)\right]$$

In the limit of $q^2 \gg m^2$,

$$\ln\Delta(x) = \ln[m^2 - x(1-x)q^2]$$

$$= \ln\left[x(1-x)q^2\left(\frac{m^2}{x(1-x)q^2} - 1\right)\right]$$

$$= \ln[x(1-x)q^2] + \ln\Big(\underbrace{\frac{m^2}{x(1-x)q^2} - 1}_{<0\ !}\Big)$$

$$= \ln[x(1-x)q^2] + i\pi - \frac{m^2}{x(1-x)q^2} + \mathcal{O}\left(\left(\frac{m^2}{q^2}\right)^2\right)$$

$$I = \int_0^1 dx\cdot x(1-x)\ln\Delta(x)$$

$$= \int_0^1 dx\cdot x(1-x)\left\{\ln[x(1-x)q^2] + i\pi - \frac{m^2}{x(1-x)q^2} + \mathcal{O}\left(\left(\frac{m^2}{q^2}\right)^2\right)\right\}$$

$$= -\frac{5}{18} + \frac{1}{6}\ln q^2 + \frac{i\pi}{6} - \frac{m^2}{q^2} + \mathcal{O}(\cdots)$$

$$\Rightarrow \tilde{\Pi}(q^2) = \frac{e_0^2}{2\pi^2}\left[-\left(-\frac{5}{18} + \frac{1}{6}\ln q^2 + \frac{i\pi}{6} - \frac{m^2}{q^2} + \mathcal{O}(\cdots)\right) + \frac{1}{6}\ln(m^2)\right]$$

$$\Rightarrow \mathrm{Re}\,\tilde{\Pi}(q^2) = \frac{e_0^2}{2\pi^2}\left[\frac{5}{18} - \frac{1}{6}\ln q^2 + \frac{m^2}{q^2} + \frac{\ln(m^2)}{6} + \mathcal{O}(\cdots)\right]$$

$$Z_3 = 1 - \Pi(0) = 1 - \frac{e_0^2}{6\pi^2}\frac{1}{\epsilon} + \text{finite}$$

$$e^2(-\mu_R'^2) - e^2(-\mu_R^2)$$ ← should be finite

$$\simeq e_0^2\left(1 - \frac{e_0^2}{6\pi^2}\cancel{\frac{1}{\epsilon}}\right)\left(\cancel{1} - \mathrm{Re}\,\tilde{\Pi}(-\mu_R'^2) - \cancel{1} + \mathrm{Re}\,\tilde{\Pi}(-\mu_R^2)\right)$$

$$e^2 = Z_3 e_0^2 \simeq (1-\Pi(0))\, e_0^2$$
$$= \left\{1 - \frac{e_0^2}{2\pi^2}\left[\frac{1}{3}\cdot\frac{1}{\epsilon} + \frac{1}{6}\ln(4\pi) + \frac{1}{3}\ln\mu_R - \frac{1}{6}\ln(m^2) + \mathcal{O}(\epsilon)\right]\right\} e_0^2$$

$$\rightarrow \quad \beta = \mu_R \frac{\partial e}{\partial \mu_R} = \mu_R \frac{\partial e^2}{\partial \mu_R}\,\frac{\partial e}{\partial e^2} = \cancel{\mu_R}\left(e_0^2\,\frac{e_0^2}{2\pi^2}\cdot\frac{1}{3}\,\frac{1}{\cancel{\mu_R}}\right)\frac{1}{2e}$$
$$= \frac{e_0^4}{6\pi^2}\cdot\frac{1}{2e} \sim \frac{e^3}{12\pi^2} \qquad (e \sim e_0\ ?)$$

$$e^2(q^2) = \frac{e^2}{1+\mathrm{Re}\,\bar{\pi}(q^2)}$$

$$\bar{\pi}(q^2) = \pi(q^2) - \pi(0)$$
$$= -\frac{e_0^2}{2\pi^2}\int_0^1 dx\; x(1-x)\,\underbrace{\left(\ln\frac{m^2 - x(1-x)q^2}{\mu^2} - \ln\frac{m^2}{\mu^2}\right)}_{=\ \ln\frac{m^2 - x(1-x)q^2}{m^2}}$$

$$= -\frac{e_0^2}{2\pi^2}\int_0^1 dx\, x(1-x)\left[\ln\frac{-q^2}{m^2} + \ln x(1-x)\right] \qquad q^2 \gg m^2$$
$$= -\frac{e_0^2}{12\pi^2}\left[\ln\left(\frac{-q^2}{m^2}\right)\cdot\frac{1}{6} - \frac{5}{3}\right] \qquad \text{const}$$

$$e^2(q^2) = \frac{e^2}{1+\mathrm{Re}\,\bar{\pi}(q^2)} \simeq e^2(1-\mathrm{Re}\,\bar{\pi}(q^2))$$

$$e^2(-\mu_R'^2) - e^2(-\mu_R^2)$$
$$= e^2\left(+\frac{e_0^2}{12\pi^2}\right)\left[\ln\left(\frac{\mu_R'^2}{m^2}\right) - \ln\left(\frac{\mu_R^2}{m^2}\right)\right]$$

$$\left(\begin{array}{l} e^2 \simeq e_0^2 + \mathcal{O}(e_0^4) \\ \text{b.c.}\quad e^2 = Z_3\, e_0^2 \end{array}\right.$$

$$= \frac{e_0^4}{12\pi^2}\ln\frac{\mu_R'^2}{\mu_R^2}$$

$$\beta = \mu_R\frac{\partial e}{\partial \mu_R} = \mu_R'\frac{\partial}{\mu_R'}\left[e^2(-\mu_R^2) + \frac{e_0^4}{12\pi^2}\log\left(\frac{\mu_R'^2}{\mu_R^2}\right)\right]^{1/2}$$
$$= \mu_R'\,\frac{1}{2}\,\frac{2\,\frac{e_0^4}{12\pi^2}\,\frac{\mu_R}{\mu_R'}\,\frac{1}{\mu_R}}{[e(-\mu_R'^2)]^{1/2}} \quad \cdots$$
$$= \frac{e_0^4}{18\pi^2 e_0\sqrt{1-\pi(0)}} = \frac{e_0^3}{12\pi^2(1-\pi(0))^{3/2}}(1-\pi(0)) = \frac{e^3}{12\pi^2}(1-\pi(0)) \quad e_0^2 \sim e^2$$
$$= \frac{e^3}{12\pi^2} + \mathcal{O}(e^5)$$

c) On-shell subtraction $q^2 = -M^2$

$$\tilde{\Pi}(q^2) = \Pi(q^2) - \Pi(-M^2) \qquad \left(\Delta(x) = m^2 + x(1-x)M^2 \right.$$

$$\Pi(-M^2) = \frac{e_0^2}{2\pi^2}\left[\frac{1}{3}\cdot\frac{1}{\epsilon} + \frac{\ln(4\pi)}{6} - \frac{\gamma_E}{6} + \frac{1}{3}\ln\mu - \underbrace{\int_0^1 dx\, x(1-x)\ln\Delta(x)}_{=:I} + \mathcal{O}(\epsilon)\right]$$

$$I = \int_0^1 dx\, x(1-x)\ln(m^2 + x(1-x)M^2)$$

$$= \begin{cases} \int_0^1 dx\, x(1-x)\left[\ln\left(\frac{m^2}{x(1-x)M^2}+1\right) + \ln(x(1-x)M^2)\right], & M^2 \gg m^2 \\ \int_0^1 dx\, x(1-x)\left[\ln\left(1+\frac{x(1-x)M^2}{m^2}\right) + \ln(m^2)\right], & m^2 \gg M^2 \end{cases}$$

$$\simeq \begin{cases} \int_0^1 dx\, x(1-x)\left[\frac{m^2}{M^2}\frac{1}{x(1-x)} + \ln(x(1-x)) + \ln M^2\right], & M^2 \gg m^2 \\ \int_0^1 dx\, x(1-x)\left[\frac{x(1-x)M^2}{m^2} + \ln m^2\right], & m^2 \gg M^2 \end{cases}$$

$$= \begin{cases} \frac{m^2}{M^2} - \frac{5}{18} + \frac{1}{6}\ln M^2, & M^2 \gg m^2 \\ \frac{1}{30}\frac{M^2}{m^2} + \frac{1}{6}\ln m^2, & m^2 \gg M^2 \end{cases}$$

$$\Pi(q^2) = \frac{e_0^2}{2\pi^2}\left[\frac{1}{3}\cdot\frac{1}{\epsilon} + \frac{\ln(4\pi)}{6} - \frac{\gamma_E}{6} + \frac{1}{3}\ln\mu - \int_0^1 dx\, x(1-x)\ln\Delta(x) + \mathcal{O}(\epsilon)\right]$$

$$\int_0^1 dx\, x(1-x)\ln[m^2 - x(1-x)q^2]$$

$$= \int_0^1 dx\, x(1-x)\cdot \begin{cases} \ln\left[\frac{m^2}{x(1-x)q^2} - 1\right], & q^2 \gg m^2 \\ \ln\left[1 - \frac{x(1-x)q^2}{m^2}\right], & m^2 \gg q^2 \end{cases}$$

$$\simeq \int_0^1 dx\, x(1-x) \begin{cases} i\pi - \frac{m^2}{x(1-x)q^2}, & q^2 \gg m^2 \\ -\frac{x(1-x)q^2}{m^2}, & m^2 \gg q^2 \end{cases}$$

$$= \begin{cases} \frac{i\pi}{6} - \frac{m^2}{q^2}, & q^2 \gg m^2 \\ -\frac{1}{30}\frac{q^2}{m^2}, & m^2 \gg q^2 \end{cases}$$

$$\tilde{\pi}(q^2) \propto \ln\left[\frac{m^2 - x(1-x)q^2}{m^2 + x(1-x)M^2}\right]$$

$$\Rightarrow \tilde{\pi}(q^2) = \frac{e_0^2}{2\pi^2}\left\{-\int_0^1 dx\, x(1-x)\ln[m^2 - x(1-x)q^2] + \int_0^1 dx\, x(1-x)\ln[m^2 + x(1-x)M^2]\right\}$$

$$= \begin{cases} \frac{e_0^2}{2\pi^2}\left[-\frac{i\pi}{6} + \frac{m^2}{q^2} + \frac{m^2}{M^2} - \frac{5}{18} + \frac{1}{6}\ln M^2\right], & q^2, M^2 \gg m^2 \\ \frac{e_0^2}{2\pi^2}\left[+\frac{1}{30}\frac{q^2}{m^2} + \frac{1}{30}\frac{M^2}{m^2} + \frac{1}{6}\ln m^2\right], & m^2 \gg q^2, M^2 \end{cases}$$

$$\ln(-q^2/M^2)$$

$$\rightarrow \tilde{\pi}(q^2) \sim 1/q^2 + \text{const}, \qquad q^2, M^2 \gg m^2$$

$$\sim q^2 + \text{const}, \qquad m^2 \gg q^2, M^2$$

$$\longrightarrow 0$$

c) $\tilde{\pi}(q^2) = \pi(q^2) - \pi(-M^2)$

$$= -\frac{e_0^2}{2\pi^2}\int_0^1 dx\, x(1-x)\ln\frac{m^2 - x(1-x)q^2}{m^2 + x(1-x)M^2}$$

$m^2 \ll M^2, q^2$:

$$= \dots \ln\left(\frac{-q^2}{M^2}\right) \longrightarrow \text{part (b)}$$

$m^2 \gg M^2, q^2$

$$= \underset{\substack{\uparrow \\ \text{leading order}}}{\underbrace{\ln\frac{m^2}{m^2}}_{=0}} + \frac{e_0^2}{2\pi^2}\frac{q^2+M^2}{m^2}\underbrace{\int_0^1 dx\, x^2(1-x)^2}_{\frac{1}{30}} + \mathcal{O}(m^{-4})$$

$$\Rightarrow m \to \infty, \quad \tilde{\pi}(q^2) \to 0, \quad e^2(q^2) = \frac{e^2}{1 + \mathrm{Re}\,\tilde{\pi}(q^2)} \xrightarrow{m \to \infty} e^2$$

a constant

d) In $\overline{MS}$,

$$\tilde{\Pi}(q^2) = \Pi(q^2) - \overset{\text{arbitrary const.}}{A}\left[\frac{1}{\epsilon} - (\gamma_E - \log 4\pi)/2\right]$$

$$= \frac{e_0^2}{2\pi^2}\left[\frac{1}{3}\ln\mu - \int_0^1 dx\, x(1-x)\ln\Delta(x) + \mathcal{O}(\epsilon)\right]$$

$m^2 \gg q^2, \mu^2$

$$= \frac{e_0^2}{2\pi^2}\left(\underbrace{\frac{1}{3}\ln\mu}_{\frac{1}{6}\ln\frac{m^2}{\mu^2}} + \cancel{\frac{1}{30}\frac{q^2}{m^2}}\right) \sim q^2 + \text{const}$$

In $\overline{MS}$: $\quad \tilde{\Pi}(q^2) = \cdots \int_0^1 dx\, x(1-x)\ln\frac{m^2 - x(1-x)q^2}{\mu^2}$

$$\xrightarrow{m^2\to\infty} \cdots \ln\frac{m^2}{\mu^2} \neq 0$$

different from part c)

Good quantity is the difference of coupling at difference scale.