H.12 $\qquad \mathcal{L} = -\frac{1}{2} H(\partial^2 + M^2)H + \bar{\Psi}(i\slashed{\partial} - m)\Psi - g\bar{\Psi}\Psi H$

a)

Introduce sources j and η of H and Ψ

$$\mathcal{L} \rightarrow \mathcal{L} + Hj + \bar{\Psi}\eta + \bar{\eta}\Psi ,$$

$$\mathcal{L}_0 = -\frac{1}{2} H(\partial^2 + M^2)H + \bar{\Psi}(i\slashed{\partial} - m)\Psi + Hj + \bar{\Psi}\eta + \bar{\eta}\Psi$$

the free Lagrangian

$$W[J] = \frac{\langle 0|0\rangle_{j,\eta,\bar{\eta}}}{\langle 0|0\rangle_{j=\eta=\bar{\eta}=0}}$$

$$\langle 0|0\rangle_{j,\eta,\bar{\eta}} = \int \mathcal{D}\bar{\Psi}\,\mathcal{D}\Psi\,\mathcal{D}H \exp\left(i\int d^4x\; \mathcal{L}[\bar{\Psi},\Psi,H,\bar{\eta},\eta,j]\right)$$

$$\qquad\qquad = \int \mathcal{D}\bar{\Psi}\,\mathcal{D}\Psi\,\mathcal{D}H \exp\left(i\int d^4x \mathcal{L}_0\right) \exp\left(-i\int d^4x\; gH\bar{\Psi}\Psi\right)$$

$$\langle 0|0\rangle_{j=\eta=\bar{\eta}=0} = \int \mathcal{D}\bar{\Psi}\,\mathcal{D}\Psi\,\mathcal{D}H \exp\left(i\int d^4x\; \mathcal{L}[\bar{\Psi},\Psi,H,\bar{\eta}=\eta=j=0]\right)$$

$$\qquad\qquad = \int \mathcal{D}\bar{\Psi}\,\mathcal{D}\Psi\,\mathcal{D}H \exp\left(i\int d^4x\, \mathcal{L}_0|_{j=\eta=\bar{\eta}=0}\right) \exp\left(-i\int d^4x\; gH\bar{\Psi}\Psi\right)$$

$$H \rightarrow \frac{1}{i}\frac{\delta}{\delta j} \quad , \quad \Psi \rightarrow \frac{1}{i}\frac{\delta}{\delta\bar{\eta}} \quad , \quad \bar{\Psi} \rightarrow -\frac{1}{i}\frac{\delta}{\delta\eta}$$

$$\exp\left(-i\int d^4x\, gH\bar{\Psi}\Psi\right) \rightarrow \exp\left[-ig\int d^4x \frac{1}{i}\frac{\delta}{\delta j(x)}\frac{1}{i}\frac{\delta}{\delta\eta(x)}\frac{1}{i}\frac{\delta}{\delta\bar{\eta}(x)}\right] \checkmark$$

(use Gaussian integral to evaluate $\mathcal{D}H\,\mathcal{D}\bar{\Psi}\,\mathcal{D}\Psi$)

b) $\qquad F = -g\bar{\Psi}\Psi ,\qquad$ no external sources for H

$$\langle 0|0\rangle_{j,\bar{\eta},\eta} = \int \mathcal{D}\bar{\Psi}\,\mathcal{D}\Psi\,\mathcal{D}H \exp\left(i\int d^4x\,(-\frac{1}{2}H\hat{D}_H H + \bar{\Psi}\hat{D}_\Psi\Psi + FH)\right)$$

$+\bar{\Psi}\eta + \bar{\eta}\Psi$

Shift $\qquad H \rightarrow H' = H - i\int d^4y\; D_F(x-y)\,F(y)$

$$F(y) = -g\,\bar{\Psi}(y)\,\Psi(y)\,H(y)$$

$$S_H = \int d^4x \left(-\tfrac{1}{2} H \hat{D}_H H + FH \right)$$

$$\longrightarrow S_{H'} = \int d^4x \left(-\tfrac{1}{2} H' \hat{D}_H H' + FH' \right)$$

$$= \int d^4x \left\{ -\tfrac{1}{2} H' \hat{D}_H H' + \boxed{FH} \right.$$

it gets canceled out with $\tfrac{1}{2}(*)$

$$\left. - \underbrace{i \int d^4y \; F(x) \, D_F(x-y) F(y)}_{(*)} \right\}$$

$$\Rightarrow \langle 0|0 \rangle_{j, n, \bar{n}} = \int \mathcal{D} H' \exp\left(-\tfrac{1}{2} \int d^4x \; H' \hat{D}_H H' \right)$$

$$\times \int \mathcal{D}\bar{\Psi} \, \mathcal{D}\Psi \, \exp\left(i \int d^4x \, (\bar{\Psi} \hat{D}_\Psi \Psi - i \int d^4y \; F(x) D_F(x-y) F(y)) \right)$$

$$\begin{pmatrix} \text{First factor} \quad \text{is} \quad \text{Gaussian integral} \\[4pt] \int \mathcal{D} H' \exp\left(-\tfrac{1}{2} \int d^4x \, H' \hat{D}_H H' \right) = (\det \hat{D}_H)^{-\tfrac{1}{2}} \\[4pt] \text{It gets cancelled out in} \quad W \end{pmatrix} \leftarrow \text{\color{red}Wick rotation}$$

$$= \tilde{N} \int \mathcal{D}\bar{\Psi} \mathcal{D}\Psi \, \exp\left(i \int d^4x \, (\bar{\Psi} \hat{D}_\Psi \Psi - \tfrac{i}{2} \int d^4y \; F(x) D_F(x-y) F(y)) \right)$$
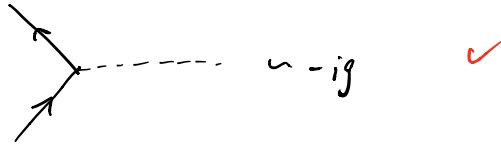
$$\longrightarrow W = \frac{1}{\langle 0|0\rangle_{j=\bar{\eta}=\eta=0}} \int \mathcal{D}\bar{\Psi} \mathcal{D}\Psi \, \exp\left(i \int d^4x \, (\bar{\Psi} \hat{D}_\Psi \Psi - \tfrac{i}{2} \int d^4y \; F(x) D_F(x-y) F(y)) \right)$$

$$\mathcal{L}_{eff} = \bar{\Psi} \hat{D}_\Psi \Psi - i \int d^4y \; F(x) D_F(x-y) F(y)$$

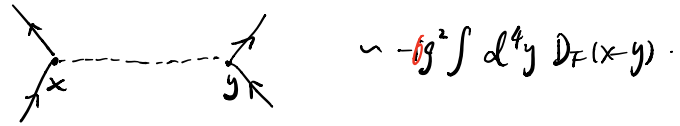c) $\quad P^2 \ll M^2, \quad D_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - M^2 + i\varepsilon} e^{-ip(x-y)}$

$$= \int_{|p|^2 < \Lambda^2} \frac{d^4p}{(2\pi)^4} \underbrace{\frac{i}{p^2/M^2 - 1} \cdot \frac{1}{M^2}}_{} e^{-ip(x-y)}$$

$$= i \left(-1 + \mathcal{O}(P^2/M^2) \right)$$

$$\longrightarrow \mathcal{L}_{eff} \cong \bar{\Psi} \hat{D}_\Psi \Psi + \underbrace{\tfrac{i}{2} \int d^4y \; F(x) \frac{i}{M^2} \delta^{(4)}(x-y) F(y)}_{}$$

$$= -F^2(x) \frac{1}{2M^2}$$

$$= -\frac{g^2}{2M^2} (\bar{\Psi}\Psi)^2$$

d) i)

$\sim -ig$ ✓

ii)

$\sim -ig^2 \int d^4y \, D_F(x-y) \, .$

iii)

$\sim i\frac{g^2}{2M^2}$

e) We went from small scale physics to large scale by integrating field in generating function out.