

H.6

$$a) \quad \theta_i^2 = 0, \quad \forall i, \quad \theta_i \in G$$

$\Rightarrow$  Terms with Grassmann variable with power higher than 2 is automatically zero. Thus any function can be decomposed into monomial in  $\theta_i$ :

Here:

$$\begin{aligned} F(\tilde{\xi}) &= F(\tilde{\xi}^*, \tilde{\xi}) \\ &= F^{(0)} + \sum_i F_i^{(1)} \tilde{\xi}_i + \dots + \sum_{i, \dots, k} F_{i, \dots, k}^{(2n)} \tilde{\xi}_i \dots \tilde{\xi}_j \dots \tilde{\xi}_k \end{aligned}$$

$$F(\tilde{\xi}(\tilde{\eta})) = F^{(0)} + \sum_i F_i^{(1)} M_{i\mu} \tilde{\eta}_\mu + \dots + \sum_{i, \dots, k} F_{i, \dots, k}^{(2n)} M_{i\mu} \tilde{\eta}_\mu \dots M_{j\nu} \tilde{\eta}_\nu \dots M_{k\lambda} \tilde{\eta}_\lambda$$

In the integral only the terms with  $(2n)$  (unique) Grassmann variables are contributing. All other terms vanish because of  $\int d\theta_i = 0$ .

$$\begin{aligned} \int d\tilde{\xi}_1 \dots d\tilde{\xi}_{2n} F(\tilde{\xi}) &= \int d\tilde{\xi}_1 \dots d\tilde{\xi}_{2n} \sum_{i, \dots, k} F_{i, \dots, k}^{(2n)} \tilde{\xi}_i \dots \tilde{\xi}_k \\ &= \sum_{\alpha, \dots, \beta} \int M_{1\alpha} d\tilde{\eta}_\alpha \dots M_{2n, \beta} d\tilde{\eta}_\beta \\ &\quad \times \underbrace{\sum_{i, \dots, k} F_{i, \dots, k}^{(2n)} \sum_{\mu, \dots, \nu} M_{i\mu} \tilde{\eta}_\mu \dots M_{k\nu} \tilde{\eta}_\nu}_{\sim F(\tilde{\xi}(\tilde{\eta}))} \quad \checkmark \end{aligned}$$

Need to take care of order of  $\alpha \dots \beta$ .

$$\begin{aligned} &= \sum_{\alpha, \dots, \beta} M_{1\alpha} \dots M_{2n, \beta} \epsilon_{\alpha, \dots, \beta} \int d\tilde{\eta}_1 \dots d\tilde{\eta}_{2n} F(\tilde{\xi}(\tilde{\eta})) \\ &= \underbrace{(\det M)^{-1}}_{\left| \frac{\partial(\eta, \eta^*)}{\partial(\tilde{\xi}, \tilde{\xi}^*)} \right|} \int d\tilde{\eta}_1 \dots d\tilde{\eta}_{2n} F(\tilde{\xi}(\tilde{\eta})) \quad \checkmark \end{aligned}$$

$$b) \int \prod_{i=1}^n d\eta_i^* d\eta_i \exp(-\underbrace{\eta_k^* H_{ke} \eta_e + \xi_k^* \eta_k + \eta_k^* \xi_k}_{\text{...}})$$

$$= -\eta^T H \eta + \xi^T \eta + \eta^T \xi$$

$$= -\omega^T \Delta \omega + \underbrace{J^T H^{-1} J}_{\text{...}}$$

$$\text{with } \Delta = U H U^{-1} \quad \Delta = \text{diag}(\lambda_{11}, \dots, \lambda_{nn})$$

$$\omega = U(\eta - H^{-1} \xi), \quad \det(U) = 1, \quad U^T U = \mathbb{1}_n$$

probably  
no need  
to diagonalize

$$-(\underbrace{\eta - H^{-1} \xi}_{}^T \underbrace{U^T \Delta U}_{})(\eta - H^{-1} \xi)$$

$$= (\eta^T - \xi^T H)$$

$$= -(\eta^T - \xi^T H) H (\eta - H^{-1} \xi)$$

$$= -\eta^T H \eta - \xi^T H H H^{-1} \xi + \xi^T H H \eta + \eta^T H H^{-1} \xi$$

$$= \left| \frac{\partial(\omega^*, \omega)}{\partial(\eta^*, \eta)} \right| \int \prod_{i=1}^n d\omega_i^* d\omega_i \exp(-\omega^T \Delta \omega + \xi^T H^{-1} \xi)$$

$$\left( \begin{array}{l} = |\det U| \\ = 1 \end{array} \right)$$

$$= \exp(\xi^T H^{-1} \xi) \int \prod_{i=1}^n d\omega_i^* d\omega_i \exp(-\omega^T \underbrace{\Delta \omega}_{\text{diagonal}})$$

$$= -\omega_1^* \lambda_{11} \omega_1 - \dots - \omega_n^* \lambda_{nn} \omega_n$$

$$\left( \int d\theta^* d\theta \underbrace{\exp(-\theta^* \theta)}_{= 1 - \theta^* \theta} = - \int d\theta^* d\theta \theta^* \theta = +1 \right)$$

$$= \exp(\xi^T H^{-1} \xi) \int \prod d\omega_i^* d\omega_i (1 - \omega_1^* \lambda_{11} \omega_1 - \dots - \omega_n^* \lambda_{nn} \omega_n)$$

$$= \exp(\xi^T H^{-1} \xi) \underbrace{\lambda_{11} \cdot \dots \cdot \lambda_{nn}}_{= \det \Delta = \det H}$$



$$= \sqrt{2E_k} e^{-\frac{1}{2}\lambda} \langle 0 | a_{\vec{k}} \exp(-i \int d^4x \int \frac{d^3\vec{p}}{(2\pi)^3 2E_p} e^{+ipx} a_{\vec{p}}^\dagger j(x)) \exp(-i \int d^4x \int \frac{d^3\vec{p}}{(2\pi)^3 2E_p} e^{-ipx} a_{\vec{p}} j(x)) | 0 \rangle$$

$$\langle 0 | \exp(\dots) -i \int d^4x \int \frac{d^3\vec{p}}{(2\pi)^3 2E_p} e^{+ipx} \cdot (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{k}) j(x) | 0 \rangle$$

$$\langle 0 | -i \int d^4x \frac{1}{\sqrt{2E_k}} e^{ikx} j(x) | 0 \rangle$$

$$= e^{-\frac{1}{2}\lambda} \cdot -i \int d^4x e^{ikx} j(x)$$

$$\Rightarrow \langle n | S | 0 \rangle = e^{-\frac{1}{2}\lambda} \frac{1}{n!} (\tilde{j}(k))^n \quad \checkmark$$

$$\langle k_1 \dots k_n | S | 0 \rangle$$

$$= \langle k_1 \dots k_n | \exp(-i \int d^4x \phi_1^-(x) j(x)) | 0 \rangle \exp(-\frac{1}{2}\lambda)$$

$$= \frac{(-i)^n}{n!} \int d^4x_1 \dots d^4x_n \langle k_1 \dots k_n | \phi_1^-(x_1) j(x_1) \dots \phi_1^-(x_n) j(x_n) | 0 \rangle W[j]$$

$$= (-i)^n \left( \int d^4x_1 e^{ik_1 x_1} j(x_1) \right) \dots \left( \int d^4x_n e^{ik_n x_n} j(x_n) \right) W[j]$$

$$= (-i)^n \tilde{j}(k_1) \dots \tilde{j}(k_n) W[j]$$

$$P(0 \rightarrow n) = \frac{1}{n!} \int \frac{d^3k_1}{(2\pi)^3 2E_1} \dots \int \frac{d^3k_n}{(2\pi)^3 2E_n} |\langle k_1 \dots k_n | S | 0 \rangle|^2$$

$$= \frac{\lambda^n e^{-\lambda}}{n!} \rightarrow \text{Poisson distribution}$$

$$\sum_{n=0}^{\infty} P(0 \rightarrow n) = e^{-\lambda} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} = 1$$

$$\sum_{n=0}^{\infty} n P(0 \rightarrow n) = e^{-\lambda} \sum_{n=0}^{\infty} \frac{\lambda^n}{(n-1)!} = \lambda$$