H.13

a) $$\frac{\partial}{\partial t}\rho(t,\vec{x}) = -v(x)\frac{\partial}{\partial x}\rho(t,\vec{x}) + g(x)\rho(t,x)$$

proof ?

rate of change — decrease due to flow — increase due to growth

3p

$$\rightarrow \left[\frac{\partial}{\partial t} + v(x)\frac{\partial}{\partial x} - g(x)\right]\rho(t,\vec{x}) = 0$$

$$\left(\frac{\partial}{\partial t} + v(x)\frac{\partial}{\partial x} - g(x)\right)\rho(t,x) = 0$$

def: growth rate: $\frac{d}{dt}\rho(t,x) \doteq g(x)\rho(t,x)$

$$\rightarrow \frac{\partial}{\partial t}\rho(t,x) + \underbrace{\frac{dx}{dt}}_{v(x)}\frac{\partial}{\partial x}\rho(t,x)$$

b) $\bar{x}(0,x) = x$

$$\frac{d}{dt}\bar{x}(t,x) = v(\bar{x})$$

Only consider $x = \bar{x}$

$$\rightarrow \left[\frac{\partial}{\partial t} + \underbrace{v(\bar{x})}_{=\frac{d\bar{x}(t,x)}{dt}}\frac{\partial}{\partial \bar{x}} - g(\bar{x})\right]\rho(t,\bar{x}) = 0$$

$$\frac{\partial}{\partial t}\rho(t,\vec{x}) - \frac{\partial}{\partial t}\rho(0,\bar{x}) = g(\bar{x})\rho(t,\bar{x}) \quad \Big| \int dt$$

$$\rho(t,\bar{x}) - \rho(0,\bar{x}) = \int_0^t dt\, g(\bar{x})\,\rho(t,\bar{x})$$

2p

$$\rho(t,\vec{x}) = \rho_0(\bar{x}(-t,x))\exp\left(\int_0^t dt'\, g(\bar{x}(-t',x))\right)$$

$$\frac{\partial}{\partial t}\rho(t,x) = \Big(\underbrace{\frac{\partial}{\partial t}\rho_0(\bar{x}(-t,x))}_{=\frac{\partial}{\partial \bar{x}}\rho_0(\bar{x})\underbrace{\frac{d\bar{x}}{dt}}_{=-v(\bar{x})}} + \rho_0(\bar{x}(-t,x))\,g(\bar{x}(-t,x))\Big)\exp(\cdots)$$

$$= \Big(\underbrace{\frac{\partial}{\partial x}\rho_0(\bar{x}(-t,x))}_{=\frac{\partial}{\partial \bar{x}}\rho_0(\bar{x})\frac{\partial \bar{x}}{\partial x} = \frac{v(\bar{x})}{v(x)}\frac{\partial}{\partial \bar{x}}\rho_0(\bar{x})} + \rho_0(\bar{x}(-t,x))\frac{\partial}{\partial x}\int_0^t dt'\, g(\bar{x}(-t,x))\Big)\exp(\cdots)$$

$$\begin{cases} \frac{d}{dt}\bar{x}(t,x) = v(\bar{x}) & \Rightarrow \int_x^{\bar{x}} \frac{dx'}{v(x')} = \int_0^t dt' \\ & \Rightarrow \frac{1}{v(\bar{x})}\frac{\partial \bar{x}}{\partial x} - \frac{1}{v(x)} = 0 \qquad (\rho.i) \\ & \Rightarrow \frac{\partial \bar{x}}{\partial x} = \frac{v(\bar{x})}{v(x)} \end{cases}$$

$$\frac{\partial}{\partial x}\int_0^t dt'\, g(\bar{x}(-t,x)) = \frac{1}{v(x)}\int_0^t dt'\, v(\bar{x})\frac{\partial}{\partial \bar{x}} g(\bar{x})$$

$$\Rightarrow \left[\frac{\partial}{\partial t} + v(x)\frac{\partial}{\partial x}\right]\rho(t,x)$$

$$= \rho_0(\bar{x}(-t,x))\left[g(\bar{x}(-t,x)) + \int_0^t dt'\, v(\bar{x}(-t',x))\frac{\partial}{\partial \bar{x}} g(\bar{x}(-t',x))\right]\exp(\cdots)$$

Hint: $f(a) - f(0) = \int_0^a dx \frac{d}{dx} f(x)$

$$g(\bar{x}(-t,x)) = g(\bar{x}(0,x)) + \int_0^{-t} dt' \frac{\partial}{\partial t'} g(\bar{x}(t',x))$$

$$= g(x) - \int_0^t dt' \frac{\partial}{\partial t'} g(\bar{x}(-t',x))$$

$$\left|\ \frac{\partial}{\partial t'} = \frac{\partial \bar{x}}{\partial t'}\frac{\partial}{\partial \bar{x}}\right.$$

$$= g(x) - \int_0^t dt' \frac{\partial \bar{x}}{\partial t'}\frac{\partial}{\partial \bar{x}} g(\bar{x}(-t',x))$$

$$= g(x) - \int_0^t dt'\, v(\bar{x}(-t,x))\frac{\partial}{\partial \bar{x}} g(\bar{x}(-t',x))$$

$$\left[\frac{\partial}{\partial t} + v(x)\frac{\partial}{\partial x}\right]\rho(t,x) = g(x)\underbrace{\rho_0(\bar{x}(-t,x))\exp(\cdots)}_{=\rho(x)}$$

c) $x \to \vec{x}$

$$\Rightarrow \rho(t,\vec{x}) = \rho_0(\vec{\bar{x}}(-t,\vec{x}))\exp\left[\int_0^t dt'\, g(\vec{\bar{x}}(-t',\vec{x}))\right]$$

Generalization to n-dimension,

$$t \to t, \quad x \to \vec{x}, \quad v \to \vec{v}, \quad \frac{\partial}{\partial x} \to \vec{\nabla}$$

new equation: $\left[\frac{\partial}{\partial t} + \vec{v}(t,\vec{x})\cdot\vec{\nabla} - g(t,\vec{x})\right]\rho(t,\vec{x}) = 0$

H.14

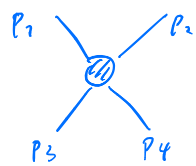
a) $\Gamma^{(n)}(\{\lambda p_i\}, g, \mu) = \Gamma^{(n)}(\lambda\{p_i\}, g, \mu\}$

$$= \underbrace{\mu^D}_{\substack{\text{preserve}\\ \text{the dimension}\\ \text{of } P^\mu}} \underbrace{f(\{\lambda^2 p_i\cdot p_j/\mu^2\}, g)}_{\text{dimensionless}}$$

No! G or Γ is a Lorentz scalar. Only p_i^2, $p_i p_j$ are legal combinations of momenta!

(not unique? can also use $p_i^D f(\cdots)$)

Example (ϕ^4):

$i\Gamma^{(4)} =$ 

$$= -ig - \frac{ig^2}{32\pi^2}\left(6 + \sum_{q^2 = s,t,u} \int_0^1 dx \ln \frac{-q^2 x(1-x) + i\epsilon}{\mu^2}\right)$$

$\uparrow$ q^2/μ^2 structure as in general form

b) $d\Gamma^{(n)}(\{\lambda p_i\}, g, \mu\} = \frac{\partial \Gamma^{(n)}}{\partial \mu} d\mu + \frac{\partial \Gamma^{(n)}}{\partial \lambda} d\lambda \left(+ \frac{\partial \Gamma^{(n)}}{\partial g} dg\right)$

Not eq. from the sheet!

$$\stackrel{!}{=} D\mu^{D-1} d\mu\, f(\{\lambda^2 p_i\cdot p_j/\mu^2\}, g)$$

$$\Rightarrow \left(\frac{\partial}{\partial\mu} + \frac{\partial\lambda}{\partial\mu}\frac{\partial}{\partial\lambda}\right)\Gamma^{(n)}(\{\lambda p_i\}, g, \mu) = D\mu^{D-1} f(\cdots) = D\mu^{-1}\Gamma^{(n)}(\{\lambda p_i\}, g, \mu)$$

$$\left[\mu\frac{\partial}{\partial\mu} + \underbrace{\mu\frac{\partial\lambda}{\partial\mu}}_{\beta}\frac{\partial}{\partial\lambda} - D\right]\Gamma^{(n)}(\{\lambda p_i\}, g, \mu) = 0$$

$$\beta = \lambda\ ?$$

$$\left[\mu\frac{\partial}{\partial\mu} + \frac{\partial}{\partial t} - D\right]\Gamma^{(n)}(\{\lambda^2 p_i\cdot p_j\}, g, \mu) = 0$$

$$\mu\frac{\partial}{\partial\mu}\Gamma^{(n)} = \mu\frac{\partial}{\partial\mu}\left(\mu^D f(\{\lambda^2 p_i p_j\}/\mu^2, g)\right)$$

$$= D\Gamma^{(n)} + \underbrace{\mu^{D} \sum_{\{\lambda^2 p_i p_j\}} f'(\{\lambda^2 p_i p_j\}/\mu^2, g)\, \frac{\lambda^2 p_i p_j}{\mu^2}\,(-2)}_{= -\lambda \frac{\partial}{\partial \lambda} f(\{\lambda^2 p_i p_j\}/\mu^2,\, g)}$$

$$= D\Gamma^{(n)} \underbrace{- \mu^{D} \frac{\partial}{\partial t} f(\{\lambda^2 p_i p_j\}/\mu^2, g)}_{= \frac{\partial}{\partial t}\Gamma^{(n)}}$$