H.2 $$L = \frac{m}{2}\dot{x}^2 - \frac{m}{2}\omega^2 x^2 + f(t)x = L_0 + f(t)x$$

a) EOM:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_{cl}} - \frac{\partial L}{\partial x_{cl}} = 0$$

$$\frac{d}{dt}\left(\frac{m}{2}\cdot 2\dot{x}_{cl}\right) - \left(-\frac{m\omega^2}{2}\cdot 2x_{cl} + f(t)\right) = 0$$

$$m\ddot{x}_{cl}(t) + m\omega^2 x_{cl}(t) - f(t) = 0$$

$$\ddot{x}_{cl}(t) + \omega^2 x_{cl}(t) = \frac{1}{m} f(t)$$

write $x(t) = x_{cl}(t) + y(t)$

$$L = \frac{m}{2}\left(\dot{x}_{cl}(t) + \dot{y}(t)\right)^2 - \frac{m}{2}\omega^2\left(x_{cl}(t) + y(t)\right)^2 + f(t)\left(x_{cl}(t) + y(t)\right)$$

$$= L_{cl,0} + L_y + m\dot{x}_{cl}(t)\dot{y}(t) - m\omega^2 x_{cl}(t)y(t) + f(t)x_{cl}(t)$$

$$\int_{t_a}^{t_b} dt\, L - \int_{t_a}^{t_b} dt\,(L_{cl} + L_y)$$

$$= m\int_{t_a}^{t_b} dt\,\left(\dot{x}_{cl}\dot{y} - \omega^2 x_{cl} y + \frac{1}{m} f x_{cl}\right)$$

$$= m\underbrace{\int_{t_a}^{t_b} dt\,\left(\underbrace{-\ddot{x}_{cl} y - \omega^2 x_{cl} y}_{=\frac{1}{m} f x_{cl}} + \frac{1}{m} f x_{cl}\right)}_{=0} + \overbrace{\left[x_{cl}\dot{y}\right]_{t=t_a}^{t=t_b}}^{=0}$$

$$\langle 0\, t_a | 0\, t_b\rangle_H^f$$

$$= \langle 0 | e^{iH(t_b - t_a)} | 0\rangle_S$$

$$= \int \mathcal{D}x\; e^{i\int_{t_a}^{t_b} dt\, L}$$

$\mathcal{D}x = \mathcal{D}x_{cl} + \mathcal{D}y$

$$= \int \mathcal{D}x\, e^{i\int_{t_a}^{t_b} dt\,(L_{cl,0} + L_y)}$$

$$= e^{iS_{cl}} \int \mathcal{D}y\; e^{i\int_{t_a}^{t_b} dt\, L_y}$$

$$\langle 0\, t_a | 0\, t_b\rangle^{f=0} = \int \mathcal{D}x\, e^{i\int_{t_a}^{t_b} dt\, L_0}$$

$$\Rightarrow \langle 0\, t_a | 0\, t_b\rangle^f = e^{iS_{cl}} \langle 0\, t_a | 0\, t_b\rangle^{f=0}$$

$$S_{cl} = m\int_{t_a}^{t_b} dt\ \frac{1}{2}\dot{x}_{cl}^2 - \frac{1}{2}\omega^2 x_{cl}^2 \underset{\substack{\uparrow \\ \text{EOM}}}{=} m\int_{t_a}^{t_b} dt\ \frac{1}{2} f(t)\, x_{cl}(t)$$

b)

$$G(t-t') = \langle 0, t_b | T\, x(t)\, x(t') | 0, t_a\rangle^f$$

$$= \frac{1}{i}\frac{\delta}{\delta f(t')}\,\frac{1}{i}\frac{\delta}{\delta f(t)}\,\langle 0, t_b | 0, t_a\rangle^f$$

$$= \int \mathcal{D}x\ \ x(t)\, x(t')\ e^{i\int_{t_b}^{t_a} dt' (L_0 + f x)}$$

H.3 $\quad H = \frac{\hbar^2 \hat{k}^2}{2m} + V(\hat{x})$

a) $\langle x_b, t_b | x_a, t_a \rangle_H = \langle x_b | e^{-iH(t_b - t_a)} | x_a \rangle_S$

$= \langle x_b | e^{-iH(t_b - t')} e^{-iH(t' - t_a)} | x_a \rangle_S$

$= \int dx' \langle x_b | e^{-iH(t_b - t')} | x' \rangle \langle x' | e^{-iH(t' - t_a)} | x_a \rangle$

$= \int dx' \langle x_b, t_b | x', t' \rangle \langle x', t' | x_a, t_a \rangle$

especially for $t_b > t' > t_a$

(the same valid for $t' > t_b$ or $t' < t_a$?)

b) $\langle x, t+\Delta t | x_a, t_a \rangle = \int dx' \langle x, t+\Delta t | x', t' \rangle \langle x', t' | x_a, t_a \rangle$

$LHS = \langle x | e^{-iH(t+\Delta t - t_a)} | x_a \rangle$

$= \langle x | e^{-iH(t-t_a)} \underbrace{e^{-iH\Delta t}}_{= 1 - iH\Delta t} | x_a \rangle$

$= \langle x, t | x_a, t_a \rangle + \Delta t \frac{\partial}{\partial t} \langle x, t | x_a, t_a \rangle$

$RHS = \int dx' \underbrace{\langle x | e^{-iH(t+\Delta t - t')} | x' \rangle} \langle x' | e^{-iH(t' - t_a)} | x_a \rangle$

$= \int \frac{dk}{2\pi} \langle x | k \rangle \langle k | e^{-iH(t+\Delta t - t')} | x' \rangle$

$= \frac{1}{2\pi} \int dk \, e^{ixp/\hbar} \langle p | e^{-i\left(\frac{\hbar^2 \hat{k}^2}{2m} + V(\hat{x})\right)(t+\Delta t - t')} | x' \rangle$

$= \frac{1}{2\pi} \int dk \, e^{i(x-x')p/\hbar} e^{-ip^2/2m \, (t+\Delta t - t')} e^{-iV(\hat{x})(t+\Delta t - t')}$

$= \frac{1}{2\pi} e^{-iV(x')\Delta t/\hbar} \underbrace{\int dk e^{\frac{-i}{2m}\Delta t p^2 + i(x-x')p/\hbar}}$, $t = t'$

$= \sqrt{\frac{2\pi m}{+i\hbar^2 \Delta t}} \, e^{\frac{-(x-x')^2}{2i\hbar \Delta t} \cdot m}$ $\quad a = \frac{+i\hbar^2 \Delta t}{m}$, $b = i(x-x')$

$$= \sqrt{\frac{2\pi m}{i\hbar \Delta t}}\, e^{\frac{i(x-x')^2 m}{2\hbar\, \Delta t}}$$

$$= \int dx' \sqrt{\frac{m}{2\pi i\hbar \Delta t}}\, e^{\frac{im(x-x')^2}{2\hbar\Delta t}} \left(1-\frac{i}{\hbar} V(x')\Delta t\right) \langle x' t | x_a t_a \rangle$$

c) Main contribution of integral : $x' \approx x$

$$\langle x', t | x_a, t_a \rangle = \langle x, t | x_a, t_a \rangle + \frac{\partial}{\partial x} \langle x, t | x_a, t_a \rangle (x' - x) + \frac{\partial^2}{\partial x^2} \langle x, t | x_a, t_a \rangle (x'-x)^2 + \mathcal{O}(x'^3)$$

falls off in the integral since it is odd

$$\text{First term} = \int_{-\infty}^{\infty} dx'\, e^{\frac{im}{2\hbar\Delta t}(x-x')^2} \cdot \sqrt{\frac{m}{2\pi i\hbar\Delta t}} \left(1-\frac{i}{\hbar}V(x)\Delta t\right) \langle x, t | x_a, t_a \rangle$$

$$= \underbrace{\sqrt{2\pi \cdot \frac{i\hbar\Delta t}{m}}\sqrt{\frac{m}{2\pi i\hbar\Delta t}}}_{} \left(1-\frac{i}{\hbar}V(x)\Delta t\right) \langle x, t | x_a, t_a \rangle$$

$$\text{Third term} = \underbrace{\sqrt{\frac{m}{2\pi i\hbar\Delta t}}}_{} \overset{1}{\left(1-\frac{i}{\hbar}V(x)\Delta t\right)} \frac{\partial^2}{\partial x^2} \langle x, t | x_a, t_a \rangle \underbrace{\int_{-\infty}^{\infty} dx'\, e^{\frac{im(x-x')^2}{2\hbar\Delta t}} (x'-x)^2}_{}$$

$$= \left(\frac{i\hbar\Delta t}{m}\right)^{3/2} \int dy\, e^{y^2}\, y^2 \cdot \frac{1}{2}$$

$$= \left(\frac{i\hbar\Delta t}{m}\right)^{3/2} \sqrt{\pi} \cdot \frac{1}{2}$$

$$= \underbrace{\left(\frac{i\hbar\Delta t}{m}\right)^{3/2}}_{} \sqrt{\pi} \cdot \frac{1}{2}$$

$$= \left(1-\frac{i}{\hbar}V(x)\Delta t\right) \frac{\partial^2}{\partial x^2} \langle x, t | x_a, t_a \rangle \frac{i\hbar\Delta t}{2m}$$

$$\Rightarrow \langle x, t | x_a, t_a \rangle + \Delta t \frac{\partial}{\partial t} \langle x, t | x_a, t_a \rangle = \left(1-\frac{i}{\hbar}V(x)\Delta t\right)\left(\langle x, t | x_a, t_a \rangle + \frac{\partial^2}{\partial x^2} \langle x, t | x_a, t_a \rangle\right)$$

$$\Delta t \frac{\partial}{\partial t} \langle x, t | x_a, t_a \rangle = -\frac{i}{\hbar} V(x)\Delta t \langle x, t | x_a, t_a \rangle$$

$$+\left(1-\frac{i}{\hbar}\underbrace{\cancel{V(x)\delta t}}_{\mathcal{O}(\delta t^2)}\right)\frac{\partial^2}{\partial x^2}\langle x,t|x_a,t_a\rangle\frac{i\hbar\delta t}{2m}$$

$$\Rightarrow \frac{\partial}{\partial t}\langle x,t|x_a,t_a\rangle = -\frac{i}{\hbar}V(x)\langle x,t|x_a,t_a\rangle + \frac{i\hbar}{2m}\frac{\partial^2}{\partial x^2}\langle x,t|x_a,t_a\rangle$$

$\Rightarrow$ (C)

Substitute $|x_a,t_b\rangle = |\psi\rangle$

$$\Rightarrow i\hbar\frac{\partial}{\partial t}\psi(x,t) = -\frac{\hbar^2}{2m}\partial_x^2\psi(x,t) + V(x)\psi(x,t)$$