H.1 $\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - c\phi - \frac{m^2}{2}\phi^2 - \frac{g}{3!}\phi^3$

a) $[\mathcal{L}] = d$

$[\phi] = \frac{d}{2} - 1$

$\Rightarrow \quad [g] + 3[\phi] = [g] + \frac{3}{2}d - 3 \overset{!}{=} d \quad \Leftrightarrow \quad [g] = 3 - \frac{1}{2}d$

$d = 6 \quad \Rightarrow \quad [g] = 3 - \frac{1}{2}6 = 0$

Now $d = 6 - 2\epsilon, \quad g \to \mu^x g, \quad c \to \mu^y c$

$\Rightarrow \quad x = 3 - \frac{1}{2}d \quad ,$

$[c] + [\phi] = [c] + \frac{d}{2} - 1 = d$

$\Rightarrow [c] = 1 + \frac{d}{2}$

$\Rightarrow \quad y = 1 + \frac{d}{2}$

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \mu^{1+\frac{d}{2}} c\phi - \frac{m^2}{2}\phi - \mu^{3-\frac{1}{2}d}\frac{g}{3!}\phi^3$$

b) $d = 6, \quad x = 0, \quad y = 4 > 0$

$\Rightarrow$ renormalizable

$D = dL - 2I \quad \leftarrow$ from expression of integrals

$L = I - V + 1$

$3V = 2I + E \quad \Rightarrow \quad (3V - E)\cdot\frac{1}{2} = I$

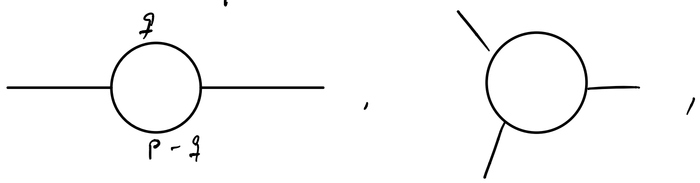
$$\Rightarrow \quad D = d\left(\tfrac{1}{2}(3V - E) - V + 1\right) - 2\cdot\tfrac{1}{2}(3V - E)$$

$$= d\left(\tfrac{1}{2}V - \tfrac{1}{2}E + 1\right) - 3V + E$$

$$= d + \left(\tfrac{1}{2}d - 3\right)V + \left(1 - \tfrac{1}{2}d\right)E$$

$d = 6 \quad \Rightarrow \quad D = 6 - 2E$

1PI one-loop diagram



$$(-ig)^2 \int \frac{d^4 q}{(q^2-m^2)((p-q)^2-m^2)}$$

$D = 6 - 2 \cdot 2 = 2$ $\qquad D = 0$

$$\Sigma_{2p}(p^2) = \Sigma_{2p}(0) + p^2 \Sigma'_{2p}(0) + \tilde{\Sigma}_{2p}(p^2)$$

$$\Gamma_{3p}(p^2, p'^2) = \Gamma_{3p}(0,0) + p^2 \Gamma'_{3p}(0,0) + \tilde{\Gamma}_{3p}(p^2, p'^2)$$

c)

$$-i\Sigma_{2p}(p^2) = \frac{(-i\mu^{3-d/2} g)^2}{2} \int \frac{d^d q}{(2\pi)^d} \frac{(i)^2}{[(p-q)^2-m^2][q^2-m^2]}$$

$$= \frac{\mu^{6-d} g^2}{2(2\pi)^d} \int_0^1 dx \int d^d q \frac{1}{[\underbrace{x((p-q)^2-m^2) + (1-x)(q^2-m^2)}]^2}$$

$$= p^2 x - 2pqx + \cancel{q^2 x} - \cancel{m^2 x}$$
$$q^2 - m^2 - \cancel{q^2 x} + \cancel{m^2 x}$$
$$= q^2 - 2pqx - m^2 + p^2 x$$
$$= (q - px)^2 - p^2 x^2 - m^2 + p^2 x$$

$$= \frac{\mu^{6-d} g^2}{2(2\pi)^d} \int_0^1 dx \int d^d k \frac{1}{(k^2 - \Delta)^2}$$

$\Delta = +m^2 + p^2 x(x-1)$, $k = q - px$

$k_0 = i k_{0,E}$, $\vec{k} = \vec{k}_E$

$d = 6 - 2\epsilon$

$$= \frac{i\mu^{6-d} g^2}{2(2\pi)^d} \int_0^1 dx \int d^d k_E \frac{1}{(k_E^2 + \Delta)^2}$$

$$= \frac{ig^2}{2(2\pi)^d} \int_0^1 dx \; \pi^{3-\epsilon} \; \Delta(x)^{3-\epsilon-2} \frac{\Gamma(2-3+\epsilon)}{\Gamma(2)}$$

$$= \frac{ig^2}{2(2\pi)^d} \pi^{3-\epsilon} \Gamma(\epsilon - 1) \int_0^1 dx \, [m^2 + p^2 x(x-1)]^{1-\epsilon}$$

$$\overbrace{}$$

$$= \frac{(-1)^1}{1!}\left(\frac{1}{\epsilon} + \psi(1+1) + \mathcal{O}(\epsilon)\right)$$

$$= -\left(\frac{1}{\epsilon} + \psi(1) + 1 + \mathcal{O}(\epsilon)\right)$$

$$= -\left(\frac{1}{\epsilon} - \gamma_E + 1 + \mathcal{O}(\epsilon)\right)$$

$\epsilon \to 0$, $a^\epsilon = 1 + \epsilon \ln a$

$$= \frac{ig^2}{2(2\pi)^6}\pi^3\left(1 - \epsilon\ln\pi\right)\left(-\frac{1}{\epsilon} + \gamma_E - 1 + \mathcal{O}(\epsilon)\right)$$

$2\cdot 8^2 =$

$$\times \int_0^1 dx\left(m^2 + p^2 x(x-1)\right)\left(1 - \epsilon\ln(m^2 + p^2 x(x-1))\right)$$

$$= \frac{-ig^2}{2\cdot 64\pi^3}\frac{1}{\epsilon}\int_0^1 dx\left(m^2 + p^2(x^2 - x)\right)$$

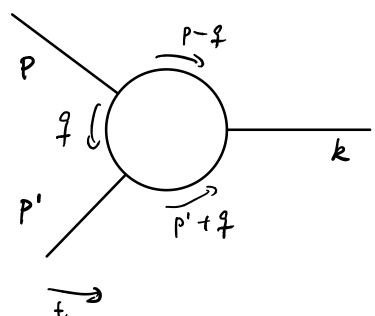
$$\Sigma_{2p}(0) = \frac{g^2}{64\pi^3}\frac{m^2}{2\epsilon}$$

$$\Sigma'_{2p}(0) = \frac{g^2}{2\cdot 64\pi^3}\frac{1}{\epsilon}\underbrace{\frac{d}{dp^2}\int_0^1 dx\left(m^2 + p^2(x^2 - x)\right)\Big|_{p=0}}$$

$$= \int_0^1 dx(x^2 - x)$$

$$= \frac{1}{6}$$

$$= \frac{g^2}{6\cdot 64\pi^3}\frac{1}{2\epsilon}$$



$$-i\Gamma_{3p}(p^2, p'^2) = (-i\mu^{3-d/2}g)^3\int\frac{d^dq}{(2\pi)^d}\frac{(i)^3}{\underbrace{(q^2-m^2)}_{A_1}\underbrace{((p-q)^2-m^2)}_{A_2}\underbrace{((p'+q)^2-m^2)}_{A_3}}$$

$$\Gamma(p^2, p'^2) = \frac{1}{i} \frac{\mu^{9-3d/2} g^3}{(2\pi)^d} \int d^d q \int_0^1 dx \int_0^1 dy \int_0^1 dz \; \delta(x+y+z-1) \frac{2!}{(xA_1 + yA_2 + zA_3)^3}$$

$$x(q^2 - m^2) + y((p-q)^2 - m^2) + z((p'+q)^2 - m^2)$$
$$= (x+y+z)(q^2 - m^2) - 2pqy + p^2 y + 2p'q z + p'^2 z$$

$$= \frac{2\mu^{9-3d/2} g^3}{i(2\pi)^d} \int d^d q \int_0^1 dy\, dz \frac{1}{[q^2 - m^2 - 2pqy + p^2 y + 2p'qz + p'^2 z]^3}$$

$$\text{Denominator} = q^2 + q(-2py + 2p'z) - m^2 + p^2 y + p'^2 z$$
$$= \underbrace{[q + (2p'z - 2py)]^2}_{\tilde{q}} \underbrace{- (2p'z - 2py)^2 - m^2 + p^2 y + p'^2 z}$$
$$= -4p'^2 z^2 - 4p^2 y^2 + 8p'pyz - m^2 + p^2 y + p'^2 z$$
$$= p'^2 z(1 - 4z) + p^2 y(1 - 4y) + 8p'pyz - m^2$$
$$=: \Delta^2$$

$$= \frac{2\mu^{9-3d/2} g^3}{i(2\pi)^d} \int d^d\tilde{q} \int_0^1 dy\, dz \frac{1}{(\tilde{q}^2 - \Delta^2)^3}$$

Wick-rotation

$$= \frac{2\mu^{9-3d/2} g^3}{(2\pi)^d} \int_0^1 dy\, dz \underbrace{\int \frac{d^d \tilde{q}_E}{(\tilde{q}_E^2 + \Delta^2)^3}}_{= \pi^{d/2} (\Delta^2)^{d/2-3} \frac{\Gamma(3-d/2)}{\Gamma(3)}}$$

$d = 6 - 2\epsilon$

$$= \cancel{2} \frac{\mu^{3\epsilon} g^3}{(2\pi)^6} \int_0^1 dy\, dz\, \pi^3 \underbrace{(\Delta^2)^{-\epsilon}}_{= 1 - \epsilon \ln \Delta^2} \frac{\Gamma(\epsilon)}{\cancel{2!}}$$

the divergent part

$$= \frac{\mu^{3\epsilon} g^3 \pi^3}{(2\pi)^6} \underbrace{\int_0^1 dy\, dz}_{= 1/2} \cdot \frac{1}{\epsilon} = \frac{\mu^{3\epsilon} g^3}{4 \cdot 16\, \pi^3} \frac{1}{2\epsilon} = \frac{\mu^{3\epsilon} g^3}{64 \pi^3} \frac{1}{2\epsilon}$$

$$\Gamma'(0,0) = \frac{\mu^{3\epsilon} g^3}{(2\pi)^6} \frac{d}{dp^2} \int_0^1 dy\, dz \left(p'^2 z(1-4z) + p^2 y(1-4y) + 8p'p\, y\, z - m^2\right)^{-\epsilon} \Gamma(\epsilon)$$

$$= \frac{\mu^{3\epsilon} g^3}{(2\pi)^6} \int_0^1 dy\, dz\, (\Delta^2)^{-\epsilon-1}\, y(1-4y)\, \Gamma(\epsilon) \cdot (-\epsilon) \Big|_{p=p'=0}$$

$$= \frac{\mu^{3\epsilon} g^3}{(2\pi)^6} \int_0^1 dy\, dz\, (-m^2)^{-\epsilon-1}\, y(1-4y)\, \Gamma(\epsilon) \cdot (-\epsilon)$$

$$= \int_0^1 dy\; y(1-4y) \cdot y$$

$$= \int_0^1 dy\, (y^2 - 4y^3)$$

$$= \left[\tfrac{1}{3} y^3 - y^4\right]_0^1$$

$$= \frac{1}{3} - 1 = -\frac{2}{3}$$

$$= \frac{\mu^{3\epsilon} g^3}{(2\pi)^6} \left(-\frac{1}{m^2}\right) \cdot \left(-\frac{2}{3}\right) \left(\frac{1}{\epsilon} + \mathcal{O}(\epsilon)\right)(-\epsilon)$$

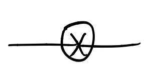
$$= -\frac{\mu^{3\epsilon} g^3}{(4\pi)^3} \frac{2}{3m^2} = -\frac{\mu^{3\epsilon} g^3}{3 \cdot 32 \cdot \pi^3\, m^2}$$

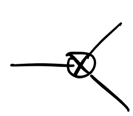
d) $\mathcal{L}_R = \mathcal{L}_0 + \mathcal{L}_{c.t}$

$$= \frac{1}{2}(\partial\phi)^2 - c\phi - \frac{m^2}{2}\phi^2 - \frac{g}{3!}\phi^3$$

$$+ \frac{\delta Z}{2}(\partial\phi)^2 - \delta c\phi - \frac{\delta m}{2}\phi^2 - \frac{\delta g}{3!}\phi^3$$

Feynman rules for counter-term interactions:

 $= i(\delta z\, p^2 - \delta m) = i\Sigma(p^2)$

 $= -i\delta g = -i\Gamma(0,0)$

$$\Rightarrow \quad \delta g = \frac{\mu^{3\epsilon} g^3}{64\pi^3} \frac{1}{2\epsilon}$$

$$\delta z = \frac{g^2}{6 \cdot 64\pi^3} \frac{1}{2\epsilon}$$

$$\delta m = \frac{g^2}{64\pi^3} \frac{m^2}{2\epsilon}$$

$$\delta c = \quad ?$$