Advanced Quantum Field Theory

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see Peskin

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2.1 Reminder: Path integrals in Quantum Mechanics

Transition amplitude is given by

$$\langle x_b|\mathrm{e}^{-iH(t_b-t_a)}|x_a\rangle_S = \langle x_b, t_b|x_a, t_a\rangle_H \tag{2.1.1}$$

Here we denotes the Schrödinger picture states by $_S$ and Heisenberg picture states by $_H$.

$$\begin{aligned}
|x_a, t_a\rangle &= \mathrm{e}^{iHt_a}\,|x_a\rangle && (2.1.2)\\
\hat{H}_a(t_a) &= \mathrm{e}^{iHt_a}\,\hat{x}_S\,\mathrm{e}^{-iHt_a} && (2.1.3)\\
\hat{x}_H(t_a)\,|x_a, t_a\rangle &= \mathrm{e}^{iHt_a}\,\hat{x}_s\,|x_a\rangle = \mathrm{e}^{iHt_a}\,x_a\,|x_a\rangle &&\\
&= x_a\mathrm{e}^{iHt_a}\,|x_a\rangle = x_a\,|x_a, t_a\rangle && (2.1.4)
\end{aligned}$$

We are looking at time evolution in position space.

It can be calculated directly for free particle with Hamiltonian $H = H_0 = \frac{\hat{p}^2}{2m}$

$$\langle x_b|\mathrm{e}^{-i\frac{\hat{p}^2}{2m}(t_b-t_a)}|x_a\rangle = \sqrt{\frac{m}{2\pi i(t_b-t_a)}}\,\mathrm{e}^{i(x_b-x_a)^2\frac{m}{2(t_b-t_a)}} \tag{2.1.5}$$

We are going to insert $1 = \int \mathrm{d}^3p\,|p\rangle\,\langle p|$ and use $\langle x|p\rangle$ is the plane wave

For general Hamiltonian $H = H_0 + V$ and $[H_0, V] \neq 0$ the procedure is as following

- divide t into N small intervals $t = N\cdot\epsilon$
- use Lie-Kato-Trotter product formula

$$\mathrm{e}^{A+B} = \lim_{N\to 0}\left(\mathrm{e}^{A/N}\mathrm{e}^{B/N}\right)^N \qquad A, B \in GL(n, \mathbb{C}) \tag{2.1.6}$$

Then we get a functional for path $x(t')$

$$\langle x_b|\mathrm{e}^{-iH(t_b-t_a)}|x_a\rangle = \int \mathcal{D}x\mathrm{e}^{iS[x]/\hbar} \tag{2.1.7}$$

with $S[x] = \int_{t_a}^{t_b}\mathrm{d}t'\left[\frac{m}{2}\dot{x}(t') - V(x(t'))\right]$

†see also in Peskin and Schroeder Ch 9.1, Ryder Ch 5.1, L.S.Brown Ch1 1-3

Definition integration measure

$$\mathcal{D}x = D[x(t)] = \lim_{N\to\infty}\left(\frac{mN}{2\pi i\Delta t}\right)^{N/2} \mathrm{d}x(t_1)\ldots\mathrm{d}x(t_{N-1}) \tag{2.1.8}$$

with $\Delta t = (t_b - t_a)/N$

Pictorially we sum over all paths (i.e. amplitudes). Remember the superposition principle in quantum mechanics!

Classical path comes from Hamilton principle $\delta S = 0$

$$\left.\frac{\delta S[x]}{\delta x(t)}\right|_{x=x_{\mathrm{cl}}} = 0 \tag{2.1.9}$$

Classical path dominates the transition probability in the limit $\hbar \to 0$. It is the contribution with fewest oscillations in the path integral. Others interfere destructively (averaged out). This is essentially stationary phase approximation.

Example harmonic oscillation

$$L = \frac{m}{2}\left(\dot{x}^2 - \omega^2 x^2\right) \tag{2.1.10}$$

Then the classical path obeys the equation of motion

$$\ddot{x}_{\mathrm{cl}}(t) + \omega^2 x_{\mathrm{cl}}(t) = 0 \tag{2.1.11}$$

Split a general path into classical and fluctuations $x(t) = x_{\mathrm{cl}}(t) + y(t)$. The action turns into

$$S[x] = S[x_{\mathrm{cl}}] + \underbrace{\int \mathrm{d}t\, \frac{\delta s}{\delta x(t)}|_{x=x_{\mathrm{cl}}} y(t)}_{=0} + \frac{1}{2}\int \mathrm{d}t \int \mathrm{d}t'\, \frac{\delta^2 S}{\delta x(t)\delta x(t')}|_{x=x_{\mathrm{cl}}} y(t)y(t') + \ldots$$

Then we can factor out the classical path contribution in transition probability

$$\langle x_b|\mathrm{e}^{-iHT}|x_a\rangle = \int \mathcal{D}x \mathrm{e}^{\frac{i}{\hbar}S[x]} = \mathrm{e}^{\frac{i}{\hbar}S[x_a]} \int \mathcal{D}x \mathrm{e}^{\frac{i}{\hbar}S[y]}$$

The integral is to sum over fluctuations around the classical path. Ideally suited to treat fluctuations (quantum and thermal). The explicit calculation for harmonics oscillator can be found in AQT course.

Physical Interpretation the transition probability is the propagator

$$\langle x_b|\mathrm{e}^{-iH(t_b-t_a)}|x_a\rangle = U(x_b t_b; x_a t_a) \tag{2.1.12}$$

Superposition principle takes the form

$$\begin{aligned}
\psi(x_b, t_b) &= \langle x_b|\psi(t_b)\rangle = \langle x_b|\mathrm{e}^{-iHt_b}|\psi\rangle \\
&= \int \mathrm{d}x_a \,\langle x_b|\mathrm{e}^{-iH(t_b-t_a)}|x_a\rangle \,\langle x_a|\mathrm{e}^{-iHt_a}|\psi\rangle \\
&= \int \mathrm{d}x_a\, U(x_b t_b; x_a t_a) \underbrace{\langle x_a|\psi(t_a)\rangle}_{\psi(x_a,t_a)}
\end{aligned}$$

2.2 Quantum Mechanical Path Integrals and External Forces

Definition Time evolution operator in path integral representation

$$U(x_b, t_b; x_a, t_a) = \langle x_b, t_b | x_a, t_a \rangle = \int \mathcal{D}x(t) \mathrm{e}^{iS[x]} = \int \mathcal{D}x(t) \mathrm{e}^{i \int_{t_a}^{t_b} \mathrm{d}t L(x, \dot{x})} \tag{2.2.1}$$

Add coupling to an external force (source) $f(t)$

$$L = L_0 + f(t)x(t) \tag{2.2.2}$$

Definition functional derivatives with respect to $if(t)$

$$\frac{\delta}{\delta f(t)} \int \mathrm{d}t' \, f(t')g(t') = g(t) \tag{2.2.3}$$

For a general functional of external forces

$$F[f] = \int \mathrm{d}t_1 \, K_1(f_1) f(t_1) + \frac{1}{2!} \int \mathrm{d}t_1 \, \mathrm{d}t_2 \, K_2(t_1, t_2) f(t_1) f(t_2) + \dots \tag{2.2.4}$$

with the $K_n(t_1, \dots, t_n)$ totally symmetric in the arguments $t_1, \dots, t_n$, since antisymmetric contributions drop automatically upon integration. The functional derivatives is then

$$\frac{\delta F}{\delta f(t)} = K_1(t) + \int \mathrm{d}t_2 \, K_2(t, t_2) f(t_2) + \frac{1}{2!} \int \mathrm{d}t_2 \, \mathrm{d}t_1 \, K_3(t, t_2, t_3) f(t_2) f(t_3) + \dots \tag{2.2.5}$$

Consider functional derivative of time evolution operator

$$\begin{aligned} \frac{1}{i} \frac{\delta}{\delta f(t)} \langle x_b, t_b | x_a, t_a \rangle^f &= \int \mathcal{D}x \exp\left(i \int_{t_a}^{t_b} \mathrm{d}t' \, L_0 \right) \frac{1}{i} \frac{\delta}{\delta f(t)} \exp\left(i \int_{t_a}^{t_b} \mathrm{d}t' \, f(t')x(t') \right) \\ &= \int \mathcal{D}x \, x(t) \exp\left(i \int_{t_a}^{t_b} \mathrm{d}t' \, [L_0 + f(t')x(t')] \right) \end{aligned}$$

To split the path integral into two parts, time before and after t (superposition principle). M steps before t and $N - M - 1$ steps after t. The integration over $x(t)$ is to sum over all possible positions at time t.

$$\int_{t_a}^{t_b} \mathcal{D}x = \int \mathrm{d}x(t) \int_t^{t_b} \mathcal{D}x \int_{t_a}^{t} \mathcal{D}x$$

Then

$$\begin{aligned} \frac{1}{i} \frac{\delta}{\delta f(t)} \langle x_b, t_b | x_a, t_a \rangle^f &= \int \mathrm{d}x(t) \underbrace{\int \mathcal{D}x \exp\left(i \int_t^{t_b} \mathrm{d}t' \, (L_0 + fx) \right)}_{N-M-1 \text{ factor}} x(t) \underbrace{\int \mathcal{D}x \exp\left(i \int_{t_a}^{t_b} \mathrm{d}t' \, (L_0 + fx) \right)}_{M \text{ factor}} \\ &= \int \mathrm{d}x(t) \, \langle x_b, t_b | x(t), t \rangle^f \, x(t) \, \langle x(t), t | x_a, t_a \rangle^f \end{aligned}$$

Here $x(t)$ is an eigenvalue, not an operator, so we write $x(t) = \bar{x}$ with

$$\int \mathrm{d}\bar{x}\, |\bar{x}, t\rangle\, \bar{x}\, \langle \bar{x}, t| = \int \mathrm{d}\bar{x}\, \bar{x}\, \langle \bar{x}, t|\bar{x}, t\rangle = x(t)$$

the Heisenberg operator.

We get

$$\frac{1}{i}\frac{\delta}{\delta f(t)}\, \langle x_b t_b | x_a t_a\rangle^f = \langle x_b, t_b | x(t) | x_a, t_a\rangle \tag{2.2.6}$$

The functional derivative with respect to the external force $f(t)$ which couples to $x(t)$, to "insert" the operator $x(t)$ into the matrix element.

Now consider *two* functional derivatives with $t_b \geq t, t' \geq t_a$

$$\frac{1}{i}\frac{\delta}{\delta f(t)}\frac{1}{i}\frac{\delta}{\delta f(t')}\, \langle x_b, t_b | x_a, t_a\rangle^f = \int \mathcal{D}x\, x(t) x(t') \mathrm{e}^{i\int_{t_a}^{t_b} \mathrm{d}t' [L_0 + f\cdot x]} \tag{2.2.7}$$

In general

$$\begin{aligned}
\frac{1}{i}\frac{\delta}{\delta f(t)}\frac{1}{i}\frac{\delta}{\delta f(t')}\, \langle x_b, t_b | x_a, t_a\rangle^f &= \frac{1}{i}\frac{\delta}{\delta f(t)}\, \langle x_b, t_b | x(t') | x_a, t_a\rangle^f \\
&= \frac{1}{i}\frac{\delta}{\delta f(t)} \int \mathrm{d}\bar{x}'\, \langle x_b, t_b | \bar{x}', t\rangle^f\, \bar{x}'\, \langle \bar{x}', t' | x_a, t_a\rangle^f \\
&= \int \mathrm{d}\bar{x}' \left(\frac{1}{i}\frac{\delta}{\delta f(t)}\, \langle x_b, t_b | \bar{x}', t\rangle^f\right) \bar{x}'\, \langle \bar{x}', t' | x_a, t_a\rangle^f \\
&\quad + \int \mathrm{d}\bar{x}'\, \langle x_b, t_b | \bar{x}', t\rangle^f\, \bar{x}' \left(\frac{1}{i}\frac{\delta}{\delta f(t)}\, \langle \bar{x}', t' | x_a, t_a\rangle^f\right)
\end{aligned}$$

Then transition amplitudes only depend on the time interval, where the external forces actually act

$$\frac{1}{i}\frac{\delta}{\delta f(t)}\, \langle x_b, t_b | \bar{x}', t'\rangle^f = \begin{cases} \langle x_b, t_b | x(t) | \bar{x}', t'\rangle^f & t > t' \\ 0 & t < t' \end{cases}$$

$$\frac{1}{i}\frac{\delta}{\delta f(t)}\, \langle \bar{x}', t' | x_b, t_b\rangle^f = \begin{cases} 0 & t > t' \\ \langle \bar{x}', t' | x(t) | x_b, t_b\rangle^f & t < t' \end{cases}$$

Eliminate $\bar{x}'$ integration as before

$$\frac{1}{i}\frac{\delta}{\delta f(t)}\frac{1}{i}\frac{\delta}{\delta f(t')}\, \langle x_b, t_b | x_a, t_a\rangle^f = \langle x_b, t_b | T\left[x(t), x(t')\right] | x_a, t_a\rangle^f \tag{2.2.8}$$

This can be easily generalised

$$\begin{aligned}
\frac{1}{i}\frac{\delta}{\delta f(t')}\frac{1}{i}\frac{\delta}{\delta f(t'')} \ldots \langle x_b, t_b | x_a, t_a\rangle^f &= \langle x_b, t_b | T\left[x(t') x(t'') \ldots\right] | x_a, t_a\rangle^f &\quad (2.2.9) \\
&= \int \mathcal{D}x\, x(t') x(t'') \ldots \exp\left(i\int_{t_a}^{t_b} \mathrm{d}t\, (L_0(x, \dot{x}) + f(t) x(t))\right) &\quad (2.2.10)
\end{aligned}$$

Interpretation the addition of external force to the Lagrangian of a path integral produces a "generating functional" for a matrix element which contain time-ordered products of arbitrary many position operators. The functional derivative is just a trick to generate the matrix element in the propagator. This is called Schwinger source theory.

Now we can set $f = 0$

$$\langle x_b, t_b | T\left[x(t')x(t'')\dots\right] | x_a, t_a\rangle^{f=0} = \int \mathcal{D}x x(t')x(t'')\dots \exp\left(i\int_{t_a}^{t_b} L_0(x,\dot{x})\right) \tag{2.2.11}$$

or in case of an arbitrary generating functional $F[x]$

$$\langle x_b, t_b | T\left\{F[x]\right\} | x_a, t_a\rangle^{f=0} = \int \mathcal{D}x F[x] \exp\left(i\int_{t_a}^{t_b} L_0(x,\dot{x})\right) \tag{2.2.12}$$

for example

$$\langle x_b, t_b | x_a, t_a\rangle^{f} = \langle x_b, t_b | T \mathrm{e}^{i\int_{t_a}^{t_b}\mathrm{d}t' q(t')f(t')} | x_a, t_a\rangle^{f=0}$$

2.3 Scalar Field Theories and Feynman Rules

We are going to generalise the concept of path integral to field theories. Simplest example is a neutral (real) scalar field $\phi(x)$ coupled to an external classical "current"/source $j(x)$

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi\right)^2 - \frac{1}{2}m^2\phi^2 + \phi j(x) = \mathcal{L}_0 + \phi(x)j(x) \tag{2.3.1}$$

Proceed along the lines of quantum mechanical path integral with external forces

- construct a generating functional
- using the functional-integral-representation derive expressions for the correlation functions $\tilde{=}$ Feynman rules

Sufficient to consider vacuum-to-vacuum amplitudes in the presence of $j(x)$. Consider $t_a = -\infty(1-i\epsilon)$, $t_b = +\infty(1-i\epsilon)$ and $j(x) = 0$ for $t \mapsto \pm\infty$

$$\langle 0|0\rangle^j = \int \mathcal{D}x\, \phi(x) \exp\left(i\int \mathrm{d}^4x\, \mathcal{L}(\phi, \partial_\mu\phi)\right)$$

where $\mathcal{D}\phi(x)$ in the generalization $\mathcal{D}x \mapsto \mathcal{D}(\text{field})$

Compute $\langle 0|0\rangle^j$ (exact for a free field theory). First to solve with classical action

$$\delta\int \mathrm{d}^4x\left[\frac{1}{2}\left(\partial_\mu\phi_{\mathrm{cl}}\right)^2 - \frac{1}{2}m^2\phi_{\mathrm{cl}}^2 + \phi_{\mathrm{cl}}j\right] = 0$$

$$\left(\partial^2 + m^2\right)\phi_{\mathrm{cl}}(x) = j(x)$$

Solution

$$\phi_{\mathrm{cl}}(x) = i\int \mathrm{d}^4y\, D_F(x-y)j(y) \tag{2.3.2}$$

since Feynman-propagator is the Green's function of the KG operator.

$$\left(\partial^2 + m^2\right)D_F(x-y) = -i\delta^{(4)}(x-y) \tag{2.3.3}$$

To define the "fluctuation" field $\phi'(x)$ via $\phi(x) = \phi_{\text{cl}}(x) + \phi'(x)$. Then the Lagrangian is

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\left(\partial_\mu\phi_{\text{cl}} + \partial_\mu\phi'\right)^2 - \frac{m^2}{2}\left(\phi_{\text{cl}} + \phi'\right)^2 + \left(\phi_{\text{cl}} + \phi'\right)\cdot j(x) \\ &= \mathcal{L}_{\text{cl}} + \mathcal{L}' + \left[(\partial_\mu\phi_{\text{cl}})(\partial^\mu\phi') - m^2\phi_{\text{cl}}\phi' + j\phi\right]\end{aligned}$$

after integration by parts and using equation of motion the last part vanishes. Then ϕ' (per construction) is a free field. Thus

$$\langle 0|0\rangle^j = \int \mathcal{D}\phi' \exp\left(i\int \mathrm{d}^4x\,(\mathcal{L}_{\text{cl}} + \mathcal{L}')\right) = \mathrm{e}^{iS_{\text{cl}}}\,\langle 0|0\rangle^{j=0} \tag{2.3.4}$$

On the other hand, iS_{cl} can be rewritten as

$$\begin{aligned}iS_{\text{cl}} &= i\int \mathrm{d}^4x\left[\frac{1}{2} - \frac{m}{2}\phi_{\text{cl}}^2 + \phi_{\text{cl}}j\right] \\ &= i\int \mathrm{d}^4x\left[-\frac{1}{2}\phi_{\text{cl}}\underbrace{\left(\partial^2 + m^2\right)\phi_{\text{cl}}}_{=j \text{ from e.o.m.}} + \phi_{\text{cl}}j\right] \\ &= \frac{i}{2}\int \mathrm{d}^4x\,\phi_{\text{cl}}(x)j(x) \\ &= -\frac{1}{2}\int \mathrm{d}^4x\,\mathrm{d}^4y\;j(x)D_F(x-y)j(y)\end{aligned}$$

Definition <u>generating functional</u> in the free scalar field theory

$$\begin{aligned}W_0[j] &= \frac{Z[j]}{Z[j=0]} = \frac{\langle 0|0\rangle^j}{\langle 0|0\rangle^{j=0}} \\ &= \exp\left(-\frac{1}{2}\int \mathrm{d}^4x\,\mathrm{d}^4y\;j(x)D_F(x-y)j(y)\right)\end{aligned} \tag{2.3.5}$$

Connection to the S-matrix

$$\begin{aligned}S &= U(-\infty,\infty) \\ &= \lim_{t_i\mapsto-\infty(1-i\epsilon)}\;\lim_{t_f\mapsto+\infty(1-i\epsilon)} T\exp\left(-i\int_{t_i}^{t_f}\mathrm{d}t\,\mathcal{H}_{int}(t)\right) \\ &= T\exp\left(-i\int \mathrm{d}^4x\,\mathcal{H}_{int}(x)\right) \\ &= T\exp\left(i\int \mathrm{d}^4x\,\phi(x)j(x)\right) \\ &= T\sum_{n=0}^{\infty}\frac{i^n}{n!}\int \mathrm{d}^4x_1\ldots\mathrm{d}^4x_n\;j(x_1)\ldots j(x_n)\phi(x_1)\ldots\phi(x_n)\end{aligned}$$

$$\langle 0|S|0\rangle = \sum_{n=0}^{\infty}\frac{i^n}{n!}\int \mathrm{d}^4x_1\ldots\mathrm{d}^4x_n\;j(x_1)\ldots j(x_n)G_n^0(x_1,\ldots,x_n) \tag{2.3.6}$$

where $G_n^0(x_1,\ldots,x_n) = \langle 0|T[\phi(x_1)\ldots\phi(x_n)]|0\rangle$ the n-point-Green's function of the free scalar field theory.

We can calculate the Green's function as for the quantum mechanical path integral with external forces via functional derivatives of the generating functional

$$W_0[j] = \frac{\int \mathcal{D}\phi \exp\Big(i \int \mathrm{d}^4 x(\mathcal{L}_0(\phi, \partial_\mu \phi) + \phi j)\Big)}{\int \mathcal{D}\phi \exp\Big(i \int \mathrm{d}^4 x(\mathcal{L}_0(\phi, \partial_\mu \phi))\Big)} \tag{2.3.7}$$

$$\begin{aligned} G_n^0(x_1, \dots, x_n) &= \frac{1}{i}\frac{\delta}{\delta j(x_1)} \dots \frac{1}{i}\frac{\delta}{\delta j(x_n)} W_0[j]|_{j=0} \\ &= \frac{\int \mathcal{D}\phi \exp\Big(i \int \mathrm{d}^4 x(\mathcal{L}_0(\phi, \partial_\mu \phi))\Big)\phi(x_1)\dots\phi(x_n)}{\int \mathcal{D}\phi \exp\Big(i \int \mathrm{d}^4 x(\mathcal{L}_0(\phi, \partial_\mu \phi))\Big)} \\ &= \langle 0|T\phi(x_1)\dots\phi(x_n)|0\rangle \end{aligned} \tag{2.3.8}$$

The central result here is that these three things are closely related: S-matrix $\leftrightarrow$ Green's function $\leftrightarrow$ Path integral

Special case, two-point function

$$\begin{aligned} G_2^0(x_1, x_2) &= \langle 0|T\phi(x_1)\phi(x_2)|0\rangle \\ &= \frac{1}{i}\frac{\delta}{\delta j(x_1)}\frac{1}{i}\frac{\delta}{\delta j(x_2)} \exp\left[-\frac{1}{2}\int \mathrm{d}^4x\, \mathrm{d}^4y\, j(x)D_F(x-y)j(y)\right]\Bigg|_{j=0} \end{aligned}$$

The exponential is the generating functional.

$$\begin{aligned} &= -\frac{1}{\delta j(x_1)}\left[-\frac{1}{2}\int \mathrm{d}^4y\, D_F(x_2-y)j(y) - \frac{1}{2}\int \mathrm{d}^4x\, j(x)D_F(x-x_2)\right] W_0[j]\Bigg|_{j=0} \\ &= D_F(x_1 - x_2) \end{aligned}$$

Four-point function. Use abbreviations $\phi_i = \phi(x_i)$, $j_x = j(x)$, $D_{x_i} = D_F(x - x_i)$ and integration over the repeated index is implied.

$$\begin{aligned} G_4^0(x_1, x_2, x_3, x_4) &= \langle 0|T\phi_1\phi_2\phi_3\phi_4|0\rangle \\ &= \left(\frac{1}{i}\right)^4 \frac{\delta}{\delta j_1}\frac{\delta}{\delta j_2}\frac{\delta}{\delta j_3}\frac{\delta}{\delta j_4}\ \mathrm{e}^{-\frac{1}{2}j_x D_{xy} j_y}\Big|_{j=0} \end{aligned}$$

Since $D(x-y) = D(y-x)$, one can combine two integrals after substituition into one.

$$\begin{aligned} &= \frac{\delta}{\delta j_1}\frac{\delta}{\delta j_2}\frac{\delta}{\delta j_3}\,[-j_{\tilde{x}}D_{\tilde{x}4}]\ \mathrm{e}^{-\frac{1}{2}j_x D_{xy} j_y}\Big|_{j=0} \\ &= \frac{\delta}{\delta j_1}\frac{\delta}{\delta j_2}\Big[-D_{34} + j_{\tilde{x}}D_{\tilde{x}4}j_{\tilde{y}}D_{\tilde{y}3}\Big]\,\mathrm{e}^{-\frac{1}{2}j_x D_{xy} j_y}|_{j=0} \\ &= \frac{\delta}{\delta j_1}\Big[D_{34}j_{\tilde{x}}D_{\tilde{x}2} + D_{24}j_{\tilde{y}}D_{\tilde{y}3} + j_{\tilde{x}}D_{\tilde{x}4}D_{23} + \dots\Big]\,\mathrm{e}^{-\frac{1}{2}j_x D_{xy} j_y}|_{j=0} \end{aligned}$$

Dots are the terms contains j.

$$= D_{34}D_{12} + D_{24}D_{13} + D_{14}D_{23}$$

$$\begin{aligned} G_4^0(x_1, x_2, x_3, x_4) =& D_F(x_3 - x_4)D_F(x_1 - x_2) \\ &+ D_F(x_2 - x_4)D_F(x_1 - x_3) + D_F(x_1 - x_4)D_F(x_2 - x_3) \end{aligned} \tag{2.3.9}$$

So we recovered Wick's theorem in path integral representation.

$$\frac{1}{N}\int \mathcal{D}\phi e^{i\int d^4x\mathcal{L}_0[\phi]}\phi(x_1)\phi(x_2) = \wick{\c\phi(x_1)\c\phi(x_2)}$$

with $N = \int \mathcal{D}\phi e^{i\int d^4x\mathcal{L}_0[\phi]}$

$$\frac{1}{N}\int \mathcal{D}\phi e^{i\int d^4x\mathcal{L}_0[\phi]}\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) = \wick{\c1\phi(x_1)\c1\phi(x_2)\c2\phi(x_3)\c2\phi(x_4)} + \wick{\c1\phi(x_1)\c2\phi(x_2)\c1\phi(x_3)\c2\phi(x_4)} + \wick{\c1\phi(x_1)\c2\phi(x_2)\c2\phi(x_3)\c1\phi(x_4)}$$

Interacting field Consider now an interacting field theory, e.g. ϕ^4

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 + \phi j \\ &= \mathcal{L}_0 - \frac{\lambda}{4!}\phi^4 + \phi j\end{aligned}$$

The generating functional or the normalized vacuum-to-vacuum transition amplitude is given

$$W[j] = \frac{\langle 0|0\rangle^j}{\langle 0|0\rangle^0} = \sum_{n=0}^{\infty}\frac{i^n}{n!}\int d^4x_1\dots d^4x_n\, j(x_1)\dots j(x_n)G_n(x_1,\dots,x_n) \tag{2.3.10}$$

$$\langle 0|0\rangle^j = \int \mathcal{D}\phi \exp\left\{i\int d^4x\left[\frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 + \phi j\right]\right\} \tag{2.3.11}$$

thus

$$G_n(x_1,\dots,x_n) = \frac{\int \mathcal{D}\phi\phi(x_1)\dots\phi(x_n)\exp\left\{i\int d^4x\,\frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4\right\}}{\int \mathcal{D}\phi \exp\left\{i\int d^4x\,\frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4\right\}} \tag{2.3.12}$$

Our aim is to get perturbative expansion of G_n and thus for the S-matrix. For this purpose, expanse $\lambda\phi^4$ term!

$$\sum_{n=0}^{\infty}\int \mathcal{D}\phi\phi(x_1)\dots\phi(x_n)\frac{1}{n!}\left(\int d^4y\,\frac{-i\lambda}{4!}\phi^4(y)\right)^n e^{i\int d^4x\mathcal{L}_0}$$

This expansion is equivalent to the Dyson-Wick expansion of the S-matrix in powers of $\mathcal{H}_{\text{int}}$.

Two-point Green's function

Note that denominator cancels the vacuum diagrams, so we only have perturbation theory for connected graphs.

- $\mathcal{O}(\lambda^0)$

$$G_2^0 = \wick{\c\phi(x_1)\c\phi(x_2)} = D_F(x_1 - x_2) \tag{2.3.13}$$

- $\mathcal{O}(\lambda^1)$

$$\begin{aligned}G(x_1,x_2) &= \frac{\int \mathcal{D}\phi\phi(x_1)\phi(x_2)e^{i\int d^4x(\mathcal{L}_0(x) - \frac{\lambda}{4!}\phi^4(x))}}{\int \mathcal{D}\phi e^{i\int d^4x(\mathcal{L}_0(x) - \frac{\lambda}{4!}\phi^4(x))})} \\ &= \frac{\int \mathcal{D}\phi\phi(x_1)\phi(x_2)\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{i\lambda}{4!}\int d^4y\,\phi^4(y)\right)^n e^{i\int d^4x\mathcal{L}_0(x)}}{\int \mathcal{D}\phi\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{i\lambda}{4!}\int d^4y\,\phi^4(y)\right)^n e^{i\int d^4x\mathcal{L}_0(x)}}\end{aligned}$$

$$
\begin{aligned}
G_2^1(x_1,x_2) &= \frac{1}{N}\int \mathcal{D}\phi\phi(x_1)\phi(x_2)\left(-\frac{i\lambda}{4!}\int \mathrm{d}^4y\,\phi^4(y)\right)\mathrm{e}^{i\int \mathrm{d}^4x\mathcal{L}_0(x)} \\
&= \frac{1}{N}\left(\frac{-i\lambda}{4!}\right)\int \mathcal{D}\phi\phi(x_1)\phi(x_2)\phi(y)\phi(y)\phi(y)\phi(y)\mathrm{e}^{i\int \mathrm{d}^4x\mathcal{L}_0(x)} \\
&= \frac{1}{N}\left(\frac{-i\lambda}{4!}\right)\int \mathrm{d}^4y\left[\phi(x_1)\phi(x_2)\left(\phi(y)\phi(y)\phi(y)\phi(y)+\phi(y)\phi(y)\phi(y)\phi(y)+\phi(y)\phi(y)\phi(y)\phi(y)\right)\right. \\
&\quad \left. +\,\phi(x_1)\phi(x_2)\phi(y)\phi(y)\phi(y)\phi(y) + 11 \text{ more terms}\right] \\
&= \frac{1}{N}\left\{-\frac{i\lambda}{8}\int \mathrm{d}^4y\,\phi(x_1)\phi(x_2)\left(\phi(y)\phi(y)\right)^2 - \frac{i\lambda}{2}\phi(x_1)\phi(y)\phi(x_2)\phi(y)\phi(y)\phi(y)\right\}
\end{aligned}
$$

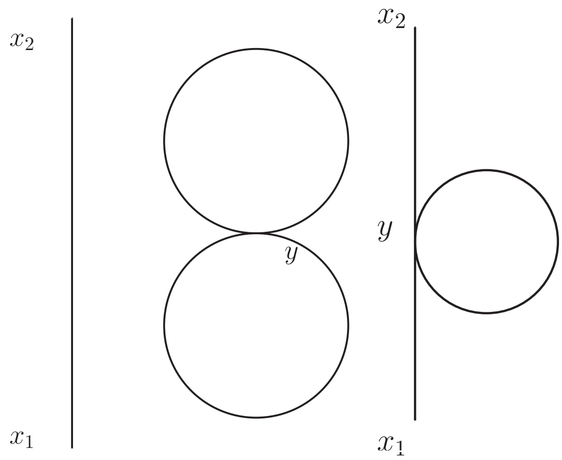


Figure 2.1: Feynman diagrams

General case: common (position space) Feynman rules for interacting Green's functions $G_n(x_1,\ldots,x_n)$

2.4 Photon Propagator in Path Integrals*

How can we derive the Feynman rule for photon propagator?

$$\frac{-ig^{\mu\nu}}{k^2+i\epsilon} \tag{2.4.1}$$

What is the problem? The functional integral $\int \mathcal{D}A_\mu \mathrm{e}^{iS[A]}$ incorporated the action

$$
\begin{aligned}
S &= \int \mathrm{d}^4x\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) \\
&= \frac{1}{2}\int \mathrm{d}^4x\,A_\mu(x)\left(\partial^2 g^{\mu\nu}-\partial^\mu\partial^\nu\right)A_\nu(x) \\
&= \frac{1}{2}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\tilde{A}_\mu(k)(-k^2g^{\mu\nu}+k^\mu k^\nu)\tilde{A}_\nu(-k)
\end{aligned} \tag{2.4.2}
$$

This expression has a great deal of (interconnected) problems

*see also Ryder, Chap 7.1-2; P & S, Chap 9.4

1. Assume the photon propagator $D_{\mu\nu}(x-y)$ to be solution of
$$\left(\partial^2 g^{\mu\nu} - \partial^\mu\partial^\nu\right) D_{\nu\lambda}(x-y) = i g^\mu_\lambda \delta^{(4)}(x-y)$$
multiply with partial derivative
$$\left(\partial^\nu\partial^2 - \partial^2\partial^\nu\right) D_{\nu\lambda} = 0 \cdot \partial^\nu D_{\nu\lambda} \neq i\partial^\nu \delta^{(4)}(x-y)$$
$D_{\nu\lambda}$ has no inverse (formally singular). The same holds in momentum space.
2. We need this inverse for the derivation of the generating function with external currents.
3. $\frac{1}{2}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\tilde{A}_\mu(k)(-k^2 g^{\mu\nu} + k^\mu k^\nu)\tilde{A}_\nu(-k)$ vanished for all $\tilde{A}_\nu(k) = k_\mu \alpha(k)$ all these field configurations have the same weight 1 in $\int \mathcal{D}A_\mu \mathrm{e}^{iS[A]}$. It will terribly diverge.
4. These configurations correspond to gauge transformation $\mathcal{L}$ (also S) invariant under $A_\mu(x) \mapsto A_\mu(x) + \frac{1}{\mathrm{e}}\partial_\mu\alpha(x)$. Thus redundant (path-integral) integration over gauge equivalent configurations.

Schematically to split each A_μ into some fixed $\bar{A}_\mu$ and a gauge transformation α
$$A_\mu(x) \mapsto \bar{A}_\mu(x), \alpha(x)$$
then the path integral is given by
$$Z = \int \mathcal{D}A_\mu \mathrm{e}^{iS} \sim \int \mathcal{D}\bar{A}_\mu e^{iS} \int \mathcal{D}\alpha$$
since S is gauge invariant, i.e. independent of α.

The divergence stems from $\int \mathcal{D}\alpha$ which cancels in the ratio $W[j] = \frac{Z[j]}{Z[0]}$

The aim is to factorize of the gauge part in the path integral via a "smart integration" of the gauge-fixing as a constraint. This is called Faddeev-Popov method.

2.4.1 Factorization of Constraints*

Externally simple example
$$I = \int \mathrm{d}x\,\mathrm{d}y\,\mathrm{e}^{-(x^2+y^2)}$$
it is rotation invariant, use polar coordinates
$$= \int \mathrm{d}\theta\,\mathrm{d}r\, re^{-r^2}$$
$\int \mathrm{d}\theta = 2\pi$ corresponds to $\int \mathcal{D}\alpha$ in the path integral.

A more general expansion for this separation
$$I = \int \mathrm{d}\theta' \int \mathrm{d}r \int \mathrm{d}\theta\, r\mathrm{e}^{-r^2}\delta(\theta)$$
The delta function reduced the integration path to one along the x-axis ($\theta = 0$).

*Ryder, Chap 7.2

A more general path

$$f(\theta) = y\cos\theta - x\sin\theta = 0 \tag{2.4.3}$$

i.e. $\theta \neq 0$.

How can we include this constraint in path integral?

$$\delta(f(\theta)) = \sum_i \left|\frac{\partial f(\theta_i)}{\partial\theta}\right|^{-1} \delta(\theta - \theta_i) \tag{2.4.4}$$

with θ_i the roots of $f(\theta)$.

$$\begin{aligned} \theta_1 &= \arctan\left(\frac{y}{x}\right) \quad \theta_2 = \pi + \arctan\left(\frac{y}{x}\right) \\ \left|\frac{\partial f}{\partial\theta}\right| &= y\sin\theta + x\cos\theta = r = \left|\frac{\partial f}{\partial\theta}\right|_{\theta_1,\theta_2} \end{aligned}$$

thus

$$\begin{aligned} \delta(f(\theta)) &= \frac{1}{r}\left(\delta(\theta-\theta_1) + \delta(\theta-\theta_2)\right) \\ \int \delta(f(\theta))\mathrm{d}\theta &= \frac{2}{r} = \frac{2}{\sqrt{x^2+y^2}} \end{aligned}$$

rewrite this as $\Delta(r)\int\delta(f(\theta))\,\mathrm{d}\theta = 1$, i.e.

$$\Delta(r) = \frac{r}{2} = \frac{\sqrt{x^2+y^2}}{2} \tag{2.4.5}$$

Note that $f(\theta)$ can simply be obtained by a rotation from y-axis

$$\begin{aligned} y' &= y\cos\theta - x\sin\theta \\ x' &= x\cos\theta + y\sin\theta \\ x^2 + y^2 &= x'^2 + y'^2 \end{aligned}$$

$$\begin{aligned} \Delta(r)\int \mathrm{d}\theta\,\delta(f(\theta)) &= 1 \\ \Delta\left(\sqrt{x'^2+y'^2}\right)\int \mathrm{d}\theta\,\delta(y') &= 1 \end{aligned}$$

remember $y' = f(\theta) = y\cos\theta - x\sin\theta$.

Insert this unity into $I = \int \mathrm{d}x\,\mathrm{d}y\,\mathrm{e}^{-(x^2+y^2)}$

$$I = \int \mathrm{d}\theta \int \mathrm{d}x'\,\mathrm{d}y'\,\mathrm{e}^{-(x'^2+y'^2)}\Delta\left(\sqrt{x'^2+y'^2}\right)\delta(y')$$

It exhibits separation of variables made possible by the rotation invariance of the integral. The integral $\int \mathrm{d}x'\,\mathrm{d}y' \ldots$ is independent of θ, so $\int \mathrm{d}\theta$ is simply an overall multiplication factor in integral.

Finally $\Delta(r)$ can also be rewritten as

$$\begin{aligned} \Delta(r)^{-1} &= \int \mathrm{d}\theta\,\delta(f(\theta)) \\ &= \int \delta(f(\theta))\det\left|\frac{\mathrm{d}\theta}{\mathrm{d}f}\right|\mathrm{d}f \\ &= \det\left|\frac{\mathrm{d}\theta}{\mathrm{d}f}\right|_{f=0} \end{aligned} \tag{2.4.6}$$

then we have the functional determinant

$$\Delta(r) = \det\left|\frac{\mathrm{d}f}{\mathrm{d}\theta}\right|_{f=0} \tag{2.4.7}$$

2.4.2 Gauge Fixing in a Path Integral

Consider a gauge transformation

$$A_\mu(x) \mapsto A_\mu^\alpha(x) = A_\mu(x) + \frac{1}{e}\partial_\mu\alpha(x) \tag{2.4.8}$$

and a gauge-fixing condition

$$F[A_\mu] = 0 \tag{2.4.9}$$

In analogy to the condition 2.4.6

$$\Delta_F^{-1}[A_\mu] = \int \mathcal{D}\alpha\delta\left(F[A_\mu^\alpha]\right) \tag{2.4.10}$$

where δ is a δ-functional.
$\Delta_F^{-1}[A_\mu]$ is gauge invariant

$$\Delta_F^{-1}\left[A_\mu^{\alpha'}\right] = \int \mathcal{D}\alpha\delta\left(F[A_\mu^{\alpha+\alpha'}]\right)$$

group invariant measure $\alpha'' = \alpha + \alpha'$

$$= \int \mathcal{D}\alpha''\delta\left(F[A_\mu^{\alpha''}]\right) = \Delta_F^{-1}[A_\mu]$$

Insert $1 = \Delta_F[A_\mu]\int \mathcal{D}\alpha\delta(F[A_\mu^\alpha])$ into the path integral Z

$$\begin{aligned} Z &= \int \mathcal{D}A_\mu \mathrm{e}^{iS[A_\mu]} \\ &= \int \mathcal{D}A_\mu\Delta_F[A_\mu]\int \mathcal{D}\alpha\delta\left(F[A_\mu^\alpha]\right)\mathrm{e}^{iS[A_\mu]} \end{aligned}$$

using the gauge transformation $A_\mu \mapsto A_\mu^\alpha$

$$= \int \mathcal{D}A_\mu^\alpha\Delta[A_\mu^\alpha]\int \mathcal{D}\alpha\delta\left(F[A_\mu^\alpha]\right)\mathrm{e}^{iS[A_\mu^\alpha]}$$

rename $A_\mu^\alpha = A_\mu$

$$= \int \mathcal{D}\alpha\int \mathcal{D}A_\mu\Delta_F[A_\mu]\delta\left(F[A_\mu]\right)\mathrm{e}^{iS[A_\mu]} \tag{2.4.11}$$

S and $\Delta_F(\dots)$ are gauge invariant and $\mathcal{D}A_\mu^\alpha = \mathcal{D}A_\mu$, since

$$\begin{aligned} Z &= \int \mathcal{D}A_\mu \mathrm{e}^{iS[A_\mu]} \\ &= \int \mathcal{D}A_\mu^{\tilde\alpha}\mathrm{e}^{iS[A_\mu^{\tilde\alpha}]} \\ &= \int \mathcal{D}A_\mu^{\tilde\alpha}\mathrm{e}^{iS[A_\mu]} \end{aligned}$$

or $A_\mu \mapsto A_\mu^{\tilde{\alpha}}$ is just a shift of integration. Integrand of equation 2.4.11 independent of gauge α. Thus $\int \mathcal{D}\alpha$ can be moved in front and is therefore already separated!

Now use

$$\Delta_F[A_\mu] = \det\left|\frac{\delta F}{\delta\alpha}\right|_{F=0} \tag{2.4.12}$$

We will apply gauge-fixing conditions of the form (generalization of Lorenz gauge)

$$F[A_\mu] = \partial^\mu A_\mu + C(x) = 0 \tag{2.4.13}$$

with $c(x)$ any scalar function. Thus

$$F[A_\mu^\alpha] = F[A_\mu] + \frac{1}{e}\partial^2\alpha(x) \tag{2.4.14}$$

$$\Delta_F[A_\mu] = \det\left|\frac{1}{e}\partial^2\right| \tag{2.4.15}$$

independent of A_μ. $\Delta_F[A_\mu]$ can be moved in front of the path integral. (Not valid in the non-abelian case.)

Since Z and in the end, the physics does not depend on value of $C(x)$, we are free to have a linear combination of different $C(x)$. Now multiply equation (2.4.11) with a weight$\int \mathcal{D}C \exp\left(\frac{-i}{2\xi}C^2\,\mathrm{d}x\right)$. After integrating $\int \mathcal{D}C$) and using $\delta\left(F[A_\mu]\right) = \delta\left(\partial^\mu A_\mu - C(x)\right)$

$$Z = N\int \mathcal{D}A_\mu \exp\left\{i\int \mathrm{d}^4x\left(\mathcal{L} - \frac{1}{2\xi}\left(\partial_\mu A^\mu\right)^2\right)\right\} \tag{2.4.16}$$

$$\mathcal{L}_{\text{eff}} = \mathcal{L} + \mathcal{L}_{\text{GF}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 \tag{2.4.17}$$

with the gauge-fixing term in Lagrangian.

Now consider an n-point Green's function a la

$$\langle 0|T\left(O[A_\mu]\right)|0\rangle$$

Coming back to our starting problem, to find photon propagator

$$\left(-k^2 g_{\mu\nu} + \left(1 - \frac{1}{\xi}\right)k_\mu k_\nu\right)\tilde{D}_F^{\nu\lambda}(k) = i\delta_\mu^\lambda$$

possess the solution

$$\tilde{D}_F^{\mu\nu}(k) = \frac{-i}{k^2 + i\epsilon}\left(g^\mu\nu - (1-\xi)\frac{k^\mu k^\nu}{k^2}\right) \tag{2.4.18}$$

Most commonly used gauges Feynman gauge $\xi = 1$ and Landau gauge $\xi = 0$

2.5 Path Integral Quantization of Fermion Fields*

The fermionic fields anti-commute, therefore the integration over (complexed-valued) fermion fields is non-trivial.

*see also Ryder Chap 6,7, P& S, Chap 9.5

2.5.1 Grassmann Algebra[*]

Consider an algebra $\mathcal{G}_n$ generated by n anticommuting generators $\theta_1, \ldots, \theta_n$.

$$\theta_i\theta_j + \theta_j\theta_i = \left\{\theta_i, \theta_j\right\} = 0 \tag{2.5.1}$$

They are variables, not field operators (yet)!

Implicitly $\theta_i^2 = 0$ for all i Thus a basis is given monomials (polynomial of first order of one term) $1; \theta_1, \ldots, \theta_n; \theta_1\theta_2, \ldots \theta_{n-1}\theta_n; \ldots; \theta_1 \ldots \theta_n$

Each element $F(\theta) \in \mathcal{G}_n$ can be expressed as a linear combination of these monomials.

$$F(\theta) = F^{(0)} + \sum_i F_i^{(1)}\theta_i + \cdots + \sum_{i\ldots k} F_{i,\ldots,k}^{(n)}\theta_i \ldots \theta_j \ldots \theta_k \tag{2.5.2}$$

All coefficient are totally antisymmetric under exchange of the indices. We call elements with even (odd) monomials even (odd) algebra. Every $F(\theta)$ can be uniquely decomposed into a sum of even and odd monomials. Even elements commute with each other and odd elements anti-commute with each other.

In $\mathcal{G}_n$ we can define sums $F(\theta)+G(\theta)$ products $F(\theta)\cdot G(\theta)$ and functions $\mathrm{e}^{F(\theta)} = 1+F(\theta)+\frac{1}{2!}(F(\theta))^2+\ldots$. All these therms can easily be expressed as linear combinations of moments.

Differentiation

$$\frac{\partial}{\partial\theta_j}\theta_i = \delta_{ij} \tag{2.5.3}$$

$$\frac{\partial}{\partial\theta_i}c = 0, \quad c \in \mathbb{C} \tag{2.5.4}$$

from anti-commutation relation and sign convention and derivative always acts on variable directly following

$$\frac{\partial}{\partial\theta_i}(\theta_1 \ldots \theta_n) = \delta_{i1}\theta_2 \ldots \theta_n - \delta_{i2}\theta_1\theta_3 \ldots \theta_n + \cdots + (-1)^{n-1}\theta_1 \ldots \theta_{n-1} \tag{2.5.5}$$

it has the consequence

$$\left\{\frac{\partial}{\partial\theta_i}, \frac{\partial}{\partial\theta_j}\right\} = 0 \tag{2.5.6}$$

$$\left\{\frac{\partial}{\partial\theta_i}, \theta_j\right\} = \delta ij \tag{2.5.7}$$

We can then read θ_i and $\frac{\partial}{\partial\theta_j}$ as a representation of fermion creation and annihilation operators.

Integration The goal is to find generalization of functional integrals, so we only need the analogue of $\int_{-\infty}^{\infty} \mathrm{d}x$, no need of finite integrals. First consider one single Grassmann variable θ

$$\int \mathrm{d}\theta\, f(\theta) = \int \mathrm{d}\theta\,(A + B\theta)$$

it should be a linear function of A and B because of linearity of integration. To enable variable shift $\theta \mapsto \theta + \eta$

$$\int \mathrm{d}\theta\,(A + B\theta) = \int \mathrm{d}\theta\,((A + b\eta) + B\theta)$$

$$\Rightarrow \int 1 \cdot \mathrm{d}\theta = 0 \tag{2.5.8}$$

[*]see also F.A.Berezin, the method of second quantization, 1966

in addition we define

$$\int \mathrm{d}\theta\, \theta = 1 \tag{2.5.9}$$

in general

$$\int \mathrm{d}\theta_i = 0 \tag{2.5.10}$$

$$\int \mathrm{d}\theta_i\, \theta_j = \delta_{ij} \tag{2.5.11}$$

$$\left\{\mathrm{d}\theta_i\,, \mathrm{d}\theta_j\right\} = 0 = \left\{\theta_i, \mathrm{d}\theta_j\right\} \tag{2.5.12}$$

multiple integrals

$$\int \mathrm{d}\theta_n \ldots \mathrm{d}\theta_1\, F(\theta) = \int \mathrm{d}\theta_n \ldots \mathrm{d}\theta_1 \left(\sum_{i,\ldots,k}^{n} F^{(n)}_{i\ldots k} \theta_i \ldots \theta_k \right) = n! F^{(n)}_{12\ldots n} \tag{2.5.13}$$

All terms with $k < n$ vanish due to $\int \mathrm{d}\theta_i = 0^n$ Note that differentiation and integration with respect to Grassmann variables yield same result.

Gaussian integrals for <u>even</u> numbers of generators and A skew-symmetric matrix.

$$\int \mathrm{d}\theta_1 \ldots \mathrm{d}\theta_n\, \mathrm{e}^{-\frac{1}{2}\theta_i A_{ij} \theta_j} = \sqrt{\det\{A\}} \tag{2.5.14}$$

Here consider only example for $n = 2$, i.e. $A_{11} = A_{22} = 0$ and $A_{12} = -A_{21}$

$$\begin{aligned} \mathrm{e}^{-\frac{1}{2}\theta_i A_{ij} \theta_j} &= \mathrm{e}^{-\frac{1}{2}(\theta_1\theta_2 A_{12} + \theta_2\theta_1 A_{21})} \\ &= \mathrm{e}^{-A_{12}\theta_1\theta_2} \\ &= 1 - A_{12}\theta_1\theta_2 \end{aligned}$$

hence

$$\begin{aligned} \int \mathrm{d}\theta_1\, \mathrm{d}\theta_2\, \mathrm{e}^{-\frac{1}{2}\theta_i A_{ij} \theta_j} &= \int \mathrm{d}\theta_1\, \mathrm{d}\theta_2\, (1 - A_{12}\theta_1\theta_2) = A_{12} \\ &= \sqrt{\det\{A\}} \end{aligned}$$

For each skew-symmetric matrix of even rank, the determinant is a perfect square while for each skew-symmetric matrix of odd rank, $\det\{A\} = 0$. $\sqrt{\det\{A\}}$ = Pfaffian form

$$\begin{aligned} n &= 2 & P &= A_{12} = \frac{1}{2}\epsilon A_{ij} \\ n &= 4 & P &= A_{12}A_{34} - A_{13}A_{24} + A_{14}A_{23} = \frac{1}{8}\epsilon_{ijkl} A_{ij} A_{kl} \end{aligned}$$

2.5.2 Fermion Fields

Definition of Grassmann fields as functions of space-time, whose values are anti-commuting numbers, .e.g.

$$\psi(x) = \sum_i \psi_i \phi_i(x) \tag{2.5.15}$$

with $\psi_i \in \mathcal{G}, \phi_i \in \mathbb{C}$. For Dirac fields, ϕ_i are 4-component spinors.

As in the scalar case, to add external sources $\eta, \bar{\eta}$ to the Dirac Lagrangian

$$\mathcal{L} = \bar{\psi}\,(i\partial\!\!\!/ - m)\,\psi + \bar{\psi}\eta + \bar{\eta}\psi \tag{2.5.16}$$

Obviously the sources must be Grassmann valued $\{\eta,\eta\} = \{\eta,\bar{\eta}\} = \{\bar{\eta},\bar{\eta}\} = \{\psi,\eta\} = \{\bar{\psi},\eta\} = \{\psi,\bar{\eta}\} = \{\bar{\psi},\bar{\eta}\} = 0$.

Vacuum to vacuum transition amplitude in presence of external sources

$$\langle 0|0\rangle^{\eta\bar{\eta}} = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left\{ i \int \mathrm{d}^4x\,[\bar{\psi}(i\partial\!\!\!/ - m)\psi + \bar{\eta}\eta + \bar{\psi}\eta] \right\} \tag{2.5.17}$$

Determine classical solution from the least-action principle

$$\psi_{\mathrm{cl}}(x) = i \int \mathrm{d}^4y\, S_F(x-y)\eta(y) \tag{2.5.18}$$

$$\bar{\psi}_{\mathrm{cl}}(x) = i \int \mathrm{d}^4y\, \bar{\eta}(y) S_F(x-y) \tag{2.5.19}$$

with already known Dirac propagator

$$S_F(x-y) = \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{i(k\!\!\!/ + m)}{k^2 - m^2 + i\epsilon} \mathrm{e}^{-ik(x-y)} \tag{2.5.20}$$

as the Green's function of the Dirac operator $(i\partial\!\!\!/ - m)$.

By expansion of $\psi(x)$ around the classical solution $\psi_{\mathrm{cl}}(x)$ we find, like in the scalar case,

$$\langle 0|0\rangle^{\eta,\bar{\eta}} = \mathrm{e}^{iS_{\mathrm{cl}}}\,\langle 0|0\rangle^{\eta=\bar{\eta}=0} \tag{2.5.21}$$

and classical action can be rewritten as

$$iS_{\mathrm{cl}} = i \int \mathrm{d}^4x\,[\bar{\psi}_{\mathrm{cl}}\,(i\partial\!\!\!/ - m)\,\psi_{\mathrm{cl}} + \bar{\psi}_{\mathrm{cl}}\eta + \bar{\eta}\psi_{\mathrm{cl}}] \tag{2.5.22}$$

$$= -\int \mathrm{d}^4x\,\mathrm{d}^4y\, \bar{\eta}(x) S_F(x-y)\eta(y) \tag{2.5.23}$$

hence the generating functional for the free Dirac field is given as

$$W_0[\eta,\bar{\eta}] = \frac{\langle 0|0\rangle^{\eta,\bar{\eta}}}{\langle 0|0\rangle^{\eta=\bar{\eta}=0}} = \exp\left\{ -i \int \mathrm{d}^4x\,\mathrm{d}^4y\, \bar{\eta}(x) S_F(x-y)\eta(y) \right\} \tag{2.5.24}$$

Derive n-point functions from generating functional

$$\langle 0|T\psi(x)\bar{\psi}(y)|0\rangle = \frac{1}{i}\frac{\delta}{\delta\eta(y)}\frac{1}{i}\frac{\delta}{\delta\bar{\eta}(x)}\; W_0[\eta,\bar{\eta}]\Big|_{\eta=\bar{\eta}=0} \tag{2.5.25}$$

2.5.3 QED

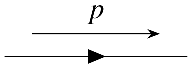
Lagrangian given as

$$\begin{aligned}
\mathcal{L}_{\mathrm{QED}} &= \bar{\psi}\,(iD\!\!\!\!/ - m)\,\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2\xi}\left(\partial_\mu A^\mu\right)^2 \\
&= \bar{\psi}\,(i\partial\!\!\!/ - m)\,\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2\xi}\left(\partial_\mu A^\mu\right)^2 - e\bar{\psi}\gamma^\mu\psi A_\mu \\
&= \mathcal{L}_0 + \mathcal{L}_{\mathrm{GF}} - e\bar{\psi}\gamma^\mu\psi A_\mu
\end{aligned} \tag{2.5.26}$$

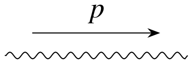
Expand the exponential of the interaction term.

$$\exp\left\{i\int \mathrm{d}^4x\,\mathcal{L}_{\mathrm{QED}}\right\} = \exp\left\{i\int \mathrm{d}^4x\,(\mathcal{L}_0 + \mathcal{L}_{\mathrm{GF}})\right\}\cdot\left[1 - ie\int \mathrm{d}^4x\,\bar{\psi}\gamma^\mu\psi A_\mu + \dots\right]$$

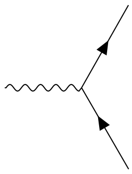
Feynman rules (position space)



$$= \int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{i(\not{p}+m)}{p^2-m^2+i\epsilon}\mathrm{e}^{-ip(x-y)} \tag{2.5.27}$$



$$= \int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{-i}{q^2+i\epsilon}\left(g^{\mu\nu} - (1-\xi)\frac{q^\mu q^\nu}{q^2}\right)\mathrm{e}^{-ip(x-y)} \tag{2.5.28}$$

μ 

$$= -ie\gamma^\mu\int \mathrm{d}^4x \tag{2.5.29}$$

Generating functional for QED

$$Z[j_\mu,\eta,\bar{\eta}] = \int \mathcal{D}A_\mu\mathcal{D}\bar{\psi}\mathcal{D}\psi\,\exp\left\{i\int \mathrm{d}^4x\left(\mathcal{L}_0 + \mathcal{L}_{\mathrm{GF}} + \mathcal{L}_{\mathrm{int}} + j_\mu A^\mu + \bar{\psi}\eta + \bar{\eta}\psi\right)\right\} \tag{2.5.30}$$

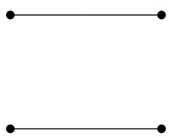
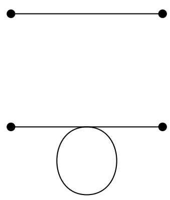
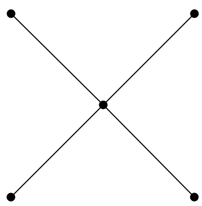
$$Z[j_\mu,\eta,\bar{\eta}] = \frac{Z[j_\mu,\eta,\bar{\eta}]}{Z[j_\mu = 0,\eta=\bar{\eta}=0]} \tag{2.5.31}$$

2.6 Generating Functional for Fully Connected Green's Functions

Return to scalar theory for simplicity. In section 2.2 we calculated 4-point function to λ^0

$$\langle 0|T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|0\rangle = D_F(x_3-x_4)D_F(x_1-x_2) + D_F(x_2-x_4)D_F(x_1-x_3) \tag{2.6.1}$$
$$+ D_F(x_1-x_4)D_F(x_2-x_3) \tag{2.6.2}$$

To next order in perturbation theory $O(\lambda^1)$

$\langle 0|T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|0\rangle = 3$  $-\,3i\lambda$  $-\,i\lambda$  $+\,O\left(\lambda^2\right)$ (2.6.3)

only the last term is fully connected and contributes to T-matrix!

There exists a generating functional $E[j]$ that only generates the fully connected diagrams

$$iE[j] = \ln(Z[j]) \tag{2.6.4}$$

Note that often in the literature, $E[j]$ is often called $W[j]$!

To show that $E[j]$ in $\lambda\phi^4$ generates no disconnected contributions in the 2- and 4-point functions!

2-point function

$$\frac{\delta^2 iE[j]}{i\delta j(x_1) i\delta j(x_2)} = \frac{1}{Z}\frac{\delta^2 Z}{i\delta j(x_1) i\delta j(x_2)} - \frac{1}{Z^2}\frac{\delta Z}{i\delta j(x_1)}\frac{\delta Z}{i\delta j(x_2)}$$

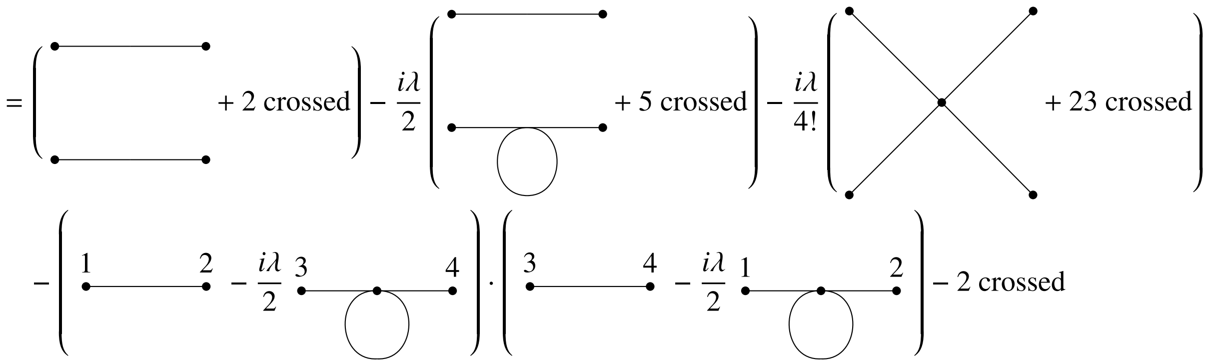
since Z is quadratic in j, for $j = 0$ term with one derivative drops put. Hence

$$\begin{aligned}\left.\frac{\delta^2 iE[j]}{i\delta j(x_1) i\delta j(x_2)}\right|_{j=0} &= \left.\frac{1}{Z}\frac{\delta^2 Z}{i\delta j(x_1) i\delta j(x_2)}\right|_{j=0} \\ &= D_F(x-y)\end{aligned}$$

there is (to arbitrary order in λ) the propagator. It doesn't have disconnected poles.

4-point function

$$\begin{aligned}&\left.\frac{\delta^4 iE[j]}{i\delta j(x_1) i\delta(x_2) i\delta j(x_3) i\delta(x_4)}\right|_{j=0} \\ &= \frac{1}{Z}\left.\frac{\delta^4 Z[j]}{\delta j(x_1)\delta(x_2)\delta j(x_3)\delta(x_4)}\right|_{j=0} - \frac{1}{Z^2}\left.\frac{\delta^2 Z}{\delta j(x_1) j(x_2)}\frac{\delta^2 Z}{\delta j(x_3)\delta j(x_4)}\right|_{j=0} - \frac{1}{Z^2}\left.\frac{\delta^2 Z}{\delta j(x_1) j(x_3)}\frac{\delta^2 Z}{\delta j(x_2)\delta j(x_4)}\right|_{j=0} \\ &\quad - \frac{1}{Z^2}\left.\frac{\delta^2 Z}{\delta j(x_1) j(x_4)}\frac{\delta^2 Z}{\delta j(x_2)\delta j(x_3)}\right|_{j=0} \\ &= \langle 0|T\phi_1\phi_2\phi_3\phi_4|0\rangle - \langle 0|T\phi_1\phi_2|0\rangle\langle 0|T\phi_3\phi_4|0\rangle - \langle 0|T\phi_1\phi_3|0\rangle\langle 0|T\phi_2\phi_4|0\rangle - \langle 0|T\phi_1\phi_4|0\rangle\langle 0|T\phi_2\phi_3|0\rangle\end{aligned}$$



Indeed only the fully connected term survive!

So define in general the "connected" or "irreducible" n-point function by

$$\langle 0|T\phi(x_1)\dots\phi(x_n)|0\rangle_{\mathrm{c}} = \frac{1}{i}\frac{\delta}{\delta j(x_1)}\dots\frac{1}{i}\frac{\delta}{\delta j(x_n)} iE[j] \tag{2.6.5}$$

We have shown that

$$\langle 0|T\phi_1\phi_2\phi_3\phi_4|0\rangle = \langle 0|T\phi_1\phi_2\phi_3\phi_4|0\rangle_c + \sum_P \langle 0|T\phi_{i_1}\phi_{i_2}\rangle_{\mathrm{c}}\langle 0|T\phi_{i_3}\phi_{i_4}|0\rangle_{\mathrm{c}}$$

Example for 6-point function

2.7 Effective action and Legendre Transform

$iE[j]$ is generating functional for irreducible Green's functions. Formally

$$iE[j] = \sum_n \frac{i^n}{n!}\,\mathrm{d}^4x_1\dots\mathrm{d}^4x_n\,G_{\mathrm{c}}(x_1,\dots,x_n)j(x_1)\dots j(x_n) \tag{2.7.1}$$

Remember the LSZ reduction formula. It is the relation between S-matrix and Green's function

$$_{\text{out}}\langle k_1 \dots k_m | k_{m+1} \dots k_n \rangle_{\text{in}} = \text{disconnected terms} + \prod_{j=1}^{n} \left(\frac{iZ}{k_j^2 - m^2 + i\epsilon} \right)^{-1} \sqrt{Z}^n G(k_1, \dots, k_n) \tag{2.7.2}$$

Conclusion is that n-point Green's function contain poles in all external legs. S-matrix elements are amputated Green's functions. In the following to derive generating functional for amputated, fully connected (one-particle-irreducible) Green's functions. It leads to "effective action".

Define the classical field

$$\phi(x) = \frac{\delta}{\delta j(x)} E[j] \tag{2.7.3}$$

For $j \neq 0$, $\phi = \phi[j]$ can in principle be inverted to $j = j[\phi]$.

Effective action is Legendre transform of $E[j]$

$$\Gamma[\phi] = E - \int \mathrm{d}^4 x\, j(x)\phi(x) \Big|_{j=j[\phi]} \tag{2.7.4}$$

$j(x)$ can be recovered from Γ by functional derivation with respect to ϕ

$$\frac{\delta}{\delta\phi(x)} \Gamma[\phi] = -j(x) \tag{2.7.5}$$

Note that $E = E[j[\phi]]$, so this calculation is not quite as trivial as it seems.

Define $\Gamma(x_1, \dots, x_n)$ through the formal expansion

$$\Gamma[\phi] = -\sum \frac{(-i)^n}{n!} \int \mathrm{d}x_1 \dots \mathrm{d}x_n\, \Gamma(x_1, \dots, x_n)\phi(x_1) \dots \phi(x_n) \tag{2.7.6}$$

$\Gamma(x_1, \dots, x_n)$ can be obtained from $\Gamma[\phi]$ by repeated function derivative with respect to $\phi(x_i)$.

Calculate first

$$\begin{aligned} \frac{\delta}{\delta\phi(y)} \phi(y) &= \delta(x - y) \\ &= \frac{\delta}{\delta\phi(y)} \left(\frac{\delta}{\delta j(x)} E[j] \right) \end{aligned}$$

use the chain rule

$$\begin{aligned} &= \int \mathrm{d}^4 z\, \frac{\delta^2 E}{\delta j(x)\delta j(z)} \frac{\delta}{\delta\phi(y)} j(z) \\ &= -i \int \mathrm{d}^4 z\, G_{\mathrm{c}}(x - z)\Gamma(y - z) \end{aligned}$$

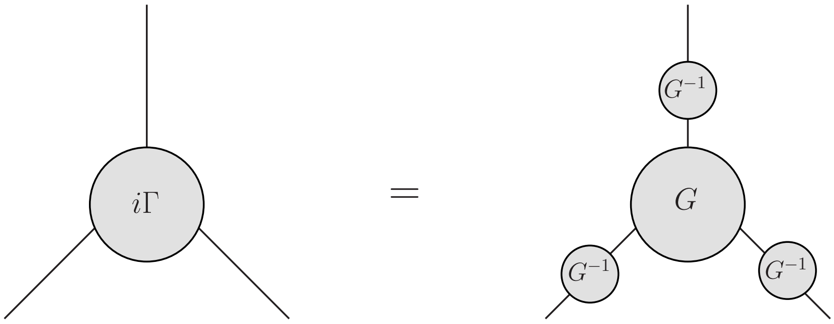
Fourier transform the result we have $i = \tilde{G}(p)\tilde{\Gamma}(p)$, or $\tilde{\Gamma}(p) = Z^{-1}(p^2 - m^2)$. Differentiate this once again

$$\begin{aligned} \frac{\delta^2\phi(x)}{\delta\phi(x)\delta(y)} &= 0 \\ &= \int \mathrm{d}^4 z\, \mathrm{d}^4[v] \left[\frac{\delta^3 E}{\delta j(x)\delta j(z)\delta(v)} \frac{\delta j(z)}{\delta\phi(y)} \frac{\delta j(v)}{\delta\phi(n)} \right] + \int \mathrm{d}^4 z\, \frac{\delta^2 E}{\delta j(x)\delta j(z)} \frac{\delta^2 j(z)}{\delta\phi(y)\delta\phi(x)} \\ &= -\int \mathrm{d}^4 z\, \mathrm{d}^4 v\, G(x, z, v)\Gamma(z, y)\Gamma(v, u) - \int \mathrm{d}^4 z\, G(x, z)\Gamma(z, y, x) = 0 \end{aligned}$$

multiply with $\int \mathrm{d}^4x\, \Gamma(x,\omega)$, use $\int \mathrm{d}^4x\, \Gamma(x-\omega)G(x-z) = i\delta(\omega - z)$

$$i\Gamma(\omega, y, u) = \int \mathrm{d}^4z\, \mathrm{d}^4v\, \mathrm{d}^4x\, G(x,z,v)\Gamma(z,y)\Gamma(x,\omega)\Gamma(u,v)$$

graphically



In word, $\Gamma(\omega, y, u)$ is the 1-particle irreducible version of $G(x,z,v)$!

2.8 Ward-Takahashi Identity for QED

Ward Takahashi identities are relations between one-particle irreducible vertex functions and propagators that hold to all orders in perturbation theory. It is in fact consequence of gauge invariance. It also plays key role in the proof of renormalizability of QED.

Generating functional of QED

$$Z[j_\mu, \eta, \bar\eta] = N \int \mathcal{D}A_\mu \mathcal{D}\phi \mathcal{D}\bar\psi \exp\left\{ i \int \mathrm{d}^4x\, \mathcal{L}_{\text{eff}} \right\} \tag{2.8.1}$$

$$\begin{aligned} \mathcal{L}_{\text{eff}} &= \underbrace{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar\psi\left[i\left(\not\partial + ie\not A\right) - m\right]\psi}_{\mathcal{L}} \\ &\quad \underbrace{-\frac{1}{2\xi}(\partial_\mu A^\mu)^2 + j_\mu A^\mu + \bar\psi\eta + \bar\eta\psi}_{\mathcal{L}_1} \end{aligned} \tag{2.8.2}$$

Two observations are seemly contradicting to each other

- $\mathcal{L}_{\text{eff}}$ is obviously not gauge invariant, since we have introduced a gauge fixing term.
- On the other hand, physics as expressed through Green's functions must be independent of gauge.

This non-trivial connection leads to differential equation for Z!

Consider infinitesimal gauge transformation

$$\begin{aligned} A_\mu &\mapsto A_\mu + \partial_\mu\Lambda \\ \psi &\mapsto \psi - ie\Lambda\psi \\ \bar\psi &\mapsto \bar\psi + ie\Lambda\bar\psi \end{aligned}$$

In the decomposition $\mathcal{L} + \mathcal{L}_1$, all the changes are induced via $\mathcal{L}_1$ ($\mathcal{L}$ is gauge invariant)

$$\delta \int \mathrm{d}^4x\, \mathcal{L}_{\text{eff}} = \delta \int \mathrm{d}^4x\, \mathcal{L}_1 = \int \mathrm{d}^4x \left[-\frac{1}{\xi}(\partial_\mu A^\mu)\partial^2\Lambda + j_\mu\partial^\mu\Lambda - ie\Lambda(\bar\eta\psi - \bar\psi\eta) \right]$$

Hence the change in $Z[j,\eta,\bar\eta]$ is

$$\delta Z[j,\eta,\bar\eta] = N\int \mathcal{D}A_\mu \mathcal{D}\psi\mathcal{D}\bar\psi \exp\left\{i\int \mathrm{d}^4x\,\mathcal{L}_{\text{eff}}\right\}$$
$$\times\, i\int \mathrm{d}^4x \left[-\frac{1}{\xi}\partial^2(\partial_\mu A^\mu) - \partial_\mu j^\mu - ie(\bar\eta\psi - \bar\psi\eta)\right]\Lambda(x)$$

As $\Lambda(x)$ is arbitrary, the term is bracket needs to vanish. Take this bracket in front of the functional Z using

$$\psi \mapsto \frac{1}{i}\frac{\delta}{\delta\bar\eta}$$
$$\bar\psi \mapsto -\frac{1}{i}\frac{\delta}{\delta\eta}$$
$$A_\mu \mapsto \frac{1}{i}\frac{\delta}{\delta j^\mu}$$

$$\left[\frac{i}{\xi}\partial^2(\partial_\mu\frac{\delta}{\delta j_\mu}) - \partial_\mu j^\mu - e\left(\bar\eta\frac{\delta}{\delta\bar\eta} - \eta\frac{\delta}{\delta\eta}\right)\right]Z[j,\eta,\bar\eta] = 0$$

Transform this into PDE for generating functional of irreducible Green's functions $Z = \mathrm{e}^{iE}$

$$-\frac{1}{\xi}\partial^2\left(\partial_\mu\frac{\delta E}{\delta j_\mu}\right) - \partial_\mu j^\mu - ie\left(\bar\eta\frac{\delta E}{\delta\bar\eta} - \eta\frac{\delta E}{\delta\eta}\right) = 0 \tag{2.8.3}$$

Finally use the effective action to derive relations for irreducible amputated vertex functions

$$\Gamma[A_\mu,\psi,\bar\psi] = E[j_\mu,\eta,\bar\eta] - \int \mathrm{d}^4x\left(j_\mu A^\mu + \bar\eta\psi + \bar\psi\eta\right) \tag{2.8.4}$$

$$\frac{\delta\Gamma}{\delta A_\mu} = -j^\mu \quad \frac{\delta E}{\delta j^\mu} = A_\mu$$
$$\frac{\delta\Gamma}{\delta\psi} = +\bar\eta \quad \frac{\delta E}{\delta\bar\eta} = \psi$$
$$\frac{\delta\Gamma}{\delta\bar\psi} = -\eta \quad \frac{\delta E}{\delta\eta} = -\bar\psi$$

After the replacement, equation 2.8.3 becomes

$$\left[\frac{1}{\xi}\partial^2\partial_\mu A^\mu - \partial_\mu\frac{\delta\Gamma}{\delta A_\mu} + ie\left(\bar\psi\frac{\delta\Gamma}{\delta\bar\psi} + \frac{\delta\Gamma}{\delta\psi}\psi\right)\right] = 0 \tag{2.8.5}$$

Take functional derivative $\frac{\delta}{\delta\bar\psi}\frac{\delta}{\delta\psi}$ and subsequently put $\bar\psi = \psi = A_\mu = 0$

$$-\partial_x^\mu\frac{\delta^3\Gamma}{\delta\bar\psi(x_1)\delta\psi(y_1)\delta A^\mu(x)} = -ie\delta^{(4)}(x-x_1)\frac{\delta^2\Gamma}{\delta\bar\psi(x_1)\delta\psi(y_1)} + ie\delta^{(4)}(x-y_1)\frac{\delta^2\Gamma}{\delta\bar\psi(x_1)\delta\psi(y_1)} \tag{2.8.6}$$

This becomes more intuitive in momentum space. First define

$$\int \mathrm{d}^4x\,\mathrm{d}^4x_1\,\mathrm{d}^4y_1\exp[i(p'x_1 - py_1 - qx)]\frac{\delta^3\Gamma}{\delta\bar\psi(x_1)\delta\psi(y_1)\delta A^\mu(x)} = e(2\pi)^4(p'-p-q)\Gamma(p,q,p') \tag{2.8.7}$$

Know already the one-particle irreducible two-point function

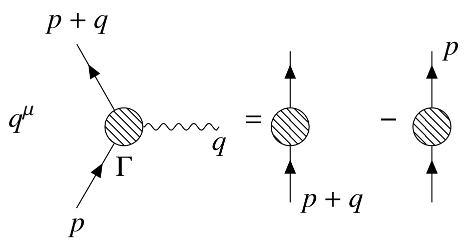
$$\int \mathrm{d}^4x_1\, \mathrm{d}^4y_1 \exp[i(p'x_1 - py_1)] \frac{\delta^2\Gamma}{\delta\bar{\psi}(x_1)\delta\psi(y_1)} = -(2\pi)^4\delta^{(4)}(p'-p)iS_F^{-1}(p) \tag{2.8.8}$$

Multiply 2.8.6 with $\exp[i(p'x_1 - py_1 - qx)]$ and integrate over x, x_1 and y_1

$$q_\mu\Gamma^\mu(p,q,p+q) = iS_F^{-1}(p+q) - iS_F^{-1}(p) \tag{2.8.9}$$

This is Ward Takahashi identity.

Graphically



At lowest order in QED

$$S_F(p) = \frac{i}{\not{p} - m}$$
$$\Gamma^\mu(p,q,p+q) = \gamma^\mu$$
$$\not{q} = (\not{p} - \not{q} - m) - (\not{p} - m)$$

In the limit $q^\mu \to 0$, we obtain Ward identity

$$\Gamma^\mu(p,0,p) = \frac{\partial iS_F^{-1}}{\partial p_\mu} \tag{2.8.10}$$

There are more Ward identities that can be derived using different functional derivatives. Start again with 2.8.3 and differentiate with respect to $j_0(y)$, then put $\eta = \bar{\eta} = j = 0$

$$-\frac{1}{\xi}\partial^2\partial_x^\mu \frac{\delta^2 E}{\delta j^\mu(x)\delta j^\nu(y)} = \partial_x^\mu g_{\mu\nu}\delta^{(4)}(x-y)$$

Remember photon propagtor is given by

$$\frac{i\delta^2 E}{i\delta j^\mu(x) i\delta j^\nu(y)} = \langle 0|TA_\mu(x)A_\nu(y)|0\rangle = G_{\mu\nu}(x-y)$$

$$\frac{1}{\xi}\partial^2\partial_x^\mu G_{\mu\nu}(x-y) = i\partial_x^\mu g_{\mu\nu}\delta^{(4)}(x-y) \tag{2.8.11}$$

After Fourier transform

$$\frac{i}{\xi}k^2k^\mu\tilde{G}_{\mu\nu}(k) = k_\nu \tag{2.8.12}$$

Again it is true to all orders in perturbation theory. To say that the longitudinal component of $G_{\mu\nu}$ is fixed and not modified by interactions

$$\tilde{G}_{\mu\nu}(k) = \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)G_T(k^2) + \frac{k_\mu k_\nu}{k^2}G_L(k^2) \tag{2.8.13}$$

with Ward identity

$$\frac{i}{\xi}k^2 G_L(k^2)k_\nu = k_\nu$$
$$G_L(k^2) = \frac{-i\xi}{k^2}$$

Propagator at leading order

$$\hat{G}_{\mu\nu}(k) = \frac{-i}{k^2}\left(g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right)$$
$$G_T(k^2) = \frac{-i}{k^2}$$
$$G_L(k^2) = -\frac{i\xi}{k^2}$$

We will make heavy use of the fact Ward Takahashi identities hold to all orders in the following sections on the renormalization of QED.

2.9 Renormalization of QED I: Divergences and Dimensional Analysis

It is known from QFT course last term that loop diagrams are often (UV-)divergent. To obtain a sensible theory, need to regularise these divergences and remove or absorb them by renormalization.

Analyse divergences structure of QED by dimensional analysis. Superficial degree of divergences D of a Feynman diagram with

- d space dimension
- L number of loops
- P_γ number of photon propagators
- P_e number of electron propagators
- N_γ number of external photons
- N_e number of external electrons
- V number of vertices

An arbitrary diagram contains the integral like

$$\int \frac{\mathrm{d}^d k_1 \dots \mathrm{d}^d k_L}{(\not{k}_{i_1} - m)\dots(\not{k}_{i_{P_e}})k^2_{j_1}\dots k^2_{j_{P_\gamma}}} \sim k^D$$

thus

$$D = dL - 2P_\gamma - P_e \tag{2.9.1}$$

We want to eliminate L, P_γ and P_e in favour of V, N_γ and N_e

- L is the number of undetermined momenta

$$L = P - V + 1 = P_\gamma + P_e - V + 1 \tag{2.9.2}$$

- Each vertex is connected to 2 electron and 1 photon line. External lines are attached to 1 vertex, internal to 2 vertices.

$$V = 2P_\gamma + N_\gamma = \frac{1}{2}(2P_e + N_e) \tag{2.9.3}$$

Put together

$$D = d + V\left(\frac{d-4}{2}\right) - N_e\left(\frac{d-1}{2}\right) - N_\gamma\left(\frac{d-2}{2}\right) \tag{2.9.4}$$

for $d = 4$

$$D = 4 - \frac{3}{2}N_e - N_\gamma \tag{2.9.5}$$

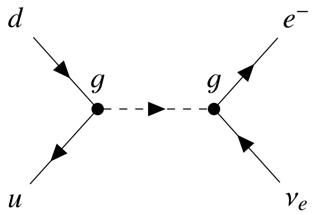
In four dimension, D is independent of number of vertices, only dependent on N_e and N_γ. $D \geq 0$ only for certain, finite "small" N_e and N_γ.

There are three different categories

- $d < 4$ super-renormalizable $\Leftrightarrow [e] > 0$
- $d = 4$ renormalizable $\Leftrightarrow [e] = 0$
- $d > 4$ non-renormalizable$\Leftrightarrow [e] < 0$

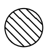
Mass dimension of coupling constant $[\psi] = (d-1)/2$ and $[A_\mu] = (d-2)/2$. Interaction $eA_\mu\bar{\psi}\psi$ leads to $[e] = 2 - d/2$

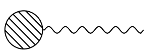
Fermi theory of weak interaction contains the interaction term $G_F\left(\bar{\psi}\gamma_\mu(1-\gamma_5)\psi\right)(\bar{\psi}\gamma^\mu(1-\gamma_5)\psi)$. Coupling constant has negative mass dimension $[G_F] = -2$, non-renormalizable.

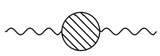


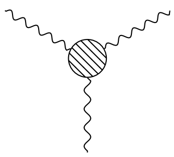
$$\sim \frac{g^2}{q^2 - M_W^2} \approx \frac{g^2}{M_W^2} = G_F$$

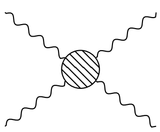
Back to QED in $d = 4$; divergent amplitudes

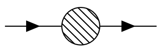
(a)  $D = 4$

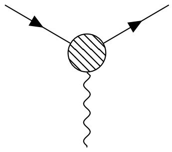
(b)  $D = 3$

(c)  $D = 2$

(d)  $D = 1$

(e)  $D = 0$

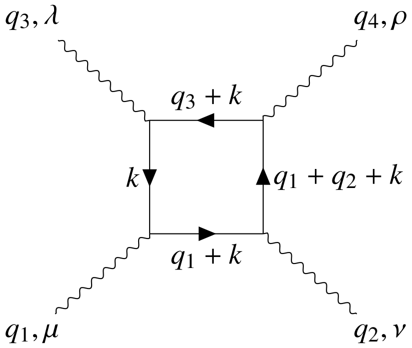
(f)  $D = 1$

(g)  $D = 0$

Need to show that all divergences can be absorbed into renormalization of the parameters of the theory ($e_0 \to e$, $m_0 \to m$) and by adjusting the field strength $\psi \mapsto Z_2^{-1/2}\psi$ and $A_\mu \mapsto Z_3^{-1/2} A_\mu A$

To ignore (a). QED is C-invariant, $A_\mu \mapsto -A_\mu$, correlation functions of <u>odd</u> numbers of photons vanish. (Furry's theorem) Then ignore ((b)) ((d)).

((e)) could be potentially dangerous. Need counter-terms like $(F_{\mu\nu}F^{\mu\nu})^2$, $(F_{\mu\nu}\tilde{F}^{\mu\nu})^2$, but they have dimension 8, i.e. need -4 mass dimension coupling constant. Then the theory becomes non-renormalizable! Gauge invariance saves us.



$$= \frac{e^4}{i} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{\gamma_\mu(\not{k} + m)\gamma_\lambda(\not{q}_3 + \not{k} + m)\gamma_\rho(\not{q}_1 + \not{q}_2 + \not{k} + m)\gamma_\nu(\not{q}_1 + \not{k} + m)}{(k^2 - m^2)((q_3 + k)^2 - m^2)((q_1 + q_2 + k)^2 - m^2)((q_1 + k)^2 - m^2)}$$

We can evaluate the divergent part by putting the external momenta to zero, since the part dependent on momenta is finite. This divergence is UV-divergence, so we can also put mass to zero in the limit of large momentum.

$$\begin{aligned} &= \frac{e^4}{i} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{\gamma_\mu \not{k} \gamma_\lambda \not{k} \gamma_\rho \not{k} \gamma_\nu \not{k}}{(k^2)^4} + \text{finite} \\ &= e^4 I(g_{\mu\lambda}g_{\nu\rho} + g_{\mu\rho}g_{\nu\lambda} + g_{\mu\nu}g_{\rho\lambda}) + \text{finite} \end{aligned}$$

From gauge invariance or another Ward identity, one can show $q_1^\mu(\dots) = 0$

$$e^4 I(q_{1\lambda}g_{\rho\nu} + q_{1\rho}g_{\lambda\nu} + q_{1\nu}g_{\lambda\rho}) + \text{finite} = 0$$

Thus I has to be finite! Symmetries can renders amplitudes more convergent than they appear superficially!

Conclusion: primitively divergent amplitudes are ((c)) photon self energy, ((f)) electron self energy and ((g)) the vertex graph.

3 IDK