H8.1 Chenhuan Wang

a) $i\hbar \gamma^\mu \partial_\mu \Psi = mc\Psi$ $\Psi(x^\mu) = u(\epsilon, \vec{p}) e^{i/\hbar(\vec{p}\cdot\vec{x} - \epsilon t)}$

$$\Rightarrow i\hbar(\beta\partial_0 + \alpha^i\partial_i)\Psi = mc\Psi$$

$$\Rightarrow i\hbar\partial_t \Psi = (c\vec{\alpha}\cdot\vec{p} + \beta mc^2)\Psi$$

Plug in $\Psi(x^\mu)$

$$i\hbar\partial_t u(\epsilon, \vec{p}) e^{i/\hbar(\vec{p}\cdot\vec{x} - \epsilon t)} = \begin{bmatrix} mc^2 & c\vec{\sigma}\cdot\vec{p} \\ c\vec{\sigma}\cdot\vec{p} & -mc^2 \end{bmatrix} u(\epsilon, \vec{p}) e^{i/\hbar(\vec{p}\cdot\vec{x} - \epsilon t)}$$

$$i\hbar \cdot \frac{i}{\hbar}\cdot(-\epsilon)\, u(\epsilon, \vec{p}) e^{\cdots} = \begin{bmatrix} mc^2 & c\vec{\sigma}\cdot\vec{p} \\ c\vec{\sigma}\cdot\vec{p} & -mc^2 \end{bmatrix} u(\epsilon, \vec{p}) e^{\cdots}$$

$$\Rightarrow \epsilon\, u(\epsilon, \vec{p}) = \begin{bmatrix} mc^2 & c\vec{\sigma}\cdot\vec{p} \\ c\vec{\sigma}\cdot\vec{p} & -mc^2 \end{bmatrix} u(\epsilon, \vec{p})$$

not sure why we can eliminate $e^{\cdots}$ here ...

$$\Rightarrow \begin{bmatrix} mc^2 - \epsilon & c\vec{\sigma}\cdot\vec{p} \\ c\vec{\sigma}\cdot\vec{p} & -mc^2 - \epsilon \end{bmatrix} u(\epsilon, \vec{p}) = 0$$

determinant: $-(m^2c^4 - \epsilon^2) - c^2(\vec{\sigma}\cdot\vec{p})^2 = 0$

$$\epsilon^2 - m^2c^4 - c^2\vec{p}^{\,2} = 0$$

$$\left((\vec{\sigma}\cdot\vec{p})(\vec{\sigma}\cdot\vec{p}) = \mathbb{1}(\vec{p}\cdot\vec{p}) + i\vec{\sigma}(\vec{p}\times\vec{p}) = \vec{p}\cdot\vec{p} \right)$$

$$\Rightarrow \epsilon = \pm\sqrt{m^2c^4 + c^2\vec{p}^{\,2}} = \pm E$$

Return to the equation:

$$\begin{bmatrix} mc^2 - \epsilon & c\vec{\sigma}\cdot\vec{p} \\ c\vec{\sigma}\cdot\vec{p} & -mc^2 - \epsilon \end{bmatrix} u(\epsilon, \vec{p}) = 0$$

$\epsilon = +E$, particle: $u(\epsilon = E) = \begin{pmatrix} \varphi \\ \frac{c\vec{\sigma}\cdot\vec{p}}{E + mc^2}\varphi \end{pmatrix}$

$$(mc^2 - E)\varphi + c\vec{\sigma}\cdot\vec{p}\left(\frac{c\vec{\sigma}\cdot\vec{p}}{E + mc^2}\right)\varphi$$

$$= \left(mc^2 - E + \frac{c^2|\vec{p}|^2}{E + mc^2}\right)\varphi = \frac{(E + mc^2)(-E + mc^2) + c^2p^2}{E + mc^2} = 0 \Rightarrow$$

b) $\vec{p} = (p, 0, 0)^T$

$$(\vec{\sigma}\cdot\vec{p})\,\varphi = \pm|\vec{p}|\,\varphi \quad\Rightarrow\quad \begin{pmatrix} 0 & p \\ p & 0 \end{pmatrix}\varphi = \pm p\,\varphi$$

$$\Rightarrow \begin{pmatrix} 0 & p \\ p & 0 \end{pmatrix}\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \pm p \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$$

$$\begin{cases} p\varphi_2 = \pm p\,\varphi_1 \\ p\,\varphi_1 = \pm p\,\varphi_2 \end{cases}$$

$$\Rightarrow \varphi_1 = \pm\varphi_2 . \qquad \varphi = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ or } \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

c) $N^2 = (E + mc^2)/2E$

$$u^\dagger u = \left[N\begin{pmatrix} \varphi \\ \frac{c\vec{\sigma}\cdot\vec{p}}{E+mc^2}\varphi \end{pmatrix}\right]^\dagger \left[N\begin{pmatrix} \varphi \\ \frac{c\vec{\sigma}\cdot\vec{p}}{E+mc^2}\varphi \end{pmatrix}\right]$$

$$= \frac{E+mc^2}{2E}\left[\varphi^\dagger\varphi + \left(\frac{c\vec{\sigma}\cdot\vec{p}}{E+mc^2}\varphi\right)^\dagger\left(\frac{c\vec{\sigma}\cdot\vec{p}}{E+mc^2}\varphi\right)\right]$$

$$\left[(\vec{\sigma}\cdot\vec{p})^\dagger = \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & p_z \end{pmatrix}^\dagger = (\vec{\sigma}\cdot\vec{p})\right]$$

$$= \frac{E+mc^2}{2E}\varphi^\dagger\varphi\left(1 + \frac{c^2|\vec{p}|^2}{(E+mc^2)^2}\right)$$

$$= \varphi^\dagger\varphi\,\frac{\cancel{E+mc^2}}{2E}\cdot\frac{(E+mc^2)^2 + p^2c^2}{(E+mc^2)^{\cancel{2}}}$$

$$= \varphi^\dagger\varphi\,\frac{1}{2E}\underbrace{\frac{E^2 + 2Emc^2 + E^2}{E+mc^2}}$$

$$= \frac{1}{2E}\,\frac{2E\,(E+mc^2)}{E+mc^2} = 1$$

$$= \varphi^\dagger\varphi$$

d) $\rho = \Psi^{+} \Psi \qquad \vec{j} = \Psi^{+} (c\vec{\alpha}) \Psi$

$$\vec{j} = u(\epsilon)^{+} (c\vec{\alpha})\, u(\epsilon)$$

$$j^{i} = c N^{2} \begin{pmatrix} \varphi \\ \frac{c\vec{\sigma}\cdot\vec{p}}{\epsilon + mc^{2}} \varphi \end{pmatrix}^{+} \begin{pmatrix} 0 & \sigma^{i} \\ \sigma^{i} & 0 \end{pmatrix} \begin{pmatrix} \varphi \\ \frac{c\vec{\sigma}\cdot\vec{p}}{\epsilon + mc^{2}} \varphi \end{pmatrix}$$

$$= c N^{2} \left(\frac{c\vec{\sigma}\cdot\vec{p}}{\epsilon + mc^{2}} \varphi^{+} \cdot \sigma^{i} ,\ \varphi^{+} \sigma^{i} \right) \begin{pmatrix} \varphi \\ \frac{c\vec{\sigma}\cdot\vec{p}}{\epsilon + mc^{2}} \varphi \end{pmatrix}$$

$$= c N^{2} \left(\frac{c\vec{\sigma}\cdot\vec{p}}{\epsilon + mc^{2}} \varphi^{+} \cdot \sigma^{i} \varphi + \varphi^{+} \sigma^{i} \frac{c\vec{\sigma}\cdot\vec{p}}{\epsilon + mc^{2}} \varphi \right)$$

$$= c N^{2} \left(\varphi^{+} \sigma^{i} \frac{c\vec{\sigma}\cdot\vec{p}}{\epsilon + mc^{2}} \varphi + \varphi^{+} \sigma^{i} \frac{c\vec{\sigma}\cdot\vec{p}}{\epsilon + mc^{2}} \varphi \right)$$

$$= N^{2} \cdot 2c\, \varphi^{+} \sigma^{i} \frac{c\vec{\sigma}\cdot\vec{p}}{\epsilon + mc^{2}} \varphi$$

$$= \frac{\cancel{\epsilon + mc^{2}}}{\cancel{2}\epsilon} \cdot \cancel{2} c\, \varphi^{+} \sigma^{i} \frac{c\vec{\sigma}\cdot\vec{p}}{\cancel{\epsilon + mc^{2}}} \varphi$$

$$\Rightarrow \vec{j} = \frac{c^{2}}{\epsilon} \varphi^{+} \vec{\sigma} (\vec{\sigma}\cdot\vec{p}) \varphi = \frac{c^{2}}{\epsilon} \varphi^{+} \vec{p} \varphi = \frac{c^{2} \vec{p}}{\epsilon} \varphi^{+} \varphi$$

using $(\vec{\sigma}\cdot\vec{p}) \varphi = \pm p \varphi$

$$\rho = \Psi^{+} \Psi = N^{2} \left(\varphi^{+} ,\ \frac{c\vec{\sigma}\cdot\vec{p}}{\epsilon + mc^{2}} \varphi^{+} \right) \cdot \begin{pmatrix} \varphi \\ \frac{c\vec{\sigma}\cdot\vec{p}}{\epsilon + mc^{2}} \varphi \end{pmatrix}$$

$$= \frac{\epsilon + mc^{2}}{2\epsilon} \varphi^{+} \varphi \left(1 + \frac{c^{2} p^{2}}{\epsilon + mc^{2}} \right) = \varphi^{+} \varphi$$

just like last part

$$\Rightarrow \vec{j} = \frac{c^{2} \vec{p}}{\epsilon} \varphi^{+} \varphi = \frac{c^{2} \vec{p}}{\epsilon} \rho = \pm \frac{c^{2} \vec{p}}{E} \rho = \pm \vec{v} \rho$$

d) $j_i = \bar{\psi}(c\vec{\alpha})\psi = \left(\varphi^{+} \quad \varphi^{+}\frac{c\vec{\sigma}\cdot\vec{p}}{E+mc^2}\right)\begin{pmatrix} 0 & c\sigma_i \\ c\sigma_i & 0 \end{pmatrix}\begin{pmatrix} \varphi \\ \frac{c\vec{\sigma}\cdot\vec{p}}{E+mc}\varphi \end{pmatrix}$

$$= N^2\left(\varphi^{+} \quad \varphi^{+}\frac{c\vec{\sigma}\cdot\vec{p}}{E+mc^2}\right)\begin{pmatrix} \frac{c^2\sigma_i(\vec{\sigma}\cdot\vec{p})\varphi}{E+mc^2} \\ c\sigma_i\varphi \end{pmatrix}$$

$$= N^2\left(\varphi^{+}\frac{c^2\sigma_i(\vec{\sigma}\cdot\vec{p})}{E+mc^2}\varphi + \varphi^{+}\frac{c^2(\vec{\sigma}\cdot\vec{p})\sigma_i}{E+mc^2}\varphi\right)$$

$$\left[\{\sigma_i,\sigma_j\} = 2\delta_{ij} \iff \sigma_i\sigma_j = 2\delta_{ij} - \sigma_j\sigma_i\right]$$

$$= N^2\left(\frac{2p_i c^2}{E+mc^2}\varphi^{+}\varphi\right) = \frac{c^2\vec{p}}{E}\varphi^{+}\varphi$$

H8.2

a) $z < -\frac{a}{2}$: $\Psi_A(z) = A_+ e^{ipz/\hbar} \begin{pmatrix} 1 \\ 0 \\ \frac{pc}{E+mc^2} \\ 0 \end{pmatrix} + A_- e^{-ipz/\hbar} \begin{pmatrix} 1 \\ 0 \\ \frac{-pc}{E+mc^2} \\ 0 \end{pmatrix}$

$z \in [-\frac{a}{2}, \frac{a}{2}]$: $\Psi_B(z) = B_+ e^{ip'z/\hbar} \begin{pmatrix} 1 \\ 0 \\ \frac{p'c}{E-V_0+mc^2} \\ 0 \end{pmatrix} + B_- e^{-ip'z/\hbar} \begin{pmatrix} 1 \\ 0 \\ \frac{-p'c}{E-V_0+mc^2} \\ 0 \end{pmatrix}$

$z > \frac{a}{2}$: $\Psi_C(z) = C_+ e^{ipz/\hbar} \begin{pmatrix} 1 \\ 0 \\ \frac{pc}{E+mc^2} \\ 0 \end{pmatrix} + C_- e^{-ipz/\hbar} \begin{pmatrix} 1 \\ 0 \\ \frac{-pc}{E+mc^2} \\ 0 \end{pmatrix}$

taking: $c = \hbar = 1$

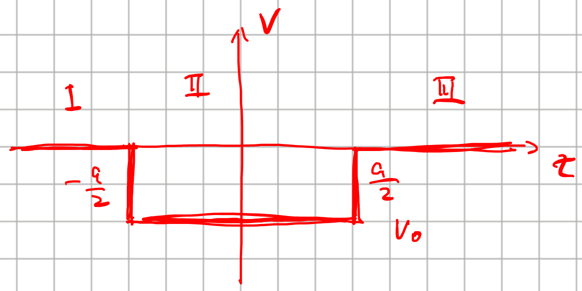
Continuity:

$$\begin{cases} \Psi_A(-\frac{a}{2}) = \Psi_B(-\frac{a}{2}) \\ \Psi_B(\frac{a}{2}) = \Psi_C(\frac{a}{2}) \end{cases}$$

$$\to \quad A_+ e^{-ipa/2} \begin{pmatrix} 1 \\ 0 \\ \frac{pc}{E+mc^2} \\ 0 \end{pmatrix} + A_- e^{+ipa/2} \begin{pmatrix} 1 \\ 0 \\ \frac{-pc}{E+mc^2} \\ 0 \end{pmatrix}$$

$$= B_+ e^{-ip'a/2} \begin{pmatrix} 1 \\ 0 \\ \frac{p'c}{E-V_0+mc^2} \\ 0 \end{pmatrix} + B_- e^{+ip'a/2} \begin{pmatrix} 1 \\ 0 \\ \frac{-p'c}{E-V_0+mc^2} \\ 0 \end{pmatrix}$$

a)



Greiner C.9

for I, III

$$(\vec{\alpha}\cdot\vec{p} + \beta m)\psi = E\psi$$

for II

$$(\vec{\alpha}\cdot\vec{p} + \beta m)\psi = (E - V_0)\psi$$

Take Spin up:

$$\psi_I(z) = A e^{ip z}\begin{pmatrix}1\\0\\\frac{p_1}{E+m}\\0\end{pmatrix} + A' e^{-ip_1 z}\begin{pmatrix}1\\0\\\frac{-p_1}{E+m}\\0\end{pmatrix}$$

$$\psi_{II}(z) = B e^{ip_2 z}\begin{pmatrix}1\\\frac{p_2}{E-V_0+m}\end{pmatrix} + B' e^{-ip_2 z}\begin{pmatrix}1\\\frac{-p_2}{E-V_0+m}\end{pmatrix}$$

omitting 0

$$\psi_{III}(z) = C e^{ip_1 z}\begin{pmatrix}1\\\frac{p_1}{E+m}\end{pmatrix} + C' e^{-ip_1 z}\begin{pmatrix}1\\\frac{-p_1}{E+m}\end{pmatrix}$$

"Current" comes from the other side $\longrightarrow$ anti-particle!

continuity at borders:

$$\psi_I(-\tfrac{a}{2}) = \psi_{II}(-\tfrac{a}{2}) \qquad \psi_{II}(\tfrac{a}{2}) = \psi_{III}(\tfrac{a}{2})$$

$\Rightarrow \quad z = -\frac{a}{2}$:

① $A e^{-ip_1 a/2} + A' e^{ip_1 a/2} = B e^{-ip_2 a/2} + B' e^{ip_2 a/2}$

② $\left(A e^{-ip_1 a/2} - A' e^{ip_1 a/2}\right)\frac{p_1}{E+m} = \left(B e^{-ip_2 a/2} - B' e^{ip_2 a/2}\right)\frac{p_2}{E-V_0+m}$

$z = \frac{a}{2}$:

③ $$B e^{ip_2 a/2} + B' e^{-ip_2 a/2} = C e^{ip_1 a/2} + C' e^{-ip_1 a/2}$$

④ $$\left(B e^{ip_2 a/2} - B' e^{-ip_2 a/2}\right) \frac{p_2}{E - V_0 + m} = \left(C e^{ip_1 a/2} - C' e^{-ip_1 a/2}\right) \frac{p_1}{E + m}$$

Write : $$\gamma = \frac{\frac{p_1}{E+m}}{\frac{p_2}{E-V_0+m}} = \frac{\sqrt{E^2 - m^2}}{E+m} \cdot \frac{E - V_0 + m}{\sqrt{(E-V_0)^2 - m^2}}$$

$$= \sqrt{\frac{E-m}{E+m} \cdot \frac{(E-V_0)-m}{(E-V_0)+m}}$$

in Matrix form :

$z = -\frac{a}{2}$:

① + ② : $$2 A e^{-ip_1 a/2} = B e^{\cdot\cdot} \left(1 + \frac{1}{\gamma}\right) + B' e^{\cdot\cdot} \left(1 - \frac{1}{\gamma}\right)$$

$$A e^{-ip_1 a/2} = \frac{1}{2} B e^{\cdot\cdot} \frac{\gamma+1}{\gamma} + \frac{1}{2} B' e^{\cdot\cdot} \left(\frac{\gamma - 1}{\gamma}\right)$$ then multiply $e^{+ip_1 a/2}$

$\rightarrow$ do the same to A' : ① − ②

$z = \frac{a}{2}$:

③ + ④ : $$2 B e^{\cdot\cdot} = C e^{\cdot\cdot} (1+\gamma) + C' e^{\cdot\cdot} (1 - \gamma)$$ multiply by $e^{ip_2 a/2}$

Same to get B'

③ − ④

$$\Rightarrow \begin{pmatrix} A \\ A' \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \frac{\gamma+1}{\gamma} e^{i(p_1 - p_2) a/2} & \frac{\gamma-1}{\gamma} e^{i(p_1 + p_2) a/2} \\ \frac{\gamma-1}{\gamma} e^{-i(p_1 + p_2) a/2} & \frac{\gamma+1}{\gamma} e^{i(p_2 - p_1) a/2} \end{pmatrix} \begin{pmatrix} B \\ B' \end{pmatrix}$$

$$\begin{pmatrix} B \\ B' \end{pmatrix} = \frac{1}{2} \begin{pmatrix} (1+\gamma) e^{i(p_1 - p_2) a/2} & (1-\gamma) e^{-i(p_1 + p_2) a/2} \\ (1-\gamma) e^{i(p_1 + p_2) a/2} & (1+\gamma) e^{i(p_2 - p_1) a/2} \end{pmatrix} \begin{pmatrix} C \\ C' \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} A \\ A' \end{pmatrix} = \frac{1}{4\gamma}\begin{pmatrix} (1+\gamma)^2 e^{i(p_1-p_2)a} - (1-\gamma)^2 e^{i(p_1+p_2)a}, & (1-\gamma^2)(e^{-ip_2 a} - e^{ip_2 a}) \\ -(1-\gamma)^2(e^{ip_2 a} - e^{ip_2 a}) & (1+\gamma)^2 e^{-i(p_1-p_2)a} - (1-\gamma)^2 e^{-i(p_1+p_2)a} \end{pmatrix} \begin{pmatrix} C \\ C' \end{pmatrix}$$

$$\int dz\, \psi^\dagger \psi = 1$$

b) $E^2 = m^2 + p_1^2 \qquad (E-V_0)^2 = m^2 + p_2^2$

p_i could be imaginary / real

$$\begin{cases} E^2 > m^2 \longrightarrow \begin{cases} E > m \\ E < -m \end{cases} \\ E^2 < m^2 \;\&\; (E-V_0)^2 > m^2 \longrightarrow -m < E < m \;\&\; \begin{cases} E - V_0 > m \rightarrow E > m + V_0 \\ E - V_0 < -m \rightarrow E < -m + V_0 \end{cases} \\ E^2,\ (E-V_0)^2 < m^2 \longrightarrow \text{in } (-\infty, \infty)\ \psi = 0 \end{cases}$$

oszillatory in II, damping in I, III