H 9.1 Chenhuan Wang

a) $\Lambda = -(1+2\vec{L}\cdot\vec{S}) - \frac{i\gamma}{r}(\vec{\alpha}\cdot\vec{r})$, $\gamma = Z\alpha$

$$\Lambda^2 = +(1+2\vec{L}\cdot\vec{S})^2 + \left[\frac{i\gamma}{r}(\vec{\alpha}\cdot\vec{r})\right]^2 + (1+2\vec{L}\cdot\vec{S})\frac{i\gamma}{r}(\vec{\alpha}\cdot\vec{r})$$

$$+ \frac{i\gamma}{r}(\vec{\alpha}\cdot\vec{r})(1+2\vec{L}\cdot\vec{S})$$

$$= 1 + 4\vec{L}\cdot\vec{S} + 4(\vec{L}\cdot\vec{S})^2 - \frac{\gamma^2}{r^2}(\vec{\alpha}\cdot\vec{r})^2 + \frac{2i\gamma}{r}\vec{\alpha}\cdot\vec{r}$$

$$+ 2i\gamma\left[(\vec{L}\cdot\vec{S})\cdot\frac{\vec{\alpha}\cdot\vec{r}}{r} + \frac{\vec{\alpha}\cdot\vec{r}}{r}\cdot\vec{L}\cdot\vec{S}\right]$$

$$\left(\frac{\vec{\alpha}\cdot\vec{r}}{r}\right)^2 = \frac{1}{r^2}(\vec{\alpha}\cdot\vec{r})^2 = \frac{1}{r^2}\alpha_i r^i \alpha_j r^j = \frac{1}{r^2}\begin{pmatrix}0 & \sigma_i\\ \sigma_i & 0\end{pmatrix} r^i \begin{pmatrix}0 & \sigma_j\\ \sigma_j & 0\end{pmatrix} r^j$$

$$= \frac{1}{r^2}\begin{pmatrix}\sigma_i\sigma_j & 0\\ 0 & \sigma_i\sigma_j\end{pmatrix} r^i r^j = \frac{1}{r^2} r^i r^j \left(\delta_{ij}\mathbb{1} + i\epsilon_{ijk}\begin{pmatrix}\sigma_k & 0\\ 0 & \sigma_k\end{pmatrix}\right)$$

$$= \mathbb{1} + i\frac{1}{r^2} r^i r^j \epsilon_{ijk}\, 2S^k = \mathbb{1} + \frac{2i}{r^2}(\vec{r}\times\vec{r})\cdot\vec{S} = \mathbb{1}$$

$$4(\vec{L}\cdot\vec{S})^2 = 4\vec{L}\cdot\vec{S}\cdot\vec{L}\cdot\vec{S} = 4\vec{L}^2\cdot\vec{S}^2 = 4\vec{L}^2\cdot\frac{1}{4}\mathbb{1} = \vec{L}^2 \quad \times$$

$$L_i S^i L_j S^j = L_i L_j \underbrace{S^i S^j}_{\neq S^2}$$

$$(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) =$$

$$= 1 + 4\vec{L}\cdot\vec{S} + \vec{L}^2 - \gamma^2 + \text{something}$$

$$(\vec{L}\cdot\vec{S})\cdot\frac{\vec{\alpha}\cdot\vec{r}}{r} = \frac{1}{2}\begin{pmatrix}\vec{\sigma}\cdot\vec{L} & 0\\ 0 & \vec{\sigma}\cdot\vec{L}\end{pmatrix}\begin{pmatrix}0 & \vec{\sigma}\cdot\vec{r}/r\\ \vec{\sigma}\cdot\vec{r}/r & 0\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}0 & \vec{L}\cdot\hat{r} + i(\vec{L}\times\hat{r})\vec{\sigma}\\ \vec{L}\cdot\hat{r} + i(\vec{L}\times\hat{r})\vec{\sigma} & 0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}0 & \vec{L}\cdot\hat{r}\\ \vec{L}\cdot\hat{r} & 0\end{pmatrix}$$

$$\text{Sth} = 2i\gamma\begin{pmatrix}0 & (\vec{L}+\vec{\sigma})\cdot\hat{r}\\ (\vec{L}+\vec{\sigma})\hat{r} & 0\end{pmatrix}$$

a) $(\vec{L}\cdot\vec{S})^2 = \frac{1}{4}\left((\vec{L}\cdot\vec{\sigma})^2 \quad (\vec{L}\cdot\vec{\sigma})^2 \right) = \frac{1}{2}(\vec{L} - 2\vec{L}\cdot\vec{S})^2$

$$(\vec{L}\cdot\vec{\sigma})^2 = (L_i\sigma_i)(L_j\sigma_j) = \frac{1}{2}(L_iL_j\sigma_i\sigma_j + \underbrace{L_jL_i}_{= L_iL_j - [L_i, L_j]}\sigma_j\sigma_i)$$

$$= \frac{1}{2}\left(L_iL_j \overbrace{\{\sigma_i,\sigma_j\}}^{2\delta_{ij}} - i\varepsilon_{ijk}L_k\sigma_j\sigma_i\right)$$

$$= L^2 + \frac{i}{4}\varepsilon_{ijk}L_k\sigma_i\sigma_j - \frac{i}{4}(\varepsilon_{ijk}L_k\sigma_j\sigma_i)$$

$$= L^2 + \frac{i}{4}\varepsilon_{ijk}L_k(\sigma_i\sigma_j - \sigma_j\sigma_i)$$

$$= L^2 + \frac{i}{4}\varepsilon_{ijk}L_k[\sigma_i, \sigma_j]$$

$$= \vec{L}^2 - 2\vec{L}\cdot\vec{S}$$

Or $(\vec{L}\cdot\vec{\sigma})^2 = L_i\sigma_iL_j\sigma_j$

$$= L_iL_j(\mathbb{1}\delta_{ij} + i\epsilon_{ijk}\sigma_k)$$

$$= \vec{L}^2 + i\underbrace{\epsilon_{ijk}L_iL_j\sigma_k}$$

$$= \frac{i}{2}(\epsilon_{ijk}L_iL_j - \epsilon_{ijk}L_jL_i)\sigma_j$$

$$= \frac{i}{2}\epsilon_{ijk}\cdot i\epsilon_{ijm}L_m\sigma_j$$

$$= i\delta_{km}L_m\sigma_j$$

$$= i L_k\sigma_j = 2iL_kS_i$$

$$(\vec{L}\cdot\vec{S})(\vec{\alpha}\cdot\vec{r})\cdot\frac{1}{r} + \frac{1}{r}(\vec{\alpha}\cdot\vec{r})(\vec{L}\cdot\vec{S})$$

$$= \frac{1}{2}\begin{pmatrix} 0 & \frac{\sigma_i\sigma_jL_ir_j}{r} + \sigma_j\sigma_i\frac{r_j}{r}L_i \\ \sigma_i\sigma_j\frac{L_ir_j}{r} + \sigma_j\sigma_i\frac{r_j}{r}L_i & 0 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 0 & -2\frac{\vec{\sigma}\cdot\vec{r}}{r} \\ -\frac{2\vec{\sigma}\cdot\vec{r}}{r} & 0 \end{pmatrix}$$

$$(1+2\vec{L}\cdot\vec{S})\frac{i\gamma}{r}(\vec{\alpha}\cdot\vec{r}) + \frac{i\gamma}{r}(\vec{\alpha}\cdot\vec{r})(1+2\vec{L}\cdot\vec{S}) = 2\frac{i\gamma}{r}(\vec{\alpha}\cdot\vec{r}) - 2\frac{i\gamma}{r}(\vec{\alpha}\cdot\vec{r})$$

$$= 0$$

c) $\left[-\frac{1}{r}\partial_r^2 r, \Lambda\right]$

$$= (1+2\vec{L}\cdot\vec{S})\frac{1}{r}\partial_r^2 r + \frac{i\gamma}{r}(\vec{\alpha}\cdot\vec{r})\frac{1}{r}\partial_r^2 r - \frac{1}{r}\partial_r^2 r(1+2\vec{L}\cdot\vec{S}) - \frac{1}{r}\partial_r^2 r\frac{i\gamma}{r}(\vec{\alpha}\cdot\vec{r})$$

$$= \frac{i\gamma}{r}\left[(\vec{\alpha}\cdot\vec{r})\frac{1}{r}\frac{\partial^2}{\partial r^2}r - \frac{\partial^2}{\partial r^2}r\left(\frac{1}{r}(\vec{\alpha}\cdot\vec{r})\right)\right]$$

$$= \frac{i\gamma}{r}\left((\vec{\alpha}\cdot\vec{r})\frac{1}{r}\partial_r^2 r - \frac{1}{r}(\vec{\alpha}\cdot\vec{r})\partial_r^2 r - \left(\frac{\partial^2}{\partial r^2}r - \frac{1}{r}\vec{\alpha}\cdot\vec{r}\right)\right)$$

b) $$\Lambda(\Lambda+1) = 1+\vec{L}^2 + 2\vec{L}\cdot\vec{S} - \gamma^2 - (1+2\vec{L}\cdot\vec{S}) - \frac{i\gamma}{r}(\vec{\alpha}\cdot\vec{r})$$
$$= \vec{L}^2 - \gamma^2 - \frac{i\gamma}{r}\vec{\alpha}\cdot\vec{r}$$

$$\frac{1}{r^2}\vec{L}^2 - (E+\frac{\gamma}{r})^2 - \frac{i\gamma}{r^3}(\vec{\alpha}\cdot\vec{r})$$
$$= \frac{1}{r^2}\left\{\vec{L}^2 - \gamma^2 - \frac{i\gamma}{r}\vec{\alpha}\cdot\vec{r}\right\} - E^2 - 2E\frac{\gamma}{r}$$
$$= \frac{1}{r^2}\Lambda(\Lambda+1) - E^2 - 2E\frac{\gamma}{r}$$

$\Rightarrow$ (2)

c) $$\Lambda = -(1+2\vec{L}\cdot\vec{S}) - \frac{i\gamma}{r}(\vec{\alpha}\cdot\vec{r}) \quad \propto r \quad \longrightarrow \partial_r^2 r = 0$$

$\Rightarrow$ Λ and ∂_r^2 commute

$\Rightarrow$ Λ and the whole terms in brackets commute

d) $\quad \Lambda^2 \psi = \lambda^2 \psi$

$$\Rightarrow \quad \left(1 - \gamma^2 + \vec{L}^2 + 2\vec{L}\cdot\vec{S}\right)\psi = \lambda^2 \psi$$

$$\left(1 - \gamma^2 + \cancel{\vec{L}^2} + \vec{J}^2 - \cancel{\vec{L}^2} - \vec{S}^2\right)\psi = \lambda^2\psi$$

$$\Rightarrow \lambda^2 = 1 - \gamma^2 + J(J+1) - S(S+1) = -\gamma^2 + j(j+1) + \frac{1}{4}$$
$$= -\gamma^2 + (j+\tfrac{1}{2})^2 \quad , \quad \lambda = \pm|\lambda|$$

$$\Lambda(\Lambda+1)\psi = \Lambda^2\psi + \Lambda\psi = \lambda^2\psi \pm |\lambda|\psi$$

$$= |\lambda|(|\lambda| \pm 1)\psi$$

$$= \begin{cases} |\lambda|(|\lambda|+1)\psi \, , & \lambda > 0 \\ |\lambda|(|\lambda|-1)\psi & \lambda < 0 \end{cases}$$

$$\Rightarrow \text{define } \check{\ell} = \begin{cases} |\lambda| & , \lambda > 0 \\ |\lambda| - 1 & , \lambda < 0 \end{cases}$$

$$\Rightarrow \quad \Lambda(\Lambda+1)\psi = \check{\ell}(\check{\ell}+1)\psi = \begin{cases} |\lambda|(|\lambda|+1)\psi & , \lambda > 0 \\ (|\lambda|-1)|\lambda|\psi & , \lambda < 0 \end{cases}$$

$$\Rightarrow (2) \longrightarrow \left(-\frac{1}{r^2}\partial_r^2 r + \frac{\check{\ell}(\check{\ell}+1)}{r^2} - \frac{2E\gamma}{r} - E^2 + m^2\right)\psi = 0$$

e) non-relativistic:

$$\left(-\frac{1}{r}\frac{\partial^2}{\partial r^2}r+\frac{l(l+1)}{r^2}-\frac{2m\gamma}{r}-2mE\right)\psi=0$$

$$\Rightarrow \left\{\frac{1}{2m}\left[-\frac{1}{r}\frac{\partial}{\partial r^2}r+\frac{l(l+1)}{r^2}\right]-\frac{\gamma}{r}\right\}\psi=E\psi$$

with energy: $E=-E_0\left(\frac{Z}{n}\right)^2=-\frac{1}{2}mc^2\alpha^2\left(\frac{Z}{n}\right)^2=-\frac{1}{2}m\left(\frac{\gamma}{n}\right)^2$

with $n=k_0+l+1$

relativistic:

$$\left(-\frac{1}{r}\frac{\partial^2}{\partial r^2}r+\frac{\tilde{l}(\tilde{l}+1)}{r^2}-\frac{2E\gamma}{r}-E^2+m^2\right)\psi=0$$

$$\left\{\frac{1}{2E}\left[\left(-\frac{1}{r}\partial_r^2 r+\frac{\tilde{l}(\tilde{l}+1)}{r^2}\right]-\frac{\gamma}{r}\right\}\psi=\frac{E^2-m^2}{2E}\psi$$

$m\to E$, $l\to\tilde{l}$, $n\to\tilde{n}=k_0+\tilde{l}+1=n+\tilde{l}-l$

$$=\begin{cases}n+|\lambda|-l, & \lambda>0\\ n+|\lambda|-1-l, & \lambda<0\end{cases}$$

somehow get neglected

Don't have to

$n=N+1+j\pm\frac{1}{2}$

$$\frac{E^2-m^2}{2E}=-\frac{1}{2}E\left(\frac{\gamma}{\tilde{n}}\right)^2,$$

$$E^2-m^2=-E^2\frac{\gamma^2}{\tilde{n}^2}$$

$$\Rightarrow E^2\left(1+\frac{\gamma^2}{\tilde{n}^2}\right)=m^2$$

$$\Rightarrow E_{nj}=m\frac{1}{\sqrt{1+\gamma^2\left(n+\underbrace{\sqrt{-\gamma^2+(j+\frac{1}{2})^2}}_{|\lambda|}-(j+\frac{1}{2})\right)^{-2}}}$$

e)

$$\left(-\frac{1}{r}\partial_r^2 r + \frac{\ell(\ell+1)}{r^2} - 2\frac{\overbrace{m\gamma}^{a}}{r} \overbrace{-2mE}^{b}\right)\psi = 0$$

$$\left(-\frac{1}{r}\partial_r^2 r + \frac{\hat{\ell}(\hat{\ell}+1)}{r^2} - 2\frac{m\gamma}{r} - E^2 + m^2\right)\psi = 0$$

f) $\gamma = Z\alpha << 1$

$$\frac{\partial|\lambda|}{\partial\gamma} = \frac{\partial}{\partial\gamma}\sqrt{-\gamma^2+(j+\tfrac{1}{2})^2} = \frac{1}{2}\sqrt{-\gamma^2+(j+\tfrac{1}{2})}^{-1}\cdot -2\gamma = -\gamma/|\lambda|$$

$$\frac{\partial\hat{n}}{\partial\gamma} = \frac{\partial}{\partial\gamma}\left(n-j-\tfrac{1}{2}+|\lambda|\right) = -\frac{\gamma}{|\lambda|}$$

$$\frac{\partial E_{nj}}{\partial\gamma} = \frac{\partial}{\partial\gamma}\frac{m}{\sqrt{1+\gamma^2\hat{n}^{-2}}} = -\frac{m}{2}\left(1+\gamma^2\hat{n}^{-2}\right)^{-3/2}\cdot\left\{2\gamma\hat{n}^{-2}-2\gamma^2\hat{n}^{-3}\cdot\frac{-\gamma}{|\lambda|}\right\}$$

$$= -\frac{m}{2}\left(1+\gamma^2\hat{n}^{-2}\right)^{-3/2}2\gamma\hat{n}^{-2}\left(1+2\gamma^2/\hat{n}|\lambda|\right)$$

$$\left.\frac{\partial E_{nj}}{\partial\gamma}\right|_{\gamma=0} = 0$$

$$\frac{\partial^2 E_{nj}}{\partial\gamma^2} = +\frac{3m}{4}\left(1+\gamma^2\hat{n}^{-2}\right)^{-5/2}\left[2\gamma\hat{n}^{-2}\left(1+2\gamma^2/\hat{n}|\lambda|\right)\right]^2$$

$$-\frac{m}{2}\left(1+\gamma^2\hat{n}^{-2}\right)^{-3/2}\Big\{2\hat{n}^{-2}\left(1+2\gamma^2/\hat{n}|\lambda|\right)-4\gamma\hat{n}^{-3}\cdot\frac{-\gamma}{|\lambda|}\left(1+2\gamma^2/\hat{n}|\lambda|\right)$$

$$+2\gamma\hat{n}^{-2}\left[4\gamma/\hat{n}|\lambda|-2\gamma^2/\hat{n}^2|\lambda|\cdot\frac{-\gamma}{|\lambda|}-2\gamma^2/\hat{n}|\lambda|^2\,\frac{-\gamma}{|\lambda|}\right]\Big\}$$

$$\left.\frac{\partial^2}{\partial\gamma^2}E_{nj}\right|_{\gamma=0} = -\frac{m}{2}\cdot 2\hat{n}^{-2} = -\frac{m}{\hat{n}^2} = -\frac{m}{n^2}$$

$$\Rightarrow\quad E_{nj} = m\left(1-\frac{\gamma^2}{2n^2}+\mathcal{O}(\gamma^4)\right)$$

H9.2

$$[AB, C] = A[B, C] + [A, C]B$$

$$[A, BC] = [A, B]C + B[A, C]$$

a)

$$[\hat{a}_i, \hat{a}_j] = 0, \quad [\hat{a}_i^+, \hat{a}_j^+] = 0$$

$$[\hat{a}_i \hat{a}_j, \hat{a}_k] = \hat{a}_i [\hat{a}_j, \hat{a}_k] + [\hat{a}_i, \hat{a}_k]\hat{a}_j = 0$$

$$[\hat{a}_i^+ \hat{a}_j, \hat{a}_k] = 0$$

$$[\hat{a}_i \hat{a}_j, \hat{a}_k^+] = \hat{a}_i [\hat{a}_j, \hat{a}_k^+] + [\hat{a}_i, \hat{a}_k^+]\hat{a}_j$$

$$= \hat{a}_i \delta_{jk} + \delta_{ik} \hat{a}_j$$

$$[\hat{a}_i^+ \hat{a}_j^+, \hat{a}_k] = \hat{a}_i^+ \underbrace{[\hat{a}_j^+, \hat{a}_k]}_{-\delta_{jk}} + \underbrace{[\hat{a}_i^+, \hat{a}_k]}_{-\delta_{ik}} \hat{a}_j^+$$

$$= -\hat{a}_i^+ \delta_{jk} + \delta_{ik} \hat{a}_j^+$$

b)

$$\hat{H} = \sum_{i,j} t_{ij} \hat{a}_i^+ \hat{a}_j + \frac{1}{2} \sum_{i,j,k,l} V_{ijkl} \hat{a}_i^+ \hat{a}_j^+ \hat{a}_k \hat{a}_l$$

$$\left[\sum_{i,j} t_{ij} \hat{a}_i^+ \hat{a}_j, \sum_i \hat{a}_i^+ \hat{a}_i\right]$$

$$= \sum_{i,j,k} t_{ij} [\hat{a}_i^+ \hat{a}_j, \hat{a}_k^+ \hat{a}_k] = \sum t_{ij} \left([\hat{a}_i^+ \hat{a}_j, \hat{a}_k^+]\hat{a}_k + \hat{a}_k^+ [\hat{a}_i^+ \hat{a}_j, \hat{a}_k] \right)$$

$$= \sum_{i,j,k} t_{ij} \left\{ \left(\hat{a}_i^+ [\hat{a}_j, \hat{a}_k^+] + [\hat{a}_i^+, \hat{a}_k^+]\hat{a}_j \right) \hat{a}_k + \hat{a}_k^+ \left(\hat{a}_i^+ [\hat{a}_j, \hat{a}_k] + \underbrace{[\hat{a}_i^+, \hat{a}_k]}\hat{a}_j \right) \right\}$$

$$= \sum_{i,j,k} t_{ij} \left(\hat{a}_i^+ \delta_{jk} \hat{a}_k - \hat{a}_k^+ \delta_{ik} \hat{a}_j \right)$$

$$= \sum_{i,j} t_{ij} \left(\hat{a}_i^+ \hat{a}_j - \hat{a}_i^+ \hat{a}_j \right) = 0$$

$$[\hat{N}, \sum V_{ikjl}\, a_i^\dagger a_j^\dagger a_\alpha a_l]$$

$$= \sum V\, [a_k^\dagger a_k, a_i^\dagger a_j^\dagger a_\alpha a_l]$$

$$= \sum V \{ a_k^\dagger [a_k, a_i^\dagger a_j^\dagger a_\alpha a_l] + [a_k^\dagger, a_i^\dagger a_j^\dagger a_\alpha a_l]\, a_k \}$$

$$= \sum V \{ a_k^\dagger [a_k, a_i^\dagger a_j^\dagger] a_\alpha a_l + a_i^\dagger a_j^\dagger [a_k^\dagger, a_\alpha a_l]\, a_k \}$$

H9.3

a) $\hat{T} = \sum \hat{t} = \sum_{i,j} t_{ij} \sum_{\alpha} |i\rangle_\alpha \langle j|_\alpha$ acting on an arbitary state

$\hat{T} C |n_1, n_2, \dots \rangle = \sum_{i,j} t_{ij} \sum_{\alpha} |i\rangle_\alpha \langle j|_\alpha \, |i_1\rangle_1 \, |i_2\rangle_2 \cdots |i_N\rangle_N$ for every particls

normalisation

$= C \sum_{i,j} t_{ij} \sum_{\alpha} |i_1\rangle_1 |i_2\rangle_2 \cdots |i\rangle_\alpha |i_N\rangle_N \, \delta_{j i_\alpha}$

$\langle j|_\alpha{}_\alpha |i_\alpha\rangle = \delta_{j i_\alpha}$

in the process a particle with state $j = i_\alpha$ is replaced by state i, i.e. $\hat{a}_i^+ \hat{a}_j$

comes from $\sum_\alpha \delta_{j i_\alpha}$, acts on n_j - particles

$= \sum t_{ij} n_j \tilde{C} | n_1, \dots, n_i + 1, \dots, n_j - 1, \dots \rangle$

($C = \frac{1}{\sqrt{n_1! \, n_2! \cdots}}$, since the occupation number of i, j states have changed the C has to change to keep the wave function normalised

$= \sum t_{ij} n_j \sqrt{n_i + 1} \frac{1}{\sqrt{n_j}} C | n_1, \dots, n_i + 1, \dots n_j - 1, \dots \rangle$

$= \sum_{i,j} t_{ij} \hat{a}_i^+ \hat{a}_j \, | i_1, i_2, \dots, i_N \rangle$ — the original ket vektor(s)

$\Rightarrow \hat{T} = \sum_{i,j} t_{ij} \hat{a}_i^+ \hat{a}_j$

(2) $\hat{F} = \frac{1}{2} \sum_{\alpha \neq \beta} \hat{f}^{(2)}(x_\alpha, x_\beta) = \frac{1}{2} \sum_{\alpha,\beta,\gamma,\delta} \hat{f}^{(2)}_{\alpha\beta\gamma\delta} \, |\alpha\rangle |\beta\rangle \langle\gamma| \langle\delta|$

$= \frac{1}{2} \sum_{\alpha,\beta,\gamma,\delta} \hat{f}^{(2)}_{\alpha\beta\gamma\delta} \, \hat{a}_\alpha^+ \hat{a}_\beta^+ \hat{a}_\gamma \hat{a}_\delta$

$F = \frac{1}{2} \sum_{\alpha \neq \beta} f^{(2)}(x_\alpha, x_\beta) = \frac{1}{2} \sum_{\substack{\alpha \neq \beta \\ i,j,k,l}} |i\rangle_\alpha |j\rangle_\beta \langle ij | f^{(2)} | kl \rangle \langle l|_\beta \langle l|_\alpha$

$= \frac{1}{2} \sum_{i,k,j,l} f^{(2)}_{ijkl} \, \hat{a}_i^+ \hat{a}_j^+ a_k a_l \sum_{\alpha \neq \beta} |0\rangle_\alpha |0\rangle_\beta \langle 0|_\beta \langle 0|_\alpha$