C6.1

1) $$x^\mu \longrightarrow x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

$$x_\mu = g_{\mu\nu} x^\nu \longrightarrow g^{\mu\nu} x_\mu = x^\nu$$

to show $$x'_\mu = x_\nu (\Lambda^{-1})^\nu{}_\mu$$

$$\Leftrightarrow \quad x'_\mu \Lambda^\mu{}_\nu = x_\nu$$

$$LHS = g_{\mu\alpha} x'^\alpha \Lambda^\mu{}_\nu$$

$$= \Lambda^\alpha{}_\beta x^\beta g_{\mu\alpha} \Lambda^\mu{}_\nu$$

$$= x^\beta \Lambda^\alpha{}_\beta g_{\mu\alpha} \Lambda^\mu{}_\nu$$

$$\overset{\text{def}}{=} x^\beta g_{\beta\nu} = x_\nu = RHS$$

2) $$x^\mu y_\mu \longrightarrow x'^\mu y'_\mu = x'^\mu g_{\mu\nu} y'^\nu$$

$$= \underbrace{\Lambda^\mu{}_\alpha x^\alpha g_{\mu\nu} \Lambda^\nu{}_\beta}_{\hat{=}\ \Lambda^T g \Lambda = g} y^\beta$$

$$= x^\alpha g_{\alpha\beta} y^\beta$$

$$= x^\alpha y_\alpha$$

C6.2

a) $$\{\gamma^\mu, \gamma^\nu\} = 2 g^{\mu\nu} \mathbb{1}_4 \qquad \gamma_5 := i\gamma^0\gamma^1\gamma^2\gamma^3$$

i) $$\gamma_\mu \gamma^\mu = g_{\mu\nu} \gamma^\nu \gamma^\mu$$

$$= \frac{1}{2}(g_{\mu\nu} + g_{\nu\mu}) \gamma^\nu \gamma^\mu$$

$$= \frac{1}{2}(g_{\mu\nu} \gamma^\nu \gamma^\mu + g_{\nu\mu} \gamma^\nu \gamma^\mu)$$

$$= \frac{1}{2}(g_{\mu\nu} \gamma^\nu \gamma^\mu + g_{\mu\nu} \gamma^\mu \gamma^\nu)$$

$$= \frac{1}{2} g_{\mu\nu} \{\gamma^\mu, \gamma^\nu\} = \frac{1}{2} g_{\mu\nu} \cdot 2 g^{\mu\nu} \mathbb{1}$$

$$= g_{\mu\nu} g^{\mu\nu} \mathbb{1}_4 = 4 \mathbb{1}_4$$

ii) $\gamma_\mu \gamma^\nu \gamma^\mu = \gamma_\mu \gamma^\nu \gamma^\mu = \gamma_\mu \{\{\gamma^\mu, \gamma^\nu\} - \gamma^\mu \gamma^\nu\}$

$$\overset{i)}{=} \gamma_\mu \cdot 2 g^{\mu\nu} \mathbb{1}_4 - \gamma_\mu \gamma^\mu \gamma^\nu$$

$$= 2\gamma^\nu - 4\gamma^\nu = -2\gamma^\nu$$

iii) $\gamma_\mu \gamma^\nu \gamma^\lambda \gamma^\rho \gamma^\mu$

$$= \gamma_\mu \gamma^\nu \gamma^\lambda (\{\gamma^\rho, \gamma^\mu\} - \gamma^\mu \gamma^\rho)$$

$$= \gamma_\mu \gamma^\nu \gamma^\lambda (2 g^{\rho\mu} \mathbb{1}_4 - \gamma^\mu \gamma^\rho)$$

$$= 2\gamma^\rho \gamma^\nu \gamma^\lambda \underbrace{- \gamma_\mu \gamma^\nu \gamma^\lambda \gamma^\mu \gamma^\rho}$$

$$= -\gamma_\mu \gamma^\nu (\{\gamma^\lambda, \gamma^\mu\} - \gamma^\mu \gamma^\lambda) \gamma^\rho$$

$$= -\gamma_\mu \gamma^\nu (2 g^{\lambda\mu} \mathbb{1}_4 - \gamma^\mu \gamma^\lambda) \gamma^\rho$$

$$= -2\gamma^\lambda \gamma^\nu \gamma^\rho + \underbrace{\gamma_\mu \gamma^\nu \gamma^\mu \gamma^\lambda \gamma^\rho}_{= -2\gamma^\nu \gamma^\lambda \gamma^\rho}$$

$$= 2\gamma^\rho \gamma^\nu \gamma^\lambda - 2\gamma^\lambda \gamma^\nu \gamma^\rho - 2\gamma^\nu \gamma^\lambda \gamma^\rho$$

$$= 2\gamma^\rho \gamma^\nu \gamma^\lambda - 2(\gamma^\lambda \gamma^\nu + \gamma^\nu \gamma^\lambda)\gamma^\rho$$

$$= 2\gamma^\rho \gamma^\nu \gamma^\lambda - 2 \cdot 2 g^{\lambda\nu} \mathbb{1}_4 \gamma^\rho$$

$$= \gamma^\rho (2\gamma^\nu \gamma^\lambda - 4 g^{\lambda\nu} \mathbb{1}_4)$$

$$= \gamma^\rho (2\gamma^\nu \gamma^\lambda - 2\{\gamma^\lambda, \gamma^\nu\})$$

$$= \gamma^\rho (\cancel{2\gamma^\nu \gamma^\lambda} - 2\gamma^\lambda \gamma^\nu - \cancel{2\gamma^\nu \gamma^\lambda})$$

$$= -2\gamma^\rho \gamma^\lambda \gamma^\nu$$

v) $\mathrm{Tr}(\gamma^\mu \gamma^\nu) = \mathrm{Tr}(-\gamma^\nu \gamma^\mu + 2 g^{\mu\nu} \mathbb{1}_4)$

$$= -\mathrm{Tr}(\gamma^\mu \gamma^\nu) + 2 \cdot g^{\mu\nu} \mathrm{Tr}(\mathbb{1}_4)$$

$$\Rightarrow \mathrm{Tr}(\gamma^\mu \gamma^\nu) = 4 g^{\mu\nu}$$

vi) $\mathrm{Tr}(\not{p} \not{k}) = \mathrm{Tr}(\gamma_\mu p^\mu \gamma_\nu k^\nu) = \mathrm{Tr}(\gamma^\mu \gamma^\nu p_\mu k_\nu)$

$$= p_\mu k_\nu 4 g^{\mu\nu} = 4(k \cdot p)$$

vii) $\{\gamma_5, \gamma^\mu\} = i\{\gamma^0\gamma^1\gamma^2\gamma^3, \gamma^\mu\}$

viii) $Tr(\gamma_5) = Tr(\gamma^0\gamma^0\gamma^5) \overset{vii)}{=} -Tr(\gamma^0\gamma^5\gamma^0) = -Tr(\gamma^0\gamma^0\gamma^5)$

$\Rightarrow Tr(\gamma^0\gamma^0\gamma^5) = Tr(\gamma^5) = 0$

ix) $$(\gamma_5)^2 = -\underline{\gamma^0\gamma^1\gamma^2\gamma^3}\,\underline{\gamma^0}\gamma^1\gamma^2\gamma^3$$
$$= -\gamma^0\gamma^0\gamma^1\gamma^2\gamma^3\gamma^1\gamma^2\gamma^3$$
$$= -\gamma^1\gamma^2\gamma^3\gamma^1\gamma^2\gamma^3$$
$$= -(-\mathbb{1})^3 = \mathbb{1}_4$$

x) $Tr(\gamma^\mu\gamma^\nu \dots \gamma^\lambda) =$