

C6.1

$$1) \quad x^\mu \longrightarrow x'^\mu = \Lambda^\mu_\nu x^\nu$$

$$x_\mu = g_{\mu\nu} x^\nu \longrightarrow g^{\mu\nu} x_\mu = x^\nu$$

to show  $x'_\mu = x_\nu (\Lambda^{-1})^\nu_\mu$

$$\Leftrightarrow x'_\mu \Lambda^\mu_\nu = x_\nu$$

$$\begin{aligned} \text{LHS} &= g_{\mu\alpha} x'^\alpha \Lambda^\mu_\nu \\ &= \Lambda^\alpha_\beta x^\beta g_{\mu\alpha} \Lambda^\mu_\nu \\ &= x^\beta \Lambda^\alpha_\beta g_{\mu\alpha} \Lambda^\mu_\nu \\ &\stackrel{\text{def}}{=} x^\beta g_{\beta\nu} = x_\nu = \text{RHS} \end{aligned}$$

$$2) \quad x^\mu y_\mu \longrightarrow x'^\mu y'_\mu = x'^\mu g_{\mu\nu} y'^\nu$$

$$= \underbrace{\Lambda^\mu_\alpha x^\alpha g_{\mu\nu} \Lambda^\nu_\beta y^\beta}_{\hat{=} \Lambda^\tau g \Lambda = g}$$

$$= x^\alpha g_{\alpha\beta} y^\beta$$

$$= x^\alpha y_\alpha$$

C6.2

$$a) \quad \{ \gamma^\mu, \gamma^\nu \} = 2 g^{\mu\nu} \mathbb{1}_4 \quad \gamma_5 := i \gamma^0 \gamma^1 \gamma^2 \gamma^3$$

$$\begin{aligned} i) \quad \gamma_\mu \gamma^\mu &= g_{\mu\nu} \gamma^\nu \gamma^\mu \\ &= \frac{1}{2} (g_{\mu\nu} + g_{\nu\mu}) \gamma^\nu \gamma^\mu \\ &= \frac{1}{2} (g_{\mu\nu} \gamma^\nu \gamma^\mu + g_{\nu\mu} \gamma^\nu \gamma^\mu) \\ &= \frac{1}{2} (g_{\mu\nu} \gamma^\nu \gamma^\mu + g_{\mu\nu} \gamma^\mu \gamma^\nu) \\ &= \frac{1}{2} g_{\mu\nu} \{ \gamma^\mu, \gamma^\nu \} = \frac{1}{2} g_{\mu\nu} \cdot 2 g^{\mu\nu} \mathbb{1} \end{aligned}$$

$$= g_{\mu\nu} g^{\mu\nu} \mathbb{1}_4 = 4 \mathbb{1}_4$$

$$\begin{aligned} \text{ii)} \quad \gamma_\mu \gamma^\nu \gamma^\mu &= \gamma_\mu \gamma^\nu \gamma^\mu = \gamma_\mu \{ \gamma^\mu, \gamma^\nu \} - \gamma^\mu \gamma^\nu \gamma^\mu \\ &\stackrel{\text{ii)}}{=} \gamma_\mu \cdot 2 g^{\mu\nu} \mathbb{1}_4 - \gamma^\mu \gamma^\nu \gamma^\mu \\ &= 2 \gamma^\nu - 4 \gamma^\nu = -2 \gamma^\nu \end{aligned}$$

$$\begin{aligned} \text{iii)} \quad \gamma_\mu \gamma^\nu \gamma^\lambda \gamma^\rho \gamma^\mu &= \gamma_\mu \gamma^\nu \gamma^\lambda ( \{ \gamma^\rho, \gamma^\mu \} - \gamma^\mu \gamma^\rho ) \\ &= \gamma_\mu \gamma^\nu \gamma^\lambda ( 2 g^{\rho\mu} \mathbb{1}_4 - \gamma^\mu \gamma^\rho ) \\ &= 2 \gamma^\rho \gamma^\nu \gamma^\lambda - \underbrace{\gamma_\mu \gamma^\nu \gamma^\lambda \gamma^\mu \gamma^\rho}_{= -\gamma_\mu \gamma^\nu ( \{ \gamma^\lambda, \gamma^\mu \} - \gamma^\mu \gamma^\lambda ) \gamma^\rho} \\ &= -\gamma_\mu \gamma^\nu ( 2 g^{\lambda\mu} \mathbb{1}_4 - \gamma^\mu \gamma^\lambda ) \gamma^\rho \\ &= -2 \gamma^\lambda \gamma^\nu \gamma^\rho + \underbrace{\gamma_\mu \gamma^\nu \gamma^\mu \gamma^\lambda \gamma^\rho}_{= -2 \gamma^\nu \gamma^\lambda \gamma^\rho} \end{aligned}$$

$$\begin{aligned} &= 2 \gamma^\rho \gamma^\nu \gamma^\lambda - 2 \gamma^\lambda \gamma^\nu \gamma^\rho - 2 \gamma^\nu \gamma^\lambda \gamma^\rho \\ &= 2 \gamma^\rho \gamma^\nu \gamma^\lambda - 2 ( \gamma^\lambda \gamma^\nu + \gamma^\nu \gamma^\lambda ) \gamma^\rho \\ &= 2 \gamma^\rho \gamma^\nu \gamma^\lambda - 2 \cdot 2 g^{\lambda\nu} \mathbb{1}_4 \gamma^\rho \\ &= \gamma^\rho ( 2 \gamma^\nu \gamma^\lambda - 4 g^{\lambda\nu} \mathbb{1}_4 ) \\ &= \gamma^\rho ( 2 \gamma^\nu \gamma^\lambda - 2 \{ \gamma^\lambda, \gamma^\nu \} ) \\ &= \gamma^\rho ( \cancel{2 \gamma^\nu \gamma^\lambda} - 2 \gamma^\lambda \gamma^\nu - \cancel{2 \gamma^\nu \gamma^\lambda} ) \\ &= -2 \gamma^\rho \gamma^\lambda \gamma^\nu \end{aligned}$$

$$\begin{aligned} \text{v)} \quad \text{Tr} ( \gamma^\mu \gamma^\nu ) &= \text{Tr} ( -\gamma^\nu \gamma^\mu + 2 g^{\mu\nu} \mathbb{1}_4 ) \\ &= -\text{Tr} ( \gamma^\mu \gamma^\nu ) + 2 \cdot g^{\mu\nu} \text{Tr} ( \mathbb{1}_4 ) \end{aligned}$$

$$\Rightarrow \text{Tr} ( \gamma^\mu \gamma^\nu ) = 4 g^{\mu\nu}$$

$$\begin{aligned} \text{vi)} \quad \text{Tr} ( \not{\epsilon} \not{k} ) &= \text{Tr} ( \gamma_\mu \epsilon^\mu \gamma_\nu k^\nu ) = \text{Tr} ( \gamma^\mu \gamma^\nu \epsilon_\mu k_\nu ) \\ &= \epsilon_\mu k_\nu 4 g^{\mu\nu} = 4 (k \cdot \epsilon) \end{aligned}$$

$$\text{vii)} \quad \{\gamma_5, \gamma^\mu\} = i\{\gamma^0\gamma^1\gamma^2\gamma^3, \gamma^\mu\}$$

$$\begin{aligned} \text{viii)} \quad \text{Tr}(\gamma_5) &= \text{Tr}(\gamma^0\gamma^0\gamma^5) \stackrel{\text{vii)}}{=} -\text{Tr}(\gamma^0\gamma^5\gamma^0) = -\text{Tr}(\gamma^0\gamma^0\gamma^5) \\ &\Rightarrow \text{Tr}(\gamma^0\gamma^0\gamma^5) = \text{Tr}(\gamma^5) = 0 \end{aligned}$$

$$\begin{aligned} \text{ix)} \quad (\gamma_5)^2 &= -\underbrace{\gamma^0\gamma^1\gamma^2\gamma^3}_{\gamma^0\gamma^0\gamma^1\gamma^2\gamma^3}\underbrace{\gamma^0\gamma^1\gamma^2\gamma^3}_{\gamma^1\gamma^2\gamma^3} \\ &= -\gamma^0\gamma^0\gamma^1\gamma^2\gamma^3\gamma^1\gamma^2\gamma^3 \\ &= -\gamma^1\gamma^2\gamma^3\gamma^1\gamma^2\gamma^3 \\ &= -(-1)^3 = \mathbb{1}_4 \end{aligned}$$

$$\text{x)} \quad \text{Tr}(\gamma^\mu\gamma^\nu\cdots\gamma^\lambda) =$$