H1.1

Schrödinger equation: $E\psi_{klm}(\vec{r}) = -\frac{\hbar^2}{2m}\nabla^2 \psi_{klm}(\vec{r})$

with $\psi_{klm}(\vec{r}) = R_{kl}(r)\, Y_{lm}(\theta,\varphi)$

a) $\Delta = \nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$

$$\Rightarrow -\frac{\hbar^2}{2m}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right) R_{kl}(r)\, Y_{lm}(\theta,\varphi)$$

$$= E\, R_{kl}(r)\, Y_{lm}(\theta,\varphi)$$

[using $\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) = \sin\theta\frac{\partial^2}{\partial\theta^2} + \cos\theta\frac{\partial}{\partial\theta}$]

$$-\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)R_{kl}(r)\,\cancel{Y_{lm}(\theta,\varphi)} - \frac{l(l+1)}{r^2}R_{kl}(r)\,\cancel{Y_{lm}(\theta,\varphi)}\right] = E\,R_{kl}(r)\,\cancel{Y_{lm}(\theta,\varphi)}$$

$$\Rightarrow \quad E R_{kl} = -\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{\partial}{\partial r} r^2\frac{\partial}{\partial r}R_{kl} + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}R_{kl}, \quad R_{kl} = R_{kl}(r)$$

[using $k = \sqrt{2mE}/\hbar$]

$$k^2 R_{kl} = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}R_{kl}\right) + \frac{l(l+1)}{r^2}R_{kl}$$

[using $-\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 R'_{kl}) = -\frac{1}{r^2}(2r R'_{kl} + r^2 R''_{kl})$]

$$\Rightarrow \quad k^2 R_{kl} = -\frac{1}{r^2}(2r R'_{kl} + r^2 R''_{kl}) + \frac{l(l+1)}{r^2}R_{kl}$$

$$R''_{kl} + \frac{2}{r}R'_{kl} + \left(k^2 - \frac{l(l+1)}{r^2}\right)R_{kl} = 0$$

b) $l = 0$

$$\int d^3r\ \psi^*_{k'om'}\psi_{kom} = \delta_{mm'}\,\delta\left(\frac{k'-k}{2m}\right)$$

The spherical harmonics are already orthogonal

$$\Rightarrow \int_0^\infty dr\, r^2 R^*_{k'0} R_{k0} = \delta\left(\frac{k'-k}{2m}\right)$$

Make the ansatz $R_{k0} = f(kr)/r$ and substitute in the equation from a):

$$\frac{\partial^2}{\partial r^2}\left(\frac{f(kr)}{r}\right) + \frac{2}{r}\frac{\partial}{\partial r}\left(\frac{f(kr)}{r}\right) + \left(k^2 - \cancel{\frac{l(l+1)}{r^2}}_{\,l=0}\right)\left(\frac{f(kr)}{r}\right) = 0$$

$$\frac{\partial}{\partial r}\left(-\frac{f(kr)}{r^2}+\frac{f'(kr)}{r}\right)=\frac{2f(kr)}{r^3}-\frac{f'(kr)}{r^2}-\frac{f'(kr)}{r^2}+\frac{f''(kr)}{r}$$

$$=\frac{2f(kr)}{r^3}-\frac{2f'(kr)}{r^2}+\frac{f''(kr)}{r}$$

$$\frac{2}{r}\frac{\partial}{\partial r}\left(\frac{f(kr)}{r}\right)=\frac{2}{r}\left(-\frac{f(kr)}{r^2}+\frac{f'(kr)}{r}\right)=-\frac{2f(kr)}{r^3}+\frac{2f'(kr)}{r^2}$$

$$\Rightarrow \quad \frac{f''(kr)}{r}+k^2\frac{f(kr)}{r}=0$$

$$f''(kr)+k^2f(kr)=0 \quad\Rightarrow\quad f(kr)=Ae^{+ikr}+Be^{-ikr}$$

in order to satisfy the orthonality condition

$$f(kr)=Be^{-ikr}$$

$$\int_0^\infty dr\, r^2 R^*_{k'0}R^*_{k0}=\int_0^\infty dr\,|B|^2 e^{+ik'r}e^{-ikr}\overset{!}{=}\delta\left(\frac{k'-k}{2\pi}\right)$$

$$=2\pi\,\delta(k'-k)$$

$$\Rightarrow \quad |B|^2\cdot\frac{1}{2}\cdot\cancel{2\pi\delta(k'-k)}=\cancel{2\pi\delta(k'-k)}$$

$$\Rightarrow \quad |B|=\sqrt{2}$$

$$\Rightarrow \quad R_{k0}=\sqrt{2}\,e^{-ikr}/r$$

c) $R_{kl}=r^l\chi_{kl}$ into the equation from a)

$$\frac{\partial^2}{\partial r^2}(r^l\chi_{kl})+\frac{2}{r}\frac{\partial}{\partial r}(r^l\chi_{kl})+\left(k^2-\frac{l(l+1)}{r}\right)r^l\chi_{kl}=0$$

$$\frac{\partial}{\partial r}(lr^{l-1}\chi_{kl}+r^l\chi'_{kl})+\frac{2}{r}(lr^{l-1}\chi_{kl}+r^l\chi'_{kl})+\left(k^2-\frac{l(l+1)}{r}\right)r^l\chi_{kl}=0$$

$$\Rightarrow \quad l(l-1)r^{l-2}\chi_{kl}+2lr^{l-1}\chi'_{kl}+r^l\chi''_{kl}+2lr^{l-2}\chi_{kl}+2r^{l-1}\chi'_{kl}$$

$$+k^2r^l\chi_{kl}-l(l+1)r^{l-2}\chi_{kl}=0$$

$$\Rightarrow \quad r^l\chi''_{kl}+(2r^{l-1}+2lr^{l-1})\chi'_{kl}+\left[\cancel{l(l-1)r^{l-2}+2lr^{l-2}}+k^2r^l-\cancel{l(l+1)r^{l-2}}\right]\chi_{kl}=0$$

$$\Rightarrow \quad \chi_{kl}'' + \frac{2(l+1)}{r}\chi_{kl}' + k^2\chi_{kl} = 0$$

d) using $\chi_{kl}' = r\chi_{k,l+1}$

$$\frac{\partial}{\partial r}(r\chi_{k,l+1}) + \frac{2(l+1)}{r} r\chi_{k,l+1} + k^2\chi_{kl} = 0$$

$$\chi_{k,l+1} + r^2\chi_{k,l+2} + 2(l+1)\chi_{k,l+1} + k^2\chi_{kl} = 0$$

$$\chi_{k,l+2} + \frac{2l+3}{r^2}\chi_{k,l+1} + \frac{k^2}{r^2}\chi_{kl} = 0$$

d) $\chi_{kl}' = r\chi_{k,l+1}$; $\chi_{kl}'' = \chi_{k,l+1} + r\chi_{k,l+1}'$ replacing.

c) : $\chi_{kl}'' + \frac{2(l+1)}{r}\chi_{kl}' + k^2\chi_{kl} = 0$

$$\chi_{k,l+1} + r\chi_{k,l+1}' + \frac{2(l+1)}{\cancel{r}}\cancel{r}\chi_{k,l+1} + k^2\chi_{kl} = 0 \quad \Big| \frac{d}{dr}$$

$$\chi_{k,l+1}' + \chi_{k,l+1}' + r\chi_{k,l+1}'' + 2(l+1)\chi_{k,l+1}' + k^2 r\chi_{k,l+1} = 0$$

$$\chi_{k,l+1}'' + \frac{2(l+1+1)}{r}\chi_{k,l+1}' + k^2\chi_{k,l+1}$$

e) $R_{kl}|_{l=0} = 2\frac{\sin(kr)}{r} = Re(2\, e^{ikr}/r)$ ✓

$$\int_0^\infty dr\, r^2 |R_{kl}|^2$$

$$= \int_0^\infty dr\, r^2 |2k\, j_l(kr)|^2 = 4k^2\int_0^\infty dr\, r^2 j_l^2(kr)$$

$$= 4k^2\int_0^\infty dr\, r^2 j_l^2(kr) = 4k^2 \cdot \frac{\pi}{2k^2} = 2\pi$$

f) $R_{kl}^{\pm} = (-1)^l\, 2\frac{r^l}{k^l}\left(\frac{d}{rdr}\right)^l \frac{\pm \sin(kr)}{r}$

e) $R_{kl} = (-1)^l \frac{2r^l}{k^l}\left(\frac{d}{rdr}\right)^l \frac{\sin(kr)}{r}$

$R_{kl} = r^l\chi_{kl}$ $\qquad \chi_{kl} = \left(\frac{1}{r}\frac{d}{dr}\right)^l \chi_{k0} = 2\left(\frac{1}{r}\frac{d}{dr}\right)^l \frac{\sin(kr)}{r}$

$\Rightarrow R_{kl}R_{kl}^* = (-1)^{2l}$

H1.2

a) If a fluid is flowing in some area, then the rate at which fluid flows out of a certain region within that area can be calculated by adding up the sources inside the region and subtracting the sinks.

$$\underbrace{\int_V (\vec{\nabla} \cdot \vec{A})\, dV}_{\text{strength of the sink (flow)}} = \underbrace{\int_S \vec{A} \cdot \vec{n}\, dS}_{\text{flux of the fluid}}$$

b) $$\text{RHS} = \int_S dS\, \phi\, (\vec{\nabla} \psi \cdot \hat{n}) = \int_V dV\, \vec{\nabla} (\phi \cdot \vec{\nabla} \psi)$$
$$= \int_V dV\, \vec{\nabla}\phi \cdot \vec{\nabla}\psi + \phi \cdot \vec{\nabla}^2 \psi$$
$$= \text{LHS}$$

c) $$\text{RHS} = \int_S dS \left(\phi \frac{\partial \psi}{\partial \vec{n}} - \psi \frac{\partial \phi}{\partial \vec{n}}\right)$$
$$= \int_S dS \left(\phi\, \vec{\nabla}\psi \cdot \hat{n} - \psi\, \vec{\nabla}\phi \cdot \hat{n}\right)$$
$$= \int_V dV\, \vec{\nabla}\left(\phi\, \vec{\nabla}\psi - \psi\, \vec{\nabla}\phi\right) = \int_V dV \left(\phi\, \Delta\psi - \psi\, \Delta\phi\right)$$
$$= \text{LHS}$$

d)

$$\int_V dV\, \Delta \frac{1}{r} = \int_S dS\, \vec{\nabla}\frac{1}{r} \cdot \hat{n},$$
$$= -\int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} d\varphi \cdot r^2 \frac{\vec{r}}{|\vec{r}|^3} \cdot \hat{n}$$
$$= -4\pi$$

$V = S^2$ with a as radius, 0 as centre

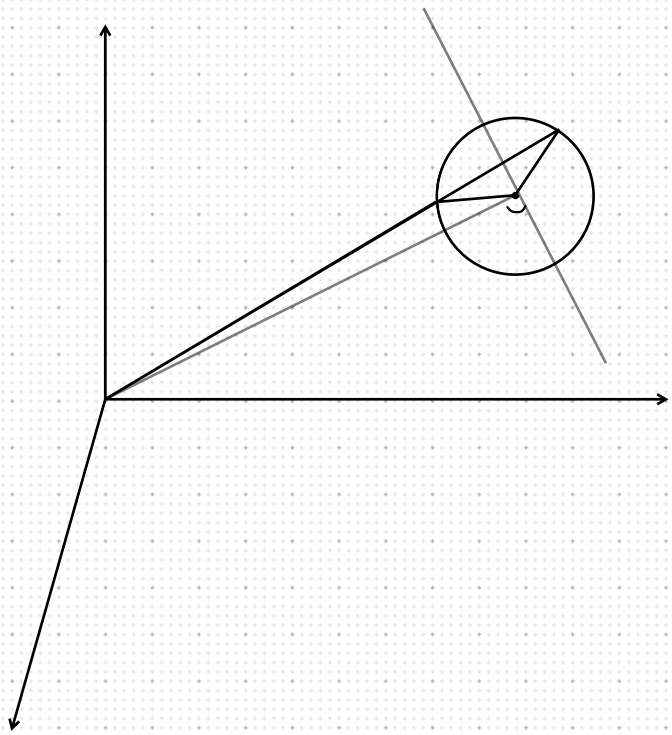
But if the centre of $V = S^2$ is not the origin, and in the limit $a \to 0$;

$$\int_V dV\, \Delta \frac{1}{r} = -\int d\Omega\, r^2 \frac{\vec{r}}{|\vec{r}|^3} \cdot \hat{n}, \quad \hat{n} = \frac{1}{a}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= 0,$$

with $(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = a^2$

For we can decompose the spherical volume into two parts, $\hat{e}_r \cdot \hat{n}$ of one part always corresponds the same scalar product of the other part with a negative sign. In the end the terms cancel each other out.



$$\Rightarrow \quad \int_V dV \, \Delta \frac{1}{r} = -4\pi \quad \text{oder} \quad 0$$

$$\Rightarrow \quad \Delta \frac{1}{r} = -4\pi \, \delta^{(3)}(\vec{r})$$

e) $\Psi = G(\vec{x}, \vec{x}')$, $\phi = \Phi(\vec{x}')$ in Green's 2nd

$$\int_V dV \left(G(\vec{x}, \vec{x}') \, \Delta \Phi(\vec{x}') - \Phi(\vec{x}') \, \Delta G(\vec{x}, \vec{x}') \right)$$

$$= \oint_S dS \left(G(\vec{x}, \vec{x}') \frac{\partial \Phi(\vec{x}')}{\partial \hat{n}} - \Phi(\vec{x}') \frac{\partial G(\vec{x}', \vec{x})}{\partial \hat{n}} \right)$$

use: $\Delta G(\vec{x}, \vec{x}') = -4\pi \, \delta^{(3)}(\vec{x} - \vec{x}')$ $\quad \Delta \Phi(\vec{x}') = -\rho(\vec{x}')/\varepsilon_0$

$$\Rightarrow \int_V dV' \left[-\frac{1}{4\pi |\vec{x}' - \vec{x}|} \cdot \left(-\frac{\rho(\vec{x}')}{\varepsilon_0} \right) - \Phi(\vec{x}') \cdot \delta^{(3)}(\vec{x} - \vec{x}') \right]$$

$$= 0$$

if $\vec{x}$ lie out side of S

$$\Rightarrow \frac{1}{\varepsilon_0}\int_V dV' \frac{\rho(\vec{x}')}{4\pi|\vec{x}'-\vec{x}|} = \Phi(\vec{x})$$

$$\Rightarrow \Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int_V dV' \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}$$