C12.1 Normal order of operator $:O(\hat{A},\hat{B}):$, $\hat{A}$ appears before $\hat{B}$.
(excludi addition ??) Yes !

a)

$$\exp\left(-i\frac{\epsilon}{\hbar}\hat{H}\right) = 1 + \left(-i\frac{\epsilon}{\hbar}\right)\hat{H} + \sum_{n=2}^{\infty}\frac{\left(-i\frac{\epsilon}{\hbar}\right)^n}{n!}\hat{H}^n$$

$$:\exp\left(-i\frac{\epsilon}{\hbar}\hat{H}\right): = 1 + \left(-i\frac{\epsilon}{\hbar}\right):\hat{H}: + \sum_{n=2}^{\infty}\frac{\left(-i\frac{\epsilon}{\hbar}\right)^n}{n!}:\hat{H}^n:$$

$$\hat{H}^n \neq :\hat{H}^n: \quad \text{except } n=1$$

$$\hat{H} = :\hat{H}: \quad ???$$

b) $\exp(-i\alpha\hat{H}) \simeq :\exp(-i\alpha\hat{H}):$

if we don't care about $O(\hat{H}^2)$

c) $\hat{H}_V(\hat{p},\hat{x}) = \frac{\hat{p}^2}{2m} + V(\hat{x}) = :\hat{H}_V(\hat{p},\hat{x}):$

$$\hat{H}_A(\hat{p},\hat{x}) = \frac{1}{2m}\left(\hat{p} - \frac{e}{c}\hat{A}(\hat{x})\right)^2$$
$$= \frac{1}{2m}\left[\hat{p}^2 - \frac{e}{c}\hat{A}(\hat{x})\hat{p} - \frac{e}{c}\hat{p}\hat{A}(\hat{x}) + \frac{e^2}{c^2}\hat{A}^2(\hat{x})\right]$$

$$:\hat{H}_A(\hat{p},\hat{x}): = \frac{1}{2m}\left[\hat{p}^2 - \frac{e}{c}\hat{p}\hat{A}(\hat{x}) + \frac{e^2}{c^2}\hat{A}^2(\hat{x})\right]$$

d) $$\hat{H}_V^2(\hat{p},\hat{x}) = \left(\frac{\hat{p}^2}{2m} + V(\hat{x})\right)^2 = \frac{\hat{p}^4}{4m^2} + \frac{\hat{p}^2}{2m}V(\hat{x}) + V(\hat{x})\frac{\hat{p}^2}{2m} + V^2(\hat{x})$$

$$:\hat{H}_V^2(\hat{p},\hat{x}): = \frac{\hat{p}^4}{4m^2} + 2\frac{\hat{p}^2}{2m}V(\hat{x}) + V^2(\hat{x})$$

$$\Rightarrow \quad \hat{H}_V^2 - :\hat{H}_V^2: = V(\hat{x})\frac{\hat{p}^2}{2m} - \frac{\hat{p}^2}{2m}V(\hat{x})$$

$$= -2\frac{\hat{p}}{2m}V(\hat{x})\hat{p}\cdot - \frac{\hat{p}^2}{2m}V(\hat{x})\cdot$$

$$= -\frac{2\hbar}{2mi}V'(\hat{x})\hat{p} + \frac{\hbar^2}{2m}V''$$

$$\Rightarrow \text{correction} = -\left(\frac{\epsilon}{\hbar}\right)^2 \cdot \left(+2\frac{\hbar^2}{2m}V'(\hat{x})\partial_x + \frac{\hbar^2}{2m}V''\right)$$

$$= -\frac{\epsilon^2}{2m}\left(2V'(\hat{x})\partial_x + V''\right)$$