12.11.2018

fundamental principles:

1) The system is described by $|\psi\rangle$, which is an element in a linear space (Hilbert)
2) Observable are represented by <u>hermitian</u> operators
3) Expectation values of an observable $\hat{A}$ in the state $|\psi\rangle$ are given by $\langle\psi|\hat{A}|\psi\rangle$
4) Time dependence of the state is given by $\hat{H}|\psi\rangle = i\hbar\partial_t|\psi\rangle$
5) The eigenvalues a_n of the operator $\hat{A}$ are the possible – values of $\hat{A}$. In the state $|n\rangle$ with $\hat{A}|n\rangle = a_n|n\rangle$, $\hat{A}$ takes on the value $a_n = \langle n|\hat{A}|n\rangle / \langle n|n\rangle$

- Notation of SRT, differential of the eigentime
- Klein-Gordon equation, continuity equation

Problem
  - choose $\psi = e^{i\varphi}$ $\qquad \partial_t\psi = e^{i\theta}$

$$\Rightarrow \rho = \frac{i\hbar}{2mc^2}\left(e^{i(\theta-\varphi)} - e^{-i(\theta-\varphi)}\right) = -\frac{\hbar}{2mc^2}\sin(\theta-\varphi)$$

↳ negative density!

  - Spin
  - Probability (is not ψ^2)

- Dirac equation

19.11.2018

- non-relativistic limit of Dirac equation
  - $\vec{p}=0$. rest particle
  - em-field coupling

$$\vec{p} \longrightarrow \vec{p} - \frac{e}{c}\vec{A} \qquad \text{minimal substitution}$$

still classical fields!

$$i\hbar\partial_t\psi = (c\vec{\alpha}\cdot\vec{p} + \beta mc^2)\psi \longrightarrow i\hbar\partial_t\psi = (c\vec{\alpha}\underbrace{(\vec{p}-\tfrac{e}{c}\vec{A})}_{\vec{\pi}} + \beta mc^2 + e\phi)\psi$$

$$\psi = \begin{pmatrix}\bar{\varphi}\\ \bar{\chi}\end{pmatrix} \qquad \bar{\varphi} = \begin{pmatrix}\bar{\varphi}_1\\ \bar{\varphi}_2\end{pmatrix} \qquad \bar{\chi} = \begin{pmatrix}\bar{\chi}_1\\ \bar{\chi}_2\end{pmatrix}$$

$$\Rightarrow i\hbar\partial_t \begin{pmatrix}\bar\varphi\\ \bar\chi\end{pmatrix} = c\begin{pmatrix}(\vec\sigma\cdot\vec\pi)\,\bar\chi\\ (\vec\sigma\cdot\vec\pi)\,\bar\varphi\end{pmatrix} + e\phi\begin{pmatrix}\bar\varphi\\ \bar\chi\end{pmatrix} + mc^2\begin{pmatrix}\bar\varphi\\ -\bar\chi\end{pmatrix}$$

$$\begin{pmatrix}\bar\varphi\\ \bar\chi\end{pmatrix} = \exp\left(-i\frac{mc^2}{\hbar}t\right)\begin{pmatrix}\varphi\\ \chi\end{pmatrix}$$ pull out the most important time scale!

$$\rightarrow i\hbar\partial_t\begin{pmatrix}\varphi\\ \chi\end{pmatrix} = c\begin{pmatrix}\vec\sigma\cdot\vec\pi\ \chi\\ \vec\sigma\cdot\vec\pi\ \varphi\end{pmatrix} + e\phi\begin{pmatrix}\varphi\\ \chi\end{pmatrix} - 2mc^2\begin{pmatrix}0\\ \chi\end{pmatrix}$$ ← from time dependence $e^{-i\frac{mc^2}{\hbar}t}$

$$i\hbar\partial_t\left(e^{-i\frac{mc^2}{\hbar}t}\begin{pmatrix}\varphi\\ \chi\end{pmatrix}\right) = i\hbar\left(-i\frac{mc^2}{\hbar}\right)e^{\sim}\begin{pmatrix}\varphi\\ \chi\end{pmatrix} + i\hbar\left(\partial_t\begin{pmatrix}\varphi\\ \chi\end{pmatrix}\right)e^{\sim}$$

2. equation: $$i\hbar\partial_t\chi = c\,\vec\sigma\cdot\vec\pi\,\varphi + e\phi\chi - 2mc^2\chi$$

Assume $|i\hbar\partial_t\chi| \ll |2mc^2\chi|$

$$\Rightarrow \chi \simeq \frac{\vec\sigma\cdot\vec\pi}{2mc}\varphi,\quad |\vec\pi| \sim mv,\quad \chi \simeq (v/c)\,\varphi$$

1. equation becomes:

$$i\hbar\partial_t\varphi = c(\vec\sigma\cdot\vec\pi)\frac{\vec\sigma\cdot\vec\pi}{2mc}\varphi + e\phi\varphi$$
$$= \frac{1}{2m}(\vec\sigma\cdot\vec\pi)(\vec\sigma\cdot\vec\pi)\varphi + e\phi\varphi$$

use the identity: $(\vec\sigma\cdot\vec a)(\vec\sigma\cdot\vec b) = (\vec a\cdot\vec b) + i\vec\sigma\cdot(\vec a\times\vec b)$

$$(\vec\sigma\cdot\vec\pi)(\vec\sigma\cdot\vec\pi) = \vec\pi^2 + i\vec\sigma(\vec\pi\times\vec\pi)$$ ← $\hat\pi$ an operator

$$(\vec\pi\times\vec\pi) = (\vec p - \tfrac{e}{c}\vec A)\times(\vec p - \tfrac{e}{c}\vec A) = -\frac{e}{c}(\vec A\times\vec p + \vec p\times\vec A)$$
$$\vec\nabla\times(\vec A\varphi) = \underbrace{(\vec\nabla\times\vec A)}_{\vec B}\varphi - \vec A\times(\nabla\varphi)$$

$$\Rightarrow (\vec\sigma\cdot\vec\pi)(\vec\sigma\cdot\vec\pi) = \vec\pi^2 - \frac{\hbar e}{c}\vec\sigma\cdot\vec B$$

<u>Pauli equation:</u>

$$i\hbar\partial_t\varphi = \left[\frac{1}{2m}(\vec p - \tfrac{e}{c}\vec A)^2 - \frac{e\hbar}{2mc}\vec\sigma\cdot\vec B + e\phi\right]\varphi$$

in homogeneous field $\vec A = \frac{1}{2}\vec B\times\vec r \Rightarrow \vec\nabla\times\vec A = \vec B$ $\quad \vec S = \frac{1}{2}\hbar\vec\sigma$

$\pm\hbar/2$ eigenvalues

$$(\vec p - \tfrac{e}{c}\vec A)^2 = \vec p^2 - \frac{e}{c}(\vec A\cdot\vec p + \vec p\cdot\vec A) + \frac{e^2}{c^2}\vec A^2$$
$$(\vec p\cdot\vec A)\varphi = \left(\frac{\hbar}{i}\vec\nabla\cdot\vec A\right)\varphi = \vec A\cdot\vec p\,\varphi \quad , \quad \vec\nabla\cdot\vec A = 0$$

$$\vec{p}\cdot\vec{A} + \vec{A}\cdot\vec{p} = 2\vec{A}\vec{p} = (\vec{B}\times\vec{r})\vec{p} = \vec{B}\cdot\underbrace{(\vec{r}\times\vec{p})}_{\text{Orbital angular momentum}}$$

$$\Rightarrow \quad \boxed{\left(\frac{1}{2m}\vec{p}^{\,2} - \frac{e}{2mc}(\vec{L}+2\vec{S})\cdot\vec{B} + \frac{e^2}{2mc^2}\vec{A}^{\,2} + e\phi\right)\varphi = i\hbar\partial_t\varphi}$$

Landé factor

in analogy with classical one

$$\hat{H} = \frac{1}{2m}\vec{p}^{\,2} - \vec{\mu}\cdot\vec{B} + \frac{e^2}{2mc}\vec{A} + e\phi \,.$$ $\vec{\mu}$: magnetic moment

$$\vec{\mu} = \vec{\mu}_{orbital} + \vec{\mu}_{spin} = \frac{e}{2mc}(\vec{L}+2\vec{S})$$

Minimal substituition:

$p_\mu = i\hbar\partial_\mu$ momentum operator

coupling to the em field $A_\mu = (\phi, \vec{A})$

$p_\mu \longrightarrow p_\mu - \frac{e}{c}A_\mu \quad \Rightarrow \quad (\not{p} - \frac{e}{c}\not{A} - mc)\psi = 0$

Dirac in the em-field $(\gamma^\mu(i\hbar\partial_\mu - \frac{e}{c}A_\mu) - mc)\psi = 0$

21.11.2018 QM and EM

$$\hat{\psi}(\vec{x},t) = e^{-i\chi(\vec{x},t)}\psi(\vec{x},t) \qquad e^{-i\chi(\vec{x},t)} \quad \text{Gauge groupe } U(1)$$

$$\vec{p}\,\hat{\psi}(\vec{r},t) = \frac{\hbar}{i}\vec{\nabla}(e^{-i\chi(\vec{r},t)}\psi) = e^{-i\chi(\vec{r},t)}\left(\frac{\hbar}{i}\vec{\nabla} - \hbar(\vec{\nabla}\chi)\right)\psi(\vec{r},t)$$

$$\vec{p}^{\,2}/2m\,\hat{\psi}(\vec{r},t) = e^{-\chi(\vec{r},t)}\frac{1}{2m}\left(\frac{\hbar}{i}\vec{\nabla} - \hbar\vec{\nabla}\chi\right)^2\psi$$

$$\partial_t\hat{\psi}(\vec{r},t) = i\hbar e^{-i\chi(\vec{r},t)}\partial_t\psi(\vec{r},t) + \hbar e^{-i\chi(\vec{r},t)}(\partial_t\chi(\vec{r},t))\cdot\psi(\vec{r},t)$$

$\Rightarrow$ SE: of free particle

$$\frac{1}{2m}(\vec{p} - \hbar\vec{\nabla}\chi)^2 - \hbar(\nabla\chi(\vec{r},t))\psi = i\hbar\partial_t\psi$$

$$H = \frac{1}{2m}(\vec{p} - \frac{e}{c}\vec{A})^2 + e\phi \qquad \vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\partial_t A$$

$$\vec{B} = \vec{\nabla}\times\vec{A}$$

can identify : $\frac{e}{c}\vec{A} = \hbar\vec{\nabla}\chi \qquad e\phi = -\hbar\partial_t\chi$

$\Rightarrow$ Fields : $\vec{B} = \vec{\nabla} \times \vec{A} = \frac{e}{c} (\vec{\nabla} \times \vec{\nabla} \chi) = 0$

$$\vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\partial_t \vec{A} = \frac{\hbar}{e} (\vec{\nabla} \partial_t \chi - \frac{1}{c} \partial_t \vec{\nabla} c(\chi)) = 0$$

$\Rightarrow$ Free particle $\Rightarrow$ no fields

The requirement of local gauge invariance leads automatically to the introduction of em-potential $U(1)$. $SU(3)$

$$(\vec{p} - \frac{e}{c}\vec{A})^2 = \vec{p}^2 - \frac{e}{c}(\vec{p}\cdot\vec{A} + \vec{A}\cdot\vec{p}) + \frac{e^2}{c^2}\vec{A}^2$$

In Heisenberg Picture:

Equation of motion: $\partial_t \hat{X}_i = \frac{i}{\hbar}[\hat{H}, \hat{X}_i] = \frac{1}{m}(p_i - \frac{e}{c}A_i)$

Kinetic momentum: $\vec{\pi} = \vec{p} - e/c \vec{A}$, $[\pi_i, \pi_j] = i\frac{\hbar e}{c}\varepsilon_{ijk}B_k$

$$\hat{H} = \frac{\vec{\pi}^2}{2m} + e\phi$$

From the 2nd derivative we obtain:

$$m\partial_t^2 \vec{r} = e[\vec{E} + \frac{1}{2c}(\partial_t\vec{r}\times\vec{B} - \vec{B}\times\partial_t\vec{r})]$$

26.11.2018

System of particles $(n > 2)$ is in general not Lorentz invariante!

($n = 2 \rightarrow$ CMS, Lab frame)

Solutions of Dirac equation:

$$(c\vec{\alpha}\cdot\vec{p} + \beta mc^2)\psi = i\hbar\partial_t\psi$$

$$\psi = e^{i(\vec{p}\cdot\vec{r} - E_p t)/\hbar} U$$

$$\begin{bmatrix} mc^2 & 0 & cp_z & c(p_x - ip_y) \\ 0 & mc^2 & c(p_x+ip_y) & -cp_z \\ cp_z & c(p_x - ip_y) & -mc^2 & 0 \\ c(p_x+ip_y) & -cp_z & 0 & -mc^2 \end{bmatrix} U = E_p U \Rightarrow \begin{bmatrix} mc^2 & c\vec{\sigma}\cdot\vec{p} \\ c\vec{\sigma}\cdot\vec{p} & -mc^2 \end{bmatrix} U = E_p U$$

$$\det\begin{bmatrix}(mc^2 - E_p) & c\vec{\sigma}\cdot\vec{p} \\ c\vec{\sigma}\cdot\vec{p} & -(mc^2+E_p)\end{bmatrix} = 0 \Rightarrow \text{dispersion } E_p$$

$$(\vec{\sigma}\cdot\vec{p})(\vec{\sigma}\cdot\vec{p}) = \vec{p}^{\,2}$$

$$\begin{bmatrix}a & b\\ c & d\end{bmatrix}^{-1} = \begin{bmatrix}d & -b\\ -c & a\end{bmatrix}\frac{1}{ad-bc}$$

Ansatz: $$\begin{bmatrix}(mc^2+E_p) & c\vec{\sigma}\cdot\vec{p} \\ c\vec{\sigma}\cdot\vec{p} & -(mc^2-E_p)\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}mc^2-E_p & c\vec{\sigma}\cdot\vec{p} \\ c\vec{\sigma}\cdot\vec{p} & -(mc^2+E_p)\end{bmatrix}\begin{bmatrix}mc^2+E_p & c\vec{\sigma}\cdot\vec{p} \\ c\vec{\sigma}\cdot\vec{p} & -(mc^2-E_p)\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}m^2c^4 - E_p^2 + c^2p^2 & 0 \\ 0 & m^2c^4 - E_p^2 + c^2p^2\end{bmatrix} \leftarrow \mathbb{1}\cdot\frac{1}{\det}$$

$$\Rightarrow E_p = \pm\sqrt{m^2c^4 + c^2p^2}$$

$$UU^+ = \begin{bmatrix}mc^2+E_p & c\vec{\sigma}\cdot\vec{p} \\ c\vec{\sigma}\cdot\vec{p} & -(mc^2-E_p)\end{bmatrix}\begin{bmatrix}mc^2+E_p & c\vec{\sigma}\cdot\vec{p} \\ c\vec{\sigma}\cdot\vec{p} & -(mc^2-E_p)\end{bmatrix}$$

$$= \begin{bmatrix}(mc^2+E_p)^2 + c\vec{p}^{\,2} & 2E_p mc^2\, c\vec{\sigma}\cdot\vec{p} \\ 2E_p m c^2 c\vec{\sigma}\cdot\vec{p} & (mc^2-E_p)^2 + c^2p^2\end{bmatrix}$$

$$\Rightarrow c\vec{\sigma}\cdot\vec{p}\begin{pmatrix}u_1\\u_2\end{pmatrix} = (E_p + mc^2)\begin{pmatrix}u_3\\u_4\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}u_3\\u_4\end{pmatrix} = \frac{c\vec{\sigma}\cdot\vec{p}}{E_p+mc^2}\begin{pmatrix}u_1\\u_2\end{pmatrix}$$

$$= \frac{c\begin{pmatrix}p_z & p_x - ip_y\\ p_x+ip_y & -p_z\end{pmatrix}}{E_p + mc^2}\begin{pmatrix}u_1\\u_2\end{pmatrix}$$

$$|u_1|^2 + |u_2|^2 + |u_3|^2 + |u_4|^2 = 1 + \frac{c}{(E_p^2+mc^2)^2}(p_x^2+p_y^2+p_z^2)$$

$$= \frac{E_p^2 + 2E_pmc^2 + \overbrace{m^2c^4 + c^2p^2}^{E}}{(E_p+mc^2)^2}$$

$$= \frac{2E_p(E_p+mc^2)}{(E_p+mc^2)^2} = \frac{2E_p}{E_p+mc^2}$$

Normalization of the wave function?

S transforms the wave function ψ

In the rest frame: $E_p = mc^2$ $\quad E_p/mc^2 = 1$

The normalised wave function is given by:

$$U^{(+)}_{p\uparrow} = \sqrt{\frac{E_p}{mc^2}}\sqrt{\frac{E_p+mc^2}{2E_p}} \begin{pmatrix} 1 \\ 0 \\ cp_z/(E_p+mc^2) \\ c(p_x+ip_y)/(E_p+mc^2) \end{pmatrix}$$

$$U^{(+)}_{p\downarrow} = \sqrt{\frac{E_p+mc^2}{2mc^2}} \begin{pmatrix} 0 \\ 1 \\ c(p_x-ip_y)/(E_p+mc^2) \\ -cp_z/(E_p+mc^2) \end{pmatrix}$$

negative energies:

$$U^{(-)}_{p\uparrow} = \sqrt{\frac{E_p+mc^2}{mc^2}} \begin{pmatrix} p_z c/(E_p+mc^2) \\ c(p_x+ip_y)/(E_p+mc^2) \\ 1 \\ 0 \end{pmatrix}$$

$$U^{(-)}_{p\downarrow} = \sqrt{\frac{E_p+mc^2}{mc^2}} \begin{pmatrix} c(p_x-ip_y)/(E_p+mc^2) \\ -cp_z \\ 0 \\ 1 \end{pmatrix}$$

Dirac equation: covariance

Lorentz-transformation a: $(x^\nu)' = a^\nu{}_\mu x^\mu$

The distance is invariant under any LT

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$$

$$\Rightarrow a_\mu{}^\nu a^\mu{}_\lambda = \delta^\nu{}_\lambda$$

$\det(a) = +1$ $\quad$ proper

$ -1$ $\quad \rightarrow$ mirror / Inversion

$$(i\hbar \gamma^\mu \partial_{x^\mu} - mc)\,\psi(x) = 0$$

$(i\hbar \gamma^\mu \partial_{x^{\mu\prime}} - mc)\,\psi'(x') = 0$ need to be valid

$$\psi'(x') = S(a)\,\psi(x) \qquad \psi(x) = S(a^{-1})\,\psi'(x')$$

$\Rightarrow S^{-1}(a) = S(a^{-1})$. linear transformation

$$\partial_{x^\mu} = \frac{\partial x^{\nu\prime}}{\partial x^\mu}\,\partial_{x'^\nu} = a^\nu{}_\mu\,\partial_{x'^\nu}$$

Plug in:

$$(i\hbar \gamma^\mu a^\nu{}_\mu \partial_{x^{\nu\prime}} - mc)\,S(a^{-1})\,\psi'(x') = 0$$

$$\Rightarrow (i\hbar\, S(a)\gamma^\mu S^{-1}(a)\, a^\nu{}_\mu \partial_{x^{\nu\prime}} - mc)\,\psi'(x') = 0$$

$\Rightarrow$ Dirac equation invariant if $\boxed{S(a)\,\gamma^\mu S(a^{-1})\,a^\nu{}_\mu = \gamma^\nu}$

Using an infinitesimal Lorentz transformation

$$a^\nu{}_\mu = g^\nu{}_\mu + \underbrace{\Delta\omega^\nu{}_\mu}_{\text{small}} \qquad \Delta\omega^{\nu\mu} = -\Delta\omega^{\mu\nu}$$

$$a_\mu{}^\nu a^\mu{}_\sigma = \delta^\nu_\sigma$$

$\Rightarrow$ 6 independent components of $\Delta\omega$

Expand S in terms of $\Delta\omega$

$$S = \mathbb{1} - \frac{i}{4}\sigma_{\mu\nu}\Delta\omega^{\mu\nu} + \mathcal{O}(\Delta^2)$$

$$\sigma_{\mu\nu} = -\sigma_{\nu\mu}$$

$$\Delta\omega^\nu{}_\mu \gamma^\mu = -\frac{i}{4}\Delta\omega^{\alpha\beta}(\gamma^\nu\sigma_{\alpha\beta} - \sigma_{\alpha\beta}\gamma^\nu)$$

Task: Find 6 matrices $\sigma_{\mu\nu}$ which obey the equation

$\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$ has the desired property

The transformation is the given by

$$S = \mathbb{1} + \frac{1}{8}[\gamma^\mu, \gamma^\nu]\,\Delta\omega^{\mu\nu} + \mathcal{O}(\Delta^2)$$

Example: Take a system which is propagating with a small velocity in the x-direction

$$\Delta \omega^{\nu}{}_{\mu} = \Delta \omega \, I^{\nu}{}_{\mu} \qquad I = \begin{bmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & 0 & \\ & & & 0 \end{bmatrix} \quad I^2 = \begin{bmatrix} 1 & 0 & & \\ 0 & 1 & & \\ & & 0 & \\ & & & 0 \end{bmatrix}$$

$$I^3 = I$$

Finite transformation:

$$x^{\nu'} = \lim_{N \to \infty} \left(g + \frac{\omega}{N} I \right)^N x^{\mu} = e^{\omega I} x^{\mu}$$

$$\Delta \omega = \omega / N$$

calculate $e^{\omega I}$:

$$(1 - I^2 + I^2 \operatorname{ch}\omega + I \operatorname{sh}\omega)^{\nu}{}_{\mu}$$

$$\Rightarrow \begin{bmatrix} x^{0'} \\ x^{1'} \\ x^{2'} \\ x^{3'} \end{bmatrix} = \begin{bmatrix} \operatorname{ch}\omega & -\operatorname{sh}\omega & 0 & \\ -\operatorname{sh}\omega & \operatorname{ch}\omega & & \\ & & 1 & 0 \\ 0 & & 0 & 1 \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix}$$

$$\operatorname{th}(\omega) = v/c$$

$$\Rightarrow x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}} \qquad t' = \frac{t - v/c^2 \, x}{\sqrt{1 - v^2/c^2}}$$

The transformation of Spinor has the form

$$\psi'(x') = \exp\left(-\frac{i}{4} \omega \sigma_{\mu\nu} I^{\mu\nu}\right) \psi(x)$$

This transformation leaves the continuity equation invariant!

$$\rho = \psi^{+} \psi$$

$$j^{k} = c c^{+} \alpha_k \psi \qquad \Rightarrow j^{\mu}(x) = c \psi^{+}(x) \gamma^0 \gamma^{\mu} \psi(x)$$

$$\Rightarrow \partial_\mu \psi^{\mu}(x) = 0 \qquad \text{perserved}$$

The transformed current

$$j^{\mu'}(x') = c \psi^{+}(x') \gamma^0 \gamma^{\mu} \psi'(x')$$

$$= c \psi^{+}(x) S^{+} \gamma^0 \gamma^{\mu} S \psi(x) \qquad S^{-1} = \gamma^0 S^{+} \gamma^0$$

$$= c \psi^{+}(x) \gamma^0 S^{-1} \gamma^{\mu} S \psi(x)$$

$$= a^{\mu}{}_{\nu} \, c \psi^{+}(x) \gamma^0 \gamma^{\mu} \psi(x) = a^{\mu}{}_{\nu} \, j^{\nu}(x)$$

28.11.2018

- Parity $(ct, x, y, z) \to (ct, -x, -y, -z)$

$$P^{-1}\gamma^{\nu} P = g^{\nu\nu}\gamma^{\nu}$$

- Radial form of Dirac equation

$$i\hbar\partial_t \psi = (c\vec{\alpha}\cdot\vec{p} + mc^2\beta + V(r))\psi$$

$$\psi(\vec{r},t) = \psi(\vec{r})e^{-iEt/\hbar}$$

Gradient in polar coordinates: $\vec{\nabla} = \hat{r}\partial_r + \frac{1}{r}\hat{\theta}\partial_\theta + \frac{1}{r\sin\theta}\hat{\varphi}\partial_\varphi$

$$\hat{r} = \begin{pmatrix} \cos\varphi\sin\theta \\ \sin\varphi\sin\theta \\ \cos\theta \end{pmatrix}$$

$$\hat{p}_r = \frac{\hbar}{i}\frac{1}{r}\partial_r r = \frac{\hbar}{i}\left(\partial_r + \frac{1}{r}\right) \qquad \alpha_r = \frac{\vec{\alpha}\cdot\vec{r}}{r}$$

$$(\vec{\alpha}\cdot\vec{r})(\vec{\alpha}\cdot\vec{p}) = \begin{pmatrix} 0 & \vec{\sigma}\cdot\vec{r} \\ \vec{\sigma}\cdot\vec{r} & 0 \end{pmatrix}\begin{pmatrix} 0 & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & 0 \end{pmatrix} = \begin{pmatrix} 0 & (\vec{\sigma}\cdot\vec{r})(\vec{\sigma}\cdot\vec{p}) \\ (\vec{\sigma}\cdot\vec{r})(\vec{\sigma}\cdot\vec{p}) & 0 \end{pmatrix}$$

$$(\vec{\sigma}\cdot\vec{r})(\vec{\sigma}\cdot\vec{p}) = \vec{r}\cdot\vec{p} + i\vec{\sigma}(\vec{r}\times\vec{p}) = \vec{r}\cdot\vec{p} + i\vec{\sigma}\cdot\vec{L}$$

$$\underbrace{\frac{\vec{\alpha}\cdot\vec{r}}{r}(\vec{\alpha}\cdot\vec{r})(\vec{\alpha}\cdot\vec{p})}_{=\,\vec{\alpha}\cdot\vec{p}} = \frac{\vec{\alpha}\cdot\vec{r}}{r}\left(\hat{r}\cdot\vec{p} + \frac{i\vec{\sigma}\cdot\vec{L}}{\vec{r}}\right)$$

Hamiltonian does not commute with $\vec{S}^2, \vec{L}^2$

$$\Rightarrow \vec{\alpha}\cdot\vec{p} = \frac{\vec{\alpha}\cdot\vec{r}}{r}\left(rp_r + \frac{i}{r}\left(\hbar + \frac{2}{\hbar}\vec{S}\cdot\vec{L}\right)\right)$$

$$\vec{S} = \frac{\hbar}{2}\vec{\sigma}$$

$$\vec{J} = \vec{L} + \vec{S} \qquad \vec{J}^2 = \vec{L}^2 + \vec{S}^2 + 2\vec{L}\cdot\vec{S}$$

$$\Rightarrow \vec{\alpha}\cdot\vec{p} = \frac{\vec{\alpha}\cdot\vec{r}}{r}\left(rp_r + \frac{i}{\hbar}\left(\hbar + \frac{\vec{J}^2 - \vec{L}^2 - \vec{S}^2}{\hbar}\right)\right)$$

The Hamiltonian commutes with $\vec{J}^2, J_z, \vec{L}^2, \vec{S}^2$

$$Y_{JM}^{(l)}(\hat{\Omega}) = \sum_{m+s=M} \langle lm, \tfrac{1}{2}s | JM\rangle \underbrace{Y_{lm}(\hat{\Omega})}_{|lm\rangle} |\tfrac{1}{2}s\rangle$$

$$\hat{J}^2 Y_{JM}^{(\ell)}(\Omega) = \hbar^2 J(J+1) Y_{JM}^{(\ell)}(\Omega) \qquad \hat{J}_z Y_{JM}^{(\ell)}(\Omega) = \hbar M Y_{JM}^{(\ell)}(\Omega)$$

$$\begin{pmatrix} \psi_+(\vec{r},t) \\ \psi_-(\vec{r},t) \end{pmatrix} \xrightarrow{P} e^{i\varphi} \begin{pmatrix} \psi_+(-\vec{r},t) \\ -\psi_-(-\vec{r},t) \end{pmatrix}$$

Parity on the spherical harmonics: $Y_{JM}^{(\ell)}(\pi-\theta, \varphi+\pi) = (-1)^{\ell} Y_{JM}^{(\ell)}(\theta,\varphi)$

different ℓ for particle and antiparticle

$\rightarrow$ Ansatz:

$$\psi_{JM}^{(\omega)}(r,\theta,\varphi) = \frac{1}{r} \begin{pmatrix} F_{J+\omega/2}(r) \cdot Y_{JM}^{(J+\omega/2)}(\hat{\Omega}) \\ i G_{J-\omega/2}(r) \cdot Y_{JM}^{(J-\omega/2)}(\hat{\Omega}) \end{pmatrix}, \quad \omega = \pm 1$$

$$\Rightarrow \begin{cases} \left(-\partial_r + \dfrac{\omega(J+1/2)}{r}\right) G_{J-\omega/2}(r) = \dfrac{E - mc^2 - V(r)}{c\hbar} F_{J+\omega/2}(r) \\[2ex] \left(\partial_r + \dfrac{\omega(J+1/2)}{r}\right) F_{J+\omega/2}(r) = \dfrac{E + mc^2 - V(r)}{c\hbar} G_{J-\omega/2}(r) \end{cases}$$

3.12.2018

- Dirac equation for hydrogen-like atoms
  - Asymptotic behavior for $r \to 0$ and $r \to \infty$

$$f \sim r^{\lambda} e^{-kr}$$

$$F(\rho) = \rho^{\lambda} e^{-\rho} \sum_{k=0}^{\lambda} f_k \rho^k \qquad G(\rho) = \rho^{\lambda} e^{-\rho} \sum_{k=0}^{\infty} g_k \rho^k$$

$$f_{k+1} = \frac{Z\alpha \nu^2 + 2\nu(\lambda+k) - Z\alpha}{\nu(\tau - \lambda - k + \nu Z\alpha)} f_k \qquad \nu = \sqrt{\frac{mc^2 - E}{mc^2 + E}}$$

$$g_k = \frac{Z\alpha - \nu(\tau + \lambda + k)}{\tau - \lambda - k - \nu Z\alpha} f_k \qquad V(r) = -\frac{Z}{r}$$

$$\rho = Kr$$

Power series have to truncate, i.e. for some k:

$$Z\alpha \nu^2 + Z\nu(\lambda+k) - Z\alpha = 0$$

k fixed: $\quad \nu = \dfrac{k+\lambda}{Z\alpha} + \sqrt{\dfrac{(k+\lambda)^2}{(Z\alpha)^2} + 1} = \sqrt{\dfrac{mc^2 - E}{mc^2 + E}}$

$$\Rightarrow E = mc^2 \left\{ 1 + \left(\frac{Z\alpha}{n - (J+\frac{1}{2}) + \sqrt{(J+\frac{1}{2})^2 - (Z\alpha)^2}} \right)^2 \right\}^{-\frac{1}{2}}$$

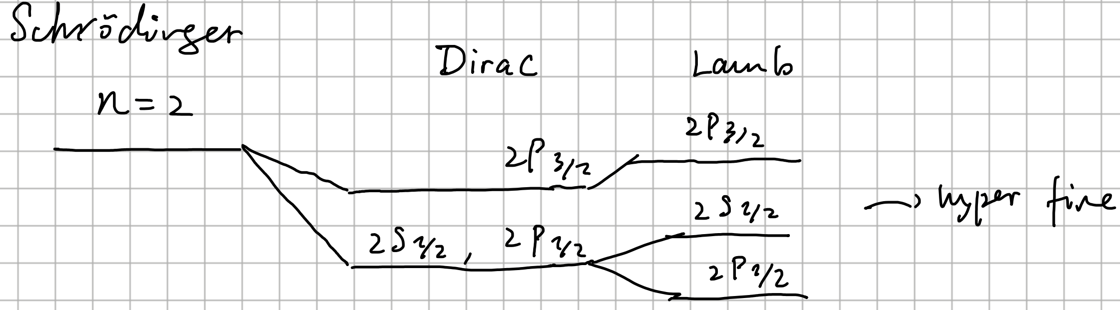
$$n = k + J + \frac{1}{2}, \quad n = 1, 2, 3 \ldots \quad J = \frac{1}{2}, \frac{3}{2}, \ldots$$

Nonrelativistic limit of it: $\alpha \longrightarrow 0$

$$E = mc^2 \left[\underbrace{1}_{\text{rest energy}} - \underbrace{\frac{(Z\alpha)^2}{2n^2}}_{\text{Schrödinger equation}} + \frac{(Z\alpha)^4}{2n^4} \left(\frac{n}{J+\frac{1}{2}} - \frac{3}{4} \right) + O(\alpha^6) \right]$$

Energie

n	J	w	l	nl_J	E/mc^2
1	$\frac{1}{2}$	$-$	0	$1S_{1/2}$	$\sqrt{1-(Z\alpha)^2}$
2	$\frac{1}{2}$	$-$	0	$2S_{1/2}$	$\sqrt{(1+\sqrt{1-(Z\alpha)^2})/2}$
2	$\frac{1}{2}$	$+$	1	$2P_{1/2}$	$-\|\|-$
2	$\frac{3}{2}$	$-$	1	$2P_{3/2}$	$\sqrt{4-(Z\alpha)^2}/2$
3	$\frac{1}{2}$	$-$	0	$3S_{1/2}$	|
3	$\frac{1}{2}$	$+$	1	$3P_{1/2}$	|
3	$\frac{3}{2}$	$-$	1	$3P_{3/2}$	|
3	$\frac{3}{2}$	$+$	2	$3d_{1/2}$	|
3	$\frac{5}{2}$	$-$	2	$3d_{3/2}$	



- Ground state of Dirac equation

$$\Psi^{(w=-1)}_{J=1/2,\, M=\frac{1}{2}}(r, \varphi, \theta) = C\,(2kr)^{\lambda-1} e^{-kr} \begin{pmatrix} 1 \\ 0 \\ \frac{i(1-\lambda)}{Z\alpha}\cos(\theta) \\ \frac{i(1-\lambda)}{Z\alpha}\sin(\theta)e^{i\varphi} \end{pmatrix}$$

Spin ↑

$$\psi^{(\omega=-1)}_{J=1/2,\, M=-\frac{1}{2}}(r,\varphi,\theta) = C(2\varkappa r)^{\lambda-1} e^{-\varkappa r} \begin{pmatrix} 0 \\ 1 \\ \frac{i(1-\lambda)}{Z\alpha}\sin\theta\, e^{-i\varphi} \\ -i\,\frac{(1-\lambda)}{Z\alpha}\cos\theta \end{pmatrix}$$

Spin $\downarrow$

$$\int d^3r\, \psi^+\psi = 1$$

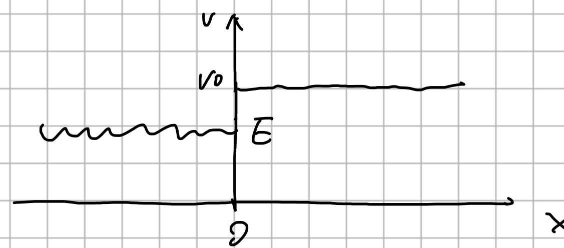
$$\rightarrow \quad C = \frac{(2\varkappa)^{3/2}}{\sqrt{4\pi}} \sqrt{\frac{1+\lambda}{2\Gamma(1+\lambda)}}$$

$$\lambda = \sqrt{1-(Z\alpha)^2}$$

$\leftarrow (\omega(J+\frac{1}{2}))^2$

- Klein paradox

$$\psi^{L,in}_{p\uparrow} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \frac{cp_x}{E+mc^2} \end{pmatrix} e^{i(p_x x - E_p t)/\hbar}$$



$E = E_p$

Dirac equation on the right side:

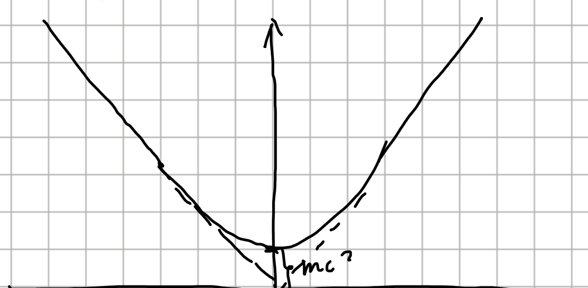
$$\begin{bmatrix} mc^2+V_0 & 0 & 0 & cp_x' \\ 0 & mc^2+V_0 & cp_x' & 0 \\ 0 & cp_x' & -mc^2+V_0 & 0 \\ cp_x' & 0 & 0 & -mc^2+V_0 \end{bmatrix} \psi^R = E\psi^R$$

$$\det\begin{pmatrix} mc^2+V_0-E & cp_x' \\ cp_x' & -mc^2+V_0-E \end{pmatrix} \overset{!}{=} 0 \quad \Rightarrow \quad (E-V_0)^2 - m^2c^4 - c^2 p_x'^2 = 0$$

$$\Rightarrow p_x' = \pm\sqrt{(E-V_0)^2 - m^2c^4}$$

$$|E_p - V_0| > mc^2$$

p_x is real



Solution right (transmitted)

$$\psi^R_{p_x'} = A \begin{pmatrix} 1 \\ 0 \\ 0 \\ \frac{cp_x'}{E_p+mc^2} \end{pmatrix} e^{i(p_x' x - E_p t)}$$

Solution left (reflected)

$$\psi^{L,out}_{p\uparrow} = B \begin{pmatrix} 1 \\ 0 \\ 0 \\ \frac{cp_x}{E_p+mc^2} \end{pmatrix} e^{i(-p_x x - E_p t)}$$

Boundary at $x=0$

$$A - B = 1$$

$$A\,(p_x'/p_x) = (1-B)$$

$$A = \frac{2}{1+p_x'/p_x} \qquad B = 1 - \frac{2\,p_x'/p_x}{\underbrace{1+p_x'/p_x}_{r}}$$

$$\frac{j_R}{j_I} = -\frac{(1+r)^2}{(1-r)^2} \qquad \frac{j_T}{j_I} = -\frac{4r}{(1-r)^2}$$

$$j_I = \frac{2p_x c^2}{E_p + mc^2} \qquad j_R = -|B|^2 \frac{2p_x c^2}{E_p + mc^2} \qquad j_T = -|A|^2 \frac{2p_x' c^2}{E' + mc^2}$$

$$\Rightarrow |j_R| \gg |j_I|$$