H2.1

a)

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We can make $\vec{k} \parallel \hat{e}_z$, because we can choose the direction of z-axis whatever we want.

$$v_{\vec{k}}(\vec{r}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \underline{c_{l,m}(\vec{k})} \cdot \underline{j_l(kr)}\, Y_l^m(\theta,\phi) = e^{i\vec{k}\vec{r}}$$

$$e^{ikr\cos\theta} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y_l^m(\theta,\phi)\, \underline{c_{l,m}(\vec{k})\, j_l(kr)} \qquad \Big|\cdot\int d\Omega\, Y_{l'}^{m'*}(\theta,\phi)$$

$$\overset{?}{=} \int d\Omega\, Y_l^{0*}(\theta)\, e^{ikr\cos(\theta)}$$

$$\Rightarrow \int d\Omega\, Y_{l'}^{m'*}(\theta,\phi)\, e^{ikr\cos\theta} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \underbrace{\int d\Omega\, Y_{l'}^{m'*}\, Y_l^m}_{=\delta_{mm'}\delta_{ll'}} \cdot c_{l,m}(\vec{k})\, j_l(kr)$$

$$\int d\Omega\, Y_l^{m*}(\theta,\phi)\, e^{ikr\cos\theta} = c_{l,m}(\vec{k})\, j_l(kr)$$

[only Y_l^{0*} appear, because $Y_l^{m*} \propto e^{-im\phi}$ and $\int d\phi\, e^{-im\phi} = 0$]

$$\Rightarrow \quad c_{l,m}(\vec{k})\, j_l(kr) = \int d\Omega\, Y_l^{0*}(\theta)\, e^{ikr\cos\theta}$$

b)

$$Y_l^0(\theta,\phi) = \frac{1}{\sqrt{(2l)!}}\left(\frac{\hat{L}_-}{\hbar}\right)^l Y_l^l(\theta,\phi)$$

$$L_\pm Y_l^m(\theta,\phi) = \hbar\sqrt{l(l+1) - m(m\pm 1)}\; Y_l^{m\pm 1}(\theta,\phi)$$

$$c_l j_l(kr) = \int d\Omega\, Y_l^{0*}(\theta)\, e^{ikr\cos\theta}$$

$$= \int d\Omega \left[\frac{1}{\sqrt{(2l)!}}\left(\frac{\hat{L}_-}{\hbar}\right)^l Y_l^l(\theta,\phi)\right]^* e^{ikr\cos\theta}$$

$$\left[\hat{L}_-^{\,l}\, Y_l^l(\theta,\phi) = \hat{L}_-^{\,l}\left[(-1)^l \sqrt{\frac{2l+1}{4\pi}\frac{1}{(2l)!}}\; P_{ll}(\cos\theta)\, e^{il\varphi}\right]\right.$$

$$\left. = (-1)^l \sqrt{\frac{2l+1}{4\pi}\frac{1}{(2l)!}} \cdot (\hbar)^l \frac{d^l}{d(\cos\theta)^l}\left[P_{ll}(\cos\theta)(\sin\theta)^{+l}\right]\right]$$

$$= \frac{1}{\sqrt{(2l)!}} \cdot \cancel{\left(\frac{1}{\hbar}\right)^l} \cdot (-1)^l \sqrt{\frac{2l+1}{2\pi(2l)!}}\; \cancel{(\hbar)^l}$$

$$\cdot \int d\Omega \frac{d^l}{d(\cos\theta)^l}\left[P_{ll}(\cos\theta)(\sin\theta)^l\right] \cdot e^{ikr\cos\theta}$$

$$= \sqrt{\frac{2l+1}{4\pi(2l)!}}\,(-1)^l \cdot \int d\varphi \underbrace{\int \sin\theta\, d\theta \frac{d^l}{d(\cos\theta)^l}\left[P_{ll}(\cos\theta)(\sin\theta)^l\right]\cdot e^{ikr\cos\theta}}$$

$$\sum_{n=0}^{l} \binom{l}{n} \frac{d^{l-n}}{d(\cos\theta)^{l-n}} P_{ll}(\cos\theta) \cdot \frac{d^n}{(d\cos\theta)^n}(\sin\theta)^l$$

not sure what to do here.

c) /

d) Using addition theorem:

$$P_l(\hat{k}\cdot\hat{r}) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y_{lm}(\hat{k})\, Y^*_{lm}(\hat{r}) \quad , \hat{k}, \hat{r} \text{ the unit vectors}$$

$$\hat{k}\cdot\hat{r} = \cos\theta$$

then we don't have $\hat{k}\cdot\hat{r}$ in the expression

$$\Rightarrow e^{i\hat{k}\cdot\hat{r}} = \sum_{l=0}^{\infty} i^l\, \cancel{(2l+1)}\, j_l(kr)\, \frac{4\pi}{\cancel{2l+1}} \sum_{m=-l}^{l} Y_{lm}(\hat{k})\, Y^*_{lm}(\hat{r})$$

$$= 4\pi \sum_{l=0}^{\infty}\sum_{m=-l}^{l} i^l\, Y_{lm}(\hat{k})\, j_l(kr)\, Y^*_{lm}(\hat{r})$$

b) in Hint. $n = l$, $m = 0$ $f(\theta) = \exp(ikr\cos\theta)$

$$C_l j_l(kr) = \int d\Omega\, Y_l^{0\,*}(\theta,\varphi)\, e^{ikr\cos\theta}$$

is the same as $*$, keep it for simplicity.

$$= \int d\Omega \frac{1}{\sqrt{(2l)!}} \left[\left(\frac{\hat{L}_-}{\hbar}\right)^l Y_l^l(\theta,\varphi)\right]^{\dagger} e^{ikr\cos(\theta)}$$

$$= \int d\Omega \frac{1}{\sqrt{(2l)!}}\, Y_l^{l\,*}(\theta,\varphi)\, \underbrace{\left(\frac{L_+}{\hbar}\right)^l e^{ikr\cos(\theta)}}$$

$$= (2^l l!) \frac{2\pi}{\sqrt{2l+1}} (ikr)^l\, Y_l^l(\theta,\varphi)\, e^{ikr\cos\theta}$$

c) $$j_\alpha(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+\alpha+1)} \left(\frac{x}{2}\right)^\alpha \xrightarrow{x \to 0} \frac{\left(\frac{x}{2}\right)^\alpha}{\Gamma(\alpha+1)}$$

$$\Gamma(x+1) = x\,\Gamma(x)$$

$$\Gamma(l + 1/2 + 1) = (l + 1/2)\,\Gamma(l + 1/2)$$

$$= \frac{\sqrt{\pi}}{2^{l+1}} (2l+1)!!$$

$$(2l+1)!! = \frac{(2l+1)!\,(2l)!}{2^l\, l!}$$

H2.2

Here I will the following def of fourier trafos:

$$\hat{f}(\omega) = \int dt\, e^{i\omega t} f(t)$$

or t ?

a) (3) in frequency space, doing fourier transformation w.r.t. $(t-t')$

$$\int d(t-t')\, e^{i\omega(t-t')}\left(i\hbar\frac{\partial}{\partial(t-t')} - \hbar\omega_0\right) G_p(t-t') = \int d(t-t')\, e^{i\omega(t-t')}\delta(t-t')$$

$$i\hbar\int \partial\, G_p(t-t')\cdot e^{i\omega(t-t')} - \hbar\omega\, G_p(\omega) = 1$$

$$i\hbar\left[\left(G_p(t-t')e^{i\omega(t-t')}\right)_{-\infty}^{\infty} - \int \underset{\substack{\| \\ dt\, e^{i\omega(t-t')}\cdot i\omega}}{de^{i\omega(t-t')}}\, G_p(t-t')\right] - \hbar\omega_0 G_p(\omega) = 1$$

$$i\hbar\cdot(-i\omega)\cdot G_p(\omega) - \hbar\omega_0 G_p(\omega) = 1$$

with $G_p(\omega) := \int d(t-t')\, e^{i\omega(t-t')} G_p(t-t')$

$$\Rightarrow (\hbar\omega - \hbar\omega_0)\, G_p(\omega) = 1$$

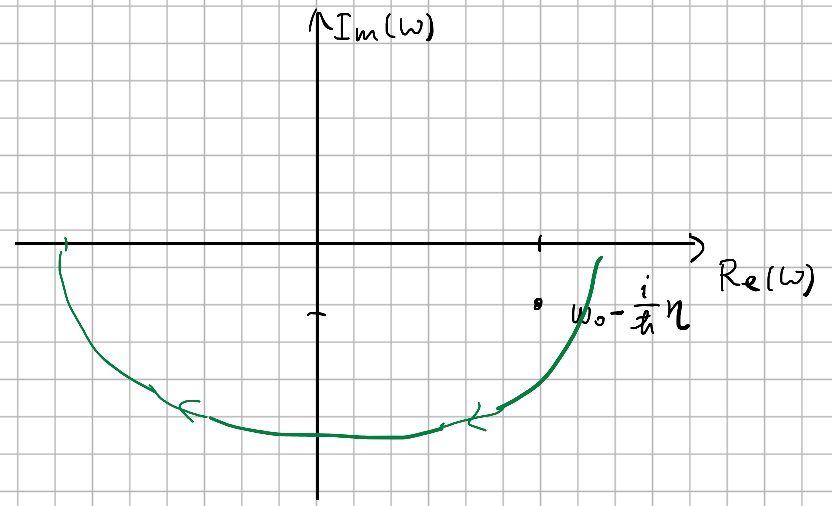
To avoid singularity we add $\pm i\eta$:

$$\Rightarrow G_p(\omega) = \frac{1}{\hbar\omega - \hbar\omega_0 \pm i\eta}$$

b) $$G_p^r(\omega) = \frac{1}{\hbar\omega - \hbar\omega_0 + i\eta} \leftarrow \text{inverse fourier}$$

$$G_p^r(t) = \int_{-\infty}^{\infty} d\omega\, e^{-i\omega t} G_p^r(\omega) = \int d\omega\, e^{-i\omega t}\frac{1}{\hbar\omega - \hbar\omega_0 + i\eta}$$

$$= \frac{1}{\hbar}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}\frac{1}{\omega - (\omega_0 - \frac{i}{\hbar}\eta)}$$



$t > 0$, lower

$t < 0$, higher $\rightarrow$ no pole

$$G_p^r(t) = \frac{1}{\hbar}\cdot 2\pi i \,\mathrm{Res}\left(e^{-i\omega t}\frac{1}{\omega-(\omega_0-\frac{i}{\hbar}\eta)},\ \omega=\omega_0-\frac{i}{\hbar}\eta\right)$$

$$= \frac{2\pi i}{\hbar}\lim_{\omega\to\omega_0-\frac{i}{\hbar}\eta} e^{-i\omega t}\frac{1}{\omega-(\omega_0-\frac{i}{\hbar}\eta)}\cdot[\omega-(\omega_0-\frac{i}{\hbar}\eta)]$$

$$= \frac{2\pi i}{\hbar}\exp(-i(\omega_0-\frac{i}{\hbar}\eta)t)$$

$$\Rightarrow\ G_p^r(t<0)=0\ .\qquad G_p^r(t>0)=\frac{2\pi i}{\hbar}\exp(-i\omega_0 t-\frac{\eta}{\hbar}t)$$

2π may not be here depending on def on Fourier trafo

c) G_p with $-i\eta$

$$G_p^a(t) = \frac{1}{\hbar}\cdot 2\pi i\,\mathrm{Res}\left(e^{-i\omega t}\frac{1}{\omega-(\omega_0+\frac{i}{\hbar}\eta)},\ \omega=\omega_0+\frac{i}{\hbar}\eta\right)$$

$$= \frac{2\pi i}{\hbar}\cdot\exp(-it(\omega_0+\frac{i}{\hbar}\eta)) = \frac{2\pi i}{\hbar}\exp(-i\omega_0 t+\frac{\eta}{\hbar}t)$$

using the positive half with $t<0$

with $t>0$. $G_p(t)=0$

d) $$\left(i\hbar\frac{\partial}{\partial t}-\frac{p^2}{2m}\right)G_p^{r,a}(t)\ ,\quad t\neq 0$$

$$=\left(i\hbar\frac{\partial}{\partial t}-\frac{p^2}{2m}\right)\cdot\frac{2\pi i}{\hbar}\exp(-i\omega_0 t\mp\frac{\eta}{\hbar}t)$$

$$=\left[-\frac{2\pi}{\hbar}\cdot\left(-i\omega_0\mp\frac{\eta}{\hbar}\right)-\frac{\pi i p^2}{\hbar m}\right]\exp(-i\omega_0 t\mp\frac{\eta}{\hbar}t)$$

$$=\left(i\frac{2\pi}{\hbar}\omega_0\pm\frac{2\pi\eta}{\hbar^2}-\frac{\pi i p^2}{m\hbar^2}\right)\exp(\ ---\ )$$

$$\underset{\hbar\omega_0=\frac{p^2}{2m}}{=}\ \pm\frac{2\pi\eta}{\hbar^2}\exp(-i\omega_0 t\mp\frac{\eta}{\hbar}t)\ ,\ \text{take } \eta=0$$

$=0$