H11.1

a) $u_n(x) = \frac{1}{\sqrt{2^n n!}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \mathrm{e}^{-m\omega x^2/(2\hbar)} H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right)$ $\quad \int_{-\infty}^{\infty} H_m(x) H_n(x) \mathrm{e}^{-x^2} \mathrm{d}x = \sqrt{\pi} 2^n n! \delta_{nm}$

$$\Rightarrow \int_{-\infty}^{\infty} dx \; u_n(x) \, u_m(x)$$

$$= \underbrace{\frac{1}{\sqrt{2^n n! \, 2^m m!}} \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}}}_{=:\alpha} \int_{-\infty}^{\infty} dx \; e^{-m\omega x^2/\hbar} \; H_n\left(\sqrt{\frac{m\omega}{\hbar}} x\right) H_m\left(\sqrt{\frac{m\omega}{\hbar}} x\right)$$

$$\left(y = \sqrt{\frac{m\omega}{\hbar}} x \quad , \quad y^2 = \frac{m\omega}{\hbar} x^2 \quad , \quad dx = \sqrt{\frac{m\omega}{\hbar}}^{-1} dy \right)$$

$$= \alpha \cdot \sqrt{\frac{\hbar}{m\omega}} \int_{-\infty}^{\infty} dy \; e^{-y^2} H_n(y) H_m(y)$$

$$= \alpha \cdot \sqrt{\frac{\hbar}{m\omega}} \sqrt{\pi} \, 2^n \, n! \, \delta_{mn}$$

$$= \delta_{mn}$$

$u_n(x) = \frac{1}{\sqrt{2^n n!}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \mathrm{e}^{-m\omega x^2/(2\hbar)} H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right)$

$$\int_{\mathbb{R}} dx \, |\psi(x,0)|^2 \Rightarrow$$

$$LHS = \int dx \left| \sum_{n=0}^{\infty} A_n u_n(x) \right|^2 = \sum_m \sum_n |A_n||A_m| \int dx \; u_n(x) \, u_m(x)$$

$$= \sum_n |A_n|^2$$

$$RHS = \int dx \, \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}} e^{-m\omega(x-a)^2/\hbar}$$

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}} \int dx \; e^{-y^2 + 2\xi_0 y - \xi_0^2} \qquad y = y(x)$$

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}} \int dx \; e^{-y^2} \left(e^{\xi_0 y - \frac{1}{2}\xi_0^2}\right)^2$$

$$e^{2y\frac{\xi_0}{2} - \left(\frac{\xi_0}{2}\right)^2}$$

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}} e^{-\xi_0^2/2} \int dx \; e^{-y^2} \sum_m H_m(y) \frac{(\xi_0/2)^m}{m!} \; \sum_n H_n(y) \frac{(\xi_0/2)^n}{n!}$$

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}} e^{-\xi_0^2/2} \sum_{m,n} \frac{(\xi_0/2)^{m+n}}{m!\,n!} \int \underbrace{dx}_{=\sqrt{\frac{\hbar}{m\omega}}\,dy} e^{-y^2} H_m(y) H_n(y)$$

$$= \cancel{\pi^{-\frac{1}{2}}} e^{-\xi_0^2/2} \sum_{m,n} \frac{(\xi_0/2)^{m+n}}{m!\,\cancel{n!}} \cancel{\sqrt{\pi}}\, 2^n \cancel{n!}\, \delta_{mn}$$

$$= e^{-\xi_0^2/2} \sum_n (\xi_0/2)^{2n} \frac{2^n}{n!} = e^{-\xi_0^2/2} \sum_n \frac{\xi_0^{2n}}{2^n n!}$$

But obviously it is a divergent series. Would this be problematic?

$$\Rightarrow |A_n|^2 = e^{-\xi_0^2/2} \frac{\xi_0^{2n}}{2^n n!},$$

if $A_n \in \mathbb{R}$. $\quad A_n = e^{-\xi_0^2/4} \dfrac{\xi_0^n}{\sqrt{2^n n!}}$

b) $$\psi(x,t) = e^{-i\omega t/2} \sum A_n u_n(x) e^{-in\omega t} \qquad (\xi = y)$$

$$= e^{-i\omega t/2} e^{-\xi_0^2/4} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{1}{2}\xi^2} \sum_{n=0} \frac{H_n(x)}{n!} \left(\frac{1}{2}\xi_0 e^{-i\omega t}\right)^n$$

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{1}{2}\xi^2 - \frac{1}{4}\xi_0^2 - \frac{i\omega t}{2} - \underline{\frac{1}{4}\xi_0^2 e^{-2i\omega t}} + \underline{\xi\xi_0 e^{-i\omega t}}\right)$$

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{1}{2}\xi^2 - \frac{1}{4}\xi_0^2 - \frac{i\omega t}{2} - \frac{1}{4}\xi_0^2\cos(2\omega t) + \frac{i}{4}\xi_0^2 \sin(2\omega t)\right.$$
$$\left. + \xi_0\xi\cos(\omega t) - i\xi\xi_0\sin(\omega t)\right)$$

$$\left[\cos(2x) = 2\cos^2(x) - 1\right]$$

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left[-\frac{1}{2}\xi_0^2 - \frac{1}{4}\xi_0^2 - \frac{1}{2}\xi_0^2\cos^2(\omega t) + \xi\xi_0\cos(\omega t)\right.$$
$$\left. + \frac{1}{4}\xi_0^2 - i\underbrace{\left(\frac{\omega t}{2} + \xi\xi_0\sin(\omega t) - \frac{1}{4}\xi_0^2\sin(2\omega t)\right)}_{\text{phase}}\right]$$

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \exp\left(-\frac{1}{2}\left(\xi - \xi_0\cos(\omega t)\right)^2\right) \cdot e^{\varphi}$$

c) $\langle x|\hat{a}|\phi\rangle = \phi\langle x|\phi\rangle$, $\phi(x) = \langle x|\phi\rangle$

$$\langle x|\sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x}+\frac{i}{m\omega}\hat{p}\right)|\phi\rangle = \phi\,\phi(x)$$

$$\sqrt{\frac{m\omega}{2\hbar}}\langle x|\hat{x}|\phi\rangle + \sqrt{\frac{\hbar}{2m\omega}}\langle x|\nabla|\phi\rangle = \phi\,\phi(x)$$

$$\Rightarrow \partial_x\langle x|\phi\rangle = \sqrt{\frac{2m\omega}{\hbar}}\phi\,\phi(x) - \frac{m\omega}{\hbar}x\phi(x)$$

$$\Rightarrow \partial_x\phi(x) = \phi(x)\left(\sqrt{\frac{2m\omega}{\hbar}}\phi - \frac{m\omega}{\hbar}x\right)$$

d) $\partial_x\phi(x) = (a-bx)\phi(x)$, $a = \sqrt{\frac{2m\omega}{\hbar}}\phi$, $b = \frac{m\omega}{\hbar}$

$$\Rightarrow \frac{\partial\phi(x)}{\partial x} = (a-bx)\phi(x)$$

$$\frac{\partial\phi(x)}{\phi(x)} = (a-bx)\partial x \quad \Big|\int$$

$$\ln|\phi(x)| = ax - \frac{b}{2}x^2 + \tilde{C}$$

$$\phi(x) = e^{\tilde{C}}e^{ax-\frac{b}{2}x^2} = e^{\tilde{C}}\exp\left(\sqrt{\frac{2m\omega}{\hbar}}\phi x - \frac{m\omega}{2\hbar}x^2\right)$$

$$\Rightarrow \phi(x) = C\exp\left(-\frac{m\omega}{2\hbar}\left(x-\sqrt{\frac{2\hbar}{m\omega}}\phi\right)^2\right)$$

e) $|\phi\rangle_{sch} = \sum \frac{\phi_n}{n!} e^{-i\omega(n+\frac{1}{2})t} |n\rangle = e^{-i\omega t} |\phi e^{-i\omega t}\rangle$

$t=0$

$$\langle x|\phi\rangle_{sch} = c\, e^{-\alpha^2(x - \frac{|\phi|}{\alpha} e^{-i\omega t})^2} e^{-i\omega t/2} \quad , \alpha = \sqrt{\frac{m\omega}{2\hbar}}$$

To find the c:

$$\int_{-\infty}^{\infty} dx\, \phi_{sch}^* \phi_{sch}$$

$$= |c|^2 \int_{\mathbb{R}} dx\, \underbrace{\exp(-\alpha^2(x - \frac{|\phi|}{\alpha} e^{-i\omega t})^2) \exp(-\alpha^2(x - \frac{|\phi|}{\alpha} e^{i\omega t})^2)}$$

$$= \exp\{-\alpha^2[(x - \frac{|\phi|}{\alpha} e^{-i\omega t})^2 + (x - \frac{|\phi|}{\alpha} e^{i\omega t})^2]\}$$

$$= \exp\{-\alpha^2[x^2 + \frac{|\phi|^2}{\alpha^2} e^{-2i\omega t} - 2x \frac{|\phi|}{\alpha} e^{-i\omega t} + x^2 + \frac{|\phi|^2}{\alpha^2} e^{2i\omega t} - 2x \frac{|\phi|}{\alpha} e^{-i\omega t}]\}$$

$$= \exp\{-\alpha^2[2x^2 + \frac{|\phi|^2}{\alpha^2} \underbrace{(e^{2i\omega t} + e^{-2i\omega t})}_{= 2\cos(2\omega t)} - 2x \frac{|\phi|}{\alpha} (e^{-i\omega t} + e^{i\omega t})$$

$$= \exp\{-\alpha^2[2x^2 + 4\frac{|\phi|^2}{\alpha^2} \cos^2(\omega t) - 2\frac{|\phi|^2}{\alpha^2} - 4x \frac{|\phi|}{\alpha} \cos(\omega t)]\}$$

$$= |c|^2 e^{2|\phi|^2 - 4|\phi|^2 \cos^2(\omega t)} \underbrace{\int_{\mathbb{R}} dx\, e^{-2\alpha^2 x^2 + 4\alpha x |\phi| \cos(\omega t)}}_{= \sqrt{\frac{\pi}{2}} \frac{1}{\alpha} e^{2|\phi|^2 \cos^2(\omega t)}}$$

$$\overset{!}{=} 1$$

$$\Rightarrow \sqrt{\frac{\pi}{2}} \frac{|c|^2}{\alpha} e^{2|\phi|^2 - 2|\phi|^2 \cos^2(\omega t)} = 1$$

$$\Leftrightarrow |c|^2 = \sqrt{\frac{2}{\pi}}\, \alpha\, e^{-2|\phi|^2} e^{2|\phi|^2 \cos^2(\omega t)}$$

$$\phi_{sch}^* \phi_{sch} = |\phi_{sch}|^2 = \sqrt{\frac{m\omega}{\pi\hbar}} \exp[-2|\phi|^2 + 2|\phi|^2 \cos^2(\omega t) +$$

$$2|\phi|^2-4|\phi|^2\cos^2(\omega t)+4x|\phi|\alpha\cos(\omega t)-2\alpha^2x^2]$$

$$=\sqrt{\frac{m\omega}{\pi\hbar}}\exp[-2\alpha^2[x^2+\frac{|\phi|^2}{\alpha^2}\cos^2(\omega t)-2x\frac{|\phi|}{\alpha}\cos(\omega t)]]|$$

$$=\sqrt{\frac{m\omega}{\pi\hbar}}\exp\left(-\frac{m\omega}{\hbar}\left(x-\sqrt{\frac{2\hbar}{m\omega}}|\phi|\cos(\omega t)\right)^2\right)$$

f) $$\langle\hat{x}\rangle=\sqrt{\frac{\hbar}{2m\omega}}\langle\phi e^{-i\omega t}|\hat{a}^\dagger+\hat{a}|\phi e^{-i\omega t}\rangle$$

coherent state!

$$=\sqrt{\frac{\hbar}{2m\omega}}\langle\phi e^{-i\omega t}|\phi e^{i\omega t}+\phi e^{-i\omega t}|\phi e^{-i\omega t}\rangle$$

$$=\sqrt{\frac{\hbar}{2m\omega}}|\phi|\cos(\omega t)$$

$$\langle\hat{p}\rangle=i\sqrt{\frac{m\omega\hbar}{2}}\langle\cdots|\hat{a}^\dagger-\hat{a}|\cdots\rangle=i\sqrt{\frac{m\omega\hbar}{2}}|\phi|(e^{i\omega t}-e^{-i\omega t})$$

$$=\sqrt{2m\omega\hbar}|\phi|\sin(-\omega t)$$

$$\langle\hat{H}\rangle=\langle\phi e^{-i\omega t}|e^{i\omega t/2}\hbar\omega(\hat{a}^\dagger\hat{a}+\tfrac{1}{2})e^{-i\omega t/2}|\phi e^{-i\omega t}\rangle$$

$$=\hbar\omega\langle\phi e^{-i\omega t}|\hat{a}^\dagger\hat{a}+\tfrac{1}{2}|\phi e^{-i\omega t}\rangle$$

✓

$$=\hbar\omega(|\phi|^2+\tfrac{1}{2})$$

g) $$P(n) = C \frac{|\phi|^{2n}}{n!} \approx C \frac{|\phi|^{2n}}{\sqrt{2\pi n}\,\left(\frac{n}{e}\right)^n} = C \frac{|\phi|^{2n} e^n}{\sqrt{2\pi n}\; n^n}$$

$$E(n) = \int_0^\infty dn \; n \, P(n) = \int_0^\infty dn \; n \; C \frac{|\phi|^{2n} e^n}{\sqrt{2\pi n}\; n^n} = C \frac{1}{\sqrt{2\pi}} \int_0^\infty dn \, \sqrt{n} \left(\frac{|\phi|^2 e}{n}\right)^n$$

$$P(n) = C \frac{|\phi|^{2n}}{n!}$$

$$\ln(P(n)) \approx 2n \ln(|\phi|) - n \ln(n) + n$$

$$\frac{\partial}{\partial n} \ln(P(n)) = 2 \ln(|\phi|) - \ln(n) + 1 - 1 = 0$$

$$\Rightarrow |\phi|^2 = n$$

H 11.2

a) ??? B: $[\hat{a}_i, \hat{a}_j] |\phi\rangle = 0$

F: $\{\hat{a}_i, \hat{a}_j\} |\phi\rangle = 0$

b) $$|\phi\rangle = \sum_{n_{\alpha_1},\dots} \phi_{n_{\alpha_1}, n_{\alpha_2} \dots} |n_{\alpha_1} n_{\alpha_2} \dots\rangle$$

$$= \sum_{n_{\alpha_1},\dots} \phi_{n_{\alpha_1}, n_{\alpha_2}, \dots} \prod_n \frac{(\hat{a}^+_{\alpha_n})^{n_{\alpha_n}}}{\sqrt{n_{\alpha_n}!}} |0\rangle$$

$$\leftarrow \phi_{n_{\alpha_1},\dots} = \frac{\phi_{\alpha_1}}{\sqrt{n_{\alpha_1}!}} \frac{\phi_{\alpha_2}}{\sqrt{n_{\alpha_2}!}} \dots$$

$$= \sum_{n_{\alpha_1}\dots} \prod_n \frac{(\phi_{\alpha_n} \hat{a}^+_{\alpha_n})^{n_{\alpha_n}}}{n_{\alpha_n}!} |0\rangle$$

$$= \prod_n \exp(\phi_{\alpha_n} \hat{a}^+_{\alpha_n}) |0\rangle$$

$$= \exp\Big(\sum_n \phi_{\alpha_n} \hat{a}^+_{\alpha_n}\Big) |0\rangle$$

c) $$\hat{a}^+_{\alpha_i} |\phi\rangle = \hat{a}^+_{\alpha_i} \exp\Big(\sum_n \phi_{\alpha_n} \hat{a}^+_{\alpha_n}\Big) |0\rangle$$

$$= \exp\Big(\sum_{n \neq i} \phi_{\alpha_n} \hat{a}^+_{\alpha_n}\Big) \underbrace{\hat{a}^+_{\alpha_i} \exp(\phi_{\alpha_i} \hat{a}^+_{\alpha_i})}_{= \partial_{\phi_{\alpha_i}} \exp(\phi_{\alpha_i} \hat{a}^+_{\alpha_i})} |0\rangle$$

$$= \partial_{\phi_i} \Big(\exp\Big(\sum_n \phi_{\alpha_n} \hat{a}^+_{\alpha_n}\Big) |0\rangle \Big) = \partial_{\phi_i} |\phi\rangle$$

d) $\langle \phi | \phi' \rangle = \langle n_{\alpha_1} n_{\alpha_2} \cdots | \sum \frac{(\phi_{\alpha_1}^*)^{n_{\alpha_1}}}{\sqrt{n_{\alpha_1}!}} \frac{(\phi_{\alpha_2})^{n_{\alpha_2}}}{\sqrt{n_{\alpha_2}!}} \cdots \sum \frac{(\phi_{\alpha_1})^{n_{\alpha_1}}}{\sqrt{n_{\alpha_1}!}} \frac{(\phi_{\alpha_2})^{n_{\alpha_2}}}{\sqrt{n_{\alpha_2}!}} \cdots | n_{\alpha_1} n_{\alpha_2} \cdots \rangle$

$$= \sum_{n_{\alpha_1}, \cdots} \frac{(\phi_{\alpha_1}^* \phi_{\alpha_1})^{n_{\alpha_1}}}{n_{\alpha_1}!} \cdots$$ $\leftarrow$ Orthogonality $\Rightarrow$ we can write two sums into one

$$= \exp(\phi_{\alpha_1}^* \phi_{\alpha_1})$$

e) consider a single state α:

$$\int \frac{d\phi^* d\phi}{2\pi i} e^{-\phi^* \phi} |\phi\rangle\langle\phi|$$

$$= \int \frac{d\rho\, d\theta}{\pi} \rho e^{-\rho^2} \sum_m \frac{(\rho e^{i\theta})^m}{\sqrt{m!}} |m\rangle \sum_n \frac{(\rho e^{-i\theta})^n}{\sqrt{n!}} \langle n|$$

$$= \frac{1}{\pi} \int_0^\infty d\rho\, \rho\, e^{-\rho^2} \underbrace{\int_0^{2\pi} d\theta \sum_{m,n} \frac{\rho^{m+n} e^{im\theta - in\theta}}{\sqrt{m!\,n!}}}_{= \pi \delta_{m,n}} |m\rangle\langle n|$$

(if $\sum_{m,n} |m\rangle\langle n| = \sum_n |n\rangle\langle n|$, not sure why it holds)

$$= 2 \int_0^\infty d\rho\, \rho\, e^{-\rho^2} \sum_n \frac{\rho^{2n}}{n!} |m\rangle\langle n|$$

$$= 2 \sum_n \frac{1}{n!} \underbrace{\int_0^\infty d\rho\, e^{-\rho^2} \rho^{2n+1}}_{} |m\rangle\langle n|$$

$(d\rho^2 = 2\rho\, d\rho)$

$$= \frac{1}{2} \int_0^\infty d\rho^2\, e^{-\rho^2} \rho^{2n} = \frac{1}{2} \Gamma(n+1) = \frac{n!}{2}$$

$$= \sum_n |n\rangle\langle n|$$

Consider all states:

$$\int \prod_{\alpha_i} \frac{d\phi^*_{\alpha_i}\, d\phi_{\alpha_i}}{2\pi i} \exp\left(-\sum_{\alpha_i} \phi^*_{\alpha_i} \phi_{\alpha_i}\right) |\phi\rangle\langle\phi|$$

$$= \sum_{n_{\alpha_1}, \ldots} |n_{\alpha_1}, n_{\alpha_2}, \ldots\rangle\langle n_{\alpha_1}, n_{\alpha_2}, \ldots| \quad = \mathbb{1}$$