Important formulae in the AQT course

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1 General maths and old stuff

transformation function

$$\langle \boldsymbol{x}' | \boldsymbol{p}' \rangle = \left[\frac{1}{(2\pi\hbar)^{3/2}} \right] \exp\left(\frac{i\boldsymbol{p}' \cdot \boldsymbol{x}'}{\hbar} \right) \tag{1.1}$$

trigonometric identities

$$\begin{aligned} \sin(\alpha \pm \beta) &= \sin\alpha \cos\beta \pm \cos\alpha \sin\beta \\ \cos(\alpha \pm \beta) &= \cos\alpha \cos\beta \mp \sin\alpha \sin\beta \end{aligned} \tag{1.2}$$

Gradient in spherical coordinates

$$\nabla f(r, \theta, \varphi) = \frac{\partial f}{\partial r} \mathbf{e}_r + \frac{1}{r} \frac{\partial f}{\partial \theta} \mathbf{e}_\theta + \frac{1}{r \sin\theta} \frac{\partial f}{\partial \varphi} \mathbf{e}_\varphi \tag{1.3}$$

Laplace operator in spherical coordinates

$$\begin{aligned} \Delta &= \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin(\theta)} \frac{\partial}{\partial \theta} \left(\sin(\theta) \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2(\theta)} \frac{\partial^2}{\partial \varphi^2} \\ &= \frac{1}{r} \frac{\partial^2}{\partial r^2} r + \frac{1}{r^2 \sin(\theta)} \frac{\partial}{\partial \theta} \left(\sin(\theta) \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2(\theta)} \frac{\partial^2}{\partial \varphi^2} \end{aligned} \tag{1.4}$$

Commutator identities

$$[A, BC] = [A, B]C + B[A, C] \tag{1.5}$$

$$[AB, C] = A[B, C] + [A, C]B \tag{1.6}$$

Canonical commutation relation

$$[\hat{r}_i, \hat{p}_j] = i\hbar\delta_{ij} \tag{1.7}$$

Green's first identity

$$\int_V \left(\phi\Delta\psi + \vec{\nabla}\phi \cdot \vec{\nabla}\psi\right) \mathrm{d}V = \int_{\partial V} \phi\left(\vec{\nabla}\psi \cdot \hat{n}\right) \mathrm{d}S \tag{1.8}$$

Green's second identity

$$\int_V (\phi\Delta\psi - \psi\Delta\phi)\, \mathrm{d}V = \int_{\partial V} \left(\phi\frac{\partial\psi}{\partial\vec{n}} - \psi\frac{\partial\phi}{\partial\vec{n}}\right) \mathrm{d}S \tag{1.9}$$

with $\frac{\partial\phi}{\partial\vec{n}} = \vec{\nabla}\phi \cdot \vec{n}$

Representations of Dirac delta function $\delta(x)$

- gaussian functions

$$\lim_{\epsilon\to 0} \sqrt{\frac{1}{\pi\epsilon}} e^{-x^2/\epsilon}$$

- fourier tranform

$$\delta^{(3)}(\vec{x}) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} e^{i\vec{k}\cdot\vec{x}} \mathrm{d}^3 k$$

- poisson kernel

$$\lim_{\eta\to 0} \frac{1}{\pi} \frac{1}{x^2 + \eta^2}$$

Composition of delta function with a function

$$\int_{-\infty}^{\infty} f(x)\delta(g(x)) = \sum_i \frac{f(x_i)}{|g'(x_i)|} \tag{1.10}$$

Saddle point approximation/Stationary phase methode/saddle-point methode

three pictures of QM For a time-independent Hamiltonian H_S:

evolution of	Heisenberg	Picture Interaction	Schrödinger
Ket state	constant	$\lvert\psi_I(t)\rangle = \mathrm{e}^{iH_{0,S}t/\hbar} \lvert\psi_S(t)\rangle$	$\lvert\psi_S(t)\rangle = \mathrm{e}^{-iH_S t/\hbar} \lvert\psi_S(0)\rangle$
Observable	$A_H(t) = \mathrm{e}^{iH_S t/\hbar} A_S \mathrm{e}^{-iH_S t/\hbar}$	$A_I(t) = \mathrm{e}^{iH_{0,S}t/\hbar} A_S \mathrm{e}^{-iH_{0,S}t/\hbar}$	constant
Density matrix	constant	$\rho_I(t) = e^{iH_{0,S}\ t/\hbar} \rho_S(t) e^{-iH_{0,S}\ t/\hbar}$	$\rho_S(t) = e^{-iH_S\ t/\hbar} \rho_S(0) e^{iH_S\ t/\hbar}$

Table 1: three pictures of QM

2 Scattering

Lippmann-Schwinger equation with $\lvert\varphi\rangle$ the incident wave $\lvert\psi^{(\pm)}\rangle$ the scattered wave

$$\lvert\psi^{(\pm)}\rangle = \lvert\varphi\rangle + \frac{V}{E - H_0 \pm i\epsilon} \lvert\psi^{(\pm)}\rangle \tag{2.1}$$

Born series comes from iterating the Lippmann-Schwinger equation

$$\begin{aligned} |\psi\rangle &= |\phi\rangle + G_0(E)V\,|\phi\rangle + [G_0(E)V]^2\,|\phi\rangle + \dots \\ \text{with } G_0(E) &= \frac{1}{E_i - H_0 \pm i\varepsilon} \end{aligned} \tag{2.2}$$

transition (rate) matrix:

$$T = V + VG_0(E)V + VG_0(E)VG_0(E)V + \dots \tag{2.3}$$

Scattering amplitude $f(\boldsymbol{k}, \boldsymbol{k'})$
(has dimension of length)

$$\begin{aligned} \langle \boldsymbol{x}|\psi^{(+)}\rangle &= \langle x|i\rangle - \frac{2m}{\hbar^2}\int \mathrm{d}^3x'\,\underbrace{\frac{e^{\pm ik|\boldsymbol{x}-\boldsymbol{x}'|}}{4\pi|\boldsymbol{x}-\boldsymbol{x}'|}}_{G_\pm(\boldsymbol{x},\boldsymbol{x}')}\,V(\boldsymbol{x}')\,\langle \boldsymbol{x}'|\psi^{(\pm)}\rangle \\ &\xrightarrow[\text{large } r]{} \frac{1}{L^{3/2}}\left[e^{i\boldsymbol{k}\cdot\boldsymbol{x}} + \frac{e^{ikr}}{r}f(\boldsymbol{k},\boldsymbol{k'})\right] \\ \text{with } f(\boldsymbol{k}',\boldsymbol{k}) &= -\frac{mL^3}{2\pi\hbar^2}\left\langle \boldsymbol{k}'\middle|V\middle|\psi^{(+)}\right\rangle = -\frac{mL^3}{2\pi\hbar^2}\,\langle \boldsymbol{k}'|T|\boldsymbol{k}\rangle \end{aligned} \tag{2.4}$$

Differential cross section

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = |f(\boldsymbol{k}',\boldsymbol{k})|^2 \tag{2.5}$$

Optical theorem

$$\operatorname{Im} f(\theta = 0) = \frac{k\sigma_{tot}}{4\pi} \tag{2.6}$$

Residue Theorem with the $\mathrm{I}(\gamma, a_k)$ the winding number:

$$\oint_\gamma f(z)\,dz = 2\pi i \sum_{k=1}^{n} \mathrm{I}(\gamma, a_k)\,\mathrm{Res}(f, a_k) \tag{2.7}$$

Calculating the residues

- Simple poles

$$\mathrm{Res}(f,c) = \lim_{z\to c}(z-c)f(z) \tag{2.8}$$

- Limit formula for higher order poles

$$\mathrm{Res}(f,c) = \frac{1}{(n-1)!}\lim_{z\to c}\frac{d^{n-1}}{dz^{n-1}}\left((z-c)^n f(z)\right) \tag{2.9}$$

Jordan's Lemma This lemma states the convergence condition of integral containing $f(z) = g(z)e^{iaz}$ with $z \in C_R$, $a > 0$ over an arc in complex plane. C_R is the upper half-plane, i.e. $C_R = \{Re^{i\theta} | \theta \in [0, \pi]\}$. Then the upper bound of the the integral is:

$$\left| \int_{C_R} f(z) \mathrm{d}z \right| \leq \frac{\pi}{a} M_R \text{ where } M_R := \max_{\theta \in [0,\pi]} \left| g\left(Re^{i\theta}\right) \right| \tag{2.10}$$

An analogous statement for a semicircular contour in the lower half-plane holds when $a < 0$.

Born Approximation applicable when the scattered field is small compared to incident fielf of scatterer

$$f^{(1)}(\boldsymbol{k}', \boldsymbol{k}) = -\frac{m}{2\pi\hbar^2} \langle \boldsymbol{k}' | V | \boldsymbol{k} \rangle = -\frac{m}{2\pi\hbar^2} \int \mathrm{d}^3 x' e^{i(\boldsymbol{k}-\boldsymbol{k}')\cdot \boldsymbol{x}'} V(\boldsymbol{x}') \quad \propto V(\boldsymbol{k}' - \boldsymbol{k}) \tag{2.11}$$

$$f^{(2)} = -\frac{m}{2\pi\hbar^2} \langle \boldsymbol{k}' | V G_0(E) V | \boldsymbol{k} \rangle \tag{2.12}$$

Eikonal Approximation applicable when the potential $V(\boldsymbol{x})$ varies very little over a distance of order of wavelength λ

$$\begin{aligned} f(\boldsymbol{k}', \boldsymbol{k}) &= -ik \int_0^\infty \mathrm{d}b b J_0(kb\theta)[e^{2i\Delta(b)} - 1] \\ \Delta(b) &= \frac{-m}{2k\hbar^2} \int_{-\infty}^{+\infty} \mathrm{d}z V(\sqrt{b^2 + z^2}) \end{aligned} \tag{2.13}$$

(Spherical) Bessel functions

$$j_l(x) = (-x)^l \left(\frac{1}{x} \frac{\mathrm{d}}{\mathrm{d}x} \right)^l \frac{\sin(x)}{x} \tag{2.14}$$

$$y_l(x) = -(-x)^l \left(\frac{1}{x} \frac{\mathrm{d}}{\mathrm{d}x} \right)^l \frac{\cos(x)}{x} \tag{2.15}$$

$$J_{l+1/2}(x) = \sqrt{\frac{2x}{\pi}} j_l(x) \tag{2.16}$$

with

$$\begin{gathered} j_0 = \frac{\sin x}{x} \; ; \; y_0 = -\frac{\cos x}{x} \\ j_1 = \frac{\sin x}{x^2} - \frac{\cos x}{x} \; ; \; y_1 = -\frac{\cos x}{x^2} - \frac{\sin x}{x} \\ (j, J)_0(x) \to 1 \text{ in } x \to 0 \\ (j, J)_{1,2}(x) \to 0 \text{ in } x \to 0 \end{gathered}$$

For $z \to 0$

$$j_l(z) \approx \frac{z^l}{(2l+1)!!} \qquad y_l(z) \approx -\frac{(2l-1)!!}{z^{l+1}} \tag{2.17}$$

For $z \to \infty$

$$j_l(z) \approx \frac{\sin(z - l\pi/2)}{z} \qquad y_l(z) \approx -\frac{\cos(z - l\pi/2)}{z} \tag{2.18}$$

Spherical waves The radial part of solution of free Schrödinger equation:

$$R^{\pm}_{kl} = \pm iA\sqrt{\frac{k\pi}{2r}}H^{(1,2)}_{l+1/2}(kr) \tag{2.19}$$

plane wave expansion

$$e^{i\vec{k}\cdot\vec{r}} = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}Y_l^{m*}(\theta_k,\phi_k)j_l(kr)Y_l^m(\theta,\phi) \tag{2.20}$$

Legendre polynomials

$$P_l(x) = \frac{1}{2^l l!}\frac{\mathrm{d}^l}{\mathrm{d}x^l}\left[(x^2-1)^l\right] \tag{2.21}$$

with properties

- $\int_{-1}^{1}\mathrm{d}xP_l(x)P_{l'}(x) = \frac{2}{2l+1}\delta_{ll'}$
- $P_l(-x) = (-1)^l P_l(x)$
- $P_l(1) = 1$
- $P_0x = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2-1)$

Partial-wave Expansion of the scattering amplitude

For central potential the scattering amplitude only depends on the momentum k und the scattering angle θ:

$$f_k(\theta) = \frac{1}{k}\sum_{l=0}^{\infty}(2l+1)e^{i\delta_l(k)}\sin\left[\delta_l(k)\right]P_l(\cos(\theta)) \tag{2.22}$$

$$\sigma_{tot} = \frac{4\pi}{k^2}\sum_l(2l+1)\sin^2\delta_l \tag{2.23}$$

Determination of phase shift

$$\langle x|\psi^{(+)}\rangle = \frac{1}{(2\pi)^{3/2}}\sum i^l(2l+1)A_l(r)P_l(\cos\theta)\ r > R \tag{2.24}$$

$$A_l(r) = e^{i\delta_l}\left[\cos\delta_l j_l(kr) - \sin\delta_l n_l(kr)\right]\ \ r > R \tag{2.25}$$

$$\tan\delta_l = \frac{kRj_l'(kR) - \beta_l j_l(kR)}{kRn_l'(kR) - \beta_l n_l(kR)} \tag{2.26}$$

Breit-Wigner-Equation

$$\sigma_l = \underbrace{\frac{4\pi(2l+1)}{k^2}}_{\sigma_{max}}\frac{\gamma_l^2(kR)^{4l+2}}{1+\gamma_l^2(kR)^{4l+2}} \tag{2.27}$$

3 Relativistic quantum mechanics

Minkowski metric

$$(+,-,-,-)$$

Klein-Gordon equation

$$(\Box + m^2)\phi = 0 \tag{3.1}$$

four-current and continuity equation

$$j^\mu := \frac{i}{2m}\left[\phi^*(\partial^\mu\phi) - (\partial^\mu\phi^*)\phi\right] \tag{3.2}$$

$$\partial_\mu j^\mu = 0 \tag{3.3}$$

Dirac equation

$$i\hbar\frac{\partial}{\partial t}\Psi(x^\mu) = \left(c\vec{\alpha}\cdot\vec{p} + \beta mc^2\right)\Psi(x^\mu) \tag{3.4}$$

$$\text{with } \alpha^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} \text{ and } \beta = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \tag{3.5}$$

covariant form:

$$i\hbar\gamma^\mu\partial_\mu\Psi(x^\mu) = mc\Psi(x^\mu) \tag{3.6}$$

$$\text{with } \gamma^0 = \beta,\ \gamma^i = \beta\alpha^i, \tag{3.7}$$

$$\{\alpha^i,\alpha^j\} = 2\delta^{ij}\mathbf{1}\ ,\ \{\alpha^i,\beta\} = 0 \tag{3.8}$$

$$\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}\mathbf{1} \tag{3.9}$$

dirac density and dirac current density;

$$\rho = \Psi^\dagger\Psi \tag{3.10}$$

$$\vec{j} = \Psi^\dagger(c\vec{\alpha})\Psi = \pm\rho\vec{v} = \pm\rho\frac{c^2\vec{p}}{E} \tag{3.11}$$

or with $\bar{\Psi} = \Psi^\dagger\gamma^0$

$$\partial_\mu(\bar{\Psi}\gamma^\mu\Psi) = 0 \tag{3.12}$$

transformation of spinor:

$$S^{-1}\gamma^\mu S = \Lambda^\mu{}_\nu\gamma^\nu \tag{3.13}$$

free particle solution:

$$\Psi = \exp\left\{i(\vec{p}\cdot\vec{r} - E_p t)/\hbar\right\}U(\varepsilon,\vec{p}) \tag{3.14}$$

$$
\begin{aligned}
U^+_{p\uparrow} &= \sqrt{\frac{E_p + mc^2}{2mc^2}} \begin{pmatrix} 1 \\ 0 \\ \frac{cp_z}{E_p+mc^2} \\ \frac{c(p_x+ip_y)}{E_p+mc^2} \end{pmatrix} \quad U^+_{p\downarrow} = \sqrt{\frac{E_p + mc^2}{2mc^2}} \begin{pmatrix} 0 \\ 1 \\ \frac{c(p_x-ip_y)}{E_p+mc^2} \\ \frac{-cp_z}{E_p+mc^2} \end{pmatrix} \\
U^-_{p\uparrow} &= \sqrt{\frac{E_p + mc^2}{2mc^2}} \begin{pmatrix} \frac{cp_z}{E_p+mc^2} \\ \frac{c(p_x+ip_y)}{E_p+mc^2} \\ 1 \\ 0 \end{pmatrix} \quad U^-_{p\downarrow} = \sqrt{\frac{E_p + mc^2}{2mc^2}} \begin{pmatrix} \frac{c(p_x-ip_y)}{E_p+mc^2} \\ \frac{-cp_z}{E_p+mc^2} \\ 0 \\ 1 \end{pmatrix}
\end{aligned}
\tag{3.15}
$$

Parity operator:

$$\hat{P}_S := \beta \hat{P} \tag{3.16}$$

Pauli matrices

$$\sigma_a = \begin{pmatrix} \delta_{a3} & \delta_{a1} - i\delta_{a2} \\ \delta_{a1} + i\delta_{a2} & -\delta_{a3} \end{pmatrix} \tag{3.17}$$

$$\sigma_i \sigma_j = \delta_{ij} \mathbf{1} + i\epsilon_{ijk}\sigma_k \tag{3.18}$$

$$(\vec{a} \cdot \vec{\sigma})(\vec{b} \cdot \vec{\sigma}) = (\vec{a} \cdot \vec{b})\, \mathbf{1} + i(\vec{a} \times \vec{b}) \cdot \vec{\sigma} \tag{3.19}$$

$$[\sigma_a, \sigma_b] = 2i\epsilon_{abc}\sigma_c \tag{3.20}$$

$$\{\sigma_a, \sigma_b\} = 2\delta_{ab}\mathbf{1} \tag{3.21}$$

$$\tag{3.22}$$

4 Second Quantization

Symmetrization/Antisymmetrization of many-particle states

$$\mathcal{P}_{(B,F)}\psi(\vec{r}_1, \ldots, \vec{r}_N) = \frac{1}{N!} \sum_P \xi^P \psi(\vec{r}_{P1}, \ldots, \vec{r}_{PN}) \tag{4.1}$$

with $\xi = +1$ for bosons or -1 for fermions.

creation and annihilation operator For bosons:

$$\hat{a}^\dagger_i \left|\ldots, n_i, \ldots\right\rangle = \sqrt{n_i + 1} \left|\ldots, n_i + 1, \ldots\right\rangle \tag{4.2}$$

$$\hat{a}_i \left|\ldots, n_i, \ldots\right\rangle = \sqrt{n_i} \left|\ldots, n_i - 1, \ldots\right\rangle \tag{4.3}$$

$$[\hat{a}_i, \hat{a}^\dagger_j] = \delta_{ij} \tag{4.4}$$

$$n_i = \hat{a}^\dagger_i \hat{a}_i \tag{4.5}$$

For fermions the commutators are replaced by anticommutator.

Hamiltonian for bosons

$$\hat{H} = \sum_{i,j} t_{ij} \hat{a}^\dagger_i \hat{a}_j + \frac{1}{2} \sum_{i,j,k,l} V_{ijkl} \hat{a}^\dagger_i \hat{a}^\dagger_j \hat{a}_k \hat{a}_l \tag{4.6}$$

Field operators

$$\hat{\psi}^\dagger(\vec{x}) = \sum_\alpha \langle\alpha|\vec{x}\rangle\, \hat{a}^\dagger_\alpha = \sum_\alpha \phi^*_\alpha(\vec{x})\hat{a}^\dagger_\alpha \tag{4.7}$$

$$\hat{\psi}(\vec{x}) = \sum_\alpha \langle\vec{x}|\alpha\rangle\, \hat{a}_\alpha = \sum_\alpha \phi_\alpha(\vec{x})\hat{a}_\alpha \tag{4.8}$$

For bosons and fermions:

$$[\psi(\vec{x}), \psi^\dagger(\vec{x}\,')] = \delta^{(3)}(\vec{x} - \vec{x}\,') \tag{4.9}$$

$$\{\psi(\vec{x}), \psi^\dagger(\vec{x}\,')\} = \delta^{(3)}(\vec{x} - \vec{x}\,') \tag{4.10}$$

Hamiltonian

$$\hat{H} = \int \mathrm{d}^3x \left[\vec{\nabla}\hat{\psi}^\dagger(\vec{x})\frac{\hbar^2}{2m}\vec{\nabla}\hat{\psi}(\vec{x}) + V(\vec{x})\hat{\psi}^\dagger(\vec{x})\hat{\psi}(\vec{x})\right] \tag{4.11}$$

$$+\frac{1}{2}\int \mathrm{d}^3x \int \mathrm{d}^3x'\,\hat{\psi}^\dagger(\vec{x})\hat{\psi}^\dagger(\vec{x}\,')U(\vec{x}-\vec{x}\,')\hat{\psi}(\vec{x})\,'\hat{\psi}(\vec{x}) \tag{4.12}$$

Bosonic coherent states

$$\hat{a}_{\alpha_i}|\phi\rangle = \phi_{\alpha_i}|\phi\rangle \tag{4.13}$$

$$|\phi\rangle = \sum_{n_{\alpha_1}, n_{\alpha_2}, \dots} \phi_{n_{\alpha_1}, n_{\alpha_2}, \dots}|n_{\alpha_1}\, n_{\alpha_2}, \dots\rangle \tag{4.14}$$

$$|n_{\alpha_1}\, n_{\alpha_2}, \dots\rangle = \frac{(\hat{a}^\dagger_{\alpha_1})^{n_{\alpha_1}}}{\sqrt{n_{\alpha_1}!}}\frac{(\hat{a}^\dagger_{\alpha_2})^{n_{\alpha_2}}}{\sqrt{n_{\alpha_2}!}}\dots|0\rangle \tag{4.15}$$

$$|\phi\rangle = \exp\left(\sum_{\alpha_i}\phi^*_{\alpha_i}\hat{a}^\dagger_{\alpha_i}\right)|0\rangle \tag{4.16}$$

$$\hat{a}^\dagger_{\alpha_i}|\phi\rangle = \frac{\partial}{\partial\phi_{\alpha_i}}|\phi\rangle \tag{4.17}$$

$$\langle\phi|\phi'\rangle = \exp\left(\sum_{\alpha_i}\phi^*_{\alpha_i}\phi'_{\alpha_i}\right) \tag{4.18}$$

Fermionic coherent states

with grassman variable ξ

$$\{\xi, \hat{c}\} = \{\xi^*, \hat{c}^\dagger\} = \{\xi^*, \hat{c}\} = \{\xi, \hat{c}^\dagger\} = 0 \tag{4.19}$$

$$|\xi\rangle = \exp\left(-\sum_\alpha \xi_\alpha\hat{c}^\dagger_\alpha\right)|0\rangle \tag{4.20}$$

5 Path integral

time evolution of wave function

$$\psi(x_f, t_f) = \int \mathrm{d}x_i U(x_f, t_f, x_i, 0)\,\psi(x_i, 0) \tag{5.1}$$

$$U(x_f, t_f, x_i, 0) = \mathcal{N}\int \mathcal{D}[x(t)]e^{\frac{i}{\hbar}S[x(t)]} \tag{5.2}$$

with $S[x(t)] = \int_0^{t_f} \mathrm{d}t \mathcal{L}(x(t), \dot{x}(t))$