H5.1 Chenhuan Wang.

a) $$u_0''(r) + k^2 u_0(r) = \frac{2m}{\hbar^2} V(r)\, u_0(r)$$

$r < R$: $u_0(r) = \alpha \sin(kr)$

$$LHS = -\alpha^2 k^2 \sin(kr) + k^2 \alpha \sin(kr) = 0 = RHS$$

$r > R$: $u_0(r) = \beta \sin(kr + \delta_0(k))$

$$LHS = -\beta k^2 \sin(kr + \delta_0(k)) + k^2 \beta \sin(kr + \delta_0(k)) = 0 = RHS.$$

$$u_0(0) = \begin{cases} \alpha \sin(0) & = 0 \\ \beta \sin(\delta_0(k)) \end{cases}$$

b) SE: $$-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial r^2} R(r) + V(r)\, R(r) = E\, R(r)$$

$$\Rightarrow \partial_r^{2} R(r) = (E - V(r)) R(r) \cdot \left(-\frac{2m}{\hbar^2}\right)$$

at $x = R$, in the limit $\varepsilon \to 0$

$$\int_{R-\varepsilon}^{R+\varepsilon} dr\, \partial_r^2 R(r) = \int_{R-\varepsilon}^{R+\varepsilon} dr (E - V(r))\, R(r) \cdot \frac{-2m}{\hbar^2}$$

$$\partial_r R(r)\Big|_{r=R+\varepsilon} - \partial_r R(r)\Big|_{r=R-\varepsilon} = \int_{R-\varepsilon}^{R+\varepsilon} dr\, V(r)\, R(r) \frac{2m}{\hbar^2}$$

$$= R(R) \frac{2m}{\hbar^2} V_0$$

Plug in the Ansatz:

$$-\partial_r(\alpha \sin(kr)/r) + \partial_r(\beta \sin(kr + \delta_0(k))/r) = \frac{u_0(R)}{r} \frac{2m}{\hbar^2} V_0$$

$$-\alpha k \frac{\cos(kR)}{R} + \alpha \frac{\sin(kR)}{R^2} + \beta k \frac{\cos(kR + \delta_0(k))}{R} - \beta \frac{\sin(kR + \delta_0(r))}{R^2}$$

$$= \frac{u_0(R)}{R} \frac{2m}{\hbar^2} V_0$$

Continuity of wave function: $\alpha \sin(kR) = \beta \sin(kR + \delta_0(k))$

this condition cancels the two terms in the previous equation

$$\Rightarrow \alpha k \cos(kR) - \beta k \cos(kR + \delta_0(k)) = -\alpha \sin(kR) \frac{2mV_0}{\hbar^2}$$

$$\alpha k \cos(kR) + \alpha \sin(kR) \frac{2mV_0}{\hbar^2} = \beta k \cos(kR + \delta_0(k)) \quad \Big| \frac{1}{\alpha \sin(kR)} = \frac{1}{\beta \sin(kR+\delta_0(k))}$$

$$\Rightarrow k \cot(kR) - \frac{2mV_0}{\hbar^2} = k \cot(kR + \delta_0(k))$$

$$\Rightarrow \cot(kR + \delta_0(k)) - \cot(kR) = \frac{2mV_0}{\hbar^2} \cdot \frac{1}{k} = \Phi / kR$$

$$\cot(\alpha \pm \beta) = \frac{\cot\alpha \cot\beta \mp 1}{\cot\beta \pm \cot\alpha}$$

$$\Rightarrow \frac{\cot(kR)\cot(\delta_0(k)) - 1}{\cot(\delta_0(k)) + \cot(kR)} - \cot(kR) = \Phi / kR$$

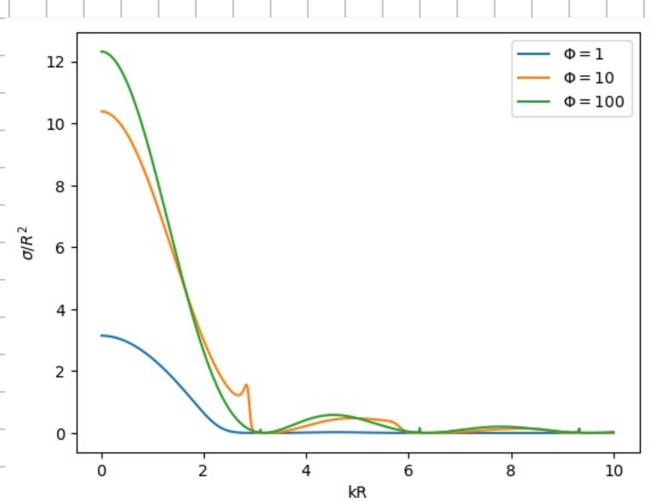
$$\frac{\cot(kR)\cancel{\cot(\delta_0(k))} - 1 - \cot^2(kR) - \cot(kR)\cancel{\cot(\delta_0(k))}}{\cot(\delta_0(k)) + \cot(kR)} = \Phi / kR$$

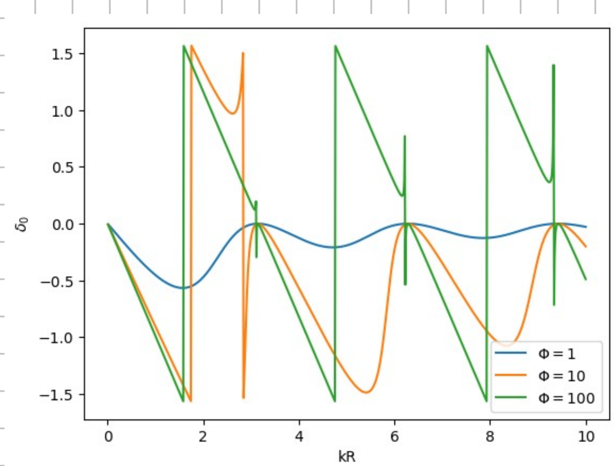
$$-1 - \cot^2(kR) = \Phi / kR \left(\cot(\delta_0(k)) + \cot(kR)\right)$$

$$\frac{kR}{\Phi}\left[-1 - \cot^2(kR)\right] - \cot(kR) = \cot(\delta_0(k))$$

$$\Rightarrow \tan(\delta_0(k)) = \frac{\sin^2(kR)}{-kR/\Phi - 2\sin(kR)\cos(kR)} = \frac{-\sin^2(kR)}{kR/\Phi + \sin(2kR)/2}$$

c)





d) $V_0 R \to \infty$, $\Phi \to \infty$ $\Rightarrow$ $\tan(\delta_0(k)) = \frac{-\sin^2(kR)}{2\sin(2kR)/2} = -\frac{\sin(kR)}{\cos(kR)}$

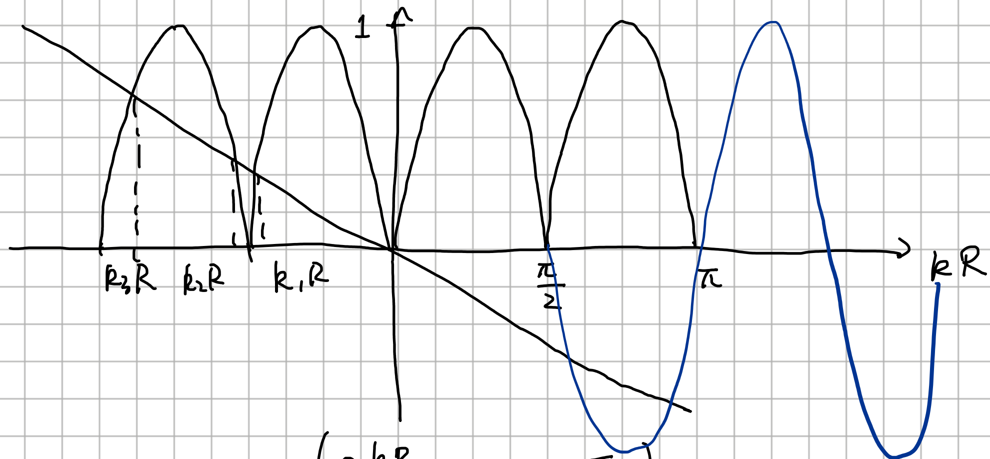
$\Rightarrow -\delta_0(k) = kR$

$\Rightarrow \sigma_0(k) = \frac{4\pi}{k^2}\sin^2(-kR) = \frac{4\pi}{k^2}\sin^2(kR) \xrightarrow{kR \ll 1} 4\pi R^2$ hard sphere scattering.

$\to$ s-wave total cross section

e) denominator zero $\to$ $kR/\Phi + \sin^{2}(2kR)/2 = 0$ $kR/\Phi + \sin(2kR)/2 = 0$

$\sin^{2}(2kR) = -\frac{2kR}{\Phi}$



$tg\,\theta = \frac{1}{\Phi} \xrightarrow{\Phi \to \infty} 0$

$k = \frac{m\pi}{2R}$

$n(V_0R) = 2 \cdot \left(\frac{2kR}{\Phi(V_0R)} \bmod \frac{\pi}{2} \right) - 1$

$\uparrow$ not taking the $kR = 0$ point into account.

Large V_0R $\to$ large Φ $\to$ small slope of $\frac{2}{\Phi}kR$

$\Rightarrow k_n \simeq -m \cdot \frac{\pi}{2}\frac{1}{R}$, $m = 1, 2, 3, \ldots$

f) in the neighborhood of resonances k_n:

$$\cot(\delta_{0,n}) = \cot(\delta_l(E))\Big|_{E=E_n} + \frac{d(\cot(\delta_l))}{dE}\Big|_{E=E_n}(E-E_n) + O((E-E_n)^2)$$

$$\Rightarrow \cot(\delta_{0,n}) = \frac{d(\cot(\delta_l))}{dE}\Big|_{E=E_n}(E-E_n)$$

$$\frac{d\cot(\delta_l)}{dE} = \frac{d}{dE}\frac{kR/\Phi + \sin(2kR)/2}{-\sin^2(kR)}$$

$$= \frac{d}{d(kR)}\frac{kR/\Phi + \sin(2kR)/2}{-\sin^2(kR)} \cdot \frac{d(kR)}{dE}$$

$$= \left\{ \frac{1}{\Phi} \cdot (-\csc^2(kR) + 2kR\cot(kR)\csc^2(kR) + \csc^2(kR) \right\} \cdot \frac{Rm}{\hbar^2 k}$$

$$\left[\;\frac{d\,kR}{dE} = \frac{d}{dE}\left(\sqrt{2mE}\cdot\frac{R}{\hbar}\right) = \frac{R}{\hbar}\,\frac{1}{2\sqrt{2mE}}\cdot\cancel{2}m = \frac{Rm}{\hbar}\,\frac{1}{\sqrt{2mE}} = \frac{Rm}{\hbar^2 k}\;\right]$$

$$= \frac{Rm}{\hbar^2 k}\,\frac{1}{\sin^2(kR)}\left\{-\frac{1}{\Phi} + 1 + 2kR\cot(kR)\right\}$$

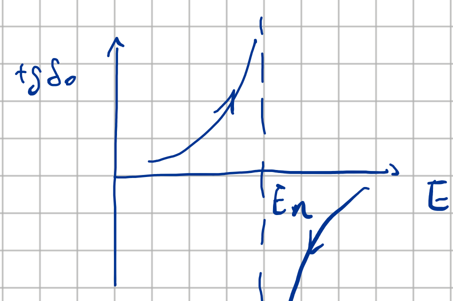
$$\Rightarrow \quad \cot(\delta_{0,n}) = (E - E_n)\cdot\frac{Rm}{\hbar^2 k}\,\frac{1}{\sin^2(kR)}\left\{1 - \frac{1}{\Phi} + 2kR\cot(kR)\right\}\Bigg|_{k=k_n}$$

$$\text{or} \quad \tan(\delta_{0,n}) = \frac{1}{E - E_n}\,\frac{\hbar^2 k_n}{Rm}\,\sin^2(k_nR)\left\{1 - \frac{1}{\Phi} + 2k_nR\cot(k_nR)\right\}^{-1}$$

$$=: \gamma_n\,\frac{k_nR}{E - E_n}$$

$$\text{with} \quad \gamma_n = \frac{\hbar^2}{R^2 m}\sin^2(k_nR)\left\{1 - \frac{1}{\Phi} + \underbrace{2k_nR\cot(k_nR)}_{\approx 0}\right\}^{-1}$$

$$\gamma_n = \frac{-\sin^2(k_nR)}{R/2V_0 + \frac{mR^2}{\hbar^2}\cos(2k_nR)}$$



$\gamma_n < 0$

$$\gamma_n = \frac{-\sin^2 k_nR}{\frac{mR^2}{\hbar^2}\left[\frac{1}{\Phi} + \cos 2k_nR\right]}$$

g) $\Rightarrow \quad \gamma_n \geq 0$

$\Rightarrow \quad \tan(\delta_0) > 0 \quad$ when $\quad E - E_n > 0$

$R\cdot k_{odd}$ are slightly smaller that $m\cdot\frac{\pi}{2}$

$R\cdot k_{even}$ " " " larger

h) $$f_k(\theta) = \frac{1}{k}\,e^{i\delta_0(k)}\sin(\delta_0(k))\,\underbrace{P_0(\cos\theta)}_{=1}$$

$$|f_k(\theta)| = \frac{1}{k}\cos(\delta_0(k))\sin(\delta_0(k))$$

$$= \frac{1}{k} \cdot \cos\left(\arctan\left(\gamma_n \frac{k_n R}{E-E_n}\right)\right) \cdot \sin\left(\arctan\left(\gamma_n \frac{k_n R}{E-E_n}\right)\right)$$

$$= \frac{1}{k^2} \cdot \frac{\gamma_n \frac{k_n R}{E-E_n}}{1+\left(\gamma_n \frac{k_n R}{E-E_n}\right)^2}$$

$$\Rightarrow \sigma_0(\theta) = |f_k(\theta)|^2 \cdot 4\pi$$ ← or use optical theorem

$$= \underbrace{\frac{4\pi}{k^2}}_{= \sigma_{max}} \left(\frac{\gamma_n \cdot \frac{k_n R}{E-E_n}}{1+\left(\gamma_n \frac{k_n R}{E-E_n}\right)^2}\right)^2$$ X

i) $V_0 R \gg 1$, strong coupling $\Rightarrow \delta_0 \approx -kR$

$$\tan(kR) = \gamma_n \frac{k_n R}{E_n - E}$$

$$\Rightarrow E_n = \frac{\gamma_n k_n R}{\tan(kR)} + E$$ X

f) $$E - E_n = \frac{\hbar^2}{2m}(k^2 - k_n^2) \approx \frac{\hbar^2 k_n}{m}(k - k_n)$$

around $k = k_n$

(2) → The resonances / singularities are caused by the denominator

$$\frac{kR}{\Phi} + \frac{1}{2}\sin(2kR) \underset{k=k_n}{\approx} \frac{R}{\Phi}(k-k_n) + R\cos(2k_n R)(k-k_n)$$

$$= \left[\frac{1}{\Phi} + \cos(2k_n R)\right] R (k - k_n)$$

$$\approx \left[\frac{1}{\Phi} + \cos(2k_n R)\right] R \cdot \frac{m}{\hbar^2 k_n}(E - E_n)$$

$$= \left[\frac{1}{2V_0 k_n} + \frac{Rm}{\hbar^2 k_n}\cos(2k_n R)\right](E - E_n)$$

$$\Rightarrow \tan(\delta_{0,n}) \simeq \frac{-\sin^2(kR)}{\left[\frac{1}{2V_0 k_n} + \frac{Rm}{\hbar^2 k_n}\cos(2k_nR)\right](E-E_n)}$$

$$= \gamma_n \frac{k_nR}{E-E_n}$$

$$\text{with } \gamma_n = \frac{-\sin^2(kR)}{\left(\frac{1}{2V_0k_n} + \frac{Rm}{\hbar^2 k_n}\cos(2k_nR)\right)(k_nR)}$$

$$= \frac{-\sin^2(kR)}{\frac{R}{2V_0} + \frac{mR^2}{\hbar^2}\cos(2k_nR)}$$

g)

$$\frac{d}{dk}\sin(2kR)\Big|_{k\simeq k_n=\frac{n\pi}{2R}} = 2R\cos(2k_nR) \begin{cases} >0 & , n=2m \\ <0 & , n=2m-1 \end{cases}$$

$$\frac{d}{dk} - \frac{2kR}{\Phi}\Big|_{k\simeq k_n} = -\frac{\hbar^2}{mV_0}$$

from plot:

$$2R\cos(2k_{2n-1}R) < -\frac{\hbar^2}{mV_0}$$

$$\Leftrightarrow \frac{mR^2}{\hbar^2}\cos(2k_{2n-1}R) < -\frac{R}{2V_0}$$

$$2R\cos(2k_{2n}R) > -\frac{\hbar^2}{mV_0}$$

$$\Leftrightarrow \frac{mR^2}{\hbar^2}\cos(2k_{2m}R) > -\frac{R}{2V_0}$$

$$\rightarrow \gamma_n : \begin{cases} >0 & , n \text{ odd} \\ <0 & , n \text{ even} \end{cases}$$

$$\Rightarrow n \text{ odd}: \quad \tan\delta_0 \begin{cases} <0 & , E \lesssim E_n \\ >0 & , E \gtrsim E_n \end{cases}$$

$$\text{even}: \quad \begin{cases} >0 & , E \lesssim E_n \\ <0 & , E \gtrsim E_n \end{cases}$$

We assume the δ_0 is a monotone function of energy E.
$E \to 0$, the phase should $\delta_0 \to 0$ (no particle)

$\Rightarrow$ odd n do not have physical meaning.

h) $$\sigma_0 = \frac{4\pi}{k^2} \sin^2(\delta_0) = \frac{4\pi}{k^2} \frac{\tan^2(\delta_0)}{1+\tan^2(\delta_0)}$$

$$\Rightarrow \sigma_{0,n} = \gamma_n^2 \frac{4\pi}{k^2} \frac{(kR)^2/(E-E_n)^2}{1+\gamma_n^2 \frac{(k_nR)^2}{(E-E_n)^2}} = \frac{4\pi}{k^2} \frac{\gamma_n^2 (kR)^2}{(E-E_n)^2 + \frac{\gamma_n^2}{(k_nR)^2}}$$

Wenn $E = E_n \Rightarrow \sigma_{0,n} = \frac{4\pi}{k^2}$

Breit - Wigner - distribution.

i) $$V_0 R \gg 1 \Rightarrow \frac{\hbar^2}{2mV_0R} = \frac{1}{\Phi} \ll 1$$

The plot of part e): $2k_nR = n\pi + \alpha \quad |\alpha| \ll 1$

$$\Rightarrow \sin(2k_nR) = \sin(n\pi + \alpha)$$
$$= \sin(n\pi)\cos(\alpha) + \cos(n\pi)\sin(\alpha)$$
$$= (-1)^n \sin(\alpha) \approx (-1)^n \alpha$$

And the resonance condition:

$$\sin(2k_nR) \overset{!}{=} -\frac{\hbar^2}{2mV_0R}(n\pi + \alpha) \approx -\frac{\hbar^2}{2mV_0R} n\pi$$

$$\to \alpha \approx (-1)^{n+1} \frac{\hbar^2}{2mV_0R} n\pi$$

$$\to 2k_nR \approx n\pi\left(1 + (-1)^{n+1} \frac{\hbar^2}{2mV_0R}\right)$$

$$E_n = \frac{\hbar^2 k_n^2}{2m} = \frac{\hbar^2 (n\pi)^2}{8mR^2}\left(1 + (-1)^{n+1}\frac{\hbar^2}{mV_0 R}\right)$$

$n = 2m$

$$\Rightarrow \quad E_n \simeq \frac{\hbar^2}{2mR^2}(m\pi)^2\left(1 - \frac{\hbar^2}{mV_0}\right)$$

j) $$\sin(k_n R) = \sin\left(\frac{n\pi}{2} + \frac{\alpha}{2}\right) = \sin\left(m\pi - \frac{\hbar^2}{2mV_0 R} m\pi\right)$$

$$= \sin(m\pi)\cos\left(\frac{\hbar^2}{2mV_0 R} m\pi\right) - \cos(m\pi)\sin\left(\frac{\hbar^2}{2mV_0 R} m\pi\right)$$

$$= (-1)^{m+1}\sin\left(\frac{\hbar^2}{2mV_0 R} m\pi\right) \simeq (-1)^{m+1}\frac{\hbar^2}{2mV_0 R} m\pi$$

$$\cos(2k_n R) \simeq \cos(n\pi) = (-1)^n = 1$$

$$\Rightarrow \gamma_n = \frac{-\sin^2(k_n R)}{\frac{R}{2V_0} + \frac{mR^2}{\hbar^2}\cos(2k_n R)}, \quad n = 2m$$

$$= \frac{-\left(\frac{\hbar^2}{2mV_0 R}\right)^2 (m\pi)^2}{\frac{R}{2V_0} + \frac{mR^2}{\hbar^2}} \simeq \frac{-\left(\frac{\hbar^2}{2mV_0 R}\right)^2 (m\pi)^2}{\frac{mR^2}{\hbar^2}\left(1 + \frac{\hbar^2}{2mV_0 R}\right)}$$

$$\simeq -\frac{\hbar^2}{mR^2}\left(\frac{\hbar^2}{2mV_0 R}\right)^2 (m\pi)^2$$

$$E_n = E_{2m} = \frac{\hbar^2}{2mR^2}(m\pi)^2\left(1 - \frac{\hbar^2}{mV_0 R}\right)$$

$$\Rightarrow -\frac{1}{2}E_n\left(\frac{\hbar^2}{mV_0 R}\right)^2 = -\frac{\hbar^2}{4mR^2}(m\pi)^2\left(\frac{\hbar^2}{mV_0 R}\right)^2 \cdot \left(1 - \frac{\hbar^2}{mV_0 R}\right)$$

$$\simeq -\frac{\hbar^2}{mR^2}(m\pi)^2\left(\frac{\hbar^2}{2mV_0 R}\right)^2 \simeq \gamma_n$$