H 7.1 Chenhuan Wang

a)
$$[\hat{J}_i, \hat{J}_j] = [\mathbb{1}_4 \hat{L}_i + S_i, \mathbb{1}_4 \hat{L}_j + S_j]$$
$$= [\mathbb{1}_4 \hat{L}_i + S_i, \mathbb{1}_4 \hat{L}_j] + [\mathbb{1}_4 \hat{L}_i + S_i, S_j]$$
$$= \mathbb{1}_4 [\hat{L}_i, \hat{L}_j] + \underbrace{[S_i, \mathbb{1}_4 \hat{L}_j]} + \underbrace{[\mathbb{1}_4 \hat{L}_i, S_j]}_{=0} + [S_i, S_j]$$
— S_j is constante
$$= \mathbb{1}_4 \, i\epsilon_{ijk} \hat{L}_k + i\epsilon_{ijk} S_k$$
$$= i\epsilon_{ijk} \hat{J}_k$$

$$[\hat{J}_i, \hat{\vec{J}}^2] = [\hat{J}_i, \hat{J}_i^2 + \hat{J}_j^2 + \hat{J}_k^2] = [\hat{J}_i, \hat{J}_j^2] + [\hat{J}_i, \hat{J}_k^2]$$
$$= [\hat{J}_i, \hat{J}_j]\hat{J}_j + \hat{J}_j[\hat{J}_i, \hat{J}_j] + [\hat{J}_i, \hat{J}_k]\hat{J}_k + \hat{J}_k[\hat{J}_i, \hat{J}_k]$$
$$\left([A, BC] = [A, B]C + B[A, C] \right)$$
$$= \underline{i\epsilon_{ijk} \hat{J}_k \hat{J}_j} + \underbrace{\hat{J}_j \hat{J}_k \cdot i\epsilon_{ijk}} + \underbrace{i\epsilon_{ikj} \hat{J}_j \hat{J}_k} + \underline{i\epsilon_{ikj} \hat{J}_k \hat{J}_j}$$
$$= 0$$

b)
$$[\hat{H}, \hat{J}_i] = [\alpha_k \hat{p}_k + \beta m + \mathbb{1}_4 V, \mathbb{1}_4 \hat{L}_i + S_i] \qquad V = V(|\vec{x}|)$$
summation! $\quad [V(|\vec{x}|), \hat{L}_i] = 0$, → Electron in hydrogen atom / or mathematically the L_i only contains the angle derivatives
$$= [\alpha_k \hat{p}_k, \mathbb{1}_4 \hat{L}_i] + [\alpha_k \hat{p}_k + \beta m, S_i]$$

$$\left[\beta S_i = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix} = \begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix} = \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = S_i \beta \right]$$

one cannot simply write α in the front!

$$= \underbrace{\alpha_k} [\hat{p}_k, \epsilon_{ijl} x_j \hat{p}_l] + [\alpha_k \hat{p}_k, S_i]$$
$$= \alpha_k \epsilon_{ijl} \{[\hat{p}_k, \hat{x}_j]\hat{p}_l + x_j[\hat{p}_k, \hat{p}_l]\} + \tfrac{1}{2}\hat{p}_k \left[\begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}, \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix} \right]$$
$$= \alpha_k \epsilon_{ijl} \hat{p}_l \delta_{kj} + \tfrac{1}{2}\hat{p}_k \left[\begin{pmatrix} 0 & \delta_{ik}\mathbb{1} + i\epsilon_{ikj}\sigma_j \\ \delta_{ik}\mathbb{1} + i\epsilon_{ikj}\sigma_j & 0 \end{pmatrix} - \begin{pmatrix} 0 & \delta_{ik}\mathbb{1} + i\epsilon_{kij}\sigma_j \\ \delta_{ik}\mathbb{1} + i\epsilon_{kij}\sigma_j & 0 \end{pmatrix} \right]$$
$$= \epsilon_{ikl}\alpha_k \hat{p}_l + \hat{p}_k \epsilon_{ikj}\alpha_j = \epsilon_{ikl}\alpha_k \hat{p}_l + \epsilon_{ikj} \hat{p}_k \alpha_j = 0$$

$$[\hat{H}, \hat{J}_i] = 0 \Rightarrow [\hat{H}, \hat{J}_i^2] = 0 \Rightarrow [\hat{H}, \hat{\vec{J}}^2] = 0$$

c) $[\hat{H}, \hat{P}_s]\psi = [\hat{H}, \beta\hat{P}]\psi = [\alpha_k \hat{p}_k + \beta m + \mathbb{1}_4 V(|\vec{x}|), \beta\hat{P}]\psi$

$= [\alpha_k \hat{p}_k, \beta\hat{P}]\psi$

$= \alpha_k \hat{p}_k \beta\hat{P}\psi - \beta\hat{P}\alpha_k\hat{p}_k\psi$? $= \underbrace{\{\alpha_k, \beta\}}_{=0} p_k P - \beta\alpha_k \underbrace{\{p_k, P\}}_{=0}$

$= 0$

$$\left[\begin{array}{ll} \alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} & \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ \alpha_k \cdot \beta = \begin{pmatrix} 0 & -\sigma_k \\ \sigma_k & 0 \end{pmatrix} & \beta \cdot \alpha_k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix} \end{array}\right]$$

$= \underline{\alpha_k} \hat{p}_k \underline{\beta} \psi(t, -\vec{x}) - \underline{\beta\alpha_k} (-\hat{p}_k)\psi(t, -\vec{x})$

$= 0$

$[\hat{J}_i, \hat{P}_S] = [\mathbb{1}_4 \hat{L}_i + S_i, \beta\hat{P}]\psi = [S_i, \beta\hat{P}]\psi$

$\begin{pmatrix} \hat{P}\hat{x}_i = -\hat{x}_i \\ \hat{P}\hat{p}_i = -\hat{p}_i \end{pmatrix}$

$= \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \psi(t, -\vec{x}) - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}\psi(t, -\vec{x})$

$= 0$

$[\hat{\vec{J}}^2, \hat{P}_S] = [\hat{J}_1^2 + \hat{J}_2^2 + \hat{J}_3^2, \hat{P}_S] = \sum_i [\hat{J}_i^2, \hat{P}_S] = \sum_i \{\hat{J}_i [\hat{J}_i, \hat{P}_S] + [\hat{J}_i, \hat{P}_S]\hat{J}_i\}$

$= [J_i J_i, P_S] = J_i [J_i, P_S] + [J_i, P_S] J_i = 0$ $=0$

d) conserved: j, m_j, P_s ← eigenvalue to $\hat{P}_s$

eigenstates: j, m_j, invariant under inverse

H7.2

a) if $\psi: \mathbb{R} \to \mathbb{C}$, then the only componente that would transform is $\partial_\mu = \frac{\partial}{\partial x^\mu} \to \partial_\mu' = \frac{\partial}{\partial x'^\mu}$. The right side is unchanged

$\Rightarrow \partial_\mu = \partial_\mu'$, i.e. the equation is then not invariant under L

b) to fix this, write $\psi: \mathbb{R}^4 \to \mathbb{C}^4$.

Lorentz transformed Dirac equation:

$$i\gamma^\mu \partial_\mu' \psi' = \frac{mc}{\hbar}\psi' \quad , \quad \psi' = S\psi, \ \partial_\mu' = \frac{\partial}{\partial x'^\mu}$$

$$i\gamma^\mu \frac{\partial}{\partial x^{\mu\prime}} S\psi = \frac{mc}{\hbar} S\psi$$

$$i\gamma^\mu \underbrace{\frac{\partial x^\nu}{\partial x'^\mu}} \frac{\partial}{\partial x^\nu} S\psi = \frac{mc}{\hbar} S\psi$$

γ^μ is scalar (even though it is matrix)

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \Rightarrow \frac{\partial x^\nu}{\partial x'^\mu} = (\Lambda^{-1})^\mu{}_\nu = \Lambda^\nu{}_\mu$$

$$i \underbrace{S^{-1}\gamma^\mu \Lambda^\nu{}_\mu S}\, \partial_\nu \psi = \frac{mc}{\hbar}\psi$$

$$= S^{-1}\gamma^\mu S \Lambda^\nu{}_\mu \overset{!}{=} 1\, \gamma^\nu$$

$$\Rightarrow S^{-1}\gamma^\mu S = \Lambda^\mu{}_\nu \gamma^\nu$$

c) $$\Lambda^\mu_\nu = [\exp(-i\vec{\omega}\cdot\vec{S} - i\vec{\zeta}\cdot\vec{K})]^\mu{}_\nu$$

$$= [\exp(-i\omega^i \cdot S^i - i\zeta^i \cdot K^i)]^\mu_\nu \quad \leftarrow \text{Summation over } i$$

$$= [\exp(-i\tfrac{1}{4}\epsilon^{ijk}\omega_{jk}\epsilon^{imn}M_{mn} - i\omega^{i0}M^{0i})]^\mu{}_\nu$$

$$\left(\epsilon^{ijk}\epsilon^{imn} = \delta^{jm}\delta^{kn} - \delta^{jn}\delta^{km} \right)$$

$$= \exp(-\tfrac{i}{2}\omega_{jk}M_{jk} - i\omega^{i0}M^{0i})^\mu{}_\nu$$

$M_{kj} = -M_{jk}$

$$= \exp(-\tfrac{i}{2}\omega_{jk}M^{jk} - \tfrac{i}{2}\underbrace{\omega^{i0}M^{0i}}_{=\omega_{i0}M^{i0}} - \tfrac{i}{2}\underbrace{\omega^{0i}M^{i0}}_{=\omega_{0i}M^{0i}})^{\mu}{}_{\nu}$$

$$\left(\begin{aligned} \omega_{\rho\sigma}M^{\rho\sigma} &= \omega_{\rho 0}M^{\rho 0} + \omega_{\rho i}M^{\rho i} \qquad [\text{Assume } \omega_{00}M^{00} = 0] \\ &= \omega_{i0}M^{i0} + \omega_{0i}M^{0i} + \omega_{ki}M^{ki} \end{aligned}\right)$$

skew symmetric tensor: diagonal elements vanish!

$$= \exp(-\tfrac{i}{2}\omega_{\rho\sigma}M^{\rho\sigma})^{\mu}{}_{\nu}$$

d) $$S = \exp(-\tfrac{i}{2}\omega_{\rho\sigma}\Sigma^{\rho\sigma})$$

Eq. (2): $$S^{-1}\gamma^{\mu}S = \Lambda^{\mu}{}_{\nu}\gamma^{\nu}$$

take the first order of the exponential in S and $\Lambda^{\mu}{}_{\nu}$

$$S = \exp(-\tfrac{i}{2}\omega_{\rho\sigma}\Sigma^{\rho\sigma})$$

$$= \sum_n \frac{1}{n!}\left(-\tfrac{i}{2}\omega_{\rho\sigma}\Sigma^{\rho\sigma}\right)^n$$

$$= -\tfrac{i}{2}\omega_{\rho\sigma}\Sigma^{\rho\sigma} + \mathcal{O}(\omega_{\rho\sigma}^2)$$

$$\Lambda^{\mu}{}_{\nu} = \exp(-\tfrac{i}{2}\omega_{\rho\sigma}M^{\rho\sigma})^{\mu}{}_{\nu} = \sum_n \frac{1}{n!}\left[(-\tfrac{i}{2}\omega_{\rho\sigma}M^{\rho\sigma})^{\mu}{}_{\nu}\right]^n$$

$$= (-\tfrac{i}{2}\omega_{\rho\sigma}M^{\rho\sigma})^{\mu}{}_{\nu} + \mathcal{O}(\omega_{\rho\sigma}^2)$$

$$\Rightarrow \left(1 - \tfrac{i}{2}\omega_{\rho\sigma}\Sigma^{\rho\sigma}\right)^{-1}\gamma^{\mu}\left(1 - \tfrac{i}{2}\omega_{\rho\sigma}\Sigma^{\rho\sigma}\right) = \left(1 - \tfrac{i}{2}\omega_{\rho\sigma}M^{\rho\sigma}\right)^{\mu}{}_{\nu}\gamma^{\nu}$$

$$\left(1 + \tfrac{i}{2}\omega_{\rho\sigma}\Sigma^{\rho\sigma}\right)\gamma^{\mu}\left(1 - \tfrac{i}{2}\omega_{\rho\sigma}\Sigma^{\rho\sigma}\right) = \left(1 - \tfrac{i}{2}\omega_{\rho\sigma}M^{\rho\sigma}\right)^{\mu}{}_{\nu}\gamma^{\nu}$$

$$\mathcal{O}(\omega^2) + \cancel{\gamma^{\mu}} + \tfrac{i}{2}\cancel{\omega_{\rho\sigma}}\Sigma^{\rho\sigma}\gamma^{\mu} - \tfrac{i}{2}\gamma^{\mu}\cancel{\omega_{\rho\sigma}}\Sigma^{\rho\sigma} = \cancel{\gamma^{\mu}} - \tfrac{i}{2}\cancel{\omega_{\rho\sigma}}(M^{\rho\sigma})^{\mu}{}_{\nu}\gamma^{\nu}$$

$$\Rightarrow \Sigma^{\rho\sigma}\gamma^{\mu} - \gamma^{\mu}\Sigma^{\rho\sigma} = -(M^{\rho\sigma})^{\mu}{}_{\nu}\gamma^{\nu}$$

$$\Rightarrow [\gamma^{\mu}, \Sigma^{\nu\sigma}] = (M^{\nu\sigma})^{\mu}{}_{\rho}\gamma^{\rho}$$

e) $g_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = g_{\rho\sigma}$, to the 1. order in exponential

$$\Rightarrow \left(1 - \frac{i}{2} \omega_{\alpha\beta} M^{\alpha\beta}\right)^\mu{}_\rho \, g_{\mu\nu} \left(1 - \frac{i}{2} \omega_{\alpha\beta} M^{\alpha\beta}\right)^\nu{}_\sigma = g_{\rho\sigma}$$

$$g_{\mu\nu} - \frac{i}{2} \mathbb{1}^\nu{}_\sigma \omega_{\alpha\beta} (M^{\alpha\beta})^\mu{}_\rho \, g_{\mu\nu} - \frac{i}{2} \mathbb{1}^\mu{}_\rho g_{\mu\nu} \omega_{\alpha\beta} (M^{\alpha\beta})^\nu{}_\sigma + \mathcal{O}(\omega^2) = g_{\rho\sigma}$$

$$g_{\mu\nu} - \frac{i}{2} \omega_{\alpha\beta} (M^{\alpha\beta})^\mu{}_\rho \, g_{\mu\sigma} - \frac{i}{2} g_{\rho\nu} \omega_{\alpha\beta} (M^{\alpha\beta})^\nu{}_\sigma = g_{\rho\sigma}$$

$$i \omega_{\alpha\beta} \left((M^{\alpha\beta})^\mu{}_\rho \, g_{\mu\sigma} + g_{\rho\nu} (M^{\alpha\beta})^\nu{}_\sigma \right) = \underline{2(g_{\mu\nu} - g_{\rho\sigma})}$$

e) to show $(M^{\rho\sigma})^\mu{}_\nu = i(g^{\rho\mu} \delta^\sigma{}_\nu - g^{\sigma\mu} \delta^\rho{}_\nu)$

$$\Lambda^T g \Lambda = g \qquad \Lambda = \Lambda_1 \Lambda_2 \cdots \Lambda_n$$

$$g_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = g_{\mu\nu}$$

$$LHS = g_{\mu\nu} \left(\delta^\mu_\rho - \frac{i}{2} \omega_{\alpha\beta} (M^{\alpha\beta})^\mu{}_\rho\right) \left(\delta^\nu_\sigma - \frac{i}{2} \omega_{\alpha\beta} (M^{\alpha\beta})^\nu{}_\sigma\right)$$

$$= g_{\rho\sigma} - \frac{i}{2} \omega_{\alpha\beta} [g_{\rho\nu} \underbrace{(M^{\alpha\beta})^\nu{}_\sigma} + g_{\mu\sigma} \underbrace{(M^{\alpha\beta})^\mu{}_\rho}]$$

plug in

f) $$\left[\gamma^\mu, \frac{i}{4} [\gamma^\nu, \gamma^\sigma]\right] = (M^{\nu\sigma})^\mu{}_\rho \gamma^\rho$$

$$LHS = \frac{i}{4} [\gamma^\mu, \gamma^\nu \gamma^\sigma] - \frac{i}{4} [\gamma^\mu, \gamma^\sigma \gamma^\nu]$$

$$= \frac{i}{4} \left\{ -\gamma^\nu \{\gamma^\sigma, \gamma^\mu\} + \{\gamma^\nu, \gamma^\mu\} \gamma^\sigma + \gamma^\sigma \{\gamma^\nu, \gamma^\mu\} - \{\gamma^\sigma, \gamma^\mu\} \gamma^\nu \right\}$$

$$= \frac{i}{4} \left\{ -\gamma^\nu \cdot 2 g^{\sigma\mu} \mathbb{1} + 2 g^{\nu\mu} \mathbb{1} \gamma^\sigma + \gamma^\sigma 2 g^{\nu\mu} \mathbb{1} - 2 g^{\sigma\mu} \mathbb{1} \gamma^\nu \right\}$$

$$= \frac{i}{4} \left(-4 g^{\sigma\mu} \gamma^\nu + 4 g^{\nu\mu} \gamma^\sigma \right)$$

$$= i \left(-g^{\sigma\mu} \gamma^\nu + g^{\nu\mu} \gamma^\sigma \right)$$

$$RHS = (M^{\nu\sigma})^\mu{}_\rho \gamma^\rho = i(g^{\nu\mu} \delta^\sigma_\rho - g^{\sigma\mu} \delta^\nu_\rho) \gamma^\rho$$

$$= i(g^{\nu\mu} \gamma^\sigma - g^{\sigma\mu} \gamma^\nu) \qquad \Rightarrow \boxtimes$$