H 12.1

a) $\mathcal{H} = \sum_i \dot{q}^i p_i - \mathcal{L}$

b) time dependent SE:

$$i\hbar \partial_t \psi(x,t) = \hat{H}\psi(x,t) = E\psi(x,t)$$

$$\Rightarrow \psi(x,t) = \psi(x) \cdot e^{-i/\hbar E t}$$

time independent SE:

$$\hat{H}\psi(x) = -\frac{\hbar^2}{2m}\partial_x^2 \psi(x) = E\psi(x)$$

$$\Rightarrow \psi(x) = \alpha e^{i\sqrt{2mE}/\hbar x} + \beta e^{-i\sqrt{2mE}/\hbar x}$$

$$\Rightarrow \psi(x,t) = e^{-i/\hbar E t}\left(\alpha e^{i/\hbar p x} + \beta e^{-i/\hbar p x}\right)$$

c) i) $x(t) := x_{cl}(t) + \eta(t)$, $\delta S[x_{cl}(t)] = 0$

$S[x_{cl}(t) + \eta(t)]$

$$= \int_0^{t_f} dt \, \mathcal{L}(x(t), \dot{x}(t))$$

$$= \int_0^{t_f} dt \, \mathcal{L}(x_{cl}(t) + \eta(t), \dot{x}_{cl}(t) + \dot{\eta}(t))$$

$$= \frac{m}{2}\int_0^{t_f} dt \, (\dot{x}_{cl}(t) + \dot{\eta}(t))^2$$

$$= \frac{m}{2}\left[\int_0^{t_f} dt \, \dot{x}_{cl}^2(t) + \int_0^{t_f} dt \, \dot{\eta}(t)^2 + \int_0^{t_f} dt \, 2\dot{x}_{cl}(t)\dot{\eta}(t)\right]$$

$$= S[x_{cl}(t)] + S[\eta(t)] + \underbrace{m\int_0^{t_f} dt \, \dot{x}_{cl}(t)\dot{\eta}(t)}$$

- integration by parts
- equation of motion

(3) $\to U(x_f, t_f, x_i, 0)$

$$= \mathcal{N} \int \mathcal{D}[x_{cl}(t) + \eta(t)] e^{\frac{i}{\hbar} S[x_{cl}(t) + \eta(t)]}$$

$$= \mathcal{N} \int \mathcal{D}[x_{cl}(t) + \eta(t)] e^{\frac{i}{\hbar} S[x_{cl}(t)] + \frac{i}{\hbar} S[\eta(t)]}$$

$$\underset{\text{change the variable}}{=} \mathcal{N} e^{i/\hbar S[x_{cl}(t)]} \int \mathcal{D}[\eta(t)] e^{i/\hbar S[\eta(t)]}$$

Q: Can we write $\mathcal{D}[x_{cl}(t) + \eta(t)] = \mathcal{D}[x_{cl}(t)] + \mathcal{D}[\eta(t)]$?

only one classical path

ii) $S[x_{cl}(t)] = \frac{m}{2} \int_0^{t_f} dt\, \dot{x}_{cl}^2(t)$

$$= \frac{m}{2} \int_{t=0}^{t=t_f} d[\dot{x}_{cl}(t)]\ \dot{x}_{cl}(t)$$

Euler equation : $\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}_{cl}(t)} = \frac{\partial \mathcal{L}}{\partial x_{cl}(t)}$, $\mathcal{L}_{cl} = \frac{1}{2} m \dot{x}_{cl}^2$

$$\Rightarrow \frac{d}{dt} \frac{\partial}{\partial \dot{x}_{cl}(t)} \left(\frac{1}{2} m \dot{x}_{cl}^2 \right) = 0$$

$$\Rightarrow \frac{d}{dt} m \dot{x}_{cl}(t) = 0 .$$

i.e. $m \dot{x}_{cl}(t) = \text{const} =: \dot{x}_0$

$$= \frac{m}{2} \dot{x}_0^2 \cdot t_f = \frac{m(x_f - x_i)^2}{2 t_f}$$

iii) Fourier series: $f(t) = \sum_{n=0}^{\infty} a_n \cos(\omega_n t) + b_n \sin(\omega_n t)$

$$\eta(0) = \eta(t_f) = 0 \quad \Rightarrow \quad a_n \equiv 0 \quad \forall n$$

$$\Rightarrow \quad \eta(t) = \sum_{n=1}^{\infty} b_n \sin(\omega_n t) \quad , \quad t \in [0, t_f]$$

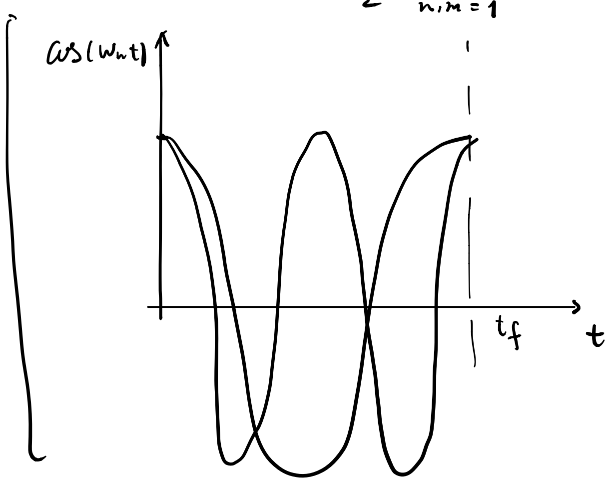
$$\omega_n = \frac{n\pi}{t_f} \leftarrow \text{periode}$$

iv) $$S[\eta(t)] = \int_0^{t_f} dt\, \mathcal{L}(\eta(t), \dot{\eta}(t))$$

$$= \frac{m}{2} \int_0^{t_f} dt \quad \dot{\eta}(t)^2 = \frac{m}{2} \int_0^{t_f} dt \left(\frac{d}{dt} \sum_{n=1}^{\infty} b_n \sin(\omega_n t) \right)^2$$

$$= \frac{m}{2} \int_0^{t_f} dt \left[\sum_{n=1}^{\infty} b_n \omega_n \cos(\omega_n t) \right]^2$$

$$= \frac{m}{2} \sum_{n,m=1}^{\infty} b_n b_m \omega_n \omega_m \int_0^{t_f} dt \cos(\omega_n t) \cos(\omega_m t)$$



$$\Rightarrow \int_0^{t_f} dt \cos(\omega_n t) \cos(\omega_m t) = \delta_{nm} \cdot \underbrace{A}_{\int_0^{t_f} \cos^2 dt}$$

$$= \frac{m}{2} \sum_{n=1}^{\infty} b_n^2 \omega_n^2 \int_0^{t_f} dt \cos^2(\omega_n t)$$

$$= \frac{m}{2} \sum_{n=1}^{\infty} b_n^2 \omega_n^2 \frac{1}{\omega_n} \left[\frac{x}{2} + \frac{\sin 2x}{2} \right]_{x=0}^{x=\omega_n t_f}$$

$$= \sum_{n=1}^{\infty} b_n^2 \frac{m n^2 \pi^2}{2 t_f^2} \frac{t_f}{2} = \sum_{n=1}^{\infty} b_n^2 \frac{m n^2 \pi^2}{4 t_f}$$

$$\Rightarrow \quad \exp\left(\frac{i}{\hbar} S[\eta(t)]\right) = \prod_{n=1}^{\infty} \exp\left(b_n^2 \frac{imn^2\pi^2}{4\hbar t_f}\right)$$

v) $$\int \mathcal{D}[\eta(t)] = \prod_{n=1}^{\infty} \int_{-\infty}^{\infty} da_n$$

$$\Rightarrow U(x_f, t_f, x_i, 0)$$

$$= \mathcal{N} e^{\frac{i}{\hbar} S[x_{cl}(t)]} \prod_{n=1}^{\infty} \int_{-\infty}^{\infty} da_n \, e^{\frac{i}{\hbar} S[\eta(t)]}$$

$$= \mathcal{N} e^{\frac{i}{\hbar} m (x_f - x_i)^2 / 2t_f} \prod_{n=1}^{\infty} \underbrace{\int_{-\infty}^{\infty} da_n \exp\left(-a_n^2 \frac{-imn^2\pi^2}{4\hbar t_f}\right)}$$

Gaussian Integral with complex number

$$\int_{-\infty}^{\infty} dx \exp\left(\frac{1}{2} iax^2 + iJx\right) = \sqrt{\frac{2\pi i}{a}} \exp\left(-\frac{iJ^2}{2a}\right)$$

$Im(a) > 0$

Gaussian integral

$$= \sqrt{\frac{\pi}{-imn^2\pi^2} \cdot 4\hbar t_f}$$

$$= \sqrt{\frac{4\hbar t_f}{-imn^2\pi}}$$

$$= \mathcal{N} \exp\left(\frac{im(x_f - x_i)^2}{2\hbar t_f}\right) \left(\prod_{n=1}^{\infty} \sqrt{\frac{4\hbar t_f \, i}{mn^2\pi}}\right)$$

vi) $$U(x_f, t_f, x_i, 0) = \langle x_f | e^{-iHt_f} | x_i \rangle$$

$$\psi(x_f, t_f) = \int dx_i \, \langle x_f | e^{-iHt_f} | x_i \rangle \langle x_i | \psi(0) \rangle$$

$$\lim_{t_f \to 0} U(x_f, t_f, x_i, 0) \overset{!}{=} \delta_{fi}$$

$$\Rightarrow \lim_{t_f \to 0} \mathcal{N} e\left(\frac{im(x_f - x_i)^2}{2\hbar t_f}\right) \sqrt{\frac{4\hbar t_f}{im\pi}} \prod_{n=1}^{\infty} \frac{1}{n} = \delta_{fi}$$

$$\delta(x) = \lim_{a \to 0} \sqrt{\frac{1}{\pi a}} e^{-x^2/a}$$

$$U(x_f, t_f, x_i, 0) \doteq \sqrt{\frac{1}{\pi}\frac{im}{2\hbar t_f}} \quad \exp\left(\frac{im(x_f-x_i)^2}{2\hbar t_f}\right)$$

d) $\psi(x,0) = \alpha e^{ipx/\hbar} + \beta e^{-ipx/\hbar}$

$$\psi(x_f, t_f) = \int dx_i \sqrt{\frac{im}{2\hbar t_f \pi}} \exp\left(\frac{im(x_f - x_i)^2}{2\hbar t_f}\right)\left(\alpha e^{ipx_i/\hbar} + \beta e^{-ipx_i/\hbar}\right)$$

$$= \sqrt{\frac{im}{2\hbar t_f \pi}} \left\{\alpha \int dx_i \exp\left(\frac{im}{2\hbar t_f} x_i^2 + \left(-\frac{im\, x_f}{\hbar t_f} + \frac{ip}{\hbar}\right) x_i + \frac{im\, x_f^2}{2\hbar t_f}\right)\right.$$

$$\left. + \beta \int dx_i \exp\left(\frac{im}{2\hbar t_f} x_i^2 + \left(-\frac{im x_f}{\hbar t_f} - \frac{ip}{\hbar}\right) x_i + \frac{im\, x_f^2}{2\hbar t_f}\right)\right.$$

$$= \sqrt{\frac{im}{2\cancel{\hbar t_f}\pi}} \; e^{im x_f^2/2\hbar t_f} \underbrace{(\alpha}_{-\text{ term}} + \underbrace{\beta)}_{+\text{ term}} \sqrt{-\frac{2\pi i}{m/\cancel{\hbar t_f}}} \exp\left[\frac{-i\left(\frac{m x_f}{\hbar t_f} \pm \frac{p}{\hbar}\right)^2}{2 \cdot m/\hbar t_f}\right]$$

$$= \cancel{e^{imx_f^2/2\hbar t_f}} (\alpha+\beta) \exp\left[-\frac{i\hbar t_f}{2m}\left(\cancel{\frac{m^2 x_f^2}{\hbar^2 t_f^2}} \pm \frac{2mx_f p}{\hbar t_f \hbar} + \frac{p^2}{\hbar^2}\right)\right]$$

$$= e^{-ip^2 t/2m\hbar}\left[\alpha e^{ix_f p/\hbar} + \beta e^{-ix_f p/\hbar}\right]$$

$$= e^{-i/\hbar\, Et}\left[\alpha e^{ix_f p/\hbar} + \beta e^{-ix_f p/\hbar}\right]$$

$\rightarrow$ (2) with $x_f = x$