H10.1 Chenhuan Wang

a)

$$\hat{a}^{\dagger}_{\tilde{\alpha}} |\tilde{\alpha}\rangle = |\check{\alpha}\rangle|\check{\alpha}\rangle = \hat{a}^{\dagger}_{\check{\alpha}} \sum_{\alpha} \langle\alpha|\check{\alpha}\rangle |\alpha\rangle = \sum_{\alpha} \langle\alpha|\check{\alpha}\rangle |\alpha\rangle \underbrace{|\check{\alpha}\rangle}$$

an $\check{\alpha}$-state particle is created

using the completeness

$$\Rightarrow \quad \hat{a}^{\dagger}_{\check{\alpha}} = \sum_{\alpha} \langle\alpha|\check{\alpha}\rangle |\alpha\rangle = \sum_{\alpha} \langle\alpha|\check{\alpha}\rangle \hat{a}^{\dagger}_{\alpha}$$

$$(\hat{a}^{\dagger}_{\check{\alpha}})^{\dagger} = \Big(\sum_{\alpha} \langle\alpha|\check{\alpha}\rangle \hat{a}^{\dagger}_{\alpha}\Big)^{\dagger} = \sum_{\alpha} \langle\check{\alpha}|\alpha\rangle \hat{a}_{\alpha} = \hat{a}_{\check{\alpha}}$$

b) Bosons:

$$[\hat{\psi}, \hat{\psi}] = \sum_{\alpha} \phi_{\alpha}(\vec{x})\hat{a}_{\alpha} \sum_{\alpha'} \phi_{\alpha'}(\vec{x}')\hat{a}_{\alpha'} - \sum_{\alpha'} \phi_{\alpha'}(\vec{x}')\hat{a}_{\alpha'} \sum_{\alpha} \phi_{\alpha}(\vec{x})\hat{a}_{\alpha}$$

$$= \sum_{\alpha} \phi_{\alpha}(\vec{x}) \sum_{\alpha'} \phi_{\alpha'}(\vec{x}') [\hat{a}_{\alpha}, \hat{a}_{\alpha'}] = 0$$

$$[\hat{\psi}^{\dagger}, \hat{\psi}^{\dagger}] = \sum_{\alpha} \phi^{*}_{\alpha}(\vec{x}) \sum_{\alpha'} \phi^{*}_{\alpha'}(\vec{x}') [\hat{a}^{\dagger}_{\alpha}, \hat{a}^{\dagger}_{\alpha'}] = 0$$

$$[\hat{\psi}, \hat{\psi}^{\dagger}] = \sum_{\alpha} \phi_{\alpha}(\vec{x})\hat{a}_{\alpha} \sum_{\alpha'} \phi^{*}_{\alpha'}(\vec{x}')\hat{a}^{\dagger}_{\alpha'} - \sum_{\alpha'} \phi^{*}_{\alpha'}(\vec{x}')\hat{a}^{\dagger}_{\alpha'} \sum_{\alpha} \phi_{\alpha}(\vec{x})\hat{a}_{\alpha}$$

$$= \sum_{\alpha} \phi_{\alpha}(\vec{x}) \sum_{\alpha'} \phi^{*}_{\alpha'}(\vec{x}') [\hat{a}_{\alpha}, \hat{a}^{\dagger}_{\alpha'}]$$

$$= \sum_{\alpha,\alpha'} \phi_{\alpha}(\vec{x}) \phi^{*}_{\alpha'}(\vec{x}') \delta_{\alpha\alpha'} = \delta^{(3)}(\vec{x} - \vec{x}')$$

c) One-particle term

$$= \sum_{\alpha\beta} \hat{a}^{\dagger}_{\alpha} \langle\alpha|T|\beta\rangle \hat{a}_{\beta} + \sum_{\alpha\beta} \langle\alpha|V|\beta\rangle \hat{a}^{\dagger}_{\alpha}\hat{a}_{\beta}$$

$$= \int d^3x' \int d^3x \sum_{\alpha\beta} \hat{a}^{\dagger}_{\alpha} \langle\alpha|\vec{x}\rangle\langle\vec{x}|T|\vec{x}'\rangle\langle\vec{x}'|\beta\rangle \hat{a}_{\beta} + \int d^3x \sum_{\alpha\beta} \hat{a}^{\dagger}_{\alpha} \varphi^{*}_{\alpha}(\vec{x}) \langle\vec{x}|V|\vec{x}'\rangle \varphi_{\beta}(\vec{x}')\hat{a}_{\beta}$$

$$= \sum_{\alpha\beta} \int d^3x' \int d^3x \Big(\hat{a}^{\dagger}_{\alpha} \varphi^{*}_{\alpha}(\vec{x}) \; -\frac{\hbar^2}{2m} \delta^{(3)}(\vec{x}-\vec{x}') \vec{\nabla}^2 \varphi^{*}_{\beta}(\vec{x}') \hat{a}_{\beta}$$

$$+ \varphi^{*}_{\alpha}(\vec{x}) \hat{a}^{\dagger}_{\alpha} \, V(\vec{x}) \, \delta^{(3)}(\vec{x}-\vec{x}') \varphi^{*}_{\beta}(\vec{x}') \hat{a}_{\beta} \Big)$$

$$= -\frac{\hbar^2}{2m} \underbrace{\int d^3x \; \hat{\psi}^{\dagger}(\vec{x}) \nabla^2 \hat{\psi}(\vec{x})} + \int d^3x \; \hat{\psi}^{\dagger} V(\vec{x}) \hat{\psi}$$

$$= \hat{\psi}^{\dagger}(\vec{x}) \vec{\nabla} \hat{\psi}(\vec{x}) \Big|_{-\infty}^{+\infty} - \int d^3x \; \nabla\hat{\psi}^{\dagger}(\vec{x}) \nabla\hat{\psi}(\vec{x})$$

$$= \frac{\hbar^2}{2m} \int d^3x \, \nabla \hat{\psi}^+(\vec{x}) \nabla \hat{\psi}(\vec{x}) + \int d^3x \, \hat{\psi}^+(\vec{x}) V(\vec{x}) \hat{\psi}(\vec{x})$$

two-particle term

$$= \frac{1}{2} \sum_{\lambda\mu\nu\rho} \langle \lambda \mu | V | \nu \rho \rangle \hat{a}_\lambda^+ \hat{a}_\mu^+ \hat{a}_\rho \hat{a}_\nu$$

$$= \frac{1}{2} \int d^3x \int d^3x' \int d^3x'' \int d^3x''' \, \delta^{(3)}(\vec{x}-\vec{x}') \delta^{(3)}(\vec{x}-\vec{x}'') \delta^{(3)}(\vec{x}'-\vec{x}''') \hat{\psi}^+(\vec{x}) \hat{\psi}^+(\vec{x}') U(\vec{x}-\vec{x}') \hat{\psi}(\vec{x}') \hat{\psi}(\vec{x})$$

$$= \frac{1}{2} \int d^3x \int d^3x' \, \hat{\psi}^+(\vec{x}) \hat{\psi}^+(\vec{x}') U(\vec{x}-\vec{x}') \hat{\psi}(\vec{x}') \hat{\psi}(\vec{x})$$

$$\Rightarrow \hat{H} = \int d^3x \left[\vec{\nabla} \psi^+(\vec{x}) \frac{\hbar^2}{2m} \vec{\nabla} \hat{\psi}(\vec{x}) + V(\vec{x}) \hat{\psi}^+(\vec{x}) \hat{\psi}(\vec{x}) \right]$$

$$+ \frac{1}{2} \int d^3x \int d^3x' \, \hat{\psi}^+(\vec{x}) \hat{\psi}^+(\vec{x}') U(\vec{x}-\vec{x}') \hat{\psi}(\vec{x}') \hat{\psi}(\vec{x})$$

d) $\underline{\text{1. Term}}$

$$= \int d^3x \, \vec{\nabla} \int \frac{d^3k}{(2\pi)^{3/2}} e^{-i\vec{k}\cdot\vec{x}} \hat{a}_k^+ \frac{\hbar^2}{2m} \vec{\nabla} \int \frac{d^3k'}{(2\pi)^{3/2}} e^{i\vec{k}'\cdot\vec{x}} \hat{a}_{k'}$$

$$= \frac{1}{(2\pi)^3} \int d^3x \, (-i\vec{k}) \int d^3k \, e^{-i\vec{k}\cdot\vec{x}} \hat{a}_k^+ \frac{\hbar^2}{2m} (i\vec{k}) \int d^3k' \, e^{i\vec{k}'\cdot\vec{x}} \hat{a}_{k'}$$

$k = |\vec{k}|$

$$= \frac{1}{(2\pi)^3} \frac{(\hbar k)^2}{2m} \int d^3x \int d^3k \, e^{-i\vec{k}\cdot\vec{x}} \hat{a}_k^+ \int d^3k' e^{i\vec{k}'\cdot\vec{x}} \hat{a}_{k'}$$

$$= \frac{(\hbar k)^2}{2m} \int d^3k \, \hat{a}_k^+ \hat{a}_k$$

$\underline{\text{2. Term}}$

$$= \int d^3x \, \hat{\psi}^+(\vec{x}) V(\vec{x}) \hat{\psi}(\vec{x})$$

$$= \int d^3x \int \frac{d^3k'}{(2\pi)^{3/2}} e^{-i\vec{k}'\cdot\vec{x}} \hat{a}_{\vec{k}'}^+ V(\vec{x}) \int \frac{d^3k}{(2\pi)^{3/2}} e^{i\vec{k}\cdot\vec{x}} \hat{a}_{\vec{k}}$$

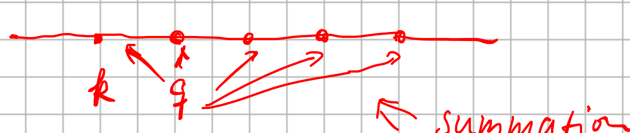
$\vec{q} = \vec{k}' - \vec{k}$

$$= \int d^3x \int d^3k' \int d^3k \, \frac{1}{(2\pi)^3} \underline{e^{i(\vec{k}-\vec{k}')\vec{x}}} \hat{a}_{k'}^+ \int \frac{d^3q}{(2\pi)^{3/2}} \underline{e^{i\vec{q}\vec{x}}} V_{\vec{q}} \hat{a}_{\vec{k}}$$

$$= \int d^3k' \int d^3k \int d^3q \, \frac{1}{(2\pi)^{3/2}} \hat{a}_{k'}^+ V_{\vec{q}} \hat{a}_k \int \frac{d^3x}{(2\pi)^3} e^{i(\vec{k}-\vec{k}')\vec{x} + i\vec{q}\cdot\vec{x}}$$

$$= \int d^3k' \int d^3k \int d^3q \, \frac{1}{(2\pi)^{3/2}} \hat{a}_{k'}^+ V_q \hat{a}_k \, \delta_{\vec{q}+\vec{k}-\vec{k}'}$$

$$= \int d^3k \int d^3q \, \frac{1}{(2\pi)^{3/2}} \hat{a}^+_{\vec{q}+\vec{k}} V_q \hat{a}_k$$



3. Term $= \frac{1}{2} \int d^3x \int d^3x' \int \frac{d^3p'}{(2\pi)^{3/2}} e^{-i\vec{p}'\cdot\vec{x}'} \hat{a}^+_{p'} \int \frac{d^3p}{(2\pi)^{3/2}} e^{-i\vec{p}\cdot\vec{x}} \hat{a}^+_p$

$$U(\vec{x}-\vec{x}') \int \frac{d^3k'}{(2\pi)^{3/2}} e^{i\vec{k}'\cdot\vec{x}'} \hat{a}_{k'} \int \frac{d^3k}{(2\pi)^{3/2}} e^{i\vec{k}\cdot\vec{x}} \hat{a}_k$$

$$= \frac{1}{2} \int d^3x \int d^3x' \frac{1}{(2\pi)^6} \int d^3p' \int d^3p \int d^3k' \int d^3k \, e^{-i\vec{p}'\cdot\vec{x}' - i\vec{p}\cdot\vec{x} + i\vec{k}'\cdot\vec{x}'}$$

$$\cdot e^{i\vec{k}\cdot\vec{x}} \int \frac{d^3q}{(2\pi)^{3/2}} e^{i\vec{q}(\vec{x}-\vec{x}')} U_{\vec{q}} \quad \hat{a}^+_{p'} \hat{a}^+_p \hat{a}_{k'} \hat{a}_k$$

Rearrange:

$$\frac{1}{(2\pi)^6} \int d^3x \int d^3x' \, e^{-i\vec{p}'\vec{x}' - i\vec{p}\cdot\vec{x} + i\vec{k}'\cdot\vec{x}' + i\vec{k}\cdot\vec{x} + i\vec{q}(\vec{x}-\vec{x}')}$$

$$= \delta_{-\vec{p}+\vec{k}+\vec{q}} \quad \delta_{-\vec{p}'+\vec{k}'-\vec{q}}$$

$$= \frac{1}{2} \int d^3p' \int d^3p \int d^3k' \int d^3k \int d^3q \frac{1}{(2\pi)^{3/2}} U_{\vec{q}} \, \delta_{-\vec{p}+\vec{k}+\vec{q}} \, \delta_{-\vec{p}'+\vec{k}'-\vec{q}}$$

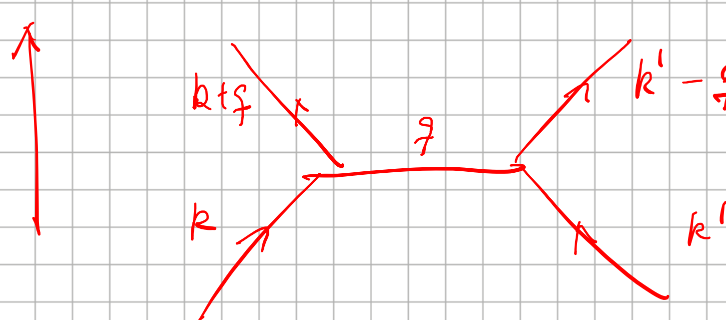
$$\cdot \hat{a}^+_{p'} \hat{a}^+_p \hat{a}_{k'} \hat{a}_k$$

$$= \frac{1}{2} \int d^3k' \int d^3k \int d^3q \frac{1}{(2\pi)^{3/2}} U_{\vec{q}} \, \hat{a}^+_{\vec{k}'-\vec{q}} \hat{a}^+_{\vec{k}+\vec{q}} \hat{a}_{\vec{k}'} \hat{a}_{\vec{k}}$$

↑ rename $\vec{k}' \Leftrightarrow \vec{k}$

$\Rightarrow$ get the expression on the sheet

V and U terms are dependent on the momentum.



H10.2

a) $|i\rangle := |0, \dots, n_i = 1, \dots 0 \dots\rangle = \hat{a}_i^+ |0\rangle$,

$$\text{RHS} = \langle i | \hat{T} | j \rangle = \langle i | \sum_{\alpha,\beta} t_{\alpha\beta} \hat{a}_\alpha^+ \hat{a}_\beta | j \rangle$$

$$= \sum_{\alpha,\beta} \langle 0, \dots 0, n_i = 1, 0, \dots 0 | \hat{a}_\alpha^+ \hat{a}_\beta | 0 \dots, n_j = 1, \dots 0\rangle \, t_{\alpha\beta}$$

$$\underset{\hat{a}|0\rangle = 0}{\overset{\uparrow}{=}} \sum_{\alpha,\beta} \delta_{\alpha i} \delta_{\beta j} t_{\alpha\beta} = t_{ij}$$

b) $$t_{ij} = \begin{cases} -\mu \ , & j = i \\ -t \ ; & j = i \pm 1 \\ 0 \ ; & \end{cases}$$

$$\hat{T} = \sum_{i,j} t_{ij} \hat{a}_i^+ \hat{a}_j = -\sum_i \mu \hat{n}_i - t \sum_i \hat{a}_i^+ \hat{a}_{i\pm 1}$$

$$= -\mu \sum_i \hat{n}_i - t \sum_i \hat{a}_i^+ \hat{a}_{i+1} + \hat{a}_{i+1}^+ \hat{a}_i$$

c) $$\hat{F} = \frac{1}{2} \sum f_{ijkl} \hat{a}_i^+ \hat{a}_j^+ \hat{a}_l \hat{a}_k = \frac{1}{2} \sum_i U \hat{a}_i^+ \hat{a}_i^+ \hat{a}_i \hat{a}_i$$

$$= \frac{U}{2} \sum_i \hat{a}_i^+ (\hat{a}_i \hat{a}_i^+ - 1) \hat{a}_i = \frac{U}{2} \sum_i \hat{n}_i (\hat{n}_i - 1)$$

d) $$\Rightarrow \hat{H} = -\mu \sum_i \hat{n}_i - t \sum_i (\hat{a}_i^+ \hat{a}_{i+1} + \hat{a}_{i+1}^+ \hat{a}_i) + \frac{U}{2} \sum_i \hat{n}_i (\hat{n}_i - 1)$$

e) $$[\hat{H}, n_j] = -t \sum_i [\hat{a}_i^+ \hat{a}_{i+1} + \hat{a}_{i+1}^+ \hat{a}_i , \hat{n}_j]$$

$$= -t \sum_i \{ [\hat{a}_i^+ \hat{a}_{i+1}, \hat{n}_j] + [\hat{a}_{i+1}^+ \hat{a}_i, \hat{n}_j] \} \neq 0$$

$$= -t \sum_i \{ \delta_{ij} \hat{a}_{i+1} [\hat{a}_i^+, \hat{n}_j] + \delta_{i+1,j} \hat{a}_i^+ [\hat{a}_{i+1}, \hat{n}_j]$$

$$+ \delta_{i+1,j} \hat{a}_i [\hat{a}_{i+1}^+, \hat{n}_j] + \delta_{ij} \hat{a}_{i+1}^+ [\hat{a}_i, \hat{n}_j] \}$$

$$= -t \{ -\hat{a}_{j+1} \hat{a}_j^+ + \hat{a}_{j-1}^+ \hat{a}_j - \hat{a}_{j-1} \hat{a}_j^+ + \hat{a}_{j+1}^+ \hat{a}_j \} \quad ✗$$

$$\left(\begin{aligned} [\hat{a}_i^+, \hat{a}_i^+ \hat{a}_l] &= [\hat{a}_i^+, \hat{a}_i^+] \hat{a}_j + \hat{a}_i^+ [\hat{a}_i^+, \hat{a}_i] = -\hat{a}_i^+ \\ [\hat{a}_i, \hat{a}_i^+ \hat{a}_i] &= [\hat{a}_i, \hat{a}_i^+] \hat{a}_i + \hat{a}_i^+ [\hat{a}_i, \hat{a}_i] = \hat{a}_i \end{aligned} \right)$$

$$[\hat{H}, \hat{N}] = -t \sum_{ij} [\hat{a}_i^+ \hat{a}_{i+1} + \hat{a}_{i+1} \hat{a}_i , n_j]$$

$$= -t \sum_j \{ -\hat{a}_{j+1} \hat{a}_j^+ + \hat{a}_{j+1}^+ \hat{a}_j + \hat{a}_{j-1}^+ \hat{a}_j - \hat{a}_{j-1} \hat{a}_j^+ \} = 0$$

f) $U > 0$

$-\mu \sum \hat{n}_i$: intrisic energy of particles ✓ chemical potential

$\frac{U}{2} \sum (\hat{n}_i - 1) \hat{n}_i$: repulsive interaction of two particles. e.g two charged particles

$-t \sum (\hat{a}_i^+ \hat{a}_{i+1} + \hat{a}_{i+1}^+ \hat{a}_i)$: energy difference when one particles get excited to another energy level ✗

hopping term