H6.1 Chenhuan Wong

a) $$\Box\phi(x) = \partial_\mu\partial^\mu \int d^4k\, \delta(k^2-m^2) g(k) e^{-i(k\cdot x)}$$

$$= \int d^4k\, \delta(k^2-m^2) g(k) \underbrace{\partial_\mu\partial^\mu e^{-i(k\cdot x)}}$$

$$= \partial_\mu \; -ik^\mu \; e^{-i(k\cdot x)}$$

$$= -k_\mu k^\mu e^{-i(k\cdot x)}$$

Should be the same if we perform the integration

$$= -\int d^4k\, \delta(k^2-m^2) g(k) k_\mu k^\mu e^{-i(k\cdot x)}$$

$$= -m^2 \int d^4k\, \delta(k^2-m^2) g(k) e^{-i(k\cdot x)}$$

$\Rightarrow$ (1)

b) $$\phi(x) = \int dk^0\, d^3k\, \delta(k^2-m^2) g(k) e^{-i(k\cdot x)}$$

$$= \int d^3k \int dk^0\, \delta(k^{0\,2} - |\vec{k}|^2 - m^2) g(k^0, k^i) e^{-i(k^0x^0 - \vec{k}\cdot\vec{x})} \quad , i = 1,2,3$$

$$\left[\int_{-\infty}^{\infty} dx\, f(x)\, \delta(g(x)) = \sum_i \frac{f(x_i)}{|g'(x_i)|} \qquad x = x_i \text{ the roots} \right]$$

$$= \int d^3k \sum_\pm \frac{1}{\left|\frac{\partial}{\partial k^0}(k^{0\,2} - |\vec{k}|^2 - m^2)\right|}\Bigg|_{k_0 = \pm E_k} g(\pm E_k, \vec{k}) e^{-i(\pm E_k x^0 - \vec{k}\cdot\vec{x})}$$

$$= \int d^3k \sum_\pm \frac{g(\pm E_k, \vec{k})}{2E_k} e^{+i(\vec{k}\cdot\vec{x} \mp E_k t)} = \phi_+ + \phi_-$$

c) $$j^\mu := \frac{i}{2m}\left[\phi^*(\partial^\mu\phi) - (\partial^\mu\phi^*)\phi\right]$$

$$LHS = \partial_\mu j^\mu = \frac{i}{2m}\left[\partial_\mu\phi^* \cdot \partial^\mu\phi + \phi^*\partial_\mu\partial^\mu\phi - \partial_\mu\partial^\mu\phi^* \cdot \phi - \partial^\mu\phi^*\,\partial_\mu\phi\right]$$

$$\propto \left[\cancel{\partial_\mu\phi^*\partial^\mu\phi} - \underbrace{\partial^\mu\phi^*\partial_\mu\phi}_{=\cancel{\partial_\mu\phi^*\partial^\mu\phi}} + \phi^*\underbrace{\partial_\mu\partial^\mu\phi}_{=m^2\phi} - \underbrace{\partial_\mu\partial^\mu\phi^*}_{m^2\phi^*}\cdot\phi\right]$$

$= 0$

d) $\|\phi\|_c^2 := \int d^3x \; j^0(t,\vec{x}) = \int d^3x \; \frac{i}{2m} [\phi^* \partial^0 \phi - \partial^0 \phi^* \cdot \phi]$

$$(\phi = \phi(t,\vec{x}))$$

$$\frac{d}{dt} \|\phi\|_c^2 = \frac{d}{dt} \int d^3x \; j^0(t,\vec{x})$$

$$= \int d^3x \; \mathrm{div} \, \vec{j}(t,\vec{x})$$

$$= \int d\vec{S} \cdot \vec{j}(t,\vec{x}) = 0 \; ?$$

← normalizebar! compact support!

e) $\|\phi\|_c^2 \Big|_{t=0} = \int d^3x \; j^0(t,\vec{x}) \Big|_{t=0}$

$$= \frac{i}{2m} \int d^3x \; (\phi^*(0,\vec{x}) \partial^0 \phi(0,\vec{x}) - \partial^0 \phi^*(0,\vec{x}) \phi(0,\vec{x})) \Big|_{t=0}$$

$$= \frac{i}{2m} \int d^3x \left[(\phi_+ + \phi_-)^* \partial^0 (\phi_+ + \phi_-) - \partial^0 (\phi_+ + \phi_-)^* (\phi_+ + \phi_-) \right] \Big|_{t=0}$$

$$= \frac{i}{2m} \int d^3x \left[\underline{\phi_+^* \partial^0 \phi_+} + \phi_+^* \partial^0 \phi_- + \phi_-^* \partial^0 \phi_+ + \underline{\phi_-^* \partial^0 \phi_-} - \underline{\partial^0 \phi_+^* \cdot \phi_+} - \partial^0 \phi_-^* \phi_+ - \partial^0 \phi_+^* \phi_- - \underline{\partial^0 \phi_-^* \cdot \phi_-} \right] \Big|_{t=0}$$

$$= \frac{i}{2m} \int d^3x \Big\{ (\phi_+^* \partial^0 \phi_+ - \partial^0 \phi_+^* \cdot \phi_+) + (\phi_-^* \partial^0 \phi_- - \partial^0 \phi_-^* \cdot \phi_-) \Big|_{t=0}$$

$$+ \cancel{\phi_+^* \partial^0 \phi_-} + \cancel{\phi_-^* \partial^0 \phi_+} - \cancel{\partial^0 \phi_-^* \cdot \phi_+} - \cancel{\partial^0 \phi_+^* \cdot \phi_-} \Big\} \Big|_{t=0}$$

$$\left[\begin{array}{l} \partial^0 \phi_\pm = \partial^0 \int d^3k \; \frac{g_\pm(\vec{k})}{2E_k} e^{i(\vec{k}\cdot\vec{x} \mp E_k t)} \qquad \phi_+^* \partial^0 \phi_- = \phi_- \partial^0 \phi_+^* \\ \quad = \int d^3k \; \frac{g_\pm(\vec{k})}{2E_k} e^{i(\vec{k}\cdot\vec{x} \mp E_k t)} (\mp i E_k) \quad \Rightarrow \\ \quad \Rightarrow \; \partial^0 \phi_+ = \partial^0 \phi_-^* \end{array} \right]$$

$$= \|\phi_+\|_c^2 + \|\phi_-\|_c^2$$

in which $\|\phi_\pm\|_c^2 = \frac{i}{2m} \int d^3x \left(\phi_\pm^* \partial^0 \phi_\pm - \partial^0 \phi_\pm^* \cdot \phi_\pm \right) \Big|_{t=0}$

$$= \frac{i}{2m}\int d^3x \left\{ \int d^3k\, \frac{g_\pm^*(\vec{k})}{2E_k} e^{-i(\vec{k}\cdot\vec{x} \mp E_k t)} \cdot (\mp i)\int d^3k \frac{g_\pm(\vec{k})}{2} e^{i(\vec{k}\cdot\vec{x} \mp E_k t)} \right.$$

$$\left. - \int d^3k\, \frac{g_\pm(\vec{k})}{2E_k} e^{i(\vec{k}\cdot\vec{x} \mp E_k t)} \cdot (\pm i)\int d^3k \frac{g_\pm^*(\vec{k})}{2} e^{-i(\vec{k}\cdot\vec{x} \mp E_k t)} \right\}\Bigg|_{t=0}$$

$$= \frac{1}{2m}\int d^3x \left\{ \pm \int d^3k\, \frac{g_\pm^*(\vec{k})}{2E_k} e^{-i\vec{k}\cdot\vec{x}} \cdot \int d^3k \frac{g_\pm(\vec{k})}{2} e^{i\vec{k}\cdot\vec{x}} \pm \int d^3k \frac{g_\pm(\vec{k})}{2E_k} e^{i\vec{k}\cdot\vec{x}} \right.$$

$$\left. \cdot \int d^3k \frac{g_\pm^*(\vec{k})}{2} e^{-i\vec{k}\cdot\vec{x}} \right\}$$

f) (6) : $\|\phi_\pm\|_c^2 = \pm \frac{(2\pi)^3}{2m}\int d^3k \int dk^0\, \delta(k^{0\,2} - |\vec{k}|^2 - m^2)\, \Theta(\pm k^0) |g(k)|^2$

$+$: $= + \frac{(2\pi)^3}{2m}\int d^3k\, \delta(k^{0\,2} - E_k^2)\, \Theta(k^0) |g_+(k)|^2$

$= \frac{(2\pi)^3}{2m}\int d^3k \frac{|g_+(k)|^2}{2E_k}$

$-$: $= - \frac{(2\pi)^3}{2m}\int d^3k \frac{|g_-(k)|^2}{2E_k}$

$\to$ (5) ∎

e) $\|\phi_c\|^2 = \frac{i}{2m}\int d^3x \int d^3q \int d^3k \frac{1}{2E_q}\frac{1}{2E_k} \Big\{$

$[g_+^*(\vec{q}) e^{-i\vec{q}\cdot\vec{x}} + g_-^*(\vec{q}) e^{-i\vec{q}\cdot\vec{x}}](-iE_k)$

$[g_+(\vec{k}) e^{i\vec{k}\cdot\vec{x}} - g_-(\vec{k}) e^{i\vec{k}\cdot\vec{x}}]$

$-(iE_q)[g_+^*(\vec{q}) e^{-i\vec{q}\cdot\vec{x}} - g_-^*(\vec{q}) e^{-i\vec{q}\cdot\vec{x}}]\cdot$

$[g_+(\vec{k}) e^{+i\vec{k}\cdot\vec{x}} + g_-(\vec{k}) e^{i\vec{k}\cdot\vec{x}}]\Big\}$

$\frac{1}{(2\pi)^3}\int e^{i(\vec{q}-\vec{k})\vec{x}} d^3x = \delta(\vec{q}-\vec{k})$

$= \frac{(2\pi)^3}{2m}\int d^3q \int d^3k \frac{1}{2E_q}\frac{1}{2E_k}\delta^{(3)}(\vec{k}-\vec{q})$

$$\{E_k [g_+^*(\vec{q})\, g_+(\vec{k}) - g_+^*(\vec{q})\, g_-(\vec{k}) +$$

$$+ g_-^*(\vec{q})\, g_+(\vec{k}) - g_-^*(\vec{q})\, g_-(\vec{k})]$$

$$+ E_q [g_+^*(\vec{q})\, g_+(\vec{k}) - g_-^*(\vec{q})\, g_+(\vec{k})$$

$$+ g_+^*(\vec{q})\, g_-(\vec{k}) - g_-^*(\vec{q})\, g_-(\vec{k})]\}$$

$$= \frac{(2\pi)^3}{2m} \int d^3k \, \frac{1}{2E_k} \{ |g_+(\vec{k})|^2 - |g_-(\vec{k})|^2 \}$$

$$= \|\phi_+\|_c^2 + \|\phi_-\|_c^2$$

H6.2 a)

$$\{\alpha^i, \alpha^j\} = \alpha^i\alpha^j + \alpha^j\alpha^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^j \\ \sigma^j & 0 \end{pmatrix} + \begin{pmatrix} 0 & \sigma^j \\ \sigma^j & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \sigma^i\sigma^j & 0 \\ 0 & \sigma^i\sigma^j \end{pmatrix} + \begin{pmatrix} \sigma^j\sigma^i & 0 \\ 0 & \sigma^j\sigma^i \end{pmatrix}$$

$$= \begin{pmatrix} \delta_{ij}\mathbb{1} + i\varepsilon_{ijk}\sigma_k + \delta_{ji}\mathbb{1} + i\varepsilon_{jik}\sigma_k & 0 \\ 0 & \delta_{ij}\mathbb{1} + i\varepsilon_{ijk}\sigma_k + \delta_{ji}\mathbb{1} + i\varepsilon_{jik}\sigma_k \end{pmatrix}$$

$$= 2\delta_{ij}\begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} = 2\delta_{ij}\mathbb{1}$$

$$\boxed{\overset{!!!}{} \quad \sigma_a\sigma_b = \delta_{ab}\mathbb{1} + i\varepsilon_{abc}\sigma_c}$$

$$\{\alpha^i, \beta\} = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} + \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}\begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix} + \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} = 0$$

- The eingenvalues of $M_{n\times n}$ must be ± 1 if the anti-commutation relations are satisfied:

$$[M^\mu, M^\mu] = 2(M^\mu)^2 = 2\delta^{\mu\mu}\mathbb{1}$$

$$\Rightarrow \quad M^{\mu\,2} = \mathbb{1}$$

in eigenbasis of M^μ $\Rightarrow \lambda = \pm 1$

- The trace of M^μ must vanish:

$$M^\mu M^\nu + M^\mu M^\nu = 0$$

$$M^\mu M^\nu = -M^\nu M^\mu$$

$$M^\mu M^\mu M^\nu = -M^\mu M^\nu M^\mu$$

$$M^\nu = -M^\mu M^\nu M^\mu$$

$$tr(M^\nu) = -tr(M^\mu M^\nu M^\mu)$$

$$= -tr(M^\nu) \qquad \Rightarrow tr(M^\nu) = 0$$

- the M^μ then must be $M_{n\times n}$. $n = 2m+1$, $m \in \mathbb{N}_0$

$$tr(M^\nu) = \sum_{i=1}^{n} \lambda_i \overset{!}{=} 0 \quad \Rightarrow \quad n \text{ has to be even}$$

b) $\gamma^0 = \beta$, $\gamma^i = \beta\alpha^i$ with $\beta = \beta^{-1}$,

$\beta\gamma^0 = \mathbb{1}$, $\beta\gamma^i = \alpha^i$

$$\Rightarrow \quad i\hbar\beta\gamma^0 \partial_t \Psi(x^\mu) = \left(-i\hbar c\beta\vec{\gamma}\cdot\vec{\nabla} + \gamma^0 mc^2\right)\Psi(x^\mu)$$

$$i\hbar\cancel{\beta}\left(\gamma^0\partial_t + c\vec{\gamma}\cdot\vec{\nabla}\right)\Psi(x^\mu) = \cancel{\gamma^0} mc^2\,\Psi(x^\mu)$$

$$i\left(\gamma^0\frac{1}{c}\partial_t + \vec{\gamma}\cdot\vec{\nabla}\right)\Psi(x^\mu) = mc/\hbar\,\Psi(x^\mu)$$

$$\Rightarrow \quad \left(-i\gamma^\mu\partial_\mu + \frac{mc}{\hbar}\right)\Psi(x^\mu) = 0$$

$$\{M^\mu, M^\nu\} = 2\delta^{\mu\nu}\mathbb{1}$$

$$(\gamma^0)^\dagger = \beta^\dagger$$

$$(\gamma^i)^\dagger = (\beta\alpha^i)^\dagger = \alpha^{i\dagger}\beta^\dagger \longrightarrow \; = ?$$

$$\{\beta, \beta\} = 2\mathbb{1} \quad \Rightarrow \quad \beta^2 = \mathbb{1} = (\gamma^0)^2$$

$$(\gamma^i)^2 = (\beta\alpha^i)^2 = \beta\alpha^i\beta\alpha^i = -\beta\beta\alpha^i\alpha^i = \mathbb{1}$$

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu$$

$$= \begin{cases} \mu = \nu = 0 & \beta\beta + \beta\beta = 2\mathbb{1} \\ \mu = \nu = 1,2,3 & \beta\alpha^i\beta\alpha^i + \beta\alpha^j\beta\alpha^i = -\alpha^i\alpha^j - \alpha^j\alpha^i = -2\delta^{ij}\mathbb{1} \end{cases}$$

$$= 2g^{\mu\nu}\mathbb{1}$$

c) $$\{\tilde{\gamma}^\mu, \tilde{\gamma}^\nu\} = \{A\gamma^\mu A^{-1}, A\gamma^\nu A^{-1}\} = A\gamma^\mu A^{-1}A\gamma^\nu A^{-1} + A\gamma^\nu A^{-1}A\gamma^\mu A^{-1}$$

$$= A\gamma^\mu\gamma^\nu A^{-1} + A\gamma^\nu\gamma^\mu A^{-1}$$

$$= A\, 2g^{\mu\nu}\mathbb{1}\, A^{-1}$$

$$= 2 g^{\mu\nu} \mathbb{1}$$

$$\tilde{\gamma}_\mu = A \gamma_\mu A^{-1} = A' \gamma_\mu A'^{-1}$$

$$\tilde{\gamma}_\mu = B \gamma_\mu B^{-1} \qquad \tilde{\gamma}_\mu = A \gamma_\mu A^{-1}, \qquad A \neq B$$

$$\gamma_\mu = B^{-1} \tilde{\gamma}_\mu B = \underbrace{B^{-1} A}_{\text{const}} \gamma_\mu \underbrace{A^{-1} B}_{1/\text{const}}$$

$$\Rightarrow \quad A = \text{const}\, B$$