CS 1

a) i) $a_\mu = g_{\mu\nu} a^\nu = (g^{\mu\nu})^{-1} a^\nu$

$\iff g^{\mu\nu} a_\mu = a^\nu$

ii) $a_\mu b^\mu = a_\mu g^{\mu\nu} b_\nu = g^{\mu\nu} a_\mu b_\nu = a^\nu b_\nu = a^\mu b_\mu$

iii) $\partial_\mu (a \cdot x) = \partial_\mu (a_\mu x^\mu) = a_\mu \partial_\mu x^\mu = a_\mu \frac{\partial}{\partial x^\mu} x^\mu = a_\mu$

$\partial^\mu (a \cdot x) = \partial^\mu (a^\mu x_\mu) = a^\mu \partial^\mu x_\mu = a^\mu$

$\partial_\mu (a \cdot x) = \partial_\mu a_\nu x^\nu$
$= a_\nu \partial_\mu x^\nu$

b) i) $(\Lambda x \cdot \Lambda y) = (\Lambda x)_\mu (\Lambda y)^\mu = (\Lambda x)_\mu g^{\mu\nu} (\Lambda y)_\nu$

$= \Lambda_\mu{}^\alpha x_\alpha g^{\mu\nu} \Lambda_\nu{}^\beta y_\beta$

$= \Lambda_\mu{}^\alpha g^{\mu\nu} \Lambda_\nu{}^\beta x_\alpha y_\beta$

$= g^{\alpha\beta} x_\alpha y_\beta$

$(x \cdot y) = x^\mu g_{\mu\nu} y^\nu$

$\rightarrow x^T g y$

$A \cdot B \rightarrow \sum_k A^{ik} B^{kj}$

$A^T \cdot B \rightarrow \sum_k A^{ki} B^{kj}$

ii) $\det(\Lambda^T g \Lambda) = \det(\Lambda^T) \det(g) \det(\Lambda) \stackrel{!}{=} \det(g)$

$\det(\Lambda^T) = \det(\Lambda)$ because $\det(A) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \prod_{i=1}^n a_{i,\sigma_i}$

$\det(A^T) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \prod_{i=1}^n a_{\sigma_i, i}$

$\Rightarrow |\det(\Lambda)| = 1$

iii) Defs of group:

Closure, $(\Lambda \cdot \Lambda') \in R^{4\times 4}$

$(\Lambda \cdot \Lambda')^T g (\Lambda \cdot \Lambda') = \Lambda^T \cdot \Lambda'^T g \Lambda \cdot \Lambda'$

$= g$

Associativity $(\Lambda \cdot \Lambda') \cdot \Lambda'' = \Lambda \cdot (\Lambda' \cdot \Lambda'')$ matrix multipli-

Identity $\mathbb{1}_4$

Inverse $\Lambda \cdot \Lambda^{-1} = \mathbb{1}_4 \leftarrow$ invertible shown before

$$(\Lambda\cdot\Lambda^{-1})^T g\,(\Lambda\cdot\Lambda^{-1})$$
$$= \Lambda^{-1T}\Lambda^T g \Lambda\ \Lambda^{-1}$$
$$= \Lambda^{-1T} g \Lambda^{-1} \quad = \quad g$$
$$\Rightarrow \Lambda^{-1} \in L$$

iv) non-abelian:

$$\Lambda\cdot\Lambda' g = \Lambda\cdot\Lambda'\cdot\Lambda'^{-1} g \Lambda'^{-1T} = \Lambda g \Lambda'^{-1T}$$

$$\Lambda'\cdot\Lambda g$$

iv) $\Lambda^\mu{}_\nu = g^\mu{}_\nu - \delta\omega^\mu{}_\nu$, $\theta(\omega^2) = 0$, $\omega < 1$

$$g_{\mu\nu} = (g^\alpha{}_\mu - \delta\omega^\alpha{}_\mu)\, g_{\alpha\beta}\, (g^\beta{}_\nu - \delta\omega^\beta{}_\nu) = g_{\mu\nu}$$

$$= g_{\mu\nu} - g_{\beta\mu}\delta\omega^\beta{}_\nu - \delta\omega^\alpha{}_\mu g_{\alpha\nu} + \theta(\omega^2)$$

$$\delta\omega_{\mu\nu} = -\delta\omega_{\nu\mu}$$

$$U(\Lambda) = e^{-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}}$$

$$U^\alpha{}_\beta = \mathbb{1} - \tfrac{i}{2}(\delta\omega_{\mu\nu})^\alpha{}_\beta \qquad U = \mathbb{1} - \tfrac{i}{2}\delta\omega_{\mu\nu}J^{\mu\nu}$$

$$(\delta\omega)^\alpha{}_\beta = g^{\alpha\mu} g_\beta{}^\nu\, \delta\omega_{\mu\nu} = \tfrac{1}{2}\delta\omega_{\mu\nu}(g^{\alpha\mu}g_\beta{}^\nu - g^{\alpha\nu}g_\beta{}^\mu)$$

$$(J^{\mu\nu})^\alpha{}_\beta = -i(g^{\alpha\mu}g_\beta^\nu - g^{\alpha\nu}g_\beta^\mu)$$

$$[J^{\mu\nu}, J^{\alpha\beta}] = -i(g^{\nu\alpha}J^{\mu\beta} - g^{\mu\alpha}J^{\nu\mu} - g^{\nu\beta}J^{\mu\alpha} + g^{\mu\beta}J^{\nu\alpha})$$