H13.1

a) $S = \int_0^{t_f} dt\, \mathcal{L}(x, \dot{x}, t)$, $x(t) = x_{cl}(t) + \eta(t)$

$$= \int_0^{t_f} dt\, \tfrac{1}{2} m (\dot{x}_{cl} + \dot{\eta})^2 - \tfrac{1}{2} m\omega^2 (x + \eta)^2$$

$$= S[x_{cl}(t)] + S[\eta(t)] + \underbrace{\int_0^{t_f} dt\, m(\dot{x}_{cl}\dot{\eta} - \omega^2 x_{cl}\eta)}$$

$$= \int_0^{t_f} dt\, m(\dot{x}_{cl}\dot{\eta} + \ddot{x}_{cl}\eta)$$

$$= m\int_0^{t_f} dt(-\ddot{x}_{cl}\eta + \ddot{x}_{cl}\eta) = 0$$

$$= S[x_{cl}(t)] + S[\eta(t)]$$

↳ in general ?

b)

ELE $\rightarrow$ $\ddot{x}_{cl} + \omega^2 x_{cl} = 0$

$x_{cl}(0) = x_i = A$

$$\Rightarrow x_{cl} = \hat{A} e^{i\omega t} + \hat{B} e^{-i\omega t}$$

$$= A\cos(\omega t) + B\sin(\omega t)$$

$x_{cl}(t_f) = x_i \cos(\omega t_f) + B\sin(\omega t_f) = x_f$

$$B = \underline{\frac{x_f - x_i\cos(\omega t_f)}{\sin(\omega t_f)}}$$

$$S[x_{cl}(t)] = \int_0^{t_f} dt \left(\tfrac{1}{2} m \dot{x}_{cl}^2 - \tfrac{1}{2} m\omega^2 x_{cl}^2\right)$$

$$= \tfrac{1}{2} m \int_0^{t_f} dt (\dot{x}_{cl}^2 + \ddot{x}_{cl} x_{cl})$$

$$\int_0^{t_f} dt\, \dot{x}_{cl}\dot{x}_{cl}$$

$$= [\dot{x}_{cl} x_{cl}]_0^{t_f} - \int_0^{t_f} d\dot{x}_{cl}\, x_{cl}$$

$$= [\dot{x}_{cl} x_{cl}]_0^{t_f} - \int_0^{t_f} dt\, \ddot{x}_{cl} x_{cl}$$

$$= \tfrac{1}{2} m [\dot{x}_{cl} x_{cl}]_0^{t_f}$$

$$\Rightarrow S[x_{cl}(t)] = \frac{1}{2} m \Big\{ x_f \left(-A\sin(\omega t_f) + B\cos(\omega t_f)\right)\omega$$
$$- x_i \omega B \Big\}$$

$$= \frac{m\omega}{2}\Big[x_f \omega \left(-x_i \sin(\omega t_f) + (x_f - x_i\cos(\omega t_f))\cot(\omega t_f)\right)$$
$$- x_i \frac{x_f - x_i\cos(\omega t_f)}{\sin(\omega t_f)} \Big]$$

$$= \frac{m\omega}{2\sin(\omega t_f)}\Big[-x_f x_i \sin^2(\omega t_f) + \cos(\omega t_f) x_f^2 - x_f x_i \cos^2(\omega t_f)$$
$$- x_i x_f - x_i^2 \cos(\omega t_f) \Big]$$

$$= \frac{m\omega}{2\sin(\omega t_f)} \left(-2x_f x_i + (x_f^2 + x_i^2)\cos(\omega t_f)\right)$$

c) $$\eta(t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{t_f} t\right)$$

$$S[\eta(t)] = \frac{m}{2}\int_0^{t_f} dt \, (\dot{\eta}^2 - \omega^2 \eta^2)$$

$$= \frac{m}{2}\int_0^{t_f} dt \left(\sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{t_f} t\right) \cdot \frac{n\pi}{t_f}\right)^2 - \omega^2 \left(\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{t_f} t\right)\right)^2$$

$$= \frac{m}{2}\int_0^{t_f} dt \left(\frac{n\pi}{t_f}\right)^2 \sum_{n=1}^{\infty} a_n^2 \cos^2\left(\frac{n\pi}{t_f} t\right) - \omega^2 \sum_{n=1}^{\infty} a_n^2 \sin^2\left(\frac{n\pi}{t_f} t\right)$$

$$= \frac{m}{2}\Big\{ \left(\frac{n\pi}{t_f}\right)^2 \sum a_n^2 \left[\frac{n\pi}{2t_f} t + \frac{1}{4}\sin\left(\frac{2n\pi}{t_f} t\right)\right]_0^{t_f}$$
$$- \omega^2 \sum_{n=1}^{\infty} a_n^2 \left[\frac{n\pi}{2t_f} t - \frac{1}{4}\sin\left(\frac{2n\pi}{t_f} t\right)\right]_0^{t_f} \Big\}$$

$$= \frac{m}{2}\left\{\left(\frac{n\pi}{t_f}\right)^2 \sum a_n^2 \frac{n\pi}{2} - \omega^2 \sum a_n^2 \frac{n\pi}{2}\right\}$$

$$= \frac{m}{2} \sum_{n=1}^{\infty} a_n^2 \left[\left(\frac{n\pi}{t_f}\right)^2 - \omega^2\right] \frac{n\pi}{2}$$

redefine a_n

$$= \frac{m}{2} \sum_{n=1}^{\infty} a_n^2 \left(\frac{n^2\pi^2}{2t_f} - \frac{\omega^2 t_f}{2}\right)$$

d) $$\int \mathcal{D}[\eta(t)] \exp\left(\frac{i}{\hbar} S[\eta(t)]\right)$$

$$= \prod_{n=1}^{\infty} \int_{-\infty}^{\infty} da_n \, \exp\left(\frac{i}{\hbar}\frac{m}{2} a_n^2 \left(\frac{n^2\pi^2}{2t_f} - \frac{\omega^2 t_f}{2}\right)\right)$$

$$= \prod_{n=1}^{\infty} \sqrt{\frac{\pi}{\frac{im}{2\hbar}\left(\frac{n^2\pi^2}{2t_f} - \frac{\omega^2 t_f}{2}\right)}} = \prod_{n=1}^{\infty} \sqrt{\frac{4\hbar\pi t_f i}{m\left(n^2\pi^2 - \omega^2 t_f^2\right)}}$$

$$= \prod_{n=1}^{\infty} \sqrt{\frac{4t_f \hbar i}{mn^2\pi}} \prod_{n=1}^{\infty} \sqrt{\frac{\omega t_f}{\left(1-\frac{\omega^2 t_f^2}{n^2\pi^2}\right)\omega t_f}}$$

$$= \prod_{n=1}^{\infty} \sqrt{\frac{4t_f\hbar i}{mn^2\pi}} \sqrt{\frac{\omega t_f}{\sin(\omega t_f)}}$$

$$\Rightarrow U(x_f, t_f, x_i, 0)$$

$$= \mathcal{N} \exp\left(\frac{i}{\hbar}\frac{m\omega}{2\sin(\omega t_f)}\left((x_f^2 + x_i^2)\cos(\omega t_f) - 2x_i x_f\right)\right)$$

$$\prod_{n=1}^{\infty} \sqrt{\frac{4t_f\hbar i}{mn^2\pi}} \sqrt{\frac{\omega t_f}{\sin(\omega t_f)}}$$

e) $\lim_{t_f \to 0} U(x_f, t_f, x_i, 0) \overset{!}{=} \delta_{x_f, x_i} \overset{!}{=} \lim_{\alpha \to 0} \sqrt{\frac{1}{\pi \alpha}}\, e^{-(x_f - x_i)^2/\alpha}$

$$\alpha = \frac{2\hbar \sin(\omega t_f)}{i m \omega}, \quad t_f \to 0 \,.\, \alpha \to 0$$

$$\cos(\omega t_f) \to 1$$

$$\lim_{t_f \to 0} N \prod_{n=1}^{\infty} \sqrt{\frac{4 t_f \hbar i}{m n^2 \pi}} \sqrt{\frac{\omega t_f}{\sin(\omega t_f)}} \exp\left(\frac{i}{\hbar} \frac{m\omega}{2\sin(\omega t_f)} \left((x_f^2 + x_i^2)\cos(\omega t_f) - 2 x_i x_f\right)\right)$$

$$\overset{!}{=} \lim_{\alpha \to 0} \sqrt{\frac{1}{\pi \alpha}}\, e^{-(x_f^2 - x_i^2)^2/\alpha}$$

Write $\alpha = \frac{2\hbar \sin(\omega t_f)}{i m \omega}$

$$LHS = \lim_{t_f \to 0} N \prod_{n=1}^{\infty} \frac{1}{n} \sqrt{\frac{4 t_f \hbar i}{m \pi}} \sqrt{\frac{\omega t_f}{\sin(\omega t_f)}} \exp\left((x_f - x_i)^2/\alpha\right)$$

$$\Rightarrow N \prod_{n=1}^{\infty} \frac{1}{n} \sqrt{\frac{4 t_f \hbar i}{m \pi}} \sqrt{\frac{\omega t_f}{\sin(\omega t_f)}} = \sqrt{\frac{i m \omega}{2\pi \hbar \sin(\omega t_f)}}$$

$$\Rightarrow U(x_f, t_f, x_i, 0) = \sqrt{\frac{i m \omega}{2\pi\hbar \sin(\omega t_f)}} \exp\left(\frac{i m \omega}{2\hbar \sin(\omega t_f)} \left((x_f^2 + x_i^2)\cos(\omega t_f) - 2 x_i x_f\right)\right)$$