H3.1 Chenhuan Wang

a) first-order Born Approximation:

$$f^{(1)}(\vec{k}',\vec{k}) = -\frac{m}{2\pi\hbar^2}\int d^3x' \, e^{i(\vec{k}'-\vec{k})\cdot\vec{x}'} V(\vec{x}')$$

$$= -\frac{m}{2\pi\hbar^2}\int d^3x' \, e^{i(\vec{k}'-\vec{k})\cdot\vec{x}'} \frac{V_0}{|\vec{x}'|} e^{-\kappa|\vec{x}'|}$$

$$= -\frac{m}{2\pi\hbar^2}\int_0^{2\pi} d\varphi \int_0^{\theta} d\theta \sin\theta \int_0^{\infty} dr\, r^2 \, e^{i(\vec{k}'-\vec{k})\cdot\vec{x}'} \frac{V_0}{r} e^{-\kappa r}$$

choose $\vec{x}$ as the z-Direction $\rightarrow$

$$= -\frac{mV_0}{\hbar^2}\int_0^{\pi} d\theta \sin\theta \int_0^{\infty} dr\, r \, e^{i(k'-k) r\cos\theta} e^{-\kappa r}$$

$$\left[\begin{aligned} &\int_{\theta=0}^{\theta=\pi} -d(\cos\theta)\, e^{i(k'-k) r\cos\theta} \\ &= -\frac{1}{i(k'-k)r}\left[e^{-i(k'-k)r} - e^{i(k'-k)r}\right] \\ &= \frac{2}{(k'-k)r}\sin((k'-k)r) \end{aligned}\right]$$

$$= -\frac{mV_0}{\hbar^2}\int_0^{\infty} dr \frac{2}{(k'-k)} \sin((k'-k)r)\, e^{-\kappa r}$$

!!!!

$$= -\frac{2mV_0}{\hbar^2(k'-k)} \mathrm{Im}\left[\int_0^{\infty} dr\, e^{i(k'-k)r-\kappa r}\right]$$

$$= -\frac{2mV_0}{\hbar^2(k'-k)} \mathrm{Im}\left[-\frac{1}{i(k'-k)-\kappa}\right]$$

$$= -\frac{2mV_0}{\hbar^2(k'-k)} \frac{(k'-k)}{\kappa^2+(k'-k)^2} = -\frac{2mV_0}{\hbar^2}\frac{1}{\kappa^2+(k'-k)^2}$$

$$\left[\begin{aligned} (k'-k) &= |\vec{k}'-\vec{k}| = \sqrt{|\vec{k}'|^2+|\vec{k}|^2-2|\vec{k}|\cdot|\vec{k}'|\cdot\cos\theta} = k\sqrt{2-2\cos\theta} \\ &= 2k\sin\left(\frac{\theta}{2}\right) \end{aligned}\right]$$

$$= -\frac{2mV_0}{\hbar^2}\frac{1}{\kappa^2+4k^2\sin^2\left(\frac{\theta}{2}\right)}$$

✓

$$\Rightarrow \frac{d\sigma^{(1)}}{d\Omega} = \frac{4m^2V_0^2}{\hbar^4}\frac{1}{\left[\kappa^2+4k^2\sin^2\left(\frac{\theta}{2}\right)\right]^2}$$

mistake on the sheet?

b) The assumption of Born approx. is that the incident field is small compared to scattered field.

If we have:

$$\langle \vec{x} | \psi^{(+)} \rangle = \frac{1}{L^{3/2}} \left[e^{i\vec{k}\cdot\vec{x}} + \frac{e^{ikr}}{r} f(\vec{k}', \vec{k}) \right]$$

then:

$$\left| e^{i\vec{k}\cdot\vec{x}} \right| \gg \left| \frac{e^{ikr}}{r} \cdot \frac{2mV_0}{\hbar^2} \frac{1}{\kappa^2 + 4k^2\sin^2(\frac{\theta}{2})} \right|$$

low energy, i.e. $e^{i\vec{k}\cdot\vec{x}} \simeq e^{ikr} \simeq 1$

$$(k \propto \sqrt{E})$$

$$\Rightarrow \quad 1 \gg \frac{2mV_0}{\hbar^2} \frac{1}{r} \frac{1}{\kappa^2 + 4k^2\sin^2(\frac{\theta}{2})}$$

$$\Rightarrow \quad \kappa^2 \gg \frac{2mV_0}{\hbar^2} \frac{1}{r} - 4k^2\sin^2(\frac{\theta}{2})$$

$$\frac{V_0}{\kappa^2} \ll \frac{\hbar^2}{2m} \cdot r$$

$$\text{or} \quad V_0 \ll \frac{\hbar^2 r [\kappa^2 + 4k^2 \sin^2(\frac{\theta}{2})]}{2mV_0}$$

c) $V(r) = \frac{V_0}{r} \exp(-\kappa r)$

when $\kappa = 0$ $\quad V(r) = \frac{V_0}{r}$

$$\Rightarrow \quad \frac{d\sigma}{d\Omega} = \frac{4m^2V_0^2}{\hbar^4} \frac{1}{[4k^2\sin^2(\frac{\theta}{2})]^2}$$

If we choose $V_0 = zz'e^2$, then we have Rutherford scattering.

The reason why one gets this result ???

d) $\vec{k}' = \vec{k}$

$$f^{2}(\vec{k}', \vec{k}) = -\frac{1}{4\pi} \frac{2m}{\hbar^2} (2\pi)^3 \int d^3x' \int d^3x'' \langle k' | x' \rangle V(x') \langle x' | \frac{1}{E - H_0 + i\varepsilon} | x'' \rangle$$
$$V(x'') \langle x'' | k \rangle$$

b) $\langle r | \psi^+ \rangle = \langle r | \varphi \rangle - \frac{2m}{\hbar^2} \int d^3r' \frac{e^{ikr}}{r'} V(r') e^{-i\vec{k}\cdot\vec{r}'}$

$$\Rightarrow \left| \frac{2m}{\hbar^2}\frac{1}{4\pi} \int d^3r' \frac{e^{ikr}}{r'} V(r') e^{-i\vec{k}\cdot\vec{r}'} \right| \ll 1.$$

$$\Rightarrow \frac{2m}{\hbar^2} \frac{|V_0|}{\kappa} \ll 1$$

d) $T = \hat{V} + \hat{V} G_0 \hat{V}$

$$f^{(2)} = -\frac{m}{2\pi} \langle \vec{k}_{out} | \hat{V} G_0 \hat{V} | \vec{k}_{in} \rangle$$

$$= -\frac{m}{2\pi} \sum_{\vec{p}',\vec{p}} \langle \vec{k}_{out} | \hat{V} | \vec{p} \rangle \underbrace{\langle \vec{p} | G_0 | \vec{p}' \rangle}_{\frac{1}{E - \frac{p^2}{2m} + i\varepsilon} \delta_{pp'}} \langle \vec{p}' | \hat{V} | \vec{k}_{in} \rangle$$

$$= -\frac{m}{2\pi} \sum_{\vec{p},\vec{p}'} V(\vec{k}_{out} - \vec{p}) \frac{1}{E - \frac{p^2}{2m} + i\varepsilon} \delta_{pp'} V(\vec{p}' - \vec{k}_{in})$$

$$= -\frac{m}{(2\pi)^3} \int \frac{dp^3}{(2\pi)^3} V^2(\vec{p} - \vec{k}) \frac{2\pi V_0}{(\vec{p} - \vec{k})^2 + \kappa^2}, \qquad V(\vec{p} - \vec{k}) = \frac{2\pi V_0}{(\vec{p} - \vec{k})^2 + \kappa^2}$$

$$\left[\int_{-1}^{1} d(\cos\theta)\, V^2(\vec{p} - \vec{k}) = (4\pi)^2 V_0^2 \int_{-1}^{1} d\cos\theta \frac{1}{(p^2 - k^2 - 2pk\cos\theta + \kappa^2)^2} \right]$$

$$= -\frac{2m(4\pi)^2}{(2\pi)^3} V_0 \int dp \frac{p^2}{[(p-k)^2 + \kappa^2][(p+k)^2 + \kappa^2]} \cdot \frac{1}{E - \frac{p^2}{2m} + i\varepsilon}$$

Poles: $k + i\kappa = p_{1,2}$, $-k \pm i\kappa = p_{3,4}$

$p_{5,6} = \pm\sqrt{2mE + i2m\varepsilon}$

$$\mathrm{Res}_1 = \frac{-ik + \kappa}{8k\kappa(k^2 - 2ik\kappa)} \qquad \mathrm{Res}_2 = \frac{ik + \kappa}{8k\kappa(k^2 + 2k\kappa)}$$

$$\mathrm{Res}_3 = \frac{\kappa}{2k^2(k^2 + 4k^2)} \qquad \Rightarrow f^{(2)} = \frac{2m^2 V_0^2}{\hbar^4} \frac{1}{\kappa^2(\kappa - 2ik)}$$

d) Second Born Approximation : $\hat{T} = \underset{\uparrow \text{ 1. Born}}{\hat{V}} + \hat{V} \underset{\uparrow \text{ 2. Born}}{G} \hat{V} + \cdots$

$$\Rightarrow f^{(2)}(\vec{k}, \vec{k}') = -\frac{mL^3}{2\pi\hbar^2} \langle \vec{k}' | \hat{V} G \hat{V} | \vec{k} \rangle$$

$$\overset{\hbar = 1}{=} -\frac{mL^3}{2\pi} \sum_p \sum_{p'} \langle \vec{k}' | \hat{V} | \vec{p}' \rangle \underbrace{\langle \vec{p}' | G | \vec{p} \rangle}_{} \langle \vec{p} | \hat{V} | \vec{k} \rangle$$

$k = p/\hbar$

$$= \frac{1}{E - p^2/2m + i\varepsilon} \delta_{pp'}$$

$V_{Yukawa}(\vec{k}) = \frac{4\pi V_0}{k^2 + K^2}$

$$= -\frac{mL^3}{2\pi} \sum_{p,p'} \frac{4\pi V_0}{(\vec{k}' - \vec{p}')^2 + K^2} \frac{1}{E - p^2/2m + i\varepsilon} \frac{4\pi V_0}{(\vec{k} - \vec{p})^2 + K^2} \delta_{pp'}$$

$\vec{k} = \vec{k}'$ is the given condition

$$= -\frac{mL^3}{2\pi} \sum_p \frac{4\pi V_0}{(\vec{k}' - \vec{p})^2 + K^2} \frac{1}{E - p^2/2m + i\varepsilon} \frac{4\pi V_0}{(\vec{k} - \vec{p})^2 + K^2}$$

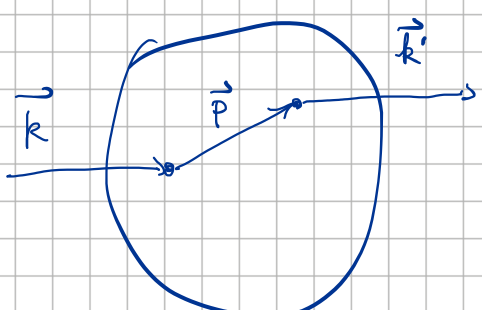
$L \to \infty$

$$= -\frac{m}{2\pi} \frac{1}{(2\pi)^3} \int d^3p \frac{(4\pi)^2 V_0^2}{[(\vec{k} - \vec{p})^2 + K^2][(\vec{k} - \vec{p})^2 + K^2]} \frac{1}{E - p^2/2m + i\varepsilon}$$

$k = 2\pi n_i / L$ in a box

$$= -\frac{m}{\pi^2} \cdot 2\pi \int_0^\infty dp\, p^2 \underbrace{\int_0^\pi d\theta \sin\theta \frac{V_0^2}{[(\vec{k} - \vec{p})^2 + K^2][(\vec{k} - \vec{p})^2 + K^2]}}_{} \frac{1}{E - p^2/2m + i\varepsilon}$$

$\theta = \angle(\vec{p}, \vec{k})$, $\vec{p}$ not $\parallel \vec{k}$!!



$$= V_0^2 \int_{-1}^{1} d(\cos\theta) \frac{1}{(k^2 + p^2 - 2kp\cos\theta + K^2)^2}$$

$$\left[\begin{array}{l} u = k^2 - p^2 - 2kp\cos\theta + K^2 \\ \frac{du}{d\cos\theta} = -2kp \end{array} \right.$$

$$= V_0^2 \frac{1}{-2kp} \int_{-1}^{1} du\, u^{-2}$$

$$= V_0^2 \frac{1}{-2kp} \left[-u^{-1} \right]_{-1}^{1}$$

$$= V_0^2 \frac{1}{-2kp} \left[\frac{1}{(k-p)^2 + K^2} - \frac{1}{(k+p)^2 + K^2} \right]$$

$$= V_0^2 \frac{1}{-2kp} \frac{(k+p)^2 + K^2 - (k-p)^2 - K^2}{[(k-p)^2 + K^2][(k+p)^2 + K^2]}$$

$$= V_0^2 (-2) \frac{1}{[(k-p)^2 + K^2][(k+p)^2 + K^2]}$$

$$= \frac{4mV_0^2}{\pi} \int_0^\infty dp \, \frac{p^2}{[(k-p)^2+\kappa^2][(k+p)^2+\kappa^2]} \, \frac{1}{E - p^2/2m + i\varepsilon}$$

Poles: $(k-p)^2+\kappa^2 = 0 \quad k-p = \pm i\kappa \quad \Rightarrow \quad p_{1,2} = k \mp i\kappa$

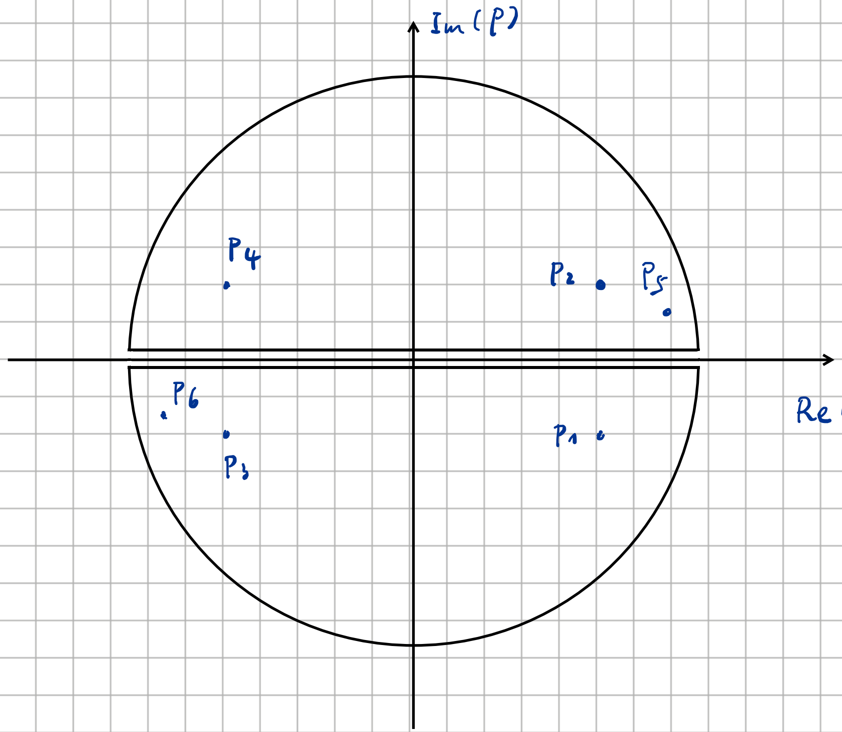
$(k+p)^2+\kappa^2 = 0 \quad \Rightarrow \quad p_{3,4} = -k \mp i\kappa$

$E - p^2/2m + i\varepsilon = 0 \quad \Rightarrow \quad p_{5,6} = \pm\sqrt{2mE + 2im\varepsilon} = \pm\sqrt{r}\, e^{i\varphi/2}$

$\tan\varphi = \frac{m\varepsilon}{mE} = \frac{\varepsilon}{E} << 1$

$2mE = r\cos\varphi \quad 2m\varepsilon = r\sin\varphi$

$r = 2m\sqrt{E^2+\varepsilon^2}$



$$\Rightarrow \mathrm{Res}(g(p), p_2 = k+i\kappa) = \lim_{p\to p_2} (p-p_2)\, g(p)$$

$$= \lim_{p\to p_2} \frac{p^2}{[p-(k-i\kappa)][(k+p)^2+\kappa^2]} \cdot \frac{1}{E - p^2/2m + i\varepsilon}$$

$$= \frac{(k+i\kappa)^2}{2i\kappa[(2k+i\kappa)^2+\kappa^2]} \cdot \frac{1}{E - \frac{1}{2m}(k+i\kappa)^2 + i\varepsilon}$$

$$= \frac{(k+i\kappa)^2}{2i\kappa(4k^2+4ik\kappa)} \cdot \frac{1}{-\frac{1}{2m}(2ik\kappa-\kappa^2)+i\varepsilon}$$

$$= \frac{(k+i\kappa)^2}{8i\kappa k(k+i\kappa)} \, \frac{-2m}{2ik\kappa-\kappa^2} \qquad \varepsilon \to 0$$

$$= \frac{(k+iK)}{4ikK}\,\frac{m}{K(K-2ik)} = \frac{m\,(k+iK)}{4ikK^2\,(K-2ik)}$$

$$\mathrm{Res}\,(g(p),\ p_4=-k+iK) = \lim_{p\to p_4} (p-p_4)\, g(p)$$

$$= \lim_{p\to p_4} \frac{p^2}{[(k-p)^2+K^2][\,p-(-k-iK)\,]} \cdot \frac{1}{E-p^2/2m+i\varepsilon}$$

$$= \frac{(-k+iK)^2}{[(2k-iK)^2+K^2][-k+iK-(-k-iK)]}\;\frac{1}{E-\frac{1}{2m}(-k+iK)^2+i\varepsilon}$$

$$= \frac{(-k+iK)^2}{(4k^2-4ikK)\cdot 2iK}\;\frac{2m}{(+K^2+2ikK)}$$

$$= \frac{k-iK}{4iKk}\;\frac{m}{K(K+2ik)} = \frac{m(k-iK)}{4iK^2k(K+2ik)}$$

$$\mathrm{Res}\,(g(p),\ p_5=+\sqrt{2mE+i2m\varepsilon}) = \lim_{p\to p_5} (p-p_5)\, g(p)$$

$$= \lim_{p\to p_5} \frac{p^2}{[(k-p)^2+K^2][(k+p)^2+K^2]}\;\frac{2m}{(p-p_6)}$$

$\varepsilon \to 0$

$$= \frac{2mE}{[k^2-2k\sqrt{2mE}+2mE+K^2][k^2+2k\sqrt{2mE}+2mE+K^2]\cdot 2\sqrt{2mE}}\;2m$$

$k=\sqrt{2mE}$

$$= \frac{k\cdot 2m}{2\,K^2\,(4k^2+K^2)}$$

$$\Rightarrow \frac{4mV_0^2}{\pi}\cdot\int_0^\infty dp\; g(p) = \frac{4mV_0^2}{\pi}\cdot\frac{1}{2}\cdot 2\pi i\sum \mathrm{Res}$$

!

$$= \frac{i4\pi\, mV_0^2}{\pi}\left\{\frac{m\,(k+iK)}{4ikK^2\,(K-2ik)} + \frac{m(k-iK)}{4iK^2k(K+2ik)} + \frac{2mk}{2K^2(4k^2+K^2)}\right\}$$

$m=1$?

$$= \frac{i\,4\pi\, mV_0^2}{\pi}\left\{\frac{(k+iK)(K-2ik)+(k-iK)(K-2ik)}{4ikK^2(K^2+4k^2)} + \frac{k\cdot 2ik\cdot 2m}{4ikK^2(K^2+4k^2)}\right\}$$

$$= \frac{\cancel{i4\pi}\; mV_0^2}{\cancel{\pi}}\cdot\frac{\cancel{iK^2}+kK-2ik^2+\cancel{2kK}+kK-\cancel{2ik^2}-\cancel{iK^2}-\cancel{2kK}+\cancel{2ik^2}}{\cancel{4i}kK^2(K^2+4k^2)}$$

$$= m^2V_0^2\,\frac{2k\,(K-2ik)}{kK^2\,(K^2+4k^2)} = 2m^2V_0^2\,\frac{1}{K^2(K-2ik)}$$

e) $f^{(1)} \in \mathbb{R} \Rightarrow \operatorname{Im} f(\theta=0) = 0$

To verify the optical theorem for Yukawa $\to$ second order Born

$$\operatorname{Im} f^{(2)} = \frac{16\pi V_0^2 m^2}{\kappa^2 (4k^2 + \kappa^2)\hbar^4}$$

H3.2

a) $$\Delta(b) = -\frac{m}{2k\hbar^2}\int_{-\infty}^{\infty} dz\, V(\sqrt{b^2+z^2}) \quad , \quad V(r) = \frac{Ze^2}{r}$$

$$= -\frac{m}{2k\hbar^2}\int_{-\infty}^{\infty} dz\, \frac{Ze^2}{\sqrt{b^2+z^2}}$$

$$= -\frac{Ze^2 m}{2k\hbar^2}\int_{-\infty}^{\infty} dz\, (b^2+z^2)^{-\frac{1}{2}}$$

The integral does not converge!

$$b^2+z^2 < z^2 \Rightarrow (b^2+z^2)^{\frac{1}{2}} < z$$

and $\int_{-\infty}^{\infty} dz\, z = \infty$

b) In scattering experiment the observation point is normally set at relatively far away from potential. The influence of V is minimal so we can just assume it has none.

c) $$\Delta(b) = -\frac{Ze^2 m}{2k\hbar^2}\underbrace{\int_{-\infty}^{\infty} dz\, (b^2+z^2)^{-\frac{1}{2}}\, F(\sqrt{b^2+z^2})}$$

$$= \int_{-\sqrt{a^2-b^2}}^{+\sqrt{a^2-b^2}} dz\, (b^2+z^2)^{-\frac{1}{2}}$$

$$= b\int_{-\sqrt{a^2-b^2}/b}^{+\sqrt{a^2-b^2}/b} d\left(\frac{z}{b}\right)\frac{1}{\sqrt{1+\left(\frac{z}{b}\right)^2}}$$

$$\frac{z}{b} = \tan u$$

$$= b\int_{-\arctan(\sqrt{a^2/b^2-1})}^{\arctan(\sqrt{a^2/b^2-1})} du\, \sec^2(u)\frac{1}{\sqrt{1+\tan^2 u}}$$

$$\frac{d\tan u}{du} = \sec^2(u)$$

$$1+\tan^2 u = \sec^2 u$$

$$= b\int_{\dots}^{\dots} du\, \sec u$$

$$= b\left[\ln|\sec u + \tan u|\right]_{-\arctan(\sqrt{a^2/b^2-1})}^{\arctan(\sqrt{a^2/b^2-1})}$$

$$= b\Big\{\ln\left|\sec\arctan\sqrt{\frac{a^2}{b^2}-1} + \sqrt{\frac{a^2}{b^2}-1}\right|$$

$$- \ln\left|\sec\left(-\arctan\sqrt{\frac{a^2}{b^2}-1}\right) - \sqrt{\frac{a^2}{b^2}-1}\right|$$

$$\left[\ \sec\arctan\sqrt{\frac{a^2}{b^2}-1} = \sec\left(-\arctan\sqrt{\frac{a^2}{b^2}-1}\right) = \sqrt{1+\sqrt{\frac{a^2}{b^2}-1}^{\,2}} = \sqrt{1+\frac{a^2}{b^2}-1} = \frac{a}{b}\ \right]$$

$$= b\left\{\ln\left|\frac{a}{b}+\sqrt{\frac{a^2}{b^2}-1}\right| - \ln\left|\frac{a}{b}-\sqrt{\frac{a^2}{b^2}-1}\right|\right\}$$

$$= b\ln\left|\frac{a+\sqrt{a^2-b^2}}{a-\sqrt{a^2-b^2}}\right| = b\ln\left(1+\frac{2\sqrt{a^2-b^2}}{a-\sqrt{a^2-b^2}}\right)$$

d) $ka \to \infty \quad \Rightarrow \quad \theta = 0$

e) $\theta \gg \frac{1}{ka}\ , \quad 0 < \theta \leq \pi$

The second term (-1) can be neglected, because

c) $$\Delta(b) = -\frac{mZe^2}{2k\hbar^2}\int_{-\sqrt{c}}^{\sqrt{c}} dz\, \sqrt{b^2+z^2} \qquad c = a^2-b^2$$

$$= A\int_0^{\sqrt{c-1}} dx\,\frac{1}{\sqrt{x^2+1}} = A\ln\left(\frac{a+\sqrt{a^2-b^2}}{b}\right)$$

d) All angles are affected

e)

1. $\theta \gg \frac{1}{ka}$

$$\int_0^\infty db\, b\, J_0(2kb\sin\theta/2) = \frac{1}{2\pi}\int d^2b\, e^{i(\vec{k}-\vec{k}')\cdot\vec{b}} \qquad \theta \neq 0$$

$$\vec{k} \neq \vec{k}' \Rightarrow 0$$

2. $\ln\left(\frac{a}{b}+\sqrt{\left(\frac{a}{b}\right)^2-1^2}\right) \simeq \ln\left(\frac{2a}{b}\right) + \vartheta\left(\frac{b^2}{a^2}\right)$

3. $\Delta(b) = \frac{mZe^2}{k\hbar^2}\ln\left(\frac{b}{2a}\right)$

$$\Rightarrow f(\vec{k},\vec{k}') = -ik\int_0^\infty db\, b\, J_0\left(2kb\sin\frac{\theta}{2}\right) e^{2i\Delta(b)}$$

$$= ik\int_0^\infty db\, b\, J_0\left(2kb\sin\frac{\theta}{2}\right)\left(\frac{b}{2a}\right)^2 \exp\left[\frac{imZe^2}{k\hbar^2}\right]$$

f) $\frac{d\sigma}{d\Omega} = |f(\theta)|^2 = \frac{4m^2 Z^2 e^4}{\hbar^2 (2k \sin\theta/2)^4}$ Rutherford scattering