H4.1 Chenlun Wang

a) • $\int_{-1}^{1} dx\, P_l(x)\, P_{l'}(x)$

$$= \left(\frac{1}{2^l l!}\right)^2 \int_{-1}^{1} dx\, \frac{d^l}{dx^l}\left[(x^2-1)^l\right] \frac{d^{l'}}{dx^{l'}}\left[(x^2-1)^{l'}\right]$$

Hint with $i=l$

$$= \left(\frac{1}{2^l l!}\right)^2 \int_{-1}^{1} dx\, \left[(x^2-1)^l\right] \frac{d^{l'+l}}{dx^{l'+l}}\left[(x^2-1)^{l'}\right]\cdot(-1)^l$$

polynomial with degree $2l'$

$\Rightarrow$ $l'+l \leq 2l'$

in order to make the integral nonzero.

One can also do it to l

$\Rightarrow$ $\delta_{ll'}$

to determine the factor:

$$\frac{(2l)!\cdot(-1)^l}{2^{2l}(l!)^2}\cdot\delta_{ll'} \underbrace{\int_{-1}^{1} dx\, (x^2-1)^l}$$

$x = 2u-1$

$dx = 2du$

$$= 2\int_0^1 du\, ((2u-1)^2-1)^l$$

$$= 2\int_0^1 du\, (4u^2-4u)^l$$

$$= 2\cdot 2^{2l}\int_0^1 du\, u^l(u-1)^l$$

$$= 2^{2l+1}\frac{1}{l+1}\int_0^1 d(u^{l+1})\,(u-1)^l$$

$$= 2^{2l+1}\frac{1}{l+1}\left\{\left[u^{l+1}(u-1)^l\right]_0^1 - \int d\left((u-1)^l\right)u^{l+1}\right\}$$

$$= 2^{2l+1}\frac{1}{l+1}\cdot(-1)\cdot l\int_0^1 du\, (u-1)^{l-1}u^{l+1}$$

l-times

$$= 2^{2l+1}(-1)^l\frac{l!}{(2l)!}\, l!\int_0^1 du\, u^{2l}$$

$$= 2^{2l+1}\cdot(-1)^l\frac{(l!)^2}{(2l)!}\,\frac{1}{2l+1}$$

Put together $\int_{-1}^{1} dx\, P_l(x)\, P_{l'}(x) = \frac{2}{2l+1}\delta_{ll'}$

- $P_\ell(-x) = (-1)^\ell P_\ell(x)$

$$P_\ell(-x) = \frac{1}{2^\ell \ell!} \frac{d^\ell}{d(-x)^\ell} \left[(x^2-1)^\ell\right]$$

$$= \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell} \frac{dx^\ell}{d(-x)^\ell} \left[(x^2-1)^\ell\right]$$

$$= (-1)^\ell P_\ell(x)$$

- $P_\ell(1) = 1$

$$P_\ell(1) = \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell} \left[(x^2-1)^\ell\right]\Big|_{x=1}$$

The generating function of $P_\ell(x)$,

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x) t^n$$

Legendre P. is first "discovered" to expand the newtonian potential

$x = 1$

$$\Rightarrow \frac{1}{\sqrt{1-2t+t^2}} = \sum P_n(1) t^n$$

$$\sum P_n(1) t^n = (t-1)^{-1} = 1 + t + t^2 + \cdots t^n + \cdots$$

$$\Rightarrow P_n(1) \equiv 1$$

- $P_0(x) = 1$

$$P_1(x) = \frac{1}{2\cdot 1} \frac{d}{dx}(x^2-1) = \frac{1}{2}\cdot 2x = x$$

$$P_2(x) = \frac{1}{4\cdot 2!} \frac{d^2}{dx^2}\left[(x^2-1)^2\right] = \frac{1}{8} \frac{d^2}{dx^2}(x^4 - 2x^2 + 1)$$

$$= \frac{1}{8}\cdot(4\cdot 3x^2 - 4) = \frac{1}{2}(3x^2-1)$$

b) S-Wave: $\ell = 0$

$$\frac{d\sigma}{d\Omega} = |f_k(\theta)|^2 = \left| \frac{1}{k} e^{i\delta_0(k)} \sin(\delta_0(k)) P_0(\cos\theta) \right|^2$$

$$= \left(\frac{1}{k} \sin(\delta_0(k)) \right)^2$$

P-Wave: $\ell = 1$

$$\frac{d\sigma}{d\Omega} = |f_k(\theta)|^2 = \left| \frac{1}{k} 3 e^{i\delta_1(k)} \sin(\delta_1(k)) P_1(\cos\theta) \right|^2$$

$$= \left[\frac{1}{k} 3 \sin(\delta_1(k)) \cos\theta \right]^2 = \left(\frac{3}{k} \sin(\delta_1(k)) \cos\theta \right)^2$$

D-Wave: $\ell = 2$

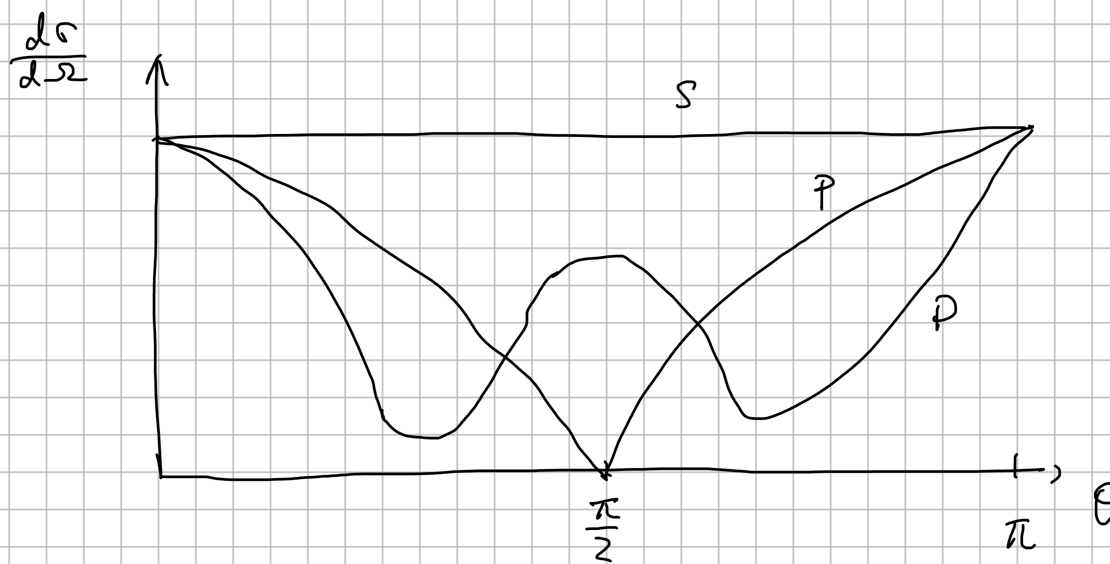
$$\frac{d\sigma}{d\Omega} = |f_k(\theta)|^2 = \left| \frac{1}{k} \cdot 5 e^{i\delta_2(k)} \sin(\delta_2(k)) P_2(\cos\theta) \right|^2$$

$$= \left[\frac{5}{2k} \sin(\delta_2(k)) (3\cos^2\theta - 1) \right]^2$$

S-P-Wave: $\ell = 0 + \ell = 1$

$$\frac{d\sigma}{d\Omega} = |f_k(\theta)|^2 = \left| \frac{1}{k} \left(e^{i\delta_0(k)} \sin(\delta_0(k)) + 3 e^{i\delta_1(k)} \sin(\delta_1(k)) \cos\theta \right) \right|^2$$

$$= \frac{1}{k^2} \Big[\sin^2(\delta_0(k)) + 9 \sin^2(\delta_1(k)) \cos^2\theta + 6 \cos(\delta_0(k) - \delta_1(k)) \cdot$$
$$\sin(\delta_0(k)) \sin(\delta_1(k)) \cos\theta \Big]$$



$\leftarrow$ normalised so that $\max\left(\frac{d\sigma}{d\Omega}\right) = 1$ (m²)

To plot the s-p-wave would be difficult, since we don't have information about $\sin(\delta_l(k))$

c) $\sigma = \int d\Omega \; |f_k(\theta)|^2$

$$= \int d\Omega \left| \frac{1}{k} \sum_{l=0}^{\infty} (2l+1) \, e^{i\delta_l(k)} \sin(\delta_l(k)) \, P_l(\cos\theta) \right|^2$$

$$=: \sum_{l=0}^{\infty} \sigma_l \, ,$$

$$= \frac{1}{k^2} \sum_{l=0}^{\infty} \left| (2l+1) \, e^{i\delta_l(k)} \sin(\delta_l(k)) \right|^2 \underbrace{\int d\Omega \, |P_l(\cos\theta)|^2}_{\text{orthogonality}}$$

$$= 2\pi \cdot \frac{2}{2l+1}$$

$$= \frac{1}{k^2} \sum_{l=0}^{\infty} \left[(2l+1) \, e^{i\delta_l(k)} \sin(\delta_l(k)) \right]^2 \cdot 4\pi \, \frac{1}{2l+1}$$

$$= \sum_{l=0}^{\infty} (2l+1) \cdot 4\pi \cdot \left[\frac{1}{k} e^{i\delta_l(k)} \sin(\delta_l(k)) \right]^2$$

$$\Rightarrow \sigma_i = 4\pi (2l+1) \, |f(k)|^2 \, , \qquad f(k) = e^{i\delta_l(k)} \sin(\delta_l(k))/k$$

$$\max(\sigma_i) = 4\pi (2l+1) \, \max(|f(k)|^2)$$

$$= 4\pi (2l+1) \, \max\left[\left(\frac{\sin(\delta_l(k))}{k} \right)^2 \right]$$

$\Rightarrow$ with phase $\delta_l(k) = \frac{\pi}{2}$ is the

σ at its maximum

d) OT : $\mathrm{Im}(f(\theta=0)) = \dfrac{k \, \sigma_{tot}}{4\pi}$

S-Wave: $LHS = \mathrm{Im}\left(\frac{1}{k} (2l+1) \sin(\delta_l(k)) \, e^{i\delta_l(k)} P_l(\cos\theta) \right) \Big|_{l=0}$

$$= \mathrm{Im}\left(\frac{1}{k} \sin(\delta_0(k)) \, e^{i\delta_0(k)} \, P_0(\cos\theta) \right)$$

$$= \frac{1}{k} \sin^2(\delta_0(k))$$

$$RHS = \frac{k}{4\pi} \int d\Omega \left(\frac{1}{k} \sin(\delta_0(k)) \right)^2$$

$$= \sin^2(\delta_0(k))/k \qquad \Rightarrow \checkmark$$

P-Wave:
$$LHS = \text{Im}\left(\frac{1}{k}\cdot 3\cdot \sin(\delta_1(k))\, e^{i\delta_1(k)} P_1(\cos\theta)\right)\Big|_{\theta=0}$$
$$= \text{Im}\left(\frac{3}{k}\sin(\delta_1(k))\, e^{i\delta_1(k)} \cos\theta\right)\Big|_{\theta=0}$$
$$= \frac{3}{k}\sin^2(\delta_1(k))\cos\theta\Big|_{\theta=0} = \frac{3}{k}\sin^2(\delta_1(k))$$
$$RHS = \frac{k}{4\pi}\int d\Omega \left(\frac{3}{k}\sin(\delta_1(k))\cos\theta\right)^2$$
$$= \frac{k}{2}\frac{9}{k^2}\sin^2(\delta_1(k))\int_{-1}^{1} d(\cos\theta)\, \cos^2\theta$$
$$= \frac{9}{2k}\sin^2(\delta_1(k))\,\frac{1}{3}\cdot 2$$
$$= \frac{3}{k}\sin^2(\delta_1(k)) \quad \Rightarrow \checkmark$$

D-Wave:
$$LHS = \text{Im}\left(\frac{1}{k}\, 5\, \sin(\delta_2(k))\, e^{i\delta_2(k)}\, \frac{1}{2}(3\cos^2\theta - 1)\right)\Big|_{\theta=0}$$
$$= \frac{5}{2k}\sin^2(\delta_2(k))\,(3\cos^2\theta - 1)\Big|_{\theta=0} = \frac{5}{k}\sin^2(\delta_2(k))$$
$$RHS = \frac{k}{4\pi}\int d\Omega \left[\frac{5}{2k}\sin(\delta_2(k))\,(3\cos^2\theta - 1)\right]^2$$
$$= \frac{k}{2}\left[\frac{5}{2k}\sin(\delta_2(k))\right]^2 \underbrace{\int_{-1}^{1} d\cos\theta\,(3\cos^2\theta - 1)^2}$$
$$= \int_{-1}^{1} dx\,(9x^4 - 6x^2 + 1)$$
$$= \frac{9}{5}\left[x^5\right]_{-1}^{1} - 2\left[x^3\right]_{-1}^{1} + 2$$
$$= \frac{18}{5} - 2\cdot 2 + 2 = \frac{8}{5}$$
$$= \frac{k}{2}\,\frac{5^2}{4k^2}\sin^2(\delta_2(k))\cdot\frac{8}{5}$$
$$= \frac{5}{k}\sin^2(\delta_2(k)) \quad \Rightarrow \checkmark$$

S-P-Wave:
$$LHS = \text{Im}(f(\theta=0)) = \frac{1}{k}\left(\sin(\delta_0(k)) + 3\sin(\delta_1(k))\right)$$

$$\mathrm{RHS} = \frac{k}{4\pi}\int d\Omega \frac{1}{k^2}\Big[\sin^2(\delta_0(k)) + 9\sin^2(\delta_1(k))\cos^2\theta + 6\cos(\delta_0(k) - \delta_1(k))\cdot \underbrace{\sin(\delta_0(k))\sin(\delta_1(k))\cos\theta}_{=0}\Big]$$

$$\int_{-1}^{1} d\cos\theta \, \cos\theta = 0$$

$$= \frac{k}{4\pi}\left(\sigma_{tot,s} + \sigma_{tot,p}\right)$$

$$\Rightarrow \checkmark$$

H 4.2

We have the potential: $V(r) = \begin{cases} \infty & : r \leq R \\ 0 & : r > R \end{cases}$

$$\tan[\delta_\ell(k)] = \frac{j_\ell(kR)}{y_\ell(kR)}$$

a) $$f_\ell(k) = \frac{1}{k\cot[\delta_\ell(k)] - ik} = \frac{1}{k\,\frac{y_\ell(kR)}{j_\ell(kR)} - ik} = \frac{1}{k}\,\frac{1}{\frac{y_\ell(kR)}{j_\ell(kR)} - i}$$

$k \ll 1$: $\quad j_\ell(kR) \sim \frac{(kR)^\ell}{(2\ell+1)!!} \qquad y_\ell(kR) \sim \frac{(2\ell-1)!!}{(kR)^{\ell+1}}$

$$\Rightarrow \frac{y_\ell(kR)}{j_\ell(kR)} = \frac{(2\ell-1)!!\,(kR)^\ell}{(2\ell+1)!!\,(kR)^{\ell+1}} \propto (kR)^{-1} \longrightarrow 0$$

b) $\ell = 0$. $\quad \tan(\delta_0(k)) = \frac{j_0(kR)}{y_\ell(kR)} = \tan(-kR) \Rightarrow \delta_0 = -kR$

$$\Rightarrow k\cot(-kR) = k\left\{\cot(0) + \frac{d(\cot(-kR))}{dk}\Big|_{k=0} k + \frac{1}{2}\frac{d^2(\cot(-kR))}{dk^2}\Big|_{k=0} k^2 + \mathcal{O}(k^3)\right\}$$

Singularity !!

$$k\cot(-kR) = k\,\frac{\cos(-kR)}{\sin(-kR)} = k\,\frac{1 - \frac{R^2k^2}{2} + \mathcal{O}((kR)^4)}{(-kR) - \frac{1}{6}(-kR)^3 + \mathcal{O}((kR)^5)}$$

$$= -\frac{1}{R} + \frac{1}{3}Rk^2$$

$$a_0 = R, \qquad r_e = \frac{2R}{3}$$

c) $kR \ll 1$. low energy

$$\frac{d\Omega}{d\sigma} = \frac{\sin^2(\delta_0)}{k^2} = \frac{\sin^2(kR)}{k^2} = \frac{\left(kR - \frac{1}{6}(kR)^3 + \mathcal{O}((kR)^4)\right)^2}{k^2}$$

$$= \frac{k^2R^2 - \frac{1}{3}(kR)^4}{k^2} + \mathcal{O}((k)^4)$$

$$= R^2 - \frac{1}{3} k^2 R^4$$

$$\Rightarrow \sigma_{tot} = 4\pi \left(R^2 - \frac{1}{3} k^2 R^4 + \mathcal{O}((k)^4) \right)$$

$$= 4\pi R^2 \left[1 - \frac{1}{3} k^2 R^2 + \mathcal{O}((kR)^4) \right]$$

d) $$\sin^2(\delta_\ell(k)) = \sin^2(\delta_\ell) = \frac{\sin^2(\delta_\ell)}{\sin^2(\delta_\ell) + \cos^2(\delta_\ell)} = \frac{\tan^2(\delta_\ell)}{1 + \tan^2(\delta_\ell)} = \frac{j_\ell^2(kR)}{y_\ell^2(kR) + j_\ell^2(kR)}$$

$$\left[kR \to \infty : \quad j_\ell(z) \simeq \frac{\sin(z - \ell\pi/2)}{z}, \quad y_\ell(z) \simeq -\frac{\cos(z - \ell\pi/2)}{z} \right]$$

$$\simeq \frac{\sin^2(kR - \ell\pi/2)}{\cos^2(kR - \ell\pi/2) + \sin^2(kR - \ell\pi/2)} = \sin^2(kR - \ell\pi/2)$$

e) $$\sin^2[\delta_{\ell+1}(k)] \simeq \sin^2\left(kR - \ell\frac{\pi}{2} - \frac{\pi}{2}\right) = \cos^2\left(kR - \ell\frac{\pi}{2}\right)$$

$$\Rightarrow \sin^2[\delta_{\ell+1}(k)] + \sin^2[\delta_\ell(k)] \simeq 1$$

$$\sigma_{high} = \frac{4\pi}{k^2} \sum_{\ell=0}^{\ell_{max}} (2\ell+1) \sin^2(\delta_\ell(k))$$

$$= \frac{4\pi}{k^2} \Big\{ \underbrace{\sin^2(\delta_0(k)) + \sin^2(\delta_1(k))}_{1} + \underbrace{2\sin^2(\delta_1(k)) + 2\sin^2(\delta_2(k))}_{=2} + \dots \Big\}$$

$$= \frac{4\pi}{k^2} \sum_{l=0}^{\ell_{max}} (\ell+1) = \frac{4\pi}{k^2} \frac{(\ell_{max}+1)\ell_{max}}{2}$$

$$= \frac{4\pi}{k^2} \frac{(kR+1)kR}{2} \simeq 2\pi R^2$$

f) $$\sigma_{cl} = \pi R^2$$

g)

σ_{low}	$4\pi R^2$
σ_{high}	$2\pi R^2$
σ_{cl}	πR^2

Classical result is when energy humongous!