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1. When ignoring problems of decoherence, neutrino oscillation can be derived heuristically from the quantum mechanical time evolution

$$\frac{d}{dt}|\nu_\alpha\rangle = -iH_{\alpha\beta}|\nu_\beta\rangle, \tag{1}$$

where ν_α ($\alpha = e, \mu$) are the interaction (or flavor) eigenstates. The mass and interaction eigenstates will be related by

$$|\nu_k\rangle = \sum_\alpha U_{k\alpha}|\nu_\alpha\rangle, \tag{2}$$

where U is the unitary matrix which diagonalizes the neutrino mass matrix. In the following we restrict ourselves to two flavors ν_e and ν_μ , and hence two mass eigenstates.

1) Assume $|\nu_k\rangle$ evolves as $|\nu_k\rangle_t = e^{-iE_kt}|\nu_k\rangle$ with $E_k = \sqrt{m_k^2+p^2}$. (As shown in class, p has to be assumed to be the same for both mass eigenstates in order to obtain oscillations.) Show that in the limit $p \gg m_k$ the free Hamiltonian for the mass eigenstates is given by

$$H_m = p\mathbf{1} + \frac{1}{2p}\begin{pmatrix} m_1^2 & 0 \\ 0 & m_2^2 \end{pmatrix}, \tag{3}$$

where $\mathbf{1}$ is the 2×2 unit matrix. Hereafter we neglect the first, leading, term which merely contributes to the overall phase. (The overall phase is irrelevant for all probabilities.)

$$|\nu_k\rangle_t = e^{-i\sqrt{m_k^2+p^2}\,t}|\nu_k\rangle \simeq \exp\left[-ip\left(1+\tfrac{1}{2}\tfrac{m_k^2}{p^2}\right)t\right]|\nu_k\rangle$$ ✓

$$\frac{d}{dt}|\nu_\alpha\rangle_t = \frac{d}{dt}\sum_k U^*_{\alpha k}|\nu_k\rangle_t$$

$$= \sum_k U^*_{\alpha k}(-ip)\left(1+\tfrac{1}{2}\tfrac{m_k^2}{p}\right)|\nu_k\rangle_t$$

$$= -ip\sum_{k,\beta} U^*_{\alpha k}\left(1+\tfrac{1}{2}\tfrac{m_k^2}{p}\right)U_{k\beta}|\nu_\beta\rangle t$$

$$= -ip\sum_\beta \delta_{\alpha\beta}\left(1+\tfrac{1}{2}\tfrac{m_k^2}{p}\right)U_{k\beta}|\nu_\beta\rangle t$$

$$\stackrel{!}{=} -i\sum_\beta H_{\alpha\beta}|\nu_\beta\rangle$$

$$\Rightarrow H_m = p\mathbb{1}_2 + \frac{1}{2p}\begin{pmatrix} m_1^2 & 0 \\ 0 & m_2^2\end{pmatrix}$$

α ~> indeed I can't follow what you're doing! U is only $\mathbb{1}$ if you're staying in the mass basis! that is where H is diagonal. You're mixing things up!

confused. Shouldn't we get H_f by doing this? (corresponds to $|\nu_\beta\rangle$)

→ that is true; but you don't get this for $H_{\alpha\beta}$! flavor basis! ← $\alpha,\beta = e,\mu$

$$H_m = \begin{pmatrix} E_1 & \\ & E_2 \end{pmatrix} = \begin{pmatrix} \sqrt{m_1^2+p^2} & \\ & \sqrt{m_2^2+p^2} \end{pmatrix} = \begin{pmatrix} p+\frac{1}{2p}m_1^2 & \\ & p+\frac{1}{2p}m_2^2 \end{pmatrix}$$

2, Show that when we parametrize

$$U = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \tag{4}$$

the free Hamiltonian for the flavor eigenstates becomes

$$H_f = \frac{1}{2p}\left[\frac{m_1^2+m_2^2}{2}\mathbf{1} + \frac{\delta m^2}{2}\begin{pmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}\right]. \tag{5}$$

where $\delta m^2 = m_2^2 - m_1^2$. Again we neglect the first term in the square bracket which merely contributes to the overall phase.

$$\langle \nu_i | H_{m,ik} | \nu_k \rangle$$

$$= \sum_{\alpha,\beta} \langle \nu_\alpha | U_{i\alpha}^* \; H_{m,ik} \; U_{k\beta} | \nu_\beta \rangle$$

$$\Rightarrow \; H_{f,\alpha\beta} = \sum_{i,k} U_{i\alpha}^* H_{m,ik} U_{k\beta}$$ ✓

✓✓ $U^\dagger H_m U = \begin{pmatrix} c & s \\ -s & c \end{pmatrix}\begin{pmatrix} \frac{m_1^2}{2p} & 0 \\ 0 & \frac{m_2^2}{2p} \end{pmatrix}\begin{pmatrix} c & -s \\ s & c \end{pmatrix}$, $c = \cos\theta$, $s = \sin\theta$

ignore the first term in H_m

$$= \begin{pmatrix} c\frac{m_1^2}{2p} & s\frac{m_2^2}{2p} \\ -s\frac{m_1^2}{2p} & c\frac{m_2^2}{2p} \end{pmatrix}\begin{pmatrix} c & -s \\ s & c \end{pmatrix}$$

$$= \begin{pmatrix} c^2\frac{m_1^2}{2p} + s^2\frac{m_2^2}{2p} & -sc\frac{m_1^2}{2p} + sc\frac{m_2^2}{2p} \\ -sc\frac{m_1^2}{2p} + sc\frac{m_2^2}{2p} & +s^2\frac{m_1^2}{2p} + c^2\frac{m_2^2}{2p} \end{pmatrix}$$

$$= \frac{1}{2p}\begin{pmatrix} c^2 m_1^2 + s^2 m_2^2 & +\frac{1}{2}\sin 2\theta \, \delta m^2 \\ \frac{1}{2}\sin 2\theta \, \delta m^2 & +s^2 m_1^2 + c^2 m_2^2 \end{pmatrix}$$

$$= \frac{1}{2p}\cdot\frac{1}{2}\begin{pmatrix} (m_1^2+m_2^2) + (m_2^2-m_1^2)(\sin^2\theta - \cos^2\theta) & +\sin 2\theta \, \delta m^2 \\ +\sin 2\theta \, \delta m^2 & (m_1^2+m_2^2) + (m_2^2-m_1^2)(\cos^2\theta - \sin^2\theta) \end{pmatrix}$$ ✓

$= (5)$ ✓✓

3. If we were to start from eq.(1) with $H_{\alpha\beta}$ given by the second term in eq.(5), we'd first have to diagonalize the Hamiltonian. Show that the first and second row of U, given in eq.(4), are ~~eigenvalues~~ eigenvectors of this reduced Hamiltonian, with eigenvalues $\pm\delta m^2/(4p)$. The solution of (1) then proceeds as presented in class, by expressing flavor eigenstates in terms of mass eigenstates which have a simple time evolution.

$$\hat{H}_f = \frac{1}{2p}\frac{\delta m^2}{2}\begin{pmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} \checkmark$$

$$\det \begin{vmatrix} -\cos 2\theta - \lambda & \sin 2\theta \\ \sin 2\theta & \cos 2\theta - \lambda \end{vmatrix} = -\cos^2 2\theta + \lambda^2 - \sin^2 2\theta \stackrel{!}{=} 0$$

$$\Rightarrow \lambda = \pm 1$$

$$\Rightarrow \hat{H}_{f,\,diag} = \frac{\delta m^2}{4p}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

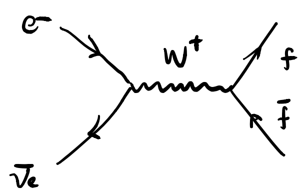
⇒ ✓ yes; but you were also asked to show that U rows are eigenvectors!

4. In the presence of matter, the effective Hamiltonian describing neutrino propagation receives additional contributions from coherent interactions between neutrinos and matter particles (forward scattering). Interactions due to neutral currents are the same for all neutrinos, and therefore drop out of the oscillation phase. However, charged current reactions lead to a term specific to electron neutrinos,

$$\Delta H = \begin{pmatrix} \sqrt{2}G_F n_e & 0 \\ 0 & 0 \end{pmatrix}, \qquad (6)$$

where n_e is the electron number density. Which Feynman diagram is responsible for this term?

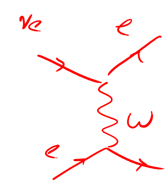
CC - interaction



⊘ Nope!

$\nu_e + e \rightarrow \nu_e + e$ is the process of interest!

(→



μ, τ too unstable

5. We diagonalize the Hamiltonian $H + \Delta H$ by

$$\begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} = \begin{pmatrix} \cos\tilde{\theta} & \sin\tilde{\theta} \\ -\sin\tilde{\theta} & \cos\tilde{\theta} \end{pmatrix} \cdot \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}, \tag{7}$$

where $\nu_{1,2}$ on the right–hand side are the new mass eigenstates (eigenstates of the Hamiltonian including ΔH). Obtain the energy eigenvalues and $\tilde{\theta}$. What is the critical number density where the resonance occurs, i.e. two masses become equal? At this critical density, what is the effective mixing angle $\tilde{\theta}$? What does this mean for the propagating neutrino, assuming the change of $\tilde{\theta}$ is sufficiently slow? *Hint:* Show that an eigenvector of $\begin{pmatrix} A & B \\ B & C \end{pmatrix}$, written as in eq.(7), satisfies $\tan 2\tilde{\theta} = 2B/(A - C)$.

This is alright!
$\rightarrow$ I would think there is a much easier solution
$\rightarrow$ to form $\tilde{U}^T (H+\Delta H)\tilde{U}$
and ask for it to be diagonal.

$$\det M = \det \begin{pmatrix} A-\lambda & B \\ B & C-\lambda \end{pmatrix} = AC + \lambda^2 - (A+C)\lambda - B^2 \stackrel{!}{=} 0$$

$$\lambda^2 - (A+C)\lambda + AC - B^2 = 0$$

$$\lambda^2 - 2\cdot\frac{A+C}{2}\lambda + \left(\frac{A+C}{2}\right)^2 - \frac{1}{4}(A^2+C^2-2AC) - B^2 = 0$$

$$\left(\lambda - \frac{A+C}{2}\right)^2 = \left(\frac{A-C}{2}\right)^2 + B^2$$

$$\lambda = \pm\sqrt{\left(\frac{A-C}{2}\right)^2 + B^2} + \frac{A+C}{2}$$

$$= \frac{1}{2}(\pm D + A + C), \quad D := \sqrt{(A-C)^2 + 4B^2}$$

$$(M - \lambda_1 \cdot \mathbb{1}) = \begin{pmatrix} A - \frac{1}{2}(D+A+C) & B \\ B & C - \frac{1}{2}(D+A+C) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}(A-C-D) & B \\ B & \frac{1}{2}(-A+C-D) \end{pmatrix}$$

the eigenvector
$$\begin{pmatrix} \frac{1}{2}(A-C-D) & B \\ B & \frac{1}{2}(-A+C-D) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} x_1 \cdot \frac{1}{2}(A-C-D) + x_2 \cdot B = 0 \\ x_1 \cdot B + \frac{1}{2}(-A+C-D)\cdot x_2 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x_2 = -\frac{1}{2B}(A-C-D)\,x_1 \\ x_1 = \frac{1}{2B}(A-C+D)\,x_2 \end{cases}$$

$$\rightarrow \quad x_1 = -\frac{1}{2B}(A-C+D) \qquad x_2 = 1$$

$$\Rightarrow \quad x_1 \overset{!}{=} \cos\hat{\theta}\,, \quad x_2 \overset{!}{=} -\sin\hat{\theta}$$

$$\Rightarrow \quad \tan 2\hat{\theta} = \frac{2\tan\theta}{1-\tan^2\theta}$$

$$= \frac{2\cdot\frac{-x_2}{x_1}}{1-\left(\frac{x_2}{x_1}\right)^2}$$

$$= 2\,\frac{-x_1x_2}{x_1^2-x_2^2}$$

$$= 2\,\frac{\frac{1}{2B}(A-C+D)}{\frac{1}{4B^2}(A-C+D)^2-1}$$

$$= 4B\,\frac{A-C+D}{(A-C+D)^2-4B^2}$$

$$= 4B\,\frac{A-C+\sqrt{(A-C)^2+4B^2}}{(A-C)^2+2(A-C)\sqrt{(A-C)^2+4B^2}+(A-C)^2+\cancel{4B^2}-\cancel{4B^2}}$$

$$= \frac{4B}{2(A-C)}\,\frac{A-C+D}{(A-C)+D}$$

$$\tan 2\hat{\theta} = \frac{2B}{A-C}$$

$$M = \begin{pmatrix} A & B \\ B & C \end{pmatrix} = H + \Delta H = \frac{1}{2p}\frac{\delta m^2}{2}\begin{pmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} + \begin{pmatrix} \sqrt{2}\,G_F n_e & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow \quad B = \frac{\delta m^2}{4p}\sin 2\theta$$

$$A = -\frac{\delta m^2}{4p}\cos 2\theta + \sqrt{2}\,G_F n_e$$

$$C = \frac{\delta m^2}{4p}\cos 2\theta$$

$$\Rightarrow \quad D = \sqrt{(A-C)^2+4B^2} = \sqrt{\left(-\frac{\delta m^2}{2p}\cos 2\theta + \sqrt{2}\,G_F n_e\right)^2 + 4\left(\frac{\delta m^2}{4p}\right)^2\sin^2 2\theta}$$

$$= \sqrt{\left(\frac{\delta m^2}{2p}\right)^2 + 2G_F^2 n_e^2 - \frac{\sqrt{2}\,\delta m^2}{p}\cos 2\theta\, G_F n_e}$$

$$\lambda_{1,2} = \frac{1}{2}(\pm D + A + C) = \frac{1}{2}(\pm D + \sqrt{2}\,G_F n_e)$$

$$= \frac{1}{2}\left(\pm\sqrt{\left(\frac{\delta m^2}{2p}\right)^2 + 2G_F^2 n_e^2 - \frac{\sqrt{2}\,\delta m^2}{p}\cos 2\theta\, G_F n_e} + \sqrt{2}\,G_F n_e\right)$$

$$\tan 2\hat{\theta} = \frac{2B}{A-C} = \frac{\frac{\delta m^2}{4p}\sin 2\theta}{-\frac{\delta m^2}{2p}\cos 2\theta + \sqrt{2}\,G_F n_e}$$

$$\Rightarrow \tilde{\Theta} = \frac{1}{2} \arctan \frac{\frac{\delta m^2}{4p} \sin 2\theta}{-\frac{\delta m^2}{2p} \cos 2\theta + \sqrt{2}\, G_F n_e}$$

Two masses identical

$\sim D = 0$ $\rightsquigarrow$ wait! what? α

$$\sim \left(\frac{\delta m^2}{2p}\right)^2 + 2 G_F^2 n_e^2 - \frac{\sqrt{2}\,\delta m^2}{p} \cos 2\theta\, G_F n_e = 0$$

$$n_e^2 = \frac{\delta m^2}{\sqrt{2} p G_F} \cos 2\theta \cdot n_e + \frac{1}{2 G_F^2} \left(\frac{\delta m^2}{2p}\right)^2 = 0$$ $\rightarrow$ $n_e^2 = 0$?! α

$$n_e = \frac{1}{2} \left(\frac{\delta m^2}{\sqrt{2} p G_F} \cos 2\theta \pm \sqrt{ \left(\frac{\delta m^2}{\sqrt{2} p G_F}\right)^2 \cos^2 2\theta - 4 \frac{1}{2 G_F^2} \left(\frac{\delta m^2}{2p}\right)^2 } \right)$$

$$= \frac{1}{2} \left(\frac{\delta m^2}{\sqrt{2} p G_F} \cos 2\theta \pm \frac{\delta m^2}{2 p G_F} \sqrt{2 \cos^2 2\theta - 2} \right)$$ α nope!

$$= \frac{1}{2} \frac{\delta m^2}{\sqrt{2} p G_F} \left(\cos 2\theta \pm \sin 2\theta \right)$$ critical density

Denominator in $\tan 2\tilde{\theta}$ $= -\frac{\delta m^2}{2p} \cancel{\cos 2\theta} + \cancel{\sqrt{2}\, G_F}\, \frac{1}{2} \frac{\delta m^2}{\cancel{\sqrt{2}}\, p\, \cancel{G_F}} \left(\cancel{\cos 2\theta} \pm \sin 2\theta \right)$

$$= \pm \frac{1}{2} \frac{\delta m^2}{p} \sin 2\theta$$

$$\Rightarrow \tan 2\tilde{\theta} = \pm \frac{1}{2}$$ $\rightsquigarrow$ Nope! α

Critical density $\rightarrow$ $\nu_1 = \nu_2$ no distinction.

Easier:

$$\tilde{U} A' \tilde{U}^{-1} = \underset{\substack{\uparrow \\ \text{diagonal}}}{A}$$

$$\rightarrow \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \begin{pmatrix} A & B \\ B & C \end{pmatrix} \begin{pmatrix} c & -s \\ s & c \end{pmatrix}$$

$$= \begin{pmatrix} cA + sB & cB + sC \\ -sA + cB & -sB + cC \end{pmatrix} \begin{pmatrix} c & -s \\ s & c \end{pmatrix}$$

$$= \begin{pmatrix} \cdots & -s(cA+sB) + c(cB+sC) \\ \cdots & \cdots \end{pmatrix} \quad \overset{!}{=} 0$$

$$\Rightarrow -scA - s^2 B + c^2 B + scC = 0$$

$$-\tan\tilde{\theta} A - \tan^2\tilde{\theta} B + B + \tan\tilde{\theta} C = 0$$

$$B(1 - \tan^2\tilde{\theta}) = \tan\tilde{\theta}(A - C)$$

$$\tan 2\tilde{\theta} = \frac{2\tan\tilde{\theta}}{1 - \tan^2\tilde{\theta}} \qquad \frac{2B}{A-C} = \frac{2\tan\tilde{\theta}}{1-\tan^2\tilde{\theta}}$$

$$\Rightarrow \tan 2\tilde{\theta} = \frac{2B}{A-C}$$

$$\begin{pmatrix} A & B \\ B & C \end{pmatrix} = H + \Delta H = \frac{\delta m^2}{4p} \begin{pmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} + \begin{pmatrix} \sqrt{2} G_F n_e & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow \tan 2\tilde{\theta} = \frac{\frac{\delta m^2}{2p} \sin 2\theta}{-\frac{\delta m^2}{4p}\cos 2\theta + \sqrt{2} G_F n_e - \frac{\delta m^2}{4p}\cos 2\theta}$$

$$= \frac{2\sin 2\theta}{-\cos 2\theta + \frac{1}{\delta m^2} 4\sqrt{2}\, p G_F n_e - \cos 2\theta}$$

$$= \frac{2\sin 2\theta}{-2\cos 2\theta + \frac{4\sqrt{2}}{\delta m^2} p G_F n_e}$$

Matrix $\quad \frac{4p}{\delta m^2}(H + \Delta H) = \begin{pmatrix} -\cos 2\theta + \overbrace{\frac{4\sqrt{2}}{\delta m^2} p G_F n_e}^{\alpha} & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}$

$$= \begin{pmatrix} -\cos 2\theta + \alpha & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}$$

$$\det \begin{vmatrix} -\cos 2\breve{\theta} + \alpha - \lambda & \sin 2\breve{\theta} \\ \sin 2\breve{\theta} & \cos 2\breve{\theta} - \lambda \end{vmatrix} \overset{!}{=} 0$$

$$\Rightarrow (-\cos 2\breve{\theta} + \alpha - \lambda)(\cos 2\breve{\theta} - \lambda) - \sin^2 2\breve{\theta} = 0$$

$$\underline{-\cos^2 2\breve{\theta}} + \cos 2\breve{\theta}\,(\cancel{\lambda} - \cancel{\lambda} + \alpha) - \alpha\lambda + \lambda^2 - \underline{\sin^2 2\breve{\theta}} = 0$$

$$\lambda^2 - \alpha\lambda - 1 + \alpha\cos 2\breve{\theta} = 0$$

$$\Rightarrow \lambda_{\pm} = \frac{1}{2}\left(\alpha \pm \sqrt{\alpha^2 - 4\cdot 1\cdot(-1 + \alpha\cos 2\breve{\theta})}\right)$$

$$= \frac{1}{2}\left(\alpha \pm \sqrt{\alpha^2 - 4(\alpha\cos 2\breve{\theta} - 1)}\right)$$

$\lambda_+ \overset{!}{=} \lambda_-$, when

$$\alpha^2 - 4(\alpha\cos 2\breve{\theta} - 1) = 0$$

$$\Rightarrow \alpha^2 - 4\alpha\cos 2\breve{\theta} + 4 = 0$$

$$\alpha_{\pm} = \frac{1}{2}\left(+4\cos 2\breve{\theta} \pm \sqrt{16\cos^2 2\breve{\theta} - 4\cdot 1\cdot 4}\right)$$

$$= 2\left(\cos 2\breve{\theta} \pm \underbrace{\sqrt{\cos^2 2\breve{\theta} - 1}}_{|\sin 2\breve{\theta}|}\right)$$

since $\alpha > 0$.

$$\Rightarrow \alpha_+ = 2(\cos 2\breve{\theta} + \sin 2\breve{\theta}) = \frac{4\sqrt{2}}{\delta m^2}\, p\, G_F\, n_e$$

2) We consider the following toy model with two real scalars ϕ, Φ, with mass terms

$$-\mathcal{L}_{\text{mass}} = \frac{1}{2}\left[(m\phi + M\Phi)^2 + m^2\Phi^2\right], \tag{8}$$

where $m \ll M$. For energy scales much smaller than M, we can integrate out the heavy mode Φ, by solving its equation of motion in the approximation $\Phi =$const., i.e. setting all derivatives of Φ to zero; the equation of motion is then simply given by $\partial\mathcal{L}/\partial\Phi = 0$. This is motivated by the consideration that derivatives of fields correspond to their kinetic energy or 3–momentum, i.e. to their motion; for a very heavy particle one would need a lot of energy to make it move, so setting Φ to constant should be a good approximation for energy scales $\ll M$. Find out the resulting mass term for ϕ after this procedure.

$$\frac{\partial \mathcal{L}}{\partial \Phi} = \frac{\partial}{\partial \Phi}\left[(M^2+m^2)\Phi^2 + 2mM\phi\Phi \right]$$

$$= (M^2+m^2)\cancel{2}\Phi + \cancel{2}mM\phi = 0$$

$$\Rightarrow \quad \Phi = -\frac{mM}{M^2+m^2}\phi$$

$$\Rightarrow -\mathcal{L}_{mass} = \frac{1}{2}\left[\left(m\phi - \frac{mM^2}{M^2+m^2}\phi\right)^2 + m^2\frac{m^2M^2}{(M^2+m^2)^2}\phi^2 \right]$$

$$= \frac{\phi^2}{2}\left[m^2 - 2\frac{m^2M^2}{M^2+m^2} + \frac{m^2M^4}{(M^2+m^2)^2} + \frac{m^4M^2}{(M^2+m^2)^2} \right]$$

$$= \phi^2 \frac{1}{2(M^2+m^2)^2}\left[m^2(M^2+m^2)^2 - 2m^2M^2(M^2+m^2) + m^2M^4 + m^4M^2 \right]$$

$$= \phi^2 \frac{1}{2(M^2+m^2)^2}\left[m^2(\cancel{M^4} + \cancel{2m^2M^2} + m^4) - \cancel{2m^2M^4} - \cancel{2m^4M^2} + \cancel{m^2M^4} + m^4M^2 \right]$$

$$= \phi^2 \frac{1}{2(M^2+m^2)^{\cancel{2}}} m^4(m^2 + M^2)$$

$$= \frac{1}{2}\frac{m^4}{m^2+M^2}\phi^2$$

$$\Rightarrow \quad \text{mass of } \phi = \frac{m^2}{\sqrt{m^2+M^2}}$$

$$m \ll M \quad \rightarrow \quad \frac{m^2}{M\sqrt{1+\left(\frac{m}{M}\right)^2}} \simeq \frac{m^2}{M}\left(1 - \frac{1}{2}\left(\frac{m}{M}\right)^2\right) \simeq \frac{m^2}{M}$$

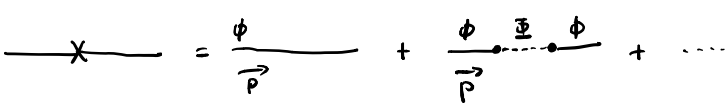
2. Directly diagonalize the squared mass matrix derived from (8). Check that the smaller eigenvalue reproduces the value obtained above in the limit $m \ll M$. *Hint:* You'll have to expand the square root up to second order!

$$-\mathcal{L}_{mass} = \frac{1}{2}\begin{pmatrix}\phi & \Phi\end{pmatrix} \underbrace{\begin{pmatrix} m^2 & mM \\ mM & M^2+m^2 \end{pmatrix}}_{=:\,\mathcal{M}} \begin{pmatrix}\phi \\ \Phi\end{pmatrix}$$

$$\det(\mathcal{M} - \lambda\cdot\mathbb{1})$$
$$= \begin{vmatrix} m^2-\lambda & mM \\ mM & m^2+M^2-\lambda \end{vmatrix} = (m^2-\lambda)(m^2+M^2-\lambda) - m^2M^2$$
$$= m^4 + \cancel{m^2M^2} - (2m^2+M^2)\lambda + \lambda^2 - \cancel{m^2M^2} \overset{!}{=} 0$$
$$\Rightarrow \lambda^2 - (2m^2+M^2)\lambda = -m^4$$
$$\lambda^2 - 2\cdot(m^2+\tfrac{1}{2}M^2)\lambda + (m^2+\tfrac{1}{2}M^2)^2 = m^2M^2 + \tfrac{1}{4}M^4$$
$$[\lambda - (m^2+\tfrac{1}{2}M^2)]^2 = M^2(m^2+\tfrac{1}{4}M^2)$$
$$\Rightarrow \lambda_{1,2} = \pm M\sqrt{m^2+\tfrac{1}{4}M^2} + (m^2+\tfrac{1}{2}M^2)$$
$$= \pm\frac{M^2}{2}\sqrt{4\left(\frac{m}{M}\right)^2+1} + (m^2+\tfrac{1}{2}M^2)$$

$$\lambda_2 = m'^2 = m^2+\tfrac{1}{2}M^2 - \frac{M^2}{2}\sqrt{1+\left(\frac{2m}{M}\right)^2}$$
$$= m^2+\tfrac{1}{2}M^2\left(1-\sqrt{1+\left(\frac{2m}{M}\right)^2}\right)$$
$$\simeq m^2+\tfrac{1}{2}M^2\left(\cancel{1}-\cancel{1}-\tfrac{1}{2}\left(\frac{2m}{M}\right)^2+\tfrac{1}{8}\left(\frac{2m}{M}\right)^4\right) = \frac{m^4}{M^2} \rightarrow m'_\phi \simeq \frac{m^2}{M}$$

3. This problem can also be solved by treating the off–diagonal mass term in (8) as a 2–point "coupling", and resumming the ϕ propagator, where the "unperturbed" propagators for ϕ and Φ are $i/(p^2-m^2)$ and $i/(p^2-M^2-m^2)$, respectively. Show that the resummed ϕ propagators, with infinitely many insertions of $\phi\Phi$ couplings and Φ propagators, has a pole at the mass derived above.



$$= \frac{i}{p^2-m^2}\left(1 + \underbrace{(-imM)^2\frac{i}{p^2-M^2-m^2}}_{-i\Sigma(p^2)}\frac{i}{p^2-m^2} + \cdots\right)$$

$$= \frac{i}{p^2 - m^2} \frac{1}{1 + i\Sigma(p^2)\frac{i}{p^2-m^2}}$$

⤳ please do mention that you are using a summation rule. ~✓

$$= \frac{i}{p^2 - m^2 - \Sigma(p^2)} .$$

$$\Sigma(p^2) = \frac{m^2 M^2}{p^2 - M^2 - m^2} \simeq \frac{m^2 M^2}{-M^2 - m^2} , \quad p^2 \ll M^2$$

$$= \frac{i}{p^2 - \left(m^2 - \frac{m^2 M^2}{M^2 + m^2}\right)}$$

✓ fair enough! :)

$$m^2 \longrightarrow m^2\left(1 - \frac{M^2}{m^2 + M^2}\right)$$

$$m \longrightarrow m\sqrt{1 - \frac{M^2}{m^2+M^2}} = m \frac{m}{\sqrt{m^2+M^2}} = \frac{m^2}{\sqrt{m^2+M^2}}$$

$$= \frac{m^2}{M} \frac{1}{\sqrt{1 + \left(\frac{m}{M}\right)^2}} \simeq \frac{m^2}{M} \frac{1}{1 + \frac{1}{2}\left(\frac{m}{M}\right)^2}$$

$$\simeq \frac{m^2}{M}\left[1 - \frac{1}{2}\left(\frac{m}{M}\right)^2\right]$$