worked on 3/5

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1) 1.

$U(1)_Y$: $\beta_1 = \frac{g_1^4}{8\pi^2}\left[0 + \frac{2}{3}\sum_{c.f.} Y^2 + \frac{1}{3}\sum_{c.s.} Y^2 \right]$

$= \frac{g_1^4}{8\pi^2}\left\{ \frac{2}{3}\left[\underbrace{\left(\frac{1}{6}\right)^2 \cdot 6 \cdot 3 + \left(\frac{2}{3}\right)^2 \cdot 3^2 + \left(\frac{1}{3}\right)^2 \cdot 3}_{\text{color!}} + \left(\frac{1}{2}\right)^2 \cdot 6 + 1^2 \cdot 3 \right] + \underbrace{\frac{1}{3} \cdot 2 \left(\frac{1}{2}\right)^2}_{\text{Higgs doublet}} \right\}$

$= \frac{g_1^4}{8\pi^2}\left\{ \frac{2}{3}\left[\underbrace{\frac{1}{2} + 4 + 1 + \frac{3}{2} + 3}_{10} \right] + \frac{1}{3} \cdot \frac{1}{2} \right\}$

(in β_1 consider ~~color but~~ in β_2 don't ?

↳ why not? they are different states!)

$= \frac{g_1^4}{8\pi^2}\left\{ \frac{20}{3} + \frac{1}{6} \right\}$

$= \frac{g_1^4}{8\pi^2} \cdot \frac{41}{6} \quad \longrightarrow b_1 = \frac{-41}{6}$

$\beta_2 = \frac{g_2^2}{8\pi^2}\left[-\frac{11}{3} \cdot 2 + \frac{2}{3} \cdot \frac{1}{2} \cdot 12 + \frac{1}{3} \cdot \frac{1}{2} \cdot 1 \right] = -\frac{g_2^2}{8\pi^2} \cdot \underbrace{\frac{19}{6}}_{=b_2}$ ✓

colors $\Leftarrow 3 \cdot 3 + 3$ — 6 quarks (L), 6 leptons (L); Higgs

3 gen. quarks; 3 gen. of leptons

$\beta_3 = \frac{g_3^2}{8\pi^2}\left[-\frac{11}{3} \cdot 3 + \frac{2}{3} \cdot \frac{1}{2} \cdot 12 \right] = -\frac{g_3^4}{8\pi^2} \cdot 7$

12 quarks (L, R), no color factor b/c q's are 3 in SU(3)

From lecture : $\frac{1}{g_i^2(Q)} = \frac{1}{g_i^2(M_Z)} + \frac{b_i}{8\pi^2} \ln \frac{Q}{M_Z}$

$g_1(M'_X) = g_2(M'_X)$

$\Rightarrow \frac{1}{g_1^2(M_Z)} + \frac{b_1}{8\pi^2} \ln \frac{M'_X}{M_Z} = \frac{1}{g_2^2(M_Z)} + \frac{b_2}{8\pi^2} \ln \frac{M'_X}{M_Z}$

$\Rightarrow \left[\frac{1}{g_1^2(M_Z)} - \frac{1}{g_2^2(M_Z)} \right] = \frac{-b_1 + b_2}{8\pi^2} \ln \frac{M'_X}{M_Z}$

$\ln \frac{M'_X}{M_Z} = \frac{8\pi^2}{b_2 - b_1}\left[\frac{1}{g_1^2(M_Z)} - \frac{1}{g_2^2(M_Z)} \right]$

$= \frac{8\pi^2}{\frac{19}{6} + \frac{41}{6}}\left[\frac{3}{5} \frac{1}{0.128} - \frac{1}{0.421} \right]$

$\left(g_1 = \sqrt{\frac{5}{3}} g_Y \right.$

$\simeq 18.3$

$\Rightarrow M_x' \simeq 7.7 \cdot 10^9 \text{ GeV} \ll M_x$

$$\frac{1}{g_3^2(M_x')} = \frac{1}{g_3^2(M_Z)} + \frac{b_3}{8\pi^2} \ln\frac{M_x'}{M_Z}$$

$$= \frac{1}{g_1^2(M_x')} = \frac{1}{g_1^2(M_Z)} + \frac{b_1}{8\pi^2} \ln\frac{M_x'}{M_Z}$$

$$\Rightarrow \frac{1}{g_3^2(M_Z)} = \frac{1}{g_1^2(M_Z)} + \frac{b_1 - b_3}{8\pi^2} \ln\frac{M_x'}{M_Z}$$

$\Rightarrow g_3^2(M_Z) = 0.67 < 1.50 \Rightarrow g_1, g_2, g_3$ cannot meet at same point

2. add new fields with $m \geq M_Z$ forming $\underline{5}$ or $\underline{10}$

$m \geq M_Z \Rightarrow$ they don't contribute $g_i^2(M_Z)$, $\Rightarrow$ well nope' you are running g's down from M_X down to M_Z; $M_Y > M_Z$ does contribute!

$g_i^2(M_Z)$ don't change because of new fields

Decomposition of $\underline{5}$ into $SU(3)_c$ and $SU(2)_L$

$\underline{5} = (3,1)_{\frac{1}{3}} + (1,2)_{-\frac{1}{2}} \sim (d, \bar{\ell})$, $\quad \bar{5} \sim (\bar{3},1)_{-\frac{1}{3}} + (1,2)_{\frac{1}{2}} \sim (\bar{d}\ \ell)$

At $Q^2 \geq M_Z^2$:

$$\beta_2 = -\frac{g_2^4}{8\pi^2}\left(b_2 - \frac{2}{3}\cdot\frac{1}{2}\cdot 1\right) = -\frac{g_2^4}{8\pi^2}\left(b_2 - \frac{2}{6}\right) = -\frac{g_2^4}{8\pi^2}\,\frac{17}{6}$$

$$\beta_3 = -\frac{g_3^2}{8\pi^2}\left(b_3 - \frac{2}{3}\cdot\frac{1}{2}\cdot 1\right) = -\frac{g_3^2}{8\pi^2}\left(b_3 - \frac{1}{3}\right) = -\frac{g_3^2}{8\pi^2}\,\frac{20}{3}$$

$$\Rightarrow \underline{b_3 - b_2 = \hat{b}_3 - \hat{b}_2}$$

$$\beta_1 = -\frac{g_2^4}{8\pi^2}\Big(b_1 - \underbrace{\tfrac{2}{3}\left(-\tfrac{1}{3}\right)^2\cdot 3}_{(3,1)} - \underbrace{\tfrac{2}{3}\left(-\tfrac{1}{2}\right)^2\cdot 2}_{(1,2)}\Big) = -\frac{g_2^4}{8\pi^2}\left(b_1 - \frac{5}{9}\right)$$

$\rightsquigarrow$ forgetting the b_1, b_Y conversion factor of $5/3$?!

$$\Rightarrow \underline{b_2 - b_1 \neq \hat{b}_2 - \hat{b}_1}$$

$b_Y \mathrel{+}= \frac{2}{3} \times \left(\frac{1}{3}\right)^2 \times 3 \times 3 + \frac{2}{3} \times \left(\frac{1}{2}\right)^2 \times 3 \times 2 = \frac{5}{3}$

hypercharge

generations

$b_3 \mathrel{+}= \frac{2}{3} \times 3 \times \frac{1}{2} = 1$

$b_2 \mathrel{+}= \frac{2}{3} \times 3 \times \frac{1}{2} = 1$

b_Y $\quad + = \frac{2}{3}\times\left(\frac{1}{6}\right)^2\times 3\times 2\times 3 + \frac{2}{3}\cdot\left(\frac{2}{3}\right)^2\times 3\times 3 + \frac{2}{3}\times 1\times 3 = 5$

color doublet generations

Same down here!

b_3 $\quad + = \frac{2}{3}\times 3\times\frac{1}{2}(2+1) = 3$

SU(2) doublet

b_2 $\quad + = \frac{2}{3}\times 3\times\frac{1}{2}\times 3 = 3$

$$\underline{10} = (3,2)_{\frac{1}{6}} + (\bar{3},1)_{-\frac{2}{3}} + (1,1)_{1} \sim Q\ \bar{u}\ R$$

$$\beta_1 = -\frac{g_1^4}{8\pi^2}\left\{b_1 - \left[\left(\frac{1}{6}\right)^2\cdot 2\cdot 3 + \left(\frac{2}{3}\right)^2\cdot 3 + (1)^2\cdot 1\right]\cdot\frac{2}{3}\right\} = -\frac{g_1^4}{8\pi^2}\left(b_1 - \frac{10}{9}\right)$$

$$\beta_2 = -\frac{g_2^4}{8\pi^2}\left(b_2 - \frac{2}{3}\cdot\frac{1}{2}\right) = -\frac{g_2^4}{8\pi^2}\left(b_2 - \frac{1}{3}\right)$$

$$\beta_3 = -\frac{g_3^4}{8\pi^2}\left(b_3 - \frac{2}{3}\cdot\left(\frac{1}{2}\cdot 2 - \frac{1}{2}\right)\right) = -\frac{g_3^2}{8\pi^2}\left(b_3 - \frac{1}{3}\right)$$

anti-fund.

$$\Rightarrow \tilde{b}_3 - \tilde{b}_2 = b_3 - b_2, \qquad \hat{b}_2 - \hat{b}_1 \neq b_2 - b_1$$

Somewhere it went wrong

3. $\quad M_X = M_X' \quad \Rightarrow \quad g_1(M_X) = g_2(M_X) = g_3(M_X)$

New fields mass $\simeq M_Z$ $\Rightarrow$ they contribute to b_i at $Q^2 \simeq M_Z$

$$\ln\frac{M_X}{M_Z} = \frac{8\pi^2}{b_2 - b_1}\left(\frac{1}{g_1^2(M_Z)} - \frac{1}{g_2^2(M_Z)}\right)$$

Not really!

$$\ln\frac{M_Y}{M_Z} = \frac{8\pi^2}{b_3 - b_2}\left(\frac{1}{g_2^2(M_Z)} - \frac{1}{g_3^2(M_Z)}\right)$$

II) 1. Define $g_2 = g_1$ at $Q = M_X'$

$$\frac{dg_i^2(Q)}{d\ln Q} = \beta_i(g_i) = -\frac{g_i^4}{8\pi^2} b_i$$

$$\Rightarrow \quad \frac{dg_i^2(Q)}{(g_i^2)^2} = \frac{-b_i}{8\pi^2} d\ln Q$$

$$\Rightarrow \quad \left[-\frac{1}{g_i^2(Q)}\right]_{M_Z}^{M_X'} = \frac{-b_i}{8\pi^2} \ln\frac{M_X'}{M_Z}$$

$$\Rightarrow \quad \frac{1}{g_i^2(M_Z)} - \frac{1}{g_i^2(M_X')} = \frac{-b_i}{8\pi^2} \ln\frac{M_X'}{M_Z}$$

with $i = 1, 2$

$$\begin{cases} \frac{1}{g_1^2(M_Z)} - \frac{1}{g_1^2(M_X')} = \frac{-b_1}{8\pi^2} \ln\frac{M_X'}{M_Z} \\ \frac{1}{g_2^2(M_Z)} - \frac{1}{g_2^2(M_X')} = \frac{-b_2}{8\pi^2} \ln\frac{M_X'}{M_Z} \\ g_1(M_X') = g_2(M_X') \end{cases}$$

$$\Rightarrow \quad \frac{1}{g_1^2(M_Z)} - \frac{1}{g_2^2(M_Z)} = \frac{b_2 - b_1}{8\pi^2} \ln\frac{M_X'}{M_Z}$$

$\Rightarrow M_X' = \dots$ is determined

$$\frac{1}{g_3^2(M_Z)} - \frac{1}{g_3^2(M_X')} = \frac{1}{g_3^2(M_Z)} - \frac{1}{g_2^2(M_X')} = \frac{-b_3}{8\pi^2} \ln\frac{M_X'}{M_Z}$$

$$\Rightarrow \quad \frac{1}{g_3^2(M_Z)} = \underbrace{\frac{1}{g_2^2(M_X')}}_{= \frac{1}{g_2^2(M_Z)} + \frac{b_2}{8\pi^2} \ln\frac{M_X'}{M_Z}} - \frac{b_3}{8\pi^2} \ln\frac{M_X'}{M_Z} = \dots$$

2. g_3 and g_2 meet at

$$\frac{1}{g_3^2(M_Z)} - \frac{1}{g_2^2(M_Z)} = \frac{b_2 - b_3}{8\pi^2} \ln\frac{M_X}{M_Z}$$

g_1 and g_2 $\longrightarrow$ $(b_2 - b_1)$

Add new fields (mass $\gtrsim M_Z$) of $\underline{5}$

$$\underline{5} = \underset{d_R}{\underset{\uparrow}{(3,1)_{-\frac{1}{3}}}} + \underset{\bar{\ell}_L}{\underset{\uparrow}{(1,\bar{2})_{\frac{1}{2}}}}$$

($\bar{2} \simeq 2$ in $SU(2)$, so we can treat them as fund. rep

$$b_Y \mathrel{+}= \frac{2}{3}\cdot\left(\frac{1}{3}\right)^2\cdot 3\cdot 3 + \frac{2}{3}\cdot\left(\frac{1}{2}\right)\cdot 3\cdot 2 = \frac{5}{3} \quad \rightarrow \quad b_1 \mathrel{+}= \underline{1}$$

$$b_2 \mathrel{+}= \frac{2}{3}\cdot\frac{1}{2}\cdot 3 = 1$$

$$b_3 \mathrel{+}= \frac{2}{3}\cdot\frac{1}{2}\cdot 3 = 1 \qquad \Rightarrow \text{ No change in } M_X, M_X' \,!$$

Add $\sim$ of $\underline{10}$

$$\underline{10} = \underset{q_L}{\underset{\uparrow}{(3,2)_{\frac{1}{6}}}} + \underset{\text{count as fund. rep.}}{\underset{\uparrow}{\underset{u^c}{\underset{\uparrow}{(\bar{3},1)_{-\frac{2}{3}}}}}} + \underset{e_R^c}{\underset{\uparrow}{(1,1)_1}}$$

$$b_Y \mathrel{+}= \frac{2}{3}\cdot\left[\left(\frac{1}{6}\right)^2\cdot 6\cdot 3 + \left(\frac{2}{3}\right)^2\cdot 3\cdot 3 + (1)^2\cdot 3\right] = \overbrace{\frac{2}{3}\left[\frac{1}{2} + 4 + 3\right]}^{\frac{25}{2}} = 5$$

$$b_2 \mathrel{+}= \frac{2}{3}\cdot\frac{1}{2}\cdot 3\cdot 3 = 3$$

$$b_3 \mathrel{+}= \frac{2}{3}\cdot\frac{1}{2}\cdot(3\cdot 2 + 3) = 3$$

$\Rightarrow$ No change !

3. If new fields $\sim \sqrt{M_X M_Z}$

different b_i, or β_i

$\Rightarrow$ $0 < Q < M_Z$: $b_i = b_i$

$M_Z < Q < \sqrt{M_Z M_X}$: $\hat{b}_i$ containing contribution from new fields

In case of g_Y

$$\Rightarrow \frac{1}{g_Y^2(M_Z)} - \frac{1}{g_Y^2(\sqrt{M_Z M_X})} = \frac{-b_Y}{8\pi^2}\ln\frac{\sqrt{M_Z M_X}}{M_Z} = \frac{-b_Y}{8\pi^2}\frac{1}{2}\ln\frac{M_X}{M_Z}$$

$$\frac{1}{g_Y^2(\sqrt{M_Z M_X})} - \frac{1}{g_Y^2(M_X)} = \frac{-\hat{b}_Y}{8\pi^2}\ln\frac{M_X}{\sqrt{M_Z M_X}} = \frac{-\hat{b}_Y}{8\pi^2}\frac{1}{2}\ln\frac{M_X}{M_Z}$$

$$\Rightarrow \frac{1}{g_Y^2(M_Z)} - \frac{1}{g_Y^2(M_X)} = -\frac{(b_Y + \hat{b}_Y)}{8\pi^2}\frac{1}{2}\ln\frac{M_X}{M_Z}$$

On the other hand, directly

$$\frac{1}{g_Y^2(M_Z)} - \frac{1}{g_Y^2(M_X)} = \frac{-B_Y}{8\pi^2} \ln\frac{M_X}{M_Z}$$

$$\Rightarrow \quad 2B_Y = b_Y + \hat{b}_Y$$

$$\Rightarrow \quad 2(\cancel{b_Y} + \Delta b_Y) = \cancel{b_Y} + (\cancel{b_Y} + \Delta \hat{b}_Y)$$

$$\Rightarrow \quad 2\Delta b_Y = \Delta \hat{b}_Y$$

$(\psi^c)_L = \psi^c_R \; , \; (\psi)_L = \psi_L$ $\qquad\qquad$ $\psi^c_L = (\psi_L)^c = (\psi^c)_R$

Wrong. c comes after L, R

$(\psi_R)^c = (\psi^c)_L$

$\overset{?}{=} \psi^c_L$

what do you mean? that ψ^c_R is L.C.? then yes!

2) 1. in (2.22)

$$\underline{10}_L = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & u_3^c & -u_2^c & -u_1 & -d_1 \\ -u_3^c & 0 & u_1^c & -u_2 & -d_2 \\ u_2^c & -u_1^c & 0 & -u_3 & -d_3 \\ u_1 & u_2 & u_3 & 0 & -e^c \\ d_1 & d_2 & d_3 & e^c & 0 \end{pmatrix}_L$$

$$\begin{pmatrix} 0 & u^c_{3R} & -u^c_{2R} & -u_{1L} & -d_{1L} \\ & 0 & u^c_{1R} & -u_{2L} & -d_{2L} \\ & - & 0 & -u_{3L} & -d_{3L} \\ & & & 0 & -e^c_R \\ & & & & 0 \end{pmatrix} = \chi_{10}$$

$$\Rightarrow -\mathrm{tr}\,(\bar{\chi}_{10}\, i\not\partial\, \chi_{10}) = -\mathrm{tr}(\chi_{10}^{+} \overbrace{\gamma_0}^{\text{spinor space}} i\not\partial\, \chi_{10})$$

Because of trace, only consider diagonal component of the product $\rightarrow$?!

11: $\overline{u_3^c}\, i\not\partial\, u_3^c + \ldots$? Not so trivial, need to do explicitely. Not same for L, R fields!

To show it is indeed like kinetic term in SM

$$\overline{q^c} \not\partial q^c$$

$$= (C\, \bar{q}^T)^+ \gamma_0 \not\partial\, C(\bar{q})^T$$

$$= (\bar{q}^T)^+ C^+ \gamma_0 \not\partial\, C(\bar{q})^T$$

$$= \bar{q}^* C^{-1} \gamma_0 \not\partial\, C\, \gamma^0 q^*$$

$$= \bar{q}^* \underbrace{C^{-1}\gamma_0 C}_{=\gamma_0^T/(-1)}\, \underbrace{C^{-1} \not\partial\, C}_{=\not\partial^T/(-1)}\, \gamma^0 q^*$$

$$= \bar{q}^* \gamma_0^T \not\partial^T \gamma^0 q^*$$

$C\Gamma^T C^{-1} = \eta_\Gamma \Gamma$

$\Rightarrow \Gamma^T/\eta_\Gamma = C^{-1}\Gamma\, C$

$C^{-1}\gamma_0\gamma_\mu C = (\gamma_0\gamma_\mu)^T/(-1)$

Take transpose of scalar

$$= q^+ \gamma^{0T} \not\partial \underbrace{\gamma_0 \bar{q}^+}$$

$$= \gamma_0 (q^+ \gamma_0)^+$$

$$= \gamma_0 \gamma_0^+ q$$

$$= \bar{q} \not\partial q \qquad\qquad \Rightarrow -\mathrm{tr}(\bar{\chi}_{10} i\not\partial \chi_{10}) \rightarrow \sum_{q,i} \bar{q}_i\, i\not\partial\, q_i$$

$\rightarrow$

$$\mathrm{tr}[\bar{\chi}_{10} \not D \chi_{10}] = \frac{1}{2}[-2\,\overline{u^c_R} \not D u^c_R - 2\bar{u}_L \not D u_L - 2\bar{d}_L \not D d_L - \bar{e}_R \not D e_R]$$

(*) The algebra is alright, but there is a serious confusion of L, R and charge conjugation. So your results are wrong. Please read my solution and let's chat on Zoom; it is important to clear this up!

2. $\mathcal{L}_{10} \subset 2 g_5 \, tr(\bar\chi_{10} V^a_\mu \gamma^\mu t^a \chi_{10})$

$\mathcal{L}_Y = 2 g_5 \, tr(\bar\chi_{10} V^{12}_\mu \gamma^\mu t^{12} \chi_{10})$

$\left(t^{12} = \frac{1}{2\sqrt{15}} \, diag(2,2,2,-3,-3) \right)$

$$= \frac{g_5}{\sqrt{15}} \underbrace{V^{12}_\mu}_{=B_\mu} tr(\bar\chi_{10} \gamma^\mu \underbrace{t^{12}\chi_{10}})$$

$$\begin{pmatrix} 0 & -\bar u_3^c & +\bar u_2^c & +\bar u_1 & +\bar d_1 \\ +\bar u_3^c & 0 & -\bar u_1^c & +\bar u_2 & +\bar d_2 \\ -\bar u_2^c & +\bar u_1^c & 0 & +\bar u_3 & +\bar d_3 \\ -\bar u_1 & -\bar u_2 & -\bar u_3 & 0 & +\bar e^c \\ -\bar d_1 & -\bar d_2 & -\bar d_3 & -\bar e^c & 0 \end{pmatrix}_L \begin{pmatrix} 0 & 2u_3^c & -2u_2^c & -2u_1 & -2d_1 \\ -2u_3^c & 0 & 2u_1^c & -2u_2 & -2d_2 \\ 2u_2^c & -2u_1^c & 0 & -2u_3 & -2d_3 \\ -3u_1 & -3u_2 & -3u_3 & 0 & +3e^c \\ -3d_1 & -3d_2 & -3d_3 & -3e^c & 0 \end{pmatrix}_L$$

take L inside, then compute the product!

$$= \frac{-g_5 B_\mu}{2\sqrt{15}} \Big[2\bar u^c_{3R}\gamma^\mu u^c_{3R} + 2\bar u^c_{2R}\gamma^\mu u^c_{2R} - 3\bar u_{1L}\gamma^\mu u_{1L} - 3\bar d_{1L}\gamma^\mu d_{1L}$$
$$+ 2\bar u_3^c\gamma^\mu u_3^c + 2\bar u_1^c\gamma^\mu u_1^c - 3\bar u_2\gamma^\mu u_2 - 3\bar d_2\gamma^\mu d_2$$
$$+ 2\bar u_2^c\gamma^\mu u_2^c + 2\bar u_1^c\gamma^\mu u_1^c - 3\bar u_3\gamma^\mu u_3 - 3\bar d_3\gamma^\mu d_3$$
$$+ 2\bar u_1\gamma^\mu u_1 + 2\bar u_2\gamma^\mu u_2 + 2\bar u_3\gamma^\mu u_3 - 3\bar e^c\gamma^\mu e^c$$
$$+ 2\bar d_1\gamma^\mu d_1 + 2\bar d_2\gamma^\mu d_2 + 2\bar d_3\gamma^\mu d_3 - 3\bar e^c\gamma^\mu e^c \Big]_L$$

and so on

$L \to R$ because of c [Nothing to do with $\bar\psi$!]

$$= \underbrace{\frac{-g_5 B_\mu}{2\sqrt{15}}}_{= -\frac{B_\mu g_Y}{6}} \Big[4 \sum_i \bar u^c_{i\cancel{L}R}\gamma^\mu u^c_{i\cancel{L}R} - \sum_i \bar u_{iL}\gamma^\mu u_{iL} - \sum_i \bar d_{iL}\gamma^\mu d_{iL} - 6\,\bar e^c_{\cancel{L}R}\gamma^\mu e^c_{\cancel{L}R} \Big]$$

$$= B_\mu g_Y \Big[-\frac{2}{3} \sum_i \bar u^c_{i\cancel{L}R}\gamma^\mu u^c_{i\cancel{L}R} + \frac{1}{6}\sum_i \bar u_{iL}\gamma^\mu u_{iL} + \frac{1}{6}\sum_i \bar d_{iL}\gamma^\mu d_{iL} + 1 \cdot \bar e^c_{\cancel{L}R}\gamma^\mu e^c_{\cancel{L}R} \Big]$$

$\Rightarrow Y(e^c_R) = +1 \qquad Y(u^c_R) = -\frac{2}{3}$

$Y(q_L) = \frac{1}{6}$

But $(u^c)_L = u^c_L$? Where does u^c_R come from?

(*)