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1)

1. LHS $= [Q_a, P_\mu]$

has a Lorentz index, a spinor index. RHS must have these index.

Q is odd, P is even (generator of Poincaré algebra, by def. even)

$\{U_{2\pi}, Q\} = 0$
fermionic

Since $[\text{even}, \text{odd}] = \text{odd}$

$\rightarrow$ RHS must contain Q, no P_μ (or $M_{\mu\nu}$) on RHS

$\rightarrow$ only way to carry the Lorentz index is by γ_μ, $\gamma_\mu\gamma^5$

$\Rightarrow [Q_a, P_\mu] = (c_1 \gamma_\mu + c_2 \gamma_\mu \gamma^5)_{ab} Q_b$

$c_1, c_2 \in \mathbb{C}$

2. $[Q, P_\mu] = (c_1\gamma_\mu + c_2 \gamma_\mu\gamma^5)_{ab} Q_b$ | take h.c. (in spinor space)

$[Q^+, P_\mu] = Q^+ \left[c_1^* \gamma_\mu^+ + c_2^* (\gamma_\mu\gamma^5)^+\right]$ | $\cdot \gamma_0$

$$[\bar{Q}, P_\mu] = Q^+ \Big(c_1^* \underbrace{\gamma_\mu^+}_{=\gamma_0\gamma_\mu\gamma_0} + c_2^* \underbrace{(\gamma_\mu\gamma^5)^+}_{=(\gamma^5)^+\gamma_0\gamma_\mu\gamma_0 \,=\, \gamma^5\gamma_0\gamma_\mu\gamma_0}\Big)\gamma_0$$

$= \bar{Q}\,(c_1^* \gamma_\mu + c_2^* \gamma^5 \gamma_0 \gamma_\mu)$ | $\{\gamma_\mu, \gamma^5\} = 0$

$= \bar{Q}\,(c_1^* \gamma_\mu + c_2^* \gamma_\mu\gamma^5)$

$Q_a = C_{ab}\bar{Q}_b \Leftrightarrow Q = C\bar{Q}^T$

$\Leftrightarrow Q^T = \bar{Q} C^T$

$\Leftrightarrow -Q^T C^{-1} = \bar{Q}$

Plug in

$$-[Q^T, P_\mu]C^{-1} = -Q^T C^{-1}(c_1^* \gamma_\mu + c_2^* \gamma_\mu \gamma^5)$$

$$[Q^T, P_\mu] = +Q^T C^{-1}(c_1^* \gamma_\mu + c_2^* \gamma_\mu \gamma^5) C$$

$$\left(\begin{array}{l} C \Gamma^T C^{-1} = \eta_\Gamma \Gamma \quad , \quad \eta_\Gamma \text{ is } 1 \text{ or } -1 \\ \Leftrightarrow \quad \eta_\Gamma \Gamma^T = C^{-1} \Gamma C \end{array} \right.$$

$$= Q^T \left(-c_1^* \gamma_\mu^T + c_2^* (\gamma_\mu \gamma^5)^T \right)$$

$$[Q, P_\mu] = (-c_1^* \gamma_\mu + c_2^* \gamma_\mu \gamma^5) Q$$

$$\Rightarrow \quad c_1 = -c_1^* \quad \Rightarrow \quad c_1 \in i\mathbb{R}$$

$$c_2 = c_2^* \quad \Rightarrow \quad c \in \mathbb{R}$$

3.

$$① = [A,[B,C]] = [A, BC - CB] = ABC - BCA - ACB + CBA$$

$$② = [B, CA - AC] = BCA - CAB - BAC + ACB$$

$$③ = [C, AB - BA] = CAB - ABC - CBA + BAC$$

$$\Rightarrow \quad ① + ② + ③ = 0$$

4.

$$[P_\mu, [P_\nu, Q_a]] + [P_\nu, [Q_a, P_\mu]] + [Q_a, \underbrace{[P_\mu, P_\nu]}_{=0}] = 0$$

$$\Rightarrow -[P_\mu, (c_1 \gamma_\nu + c_2 \gamma_\nu \gamma_5)_{ab} Q_b] + [P_\nu, (c_1 \gamma_\mu + c_2 \gamma_\mu \gamma_5)_{ab} Q_b] = 0$$

$$\Rightarrow +c_1 (\gamma_\nu)_{ab} (c_1 \gamma_\mu + c_2 \gamma_\mu \gamma_5)_{bc} Q_c + c_2 (\gamma_\nu \gamma_5)_{ab} (c_1 \gamma_\mu + c_2 \gamma_\mu \gamma_5)_{bc} Q_c$$

$$- c_1 (\gamma_\mu)_{ab} (c_1 \gamma_\nu + c_2 \gamma_\nu \gamma_5)_{bc} Q_c - c_2 (\gamma_\mu \gamma_5)_{ab} (c_1 \gamma_\nu + c_2 \gamma_\nu \gamma_5)_{bc} Q_c = 0$$

$$\Rightarrow c_1^2 (+\gamma_\nu \gamma_\mu Q - \gamma_\mu \gamma_\nu Q) + c_1 c_2 (\underbrace{+\gamma_\nu \gamma_\mu \gamma_5 Q + \gamma_\nu \gamma_5 \gamma_\mu Q - \gamma_\mu \gamma_\nu \gamma_5 Q - \gamma_\mu \gamma_5 \gamma_\nu Q}_{=0})$$

$$+ c_2^2 (+\gamma_\nu \gamma_5 \gamma_\mu \gamma_5 - \gamma_\mu \gamma_5 \gamma_\nu \gamma_5) Q = 0$$

$$\Rightarrow \quad c_1^2 = c_2^2$$

5. Since $c_1 \in i\mathbb{R}$, $c_2 \in \mathbb{R}$ and $\underbrace{c_1^2}_{\leq 0} = \underbrace{c_2^2}_{\geq 0}$

$$\Rightarrow c_1 = c_2 = 0$$

$$\Rightarrow [Q_a, P_\mu] = 0$$

6. $\{Q_a, Q_b\} \rightarrow \{\text{odd}, \text{odd}\} = \text{even}$
must contain P_μ, $M_{\mu\nu}$

$\rightarrow$ to contract Lorentz. indices. γ_μ, $\Sigma_{\mu\nu} \sim [\gamma_\mu, \gamma_\nu]$

Follow the hint,

$$\begin{aligned}(\gamma_\mu C)^T &= C^T \gamma_\mu^T = -C\gamma_\mu^T \\ &= -C\gamma_\mu^T C^{-1} C \\ &= \gamma_\mu C\end{aligned}$$

$$\begin{aligned}(\gamma^\mu \gamma^\nu C)^T &= C^T \gamma^{\nu T} \gamma^{\mu T} \\ &= -C\gamma^{\nu T} C^{-1} C \gamma^{\mu T} C^{-1} C \\ &= -\gamma^\nu \gamma^\mu C\end{aligned}$$

$$\begin{aligned}(\Sigma^{\mu\nu} C)^T &\sim ([\gamma^\mu, \gamma^\nu] C)^T \\ &= (\gamma^\mu \gamma^\nu C - \gamma^\nu \gamma^\mu C)^T \\ &= (-\gamma^\nu \gamma^\mu C + \gamma^\mu \gamma^\nu C) \\ &= [\gamma^\mu, \gamma^\nu] C \\ &\sim \Sigma^{\mu\nu} C\end{aligned}$$

$\Rightarrow$ These two are symmetric, so is the $\{Q_a, Q_b\}$

$$\Rightarrow \{Q_a, Q_b\} = c_3 (\gamma^\mu C)_{ab} P_\mu + c_4 (\Sigma^{\mu\nu} C)_{ab} M_{\mu\nu}$$

7. $① = \{A, BC - CB\} = ABC + BCA - ACB - CBA$

$② = \{B, AC - CA\} = BAC + ACB - BCA - CAB$

$③ = [C, AB + BA] = CAB - ABC + CBA - BAC$

$\rightarrow ① + ② + ③ = 0$

8. $\{Q_a, \underbrace{[Q_b, P_\mu]}_{=0}\} + \{Q_b, \underbrace{[Q_a, P_\mu]}_{=0}\} + [P_\mu, \{Q_a, Q_b\}] = 0$

$\Rightarrow \quad C_3 \gamma^\nu C \underbrace{[P_\mu, P_\nu]}_{=0} + C_4 \Sigma^{\nu\sigma} C \underbrace{[P_\mu, M_{\nu\sigma}]}_{\neq 0} = 0$

$\Rightarrow C_4 = 0$

9. $\{Q_a, Q_b\} = c_3(\gamma^\mu C)_{ab} P_\mu$

$Q \rightarrow yQ$

$\Rightarrow \quad y^2 \{Q_a, Q_b] = c_3(\gamma^\mu C)_{ab} P_\mu$

corresponds to $\quad c_3 \rightarrow c_3 / y^2$

$\rightarrow$ c_3 can be arbitrarily chosen

2)

1. $[P_\mu, Q] = 0 \quad \Rightarrow \quad [P_\mu P^\mu, Q] = P^\mu [P_\mu, Q] = 0$

$\Rightarrow \quad P^2 Q - Q P^2 = 0$

$\Rightarrow$ all members of super multiplet same mass

$(p^2 = m^2)$

2.

Assume Q_a modifies the size of bosonic sector by factor r

Q_a also modifies the size of fermionic sector by r

$\{Q_a, Q_b\} = C_3 (\gamma^\mu C)_{ab} P_\mu$ from 1)

LHS modifies the size of bosonic/fermionic sector by r^2

RHS doesn't change the size

$\Rightarrow r^2 = 1, \quad r = 1$

$\Rightarrow$ equal number of d.o.f. of bosons and fermions

3. $P_\mu |i\rangle = 0$ (vanishing momentum)

$\Rightarrow \{Q_a, Q_b\} |i\rangle = 0$

$\rightarrow$ RHS has no well-defined modification the size of sectors

$\rightarrow$ not generally true that $n_B = n_F$