worked on $3.5/6$

Chenhuan Wang

1)

1. if $\langle \Sigma \rangle = \frac{v_x'}{2\sqrt{10}} \operatorname{diag}(1,1,1,1,-4)$

$\Rightarrow SU(5) \to SU(4) \times (S)U(1)$

since the first four fields have the same couplings $\rightsquigarrow$ same couplings?! that is true for any vev-configuration!

2. $$V(\Sigma) = \mu_x^2 \, tr(\Sigma^2) + \frac{a}{4} tr(\Sigma^4) + \frac{b}{4} [tr(\Sigma^2)]^2$$

$\Sigma \to \langle \Sigma \rangle$

$\Sigma^2 \to \langle \Sigma \rangle^2$

$$= \frac{v_x'^2}{40} \operatorname{diag}(1,1,1,1,16)$$

$$tr(\langle\Sigma\rangle^2) = \frac{v_x'^2}{40}(1+1+1+1+16) = \frac{v_x'^2}{2}$$

$$tr(\langle\Sigma\rangle^4) = \frac{v_x'^4}{16 \cdot 100} \cdot \underbrace{(1+1+1+1+16 \cdot 16)}_{=260}$$

$$= \frac{13}{800} v_x'^4 \qquad \frac{13}{80}$$

$$\Rightarrow V(\Sigma) \longrightarrow V(\langle\Sigma\rangle) = \mu_x^2 \cdot \frac{1}{2} v_x'^2 + \frac{a}{4} \cdot \frac{13}{800} v_x'^4 + \frac{b}{4}\left(\frac{1}{2} v_x'^2\right)^2$$

$$= v_x'^2 \left(\frac{1}{2}\mu_x^2 + \frac{13}{3200} a \, v_x'^2 + \frac{b}{16} v_x'^2 \right)$$

$$= v_x'^2 \left[\frac{1}{2}\mu_x^2 + v_x'^2 \left(\frac{13}{3200} a + \frac{1}{16} b \right) \right]$$

For $V(\langle\Sigma\rangle)$ bounded from below.

$\lim_{v_x' \to \infty} V(\langle\Sigma\rangle) \overset{!}{>} 0$ (ignore first term, since it will eventually get out-weighted)

$$\Rightarrow v_x'^4 \left(\frac{13}{3200} a + \frac{1}{16} b \right) > 0 \quad \rightarrow 13a + 200b > 0 \ ✓$$

1) 2. $\langle \Sigma \rangle^2 = \frac{v_x'^2}{40} \operatorname{diag}(1,1,1,1,16)$

$\langle \Sigma \rangle^4 = \frac{v_x'^4}{1600} \operatorname{diag}(1,1,1,1,256)$

$$\Rightarrow V(\Sigma) = \mu_x^2 \cdot \frac{v_x'^2}{40} \cdot 20 + \frac{a}{4} \frac{v_x'^4}{1600} \cdot 260 + \frac{b}{4} \cdot \frac{v_x'^4}{1600} (20)^2$$

$$= \frac{\mu_x^2}{2} \cdot v_x'^2 + \frac{13a}{320} v_x'^4 + \frac{b}{16} v_x'^4$$

$$\lim_{v_x' \to \infty} V(\Sigma) \overset{!}{>} 0$$

$$\Rightarrow \frac{13a}{320} + \frac{b}{16} > 0$$

$$\Rightarrow 13a + 20b > 0$$

3. with (1)

$$V(\Sigma) = \mu_x^2 \cdot \frac{v_x^2}{60} \cdot \underbrace{(3 \cdot 4 + 2 \cdot 9)}_{12+18=30} + \frac{a}{4} \frac{v_x^4}{3600} \cdot \underbrace{(3 \cdot 2^4 + 2 \cdot 3^2)}_{\substack{=24+18 \\ =42}} + \frac{b}{4} \frac{v_x^4}{3600} \cdot \underbrace{(3 \cdot 4 + 2 \cdot 9)^2}_{=900}$$

$$= \frac{\mu_x^2}{2} v_x^2 + \frac{7a}{2400} v_x^4 + \frac{b}{16} v_x^4$$

Minimal at $v_x^2 = \dfrac{-\frac{\mu_x^2}{2}}{2\left(\frac{7a}{2400} + \frac{b}{16}\right)}$

2) 1. We have the DEs in the form!

(2.43a):

$$\frac{df_t}{d\ln Q} = \frac{f_t}{16\pi^2}\left[-3\left(\frac{8}{3}g_3^2+\frac{3}{4}g_2^2+\frac{17}{36}g_Y^2\right)+\underbrace{\frac{1}{2}\left(9f_t^2+3f_b^2+2f_\tau^2\right)}_{}\overset{\to 0}{}\right]$$

$$\Rightarrow \frac{df_t}{f_t} = \frac{d\ln Q}{16\pi^2}\left[-3\left(\overbrace{\frac{8}{3}}^{c_3}g_3^2+\overbrace{\frac{3}{4}}^{c_2}g_2^2+\overbrace{\frac{17}{36}}^{c_Y}g_Y^2\right)\right]$$

$$\left(g_i = g_i(Q)\ ! \right)$$

Pretend there is only one g_i

$$\frac{df_t}{f_t} = \frac{d\ln Q}{16\pi^2}\cdot c_i\, g_i^2(Q)$$

$$= \frac{c_i}{16\pi^2}\cdot\frac{-8\pi^2}{b_i}\;\frac{dg_i^2}{g_i^2}$$

$$\frac{df_t}{f_t} = \frac{-c_i}{2b_i}\;\frac{dg_i^2}{g_i^2}$$

$$\Rightarrow \ln\frac{f_t(Q)}{f_t(M_Z)}\,M_X = \frac{-c_i}{2b_i}\ln\frac{g_i^2(Q)}{g_i^2(M_Z)}\,M_X$$

$$\Leftrightarrow f_t(Q) = \Bigg[\underbrace{\frac{g_i^2(Q)}{g_i^2(M_Z)}}_{}\Bigg]^{-c_i/2b_i}_{M_X} f_t(M_Z)\ M_X$$

$$= \frac{\alpha_i(Q)}{\alpha_i(M_Z)}$$

RGE:

$$\frac{dg_i^2}{d\ln Q} = -\frac{g_i^4}{8\pi^2}b_i$$

$$\Leftrightarrow \frac{dg_i^2}{g_i^2} = -\frac{b_i}{8\pi^2}g_i^2\, d\ln Q$$

$$\ln y = a\cdot\ln x$$

$$y = e^{a\cdot\ln x} = e^{\ln x^a} = x^a$$

$\Rightarrow$ multiple g_i's are "multi"plicative!

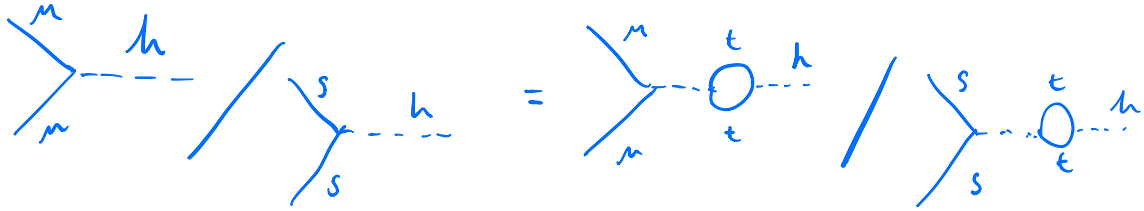
$$\Rightarrow f_t(Q) = f_t(M_Z)\,\prod\left[\frac{\alpha_i(Q)}{\alpha_i(M_Z)}\right]^{-c_i/2b_i}$$

same for f_b, f_τ

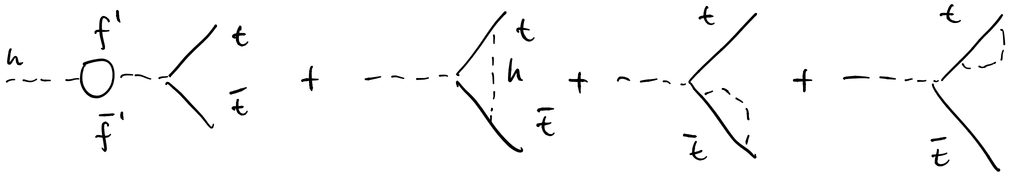
$$\Rightarrow f_t(Q) = f_t(M_Z)\left[\frac{\alpha_3(Q)}{\alpha_3(M_Z)}\right]^{\frac{4}{b_3}}\left[\frac{\alpha_2(Q)}{\alpha_2(M_Z)}\right]^{\frac{9}{8b_2}}\left[\frac{\alpha_Y(Q)}{\alpha_Y(M_Z)}\right]^{\frac{17}{24b_Y}}\ M_X$$

$$f_b(Q) = f_b(M_Z)\left[\frac{\alpha_3(Q)}{\alpha_3(M_Z)}\right]^{\frac{4}{b_3}}\left[\frac{\alpha_2(Q)}{\alpha_2(M_Z)}\right]^{\frac{9}{8b_2}}\left[\frac{\alpha_Y(Q)}{\alpha_Y(M_Z)}\right]^{\frac{5}{24b_Y}}$$

$$f_\tau(Q) = f_\tau(M_Z)\left[\frac{\alpha_3(Q)}{\alpha_3(M_Z)}\right]^{0}\left[\frac{\alpha_2(Q)}{\alpha_2(M_Z)}\right]^{\frac{9}{2b_2}}\left[\frac{\alpha_Y(Q)}{\alpha_Y(M_Z)}\right]^{\frac{15}{8b_Y}}$$



2. top Yukawa interactions



closed fermion loop ⟶ overall (−1) sign

So what? That is relevant for a cancellation among diff contributions to $\mathcal{M}$; this is a comparison between two diff processes!

3. Now:

$$\frac{df_t}{d\ln Q} = \frac{f_t}{16\pi^2}\left[-3\left(\frac{8}{3}g_3^2+\frac{3}{4}g_2^2+\frac{17}{36}g_Y^2\right)\right] + \frac{\alpha}{16\pi^2}f_t^3$$

$$\frac{df_t}{f_t} = \frac{d\ln Q}{16\pi^2}\left[-3\left(\frac{8}{3}g_3^2+\frac{3}{4}g_2^2+\frac{17}{36}g_Y^2\right)+\alpha f_t^2\right]$$

$$\ln\frac{f_t(Q)}{f_t(M_Z)} = \sum_{i=Y,2,3}\frac{-C_i}{2b_i}\ln\frac{\alpha_i(Q)}{\alpha_i(M_Z)} + \frac{\alpha}{16\pi^2}\underbrace{\int_{\ln M_Z}^{\ln Q} d\ln Q'\, f_t^2(Q')}_{f_t \simeq f_{t,app}}$$

$$f_t^2(M_Z)\int_{\ln M_Z}^{\ln Q} d\ln Q' \left[\frac{g_3^2(Q')}{g_3^2(M_Z)}\right]^{\frac{4}{b_3}\cdot 2}\left[\frac{g_2^2(Q')}{g_2^2(M_Z)}\right]^{\frac{9}{8b_2}\cdot 2}\left[\frac{g_Y^2(Q')}{g_Y^2(M_Z)}\right]^{\frac{17}{24b_Y}\cdot 2}$$

$$\propto \int_{\ln M_Z}^{\ln Q} d\ln Q' \left(\frac{1}{g_3^2(M_Z)}+\frac{b_3}{8\pi^2}\ln\frac{Q'}{M_Z}\right)^{\frac{-4}{b_3}\cdot 2}\left(\frac{1}{g_2^2(M_Z)}+\frac{b_2}{8\pi^2}\ln\frac{Q'}{M_Z}\right)^{\frac{-9}{8b_2}\cdot 2}$$

$$\times\left(\frac{1}{g_Y^2(M_Z)}+\frac{b_Y}{8\pi^2}\ln\frac{Q'}{M_Z}\right)^{\frac{-17}{24b_Y}\cdot 2}$$

pretty impossible ......

$$\frac{df_t}{f_t} = \frac{\alpha}{16\pi^2}f_t^2 \qquad \frac{df_t}{f_t^3} = \frac{\alpha}{16\pi^2}$$

Sorry, can't follow?! So now just keeping the Yukawa? How?

$$\Rightarrow \frac{1}{2}\left(\frac{1}{f_t^2(M_Z)} - \frac{1}{f_t^2(Q)}\right) = \frac{\alpha}{16\pi^2}$$

$$\Rightarrow f_t^2(Q) = \left[\frac{-\alpha}{8\pi^2} + \frac{1}{f_t^2(M_Z)}\right]^{-1}$$

$$= \frac{f_t^2(M_Z)}{1-\frac{\alpha}{8\pi^2} f_t^2(M_Z)}$$

$$\Rightarrow F(Q) = 1-\frac{\alpha}{8\pi^2} f_t^2(M_Z)$$ $\longrightarrow$ But that is a constant?!

$f_t(M_X)$ is finite,

$$\Rightarrow 1-\frac{\alpha}{8\pi^2} f_t^2(M_Z) > 0 ,$$ (cannot be negative because of square)

$\longrightarrow$ why f_t at M_Z?

$$\Rightarrow \frac{8\pi^2}{\alpha} > f_t^2(M_Z)$$

$$f_t(M_Z) < \sqrt{\frac{8}{\alpha}}\,\pi$$

$$m_t(M_Z) < \sqrt{\frac{8}{\alpha}}\,\pi \cdot 175 \text{ GeV}$$

$$\frac{df_{t,app}}{dt} = \frac{f_{t,app}}{16\pi^2} g \quad (1)$$

$$\frac{df_t}{dt} = \frac{f_t}{16\pi^2}\left(g + \frac{9}{2} f_t^2\right) \quad (2)$$

$$f_t^2 = f_{t,app}^2 / F \quad (3)$$

$t = \ln Q$

(3) in (2) $\Rightarrow$

$$\frac{d(f_{t,app}^2/F)}{dt} = \frac{f_{t,app}^2}{F}\frac{1}{16\pi^2}\left(g + \frac{9}{2}\frac{f_{t,app}^4}{F^2}\right)$$

$$LHS = \frac{2f_{t,app}}{F}\frac{df_{t,app}}{dt} - \frac{f_{t,app}^2}{F^2}\frac{dF}{dt}$$

$$= f_{t,app}^2\left(\frac{2}{Ff_{t,app}}\frac{df_{t,app}}{dt} - \frac{1}{F^2}\frac{dF}{dt}\right)$$

$$\overset{(1)}{=} f_{t,app}^2\left(\frac{1}{F}\frac{2}{\cancel{f_{t,app}}}\frac{\cancel{f_{t,app}}}{16\pi^2} g - \frac{1}{F^2}\frac{dF}{dt}\right)$$

$$= \frac{f_{t,app}^2}{8\pi^2 F}\left(g - \frac{8\pi^2}{F}\frac{dF}{dt}\right)$$

$$\Rightarrow \quad \frac{d f_t^2}{dt} = \frac{f_t^2}{8\pi^2}\left(g - \frac{8\pi^2}{F}\frac{dF}{dt}\right)$$

$$② \Rightarrow LHS = \frac{df_t}{df_t^2}\frac{df_t^2}{dt} = \frac{1}{2f_t}\frac{df_t^2}{dt} \Rightarrow \frac{df_t^2}{dt} = \frac{f_t^2}{8\pi^2}\left(g + \frac{9}{2}f_t^2\right)$$

$$\Rightarrow \quad -\frac{8\pi^2}{F}\frac{dF}{dt} = \frac{9}{2}f_t^2 = \frac{9}{2}\frac{f_{t,app}^2}{F}$$

$$\Rightarrow \quad \frac{dF}{dt} = -\frac{1}{8\pi^2}\frac{9}{2}f_{t,app}^2$$

$$\Rightarrow F(M_X) - F(Q) = -\frac{9}{16\pi^2}\int_{\ln Q}^{\ln M_X} dt\, f_{t,app}^2$$

At $t = \ln Q = \ln M_X$, no corrections because of

$F = 1$, $\Rightarrow$ $F(Q) = 1 + \frac{9}{16\pi^2}\int_{\ln Q}^{\ln M_X} dt\, f_{t,app}^2$

Now in RGE, if gauge term has same order of magnitude as Yukawa term.

$$\rightarrow \quad \frac{df}{dt} \sim f^3$$

$$\Rightarrow \quad \frac{df}{f^3} \sim dt$$

$$\Rightarrow \quad \frac{1}{f^2} \sim t$$

there will be Landau pole (not at $t=0$! There are other terms)

Instead we want f to be finte ($g_3^2(m_t) \sim g_3^2(m_Z)$ as an estimate)

$$\frac{9}{2}f^2 \sim 3 \times \frac{8}{3}g_3^2 \Big|_{t = \ln m_t}$$

(from this)

$$\rightarrow f_t^2(m_t^2) \sim \frac{16}{9}g_3^2(m_t^2) \sim 2.31 \qquad \text{with } m_t \sim 280\,\text{GeV}$$

$$\rightarrow f_t \sim 1.52$$

$$\Rightarrow f_t(m_t)\cdot v_H \sim 260\,\text{GeV}$$