Chenhuan Wang

1. a) $\mathcal{L}_{q-\text{mass}} = -\sum_{q=u,d} \overline{q_{jR}}(\mathcal{M}_q)_{jk} q_{kL} + h.c.,$ $\qquad q = u, d$

~~matrices.~~ Which combinations of these matrices appear in the interactions of the physical (mass eigenstate) quarks q'_k with W and Z bosons?

(Before SSB: $\quad \mathcal{L} \subset \sum_{q=u,d} \bar{q}_{jL}\, i\gamma_\mu D^\mu q_{jL} + \bar{u}_R\, i\gamma_\mu D^\mu u_R + \bar{d}_R\, i\gamma_\mu D^\mu d_R$)

After SSB: $\quad$ neutral current (Z) : $\propto Z^\mu\, \bar{q}_{L,R}\, \gamma_\mu q_{L,R}$

$$\rightarrow \quad (U^{-1})^{+\,L,R}_q \, (U^{-1})^{L,R}_q = \mathbb{1}$$

no effects on NC interactions.

charged current (W) : $\propto W^+_\mu\, \bar{u}_L \gamma^\mu d_L + W^-_\mu\, \bar{d}_L \gamma^\mu u_L$

$$\rightarrow \quad (U^{-1})^{+\,L}_u\, (U^{-1})^L_d \quad \text{and} \quad (U^{-1})^{+\,L}_d\, (U^{-1})^L_u$$

(Remark: not sure whether inverse in (U^{-1}) should be here, not clear if q or q' is in mass eigenstates)

b) ~~where l_j stands for one of the three charged leptons.~~ Show that in the SM one can assume without loss of generality that $\mathcal{M}_l$, and hence the matrix of lepton Yukawa couplings, is diagonal. ~~*Hint:* Diagonalize $\mathcal{M}_l$ as in the previous case for~~

$$\mathcal{L}_{l\text{-mass}} = -\overline{l_{j,R}}\, (\mathcal{M}_l)_{jk}\, l_{kL} + h.c.$$

$$l_j = e^-, \mu^-, \tau^-$$

Unitary transformation U so that $\quad l_{j\,L,R} = U \underbrace{l'_{j\,L,R}}_{\text{in mass eigenstates}}$

$$\rightarrow \mathcal{L}_{l\text{-mass}} = -\bar{\vec{l}}'_R\, \underbrace{U^+ \mathcal{M}_l U}_{\text{diagonal}}\, \vec{l}'_L$$

2. $\psi = \begin{pmatrix} \xi_L \\ \xi_R \end{pmatrix}$ $\xi_{L,R}$ 2-comp. spinor

a) A *Dirac* mass term is of the form

$$\mathcal{L}_{\text{Dirac mass}} = -m\overline{\psi_R}\psi_L + h.c.; \tag{4}$$

the quark and charged lepton mass terms discussed in the first problem are of this form. Write this mass term in terms of the two–component spinors ξ_L and ξ_R.

$$\mathcal{L}_{\text{Dirac mass}} = -m \begin{pmatrix} 0 & \xi_R^\dagger \end{pmatrix} \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix} \begin{pmatrix} \xi_L \\ 0 \end{pmatrix} - m \begin{pmatrix} \xi_L^\dagger & 0 \end{pmatrix} \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \xi_R \end{pmatrix}$$

$$= -m\, \xi_R^\dagger \xi_L - m \xi_L^\dagger \xi_R$$

b) A *Majorana spinor* has to satisfy the condition

$$\psi^C \equiv C\bar{\psi}^T = \psi, \tag{5}$$

where $C = i\gamma^2\gamma^0$ is the charge conjugation matrix and the superscript T indicates transposition. What is the physical meaning of this condition, and what does it imply for the two–component spinors $\xi_{L,R}$?

$\psi^c = \psi \Rightarrow$ The particle is its own anti-particle.

$$\psi^c = C\bar{\psi}^T = i\gamma^2\gamma^0 \begin{pmatrix} \xi_L^* \\ \xi_R^* \end{pmatrix}$$

in chiral basis

$$= i \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \xi_L^* \\ \xi_R^* \end{pmatrix}$$

$$= \begin{pmatrix} i\sigma^2 & 0 \\ 0 & -i\sigma^2 \end{pmatrix} \begin{pmatrix} \xi_L^* \\ \xi_R^* \end{pmatrix}$$

$$= \begin{pmatrix} i\sigma^2 \xi_L^* \\ -i\sigma^2 \xi_R^* \end{pmatrix}$$

$$\overset{!}{=} \begin{pmatrix} \xi_L \\ \xi_R \end{pmatrix}$$

$$\rightarrow \quad i\sigma^2 \xi_L^* = \xi_L$$

$$-i\sigma^2 \xi_R^* = \xi_R$$

c) Show that $(\psi_L)^C$ is a *right-handed* field. To that end, show that C is unitary; that $C^{-1}\gamma_5 C = \gamma_5^T = \gamma_5$ (the second relation holds in the chiral basis); and hence that $P_R(\psi_L)^C = C[\overline{\psi_L P_R}]^T$ whereas $P_L(\psi_L)^C = 0$, where $P_{L,R} = (1 \mp \gamma_5)/2$ are the chiral projectors. Do not assume that ψ satisfies the Majorana condition (5).

$$P_L(\psi_L)^C = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & 0 \end{pmatrix} (\psi_L)^C$$

$$= \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & 0 \end{pmatrix} i \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix} \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix} \overline{(P_L \psi)}^T$$

$$= i \begin{pmatrix} 0 & \sigma_2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix} \left\{ \left[\begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_L \\ \xi_R \end{pmatrix} \right]^\dagger \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix} \right\}^T$$

$$\propto \begin{pmatrix} \sigma_2 & 0 \\ 0 & 0 \end{pmatrix} \left[\begin{pmatrix} \xi_L^\dagger & 0 \end{pmatrix} \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix} \right]^T$$

$$= \begin{pmatrix} \sigma_2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \xi_L^\dagger \end{pmatrix}^T$$

$$= 0$$

Since P_R and P_L form complete orthogonal basis

$\rightarrow$ $(\psi_L)^C$ is right-handed.

$$C = i\gamma^2\gamma^0$$

$$= \begin{pmatrix} i\sigma^2 & 0 \\ 0 & -i\sigma^2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$C^\dagger = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$\rightarrow C^\dagger C = \mathbb{1} \quad \rightarrow \quad C^\dagger = C^{-1}$$

$$C^{-1}\gamma_5 C = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & 0 & 1 \\ & & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & 0 & -1 \\ & & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} = \gamma_5 = \gamma_5^T$$

$$P_R(\psi_L)^c = (\psi_L)^c = C\,\overline{\psi_L}^T = C[\overline{\psi_L}P_R]^T$$

right-handed

$P_R^2 = P_R$ $\qquad$ $P_R^2 = P_R$

d) A *Majorana* mass term is written as

$$\mathcal{L}_{\text{Majorana mass}} = -\frac{1}{2}M_L\overline{(\psi_L)^C}\psi_L + h.c. \tag{6}$$

Why do we need the factor 1/2 if M_L is to be the physical (pole) mass?

$$h.c. = -\frac{1}{2}M_L\overline{(\psi_L)}\psi_L^c = -\frac{1}{2}M_L\psi_L^+\gamma^0 C\psi_L^c$$

$$= \frac{1}{2}M_L\psi_L^+ C\gamma^0\psi_L^c$$

$$= -\frac{1}{2}M_L(\psi_L C)^+\gamma^0\psi_L^c$$

$$= -\frac{1}{2}M_L\overline{(\psi_L)^c}\psi_L$$

because the h.c. is the other half

e) In general one can also write a term anologuous to that in (6), with $L \to R$ everywhere. In general $M_L \neq M_R$ is permitted, but are these two quantities related if ψ is a Majorana spinor, i.e. if it satisfies the condition (5)?

$$\to \mathcal{L}_{MM} = -\frac{1}{2}M_L\overline{(\psi_L)^c}\psi_L - \frac{1}{2}M_R\overline{(\psi_R)^c}\psi_R + h.c.$$

$$= -M_L\overline{(\psi_L)^c}\psi_L - M_R\overline{(\psi_R)^c}\psi_R$$

ψ_L and ψ_R are two separate fields,

$\to M_L \neq M_R$ (in general), even if $\psi^c = \psi$

f) Can a Majorana mass term be introduced for any fermion in the SM, if we require gauge invariance and restrict ourselves to terms in the Lagrangian that are power–counting renormalizable? ~~And can one write a Majorana mass term for~~

No. If observe the mass term $\overline{(\psi_L)^c}\psi_L$,

e.g. hypercharge,

since "$-$" and "C" both give sign to quantum number.

If to involve higgs fields, also no. since Yukawa term is not gauge invariant ($U(1)_Y$). Since the $\overline{\psi}$ and ψ have opposite sign and higgs field have some non-zero quantum number.

~~power–counting renormalizable?~~ And can one write a Majorana mass term for the SM neutrinos involving the (vev of the) Higgs boson if non–renormalizable terms are allowed?

Yes. Instead of one scalar in Yukawa, need to have its adjoint field in interaction term $\rightarrow$ $U(1)_Y$ invariant