H.11

1. $\forall g \in U(n)$

$$K_{\bar{D}}(g) = tr[\bar{D}(g)] = tr[D(g^{*})^{t}] = tr[D(g^{*})]$$

$\forall g \in U(n)$, $\exists g_0 \in [g]$

s.t. $g_0 = diag(\lambda_1, \dots, \lambda_n)$

$K_D(g) = K_D(g_0)$

$D(g)$ is unitary under $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{K}$

$$\langle D^{-1}(g) x, y \rangle = \langle x, D(g) y \rangle = \overline{\langle D(g) y, x \rangle}$$

$$\to K_{\bar{D}}(g) = K_D(g_0^{*}) = K_D(g_0^{-1}) = K_{D^{-1}}(g_0) = K_{D^{-1}}(g)$$

$$= tr[D^{-1}(g)]$$

$$= \sum_i^n \langle e_i, D^{-1}(g) e_i \rangle$$

$$= \sum \overline{\langle e_i, D(g) e_i \rangle}$$

$$= \overline{tr(D(g))}$$

$$= \overline{K_D(g)}$$

2. $$\bar{K}(g) = K(\bar{\lambda}_1, \dots, \bar{\lambda}_n) = \frac{\det[(\bar{\lambda}_i)^{z_k}]}{\det[(\bar{\lambda}_i)^{k-1}]}, \quad z_k = l_k + k - 1$$

$$= \bar{K}(g)$$

$$\overset{!}{=} \frac{\det[(\lambda_i)^{\tilde{z}_k}]}{\det[(\lambda_i)^{k-1}]} (\lambda_1 \lambda_2 \dots \lambda_n)^{n-1-z_n}$$

$\Rightarrow$

$$\det[(\bar{\lambda}_i)^{z_k}] \det[(\lambda_i)^{k-1}] \overset{?}{=} \det[(\lambda_i)^{\tilde{z}_k}] \det[(\bar{\lambda}_i)^{k-1}] (\lambda_1 \dots \lambda_n)^{n-1-z_n}$$

$\tilde{z}_k = z_n - z_{n+1-k}$, $\quad \tilde{z}_k = \tilde{l}_k + k - 1$

since $\underbrace{(\tilde{l}_k + k - 1)}_{\tilde{z}_k} + z_n - z_{n+1-k} = \tilde{z}_k + (l_n + n - 1) - (l_{n+1-k} + n + 1 - k - 1)$

$$= \tilde{z}_k + \underbrace{\tilde{l}_k + k - 1} = 2\tilde{z}_k$$

$$\bar{\lambda}_i = \lambda_i^{-1} \qquad (|\lambda_i| = 1)$$

$$\hookrightarrow \det[(\lambda_i)^{-z_k}]\det[(\lambda_i)^{k-1}] = \det[(\lambda_i)^{z_n - z_{n+1-k}}]\det[(\lambda_i)^{1-k}]$$
$$\times (\lambda_1 \dots \lambda_n)^{n-1-z_n}$$

$$\det[(\lambda_i)^{z_n - z_{n+1-k}}] = (\lambda_1)^{z_n}(\lambda_2)^{z_n}\dots(\lambda_n)^{z_n}\underbrace{\det[(\lambda_i)^{-z_{n+1-k}}]}_{=\det[(\lambda_i)^{-z_k}](-1)^{f(n)}}$$

Need to find $f(n)$

$n = 1$, $f(1)$ even

$n = 2$, $f(2)$ odd $\qquad f(n) = \frac{n}{2}(n-1)$

$n = 3$, $f(3)$ odd $\qquad |$

$n = 4$, $f(4)$ even $\qquad f_n = \sum_{i=1}^{n-1} i = \frac{n}{2}(n-1)$

$n \geq 5$, pattern repeats

$$\det[(\lambda_i)^{-z_k}]\det[(\lambda_i)^{k-1}]$$
$$= (\lambda_1)^{z_n}(\lambda_2)^{z_n}\dots(\lambda_n)^{z_n}(-1)^{\frac{n}{2}(n-1)}\det[(\lambda_i)^{-z_k}]\det[(\bar{\lambda}_i)^{k-1}]$$
$$\times (\lambda_1 \dots \lambda_n)^{n-1-z_n}$$
$$= (-1)^{\frac{n}{2}(n-1)}\underbrace{\det[(\lambda_i)^{-z_k}]\det[(\bar{\lambda}_i)^{k-1}]}_{=\prod_{i<j}(\bar{\lambda}_i - \bar{\lambda}_j)}(\lambda_1 \dots \lambda_n)^{n-1-z_n}$$

$\leftarrow$ Vandermore det.

$$LHS = \prod_{i<j}(\lambda_j - \lambda_i)$$

$$\Rightarrow \prod_{i<j}(\lambda_j - \lambda_i) = (-1)^{\frac{n}{2}(n-1)}\prod_{i<j}(\bar{\lambda}_i - \bar{\lambda}_j)(\lambda_1 \dots \lambda_n)^{n-1}$$

$$\frac{\lambda_j - \lambda_i}{\bar{\lambda}_j - \bar{\lambda}_i} = \frac{\lambda_j - \lambda_i}{\bar{\lambda}_j - \bar{\lambda}_i}\frac{\lambda_j\lambda_i}{\lambda_j\lambda_i} = \frac{\lambda_j\lambda_i(\lambda_j - \lambda_i)}{(\bar{\lambda}_j\lambda_j\lambda_i - \lambda_i\bar{\lambda}_i\lambda_j)} = \frac{\lambda_j\lambda_i(\lambda_j - \lambda_i)}{-(\lambda_j - \lambda_i)}$$

$$= -\lambda_i \lambda_j$$

$$\underbrace{\prod_{i<j} (-\lambda_i \lambda_j)}_{} = (-1)^{\frac{n}{2}(n-1)} (\lambda_1 \dots \lambda_n)^{n-1}$$

$$= \left[\prod_{i<j} (-1)\right] \cdot \prod_{i<j} (\lambda_i \lambda_j)$$

$$= \left[\prod_{i<j} (-1)\right] \cdot \underbrace{\begin{array}{l} (\lambda_1\lambda_2)(\lambda_1\lambda_3) \dots (\lambda_1 \lambda_n) \cdot \\ \qquad (\lambda_2 \lambda_3) \dots (\lambda_2 \lambda_n) \\ \qquad\qquad \dots \\ \qquad\qquad (\lambda_{n-1}\lambda_n) \end{array}}_{}$$

$$= (\lambda_1 \dots \lambda_n)^{n-1}$$

$$\to \prod_{i<j} (-1) = (-1)^{\frac{n}{2}(n-1)}$$

$(n-1)$ for λ_i.
A pair of $\lambda_i \lambda_j$ has factor (-1)
$\to \frac{n}{2}(n-1)$

3. $g \in SU(n)$. $(\lambda_1 \dots \lambda_n) = \det(g) = 1$

$$\bar{K}(g) = \frac{\det[(\lambda_i)^{\tilde{z}_k}]}{\det[(\lambda_i)^{k-1}]}, \quad \tilde{z}_k = \tilde{l}_k + k - 1$$

$$K(g) = \frac{\det[(\lambda_i)^{z_k}]}{\det[(\lambda_i)^{k-1}]}, \quad z_k = l_k + k - 1$$

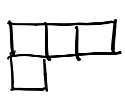
with $\tilde{l}_k = l_n - l_{n+1-k}$

e.g. $SU(3)$. $\quad z_1 < z_2 < z_3$

$$l_1 \leq l_2 \leq l_3$$

Example $\quad l_1 = 0, \; l_2 = 1, \; l_3 = 3 \quad \xrightarrow{\tilde{l}} \quad \tilde{l}_3 = l_3 - l_1 = 3$

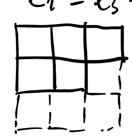
3)
2)
1)



$$\tilde{l}_2 = l_3 - l_2 = 2$$

$$\tilde{l}_1 = l_3 - l_3 = 0$$

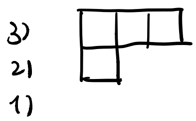
3)
2)
1)



$\tilde{l}_3 = l_3 - l_1 = 3$

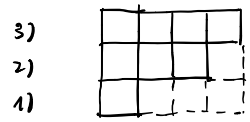
$\tilde{l}_2 = l_3 - l_2 = 1$

$\tilde{l}_1 = 0$



$\xrightarrow{l}$ (choose l_4)

$l_3 = 4$, $l_2 = 3$, $l_1 = 1$



take transpose

$g \in SU(n)$, $\det(g) = 1$ $\longrightarrow$ $[\det(g)]^{l_1} = 1$

choose $l_1 = 0$

Character Tables with Orthogonality

1) $S_2 = \{e, a\}$, $e = ()$, $a = (12)$

conjugacy classes: $C_1 = \{e\}$, $C_2 = \{a\}$

$|C_1| = 1$, $|C_2| = 1$

Rule 1: # of conj. classes $\hat{=}$ # of irrep. $\longrightarrow$ character table always square

S_2	C_1	C_2
χ_1	1	1
χ_2	x	y

Signum rep.

Rule 2: There are always a trivial representation

Rule 3 (Orthogonality):

1) Every column must be orthogonal to other columns.
2) Every row must be orthogonal to other rows with respect to the number of elements in conjugancy class.

$\rightarrow \quad 1 \cdot 1 + x \cdot y = 0 \quad \Rightarrow \quad xy = -1$

$$\underset{|C_1|}{1} \cdot (1 \cdot x) + \underset{|C_2|}{1} \cdot (1 \cdot y) = 0 \Rightarrow x = -y$$

$$\Rightarrow x = \pm 1, \quad y = \mp 1$$

But $x = -1$, then $\mathrm{Dim}\, \chi_2 = -1 \Rightarrow \times$

$$S_3 = \{\underbrace{e}_{C_1}, \underbrace{(12), (13), (23)}_{C_2}, \underbrace{(123), (132)}_{C_3}\}$$

S_3	C_1	C_2	C_3	
χ_1	1	1	1	
χ_2	1	-1	1	← Signum rep.
χ_3	x	y	z	

$$\vec{\chi}_1 \cdot \vec{\chi}_3 = 1 \cdot (1 \cdot x) + 3 \cdot (1 \cdot y) + 2(1 \cdot z) = 0$$

or take $|\vec{C}_1|^2, \ |\vec{C}_2|^2, \ |\vec{C}_3|^2 = n = 6$

$$\Rightarrow x = 2, \ y = 0, \ z = -1$$