H.13

1. $\det D(g): \quad G \to \mathbb{R}$

$$\det D(g_1 g_2) = \det[D(g_1) D(g_2)] = \det D(g_1) \cdot \det D(g_2)$$

$\to$ rep. ✓

$\mathrm{tr}\, D(g): \quad G \to \mathbb{R}$ (or take $\mathrm{tr}\, D(e) = d$)

$$\mathrm{tr}\, D(g_1 g_2) = \mathrm{tr}[D(g_1) \cdot D(g_2)] \underset{\text{in general}}{\neq} \mathrm{tr}\, D(g_1) \cdot \mathrm{tr}\, D(g_2)$$

$\to$ not rep. ✓ unless $D(g)$ is 1-dim

$D(g)^{-1}: \quad G \to GL(V)$

$$D(g_1 g_2)^{-1} = [D(g_1) \cdot D(g_2)]^{-1} = D(g_2)^{-1} \cdot D(g_1)^{-1}$$

not necessarily commute

$\to$ not rep. ✓ unless G is abelian

$D(g)^t: \quad G \to GL(V)$

$$D(g_1 g_2)^t = [D(g_1) \cdot D(g_2)]^t = D^t(g_2) \cdot D^t(g_1)$$

$\to$ not rep. ✓ unless G is abelian

$D(g^{-1})^t: \quad G \to GL(V)$

$$D((g_1 g_2)^{-1})^t = D(g_2^{-1} g_1^{-1})^t = D(g_1^{-1})^t \cdot D(g_2^{-1})^t$$

$\to$ rep. ✓

$\alpha \cdot D(g): \quad G \to GL(V)$

$$\alpha \cdot D(g_1 g_2) = \alpha D(g_1) \cdot D(g_2)$$

$\to$ not rep. ✓ unless $\alpha = 1$

2. $D_i = \sum_{g \in K_i} D(g)$, $D(g) \in K_i \to D(g) = D(h g h^{-1})$

$$\Rightarrow \quad D(g) D(h) = D(h) D(g)$$

Unitarity
finite $\xrightarrow{\text{auto}}$ unitary ?

$$\Rightarrow \quad D(g) = \lambda \mathbb{1}$$
$$\Rightarrow \quad D_i = \sum_{g \in K_i} D(g) = \lambda_i \mathbb{1}_n$$

$D(h) D_i D(h)^{-1} = \sum D(h) D(g) D(h)^{-1}$

$\tilde{g} = hgh^{-1}$ $\quad = \sum D(hgh^{-1})$

$= \sum D(\tilde{g})$

$= D_i$

$\rightarrow D_i = \lambda_i \mathbb{1}_n$

H.14

1. $A_3 = \{e, (123), (132)\}$

Muliplication Table

	e	$\overset{=a}{(123)}$	$\overset{=a^2}{(132)}$
e	e	(123)	(132)
(123)	(123)	e (132)	e
(132)	(132)	e	e (123)

LIKE SUDOKU: element can appear once in a line / column.

A_3 is abelian
$\Rightarrow$ Every element forms its conjugacy class

$$(123) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \quad (132) = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

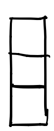
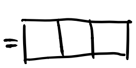
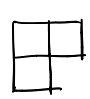
$$(123)\cdot(132) = \begin{matrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 2 & 3 & 1 \\ \downarrow & \downarrow & \downarrow \\ 1 & 2 & 3 \end{matrix} = e$$

(In retrospect, there are two elements in A_3 besides e, they must be inverse to each other)

Conjugacy Classes $\quad \{(123), (132)\}$

Charakters

$$f_Y(\sigma) = \begin{cases} 0, & \sigma \neq p \cdot q \\ \varepsilon(p), & \sigma = p \cdot q \end{cases}$$

$A =$  $\quad S =$  $\quad M =$ 

$f_A(\sigma) = \varepsilon(\sigma)$ $\quad f_S(\sigma) = +1$ $\quad f_M(123) = \sigma(13) = -1$

$f_A((132)) = f_A((123)) = +1$ $\quad f_M = (132) = 0$

$\rightarrow \quad \tilde{f}_S = \frac{1}{3} \sum f_S = 1 \quad , \quad N_S = 1$

$\tilde{f}_A = \frac{1}{3} \sum f_A = 1 \quad , \quad N_S = 1$

$\tilde{f}_M = \frac{1}{3} \sum f_M = 0 \quad , \quad N_A = \sqrt{\frac{3}{2}}$

$$N_\gamma^2 = \frac{m!}{\sum |\tilde{f}_\gamma(r)|^2}$$

Character Table

	K
$\tilde{K}_S$	1
$\tilde{K}_A$	1
$\tilde{K}_M$	0

Different from character table of S_3,

They are different group ...

$\hat{=}$ Cyclic group Z_3

$\rightarrow a^3 = 1 \quad \rightarrow D(g_k) = \exp(2\pi i k/3) \,, \quad k = 0, 1, 2$

From H.4 $\rightarrow$ irrep. of abelian group is 1-dim

	e	a	a^2
$\lvert K_i \rvert$	1	1	1
χ_1	1	1	1
χ_2	1	$\exp(2\pi i/3)$	$\exp(4\pi i/3)$
χ_3	1	$\exp(4\pi i/3)$	$\exp(2\pi i/3)$

Why we have complex number?

if there is no inverse in conjugacy class, the characters can be complex.

2. $\mathbb{Z}_4 = \{e, a, a^2, a^3\}$ $\quad K_4 = \{e, b, c, bc\}$, $b^2 = c^2 = e$

a)

	e	a	a^2	a^3
e	e	a	a^2	a^3
a	a	a^2	a^3	e
a^2	a^2	a^3	e	a
a^3	a^3	e	a	a^2

✓

	e	b	c	bc
e	e	b	c	bc
b	b	e	bc	c
c	c	bc	e	b
bc	bc	c	b	e

✓

Both are abelian

b) $H \subset G$, order of $H = h$, order of $G = g$ $\rightarrow$ h is a divisor of g

(Lagrange's theorem)

$\rightarrow 4 = 1 \cdot 4 = 2 \cdot 2$ only two ways

f

c) $\mathbb{Z}_4$:

$K_1 = [e] = \{e\}$
$K_2 = [a] = \{a\}$
$K_3 = [a^2] = \{a^2\}$
$K_4 = [a^3] = \{a^3\}$

Result of being abelian

K_4:

$K_1 = [e] = \{e\}$
$K_2 = [b] = \{b\}$
$K_3 = [c] = \{c\}$
$K_4 = [bc] = \{bc\}$

Abelian group $\rightarrow$ conjugacy class only one element

	K_1	K_2	K_3	K_4
m_r	1	1	1	1
A	1	1	1	1
B	1	1 i	-1	-1 $-i$
C	1	-1	1	-1
D	1	-1 $-i$	-1	1 i

	K_1	K_2	K_3	K_4
m_r	1	1	1	1
A	1	1	1	1
B	1	1	-1	-1
C	1	-1	1	-1
D	1	-1	-1	1

✓

self inverse elements in K_i $\rightarrow$ K must be real

$\langle K^1, K^1 \rangle$
$= \langle K^A, K^A \rangle$
$= 4$

$\mathbb{Z}_4$: $a^4 = 1$

$\exp\left(\frac{2\pi i k}{4}\right)$, $k = 0, 1, 2, 3$

$\langle K_A, K_B \rangle = 0 \Rightarrow \Sigma(B, K_{2,3,4}) = -1$

...

We can see the above configuration satisfies orthogonality.

(not unique?)

H.15

1. $K_{Y^t}(\sigma) = \underbrace{N_{Y^t}}_{N_Y} \tilde{f}_{Y^t}(\sigma) = N_Y \cdot \begin{cases} 0 & , \sigma \neq pq \\ \varepsilon(q) & , \sigma = pq \end{cases}$

$\varepsilon(q) = \underline{\varepsilon(p \cdot p \cdot q)} = \varepsilon(p) \cdot \varepsilon(\sigma)$

$\rightarrow K_{Y^t}(\sigma) = \varepsilon(\sigma) \cdot K_Y(\sigma)$

1. $\varepsilon(\sigma) \tilde{f}_Y(\sigma) = \varepsilon(\sigma) \sum_{\tau \in S_m} f_Y(\tau \sigma \tau^{-1})$

ε is rep.

$\downarrow$

$\varepsilon(\sigma) = \varepsilon(\tau \sigma \tau^{-1})$

$= \sum_{\tau \in S_m} \varepsilon(\sigma) f_Y(\tau \sigma \tau^{-1})$

$= \sum_{\tau \in S_m} \varepsilon(\tau \sigma \tau^{-1}) f_Y(\tau \sigma \tau^{-1})$

$\varepsilon(\rho) f_Y(\rho) = \begin{cases} 0 & \text{if } \rho \neq p \cdot q \\ \varepsilon(\rho)\varepsilon(p) = \varepsilon(pq)\varepsilon(p) & \text{if } \rho = p \cdot q \end{cases}$

$= \varepsilon^2(p)\varepsilon(q)$

$= \varepsilon(q)$

$= f_{Y^t}(\rho) \quad , \quad N_{Y^t} = N_Y$

$\rightarrow K_{Y^t}(\sigma) = \varepsilon(\sigma) \cdot K_Y(\sigma)$

2. compact Lie-group: $n_\ell = \frac{1}{V(G)} \int dg\, \overline{K_\ell(g)} \cdot K_{\ell_1}(g) K_{\ell_2}(g)$

finite group S_m: $n_\ell = \frac{1}{m!} \sum_{\sigma \in S_m} \overline{K_Y(\sigma)} K_{Y_1}(\sigma) K_{Y_2}(\sigma)$

Symm: $K_{Y_S}(\sigma) = 1 \quad \forall \sigma \in S_m$

$n_{Y_S} = \frac{1}{m!} \sum_{\sigma \in S_m} K_{Y_1}(\sigma) K_{Y_2}(\sigma) = \delta_{Y_1 Y_2}$

Anti-symm: $K_{Y_A}(\sigma) = \varepsilon(\sigma) \quad \forall \sigma \in S_m$

$n_{Y_A} = \frac{1}{m!} \sum \varepsilon(\sigma) K_{Y_1}(\sigma) K_{Y_2}(\sigma)$

$$\underbrace{}_{K_{\gamma_1^t}(\sigma)}$$

$$= \frac{1}{m!} \sum K_{\gamma_1^t} K_{\gamma_2} = \delta_{\gamma_1^t \gamma_2}$$