

H.5

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1. in V^i , we have the orthonormal basis

$$\langle e_\alpha^i, e_\beta^i \rangle = \delta_{\alpha\beta}$$

$$V = \bigoplus^k V^i = \bigoplus^l V^j$$

$$e_\alpha^{ki} = (e_\alpha^i, \dots, e_\alpha^i), \quad e_\beta^{lj} = (e_\beta^j, \dots, e_\beta^j)$$

$$\langle e_\alpha^{ki}, D(g) e_\beta^{lj} \rangle = \langle (e_\alpha^i, \dots, e_\alpha^i), (D^j(g) e_\beta^j, \dots, D^j(g) e_\beta^j) \rangle$$

 D^i uni. irrep in V^i , D rep in V

$$V = \tilde{V}^1 \oplus \tilde{V}^2 \oplus \dots \oplus \tilde{V}^n$$

$$= \underbrace{V_1^1 \oplus \dots \oplus V_{k_1}^1}_{\text{all have equivalent rep. } D^1} \oplus \dots \oplus V_{k_n}^n$$

regroup them into equivalent subspaces

Let $D_k^i(g)$ be the rep. D restricted to the subspace.This $D_k^i(g)$ is equivalent to D^i , which is acting on V^i \hookrightarrow Equivalent, \exists a bijective map $D_k^i(g) = A_{ki}^{-1} D^i(g) A_{ki}$ $(A_{ki}: V_k^i \rightarrow V^i \text{ isometry})$ V_k^i has a orthonormal basis $\{e_\alpha^{ki}\}$

$$e_\alpha^{ki} = A_{ki}^{-1} e_\alpha^i$$

 $(A_{ki} \text{ unitary})$
 \Leftrightarrow
bijective & isometry

$$\langle e_\alpha^{ki}, e_\beta^{ki} \rangle = \langle A_{ki}^{-1} e_\alpha^i, A_{ki}^{-1} e_\beta^i \rangle$$

$$= \langle e_\alpha^i, A_{ki} A_{ki}^{-1} e_\beta^i \rangle$$

$$= \delta_{\alpha\beta}$$

So $e_\alpha^{ki} \in V_k^i$, $e_\beta^{lj} \in V_l^j$

$$\rightarrow \langle e_\alpha^{ki}, e_\beta^{lj} \rangle_V = \delta_{kl} \delta_{ij} \langle e_\alpha^{ki}, e_\beta^{ki} \rangle$$

 \uparrow scalar product in the whole space V .
Other components are zero

$$\begin{aligned}
 \underbrace{\langle e_\alpha^{k_i}, D(g) e_\beta^{l_j} \rangle}_{V_k^i} &= \delta_{kl} \delta_{ij} \langle e_\alpha^{k_i}, D(g) e_\beta^{k_i} \rangle \\
 &= \delta_{kl} \delta_{ij} \langle A_{k_i}^{-1} e_\alpha^i, D_k^i(g) A_{k_i}^{-1} e_\beta^i \rangle \\
 &= \delta_{kl} \delta_{ij} \langle e_\alpha^i, \underbrace{A_{k_i} D_k^i(g) A_{k_i}^{-1}}_{D^i(g)} e_\beta^i \rangle
 \end{aligned}$$

$$2. \quad K^i(g) = \text{tr}[D^i(g)] = \sum_\alpha D_{\alpha\alpha}^i(g) = \sum_\alpha \langle e_\alpha^i, D^i(g) e_\alpha^i \rangle$$

$$\begin{aligned}
 \langle K^i(g), K^j(g) \rangle &= \int dg \overline{K^i(g)} K^j(g) \\
 &= \sum_{\alpha, \beta} \int dg \overline{\langle e_\alpha^i, D^i(g) e_\alpha^i \rangle} \langle e_\beta^j, D^j(g) e_\beta^j \rangle \\
 &= \sum_{\alpha, \beta} \delta^{ij} \delta_{\alpha\beta} \delta_{\alpha\beta} \frac{V(A)}{\dim V_i} \\
 &= \delta^{ij} \sum_\alpha \delta_{\alpha\alpha} \frac{V(A)}{\dim V_i}
 \end{aligned}$$

$$\tilde{K}^i = \frac{1}{\sqrt{V(A)}} K^i \quad \leftarrow$$

$$= \delta^{ij} \underbrace{V(A)}_{\text{normalisation}} \quad \checkmark$$

Peter - Weyl

$$\begin{aligned}
 f \in F_0(A) \quad , \quad f(g) &= \sum_{i, \alpha, \beta} C_{\alpha\beta}^i D_{\alpha\beta}^i(g) \\
 f(g) = f(g' g g'^{-1}) &= \sum_{i, \alpha, \beta} C_{\alpha\beta}^i D_{\alpha\beta}^i(g' g g'^{-1})
 \end{aligned}$$

equal!

$$= \sum_{\substack{i, \alpha, \beta, \\ r, s}} C_{\alpha\beta}^i D_{\alpha r}^i(g') D_{rs}^i(g) D_{s\beta}^i(g'^{-1})$$

$$\Rightarrow C_{rs}^i = \sum_{\alpha, \beta} D_{s\beta}^i(g'^{-1}) C_{\alpha\beta}^i D_{\alpha r}^i(g')$$

$$C^i = D^i(g'^{-1}) C^i D^i(g')$$

$$= (D^i(g'))^{-1} C^i D^i(g')$$

$$D^i(g') C^i = C^i D^i(g')$$

$$D^i \text{ is irrep, using Schur's lemma} \Rightarrow C^i = \lambda_i \mathbb{1}$$

$$\begin{aligned}
 f_0(g) \in F_0(G) : f_0(g) &= \sum_{i, \alpha, \beta} C_{i, \alpha, \beta}^i D_{\alpha \beta}^i(g) \\
 &= \sum_{i, \alpha, \beta} \lambda_i \delta_{\alpha \beta} D_{\alpha \beta}^i(g) \\
 &= \sum_{i, \alpha} D_{\alpha \alpha}^i(g) \\
 &= \sum_i \lambda_i \chi_i(g) \rightarrow \text{is a basis}
 \end{aligned}$$

H.6

$$\Delta_n(a_1, \dots, a_n) = \begin{vmatrix} 1 & a_1 & a_1^2 & \dots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \dots & a_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & a_n^2 & \dots & a_n^{n-1} \end{vmatrix} = \det A_{ik}$$

$$n=1, \quad \Delta_1(a_1) = |1| = 1$$

$$n=2, \quad \Delta_2(a_1, a_2) = \begin{vmatrix} 1 & a_1 \\ 1 & a_2 \end{vmatrix} = a_2 - a_1 = \prod_{1 \leq i < j \leq 2} (a_j - a_i)$$

$$\text{Assume it's true for } \Delta_n(a_1, \dots, a_n) = \prod_{1 \leq i < j \leq n} (a_j - a_i)$$

$$\rightarrow \Delta_{n+1}(a_1, \dots, a_{n+1})$$

$$= \begin{vmatrix} 1 & \dots & a_1^{n-1} & a_1^n \\ \vdots & \ddots & \vdots & \vdots \\ 1 & a_n & \dots & a_n^{n-1} & a_n^n \\ 1 & a_{n+1} & \dots & a_{n+1}^{n-1} & a_{n+1}^n \end{vmatrix}$$

$A(i)$: determinant of A without last row and n -th column

$$= a_{n+1}^n A(n) + \dots + (-1)^n A(1)$$

$$\text{if } a_{n+1} = a_i, \quad i = 1, \dots, n$$

$$= 0$$

\uparrow identical rows, exchanging rows in det gives ± 1 factor

$$\Rightarrow \Delta_{n+1}(a_1, \dots, a_{n+1}) = C \prod_{i=1}^n (a_{n+1} - a_i)$$

$$= C a_{n+1}^n + \dots$$

$$\Rightarrow C a_{n+1}^n = \Delta_n(a_1, \dots, a_n)$$

$$\Rightarrow \Delta_{n+1}(a_1, \dots, a_{n+1}) = \Delta_n(a_1, \dots, a_n) \prod_{i=1}^n (a_{n+1} - a_i)$$

$$= \prod_{1 \leq i < j \leq n} (a_j - a_i) \prod_{m=1}^n (a_{n+1} - a_m)$$

$$= \prod_{1 \leq i < j \leq n+1} (a_j - a_i)$$