

H.13

$$1. \det D(g) : G \rightarrow \mathbb{R}$$

$$\det D(g_1 g_2) = \det[D(g_1) D(g_2)] = \det D(g_1) \cdot \det D(g_2)$$

$$\text{tr } D(g) : G \rightarrow \mathbb{R} \quad (\text{or take } \text{tr } D(e) = d)$$

$$\text{tr } D(g_1 g_2) = \text{tr}[D(g_1) \cdot D(g_2)] \neq \text{tr } D(g_1) + \text{tr } D(g_2) \quad \text{in general}$$

$\rightarrow$  not rep. ✓ unless  $D(g)$  is 1-dim

$$D(g)^{-1} : G \rightarrow GL(V)$$

$$D(g_1 g_2)^{-1} = [D(g_1) \cdot D(g_2)]^{-1} = D(g_2)^{-1} \cdot D(g_1)^{-1}$$

$\curvearrowleft$  not necessarily commute

$\rightarrow$  not rep. ✓ unless  $G$  is abelian

$$D(g)^t : G \rightarrow GL(V)$$

$$D(g_1 g_2)^t = [D(g_1) \cdot D(g_2)]^t = D(g_2)^t \cdot D(g_1)^t$$

$\rightarrow$  not rep. ✓ unless  $G$  is abelian

$$D(g^{-1})^t : G \rightarrow GL(V)$$

$$D(g_1 g_2)^{-t} = D(g_2^{-1} g_1^{-1})^t = D(g_1^{-1})^t \cdot D(g_2^{-1})^t$$

$\rightarrow$  rep. ✓

$$\alpha \cdot D(g) : G \rightarrow GL(V)$$

$$\alpha \cdot D(g_1 g_2) = \alpha D(g_1) \cdot D(g_2)$$

$\rightarrow$  not rep. ✓ unless  $\alpha = 1$

$$2. \quad D_i = \sum_{g \in k_i} D(g) \quad , \quad D(g) \in k_i \rightarrow D(g) = D(h g h^{-1})$$

$$\Rightarrow D(g) D(h) = D(h) D(g)$$

Unitarity finite auto unitary?

$$\Rightarrow D(g) = \lambda \mathbb{1}$$

$$\Rightarrow D_i = \sum_{g \in k_i} D(g) = \lambda_i \mathbb{1}_n$$

$D(h)D(g)D(h)^{-1} = \sum D(h)D(g)D(h)^{-1}$   
 $\tilde{g} = hgh^{-1}$   
 $= \sum D(hgh^{-1})$   
 $= \sum D(\tilde{g})$   
 $= D_i$   
 $\rightarrow D_i = \lambda_i \mathbb{1}_n$

H.14

$$1. A_3 = \{e, (123), (132)\}$$

Multiplication  
Table

	$e$	$(123)$	$(132)$	$(132)$	$e$	$(123)$
$e$	$e$	$(123)$	$(132)$	$(132)$	$e$	$(123)$
$(123)$	$(123)$	$e$	$(132)$	$(132)$	$(123)$	$e$
$(132)$	$(132)$	$(132)$	$e$	$(123)$	$(123)$	$(123)$

$A_3$  is abelian  
 $\Rightarrow$  Every element forms its conjugacy class

$$(123) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \quad (132) = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

$$(123) \cdot (132) = \begin{array}{ccc|c} & 1 & 2 & 3 \\ & 2 & 3 & 1 \\ & 3 & 1 & 2 \\ \hline 1 & 2 & 3 & 1 \end{array} = e$$

(In retrospect, there are two elements in  $A_3$  besides  $e$ ,  
they must be inverse to each other)

Conjugacy Classes

$$\{(123), (132)\}$$

Charakters

$$f_{\chi(\sigma)} = \begin{cases} 0, & \sigma \neq p \cdot f \\ \epsilon(p), & \sigma = p \cdot f \end{cases}$$

$$A = \boxed{\phantom{0}}$$

$$S = \boxed{\phantom{0}} \boxed{\phantom{0}} \boxed{\phantom{0}}$$

$$M = \boxed{\phantom{0}} \boxed{\phantom{0}}$$

$$f_A(\sigma) = \epsilon(\sigma)$$

$$f_S(\sigma) = +1$$

$$f_M(123) = \sigma(13) = -1$$

$$f_A((132)) = f_A((123)) = +1$$

$$f_M = (132) = 0$$

$$\rightarrow \tilde{f}_S = \frac{1}{3} \sum f_S = 1 , \quad N_S = 1$$

$$N_Y^2 = \frac{m!}{\sum |f_Y(m)|^2}$$

$$\tilde{f}_A = \frac{1}{3} \sum f_A = 1 , \quad N_S = 1$$

$$\tilde{f}_M = \frac{1}{3} \sum f_M = 0 , \quad N_A = \sqrt{\frac{3}{2}}$$

Character Table

	K
$\tilde{\chi}_S$	1
$\tilde{\chi}_A$	1
$\tilde{\chi}_M$	0

Different from character table of  $S_3$ ,

They are different group ...

$\hat{=}$  Cyclic group  $\mathbb{Z}_3$

$$\rightarrow a^3 = 1 \rightarrow D(g_k) = \exp(2\pi i k/3), \quad k=0, 1, 2$$

From H.4  $\rightarrow$  Irrep. of abelian group is 1-dim

	e	a	$a^2$	Why we have complex number?
$ k $	1	1	1	
$\chi_1$	1	1	1	
$\chi_2$	1	$\exp(2\pi i/3)$	$\exp(4\pi i/3)$	
$\chi_3$	1	$\exp(4\pi i/3)$	$\exp(2\pi i/3)$	



if there is no inverse in conjugacy class,  
the characters can be complex.

$$2. \quad \mathbb{Z}_4 = \{e, a, a^2, a^3\} \quad K_4 = \{e, b, c, bc\}, \quad b^2 = c^2 = e$$

	c	a	$a^2$	$a^3$
e	e	a	$a^2$	$a^3$
a	a	$a^2$	$a^3$	e
$a^2$	$a^2$	$a^3$	e	a
$a^3$	$a^3$	e	a	$a^2$

	e	b	c	bc
e	e	b	c	bc
b	b	e	bc	c
c	c	bc	e	b
bc	bc	c	b	e

Both are abelian

b)  $H \subset G$ , order of  $H = h$   $\rightarrow h$  is a divisor of  $g$   
order of  $G = g$

(Lagrange's theorem)

$\rightarrow 4 = 1 \cdot 4 = 2 \cdot 2$  only two ways

c)  $\mathbb{Z}_4$ :

$$\left. \begin{array}{l} K_1 = [e] = \{e\} \\ K_2 = [a] = \{a\} \\ K_3 = [a^2] = \{a^2\} \\ K_4 = [a^3] = \{a^3\} \end{array} \right\} \text{Result of being abelian}$$

$K_4$ :

$$\left. \begin{array}{l} K_1 = [e] = \{e\} \\ K_2 = [b] = \{b\} \\ K_3 = [c] = \{c\} \\ K_4 = [bc] = \{bc\} \end{array} \right\}$$

Abelian group  
 $\rightarrow$  conjugacy classes  
only one element

mr	$k_1$	$k_2$	$k_3$	$k_4$
A	1	1	1	1
B	1	$\textcolor{red}{1+i}$	-1	$\textcolor{red}{-1-i}$
C	1	-1	1	-1
D	1	$\textcolor{red}{-1-i}$	-1	$\textcolor{red}{1+i}$

mr	$k_1$	$k_2$	$k_3$	$k_4$
A	1	1	1	1
B	1	1	-1	-1
C	1	-1	1	-1
D	1	-1	-1	1

$\swarrow$  self inverse elements in  $K_4$   
 $K$  must be real

$$\begin{aligned} & \langle K^1, K^1 \rangle \\ &= \langle K^A, K^A \rangle \\ &= 4 \end{aligned}$$

$$\langle K_A, K_B \rangle = 0 \Rightarrow \sum (B, K_{2,3,4}) = -1$$

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$$k_4: a^4 = 1$$

$$\exp\left(\frac{2\pi i k}{4}\right), \quad k=0, 1, 2, 3$$

We can see the above configuration  
satisfies orthogonality.  
(not unique?)

H. 15

$$1. \quad K_{Y^t}(\sigma) = \underbrace{N_{Y^t}}_{N_Y} \tilde{f}_{Y^t}(\sigma) = N_Y \cdot \begin{cases} 0 & , \sigma \neq p \cdot q \\ \varepsilon(q) & , \sigma = p \cdot q \end{cases}$$

$$\varepsilon(q) = \underline{\varepsilon(p \cdot p \cdot q)} = \varepsilon(p) \cdot \varepsilon(q)$$

$$\rightarrow K_{Y^t}(\sigma) = \varepsilon(\sigma) \cdot K_Y(\sigma)$$

$$1. \quad \varepsilon(\sigma) \tilde{f}_Y(\sigma) = \varepsilon(\sigma) \sum_{\tau \in S_m} f_Y(\tau \sigma \tau^{-1})$$

$\varepsilon$  is rep.  
↓  
 $\varepsilon(\sigma) = \varepsilon(\tau \sigma \tau^{-1})$

$$= \sum_{\tau \in S_m} \varepsilon(\sigma) f_Y(\tau \sigma \tau^{-1})$$

$$= \sum_{\tau \in S_m} \varepsilon(\tau \sigma \tau^{-1}) f_Y(\tau \sigma \tau^{-1})$$

$$\varepsilon(g) f_Y(g) = \begin{cases} 0 & \text{if } g \neq p \cdot q \\ \varepsilon(g) \varepsilon(p) = \varepsilon(p \cdot q) \varepsilon(q) & \text{if } g = p \cdot q \end{cases}$$

$$= \varepsilon^2(p) \varepsilon(q)$$

$$= \varepsilon(q)$$

$$= f_{Y^t}(g), \quad N_{Y^t} = N_Y$$

$$\rightarrow K_{Y^t}(\sigma) = \varepsilon(\sigma) \cdot K_Y(\sigma)$$

$$2. \quad \text{compact Lie-group:} \quad n_e = \frac{1}{V(G_1)} \int dg \overline{K_{e_1}(g)} \cdot K_{e_1}(g) K_{e_2}(g)$$

$$\text{finite group } S_m: \quad n_e = \frac{1}{m!} \sum_{\sigma \in S_m} \overline{K_Y(\sigma)} K_{Y_1}(\sigma) K_{Y_2}(\sigma)$$

$$\text{Symm: } K_{Y_s}(\sigma) = 1 \quad \forall \sigma \in S_m$$

$$n_{Y_s} = \frac{1}{m!} \sum_{\sigma \in S_m} K_{Y_1}(\sigma) K_{Y_2}(\sigma) = \delta_{Y_1, Y_2}$$

$$\text{Anti-symm: } K_{Y_A}(\sigma) = \varepsilon(\sigma) \quad \forall \sigma \in S_m$$

$$n_{Y_A} = \frac{1}{m!} \sum \varepsilon(\sigma) K_{Y_1}(\sigma) K_{Y_2}(\sigma)$$

$$\underbrace{k_{Y_1^+}(s)}_{= \frac{1}{m!} \sum K_{Y_1^+} K_{Y_2}^-} = \delta_{Y_1^+ Y_2^-}$$