H16.

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} b \\ c \\ a \end{pmatrix} \to (123)$$

1. $D: S_3 \to \mathbb{R}^3 \times \mathbb{R}^3$

$D(e) = \mathbb{1}_3$

$D[(123)] = Q = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

$(132) = Q^2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

$(12) = P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$(13) = QP = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

$(23) = PQ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

2. $x = \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} \in V \subset \mathbb{R}^3$ $\times$

invariant subspace $\Leftrightarrow$ reducible

take $\lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\lambda \in \mathbb{R}$

$Q\,x = \begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} x_2 \\ 0 \\ x_1 \end{pmatrix} \notin V \to$ reducible

3. $D = \varepsilon(\sigma)$ $\times$

$e = 1, \quad Q = 1, \quad P = -1$

$tr(D(e)) = 3$

$tr(D(Q)) = 0 = tr(D(***))$

$tr(D(P)) = 1 = tr(D(**))$

	e	2-cyc	3-cyc
χ_S	1	1	1
χ_A	1	-1	1
χ_M	2	0	-1

$\chi = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}}_{\chi_M} + \underbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}_{\chi_S}$

It can be split into two irreps

What does it look like? $\to$ 4

$\vec{R}$ acted on only trivial irrep.

$\vec{\rho}$ and $\vec{\lambda}$ (forms orthogonal subspace) acted on standard irrep.

(Typo on the sheet $D_{21} \longleftrightarrow D_{12}$?)

4. $Q\vec{\rho} = D(Q)_{11}\vec{\rho} + D(Q)_{21}\vec{\lambda}$, $Q = (1\,2\,3)$

$\vec{\rho} = \frac{1}{\sqrt{2}}(\vec{x}_1 - \vec{x}_2)$, $\vec{\lambda} = \frac{1}{\sqrt{6}}(\vec{x}_1 + \vec{x}_2 - 2\vec{x}_3)$

$$Q\vec{\rho} = \cancel{\tfrac{1}{\sqrt{2}}}(\vec{x}_2 - \vec{x}_3) \overset{!}{=} D(Q)_{11}\cancel{\tfrac{1}{\sqrt{2}}}(\vec{x}_1 - \vec{x}_2) + D(Q)_{21}\underset{\frac{1}{\sqrt{3}}}{\cancel{\tfrac{1}{\sqrt{6}}}}(\vec{x}_1 + \vec{x}_2 - 2\vec{x}_3)$$

$D(Q)_{21} = \frac{\sqrt{3}}{2}$, $D(Q)_{11} = -\frac{1}{2}$

$$Q\vec{\lambda} = \cancel{\tfrac{1}{\sqrt{6}}}(\vec{x}_2 + \vec{x}_3 - 2\vec{x}_1) \overset{!}{=} D(Q)_{12}\underset{\sqrt{3}}{\cancel{\tfrac{1}{\sqrt{2}}}}(\vec{x}_1 - \vec{x}_2) + D(Q)_{22}\cancel{\tfrac{1}{\sqrt{6}}}(\vec{x}_1 + \vec{x}_2 - 2\vec{x}_3)$$

$D(Q)_{22} = -\frac{1}{2}$ $\qquad$ $D(Q)_{12} = -\frac{1}{\sqrt{3}}\,\frac{3}{2} = -\frac{\sqrt{3}}{2}$

$$\Rightarrow D(Q) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \checkmark$$

$$D(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$P = (1\,2)$

$$P\vec{\rho} = \cancel{\tfrac{1}{\sqrt{2}}}(\vec{x}_2 - \vec{x}_1) \overset{!}{=} D(P)_{11}\cancel{\tfrac{1}{\sqrt{2}}}(\vec{x}_1 - \vec{x}_2) + D(P)_{21}\underset{\frac{1}{\sqrt{3}}}{\cancel{\tfrac{1}{\sqrt{6}}}}(\vec{x}_1 - \vec{x}_2 - 2\vec{x}_3)$$

$D(P)_{21} = 0$ $\quad$ no $\vec{x}_3$ on LHS

$D(P)_{11} = -1$

$$P\vec{\lambda} = \frac{1}{\sqrt{6}}(\vec{x}_1 + \vec{x}_2 - 2\vec{x}_3) \overset{!}{=} D(P)_{21}\frac{1}{\sqrt{2}}(\vec{x}_1 - \vec{x}_2) + D(P)_{22}\frac{1}{\sqrt{6}}(\cdots)$$

$D(P)_{22} = 1$, $D(P)_{21} = 0$

$$\rightarrow D(P) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \checkmark$$

$Q^2 = (132)$: $\begin{pmatrix}1 & 2 & 3\\ 3 & 1 & 2\end{pmatrix}$

$$Q^2\vec{\rho} = \cancel{\tfrac{1}{\sqrt{2}}}(\vec{x}_3 - \vec{x}_1) \overset{!}{=} D(Q^2)_{11}\cancel{\tfrac{1}{\sqrt{2}}}(\vec{x}_1-\vec{x}_2) + D(Q^2)_{21}\underset{\frac{1}{\sqrt{3}}}{\cancel{\tfrac{1}{\sqrt{6}}}}(\vec{x}_1+\vec{x}_2-2\vec{x}_3)$$

$$\rightarrow D(Q^2)_{21} = -\frac{\sqrt{3}}{2} \qquad D(Q^2)_{11} = -\frac{1}{2}$$

$$Q^2\vec{\lambda} = \cancel{\tfrac{1}{\sqrt{6}}}(\vec{x}_3+\vec{x}_1-2\vec{x}_2) \overset{!}{=} D(Q^2)_{12}\underset{\sqrt{3}}{\cancel{\tfrac{1}{\sqrt{2}}}}(\vec{x}_1-\vec{x}_2) + D(Q^2)_{22}\cancel{\tfrac{1}{\sqrt{6}}}(\vec{x}_1+\vec{x}_2-2\vec{x}_3)$$

$$D_{22} = -\frac{1}{2} \qquad D_{12} = \frac{3}{\sqrt{3}}\,\frac{1}{2} = \frac{\sqrt{3}}{2}$$

$$\rightarrow D(Q^2) = \begin{pmatrix}-\frac{1}{2} & \frac{\sqrt{3}}{2}\\ -\frac{\sqrt{3}}{2} & -\frac{1}{2}\end{pmatrix} = [D(Q)]^2$$ ✓

$QP = (13)$

$$QP\vec{\rho} = \cancel{\tfrac{1}{\sqrt{2}}}(\vec{x}_3 - \vec{x}_2) \overset{!}{=} D(QP)_{11}\cancel{\tfrac{1}{\sqrt{2}}}(\vec{x}_1-\vec{x}_2) + D(QP)_{21}\overset{\frac{1}{\sqrt{3}}}{\cancel{\tfrac{1}{\sqrt{6}}}}(\vec{x}_1+\vec{x}_2-2\vec{x}_3)$$

$$D_{21} = -\frac{\sqrt{3}}{2} \qquad D_{11} = \frac{1}{2}$$

$$QP\vec{\lambda} = \cancel{\tfrac{1}{\sqrt{6}}}(\vec{x}_3+\vec{x}_2-2\vec{x}_1) \overset{!}{=} D(QP)_{12}\underset{\sqrt{3}}{\cancel{\tfrac{1}{\sqrt{2}}}}(\vec{x}_1-\vec{x}_2) + D(QP)_{22}\cancel{\tfrac{1}{\sqrt{6}}}(\vec{x}_1+\vec{x}_2-2\vec{x}_3)$$

$$D(QP)_{22} = -\frac{1}{2} \qquad D(QP)_{12} = -\frac{\sqrt{3}}{2}$$

$$\rightarrow D(QP) = \begin{pmatrix}\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ -\frac{\sqrt{3}}{2} & -\frac{1}{2}\end{pmatrix} = D(Q)\cdot D(P)$$ ✓

$PQ = (23)$

$$PQ\vec{\rho} = \cancel{\tfrac{1}{\sqrt{2}}}(\vec{x}_1 - \vec{x}_3) = D(PQ)_{11}\cancel{\tfrac{1}{\sqrt{2}}}(\vec{x}_1-\vec{x}_2) + D(PQ)_{21}\underset{\frac{1}{\sqrt{3}}}{\cancel{\tfrac{1}{\sqrt{6}}}}(\vec{x}_1+\vec{x}_2-2\vec{x}_3)$$

$$D(PQ)_{21} = \frac{\sqrt{3}}{2}, \qquad D(PQ)_{11} = \frac{1}{2}$$

$$P Q \vec{\lambda} = \frac{1}{\sqrt{6}}(\vec{x}_1 + \vec{x}_3 - 2\vec{x}_1) = D(PQ)_{12} \frac{1}{\sqrt{2}}(\vec{x}_1 - \vec{x}_2) + D(PQ)_{22} \frac{1}{\sqrt{6}}(\vec{x}_1 + \vec{x}_2 - 2\vec{x}_3)$$

$$D_{22} = -\frac{1}{2} \quad , \quad D_{12} = \frac{3}{2} \cdot \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{2}$$

$$\to D(PQ) = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} = D(P) \cdot D(Q) \quad \checkmark$$

It is a representation!

5. $tr[D(P)] = 0$, $\quad tr[D(P)D(Q)] = tr[D(Q)D(P)] = 0$

$tr[D(Q)] = -1 \quad tr[D(Q^2)] = -1$

$tr[D(e)] = 2$

$\hookrightarrow$ mixed representation $\to \chi_M$

6. $D_\mu(\sigma)\Phi(\vec{\rho}, \vec{\lambda}) = \Phi(\sigma\vec{\rho}, \sigma\vec{\lambda})$

$\Phi_{00}(\vec{\rho}, \vec{\lambda}) = \varphi_0(\vec{\rho})\varphi_0(\vec{\lambda})$

$\varphi_0(\vec{x}) = N_0 \exp(-\frac{1}{2}|\vec{x}|^2/b^2)$

$\varphi_0(\vec{\rho})\varphi_0(\vec{\lambda}) \propto \exp(-\frac{1}{2}(|\vec{\rho}|^2 + |\vec{\lambda}|^2))$

$P(|\vec{\rho}|^2 + |\vec{\lambda}|^2) = (|D((12))\vec{\rho}|^2 + |D((12))\vec{\lambda}|^2) = |\vec{\rho}|^2 + |\vec{\lambda}|^2$

$Q(|\vec{\rho}|^2 + |\vec{\lambda}|^2) = \cdots = (|-\frac{1}{2}\vec{\rho} + \frac{\sqrt{3}}{2}\vec{\lambda}|^2 + |-\frac{1}{2}\sqrt{3}\vec{\rho} - \frac{1}{2}\vec{\lambda}|^2)$

$= (|\vec{\rho}|^2 + |\vec{\lambda}|^2)$

alle other permutations are generated by PQ

$\to \sigma \in S_3$, $\quad \sigma(|\vec{\rho}|^2 + |\vec{\lambda}|^2) = (|\vec{\rho}|^2 + |\vec{\lambda}|^2)$

$\to$ trivial rep. $\Leftrightarrow$ symmetric rep.

Φ_{00} is a basis of

$\sigma_\mu(\sigma)\Phi_{00}(\vec{\rho}, \vec{\lambda}) = \Phi_{00}(\vec{\rho}, \vec{\lambda})$

7. $\Phi_{i0}(\vec{\rho},\vec{\lambda}) = \varphi_i(\vec{\rho})\,\varphi_0(\vec{\lambda}) \propto \rho^i \exp[-\frac{1}{2}(|\vec{\rho}|^2 + |\vec{\lambda}|^2)]$

$\Phi_{0i}(\vec{\rho},\vec{\lambda}) = \varphi_0(\vec{\rho})\,\varphi_i(\vec{\lambda}) \propto \lambda^i \exp[-\frac{1}{2}(|\vec{\rho}|^2 + |\vec{\lambda}|^2)]$

$\rightarrow \begin{pmatrix} \vec{\rho} \\ \vec{\lambda} \end{pmatrix}$, mixed rep.

H17

a) $$\left.\begin{aligned} K^2_{Y_A}(\sigma) &= \epsilon^2(\sigma) \\ K^2_{Y_S}(\sigma) &= 1^2 \end{aligned}\right\} = 1 \qquad K_{Y_S}(\sigma)\, K_{Y_A} = \epsilon(\sigma)\cdot 1 = K_{Y_A}(\sigma)$$

$$K_{Y_S}(\sigma)\, K_{Y_M}(\sigma) = 1 \cdot \begin{cases} 2, & \sigma = e \\ -1, & \sigma = Q, Q^2 \\ 0, & \sigma = P_{ik} \end{cases} = K_{Y_M}(\sigma)$$

$$K_{Y_A}(\sigma)\, K_{Y_M}(\sigma) = \epsilon(\sigma) \cdot \begin{cases} 2, & \ldots \\ -1, & \ldots \\ 0, & \ldots \end{cases}$$

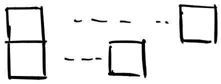
$$= \begin{cases} (-1)^0 \cdot 2, & \ldots \\ (-1)^2 \cdot (-1), & \ldots \\ (-1)^1 \cdot 0, & \ldots \end{cases} = \begin{cases} 2 \\ -1 \\ 0 \end{cases} = K_{Y_M}(\sigma)$$

$$K^2_{Y_M}(\sigma) = \begin{cases} 2^2, & \sigma = e & = \overset{1}{K_{Y_S}(e)} + \overset{1}{K_{Y_A}(e)} + \overset{2}{K_{Y_M}(e)} \\ (-1)^2, & \sigma = Q, Q^2 & = \overset{1}{K_{Y_S}(Q^{(2)})} + \overset{1}{K_{Y_A}(Q^{(2)})} + \overset{-1}{K_{Y_M}(Q^{(2)})} \\ 0^2, & \sigma = P_{ik} & = \overset{1}{K_{Y_S}(P_{ik})} + \overset{-1}{K_{Y_A}(P_{ik})} + \overset{0}{K_{Y_M}(P_{ik})} \end{cases}$$

$$= K_{Y_S} + K_{Y_A} + K_{Y_M}$$

2. $$\phi \in \mathcal{L}^2\left((\mathbb{R}^3)^2, \mathbb{C}^3_C \otimes \mathbb{C}^2_S \otimes \mathbb{C}^2_I\right)$$

$$\underset{\overset{\uparrow}{A}}{|\text{baryon}\rangle} = \underset{\overset{\uparrow}{A}}{|\text{color}\rangle} \otimes \underbrace{|\text{space}\rangle \otimes |\text{spin}\rangle \otimes |\text{isospin}\rangle}_{S}$$

3. $Y =$  $\begin{matrix} l_2 \\ l_1 \end{matrix}$ $\qquad l_2 \geq l_1$

To ~~show~~ $D_Y \simeq D_j, \quad j = \frac{1}{2}(l_2 - l_1)$

to show: $l_2 \geq l_1$, irrep. D_j where $2j = l_2 - l_1$

From H.9: $\chi_j = \dfrac{\lambda^{2j+1} - \bar{\lambda}^{2j+1}}{\lambda - \bar{\lambda}} = \dfrac{\sin[(2j+1)\varphi]}{\sin(\varphi)}, \quad \lambda = e^{i\varphi}$

$g \in SU(2), \quad z_2 - z_1 = 2j+1$

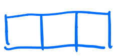
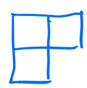
$z_1 = l_1 + 1 - 1 = l_1, \quad z_2 = l_2 + 2 - 1 = l_2 + 1$

$z_2 - z_1 = l_2 + 1 - l_1 \overset{!}{=} 2j + 1 \quad \Rightarrow l_2 - l_1 = 2j$

to show: $V = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2, \quad Y = Y_S, j = \frac{3}{2}, \quad Y = Y_M, j = \frac{1}{2}$

The dim. of the representation is $2j+1$ corresponding to states with spin $j = \frac{1}{2}(l_2 - l_1)$

↳ Only Young diag. with 3 boxes and a max. column length of 2

↳ $Y_S =$  $\qquad Y_M =$ 

$l_2 = 3, \ l_1 = 0 \qquad\qquad l_2 = 2, \ l_1 = 1$

$\rightarrow j = \frac{3}{2} \qquad\qquad \rightarrow j = \frac{1}{2}$

in SU(2): $\underbrace{\Delta^-, \Delta^0, \Delta^+, \Delta^{++},}_{16}$ (consider z-comp. of spin) $\qquad \underbrace{n, p}_{4}$

4. $|\text{color}\rangle \rightarrow$ anti-symm.

$\underset{\substack{\uparrow \\ \text{symm.}}}{|\text{space}\rangle} \otimes \underbrace{|\text{spin}\rangle \otimes |\text{isospin}\rangle}_{\text{symm.}}$

$\rightarrow$ $|\text{spin}\rangle : Y_S \qquad |\text{isospin}\rangle : Y_S \quad \Big\} \rightarrow \Delta$'s

$|\text{spin}\rangle : Y_M \qquad |\text{isospin}\rangle : Y_M \qquad \rightarrow n, p$

Only spin $\frac{1}{2}$ $\quad |\text{spin}\rangle : Y_A \quad$ not possible $\qquad |\text{isospin}\rangle : Y_A \quad$ not possible because of SU(2) only two quarks!

5. $|\text{space}\rangle$: Y_M $\quad (L = 1) \quad$ parity: (-1)

$$\chi_{Y_S} = \chi_{Y_M} \cdot \underline{\chi_{Y_A}}$$

$\lvert\text{spin}\rangle$:	$\lvert\text{isospin}\rangle$:	
Y_M	Y_M ,	$S = \frac{1}{2}$, $I = \frac{1}{2}$
Y_S	Y_M ,	$S = \frac{3}{2}$, $I = \frac{1}{2}$
Y_M	Y_S ,	$S = \frac{1}{2}$, $I = \frac{3}{2}$

- $S = \frac{3}{2}$, $I = \frac{1}{2}$, $\quad J = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$

  $$I(J^\pi) = \tfrac{1}{2}\left(\tfrac{1}{2}^-\right), \tfrac{1}{2}\left(\tfrac{3}{2}^-\right), \tfrac{1}{2}\left(\tfrac{5}{2}^-\right)$$

- $S = \frac{1}{2}$, $I = \frac{3}{2}$, $\quad J = \frac{1}{2}, \frac{3}{2}$

  $$I(J^\pi) = \tfrac{3}{2}\left(\tfrac{1}{2}^-\right), \tfrac{3}{2}\left(\tfrac{3}{2}^-\right)$$

- $S = \frac{1}{2}$, $I = \frac{1}{2}$, $\quad J = \frac{1}{2}, \frac{3}{2}$

  $$I(J^\pi) = \tfrac{1}{2}\left(\tfrac{1}{2}^-\right), \tfrac{1}{2}\left(\tfrac{3}{2}^-\right)$$

symmetric (in spin part) has higher mass.