H.7

1. $g \in SU(2)$. $\det(g) = 1$, $g g^* = \mathbb{1}$,

invertible $\det = 1$

$\boxed{u \in SU(2)}$ Suppose $u^{-1} g u = \mathrm{diag}(\lambda, \bar{\lambda})$ why does this hold?

$$\Rightarrow \det(u^{-1} g u) = \det(g) \underbrace{\det(u^{-1}) \det(u)}_{1} = \lambda \bar{\lambda} = |\lambda| = 1$$

$$g \to A^{-1} g A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = D, \quad A, g \in SU(2)$$

$$D^* D = (A^{-1} g A)^* (A^{-1} g A) = \mathbb{1}$$

$$\mathbb{1} = \begin{pmatrix} \bar{\lambda}_1 & \\ & \bar{\lambda}_2 \end{pmatrix} \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix} = \begin{pmatrix} |\lambda_1|^2 & \\ & |\lambda_2|^2 \end{pmatrix}$$

$$\Rightarrow |\lambda_1|^2 = |\lambda_2|^2 = 1$$

$$\det(D) = \lambda_1 \lambda_2 = 1$$

$$\underbrace{|\lambda_1|^2}_{1} \lambda_2 = \bar{\lambda}_1$$

$$\Rightarrow \bar{\lambda}_1 = \lambda_2$$

2. $$\chi_j(g) = tr[D_j(g)]$$
$$= tr[D_j(u g_0 u^{-1})]$$
$$= tr[D_j(u) D_j(g_0) D_j(u^{-1})]$$
$$= tr[D_j(u^{-1} u) D_j(g_0)]$$
$$= tr[D_j g_0]$$
$$= \chi_j(g_0)$$

3. $$\chi_j(g) = \chi_j(g_0) = tr[D_j(g_0)] = \sum_{m=-j}^{j} D^{mm}_{jj}(g_0)$$
$$= \sum_{m=-j}^{j} \langle P^m_j, D^j(g_0) P^m_j \rangle \to P^m_j \left(\begin{pmatrix} \bar{\lambda} & \\ & \lambda \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right)$$
$$= \sum_{m=-j}^{j} \langle D^j(g_0^{-1}) P^m_j, P^m_j \rangle$$

$$= \sum_{m=-j}^{j} \langle P_j^m(g_0 z), P_j^m(z) \rangle$$

$$= \sum_{m=-j}^{j} \frac{1}{\pi^2} \int d^4 z \; \bar{P}_j^m(g_0 z) P_j^m(z) \; e^{-|z|^2}$$

the exchange of $\bar{\lambda}, \lambda \hookrightarrow$ is not important, since we haven't impose any condition on which is which

$$= \sum_{m=-j}^{j} \frac{1}{\pi^2} \int d^4 z \; \bar{P}_j^m(z) \; \lambda^{j+m} \; \bar{\lambda}^{j-m} P_j^m(z) e^{-|z|^2}$$

$$\left[g_0 z = \begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \lambda z_1 \\ \bar{\lambda} z_2 \end{pmatrix} \right]$$

$$= \sum_{m=-j}^{j} \lambda^{j+m} \bar{\lambda}^{j-m} \underbrace{\langle P_j^m, P_j^m \rangle}_{=1}$$

$$\stackrel{?}{=} \frac{\lambda^{2j+1} - \bar{\lambda}^{2j+1}}{\lambda - \bar{\lambda}}$$

$$j = 1, \quad LHS = \sum_{m=-1}^{1} \lambda^{1+m} \bar{\lambda}^{1-m} = \lambda^0 \bar{\lambda}^2 + \lambda^1 \bar{\lambda}^1 + \lambda^2 \bar{\lambda}^0$$

$$= \bar{\lambda}^2 + \lambda\bar{\lambda} + \lambda^2$$

$$RHS = \frac{\lambda^3 - \bar{\lambda}^3}{\lambda - \bar{\lambda}}$$

$$\lambda^3 - 3\lambda^2\bar{\lambda} + 3\lambda\bar{\lambda}^2 - \bar{\lambda}^3 + 3\lambda^2\bar{\lambda} - 3\lambda\bar{\lambda}^2$$

$$= (\lambda - \bar{\lambda})^3 + 3\lambda\bar{\lambda}(\lambda - \bar{\lambda})$$

$$= (\lambda - \bar{\lambda}) \underbrace{((\lambda - \bar{\lambda})^2 + 3\lambda\bar{\lambda})}_{= \lambda^2 + \lambda\bar{\lambda} + \bar{\lambda}^2}$$

$$= \lambda^2 + \lambda\bar{\lambda} + \bar{\lambda}^2$$

$$= LHS$$

Valid for j $\rightarrow \displaystyle\sum_{m=-j}^{j} \lambda^{j+m} \bar{\lambda}^{j-m} = \frac{\lambda^{2j+1} - \bar{\lambda}^{2j+1}}{\lambda - \bar{\lambda}}$

Check for $j+1$

$$LHS = \sum_{m=-j-1}^{j+1} \lambda^{j+1+m} \bar{\lambda}^{j+1-m}$$

$$= \lambda\bar{\lambda} \sum_{m=-j-1}^{j+1} \lambda^{j+m} \bar{\lambda}^{j-m}$$

$$= \lambda\bar{\lambda} \sum_{m=-j}^{j} \lambda^{j+m} \bar{\lambda}^{j-m} + \bar{\lambda}^{2j+2} + \lambda^{2j+2}$$

$$= \lambda\bar{\lambda} \frac{\lambda^{2j+1} - \bar{\lambda}^{2j+1}}{\lambda - \bar{\lambda}} + \underbrace{\bar{\lambda}^{2j+2} + \lambda^{2j+2}}$$

$$= \frac{(\bar{\lambda}^{2j+2} + \lambda^{2j+2})(\lambda - \bar{\lambda})}{\lambda - \bar{\lambda}}$$

$$= \frac{\lambda \bar{\lambda}^{2j+2} - \bar{\lambda}^{2j+3} + \lambda^{2j+3} - \lambda^{2j+2}\bar{\lambda}}{\lambda - \bar{\lambda}}$$

$$= \frac{1}{\lambda - \bar{\lambda}} \left(\cancel{\lambda^{2j+2}\bar{\lambda}} \cancel{- \lambda \bar{\lambda}^{2j+2}} \cancel{+ \lambda \bar{\lambda}^{2j+2}} - \bar{\lambda}^{2j+3} + \lambda^{2j+3} \cancel{- \lambda^{2j+2}\bar{\lambda}} \right)$$

$$= \frac{\lambda^{2j+3} - \bar{\lambda}^{2j+3}}{\lambda - \bar{\lambda}}$$

$\Rightarrow$ Statement is correct

$$\Rightarrow \sum_{m=-j}^{j} \lambda^{j+m} \bar{\lambda}^{j-m} = \frac{\lambda^{2j+1} - \bar{\lambda}^{2j+1}}{\lambda - \bar{\lambda}}$$

Here no property / relation of $\lambda, \bar{\lambda}$ is used

$\Rightarrow$ generalized to any $\lambda \to x$, $\bar{\lambda} \to y$

Alternative:

use $$\sum_{k=0}^{m} q^k = \frac{1 - q^{m+1}}{1 - q}$$

$$\sum_{m=-j}^{j} \lambda^{j+m} \bar{\lambda}^{j-m}$$

$$= \sum_{m=0}^{j} \lambda^{j+m} \bar{\lambda}^{j-m} + \sum_{m=1}^{j} \lambda^{j-m} \bar{\lambda}^{j+m}$$

$$= (\lambda\bar{\lambda})^{j} \left[\frac{1-(\lambda/\bar{\lambda})^{j+1}}{1-(\lambda/\bar{\lambda})} + \frac{1-(\bar{\lambda}/\lambda)^{j+1}}{1-(\bar{\lambda}/\lambda)} - 1 \right]$$

$$= \frac{(\lambda\bar{\lambda})^{j}\bar{\lambda} - \lambda^{2j+1}}{\bar{\lambda}-\lambda} + \frac{\bar{\lambda}^{2j+1} - (\lambda\bar{\lambda})^{j}\lambda}{\bar{\lambda}-\lambda} - (\lambda\bar{\lambda})^{j}$$

$$= \frac{\lambda^{2j+1} - \bar{\lambda}^{2j+1}}{\lambda - \bar{\lambda}} + \underbrace{\frac{\lambda^{j}\bar{\lambda}^{j+1} - \lambda^{j+1}\bar{\lambda}^{j} - (\lambda\bar{\lambda})^{j}(\bar{\lambda}-\lambda)}{\bar{\lambda}-\lambda}}_{=0}$$

4. $$k_j(e) = \sum_{m=-j}^{j} (1)^{j+m} (1)^{j-m} = 2j+1$$

$$k_j(e) = \sum_m \langle P_j^m, D(e) P_j^m \rangle = \mathrm{Dim}\, V_j$$

5. $$g = \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}, \quad |\alpha|^2 + |\beta|^2 = 1 \iff \det(g) = 1$$

$$\det\begin{pmatrix} \alpha-\lambda & -\bar{\beta} \\ \beta & \bar{\alpha}-\lambda \end{pmatrix} = (\alpha-\lambda)(\bar{\alpha}-\lambda) + \bar{\beta}\beta = 1 + \lambda^2 - (\alpha+\bar{\alpha})\lambda = 0$$

$$\Rightarrow \lambda^2 - (\alpha+\bar{\alpha})\lambda + \tfrac{1}{4}(\alpha+\bar{\alpha})^2 = \tfrac{1}{4}(\alpha+\bar{\alpha})^2 - 1$$

$$\Rightarrow \lambda_{1,2} = \pm\frac{\sqrt{(\alpha+\bar{\alpha})^2-4}}{2} + \frac{(\alpha+\bar{\alpha})}{2}$$

$$\lambda_{1,2} = \lambda, \bar{\lambda}$$

?

$$\mathrm{tr}[g] = \alpha + \bar{\alpha} = \mathrm{tr}[g_0] = \lambda + \bar{\lambda}$$

$$k_j(\lambda) = \frac{\left[\frac{1}{2}\left(\sqrt{(\alpha+\bar{\alpha})^2-4} + (\alpha+\bar{\alpha})\right)\right]^{2j+1} - \left[\frac{1}{2}\left(-\sqrt{(\alpha+\bar{\alpha})^2-4} + (\alpha+\bar{\alpha})\right)\right]^{2j+1}}{\sqrt{(\alpha+\bar{\alpha})^2-4}}$$

$$= \frac{1}{2^{2j+1}} \frac{\left(\sqrt{(\alpha+\bar\alpha)^2-4} + (\alpha+\bar\alpha)\right)^{2j+1} \left[-\sqrt{(\alpha+\bar\alpha)^2-4} + (\alpha+\bar\alpha)\right]^{2j+1}}{\sqrt{(\alpha+\bar\alpha)^2-4}}$$

$$\sum_{k=0}^{2j+1} \binom{2j+1}{k} \left\{ \left(\sqrt{(\alpha+\bar\alpha)^2-4}\right)^k (\alpha+\bar\alpha)^{2j+1-k} + (-1)^k \sqrt{(\alpha+\bar\alpha)^2-4}^{\,k} (\alpha+\bar\alpha)^{2j+1-k} \right\}$$

for $k \in 2\mathbb{Z}+1$, the summand vanishes

$$= \frac{1}{2^{2j+1}\sqrt{(\alpha+\bar\alpha)^2-4}} \sum_{\substack{k=0 \\ k\in 2\mathbb{Z}}}^{2j} 2 \binom{2j+1}{k} \left(\sqrt{(\alpha+\bar\alpha)^2-4}\right)^k (\alpha+\bar\alpha)^{2j+1-k}$$

$$\longrightarrow \binom{2j+1}{0} (\alpha+\bar\alpha)^{2j+1} + \binom{2j+1}{2} \sqrt{(\alpha+\bar\alpha)^2-4}^{\,2} (\alpha+\bar\alpha)^{2j+1-2}$$

$$+ \cdots \binom{2j+1}{2j} (\quad)^{2j} (\quad)^1$$

$$= (2j+1)\left[\binom{2j}{0} A^0 B^{2j+1} + \binom{2j}{2} A^2 B^{2j-1} + \cdots \binom{2j}{2j} A^{2j} B \right]$$

$$= (2j+1) B \underbrace{\left[\binom{2j}{0} A^0 B^{2j} + \binom{2j}{2} A^2 B^{2j-2} + \cdots \right]}$$

$$= \frac{1}{2^{2j} A} (2j+1) B \sum_{k=0}^{j} \binom{2j}{2k} A^{2k} B^{2j-2k}$$

$\det(g - \mathbb{1}\lambda) \Rightarrow \lambda^2 - 2\lambda \,\mathrm{Re}(\alpha) + 1 \overset{!}{=} 0$

$\Rightarrow \lambda = \mathrm{Re}(\alpha) \pm i\sqrt{-\mathrm{Re}^2(\alpha) + 1}$

$\Rightarrow$ if $\mathrm{Re}(\alpha) = \pm 1$, $\quad K_j = \dim V_j$

if $Re(\alpha) \neq \pm 1$, $\quad k_j = \dfrac{\left(Re(\alpha) + i\sqrt{1-Re^2(\alpha)}\right)^{2j+1} - \left(Re(\alpha) - i\sqrt{1-Re^2(\alpha)}\right)^{2j+1}}{2i\sqrt{1-Re(\alpha)}}$

6. $\alpha = \cos\psi - i\sin\psi\cos\theta \quad \Rightarrow \quad \alpha + \bar{\alpha} = 2\cos\psi$

$\beta = \sin\psi\sin\theta(\cos\varphi + i\sin\varphi)$

$\hookrightarrow \quad \lambda_{1,2} = \cos\psi \pm i\sin\psi = e^{\pm i\psi}$

$A = \sqrt{(\alpha+\bar{\alpha})^2 - 4} = i\sin\psi$

$B = \alpha + \bar{\alpha} = 2\cos\psi$

$$\hookrightarrow = \frac{1}{2^{2j}A}(2j+1)B\sum_{k=0}^{j}\binom{2j}{2k}\underbrace{A^{2k}}_{-\sin^k\psi}\underbrace{B^{2j-2k}}_{4\cos^{j-k}\psi}$$

$$k_j(g) = \frac{e^{i(2j+1)\psi} - e^{-i(2j+1)\psi}}{e^{i\psi} - e^{-i\psi}} = \frac{\cancel{2i}\sin(2j+1)\psi}{\cancel{2i}\sin\psi}$$

7. $$g_{kl} = 2\begin{pmatrix} 1 & & \\ & \sin^2\psi & \\ & & \sin^2\psi\sin^2\theta \end{pmatrix}$$

$$dg = \sqrt{\det(g_{kl})}\, d\psi\, d\theta\, d\varphi$$

$\det(g_{kl}) = 2^3\sin^4\psi\sin^2\theta \quad \Rightarrow \quad \sqrt{\det(g)} = \sqrt{8}\sin^2\psi\sin\theta$

$\Rightarrow \quad dg = \sqrt{8}\sin^2\psi\, d\psi\, \sin\theta\, d\theta\, d\varphi$

$$\alpha = x_0^1 + i x_0^2, \qquad \beta = x_0^3 + i x_0^4$$

$$X_0 = x_0^1 e_1 + x_0^2 e_2 + x_0^3 e_3 + x_0^4 e_4, \qquad \{e_i\} \text{ Basis of } \mathbb{R}^4$$

$$g_{kl} = \mathrm{Re}\left\{ \underbrace{\mathrm{tr}\left[\left(\frac{\partial}{\partial x^k} g^*\right)\left(\frac{\partial}{\partial x^l} g\right)\right]}_{=:B} \right\}, \qquad g = \begin{pmatrix} \alpha & -\bar\beta \\ \beta & \bar\alpha \end{pmatrix}$$

$$g^* = \begin{pmatrix} \bar\alpha & \bar\beta \\ -\beta & \alpha \end{pmatrix}$$

$$\frac{\partial}{\partial x^k} = \partial_k$$

$$\Rightarrow \quad B = \mathrm{tr}\left[\begin{pmatrix} \partial_k \bar\alpha & \partial_k \bar\beta \\ -\partial_k \beta & \partial_k \alpha \end{pmatrix}\begin{pmatrix} \partial_l \alpha & -\partial_l \bar\beta \\ \partial_l \beta & \partial_l \bar\alpha \end{pmatrix}\right]$$

$$= \mathrm{tr}\begin{pmatrix} \partial_k \bar\alpha \cdot \partial_l \alpha + \partial_k \bar\beta\, \partial_l \beta & \cdots \\ \cdots & \partial_k \beta\, \partial_l \bar\beta + \partial_k \alpha\, \partial_l \bar\alpha \end{pmatrix}$$

$$= \partial_k \bar\alpha \cdot \partial_l \alpha + \partial_k \bar\beta\, \partial_l \beta + \partial_k \beta\, \partial_l \bar\beta + \partial_k \alpha\, \partial_l \bar\alpha$$

$$g_{kl} = \mathrm{Re}(B) = 2 \sum_{p=1}^{4} (\partial_k X_0^p)(\partial_l X_0^p) = 2\left(\frac{\partial}{\partial x^k} X_0 \cdot \frac{\partial}{\partial x^l} X_0\right)$$

$$|X_0|^2 = (X_0 \cdot X_0) = \mathrm{Re}^2(\alpha) + \mathrm{Im}^2(\alpha) + \mathrm{Re}^2(\beta) + \mathrm{Im}^2(\beta)$$

$$= |\alpha|^2 + |\beta|^2 = 1 \quad \longrightarrow S^3$$

$$\longrightarrow \quad g = 2\begin{pmatrix} 1 & & \\ & \sin^2\psi & \\ & & \sin^2\psi \sin^2\theta \end{pmatrix}$$

$$\longrightarrow \quad dg = \sqrt{8} \sin^2\psi \sin\theta \, d\psi \, d\theta \, d\varphi$$

$$\langle \chi_j, \chi_k \rangle = \int_0^{2\pi} d\psi \sin^2\psi \int_0^{\pi/2} d\theta \sin\theta \int_0^{2\pi} d\varphi \; \underbrace{\overline{\chi_j}(g(\psi,\theta,\varphi))}_{\frac{\sin(2j+1)\psi}{\sin\psi}} \; \underbrace{\chi_k(g(\psi,\theta,\varphi))}_{\frac{\sin(2k+1)\psi}{\sin\psi}}$$

$$= 2\pi \int_0^{2\pi} d\psi \; \sin(2j+1)\psi \; \sin(2k+1)\psi$$

$$= -\frac{2\pi}{4} \int_0^{2\pi} d\psi \left(e^{i[(2j+1)+(2k+1)]\psi} - e^{i[(2j+1)-(2k+1)]\psi} - e^{i[(2k+1)-(2j+1)]\psi}\right.$$

$$\times\, e^{-i[(2j+1)+(2k+1)]\psi} \qquad = -\frac{2\pi}{4}(-2\pi-2\pi)\delta_{kj}$$

$$= 2\pi^2\delta_{kj} = V(G)\delta_{kj}$$

H.8

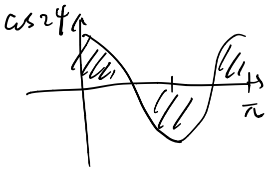
1. $$\int dg = \underbrace{\int_0^{2\pi}\sin^2\psi\, d\psi}_{} \underbrace{\int_0^{\pi/2}\sin\theta\, d\theta}_{} \underbrace{\int_0^{2\pi} d\varphi}_{2\pi} = 2\pi^2$$

$$= \frac{1}{2}\int_0^{2\pi}(1-\cos 2\psi)\,d\psi \quad\Big|\quad = -[\cos\theta]_0^{\pi/2}$$

$$= \frac{1}{2}\left(2\pi - \frac{1}{2}\cdot 0\right) \quad\Big|\quad = -(0-1)$$

$$\quad\Big|\quad = 1$$

$$= \pi$$



$$1-2\sin^2\psi = \cos 2\psi \Leftrightarrow \frac{1}{2}(1-\cos 2\psi) = \sin^2\psi$$

2. $$\sum_{k=-j}^{j} x^{j+k} y^{j-k} = \frac{x^{2j+1}-y^{2j+1}}{x-y}, \quad \text{see H7.3}$$

$$\frac{\sin(2j+1)\psi}{\sin\psi} \overset{?}{=} \sum_{k=0}^{2j} e^{2i(j-k)\psi}$$

$$RHS = \sum_{k=0}^{2j} e^{i(j-k)\psi} e^{i(j-k)\psi}$$

$$(\check{k} = k-j \Leftrightarrow k = j+\check{k})$$

$$= \sum_{\check{k}=-j}^{j} e^{-2i\check{k}\psi} = \sum_{\check{k}=-j}^{j} e^{i(j-\check{k})\psi} e^{-i(j+\check{k})\psi}$$

$$(x = e^{i\psi}, \quad y = e^{-i\psi})$$

$$= \frac{(e^{-i\psi})^{2j+1} - (e^{+i\psi})^{2j+1}}{e^{-i\psi} - e^{i\psi}}$$

$$= \frac{\cos[(2j+1)\psi] - i\sin[(2j+1)\psi] - \cos[(2j+1)\psi] - i\sin[(2j+1)\psi]}{\cos i\psi - i\sin\psi - \cos i\psi - i\sin\psi}$$

$$= \frac{\sin(2j+1)\psi}{\sin\psi}$$

$$\frac{1}{V(G)} \int_G dg\, \overline{\chi_j(g)}\, \chi_{j_1}(g)\, \chi_{j_2}(g)$$

$$= \frac{1}{2\pi^2} \cdot 2\pi \int_0^{2\pi} d\psi\, \cancel{\sin^2\psi} \underbrace{\int_0^{\pi/2} d\theta\, \sin\theta}_{=1} \frac{\sin(2j+1)\psi}{\cancel{\sin\psi}} \frac{\sin(2j_1+1)\psi}{\cancel{\sin\psi}} \frac{\sin(2j_2+1)\psi}{\sin\psi}$$

$$= \frac{1}{\pi} \int_0^{2\pi} d\psi \quad \frac{\sin(2j+1)\psi\ \sin(2j_1+1)\psi\, \sin(2j_2+1)\psi}{\sin\psi}$$

$$= \frac{1}{\pi} \int_0^{2\pi} d\psi \left(\sum_{k=0}^{2j} e^{2i(j-k)\psi} \right) \frac{e^{i(2j_1+1)\psi} - e^{-i(2j_1+1)\psi}}{2i} \frac{e^{i(2j_2+1)\psi} - e^{-i(2j_2+1)\psi}}{2i}$$

$$= -\frac{1}{4\pi} \int_0^{2\pi} d\psi \sum_{k=0}^{2j} \Big[e^{2i(j-k+j_1+j_2+1)\psi} - e^{2i(j-k+j_1-j_2)\psi} - e^{2i(j-k-j_1+j_2)\psi} - e^{2i(j-k-j_1-j_2-1)\psi} \Big]$$

$$= -\frac{1}{2} \sum_{k=0}^{2j} \left(\delta_{k,j+j_1+j_2+1} - \delta_{k,j-j_1+j_2} - \delta_{k,j+j_1-j_2} + \delta_{k,j-j_1-j_2-1} \right)$$

$$= -\frac{1}{2} \sum_{k=-j}^{j} \left(\delta_{k,j_1+j_2+1} - \delta_{k,-j_1+j_2} - \delta_{k,j_1-j_2} + \delta_{k,-j_1-j_2-1} \right)$$

$$= \begin{cases} j > j_1 + j_2 : & = 0 \\ j < |j_1 - j_2| : & = 0 \\ |j_1 - j_2| \le j \le j_1 + j_2 : & = 1 \end{cases} \qquad \longrightarrow \text{coupling of angular mom.}$$