H.5

Chenhuan Wang

1. in V^i, we have the orthonormal basis

$$\langle e^i_\alpha, e^i_\beta \rangle = \delta_{\alpha\beta}$$

$$V = \bigoplus^k V^i = \bigoplus^l V^j$$

$$e^{ki}_\alpha = (e^i_\alpha, \dots, e^i_\alpha) \quad , \quad e^{lj}_\beta = (e^j_\beta, \dots, e^j_\beta)$$

$$\langle e^{ki}_\alpha, D(g) e^{lj}_\beta \rangle = \langle (e^i_\alpha, \dots, e^i_\alpha), (D^j(g) e^j_\beta, \dots, D^j(g) e^j_\beta) \rangle$$

D^i uni. irrep in V^i, D rep in V

$$V = \hat{V}^1 \oplus \hat{V}^2 \oplus \cdots \oplus \tilde{V}^n$$

$$= \underbrace{V^1_1 \oplus \cdots \oplus V^1_{k_1}}_{\text{all have equivalent rep. } D^1} \oplus \cdots \oplus V^n_{k_n}$$

regroup them into equivalent subspaces

Let $D^i_k(g)$ be the rep. D restricted to the subspace.

This $D^i_k(g)$ is equivalent to D^i, which is acting on V^i

↳ Equivalent, $\exists$ a bijective map $D^i_k(g) = A^{-1}_{ki} D^i(g) A_{ki}$

($A_{ki} : V^i_k \to V^i$ isometry)

V^i_k has a orthonormal basis $\{ e^{ki}_\alpha \}$

(A_{ki} unitary $\Leftrightarrow$ bijective & isometry

$$e^{ki}_\alpha = A^{-1}_{ki} \, e^i_\alpha$$

$$\langle e^{ki}_\alpha , e^{ki}_\beta \rangle = \langle A^{-1}_{ki} e^i_\alpha , A^{-1}_{ki} e^i_\beta \rangle$$

$$= \langle e^i_\alpha , A_{ki} A^{-1}_{ki} e^i_\beta \rangle$$

$$= \delta_{\alpha\beta}$$

So $e^{ki}_\alpha \in V^i_k$, $e^{lj}_\beta \in V^j_l$

$$\to \langle e^{ki}_\alpha , e^{lj}_\beta \rangle_V = \delta_{kl} \delta_{ij} \langle e^{ki}_\alpha , e^{ki}_\beta \rangle$$

↖ scalar product in the whole space V.

Other components are zero

$$\langle \underbrace{e^{ki}_\alpha}_{V^i_k}, \underbrace{D(g) e^{lj}_\beta}_{V^j_l} \rangle = \delta_{kl}\delta_{ij} \langle e^{ki}_\alpha, D(g) e^{ki}_\beta \rangle$$

first they should be in the same space

$$= \delta_{kl}\delta_{ij} \langle A^{-1}_{ki}\, e^i_\alpha , D^i_k(g) A^{-1}_{ki}\, e^i_\beta \rangle$$

$$= \delta_{kl}\delta_{ij} \langle e^i_\alpha, \underbrace{A_{ki} D_k(g) A^{-1}_{ki}}_{D^i(g)}\, e^i_\beta \rangle$$

2. $$\chi^i(g) = tr[D^i(g)] = \sum_\alpha D^i_{\alpha\alpha}(g) = \sum_\alpha \langle e^i_\alpha, D^i(g) e^i_\alpha \rangle$$

$$\langle \chi^i(g), \chi^j(g) \rangle = \int dg\, \overline{\chi^i(g)}\, \chi^j(g)$$

$$= \sum_{\alpha,\beta} \int dg\, \overline{\langle e^i_\alpha, D^i(g) e^i_\alpha \rangle} \langle e^j_\beta, D^j(g) e^j_\beta \rangle$$

$$= \sum_{\alpha,\beta} \delta^{ij}\delta_{\alpha\beta}\delta_{\alpha\beta} \frac{V(G)}{dim V_i}$$

$$= \delta^{ij} \sum_\alpha \cancel{\delta_{\alpha\alpha}} \frac{V(G)}{\cancel{dim V_i}}$$

$$\tilde{\chi}^i = \frac{1}{\sqrt{V(G)}} \chi^i$$

$$= \delta^{ij} \underbrace{V(G)}_{\text{normalisation}}$$ ✓

Peter - Weyl

$f \in F_0(G)$, $f(g) = \sum_{i,\alpha,\beta} C^i_{\alpha\beta} D^i_{\alpha\beta}(g)$

$$f(g) = f(g' g g'^{-1}) = \sum_{i,\alpha,\beta} C^i_{\alpha\beta} D^i_{\alpha\beta}(g' g g'^{-1})$$

equal !

$$= \sum_{i,\alpha,\beta,\gamma,\delta} C^i_{\alpha\beta} D^i_{\alpha\gamma}(g') D^i_{\gamma\delta}(g) D^i_{\delta\beta}(g'^{-1})$$

$$\longrightarrow \quad C^i_{\gamma\delta} = \sum_{\alpha\beta} D^i_{\delta\beta}(g'^{-1}) C^i_{\beta\alpha} D^i_{\alpha\gamma}(g')$$

In Matrix $$C^i = D^i(g'^{-1}) C^i D^i(g')$$

$$= (D^i(g'))^{-1} C^i D^i(g')$$

$$D^i(g') C^i = C^i D^i(g')$$

D^i is irrep , Using schur's lemma $\Rightarrow$ $C^i = \lambda_i \mathbb{1}$

$f_0(g) \in \mathcal{F}_0(G)$: $\quad f_0(g) = \sum_{i,\alpha,\beta} C^i_{\beta\alpha} D^i_{\alpha\beta}(g)$

$$= \sum_{i,\alpha,\beta} \lambda_i \delta_{\alpha\beta} D^i_{\alpha\beta}(g)$$

$$= \sum_{i,\alpha} D^i_{\alpha\alpha}(g)$$

$$= \sum_i \lambda_i \chi_i(g) \longrightarrow \text{is a basis}$$

H.6

$$\Delta_n(a_1, \dots, a_n) = \begin{vmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ 1 & | & | & & | \\ | & | & | & \ddots & | \\ | & | & & & | \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} \end{vmatrix} = \det A_{ik}$$

$n=1, \qquad \Delta_1(a_1) = |1| = 1$

$n=2, \qquad \Delta_2(a_1, a_2) = \begin{vmatrix} 1 & a_1 \\ 1 & a_2 \end{vmatrix} = a_2 - a_1 = \prod_{1 \le i < j \le 2} (a_j - a_i)$

Assume it's true for $\Delta_n(a_1, \dots, a_n) = \prod_{1 \le i < j \le n} (a_j - a_i)$

$$\rightarrow \Delta_{n+1}(a_1, \dots, a_{n+1})$$

$$= \begin{vmatrix} 1 & \cdots & a_1^{n-1} & a_1^n \\ | & \ddots & | & | \\ | & & \ddots & | \\ 1 & a_n \cdots & a_n^{n-1} & a_1^n \\ 1 & a_{n+1} \cdots & a_{n+1}^{n-1} & a_{n+1}^n \end{vmatrix}$$

$A(i)$: determinant of A without last row and n-th column

$$= a_{n+1}^n A(n) + \cdots + (-1)^n A(0).$$

Polynomial in a_{n+1}

if $a_{n+1} = a_i, \qquad i = 1, \dots, n$

$= 0$

identical rows, exchanging rows in det gives (-1) factor

$$\Rightarrow \quad \Delta_{n+1}(a_1, \dots, a_{n+1}) = c \prod_{i=1}^{n} (a_{n+1} - a_i)$$

$$= c\, a_{n+1}^{n} + \dots$$

$$\Rightarrow \quad c\, a_n^{n} = \Delta_n(a_1, \dots, a_n) \Rightarrow c = 1$$

$$\Rightarrow \quad \Delta_{n+1}(a_1, \dots, a_{n+1}) = \Delta_n(a_1, \dots, a_n) \prod_{i=1}^{n} (a_{n+1} - a_i)$$

$$= \prod_{1 \leq i < j \leq n} (a_j - a_i) \prod_{m=1}^{n} (a_{n+1} - a_m)$$

$$= \prod_{1 \leq i < j \leq n+1} (a_j - a_i)$$