H9.

1. Take $\lambda_\ell(\varphi) = e^{i(\ell-1)\varphi}$ $\leftarrow$ cannot take $\lambda_i \equiv 1$ $\forall i$

otherwise $\det A = 0$

$$\Rightarrow \det A_{ik} = \det(\lambda_i^{k-1})$$

$$= \det\left[\left(e^{i(i-1)\varphi}\right)^{k-1}\right]$$

$$= \prod_{1\le i<j\le n}\left(e^{i(j-1)\varphi} - e^{i(i-1)\varphi}\right)$$

$$\varphi \ll 1 \qquad \hat{=} \prod \left(i(j-1)\varphi - i(i-1)\varphi\right)$$

$$= \prod i\varphi\,(j-i)$$

$$B = \begin{pmatrix} \lambda_1^{z_1} & \cdots & \lambda_1^{z_n} \\ \vdots & \ddots & \vdots \\ \lambda_n^{z_1} & \cdots & \lambda_n^{z_n} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \cdots & 1 \\ e^{iz_1\varphi} & \cdots & e^{iz_n\varphi} \\ \vdots & & \\ e^{i(n-1)z_1\varphi} & \cdots & e^{i(n-1)z_n\varphi} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & e^{iz_1\varphi} & e^{i2z_1\varphi} & \cdots & e^{i(n-1)z_1\varphi} \\ \vdots & & & & \\ 1 & e^{iz_n\varphi} & \cdots & \cdots & e^{i(n-1)z_n\varphi} \end{pmatrix}^T$$

$$= \begin{pmatrix} 1 & \tilde{\lambda}_1 & \cdots & \tilde{\lambda}_1^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & \tilde{\lambda}_n & \cdots & \tilde{\lambda}_n^{n-1} \end{pmatrix}^T \quad \text{with } \tilde{\lambda}_i = e^{iz_i\varphi}$$

$$\Rightarrow \det B = \det B^T = \det(\tilde{\lambda}_i^k)$$

$$= \prod (\tilde{\lambda}_j - \tilde{\lambda}_i)$$

$$= \prod (e^{iz_j\varphi} - e^{iz_i\varphi})$$

$\varphi \ll 1$ $$\hat{=} \prod i\varphi (z_j - z_i)$$

$$\Rightarrow K_{z_1 \dots z_n} = \frac{\prod (z_j - z_i)}{\pi (j - i)}$$

2. $g \in SU(n)$, $h \in U(n)$

$$\det g = 1 \iff \prod_{i=1}^{n} \lambda_i = 1 \longrightarrow \lambda_n = \frac{1}{\lambda_1 \dots \lambda_{n-1}}$$

$$K^{SU(n)}_{z_1, \dots z_n}(g) = K^{U(n)}_{z_1, \dots z_n}(\mathrm{diag}(\lambda_1, \dots, \lambda_{n-1}, (\lambda_1 \dots \lambda_{n-1})^{-1}))$$

3. $SU(2)$, $g_0 = \begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix}$, $|\lambda|^2 = 1$

$$A = \begin{pmatrix} 1 & \lambda \\ 1 & \bar{\lambda} \end{pmatrix}, \quad B = \begin{pmatrix} \lambda^{z_1} & \lambda^{z_2} \\ \bar{\lambda}^{z_1} & \bar{\lambda}^{z_2} \end{pmatrix}$$

$\det A = \bar{\lambda} - \lambda$

$$\det B = \lambda^{z_1} \bar{\lambda}^{z_2} - \bar{\lambda}^{z_1} \lambda^{z_2}$$

since $z_1 < z_2$
$z_2 - z_1 = 2j + 1$

$$= \bar{\lambda}^{2j+1} - \lambda^{2j+1}$$

H.10 $\quad \breve{D}$ irrep.

1. Assume $[\breve{D}(g_k), \breve{D}(g)]_- = 0 \iff \underset{A}{\breve{D}(g_k)}\,\underset{D_1}{\breve{D}(g)} - \underset{D_2}{\breve{D}(g)}\,\underset{A}{\breve{D}(g_k)} = 0$

$\Rightarrow$ 1) $A = 0$, if $\breve{D}(g) \neq \breve{D}(g)$ $\quad$ X

2) $D(\breve{g}) = D(\breve{g}) \Rightarrow A = \breve{D}(g_k) = \lambda \mathbb{1}$, $\lambda \in \mathbb{C}$

$\Rightarrow g_k$ should $\sim e$,

obviously $g_k = \lambda_k e$ satisfies this ✓

use $g_k \sim \mathbb{1}$ to show $[\ ,\ ] = 0$

2. $\breve{D}(g_k) = \lambda_k \breve{D}(e) = \lambda_k \mathbb{1}$

not true

$\breve{D}(g_k) = \alpha(\lambda_k)\mathbb{1}_V$

$g_k \cdot g_{k'} = \lambda_k \lambda_{k'} = e^{2\pi i (k+k')/n} e = g_{k+k'}$

$\breve{D}(g_k)\breve{D}(g_{k'}) = \breve{D}(g_{k+k'}) \longrightarrow \alpha(\lambda_k)\alpha(\lambda_{k'}) = \alpha(\lambda_{k+k'})$

$\rightarrow \alpha(\lambda_k) = (\lambda_k)^\ell$

$\lambda_k = e^{2\pi i k/n}$, $k = 1, \dots, n-1$

$(\lambda_k)^n = e^{2\pi i k} = 1$

What if $[\breve{D}(g_k)]^n = \breve{D}((g_k)^n) = \breve{D}(\lambda_k^n e^n) = \breve{D}(e) = \mathbb{1}_V$

$\hookrightarrow [\alpha(\lambda_k)]^n = 1 = (\lambda_k^\ell)^n = \lambda_k^{\ell \cdot n} = e^{2\pi i k \cdot \ell} \rightarrow \ell \in \mathbb{Z}$, $\ell < n$ (periodic)

3. $g \in U(n)$, $\hat{D}(g) = (\det g)^{\ell/n}\, \breve{D}(g(\det g)^{-1/n})(\det g)^m$

$= \cancel{(\det g)^{\ell/n}}\cancel{(\det g)^{-\ell/n}}\, \breve{D}(g) \det(g)^m$

$\rightarrow$ is unique

$\hat{D}(g_1 g_2) = \breve{D}(g_1 g_2)(\det g_1 g_2)^m$

$= \breve{D}(g_1)\det(g_1)^m\, \breve{D}(g_2)\det(g_2)^m$

$= \hat{D}(g_1)\hat{D}(g_2)$

$$g' = \frac{g}{\sqrt[n]{\det g}} \in SU(n) \,,$$

$$(g')^*(g') = \underbrace{g^* g}_{\mathbb{1}} \underbrace{\underbrace{\frac{1}{\sqrt[n]{\det g}}}_{= \frac{1}{\sqrt[n]{\det(1/g)}}} \frac{1}{\sqrt[n]{\det g}}}_{=1} \qquad \det(g') = \det\left(\frac{1}{\sqrt[n]{\det g}}\, g\right) = \frac{1}{\det g}\cdot \det g = 1$$

$$g \in U(n) \,, \longrightarrow \, |\det(g)|^2 = 1 \,, \quad \det(g) = e^{in\varphi} \,, \quad \varphi \in \mathbb{R}$$

$$\sqrt[n]{\det g} = e^{i\varphi} \underbrace{e^{2\pi i k/n}}_{\lambda_k} = \lambda_k e^{i\varphi}$$

$$\check{D}(g_k) = (\lambda_k)^\ell \mathbb{1}_V$$

$$\hat{D}(g) = (\lambda_k)^\ell e^{i\ell\varphi} \check{D}(g e^{i\varphi} (\lambda_k)^{-1}) e^{imn\varphi}$$
$$= e^{i(\ell+mn)\varphi} \check{D}(g_k) \check{D}(g e^{-i\varphi} (\lambda_k)^{-1})$$
$$= e^{i(\ell+mn)\varphi} \check{D}(\underbrace{\lambda_k e\, g}_{g} e^{i\varphi} (\lambda_k)^{-1}) = e^{i(\ell+m\cdot n)\varphi} \check{D}(g e^{-i\varphi})$$

$\hookrightarrow$ fixed

4. $D \to U(n)$. $\check{D} \to SU(n)$

$$\hat{D}(g) = (\det g)^{\ell/n} \check{D}(g (\det g)^{-1/n}) (\det g)^m \qquad \text{with} \quad g \in U(n)$$

?

$D(g)$ rep of $g \in U(n)$

$[D(e^{i\theta} e), D(g)] = 0$, Schur's lemma

$$\to D(e^{i\theta} e) = \mathbb{1}_V e^{i\theta\beta}$$
$$= (e^{i\theta})^\beta \mathbb{1}_V$$

$$\check{D}(g_k) = (\lambda_k)^\ell \mathbb{1}_V \,, \quad \underset{\in SU(n)}{D(g)} = D(\lambda_k e) = \check{D}(g_k) = (\lambda_k)^\ell \mathbb{1}_V$$

$$\hookrightarrow \beta = l + pn, \quad p \in \mathbb{Z}$$

$$\begin{aligned}
\hat{D}(g) &= e^{il\varphi}\, \tilde{D}(g e^{-i\varphi})\, (\det g)^m \\
&= \underbrace{e^{il\varphi}}_{e^{i(l+pn-pn)\varphi}} D(g e^{-i\varphi})\, (\det g)^m \\
&= \underbrace{\underbrace{e^{i(l+pn)\varphi}}_{\substack{= e^{i\beta\varphi} \mathbb{1}_V \\ = D(e^{i\varphi} e)}} D(g e^{-i\varphi})}_{D(g)} \underbrace{(\det g)^m\, e^{-i\varphi p \cdot n}}_{= (e^{in\varphi})^m e^{-ip\cdot n\varphi} \overset{!}{=} 1}
\end{aligned}$$

$$\longrightarrow m = p$$