H24. Lie Algebra $A$

symmetric linear form $f$ on $A$:

$$f(\alpha_1 x_1, \dots, \alpha_n x_n) = (\Pi \alpha_i) f(x_1, \dots, x_n) \quad \in A$$

$$f(x_1, \dots, x_i + \tilde{x}_i, \dots, x_n) = f(x_1, \dots, x_i, \dots, x_n) + f(x_1, \dots, \tilde{x}_i, \dots, x_n)$$

invariant if

$$f([x, x_1], \dots, x_n) + \dots f(x_1, \dots, [x, x_n]) = 0$$

basis of $A$: $\{a_l\}$

$$f_{l_1 \dots l_n} = f(a_{l_1}, \dots, a_{l_n})$$

$D$: unitary rep. of $A$

$$\hat{f} = \sum_{l_1, \dots, l_k} f_{l_1, \dots, l_k} D(a_{l_1}) \cdots D(a_{l_k})$$

1. $D$ is unitary rep. $\quad a, b \in A$ $\quad$ $\in A$, not good notation!

$$D([a,b]) = [D(a), D(b)]_- \rightarrow D([\hat{f}, x]) = [\hat{f}, D(x)]_-$$

$$[D(x), \hat{f}]_- = \sum_{l_1 \dots l_k} f_{l_1 \dots l_k} \left( D(x) D(a_{l_1}) \cdots D(a_{l_k}) - D(a_{l_1}) \cdots D(a_{l_k}) D(x) \right)$$

$$\langle a_m, [x, a_n] \rangle = \langle a_m, \mathrm{ad}(x) a_n \rangle = C^m_n$$

$$= \langle -\mathrm{ad}(x) a_m, a_n \rangle = -\langle [x, a_m], a_n \rangle = -C^l_m \overbrace{\langle a_l, a_n \rangle}^{\delta_{ln}}$$

$$= -C^n_m$$

$$[D(x), \hat{f}]_- = \sum_{l_1, \dots, l_k} f_{l_1, \dots, l_k} \{ D([x, a_{l_1}]) D(a_{l_2}) \cdots D(a_{l_k})$$

$D(x) D(a_l)$
$= D(a_l) D(x)$
$+ D([x, a_l])$

$$+ D(a_{l_1}) D([x, a_{l_2}]) \cdots D(a_{l_k})$$

$$\vdots$$

$$+ D(a_{l_1}) \cdots \underbrace{D([x, a_{l_k}])}_{= \sum_{n_1} C^m_{l_n} D(a_{n_1})} \}$$

$$= \sum_{l_1, \dots, l_k} \sum_{n_1} f_{l_1, \dots, l_k} \left[ C^{n_1}_{l_1} D(a_{n_1}) D(a_{l_2}) \cdots D(a_{l_k}) \right.$$

$$+ C^{n_1}_{l_2} D(a_{l_1}) D(a_{n_1}) D(a_{l_3}) \cdots D(a_{l_k})$$

$$\left. + \cdots \right]$$

$n_1 \Leftrightarrow l_i$

$$= \sum_{l_1 \dots l_k} \sum_{n_1} \left[ f_{n_1 l_2 \dots l_k} C^{l_1}_{n_1} D(a_{l_1}) D(a_{l_2}) \cdots D(a_{l_k}) + \cdots \right]$$

$$= -\sum_{n_1, l_2 \dots l_k} f(C^{n_1}_{l_1} a_{n_1}, \dots, a_{l_k}) D(a_{l_1}) \cdots D(a_{l_k}) + \cdots$$

$$= -\sum_{n_1, l_1 \dots l_k} \big( f([x, a_{l_1}], \dots, a_{l_k}) + f(a_{l_1}, [x, a_{l_2}], \dots, a_{l_k}) + \cdots$$
$$+ f(a_{l_1}, \dots, a_{l_{k-1}}, [x, a_{l_k}]) \big) D(a_{l_1}) \cdots D(a_{l_k})$$

$= 0$

2. irreducible, $[D(x), \hat{f}] = 0 \xrightarrow[\text{lemma}]{\text{schur's}} \hat{f} = \lambda \mathbb{1}$

$\lambda$ only depends on irrep. (space) and thus eigenvalue is unique and can be used to label rep.

3. Take $SO(3)$

general rep. given by

$$D_{SO(3)}(g_{\vec{\omega}}) = \exp\left(-\frac{i}{\hbar}(\vec{\omega} \vec{L})\right) = \exp\left(-\frac{i}{\hbar}(\vec{\omega} \cdot D(\vec{L})\right)$$

adjoint rep.

$$g_{\vec{\omega}} \vec{x} = \exp(\vec{\omega} \cdot ad(\vec{L})) \vec{x} = \exp\left(\begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix}\right) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\left( \vec{x} \in G, \quad ad(L_i) x_j = [L_i, x_j], \quad [L_i, L_j] = \varepsilon_{ijk} L_k \right)$$

$$L_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad L_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad L_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\exp(\vec{\omega} \cdot ad(\vec{L})) \vec{x} = (\mathbb{1} + \vec{\omega} \cdot ad(\vec{L}) + \cdots) \sum_j x_j L_j$$

$$= \vec{x} + \omega_i \sum_j x_j \underbrace{ad(L_i) L_j}_{= [L_i, L_j] = \varepsilon_{ijk} L_k}$$

$SO(3)$. fundamental rep. : $3 \times 3$ with real parameters and $\det = 1$ and $O^T = O^{-1}$

$$\mathrm{Lie}(SO(3)) = \mathrm{span}\left\{ ad(L_1) = \begin{pmatrix} 0 & & \\ & 0 & 1 \\ & -1 & 0 \end{pmatrix},\ ad(L_2) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},\ ad(L_3) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}$$

$$f(a,b) = \langle a, b \rangle = -\mathrm{tr}(ad(a)\, ad(b))$$

$$f_{ij} = \langle L_i, L_j \rangle = 2\delta_{ij}$$

$$\hat{f} = \sum_{i,j} f_{ij}\, D(L_i) D(L_j) = \sum_{i,j} 2\delta_{ij}\, D(L_i) D(L_j) = 2\sum_i D^2(L_i) = -\frac{2}{\hbar^2} \sum_\ell L_\ell^2$$

$$= -\frac{2}{\hbar^2} \vec{L}^2 \quad \leftarrow \text{integers}$$

H.25

$A$ is Lie algebra

$A = H \oplus H^\perp$, $H$ Cartan subalgebra with basis $\{h_i\}$

$H^\perp$ with basis $\{e_\alpha^+, e_\alpha^-\}$

know that $[h, e_\alpha^+] = \alpha(h)\, e_\alpha^-$, $h \in H$

$[h, e_\alpha^-] = -\alpha(h)\, e_\alpha^+$,

$$C = \sum_i D(x_i) D(x_i) \qquad \text{Casimir operator}$$

$$= \sum_i D^2(h_i) + \sum_\alpha D^2(e_\alpha^+) + \sum_\alpha D^2(e_\alpha^-)$$

raising / lowering operators

$$L_\alpha^\pm := D(e_\alpha^+) \pm i D(e_\alpha^-)$$

can check that $[D(x), L_\alpha^\pm] = \pm i\alpha(x)\, L_\alpha^\pm$

$x \in H$

Then $$L_\alpha^- L_\alpha^+ = D^2(e_\alpha^+) + D^2(e_\alpha^-) + \underbrace{iD([e_\alpha^+, e_\alpha^-])}$$

$$[e_\alpha^+, e_\alpha^-] = ?$$

Jacobi Identity: $\quad h \in H$

$$[h,[e_\alpha^+,e_\alpha^-]]+[e_\alpha^+,\underbrace{[e_\alpha^-,h]}_{\alpha(h)e_\alpha^+}]+[e_\alpha^-,\underbrace{[h,e_\alpha^+]]}_{+\alpha(h)e_\alpha^-}=0$$

$$\underbrace{[e_\alpha^+,\alpha(h)e_\alpha^+]+[e_\alpha^-,\alpha(h)e_\alpha^-]}_{=0}$$

$$\Rightarrow [h,[e_\alpha^+,e_\alpha^-]]=0 \Rightarrow [e_\alpha^+,e_\alpha^-]=\sum_i c_\alpha^i h_i \in H$$

$$\Rightarrow \underbrace{\langle h_i,\underbrace{[e_\alpha^+,e_\alpha^-]}_{ad(e_\alpha^+)e_\alpha^-}\rangle}_{}=c_\alpha^i=\langle[h_i,e_\alpha^+],e_\alpha^-\rangle=\alpha(h_i)\langle e_\alpha^-,e_\alpha^-\rangle=\alpha(h_i)$$

$$=-\langle ad(e_\alpha^+)h_i,e_\alpha^-\rangle$$

$$=-\langle[e_\alpha^+,h_i],e_\alpha^-\rangle$$

Let $D$ have finite dim.

$\Rightarrow \exists\ w$ s.t. $L_\alpha^+ w=0$ & $D(h)w=i\lambda(h)w$ , $h\in H$

$$Cw=\underbrace{\sum_i D^2(h_i)w}_{=-\sum_i \lambda^2(h_i)w}+\sum_\alpha L_\alpha^-\underbrace{L_\alpha^+ w}_{=0}-\underbrace{i\sum_\alpha D([e_\alpha^+,e_\alpha^-])w}_{+\sum_\alpha\sum_i \alpha(h_i)\underbrace{D(h_i)w}_{\lambda(h_i)}}$$

$$=-\left(\langle\Lambda,\Lambda\rangle-\sum_\alpha\langle\alpha,\Lambda\rangle\right)w \quad , \quad \Lambda=\begin{pmatrix}\lambda(h_1)\\ \lambda(h_2)\\ \vdots\end{pmatrix}$$

$$=-\underbrace{a}_{c}\cdot\mathbb{1}\cdot w \qquad \alpha=\begin{pmatrix}\alpha(h_1)\\ \alpha(h_2)\\ \vdots\end{pmatrix}$$