1.

Definition wrong ?

a) $$\tilde{f}_Y(\sigma) = \frac{1}{n!}\sum_{\tau \in S_n} f_Y(\tau \sigma \tau^{-1})$$

$$\tilde{f}_Y(\sigma' \sigma \sigma'^{-1}) = \sum_{\tau \in S_n} f_Y(\tau \sigma' \sigma \sigma'^{-1} \tau^{-1})$$
$$= \sum_{\tau' \in S_n} f_Y(\tau' \sigma \tau'^{-1})$$
$$= \tilde{f}_Y(\sigma)$$

b) $Y_s = \underbrace{\square \cdots \square}_{n}$ ; $\qquad Y_A = \left.\begin{array}{c}\square \\ \vdots \\ \square\end{array}\right\} n$

$$f_Y(\sigma) = \begin{cases} 0, & \sigma \neq pq \\ \varepsilon(p), & \sigma = pq \end{cases}$$

$Y_s$: $\quad P = \{e\}, \quad Q = S_n \quad \rightarrow f_{Y_s}(\sigma) = \varepsilon(p) = 1 \leftarrow$ already equivariant

$Y_A$: $\quad P = S_n, \quad Q = \{e\} \quad \rightarrow f_{Y_A}(\sigma) = \varepsilon(p) = \varepsilon(\sigma)$

$\tau \sigma \tau^{-1}$ has the same cycle structrue as $\sigma$

$$\rightarrow \varepsilon(\tau \sigma \tau^{-1}) = \varepsilon(\sigma)$$

$$K_Y(\sigma) = N_Y \tilde{f}_Y(\sigma)$$

$$\rightarrow K_{Y_s}(\sigma) = \left(\frac{n!}{n!}\right)^{1/2} \cdot \frac{1}{n!} \sum_{\tau \in S_n} f_{Y_s}(\tau \sigma \tau^{-1}) = 1 \rightarrow$$ 1-dim rep.

$$K_{Y_A}(\sigma) = \left(\frac{n!}{\underbrace{\sum(\varepsilon(\sigma))^2}_{=1}}\right)^{1/2} \cdot \frac{1}{n!} \sum_{\tau \in S_n} f_{Y_A}(\tau \sigma \tau^{-1}) = \frac{1}{n!} \sum_{\tau \in S_n} \varepsilon(\tau \sigma \tau^{-1})$$

(Since its equivarient,

$$= \frac{1}{n!} \sum \varepsilon(\sigma) = \varepsilon(\sigma)$$ $\rightarrow$ n-dim rep.

c) $S_3 = \{ e, (12), (13), (23), (123), (132) \}$

d) from b) $\chi_{Y_A}(\sigma) = \varepsilon(\sigma)$

$\chi_{Y_S}(\sigma) = 1$

$Y_M$: $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$ $\quad P = \{e, (13)\}, \quad Q = \{e, (12)\}$

$f_{Y_M}(e) = \varepsilon(e) = 1$

$f_{Y_M}((12)) = \varepsilon(e) = 1$

$f_{Y_M}((13)) = \varepsilon((13)) = -1$

$f_{Y_M}((23)) = 0$

$f_{Y_M}((123)) = 0$

$f_{Y_M}((132)) = \varepsilon((13)) = -1$

$$(12)(13) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (123)$$

$$(123) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

e) We know $\chi_{Y_S}$ and $\chi_{Y_A}$ already, just use orthogonality

| | $e$ | $(12)\,(13)\,(23)$ | $(123)\,(132)$ |
|---|---|---|---|
| $m$ | 1 | 3 | 2 |
| $\chi_S$ | 1 | 1 | 1 |
| $\chi_M$ | $x$ | $y$ | $z$ |
| $\chi_A$ | 1 | $-1$ | 1 |

Calculate $\tilde{f}_Y$, $N_Y$

$$1^2 + x^2 + 1^2 \overset{!}{=} \frac{6}{1} \Rightarrow x = 2$$

$$1 \cdot 1 + xy + 1 \cdot (-1) = 1 - 1 + 2y \overset{!}{=} 0 \Rightarrow y = 0$$

$$1 \cdot 1 + xz + 1 \cdot 1 = 0 \rightarrow z = -1$$

$\rightarrow$

| | | |
|---|---|---|
| 1 | 1 | 1 |
| 2 | 0 | $-1$ |
| 1 | $-1$ | 1 |

2. a) $\lambda_p = \mu^{p-1-j}$, $\quad p = 1, \dots, 2j+1$, $\mu = e^{i\varphi}$

$= (e^{i\varphi})^{p-1-j}$

$$A_{pq} = (\lambda_p)^{q-1} = (e^{i\varphi})^{(q-1)(p-1-j)}$$

$$A = \begin{pmatrix} 1 & (e^{i\varphi})^{-j} & \cdots & (e^{i\varphi})^{(n-1)(-j)} \\ 1 & (e^{i\varphi})^{1-j} & \cdots & \vdots \\ \vdots & \vdots & & \vdots \\ 1 & (e^{i\varphi})^{n-j} & \cdots & (e^{i\varphi})^{(n-1)(n-j)} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & (e^{i\varphi})^{-j} & \cdots & ((e^{i\varphi})^{-j})^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & (e^{i\varphi})^{n-j} & \cdots & ((e^{i\varphi})^{n-j})^{n-1} \end{pmatrix}$$

$$\rightarrow \det A = \prod_{1 \le p < q \le n} \left( (e^{i\varphi})^{q-1-j} - (e^{i\varphi})^{p-1-j} \right)$$

$$= \prod_{1 \le p < q \le n} (\lambda_q - \lambda_p)$$

$$B_{pq} = (\lambda_p)^{k_q} = (e^{i\varphi})^{k_q(p-1-j)} \quad , \quad k_q = \ell_q + q - 1$$

$$\lambda_p = (e^{i\varphi})^{p-1-j}$$

When $p = 1+j \rightarrow \lambda_{1+j} = 1$, $\quad n = 2j+1$

$$B = \begin{pmatrix} (e^{i\varphi})^{k_1(-j)} & \cdots & (e^{i\varphi})^{k_n(-j)} \\ \vdots & & \\ (e^{i\varphi})^{k_1 \cdot 0} & \cdots & (e^{i\varphi})^{k_n \cdot 0} \\ \vdots & & \\ (e^{i\varphi})^{k_1(j)} & \cdots & (e^{i\varphi})^{k_n \cdot (j)} \end{pmatrix}$$

$n - 1 - j = 2j + 1 - 1 - j = j$

$$\longrightarrow \quad K_{k_1 \dots k_n}(\hat{g}) = \frac{1}{\mu^{jN}} \frac{\prod(\mu^{k_q} - \mu^{k_p})}{\prod(\mu^{q-1} - \mu^{p-1})}$$

b) $$K_{k_1 \dots k_n}(e) = \frac{1}{\mu^{jN}} \frac{\prod\left((e^{i\varphi})^{k_q} - (e^{i\varphi})^{k_p}\right)}{\prod\left((e^{i\varphi})^{q-1} - (e^{i\varphi})^{p-1}\right)} \Bigg|_{\varphi=0}$$

$$= \frac{\prod(\cancel{1} - i\varphi k_q - \cancel{1} + i\varphi k_p)}{\prod(\cancel{1} - (q-1)i\varphi - \cancel{1} + i\varphi(p-1))} \Bigg|_{\varphi=0}$$

$$= \frac{\prod(-k_q + k_p)}{\prod(-q+p)}$$

c) $U(3)$, $n=3$.

$P = l_3 - l_2$, $Q = l_2 - l_1$, $\quad P = k_1$. $Q = k_2$

$$\dim_{PQ} = \frac{\prod_{1 \le p < q \le 3}(k_q - k_p)}{\prod_{1 \le p < q \le 3}(q-p)} = \frac{(k_3 - k_2)(k_3 - k_1)(k_2 - k_1)}{(3-2)(3-1)(2-1)}$$

$k_q = l_q + q - 1 \quad \longrightarrow \quad k_3 = l_3 + 3 - 1 = l_3 + 2$

$k_2 = l_2 + 2 - 1 = l_2 + 1$

$k_1 = l_1$

$$\longrightarrow \dim_{PQ} = \frac{(l_3 + 2 - l_2 - 1)(l_3 + 2 - l_1)(l_2 + 1 - l_1)}{2}$$

$$= \frac{1}{2}(P+1)(Q+P+2)(Q+1)$$

$Y_S$: $\quad l_3 = 3, l_2 = 0, l_2 = 0 . \Rightarrow P = 3, \quad Q = 0$

$\Rightarrow \dim = \frac{1}{2}(3+1)(0+1)(3+0+2) = 10$

$Y_M$: $\quad l_3 = 2, l_2 = 1, l_1 = 0 \Rightarrow P = 1, \quad Q = 1$

$\Rightarrow \dim = \cancel{\tfrac{1}{2}}\cancel{(1+1)}(1+1)(1+1+2) = 8$

$Y_A$: $\quad l_3 = 1, l_2 = 1, l_1 = 1 \Rightarrow P = 0, \quad Q = 0.$

$\Rightarrow \dim = \frac{1}{2} \cdot 1 \cdot 1 \cdot 2 = 1$

d) $SU(2): g = \begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix} \quad , \rightarrow \lambda_1 = \lambda, \ \lambda_2 = \bar{\lambda} \quad , \quad j = \frac{1}{2}$

$$\lambda_1 = \mu^{1-1-j} = \mu^{-1/2} = \lambda \iff \mu = \lambda^{-2}$$

$$k_2 - k_1 = 2j+1$$

$$\lambda_2 = \mu^{2-1-j} = \mu^{1/2} = \bar{\lambda} \iff \mu = \bar{\lambda}^2$$

$$K_{k_1k_2} = \frac{1}{\mu} \frac{\mu^{k_2} - \mu^{k_1}}{\mu^1 - \mu^0} = \lambda^2 \frac{\lambda^{-2k_2} - \lambda^{-2k_1}}{\lambda^{-2} - \lambda^0}$$

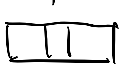
$$= \lambda^2 \frac{\lambda^{-2(k_1+2j+1)} - \lambda^{-2k_1}}{\lambda^{-2} - 1}$$

$$A = \begin{pmatrix} 1 & \lambda \\ 1 & \bar{\lambda} \end{pmatrix} \qquad B = \begin{pmatrix} \lambda^{k_1} & \lambda^{k_2} \\ \bar{\lambda}^{k_1} & \bar{\lambda}^{k_2} \end{pmatrix}$$

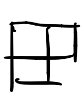
$$\rightarrow K_{k_1k_2} = \frac{\lambda^{k_1-k_2} - \lambda^{-k_1+k_2}}{\bar{\lambda} - \lambda} = \frac{\lambda^{2j+1} - \lambda^{-(2j+1)}}{\lambda - \lambda^{-1}}$$

e) $j = 1, \quad n = 3$

$k_q := l_q + q - 1$

 : $l_3 = 3 \rightarrow k_3 = 5$; $l_2 = 0 \rightarrow k_2 = 1$; $l_1 = 0 \rightarrow k_1 = 0$

 : $l_3 = 1$, $l_2 = 1$, $l_1 = 1 \rightarrow k_3 = 3$, $k_2 = 2$, $k_1 = 1$

 : $l_3 = 2$, $l_2 = 1$, $l_1 = 0 \rightarrow k_3 = 4$, $k_2 = 2$, $k_1 = 0$

$$K_S = \frac{1}{\mu^3} \frac{(\mu^5-\mu^1)(\mu^5-\mu^0)(\mu^1-\mu^0)}{(\mu^2-\mu^0)(\mu^2-\mu^1)(\mu^1-\mu^0)}$$

$$= \frac{1}{\mu^3} \frac{\cancel{\mu}(\mu^4-1)(\mu^5-1)\cancel{(\mu-1)}}{\cancel{\mu}(\mu^2-1)(\mu-1)\cancel{(\mu-1)}}$$

$$= \frac{1}{\mu^3} \frac{\cancel{(\mu^2-1)}(\mu^2+1)(\mu^5-1)}{\cancel{(\mu^2-1)}(\mu-1)}$$

$$\lambda_1 = \mu^{1-1-1} = \mu^{-1}$$

$$\lambda_2 = \mu^{2-1-1} = 1$$

$$\lambda_3 = \mu^{3-1-1} = \mu$$

$$K_A = \frac{1}{\mu^3} \frac{(\mu^3-\mu^2)(\mu^3-\mu^1)\cancel{(\mu^2-\mu)}}{\cancel{(\mu^2-\mu)}(\mu-1)(\mu^2-1)}$$

$$= \frac{1}{\mu^3} \frac{\mu^3(\mu-1)(\mu^2-1)}{(\mu-1)(\mu^2-1)}$$

$$= 1$$

$$= K^{SU(2)}_{j=0}(\lambda)$$

$$K_M = \frac{1}{\mu^3} \frac{(\mu^4-\mu^2)(\mu^4-\mu^0)(\mu^2-\mu^0)}{(\mu^2-\mu)(\mu-1)(\mu^2-1)}$$

$$= \frac{1}{\mu^3} \frac{\mu^2(\mu^2-1)(\mu^4-1)\cancel{(\mu^2-1)}}{\mu(\mu-1)(\mu-1)\cancel{(\mu^2-1)}}$$

$$= \frac{1}{\mu^2} \frac{(\mu+1)(\mu^4-1)}{\mu-1} = \frac{\mu+1}{\mu^2} \frac{(\mu^2+1)(\mu+1)\cancel{(\mu-1)}}{\cancel{\mu-1}}$$

$$= \mu^{-2}(\mu+\mu^0)(\mu^2+\mu^0)(\mu^1+\mu^0)$$

$$= (\mu^{-1}+\mu^{-2})(\mu^2+\mu^0)(\mu^1+\mu^0)$$

$$= (\mu^1+\mu^{-1}+\mu^0+\mu^{-2})(\mu^1+\mu^0)$$

$$= \mu^{2} + \mu^{1} + \mu^{0} + \mu^{-1} + \mu^{1} + \mu^{0} + \mu^{-1} + \mu^{-2}$$

$$= \mu^{2} + 2\mu^{1} + 3\mu^{0} + 2\mu^{-1} + \mu^{-2}$$

$$= \mu^{2} + \mu^{1} + \mu^{0} + \mu^{-1} + \mu^{-2} + \mu^{1} + \mu^{0} + \mu^{-1}$$

$$= \mathcal{K}^{SU(2)}_{j=2} + \mathcal{K}^{SU(2)}_{j=1}$$