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P.1

1. $$P^j_m(z) = \frac{(z_1)^{j+m}(z_2)^{j-m}}{[(j+m)!\,(j-m)!]^{1/2}}$$

$$(j+m)+(j-m) = 2j = 0, \dots, N$$

Completeness: $j+m,\ j-m = 0, \dots, N$

have to think about z_1, z_2

For every $n \leq N$, there are $n+1$ possible polynomials $P(z)$ with degree n

$$\dim(V_n) = \sum_{n=0}^{N}(n+1) = \frac{(N+1)(N+2)}{2} = \frac{N^2+3N+2}{2}$$

For each value of $j \geq 0$, there are $(2j+1)$ possible values for each j

$$\Rightarrow \dim(V_n) = \sum_{n=0}^{N/2}(2j+1) = \frac{N^2+3N+2}{2}$$

same dimension !

Linear indep.: $c_1 P^{j'}_{m'} + c_2 P^j_m = 0 \Rightarrow c_1 = c_2 = 0 \Rightarrow$ basis

when $j \neq j'$, $m \neq m'$

2. $$\langle P, P' \rangle = \frac{1}{\pi^2}\int d^4z\ \bar{P}(z)$$

$$D(g)P(z) = P(g^{-1}z), \quad g \in SU(2)$$

$$\begin{aligned} D(g_1 g_2)P(z) &= P((g_1 g_2)^{-1}z) \\ &= P(g_2^{-1} g_1^{-1} z) \\ &= D(g_2)\,P(g_1^{-1}z) \\ &= D(g_1)\,D(g_2)\,P(z) \end{aligned}$$

$$\langle D(g)P, D(g)P' \rangle = \frac{1}{\pi^2}\int d^4z\ \bar{P}(g^{-1}z)\,P'(g^{-1}z)\,e^{-|g^{-1}z|^2}, \quad z' = g^{-1}z$$

$\det(g)=1$
$g \in SU(2)$ $\longrightarrow$ $$= \frac{1}{\pi^2}\int d^4z'\ \bar{P}(z)\,P'(z)\,e^{-|z'|^2}$$

$$= \langle P, P' \rangle$$

3. $$D(g)^{-1} z = \begin{pmatrix} \bar\alpha & \bar\beta \\ -\beta & \alpha \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \bar\alpha z_1 + \bar\beta z_2 \\ -\beta z_1 + \alpha z_2 \end{pmatrix}$$

$$P(g^{-1} z)^{\ell}_{n} = \frac{(\bar\alpha z_1 + \bar\beta z_2)^{\ell+n} (-\beta z_1 + \alpha z_2)^{\ell-n}}{\cdots}$$

write $a = -\beta$, $b = \bar\alpha$, $c = \alpha$, $d = \bar\beta$

$$(a z_1 + c z_2)^{\ell-n} = \sum_{j=0}^{\ell-n} \binom{\ell-n}{j} a^j z_1^j c^{\ell-n-j} z_2^{\ell-n-j}$$

$$(b z_1 + d z_2)^{\ell+n} = \sum_{k=0}^{\ell+n} \binom{\ell+n}{j} b^k z_1^k d^{\ell+n-k} z_2^{\ell+n-k}$$

The product is

$$\sum_{j=0}^{\ell-n} \sum_{k=0}^{\ell+n} \frac{1}{\sqrt{\cdots}} \binom{\ell-n}{j} \binom{\ell+n}{j} a^j b^k c^{\ell-n-j} d^{\ell+n-k} \underbrace{z_1^{j+k}}_{z_1^{\ell-m}} z_2^{2\ell-j-k}$$

$$(j,k) \longmapsto (m,j) \quad \Rightarrow \quad j+k = \ell - m \qquad -m-n \le j \le \ell - m$$

$$\begin{cases} 0 \le j \le \ell - n \\ 0 \le k \le \ell - n \end{cases} \qquad -\ell \le m \le \ell$$

$$= \sum_{m=-\ell}^{\ell} \underbrace{\sum_{\min(0,(-m-n))}^{\max((\ell-m),(\ell-n))} \binom{\ell-n}{j} \binom{\ell+n}{\ell-m-j} a^j b^{\ell-m-j} c^{\ell-n-j} d^{n+m+j}}_{D^{\ell}_{nm}}$$

$$\times \underbrace{z_1^{\ell-m} z_2^{\ell+m} \cdot \frac{1}{\sqrt{(\ell+m)!(\ell-m)!}}}_{P^{\ell}_{m}(z)}$$

4. $\langle P^j{}_m , P^{j'}{}_{m'} \rangle \overset{?}{=} \delta_{mm'}\,\delta^{jj'}$

$(z_k = \rho_k e^{i\varphi_k}$

$$= \frac{1}{\pi^2}\int_0^\infty d\rho_1 \rho_1 \int_0^\infty d\rho_2 \rho_2 \; \frac{\rho_1^{\,j'+m'+j+m}\;\rho_2^{\,j'-m'+j-m}}{\sqrt{(j'+m')!\,(j+m)!}\;\sqrt{(j+m)!\,(j-m)!}}\; e^{-(\rho_1^2+\rho_2^2)}$$

$$\times \int_0^{2\pi} d\varphi_1 \int_0^{2\pi} d\varphi_2 \; e^{i\varphi_1(-(j+m)+j'+m')}\, e^{i\varphi_2(-(j-m)+j'-m')}$$

$$=(2\pi)^2\, \underbrace{\delta_{(j'+m')(j+m)}\,\delta_{(j-m)(j-m')}}_{=\delta_{jj'}\delta_{mm'}} \cdot \frac{1}{\pi^2}\int_0^\infty d\rho_1 \frac{\rho_1^{\,2(j+m)+1}}{(j+m)!}\int_0^\infty d_2 \frac{\rho_2^{\,2(j-m)+1}}{(j-m)!}$$

$$\times e^{-(\rho_1^2+\rho_2^2)}$$

$$= \delta_{jj'}\,\delta_{mm'}\int_0^\infty d\rho_1\, 2\rho_1 \frac{\rho_1^{\,2(j+m)}}{(j+m)!}\, e^{-\rho_1^2}\int_0^\infty d\rho_2\, 2\rho_2 \frac{\rho_1^{\,2(j-m)}}{(j-m)!}\, e^{-\rho_2^2}$$

$$= \delta_{jj'}\,\delta_{mm'}\, \cancel{\frac{(j+m)!}{(j+m)!}}\; \cancel{\frac{(j-m)!}{(j-m)!}}$$

5. $D^j(g)$ is unitary ?

$$\sum_{\mu=-j}^{j} D^j_{m'\mu}\, D^j_{m\mu} \overset{?}{=} \delta_{mm'}$$

$$\delta_{mm'} = \langle P^j_m , P^j_{m'} \rangle$$

$$= \langle D(g)P^j_m , D(g)P^j_m \rangle$$

$$= \sum_{\mu=-j}^{j} \overline{D^j_{m\mu}} \sum_{\mu'=-j}^{j} D^j_{m'\mu'} \;\underbrace{\langle P^j_\mu , P^j_{\mu'} \rangle}_{=\delta_{\mu\mu'}}$$

components

$$= \sum_{\mu=-j}^{j}\sum_{\mu'=-j}^{j} \overline{D^j_{m\mu}}\, D^j_{m'\mu'}\,\delta_{\mu\mu'}$$

$$= \sum_{\mu=-j}^{j} \overline{D^j_{m\mu}}\, D^j_{m'\mu}$$