H18.

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1. $L = \{\Lambda : \Lambda^T \eta \Lambda = \eta\}$

$(\tilde{\Lambda}_{p'}(h) \cdot \tilde{\Lambda}_{p'}(h'))$

$= ([\underline{h} - 2(\underline{p}'\cdot\underline{h})\underline{p}'] \cdot [\underline{h}' - 2(\underline{p}'\cdot\underline{h}')\underline{p}'])$

$= (\underline{h}\cdot\underline{h}) + 4(\underline{p}'\cdot\underline{h})(\underline{p}'\cdot\underline{h}')\underbrace{\underline{p}'\cdot\underline{p}'}_{=1} - 2(\underline{p}'\cdot\underline{h})(\underline{p}'\cdot\underline{h}') - 2(\underline{p}'\cdot\underline{h}')(\underline{p}'\cdot\underline{h})$

$= (\underline{h}\cdot\underline{h})$ ✓

$\to \Lambda_{p'} \in L$

$\to \tilde{\Lambda}_{p'} = \Lambda_{p'} T \in L$

$\to \det(\eta) = \det(\Lambda_{p'}^T \eta \Lambda_{p'})$

$-1 = (\det\Lambda_{p'})^2 \, {-1}$

$\det\Lambda_{p'} = \pm 1 \quad \to \quad$ if $\det\Lambda_{p'} = -1$, $\det\tilde{\Lambda}_{p'} = +1$

Further $\quad \Lambda_{p'}(\underline{h}) = \underline{h} - 2(\underline{p}'\cdot\underline{h})\underline{p}'$

$\Lambda_{p'}(\underline{h})^\mu = \underline{h}^\mu - 2(\underline{p}'\cdot\underline{h})\underline{p}'^\mu$

$= \underline{h}^\mu - 2\eta_{\alpha\beta}\underline{p}'^\alpha \underline{h}^\beta \underline{p}'^\mu$

$= \sum_{a=0}^{3} \delta^{\mu a}\underline{h}^a - 2\eta_{\alpha\beta}\underline{p}'^\alpha \underline{p}'^\mu \delta^{\beta a}\underline{h}^a$

$= \sum_{a=0}^{3} (\delta^{\mu a} - 2\eta^{\alpha\beta}\underline{p}'_\alpha \underline{p}'_\mu \delta^{\beta a})\underline{h}^a$

$\longrightarrow$ somehow $\det\Lambda_{p'} = -1$

2. $\breve{\Lambda}_{\underline{p}'}(\underline{e}_0) = \Lambda_{\underline{p}'}(-\underline{e}_0) = -\underline{e}_0 + 2(\underline{p}'\cdot\underline{e}_0)\underline{p}' \overset{!}{=} \underline{p}$

$$\to (\underline{p}'\cdot\underline{e}_0)\underline{p}' = (\underline{p}+\underline{e}_0)\tfrac{1}{2}$$

$$p'^0 \ \underline{p}' = (\underline{p}+\underline{e}_0)/2$$

0-th component: $(p'^0)^2 = 1 + |\vec{p}'|^2 = (p'^0 + 1)/2$

$$\Leftrightarrow \ x = p'^0, \quad x^2 - \tfrac{1}{2}x - \tfrac{1}{2} = 0$$

$$(x-\tfrac{1}{2})^2 = \tfrac{1}{2} + \tfrac{1}{4} = \tfrac{3}{4}$$

$$\to p'^0 = \pm\tfrac{\sqrt{3}}{2} + \tfrac{1}{2}$$

$$p'^0 > 0, \quad p'^0 = \frac{\sqrt{3}+1}{2}$$

$$|\vec{p}'|^2 = \left(\frac{\sqrt{3}+1}{2}\right)^2 - 1 = \frac{3+1+2\sqrt{3}}{4} - 1 = \frac{\sqrt{3}}{2}$$

1-th component $\quad p'^0 \ p'^1 = \tfrac{1}{2} p^1$

$$\to p'^1 = \tfrac{1}{2} p^1 \cdot \frac{2}{\sqrt{3}+1} = \frac{p^1}{\sqrt{3}+1}$$

$$\to \vec{p'} = \frac{1}{\sqrt{3}+1}\vec{p}$$

and p must be chose according to $\left|\frac{1}{\sqrt{3}+1}\vec{p}\right|^2 = \frac{3}{4}$

3. $\forall g \in L_+, \quad g = g_0 \cdot \breve{\Lambda}_{\underline{p}'}, \quad g_0 \in SO(3) \quad ?$

$g_0 \in SO(3) \implies \det(g_0) = 1 \implies g \in L_+$

$g = g_0' \breve{\Lambda}_{p'} = g_0 \breve{\Lambda}_{k}, \quad (\underline{p}'\cdot\underline{p}') = 1, \ p'^0 > 0$, same for $\underline{k}$

$$(g_0'\breve{\Lambda}_{p'})^T \eta (g_0 \breve{\Lambda}_k) = \eta$$

$LHS = \breve{\Lambda}_{p'}^T g_0'^T \eta g_0 \breve{\Lambda}_k$

$RHS = \eta = \breve{\Lambda}^T_{p'} \eta \breve{\Lambda}_k$

$$\to g_0'^T \eta g_0 = \eta$$

$$g_0'^{-1} \eta g_0 = \eta$$

$$\eta g_0 = g_0' \eta$$

$\to \eta = \lambda \mathbb{1}$?

Schur's lemma. $\eta \neq 0$

$\Rightarrow g_0 = g_0'$

$\to \breve{\Lambda}_{p'} = \breve{\Lambda}_{\underline{k}} \Rightarrow$ unique

4. $\rho(\tilde{g}_1 \tilde{g}_2) h = \sigma^{-1}\big(\tilde{g}_1\tilde{g}_2\, \sigma(\underline{h}) \underbrace{(\tilde{g}_1\tilde{g}_2)^*}_{= \tilde{g}_2^* \tilde{g}_1^*}\big)$

$\rho(\tilde{g}_1)\rho(\tilde{g}_2) h = \sigma^{-1}\big(\tilde{g}_1\, \underbrace{\sigma(\rho(\tilde{g}_2) h)}\, \tilde{g}_1^*\big)$

$= \sigma^{-1}\big(\tilde{g}_1 \;\tilde{g}_2\, \sigma(\underline{h})\, \tilde{g}_2^* \;\tilde{g}_1^*\big)$

$\rightarrow \quad \rho(\tilde{g}_1\tilde{g}_2) h = \rho(\tilde{g}_1)\rho(\tilde{g}_2) h$

$(\rho(\tilde{g})\underline{h} \cdot \rho(\tilde{g})\underline{h}) = (\underline{h}\cdot\underline{h})$ since $\rho : SL(\mathbb{C}^2) \rightarrow L_+$

$\rightarrow$ is unitary rep.

5. $\epsilon\, \overline{\sigma(\underline{h})}\, \epsilon^{-1} = \sigma(P\underline{h})$

$LHS = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \underbrace{\overline{(\sigma^\mu h_\mu)}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$= \sigma^0 \underline{h}_0 + \sigma^1 \underline{h}_1 - \sigma^2 \underline{h}_2 + \sigma_3 \underline{h}_3$

$\epsilon\sigma^0\epsilon^{-1} = \mathbb{1} = \sigma^0$

$\epsilon\sigma^1\epsilon^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

$= -\sigma^1$

$\epsilon\sigma^2\epsilon^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\sigma^2$

$\epsilon\sigma^3\epsilon^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}$

$= \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix} = -\sigma^3$

$\rightarrow LHS = \sigma(P\underline{h})$

H.18

1. $(\underline{x}\cdot\underline{y}) = x^T \eta\, y = x^0 y^0 - (\vec{x}\cdot\vec{y})$

$(\Lambda \underline{x}\cdot \Lambda \underline{y}) = (\underline{x}\cdot\underline{y}) \quad \rightarrow \quad \boxed{\eta = \Lambda^T \eta \Lambda}$

$\Lambda_{\underline{p}'}(\underline{h}) = \underline{h} - 2(\underline{p}'\cdot\underline{h})\underline{p}' \quad , \quad (\underline{p}'\cdot\underline{p}') = 1, \quad p'^0 > 0$

$T\underline{h} = -h^0 \underline{e}_0 + \sum_{k=1}^{3} h^k \underline{e}_k$

$\tilde{\Lambda}_{\underline{p}'} = \Lambda_{\underline{p}'} T \quad \in \quad L_+ \quad ?$

Check the scarlar product

$(\tilde{\Lambda}_{\underline{p}'}\underline{x} \cdot \tilde{\Lambda}_{\underline{p}'}\underline{y})$

$= ((T\underline{x} - 2(p'\cdot T\underline{x})\underline{p}')\cdot(T\underline{y} - 2(\underline{p}'\cdot T\underline{y})\underline{p}'))$

$= (T\underline{x}\cdot T\underline{y}) - 2(\underline{p}'\cdot T\underline{x})(\underline{p}'\cdot T\underline{y}) - 2(\underline{p}'\cdot T\underline{y})(\underline{p}'\cdot T\underline{x}) + 4(\underline{p}'\cdot T\underline{x})(\underline{p}'\cdot T\underline{y})\underbrace{(\underline{p}'\cdot\underline{p}')}_{=1}$

$= (T\underline{x}\cdot T\underline{y})$

$= (\underline{x}\cdot\underline{y})$

Since $\tilde{\Lambda}_{\underline{p}'} \in L$, $\det(\tilde{\Lambda}_{\underline{p}'}) = \pm 1$

$\tilde{\Lambda}_{\underline{p}'} = \Lambda_{\underline{p}'} T$

Since smoothly depends on $\underline{p}'$, $(\underline{p}'\cdot\underline{p}') = 1$, $p'^0 > 0$

we can choose $\underline{p}' = \underline{e}_0 = (1,0,0,0)$ and the determinant value is valid for all $\underline{p}'$.

$\det(\tilde{\Lambda}_{\underline{p}'}) = \det(\tilde{\Lambda}_{\underline{e}_0})$

$\Lambda_{\underline{e}_0}(\underline{h}) = \underline{h} - 2(h^0)\underline{e}_0$

$(\Lambda_{\underline{e}_0}(\underline{h}))^0 = -(h^0) \quad \rightarrow \quad \Lambda_{\underline{e}_0}(\underline{h}) = (T\underline{h})$

$(\Lambda_{\underline{e}_0}(\underline{h}))^i = h^i \quad \rightarrow \quad \tilde{\Lambda}_{\underline{e}_0}(\underline{h}) = \underline{h}$

$\rightarrow \quad \det(\tilde{\Lambda}_{\underline{e}_0}(\underline{h})) = 1$

2. $\tilde{\Lambda}_{\underline{p}'}(\underline{e}_0) = \underline{p}$

$\tilde{\Lambda}_{\underline{p}'}(\underline{e}_0) = \Lambda_{\underline{p}'} T(\underline{e}_0) = -\Lambda_{\underline{p}'}(\underline{e}_0) = -\underline{e}_0 + 2(\underline{p}')^0\underline{p}' = \underline{p}$

$$p^0 = -1 + 2(p'^0)^2 , \qquad p^i = 2(p')^0 (p')^i$$

$$\rightarrow \quad (p')^0 = \left[\tfrac{1}{2}(1+p^0)\right]^{1/2} , \qquad (p')^i = \frac{p^i}{\sqrt{2(p^0+1)}}$$

$\rightarrow$ (Lorentz-) Boost !

3. $g \in L_+$, $\quad g = g_0 \tilde{\Lambda}_{\underline{p}'}$, $\quad g_0 \in \underbrace{SO(3)}_{X}$ $\quad SO(3.1)$!

$$g_0 = \left(\begin{array}{c|c} 1 & \vec{0}^T \\ \hline \vec{0} & R \end{array}\right) \quad \text{in 4 dim} \qquad \tilde{\Lambda}_{\underline{p}'} = \left(\begin{array}{c|c} x & \vec{y}^T \\ \hline \vec{y} & z \end{array}\right)$$

$R \in SO(3)$

$$g = \left(\begin{array}{c|c} \alpha & \vec{\beta}^T \\ \hline \vec{\gamma} & \delta \end{array}\right) \overset{!}{=} g_0 \tilde{\Lambda}_{\underline{p}'} = \left(\begin{array}{c|c} x & \vec{y}^T \\ \hline R\vec{y} & Rz \end{array}\right)$$

$\rightarrow \quad x = \alpha$ fixes p^0

$\vec{y}^T = \vec{\beta}^T$

$Rz = \delta$

$R\vec{y} = \vec{\gamma}$

L-group has 6 parameters

3 boosts, $\quad$ 3 rotations

$\tilde{\Lambda}_{\underline{p}'}$ with $(\underline{p}' \cdot \underline{p}')$ condition $\qquad R \longrightarrow g_0$

$$\rightarrow \quad g = \underset{\substack{\uparrow \\ \text{rotation}}}{g_0} \cdot \underset{\substack{\nwarrow \\ \text{boost}}}{\tilde{\Lambda}_{p'}}$$

4. $\rho : SL(\mathbb{C}^2) \rightarrow L_+$

$$\rho(\tilde{g})\underline{h} = \sigma^{-1}(\tilde{g}\,\sigma(\underline{h})\,\tilde{g}^*) , \qquad \sigma(\underline{h}) = h^\mu \sigma_\mu$$

$$\rho(\tilde{g}_1 \tilde{g}_2)\underline{h} = \sigma^{-1}(\underbrace{\tilde{g}_1 \tilde{g}_2}_{\mathbb{1} = \sigma\sigma^{-1}}\ \sigma(\underline{h}) \underbrace{(\tilde{g}_1 \tilde{g}_2)^*}_{= \tilde{g}_2^* \tilde{g}_1})$$

$$= \sigma^{-1}(\tilde{g}_1\, \underbrace{\sigma(\sigma^{-1}\tilde{g}_2\, \sigma(\underline{h})\, \tilde{g}_2^*)}_{= \rho(\tilde{g}_2^*)\underline{h}}\, \tilde{g}_1)$$

$$= \sigma^{-1}(\tilde{g}_1\, \sigma(\rho(\tilde{g}_2)\underline{h})\, \tilde{g}_1^*)$$

$$= \rho(\tilde{g}_1)(\rho(\tilde{g}_2)\underline{h}) \qquad \Rightarrow \quad \rho(\tilde{g}_1 \tilde{g}_2) = \rho(\tilde{g}_1)\rho(\tilde{g}_2)$$

$$\det(\sigma(\underline{h})) = \det\begin{pmatrix} h^0+h^3 & h^1-ih^2 \\ h^1+ih^2 & h^0-h^3 \end{pmatrix} = (h^0)^2-(h^3)^2-(h^2)^2-(h^1)^2 = (\underline{h}\cdot\underline{h})$$

$$\begin{aligned}(\rho(\tilde{g})\underline{h}\cdot\rho(\tilde{g})\underline{h}) &= \det(\sigma(\rho(\tilde{g})\underline{h}))\\ &= \det(\sigma(\sigma^{-1}(\tilde{g}\,\sigma(\underline{h})\,\tilde{g}^*)))\\ &= \det(\tilde{g}\,\sigma(\underline{h})\,\tilde{g}^*)\\ &= \underbrace{\det(\tilde{g})}_{=1}\det(\sigma(\underline{h}))\underbrace{\det(\tilde{g}^*)}_{=1}\\ &= \det(\sigma(\underline{h}))\\ &= (\underline{h}\cdot\underline{h})\end{aligned}$$

5. $\varepsilon\,\overline{\sigma(\underline{h})}\,\varepsilon^{-1} = \sigma(P\underline{h})$, $\quad P = -T$ parity operator

$$\varepsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{aligned}\varepsilon\,\overline{\sigma(\underline{h})}\,\varepsilon^{-1} &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} h^0+h^3 & h^1+ih^2 \\ h^1-ih^2 & h^0-h^3 \end{pmatrix}\begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}\\ &= \begin{pmatrix} h^0-h^3 & -h^1+ih^2 \\ -h^1-ih^2 & h^0+h^3 \end{pmatrix} = \begin{pmatrix} h^0+(-h^3) & (-h^1)-i(-h^2) \\ (-h^1)+i(-h^2) & h^0-(-h^3) \end{pmatrix}\\ &= \sigma(P\underline{h})\end{aligned}$$

H.19

1. spin $\frac{3}{2}$ $\quad \square\square\square \otimes \square\square\square$ ✓ SU(3)

$$\dim = Y_S^{u(3)}(1,1,1)$$
$$= K_S^{u(3)}(1,1,1) = \sum_{i\le j\le k=1}^{3} (1)^3 = 10 , \qquad I = \frac{3}{2} , \quad I_3 = -\frac{3}{2}, \underline{-1}, -\frac{1}{2}, \underline{0}, \frac{1}{2}, \underline{1}, \frac{3}{2}$$

decuplet states

spin $\frac{1}{2}$ $\quad \begin{array}{|c|c|}\hline \ & \ \\ \hline \ & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \otimes \begin{array}{|c|c|}\hline \ & \ \\ \hline \ & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$ ✓

$$\dim = Y_M^{u(3)}(1,1,1)$$
$$= K_S^{u(3)}(1,1,1) = 8 , \qquad I = 1, \quad I_3 = -1, \underline{-\tfrac{1}{2}}, 0, \underline{\tfrac{1}{2}}, 1$$

octet states

1. Still require |spin> symm ⊗ |flavor> symm $\quad (L = 0)$

or |spin> mixed ⊗ |flavor> mixed

a) $S = \frac{3}{2}$, $\square\square\square$ in flavor

$\boxed{u|u|u}$, $\boxed{u|u|d}$, $\boxed{u|d|d}$, $\boxed{d|d|d}$, $\qquad I = \frac{3}{2}$, $S^* = 0$

4 Δ - states

$\boxed{u|d|s}$, $\boxed{u|u|s}$, $\boxed{d|d|s}$ $\qquad I = 1$, $S^* = -1$

3 Σ^* - states

$\boxed{u|s|s}$, $\boxed{d|s|s}$ $\qquad I = \frac{1}{2}$, $S^* = -2$

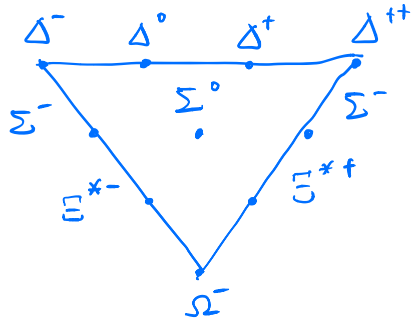
2 Ξ^* - states

$\boxed{s|s|s}$ $\qquad I = 0$, $S^* = -3$

Ω^- - state

(YT: strangeness -3)

$\longrightarrow$ 10 - states. (decuplet)

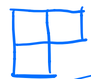


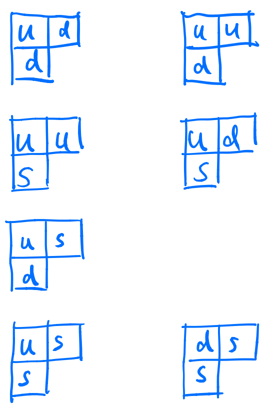
$S^* = 0$

$S^* = -1$

$S^* = -2$

$S^* = -3$

b) |spin> mixed $(S = \frac{1}{2})$, 



ordering of u, d according to symmetry of single column / row

$I = \frac{1}{2}$, $S^* = 0$
2 N-States

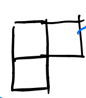
$I = 1$, $S^* = -1$
3 Σ-states

$I = 0$, $S^* = -1$
Λ-Baryon

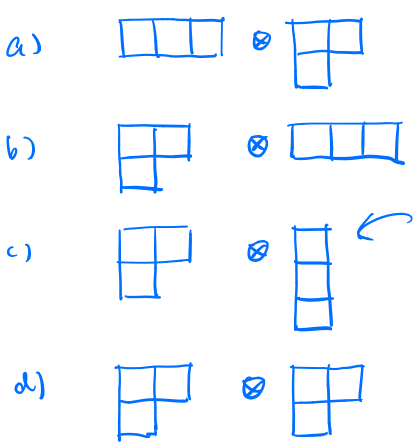
$I = \frac{1}{2}$, $S^* = -2$
2 Ξ-states

in a single column, there cannot be the same quark twice.

=> 8-states (octet)

2. $L = 1$, $\pi = -$ →  for orbital part

→ |spin> ⊗ |flavor> has to be mixed !



Y_A for flavor is now possible.
(Still Y_A for spin forbidden)

a) $(S^*; I; S; J^\pi) = (0; \frac{1}{2}, \frac{3}{2}; \frac{1}{2}^-, \frac{3}{2}^-, \frac{5}{2}^-)$,

$(-1; 1; \frac{3}{2}; \frac{1}{2}^-, \frac{3}{2}^-)$,

$$(-1;\ 0;\ \tfrac{3}{2};\ \tfrac{1}{2}^-,\ \tfrac{3}{2}^-,\ \tfrac{5}{2}^-),$$

$$(-2;\ \tfrac{1}{2};\ \tfrac{3}{2};\ \tfrac{1}{2}^-,\ \tfrac{3}{2}^-,\ \tfrac{5}{2}^-).$$

b) $(\ - - \ ) = (0;\ \tfrac{3}{2};\ \tfrac{1}{2};\ \tfrac{1}{2}^-,\ \tfrac{3}{2}^-),$

$$(-1;\ 1;\ \tfrac{1}{2};\ \tfrac{1}{2}^-,\ \tfrac{3}{2}^-),$$

$$(-2;\ \tfrac{1}{2};\ \tfrac{1}{2};\ \tfrac{1}{2}^-,\ \tfrac{3}{2}^-),$$

$$(-3;\ 0;\ \tfrac{1}{2};\ \tfrac{1}{2}^-,\ \tfrac{3}{2}^-).$$

c) $(\cdots) = (-1;\ 0;\ \tfrac{1}{2};\ \tfrac{1}{2}^-,\ \tfrac{3}{2}^-),$

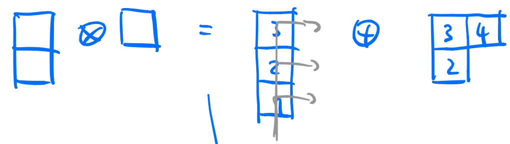
d) $- - \ = (0;\ \tfrac{1}{2};\ \tfrac{1}{2};\ \tfrac{1}{2}^-,\ \tfrac{3}{2}^-),$

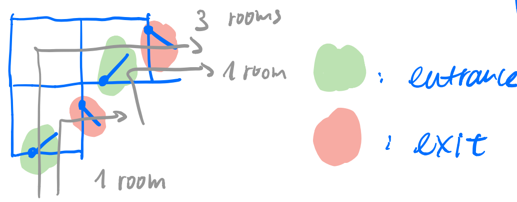
$$(-1;\ 1;\ \tfrac{1}{2};\ \tfrac{1}{2}^-,\ \tfrac{3}{2}^-),$$

$$(-2;\ \tfrac{1}{2};\ \tfrac{1}{2};\ \tfrac{1}{2}^-,\ \tfrac{3}{2}^-)$$

H.20

SU(3) : $\bar{3} \otimes 3 =$

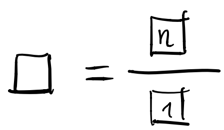




$= \frac{3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1} \oplus \frac{3 \cdot 4 \cdot 2}{3 \cdot 1 \cdot 1}$

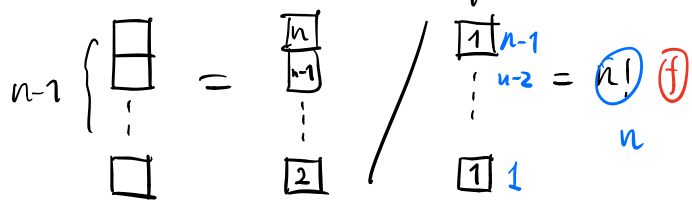
$= 1 \oplus 8$

1. fundamental reps.



$= n$ ✓

anti-fundamental reps:



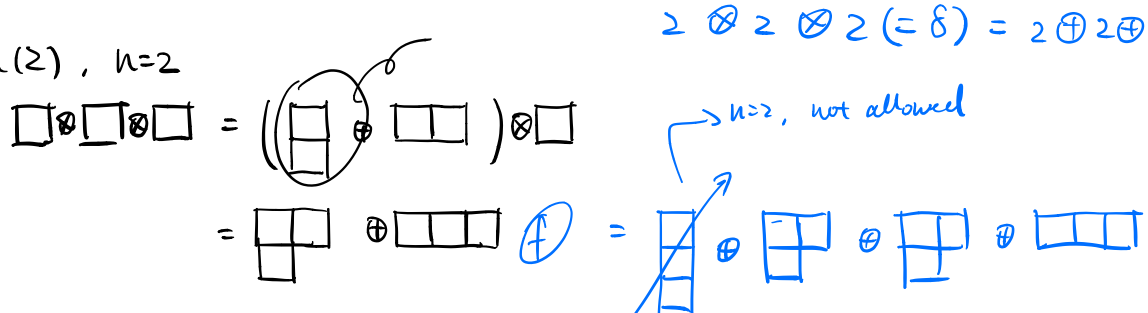
Fundamental and anti-fundamental rep. of SU(n) always have dimension n.

adjoint reps.

Adjoint has dim. n^2-1 (= # generators)

2. SU(2), n=2

$2 \otimes 2 \otimes 2 \, (=8) = 2 \oplus 2 \oplus 4$



$= \frac{2 \cdot 3 \cdot 1}{3 \cdot 1 \cdot 1} \oplus \frac{2 \cdot 3 \cdot 4}{1 \cdot 2 \cdot 3} = 2 \oplus 4$ → 6 baryons of u,d

SU(3)

$$\ydiagram{1}\otimes\ydiagram{1}\otimes\ydiagram{1} = \left(\ydiagram{1,1}\oplus\ydiagram{2}\right)\otimes\ydiagram{1}$$

$$= \ydiagram{1,1,1}\oplus\ydiagram{2,1}\oplus\ydiagram{2,1}\oplus\ydiagram{3}$$

$$\hat{=}\ 1\oplus\frac{3\cdot4\cdot2}{3\cdot1\cdot1}\oplus\frac{3\cdot4\cdot2}{3\cdot1\cdot1}\oplus\frac{3\cdot4\cdot5}{1\cdot2\cdot3}$$

$$= 1\oplus8\oplus8\oplus10$$ $\longrightarrow$ 27 baryons of u, d, s