H1. 1. $h \in H$, $h^+ = h$ and $tr[h] = 0$ Chenhuan Wang

$\Rightarrow h = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $a,b,c,d \in \mathbb{C}$

$tr[h] = 0 \Leftrightarrow h = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$

$h = h^* \Leftrightarrow \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & -\bar{a} \end{pmatrix}^T$

$\Rightarrow \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & -\bar{a} \end{pmatrix}$

$\Rightarrow a = \bar{a}$, $c = \bar{b}$, $b = \bar{c} \in \mathbb{C}$

$a \in \mathbb{R}$.

$\Rightarrow$ 3 DOF $\Rightarrow$ Dim $= 3$

2. $\langle a, b \rangle_{\mathcal{H}} := \frac{1}{2} tr[ab]$, $a, b \in \mathcal{H}$

① $\langle b, a \rangle^* = \frac{1}{2} tr[b^* a^*] = \frac{1}{2} tr[a^+ b^+] = \frac{1}{2} tr[ab] = \langle a, b \rangle$

② $\lambda \in \mathbb{C}$

$\langle a, \lambda b \rangle = \frac{1}{2} tr[a \cdot \lambda b] = \lambda \frac{1}{2} tr[ab] = \lambda \langle a, b \rangle$

$\langle a+b, c \rangle = \frac{1}{2} tr[(a+b)c] = \frac{1}{2} tr[ac + bc] = \langle a, c \rangle + \langle b, c \rangle$

③ $\langle a, a \rangle = \frac{1}{2} tr[a^2] = \frac{1}{2} tr \begin{pmatrix} a_1^2 & ? \\ ? & a_1^2 \end{pmatrix} > 0$

$\Rightarrow$ scalar product

3. **Definition 1.1.5.** If $I : G \leftrightarrow H$ is a bijective map between two groups G and H witl compositions "$*$" and "$\star$", respectively, and if for all $a, b \in G, I(a), I(b) \in H$, $I : G \leftrightarrow H$: $a \longleftrightarrow I(a)$ $b \longleftrightarrow I(b)$, the relation

$$I(a * b) = I(a) \star I(b),$$

holds, then I is called group isomorphism of G on H.

isometry $\neq$ isomorphism

$$\langle \sigma(h), \sigma(h') \rangle$$

$$= \frac{1}{2} \operatorname{tr}[\, \sigma(h)\, \sigma(h')\,]$$

$$= \frac{1}{2} \operatorname{tr}\Big[\, \sum_{i}^{3} h^{i} \sigma_i \sum_{j}^{3} h'^{j} \sigma_j \,\Big]$$

$$= \frac{1}{2} \operatorname{tr}\Big[\, \sum_{i,j}^{3} h^{i} h'^{j} \underbrace{\sigma_i \sigma_j}_{} \,\Big]$$

$$= \delta_{ij} \mathbb{1}_{\mathbb{C}^2} + i \sum_{m=1}^{3} \varepsilon_{ijm} \sigma_m$$

since $\operatorname{tr}[\sigma_i] = 0 \quad \forall\, i \in \{1,2,3\}$

$$= \frac{1}{2} \operatorname{tr}\Big[\, \underbrace{\sum_{i,j}^{3} h^{i} h'^{j} \delta_{ij}}_{\text{scalar}} \mathbb{1}_{\mathbb{C}^2} \,\Big]$$

$$= \sum_{i}^{3} h^{i} h'^{i} = (h \cdot h')$$

$$\Rightarrow \quad \sigma : \mathbb{R}^3 \longrightarrow \mathcal{H} \text{ is an isometry}$$

4. $g \in SU(2)$. $h \in \mathcal{H}$

to show $g h g^* \in \mathcal{H}$

$$(g h g^*)^* = g h g^*$$

$$\operatorname{tr}[g h g^*] = \operatorname{tr}[g^* g h]$$

$$= \operatorname{tr}[h] = 0$$

$$g = \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \qquad g^* = \begin{pmatrix} \bar{\alpha} & -\beta \\ \bar{\beta} & \alpha \end{pmatrix}^T = \begin{pmatrix} \bar{\alpha} & \bar{\beta} \\ -\beta & \alpha \end{pmatrix}$$

$$\Rightarrow g^* g = \begin{pmatrix} \bar{\alpha} & \bar{\beta} \\ -\beta & \alpha \end{pmatrix} \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

5. **Definition 2.1.2.** In general for each group G a <u>representation</u> D in a vector space V is characterised by:

$$D : G \to \mathrm{GL}(V) \text{ with } D(g_1\, g_2) = D(g_1)\, D(g_2)\,; \qquad D(e) = \mathbb{1}\,.$$

where $\mathrm{GL}(V)$ is the general linear group on V, *i.e.* the group of all linear transformations $V \to V$.

$$\lambda(g)h = g\,h\,g^* \Rightarrow \lambda : SU(2) \to GL(\mathcal{H}) \text{ is true}$$

$$LHS = \lambda(g_1 g_2)h = (g_1 g_2)\, h (g_1 g_2)^* = g_1 g_2 h g_2^* g_1^*$$

$$RHS = \lambda(g_2)h = g_2\, h\, g_2^*$$

$$\lambda(g_1)\big(\lambda(g_2)h\big) = g_1 (\lambda(g_2)h) g_1^*$$

$$= g_1\, g_2\, h\, g_2^* g_1^* = LHS$$

$\Rightarrow$ $\lambda(g)$ is a repre. of $SU(2)$ in $\mathcal{H}$

5. $\rho(g) = \sigma^{-1} \circ \lambda(g) \circ \sigma$ $\qquad (SU(2) \to O(3)\ ?)$

$$LHS = \rho(g_1 \cdot g_2) = \sigma^{-1} \circ \lambda(g_1 g_2) \circ \sigma$$

$$RHS = \rho(g_1)\rho(g_2) = \sigma^{-1} \circ \lambda(g_1) \circ \underbrace{\sigma \circ \sigma^{-1}}_{\mathbb{1}} \circ \lambda(g_2) \circ \sigma = LHS.$$

(λ is a repre. of $SU(2)$)

$\Rightarrow$ ρ is a homomorphism

6. $\rho(g)x$ $\qquad (g = \cos\varphi\,\mathbb{1} + i\sin\varphi\,\sigma(\omega))$

$$= \sigma^{-1} \circ \lambda(g) \circ \sigma\, x$$

$$= \sigma^{-1}(g\,\sigma(x)\,g^*)$$

$$= \sigma^{-1}\{(\cos\varphi\,\mathbb{1} + i\sin\varphi\,\sigma(\omega))(\sigma(x))(\cos\varphi\,\mathbb{1} - i\sin\varphi\,\sigma(\omega))\}$$

$$= \sigma^{-1}\Big(\cos^2\varphi\,\sigma(x) + \sin^2\varphi\,\sigma(\omega)\sigma(x)\sigma(\omega)$$
$$-i\cos\varphi\sin\varphi\,\sigma(x)\sigma(\omega) + \sin\varphi\cos\varphi\,\sigma(\omega)\sigma(x)\Big)$$

$$\sigma(x)\sigma(\omega) = \sum_{i,j} x^i \sigma_i\, \omega^j \sigma_j = \sum_{i,j} x^i \omega^j \Big(\delta_{ij}\mathbb{1} + \mathbb{1}\sum_k \varepsilon_{ijk}\sigma^k\Big)$$

$$= x \cdot \omega + \sum_{i,j,k} x^i \omega^j \varepsilon_{ijk}\sigma^k = x\cdot\omega\mathbb{1} + \sigma([x \times \omega])$$

$$\sigma(\omega)\sigma(x) = x\cdot\omega\underline{\mathbb{1}} + \sigma([\omega \times x])$$

$$\sigma(\omega)\,\sigma(x)\,\sigma(\omega) = \sum_i \omega^i \sigma_i \;\big(x\cdot\omega + \sigma([x\times\omega])\big)$$

$$= \sigma(\omega)\, x\cdot\omega + \sum_{ijkm} x^i \omega^j \varepsilon_{ijk} \underbrace{\sigma^k \omega^m \sigma_m}_{=\delta_{km}\mathbb{1} + \sum_n \varepsilon_{kmn}\sigma^n}$$

$$= \sigma(\omega)\, x\cdot\omega + \sum_{ijk} x^i \omega^j \varepsilon_{ijk} \omega^k \mathbb{1}$$

$$+ \sum_{ijkn} x^i \omega^j \underbrace{\varepsilon_{ijk} \omega^n \varepsilon_{kmn}}_{\substack{= \varepsilon_{kij}\varepsilon_{kmn} \\ = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}}} \sigma^n$$

$$= \sum_{ijkn} x^i \omega^j \omega^m (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})\sigma^n$$

$$= \sum_{ij} x^i \omega^j \omega_i \sigma_j - x^i \omega^j \omega_j \sigma_i$$

$$= x\cdot\omega\, \sigma(\omega) - \omega\cdot\omega\, \sigma(x)$$

$$= \sigma(\omega)\, x\cdot\omega + [x\times\omega]\cdot\omega\, \mathbb{1} + x\cdot\omega\,\sigma(\omega) - \underbrace{\omega^2}_{=1}\sigma(x)$$

$$\Rightarrow \quad \rho(g)\, x = \cos^2\varphi\, x + \sin^2\varphi\, \sigma^{-1}\big(\sigma(\omega)\cdot x\cdot\omega + \underbrace{[x\times\omega]\cdot\omega\,\mathbb{1}}_{\notin \mathcal{H}} + x\cdot\omega\,\sigma(\omega) - \sigma(x)\big)$$

$$+ i\cos\varphi\sin\varphi\, \sigma^{-1}\big(-\cancel{x\cdot\omega\mathbb{1}} - \sigma([x\times\omega]) + \cancel{x\cdot\omega\mathbb{1}} + \sigma([\omega\times x])\big)$$

$$= \cos^2\varphi\, x + \sin^2\varphi\big[(x\cdot\omega)\omega + (x\cdot\omega)\omega - x\big]$$

$$+ 2i\cos\varphi\sin\varphi\,[\omega\times x]$$

$$= \cos 2\varphi\, x + \underbrace{2\sin^2\varphi}_{(1-\cos 2\varphi)}(x\cdot\omega)\omega + 2i\underbrace{\sin\varphi\cos\varphi}_{\frac{1}{2}\sin 2\varphi}[\omega\times x]$$

$$= \cos 2\varphi\,(x - \omega(x\cdot\omega)) + \underset{\sim}{i}\,\sin 2\varphi\,[\omega\times x] + (x\cdot\omega)\,\omega$$

→ very close to (7)