

H.7

1. $g \in \mathrm{SU}(2)$. $\det(g) = 1$, $gg^* = 1$,

*invertible
 $\det = 1$*

$u \in \mathrm{SU}(2)$ Suppose $u^{-1}gu = \mathrm{diag}(\lambda, \bar{\lambda})$ why does this hold?

$$\Rightarrow \det(u^{-1}gu) = \det(g) \underbrace{\det(u^{-1})\det(u)}_1 = \lambda\bar{\lambda} = |\lambda| = 1$$

$$g \rightarrow A^{-1}gA = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = D, \quad A, g \in \mathrm{SU}(2)$$

$$D^*D = (A^{-1}gA)^*(A^{-1}gA) = 1$$

$$1 = \begin{pmatrix} \bar{\lambda}_1 & \bar{\lambda}_2 \\ \lambda_1 & \lambda_2 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} |\lambda_1|^2 & 0 \\ 0 & |\lambda_2|^2 \end{pmatrix}$$

$$\Rightarrow |\lambda_1|^2 = |\lambda_2|^2 = 1$$

$$\det(D) = \lambda_1\lambda_2 = 1$$

$$\underbrace{|\lambda_1|}_1^2 \lambda_2 = \bar{\lambda}_1$$
$$\Rightarrow \bar{\lambda}_1 = \lambda_2$$

2. $\mathcal{K}_j(g) = \mathrm{tr}[D_j(g)]$

$$= \mathrm{tr}[D_j(u g_0 u^*)]$$

$$= \mathrm{tr}[D_j(u) D_j(g_0) D_j(u^*)]$$

$$= \mathrm{tr}[D_j(u^*) D_j(g_0)]$$

$$= \mathrm{tr}[D_j g_0]$$

$$= \underset{j}{\mathcal{K}}(g_0)$$

3. $\mathcal{K}_j(g) = \mathcal{K}_j(g_0) = \mathrm{tr}[D_j(g_0)] = \sum_{m=-j}^j D_{jj}^{mm}(g_0)$

$$= \sum_{m=-j}^j \langle P_j^m, \underbrace{D_j(g_0) P_j^m}_{P_j^m} \rangle \rightarrow P_j^m (\bar{\lambda}_1 \lambda_2) \binom{\bar{\lambda}_1}{\lambda_2}$$

$$= \sum_{n=-j}^j \langle D_j(g_0^{-1}) P_j^m, P_j^m \rangle$$

$$\begin{aligned}
&= \sum_{m=-j}^j \langle P_j^m(g, z), P_j^m(z) \rangle \\
&= \sum_{m=-j}^j \frac{1}{\pi^4} \int d^4z \bar{P}_j^m(g, z) P_j^m(z) e^{-|z|^2} \\
&\quad \left. \begin{array}{l} \text{the exchange} \\ \text{of } \lambda, \bar{\lambda} \hookrightarrow \\ \text{is not important,} \\ \text{since we haven't} \\ \text{imposed any condition} \\ \text{on which is which.} \end{array} \right\} \\
&= \sum_{m=-j}^j \frac{1}{\pi^4} \int d^4z \bar{P}_j^m(z) \lambda^{j+m} \bar{\lambda}^{j-m} P_j^m(z) e^{-|z|^2} \\
&\quad \left[g, z = \begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \lambda z_1 \\ \bar{\lambda} z_2 \end{pmatrix} \right] \\
&= \sum_{m=-j}^j \lambda^{j+m} \bar{\lambda}^{j-m} \underbrace{\langle P_j^m, P_j^m \rangle}_{=1} \\
&\stackrel{?}{=} \frac{\lambda^{2j+1} - \bar{\lambda}^{2j+1}}{\lambda - \bar{\lambda}}
\end{aligned}$$

$$j=1, \quad \text{LHS} = \sum_{m=-1}^1 \lambda^{1+m} \bar{\lambda}^{1-m} = \lambda^0 \bar{\lambda}^2 + \lambda^1 \bar{\lambda}^1 + \lambda^2 \bar{\lambda}^0 \\
= \bar{\lambda}^2 + \lambda \bar{\lambda} + \lambda^2$$

$$\text{RHS} = \frac{\lambda^3 - \bar{\lambda}^3}{\lambda - \bar{\lambda}} \quad \left| \begin{array}{l} \lambda^3 - 3\lambda^2 \bar{\lambda} + 3\lambda \bar{\lambda}^2 - \bar{\lambda}^3 + 3\lambda^2 \bar{\lambda} - 3\lambda \bar{\lambda}^2 \\ = (\lambda - \bar{\lambda})^3 + 3\lambda \bar{\lambda}(\lambda - \bar{\lambda}) \\ = (\lambda - \bar{\lambda}) \underbrace{((\lambda - \bar{\lambda})^2 + 3\lambda \bar{\lambda})}_{= \lambda^2 + \lambda \bar{\lambda} + \bar{\lambda}^2} \end{array} \right.$$

$$= \lambda^2 + \lambda \bar{\lambda} + \bar{\lambda}^2$$

$$\text{Valid for } j \quad \stackrel{?}{=} \text{LHS} \quad \rightarrow \sum_{m=-j}^j \lambda^{j+m} \bar{\lambda}^{j-m} = \frac{\lambda^{2j+1} - \bar{\lambda}^{2j+1}}{\lambda - \bar{\lambda}}$$

Check for $j+1$

$$\text{LHS} = \sum_{m=-j-1}^{j+1} \lambda^{j+1+m} \bar{\lambda}^{j+1-m}$$

$$\begin{aligned}
&= \lambda \bar{\lambda} \sum_{m=-j-1}^{j+1} \lambda^{j+m} \bar{\lambda}^{j-m} \\
&= \lambda \bar{\lambda} \sum_{m=-j}^j \lambda^{j+m} \bar{\lambda}^{j-m} + \bar{\lambda}^{j+2} + \lambda^{j+2} \\
&= \lambda \bar{\lambda} \frac{\lambda^{2j+1} - \bar{\lambda}^{2j+1}}{\lambda - \bar{\lambda}} + \bar{\lambda}^{j+2} + \lambda^{j+2} \\
&= \frac{(\bar{\lambda}^{j+2} + \lambda^{j+2})(\lambda - \bar{\lambda})}{\lambda - \bar{\lambda}} \\
&= \frac{\lambda \bar{\lambda}^{j+2} - \bar{\lambda}^{j+3} + \lambda^{j+3} - \bar{\lambda}^{j+2}}{\lambda - \bar{\lambda}}
\end{aligned}$$

}

$$\begin{aligned}
&= \frac{1}{\lambda - \bar{\lambda}} (\cancel{\lambda^{j+2} \bar{\lambda}} - \cancel{\lambda \bar{\lambda}^{j+2}} + \cancel{\lambda \bar{\lambda}^{j+2}} - \cancel{\bar{\lambda}^{j+3}} + \cancel{\lambda^{j+3}} - \cancel{\lambda^{j+2} \bar{\lambda}}) \\
&= \frac{\lambda^{j+3} - \bar{\lambda}^{j+3}}{\lambda - \bar{\lambda}}
\end{aligned}$$

\Rightarrow Statement is correct

$$\Rightarrow \sum_{m=-j}^j \lambda^{j+m} \bar{\lambda}^{j-m} = \frac{\lambda^{2j+1} - \bar{\lambda}^{2j+1}}{\lambda - \bar{\lambda}}$$

Here we property / relation of $\lambda, \bar{\lambda}$ is used

\Rightarrow generalized to any $\lambda \rightarrow x, \bar{\lambda} \rightarrow y$

Alternative:

use $\sum_{k=0}^m q^k = \frac{1-q^{m+1}}{1-q}$

$$\begin{aligned}
& \sum_{m=-j}^j \lambda^{j+m} \bar{\lambda}^{j-m} \\
&= \sum_{m=0}^j \lambda^{j+m} \bar{\lambda}^{j-m} + \sum_{m=1}^j \lambda^{j-m} \bar{\lambda}^{j+m} \\
&= (\lambda \bar{\lambda})^j \left[\frac{1 - (\lambda/\bar{\lambda})^{j+1}}{1 - (\lambda/\bar{\lambda})} + \frac{1 - (\bar{\lambda}/\lambda)^{j+1}}{1 - (\bar{\lambda}/\lambda)} - 1 \right] \\
&= \frac{(\lambda \bar{\lambda})^j \bar{\lambda} - \lambda^{j+1}}{\bar{\lambda} - \lambda} + \frac{\bar{\lambda}^{j+1} - (\lambda \bar{\lambda})^j \lambda}{\bar{\lambda} - \lambda} - (\lambda \bar{\lambda})^j \\
&= \frac{\lambda^{j+1} - \bar{\lambda}^{j+1}}{\lambda - \bar{\lambda}} + \underbrace{\frac{\lambda^j \bar{\lambda}^{j+1} - \lambda^{j+1} \bar{\lambda}^j - (\lambda \bar{\lambda})^j (\bar{\lambda} - \lambda)}{\bar{\lambda} - \lambda}}_{=0}
\end{aligned}$$

4. $k_j(e) = \sum_{m=-j}^j (1)^{j+m} (1)^{j-m} = 2j+1$

$$k_j(e) = \sum_m \langle P_j^m, D(e) P_j^m \rangle = \dim V_j$$

5. $g = \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}, |\alpha|^2 + |\beta|^2 = 1 \Leftrightarrow \det(g) = 1$

$$\begin{aligned}
\det \begin{pmatrix} \alpha - \lambda & -\bar{\beta} \\ \beta & \bar{\alpha} - \lambda \end{pmatrix} &= (\alpha - \lambda)(\bar{\alpha} - \lambda) + \bar{\beta}\beta = 1 + \lambda^2 - (\alpha + \bar{\alpha})\lambda = 0 \\
&\Rightarrow \lambda^2 - (\alpha + \bar{\alpha})\lambda + \frac{1}{4}(\alpha + \bar{\alpha})^2 = \frac{1}{4}(\alpha + \bar{\alpha})^2 - 1 \\
&\Rightarrow \lambda_{1,2} = \frac{\sqrt{(\alpha + \bar{\alpha})^2 - 4}}{2} + \frac{(\alpha + \bar{\alpha})}{2} \\
&\lambda_{1,2} = \lambda, \bar{\lambda}
\end{aligned}
\right\} ?$$

$$\operatorname{tr}[g] = \alpha + \bar{\alpha} = \operatorname{tr}[g_0] = \lambda + \bar{\lambda}$$

$$k_j(\lambda) = \frac{\left[\frac{1}{2} \left(j(\alpha + \bar{\alpha})^2 - 4 \right) + (\alpha + \bar{\alpha}) \right]^{2j+1} - \left[\frac{1}{2} \left(-j(\alpha + \bar{\alpha})^2 - 4 \right) + (\alpha + \bar{\alpha}) \right]^{2j+1}}{\sqrt{(\alpha + \bar{\alpha})^2 - 4}}$$

$$= \frac{1}{2^{2j+1}} \frac{\left(\sqrt{(\alpha+\bar{\alpha})^2 - 4} + (\alpha+\bar{\alpha}) \right)^{2j+1} \left[-\sqrt{(\alpha+\bar{\alpha})^2 - 4} + (\alpha+\bar{\alpha}) \right]^{2j+1}}{\sqrt{(\alpha+\bar{\alpha})^2 - 4}}$$

$$\sum_{k=0}^{2j+1} \binom{2j+1}{k} \left\{ \left(\sqrt{(\alpha+\bar{\alpha})^2 - 4} \right)^k (\alpha+\bar{\alpha})^{2j+1-k} \right.$$

$$\left. + (-1)^k \sqrt{(\alpha+\bar{\alpha})^2 - 4}^k (\alpha+\bar{\alpha})^{2j+1-k} \right\}$$

for $k \in 2\mathbb{Z} + 1$, the summand vanishes

$$= \frac{1}{2^{2j+1} \sqrt{(\alpha+\bar{\alpha})^2 - 4}} \sum_{\substack{k=0 \\ k \in 2\mathbb{Z}}}^{2j+1} \binom{2j+1}{k} \left(\sqrt{(\alpha+\bar{\alpha})^2 - 4} \right)^k (\alpha+\bar{\alpha})^{2j+1-k}$$

$$\rightarrow \binom{2j+1}{0} (\alpha+\bar{\alpha})^{2j+1} + \binom{2j+1}{2} \sqrt{(\alpha+\bar{\alpha})^2 - 4}^2 (\alpha+\bar{\alpha})^{2j+1-2}$$

$$+ \dots \binom{2j+1}{2j} ()^{2j+1} ()^1$$

$$= (2j+1) \left[\binom{2j}{0} A^0 B^{2j+1} + \binom{2j}{2} A^2 B^{2j-1} + \dots + \binom{2j}{2j} A^{2j} B \right]$$

$$= (2j+1) B \underbrace{\left[\binom{2j}{0} A^0 B^{2j+1} + \binom{2j}{2} A^2 B^{2j-1} + \dots \right]}$$

$$= \frac{1}{2^{2j} A} (2j+1) B \sum_{k=0}^j \binom{2j}{2k} A^{2k} B^{2j-2k}$$

$$\det(g - \lambda I) \Rightarrow \lambda^2 - 2\operatorname{Re}(\alpha)\lambda + 1 = 0$$

$$\Rightarrow \lambda = \operatorname{Re}(\alpha) \pm i\sqrt{-\operatorname{Re}^2(\alpha) + 1}$$

$$\Rightarrow \text{if } \operatorname{Re}(\alpha) = \pm 1, \quad K_j = \dim V_j$$

$$\text{if } \operatorname{Re}(\alpha) \neq \pm 1, \quad k_j = \frac{(\operatorname{Re}(\alpha) + i\sqrt{1-\operatorname{Re}^2(\alpha)})^{j+1} - (\operatorname{Re}(\alpha) - i\sqrt{1-\operatorname{Re}^2(\alpha)})^{j+1}}{2i\sqrt{1-\operatorname{Re}^2(\alpha)}}$$

$$6. \quad \alpha = \cos \varphi - i \sin \varphi \cos \theta \quad \Rightarrow \quad \alpha + \bar{\alpha} = 2 \cos \varphi$$

$$\beta = \sin \varphi \sin \theta (\cos \varphi + i \sin \varphi)$$

$$\hookrightarrow \lambda_{1,2} = \cos \varphi \pm i \sin \varphi = e^{\pm i \varphi}$$

$$A = \sqrt{(\alpha + \bar{\alpha})^2 - 4} = i \sin \varphi$$

$$B = \alpha + \bar{\alpha} = 2 \cos \varphi$$

$$\hookrightarrow = \frac{1}{2^{2j} A} (2j+1) B \sum_{k=0}^j \binom{2j}{2k} \begin{matrix} A^{2k} \\ | \\ -\sin^k \varphi \end{matrix} \begin{matrix} B^{2j-2k} \\ | \\ 4 \cos^{j-k} \varphi \end{matrix}$$

$$k_j(g) = \frac{e^{i(2j+1)\varphi} - e^{-i(2j+1)\varphi}}{e^{i\varphi} - e^{-i\varphi}} = \frac{2i \sin((2j+1)\varphi)}{2i \sin \varphi}$$

$$7. \quad g_{kl} = 2 \begin{pmatrix} 1 & \sin^2 \varphi & \\ & \sin^2 \varphi \sin^2 \theta & \end{pmatrix}$$

$$dg = \sqrt{\det(g_{kl})} d\varphi d\theta d\varphi$$

$$\det(g_{kl}) = 2 \sin^4 \varphi \sin^2 \theta \quad \Rightarrow \quad \sqrt{\det(g)} = \sqrt{8} \sin^2 \varphi \sin \theta$$

$$\Rightarrow dg = \sqrt{8} \sin^2 \varphi \sin \theta d\theta d\varphi$$

$$\alpha = x_0^1 + i x_0^2, \quad \beta = x_0^3 + i x_0^4$$

$x_0 = x_0^1 e_1 + x_0^2 e_2 + x_0^3 e_3 + x_0^4 e_4$, $\{e_i\}$ Basis of \mathbb{R}^4

$$g_{kl} = \operatorname{Re} \left\{ \underbrace{\operatorname{tr} \left[\left(\frac{\partial}{\partial x^k} g^k \right) \left(\frac{\partial}{\partial x^l} g^l \right) \right] }_{=: B} \right\}, \quad g = \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}$$

$$\frac{\partial}{\partial x^k} = \partial_k$$

$$g^* = \begin{pmatrix} \bar{\alpha} & \bar{\beta} \\ -\beta & \bar{\alpha} \end{pmatrix}$$

$$\Rightarrow B = \operatorname{tr} \left[\begin{pmatrix} \partial_k \bar{\alpha} & \partial_k \bar{\beta} \\ -\partial_k \beta & \partial_k \alpha \end{pmatrix} \begin{pmatrix} \partial_l \alpha & -\partial_l \bar{\beta} \\ \partial_l \beta & \partial_l \bar{\alpha} \end{pmatrix} \right]$$

$$= \operatorname{tr} \begin{pmatrix} \partial_k \bar{\alpha} \cdot \partial_l \alpha + \partial_k \bar{\beta} \partial_l \beta & \cdots \\ \cdots & \partial_l \beta \partial_k \bar{\beta} + \partial_k \alpha \partial_l \bar{\alpha} \end{pmatrix}$$

$$= \partial_k \bar{\alpha} \cdot \partial_l \alpha + \partial_k \bar{\beta} \partial_l \beta + \partial_l \beta \partial_k \bar{\beta} + \partial_k \alpha \partial_l \bar{\alpha}$$

$$g_{kl} = \operatorname{Re}(B) = 2 \sum_{p=1}^4 (\partial_k x_0^p)(\partial_l x_0^p) = 2 \left(\frac{\partial}{\partial x^k} x_0 \cdot \frac{\partial}{\partial x^l} x_0 \right)$$

$$|x_0|^2 = (x_0 \cdot \bar{x}_0) = \operatorname{Re}^2(\alpha) + \operatorname{Im}^2(\alpha) + \operatorname{Re}^2(\beta) + \operatorname{Im}^2(\beta)$$

$$= |\alpha|^2 + |\beta|^2 = 1 \rightarrow S^3$$

$$\rightarrow g = \begin{pmatrix} 1 & \sin^2 \gamma & \sin^2 \gamma \sin^2 \theta \\ & \sin^2 \gamma & \sin^2 \gamma \sin^2 \theta \end{pmatrix}$$

$$\rightarrow dg = \sqrt{8} \sin^3 \gamma \sin \theta d\gamma d\theta d\psi$$

$$\langle k_j, k_k \rangle = \int_0^{2\pi} d\gamma \sin^2 \gamma \int_0^{2\pi} d\theta \sin \theta \int_0^\pi d\psi \underbrace{\overline{k_j}(g(\gamma, \theta, \psi))}_{\frac{\sin((j+1)\gamma)}{\sin \gamma}} \underbrace{k_k(g(\gamma, \theta, \psi))}_{\frac{\sin((k+1)\gamma)}{\sin \gamma}}$$

$$= 2\pi \int_0^\pi d\gamma \sin((j+1)\gamma) \sin((2k+1)\gamma)$$

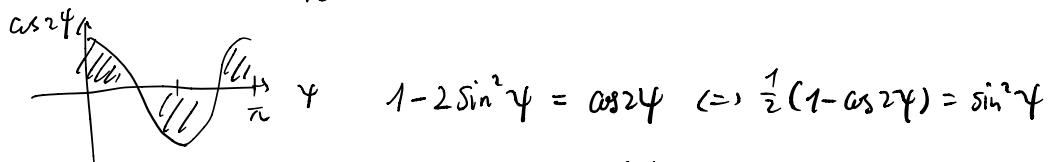
$$= -\frac{\pi}{4} \int_0^\pi d\gamma (e^{i[(2j+1)+(2k+1)]\gamma} - e^{i[(2j+1)-(2k+1)]\gamma} - e^{i[(2k+1)-(2j+1)]\gamma})$$

$$x e^{-i[(2j+1)+(2k+1)]\gamma} = -\frac{\pi}{4} (-2\pi - 2\pi) \delta_{kj}$$

$$= 2\pi^2 \delta_{kj} = V(a) \delta_{kj}$$

H.8

$$\begin{aligned} 1. \int dg &= \underbrace{\int_0^{2\pi} \sin^2 \gamma d\gamma}_{= \frac{1}{2} \int_0^{2\pi} (1 - \cos 2\gamma) d\gamma} \underbrace{\int_0^{\frac{\pi}{2}} \sin \theta d\theta}_{= -[\cos \theta]_0^{\frac{\pi}{2}}} \underbrace{\int_0^\pi d\varphi}_{= 1} = 2\pi^2 \\ &= -\left[\cos \theta \right]_0^{\frac{\pi}{2}} = -(0 - 1) \\ &= \frac{1}{2} \left(2\pi - \frac{1}{2} \cdot 0 \right) = 1 \end{aligned}$$



$$2. \sum_{k=-j}^j x^{j+k} y^{j-k} = \frac{x^{2j+1} - y^{2j+1}}{x - y}, \text{ see H7.3}$$

$$\frac{\sin(2j+1)\gamma}{\sin \gamma} \stackrel{?}{=} \sum_{k=0}^{2j} e^{i(j-k)\gamma}$$

$$\text{RHS} = \sum_{k=0}^{2j} e^{i(j-k)\gamma} e^{i(j-k)\gamma}$$

$$(\tilde{k} = k - j \Leftrightarrow k = j + \tilde{k})$$

$$= \sum_{\tilde{k}=-j}^j e^{-2i\tilde{k}\gamma} = \sum_{\tilde{k}=-j}^j e^{i(j-\tilde{k})\gamma} e^{-i(j+\tilde{k})\gamma}$$

$$(x = e^{i\gamma}, y = e^{-i\gamma})$$

$$= \frac{(e^{-i\gamma})^{2j+1} - (e^{+i\gamma})^{2j+1}}{e^{-i\gamma} - e^{+i\gamma}}$$

$$= \frac{\cos[(2j+1)\gamma] - i\sin[(2j+1)\gamma] - \cos[(2j+1)\gamma] - i\sin[(2j+1)\gamma]}{\cos i\gamma - i\sin \gamma - \cos i\gamma - i\sin \gamma}$$

$$= \frac{\sin(2j+1)\gamma}{\sin\gamma}$$

$$\frac{1}{V(a)} \int_a dg \overline{K_j(g)} K_{j_1}(g) K_{j_2}(g)$$

$$= \frac{1}{2\pi^2} \cdot 2\pi \int_0^{2\pi} d\gamma \sin^2\gamma \underbrace{\int_0^{\pi/2} d\theta \sin\theta}_{=1} \frac{\sin(2j+1)\gamma}{\sin\gamma} \frac{\sin(2j_1+1)\gamma}{\sin\gamma} \frac{\sin(2j_2+1)\gamma}{\sin\gamma}$$

$$= \frac{1}{\pi} \int_0^{2\pi} d\gamma \frac{\sin(2j+1)\gamma \sin(2j_1+1)\gamma \sin(2j_2+1)\gamma}{\sin\gamma}$$

$$= \frac{1}{\pi} \int_0^{2\pi} d\gamma \left(\sum_{k=0}^{2j} e^{2ik(j-k)\gamma} \right) \frac{e^{i(2j+1)\gamma} - e^{-i(2j+1)\gamma}}{2i} \frac{e^{i(2j_1+1)\gamma} - e^{-i(2j_1+1)\gamma}}{2i}$$

$$= -\frac{1}{4\pi} \int_0^{2\pi} d\gamma \sum_{k=0}^{2j} \left[e^{2i(j-k+j_1+j_2+1)\gamma} - e^{2i(j-k+j_1-j_2)\gamma} - e^{2i(j-k-j_1+j_2)\gamma} - e^{2i(j-k-j_1-j_2-1)\gamma} \right]$$

$$= -\frac{1}{2} \sum_{k=0}^{2j} (\delta_{k,j+j_1+j_2+1} - \delta_{k,j-j_1+j_2} - \delta_{k,j+j_1-j_2} + \delta_{k,j-j_1-j_2-1})$$

$$= -\frac{1}{2} \sum_{k=-j}^j (\delta_{k,j+j_2+1} - \delta_{k,-j_1+j_2} - \delta_{k,j_1-j_2} + \delta_{k,-j_1-j_2-1})$$

$$= \begin{cases} j > j_1 + j_2 : & = 0 \\ j < |j_1 - j_2| : & = 0 \\ |j_1 - j_2| \leq j \leq j_1 + j_2 : & = 1 \end{cases} \quad \rightarrow \text{coupling of angular mom.}$$