H.21

1. $$ad(x)\, y = [x, y]\,, \qquad x, y \in A$$

$$ad([x,y]) = [ad(x), ad(y)]_-$$

$$\begin{aligned}
ad([x,y])\, z &= [[x,y], z] \\
&= -[z, [x,y]] \\
&= [x, [y,z]] + [y, [z,x]] \\
&= ad(x)\, [y,z] - ad(y)\, [x,z] \\
&= ad(x)(ad(y)\, z) - ad(y)(ad(x)\, z)
\end{aligned}$$

$$\rightarrow \quad \begin{aligned} ad([x,y]) &= ad(x)\, ad(y) - ad(y)\, ad(x) \\ &= [ad(x), ad(y)]_- \end{aligned}$$

2. $$\langle x, y \rangle = -tr[ad(x)\, ad(y)]$$

$$\begin{aligned}
\langle x, ad(y) z \rangle &= -tr[ad(x)\, ad(ad(y) z)] \\
&= -tr[ad(x)\, ad([y,z])] \\
&= -tr[ad(x)\, [ad(y), ad(z)]_-] \\
&= -tr[ad(x)\, ad(y)\, ad(z) - ad(x)\, ad(z)\, ad(y)] \\
&= -tr[[ad(x), ad(y)]_-\, ad(z)] \\
&= -tr[ad([x,y])\, ad(z)] \\
&= -tr[ad(ad(x)\, y)\, ad(z)] \\
&= \langle ad(x)\, y, z \rangle \\
&= \langle ad(y)\, x, z \rangle
\end{aligned}$$

H.22

Let $[a,b] = [a,b]_-$

Cartan algebra H: $\quad x, y \in H\,, \quad [x,y] = 0$

Basis $\qquad h_k y = i(\langle e_k, y \rangle e_k - \frac{1}{n} y)$

$k = 1, 2, \ldots, n-1 \;;\quad e_k\,,\ k = 1, \ldots, n \quad$ Standard ONB for $\mathbb{C}^n$

Matrix elements: $\langle e_m, h_k e_\ell \rangle = i(\delta_{mk}\delta_{\ell k} - \frac{1}{n}\delta_{m\ell})$

$$A = H \oplus H_\perp$$

Basis of $H_\perp$: $\quad e^+_{k\ell}\, y = \langle e_\ell, y\rangle e_k - \langle e_k, y\rangle e_\ell$

$(1 \leq k < \ell \leq n)$ $\quad e^-_{k\ell}\, y = i(\langle e_\ell, y\rangle e_k + \langle e_k, y\rangle e_\ell)$

Matrix element: $\quad \langle e_q, e^+_{k\ell} e_r\rangle = \delta_{\ell q}\delta_{rk} - \delta_{kq}\delta_{r\ell}$

$$\langle e_q, e^-_{k\ell} e_r\rangle = i(\delta_{\ell q}\delta_{rk} + \delta_{kq}\delta_{r\ell})$$

(Definition of) the roots:

$\forall x \in H$

$$\mathrm{ad}(x)\, e^+_{k\ell} = [x, e^+_{k\ell}] = \underline{\alpha_{k\ell}(x)}\, e^-_{k\ell}$$

$$\mathrm{ad}(x)\, e^-_{k\ell} = [x, e^-_{k\ell}] = \underline{-\alpha_{k\ell}(x)}\, e^+_{k\ell}$$

α are the roots

Calculation of roots: $\quad [x, e^\pm_{k\ell}] = \pm\alpha_{k\ell}(x)\, e^\mp_{k\ell}, \qquad x \in H$

For some $x \in H$: $\quad x = \sum_{\mu=1}^{n-1} x^\mu h_\mu =: x^\mu h_\mu$

$[x, e^{+/-}_{k\ell}]\, y \qquad\qquad y \in \mathbb{C}^n$

$= x\, e^{+,-}_{k\ell}\, y - e^{+,-}_{k\ell}\, x\, y$

$= x^\mu h_\mu e^{+,-}_{k\ell}\, y - e^{+,-}_{k\ell}\, x^\mu h_\mu\, y$

$= x^\mu \underline{h_\mu}\, (1, i)\, [\langle e_\ell, y\rangle \underline{e_k} + (-,+)\langle e_k, y\rangle \underline{e_\ell}]$

$\quad - x^\mu \underline{e^{+,-}_{k\ell}}\; i(\langle e_\mu, y\rangle \underline{e_\mu} - \frac{1}{n}\underline{y})$

$= x^\mu (1,i)\langle e_\ell, y\rangle i(\langle e_\mu, e_k\rangle e_\mu \cancel{- \frac{1}{n} e_k})$

$\quad + (-,+)\, x^\mu (1,i)\langle e_k, y\rangle i(\langle e_\mu, e_\ell\rangle e_\mu \cancel{- \frac{1}{n} e_\ell})$

$\quad - x^\mu i \langle e_\mu, y\rangle (1,i)(\langle e_\ell, e_\mu\rangle e_k + (-,+)\langle e_k, e_\mu\rangle e_\ell)$

$\quad + x^\mu i \frac{1}{n}(1,i)(\cancel{\langle e_\ell, y\rangle e_k} \cancel{+ (-,+)\langle e_k, y\rangle e_\ell})$

$\langle e_\mu, e_k\rangle = \delta_{\mu k}$

$= (1,i)\, i\, x^k (\langle e_\ell, y\rangle e_k + (+,-)\langle e_k, y\rangle e_\ell)$

$\quad - (1,i)\, i x^\ell (\langle e_\ell, y\rangle e_k + (+,-)\langle e_k, y\rangle e_\ell)$

$= (x^k - x^\ell)(i, -1)(\langle e_\ell, y\rangle e_k + (+,-)\langle e_k, y\rangle e_\ell)$

$= (+,-)(x^k - x^\ell)\, e^{+,-}_{k\ell}\, y$

$\Rightarrow \quad [\,x, e^{+,-}_{k\ell}\,] = \pm (x^k - x^\ell)\, e^{\mp}_{k\ell}$

$\Rightarrow \quad \alpha_{k\ell}(x) = x^k - x^\ell$

H.23 $\quad a, b \in H \qquad\qquad A_n \quad n\times n$

$\langle a, b \rangle_K = -2n\, tr[ab] \quad \forall a, b \in H$

$\langle a, b\rangle_K = -tr[\,ad(a) \underset{\mathbb{1}}{\wedge} ad(b)\,]$, $\qquad A = H \oplus H_\perp$

$\mathbb{1} = |h_k\rangle\langle h_k| + |e^+_{k_1\ell_1}\rangle\langle e^+_{k_1\ell_1}|$
$+ |e^-_{k_2\ell_2}\rangle\langle e^-_{k_2\ell_2}|$

$$\left(\begin{array}{l} ad(a)\,|h_k\rangle = 0 \\ ad(a)\,|e^{\pm}_{k\ell}\rangle = \pm\,\alpha_{k\ell}(a)\,|e^{\mp}_{k\ell}\rangle \end{array}\right.$$

$$= -\,tr[\,\underbrace{ad(a)\big(|h_k\rangle\langle h_k|}_{=0} + \underbrace{|e^+_{k_1\ell_1}\rangle}_{\alpha_{k_1\ell_1}(a)|e^-_{k_1\ell_1}\rangle}\langle e^+_{k_1\ell_1}| + \underbrace{|e^-_{k_2\ell_2}\rangle}_{-\alpha_{k_2\ell_2}(a)|e^+_{k_2\ell_2}\rangle}\langle e^-_{k_2\ell_2}|\big)\, ad(b)\,]$$

$$= -\,tr\Big[\big(\alpha_{k_1\ell_1}(a)\,|e^-_{k_1\ell_1}\rangle\langle e^+_{k_1\ell_1}| - \alpha_{k_2\ell_2}(a)\,|e^+_{k_2\ell_2}\rangle\langle e^-_{k_2\ell_2}|\big)\, ad(b)\Big]$$

$$\Big(\ tr[A] = (\langle h_k| + \langle e^+_{k\ell}| + \langle e^-_{k\ell}|)\, A\, (|h_k\rangle + |e^+_{k\ell}\rangle + |e^-_{k\ell}\rangle)$$

$$= -\alpha_{k_1\ell_1}(a)\, \overbrace{\langle e^-_{k_3\ell_3}|e^-_{k_1\ell_1}\rangle}^{\delta_{k_3k_1}\delta_{\ell_3\ell_1}}\langle e^+_{k_1}|\, \overbrace{ad(b)\,|e^-_{k_3\ell_3}\rangle}^{= -\alpha_{k_3\ell_3}(b)\,|e^+_{k_3\ell_3}\rangle}$$

$$+\ \alpha_{k_2\ell_2}(a)\, \underbrace{\langle e^+_{k_4\ell_4}|e^+_{k_2\ell_2}\rangle}_{=\delta_{k_4k_2}\delta_{\ell_4\ell_2}}\langle e^-_{k_2\ell_2}|\, \underbrace{ad(b)\,|e^+_{k_4\ell_4}\rangle}_{=\alpha_{k_4\ell_4}(b)\,|e^-_{k_4\ell_4}\rangle}$$

$$= \alpha_{k_1\ell_1}(a)\,\alpha_{k_3\ell_3}(b)\, \delta_{k_3k_1}\delta_{\ell_3\ell_1}\delta_{k_1k_3}\delta_{\ell_1\ell_3}$$

$$+\ \alpha_{k_2\ell_2}(a)\,\alpha_{k_4\ell_4}(b)\, \delta_{k_2k_4}\delta_{\ell_2\ell_4}\delta_{k_4k_2}\delta_{\ell_4\ell_2}$$

$$= 2\sum_{1\le k<\ell\le n} \alpha_{k\ell}(a)\,\alpha_{k\ell}(b)$$

$$\Big(\ \alpha_{k\ell}(a) = a^k - a^\ell\,, \quad a = \sum_k a^k h_k$$

$$= 2 \sum_{1 \le k < l \le n} (a^k - a^l)(b^k - b^l)$$

$$= \underbrace{2 \sum_{1 \le k < l \le n} a^k b^k + 2 \sum a^l b^l}_{= 2(n-1) \sum_{k=1}^{n} a^k b^k} - 2 \sum a^k b^l - 2 \sum a^l b^k$$

$$= 2(n-1) \sum_{k=1}^{n} a^k b^k - 2 \underbrace{\sum_{k \neq l} a^k b^l}_{= \sum_{k,l} a^k b^l - \sum_{k} a^k a^k}$$

$$= 2n \sum_{k=1}^{n} a^k b^k - 2 \left(\sum_{k=1}^{n} a^k \right) \left(\sum_{l=1}^{n} b^l \right)$$

$$= \langle a, b \rangle_K$$

$$\mathrm{tr}[ab] = \sum_{i=1}^{n} (ab)_{ii} \quad ;$$

$$a_{ii} = -\frac{i}{n} \left(\sum_{k=1}^{n} a^k - n a^i \right)$$

$$b_{ii} = -\frac{i}{n} \left(\sum_{l=1}^{n} b^l - n b^i \right)$$

$$(ab)_{ii} = a_{ii} b_{ii} = -\frac{1}{n^2} \left(\left(\sum_k a^l \right) \left(\sum_l b^l \right) - n \cdot \left(\sum_k a^k \right) b^i - n \cdot a^i \left(\sum_l b^l \right) + n^2 a^i b^i \right)$$

$$= -\frac{1}{n^2} \left\{ n \left(\sum_k a^k \right) \cancel{\left(\sum_l b^l \right)} - n \left(\sum_k a^k \right) \cancel{\left(\sum b^i \right)} + n \left(\sum_i a^i \right) \left(\sum_l b^l \right) \right.$$

$$\left. + n^2 \sum_i a^i b^i \right\}$$

$$\Rightarrow \quad \mathrm{tr}[ab] = -\sum_i a^i b^i + \frac{1}{n} \left(\sum_i a^i \right) \left(\sum_l b^l \right) \qquad \Big| \cdot (-2n)$$

$$-2n \, \mathrm{tr}[ab] = 2n \sum_i a^i b^i - 2 \left(\sum_i a^i \right) \left(\sum_l b^l \right)$$

$$= \langle a, b \rangle_K$$