H2.

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1. Permutations of n objects:

There are n possibilities to choose first element. After this, only $n-1$. And so on, untill every one is ordered.

$\Rightarrow \# \sigma = n!$

S_n is a group:

- There is always an one-to-one map between elements and itself
- Inverse also exists for all $\sigma \in S_n$, since we can order the elements "inversely".

$$\sigma = \begin{pmatrix} 1 & \dots & n \\ \sigma(1) & \dots & \sigma(n) \end{pmatrix} \quad , \quad \sigma^{-1} = \begin{pmatrix} \sigma(1) & \dots & \sigma(n) \\ 1 & \dots & n \end{pmatrix}$$

- Closure is also true, since σ is a bijective mapping.

→ Identity

→ Associativity: give an example

2. $$\sigma = \begin{pmatrix} 1 & \dots & n \\ \sigma(1) & \dots & \sigma(n) \end{pmatrix}$$

$\Rightarrow \quad \sigma = \sigma(1) \circ \sigma(2) \cdots \sigma(n)$

- e: $\sigma(i) = i$, $\forall\ i \in N_n$,
- An orbitrary $x \in N_n$ and an arbitrary $\sigma \in S_n$. One can define a subset $\{k_1, \dots, k_l\} \in N_n$ with $l \leq n$ and $l_1, n, r \in N$ such that

$$k_1 = x$$

$$k_2 = \sigma(k_1) = \sigma(x)$$

$$k_3 = \sigma(k_2) = \sigma(\sigma(x))$$

$$\vdots$$

$$k_{l_1} = \sigma(k_{l_1}) = \sigma^{(l_1-1)}(x)$$

$$k_1 = \sigma^{l_1}(k_{l_1}) = x \ \wedge \ \sigma^{i}(x) \neq x \quad \forall\ r \leq (l_i - 1)$$

This corresponds to a cycle of length n

$$\sigma_{l_1} = (k_1\ k_2\ \cdots\ k_{l_1})$$

and the set

$$N_{l_1} = \{k_1, \dots, k_{l_1}\}$$

N_n is finite $\Rightarrow$ $l_1 \le n$ can be found

σ bijective $\Rightarrow$ $\sigma^i(x)$ with $0 \le i \le (l_1 - 1)$ are distinct

Since all elements of N_n are distinct, for any

$$y \in N_{l_2} = N_n \setminus N_{l_1} \subset N_n, \quad l_2 \in \mathbb{N}$$

There exists a corresponding cycle σ_{l_2}. Since $y \notin N_{l_1}$

$\Rightarrow$ $N_{l_2} \cap N_{l_1} = \emptyset$

$\Rightarrow$ Iterations till $l_1 + l_2 + \cdots = n$

3. $\sum_{l=1}^{n} n_l l < n$

$$\sum_{l=1}^{n} n_l l = n_1 + 2n_2 + \cdots + n\, n_n < n$$

f

$$\Rightarrow n_1 + \cdots + n(n_n - 1) < 0$$

$$\Rightarrow n_n > 1, \quad \text{or} \quad n_n \geq 2$$

Since $\sigma \in S_n$, n_n cannot be bigger than 1

$$\sum n_l l = n_1 + 2n_2 + \cdots + n\, n_n > n$$

$$\Rightarrow n_1 + \cdots + n(n_n - 1) > 0$$

$\sum_{l=1}^{n} n_l l \overset{?}{=} n$

(1)(2) .... (n)

$\sum_{l=1}^{n} n_l l > n$ → consider only identity, it then means σ is not bijective

$\sum_{l=1}^{n} n_l l < n$ → not bijective

4. $\sigma \in S_n$, it has cycle $(k_1, \dots, k_n)$

$\sigma = \alpha_1, \dots \alpha_n$, α_i is k_i

$\sigma' \in S_n$ such that $\sigma' = \tau \sigma \tau^{-1}$

$= \tau \alpha_1 \dots \alpha_n \tau^{-1}$

$= \tau \alpha_1 \tau^{-1} \tau \alpha_2 \dots \tau \alpha_n \tau^{-1}$

$= \alpha_{\tau(1)} \dots \alpha_{\tau(n)}$

if $\sigma(i) = j \Rightarrow \sigma'(\tau(i)) \overset{!}{=} \tau(j)$

$\sigma'(\tau(i)) = \tau \sigma \tau^{-1}(\tau(i)) = \tau \sigma(i) = \tau(j)$

$\}$ conjugacy doesn't change cycle structure

H3.

1. To show $d\tilde{V} = dV$

$$\sqrt{\det f}\sqrt{\det \tilde{g}}\, d\tilde{q}^1 \dots d\tilde{q}^{n^2} = \sqrt{\det g}\, dq^1 \dots dq^{n^2}, \quad \tilde{q}^i = f^i(q) \in \mathbb{R}$$

$$\Rightarrow \tilde{g}_{ik} = \left\langle \frac{\partial h}{\partial \tilde{q}^i}, \frac{\partial h}{\partial \tilde{q}^k} \right\rangle$$

$$= \left\langle \frac{\partial h}{\partial q^\mu}\frac{\partial q^\mu}{\partial \tilde{q}^i}, \frac{\partial h}{\partial q^\nu}\frac{\partial q^\nu}{\partial \tilde{q}^k} \right\rangle$$

$$= \left\langle \frac{\partial h}{\partial q^\mu}\left(\frac{\partial f^i(q)}{\partial q^\mu}\right)^{-1}, \frac{\partial h}{\partial q^\nu}\left(\frac{\partial f^k(q)}{\partial q^\nu}\right)^{-1} \right\rangle$$

$$= \mathrm{Re}\left(tr\left[\frac{\partial h^*}{\partial q^\mu} \frac{\partial q^\mu}{\partial f^i(q)} \frac{\partial h}{\partial q^\nu} \frac{\partial q^\nu}{\partial f^k(q)} \right]\right)$$

$$= \frac{\partial q^\mu}{\partial f^i(q)}\frac{\partial q^\nu}{\partial f^k(q)} \left\langle \frac{\partial h}{\partial q^\mu}, \frac{\partial h}{\partial q^\nu} \right\rangle$$

Jacobian $= D^l_i = \frac{\partial \tilde{q}^l}{\partial q}$

$$= \underbrace{\frac{\partial q^\mu}{\partial f^i(q)}\frac{\partial q^\nu}{\partial f^k(q)}}_{\text{~~cancels out~~ } \sqrt{\det g}} g_{\mu\nu} = D_{i\mu} D_{k\nu} g_{\mu\nu} = (D g D^T)_{ik}$$

$$\Rightarrow \det(\tilde{g}) = \det(D^{-1} g (D^{-1})^T) = (\det D)^{-2} \det g$$

$$\Rightarrow d\tilde{V} = dV$$

$$\Rightarrow d\tilde{V} = \sqrt{\det \tilde{g}}\, d\tilde{q}_1 \dots d\tilde{q}_{n^2} = |\det D|^{-1}\sqrt{\det g}\, d\tilde{q}_1 \dots d\tilde{q}_{n^2} = \sqrt{\det g}\, dq_1 \dots dq_{n^2}$$

2. $h \to h'h$

$$\det g \longrightarrow \det \tilde{g}$$

$$\tilde{g}_{ik} = \left\langle \frac{\partial (h'h)}{\partial q^i}, \frac{\partial (h'h)}{\partial q^k} \right\rangle$$

$$= \left\langle \frac{\partial (h'h)}{\partial q^i}, \frac{\partial (h'h)}{\partial q^k} \right\rangle$$

h' is fixed $$= \left\langle \cancel{\frac{\partial h'}{\partial q^i}} h + h' \frac{\partial h}{\partial q^i}, \cancel{\frac{\partial h'}{\partial q^k}} h + h' \frac{\partial h}{\partial q^k} \right\rangle$$ ✓

$$= \langle h' \frac{\partial h^*}{\partial q^i} , h' \frac{\partial h}{\partial q^k} \rangle$$

unitarity

$$= \langle \frac{\partial h^*}{\partial q^i} , \frac{\partial h}{\partial q^k} \rangle = g_{ik}$$

$h \to h h'$

$$g_{ik} \to \tilde{g}_{ik} = \langle \frac{\partial (hh')}{\partial q^i} , \frac{\partial (hh')}{\partial q^k} \rangle$$

$$= \langle \frac{\partial h}{\partial q^i} h' , \frac{\partial h}{\partial q^k} h' \rangle$$

$$= \langle \frac{\partial h}{\partial q^i} , \frac{\partial h}{\partial q^k} \rangle$$

$$= g_{ik}$$

H.4 $g_1, g_2 \in G$, an abelian group

$g_1 g_2 = g_2 g_1$

$$D(g_1) D(g_2) = D(g_1 g_2) \quad \forall g_1, g_2$$
$$= D(g_2 g_1)$$

$\Rightarrow D(g_1) D(g_2) = D(g_2) D(g_1) \quad \forall g_1, g_2$

There are two possibilities $\to D(g_1) = \lambda \mathbb{1}$, $\Rightarrow$ 1-D

1) $D: G \to \mathbb{R}$, that is $D(g) \in \mathbb{R}$

2) $D(g)$ is reducible, because of Schur's lemma

$\Rightarrow$ All irreducible repre. abelian group must be one-dimensional.