

H2.

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1. Permutations of  $n$  objects:

There are  $n$  possibilities to choose first element. After this, only  $n-1$ . And so on, until every one is ordered.

$$\Rightarrow \# \sigma = n!$$

$S_n$  is a group:

- There is always an one-to-one map between elements and itself

- Inverse also exists for all  $\sigma \in S_n$ , since we can order the elements "inversely".

$$\sigma = \begin{pmatrix} 1 & \dots & n \\ \sigma(1) & \dots & \sigma(n) \end{pmatrix}, \quad \sigma^{-1} = \begin{pmatrix} \sigma(1) & \dots & \sigma(n) \\ 1 & \dots & n \end{pmatrix}$$

- Closure is also true, since  $\sigma$  is a bijective mapping.

→ Identity

→ Associativity: give an example

$$2. \quad \sigma = \begin{pmatrix} 1 & \dots & n \\ \sigma(1) & \dots & \sigma(n) \end{pmatrix}$$

$$\Rightarrow \sigma = \sigma(1) \circ \sigma(2) \dots \sigma(n)$$

-  $e: \sigma(i) = i, \forall i \in \mathbb{N}_n$ ,

- An arbitrary  $x \in \mathbb{N}_n$  and an arbitrary  $\sigma \in S_n$ . One can define a subset  $\{k_1, \dots, k_\ell\} \in \mathbb{N}_n$  with  $\ell \leq n$  and

$\ell, n, r \in \mathbb{N}$  such that

$$k_1 = x$$

$$k_2 = \sigma(k_1) = \sigma(x)$$

$$k_3 = \sigma(k_2) = \sigma(\sigma(x))$$

$\vdots$

$$k_\ell = \sigma(k_{\ell-1}) = \sigma^{\ell-1}(x)$$

$$k_r = \sigma^{l_r}(k_{\ell_r}) = x \wedge \sigma^i(x) \neq x \quad \forall r \leq (\ell_i - 1)$$

This corresponds to a cycle of length  $n$

$$\sigma_{l_1} = (k_1 \ k_2 \ \dots \ k_{l_1})$$

and the set

$$N_{l_1} = \{k_1, \dots, k_{l_1}\}$$

$N_n$  is finite  $\Rightarrow l_1 \leq n$  can be found

$\sigma$  bijective  $\Rightarrow \sigma^i(x)$  with  $0 \leq i \leq (l_1 - 1)$  are distinct

Since all elements of  $N_n$  are distinct, for any

$$y \in N_{l_2} = N_n \setminus N_{l_1} \subset N_n, \quad l_2 \in \mathbb{N}$$

There exists a corresponding cycle  $\sigma_{l_2}$ . Since  $y \notin N_{l_1}$

$$\Rightarrow N_{l_2} \cap N_{l_1} = \emptyset$$

$$\Rightarrow \text{Iterations till } l_1 + l_2 + \dots = n$$

3. 
$$\sum_{l=1}^n n_l l < n$$

$$\sum_{l=1}^n h_l l = n_1 + 2n_2 + \dots + n n_n < n$$

f

$$\Rightarrow n_1 + \dots + n(n_n - 1) < 0$$

$$\Rightarrow n_n > 1, \text{ or } n_n \geq 2$$

Since  $\sigma \in S_n$ ,  $n_n$  cannot be bigger than 1

$$\sum n_l l = n_1 + 2n_2 + \dots + n n_n > n$$

$$\Rightarrow n_1 + \dots + n(n_n - 1) > 0$$

$$\sum_{l=1}^n n_l l \stackrel{?}{=} n$$

$\sum_{l=1}^n n_l l > n \rightarrow$  consider only identity, it then means  $\sigma$  is not bijective

$\sum_{l=1}^n n_l l < n \rightarrow$  not bijective

(1)(2)....(n)

4.  $\sigma \in S_n$ , it has cycle  $(k_1, \dots, k_n)$

$\sigma = \alpha_1 \dots \alpha_n$ ,  $\alpha_i$  is  $k_i$

$\sigma' \in S_n$  such that  $\sigma' = \tau \sigma \tau^{-1}$

$$= \tau \alpha_1 \dots \alpha_n \tau^{-1}$$

$$= \tau \alpha_1 \tau^{-1} \tau \alpha_2 \dots \tau \alpha_n \tau^{-1}$$

$$= \alpha_{\tau(1)} \dots \alpha_{\tau(n)}$$

$$\text{if } \sigma(i) = j \Rightarrow \sigma'(\tau(i)) \stackrel{?}{=} \tau(j)$$

$$\sigma'(\tau(i)) = \tau \sigma \tau^{-1}(\tau(i)) = \tau \sigma(i) = \tau(j)$$

} conjugacy doesn't  
change cycle  
structure

H3.

1. To show  $d\tilde{V} = dV$

$$\sqrt{\det f} \sqrt{\det \tilde{g}} d\tilde{q}^1 \dots d\tilde{q}^n = \sqrt{\det g} dq^1 \dots dq^n, \quad \tilde{q}^i = f^i(q) \in \mathbb{R}$$

$$\Rightarrow \tilde{g}_{ik} = \left\langle \frac{\partial h}{\partial \tilde{q}^i}, \frac{\partial h}{\partial \tilde{q}^k} \right\rangle$$

$$= \left\langle \frac{\partial h}{\partial q^\mu} \frac{\partial q^\mu}{\partial \tilde{q}^i}, \frac{\partial h}{\partial q^\nu} \frac{\partial q^\nu}{\partial \tilde{q}^k} \right\rangle$$

$$= \left\langle \frac{\partial h}{\partial q^\mu} \left( \frac{\partial f^i(q)}{\partial q^\mu} \right)^{-1}, \frac{\partial h}{\partial q^\nu} \left( \frac{\partial f^k(q)}{\partial q^\nu} \right)^{-1} \right\rangle$$

$$= \text{Re} \left( \text{tr} \left[ \frac{\partial h}{\partial q^\mu} \frac{\partial q^\mu}{\partial f^i(q)} \frac{\partial h}{\partial q^\nu} \frac{\partial q^\nu}{\partial f^k(q)} \right] \right)$$

$$= \frac{\partial q^\mu}{\partial f^i(q)} \frac{\partial q^\nu}{\partial f^k(q)} \left\langle \frac{\partial h}{\partial q^\mu}, \frac{\partial h}{\partial q^\nu} \right\rangle$$

$$\text{Jacobian} = D_i^l = \frac{\partial q^l}{\partial f^i}$$

$$= \frac{\partial q^\mu}{\partial f^i(q)} \frac{\partial q^\nu}{\partial f^k(q)} g_{\mu\nu} = D_i^\mu D_k^\nu g_{\mu\nu} = (D g D^T)_{ik}$$

$$\text{cancels out } \sqrt{\det g}$$

$$\Rightarrow \det(\tilde{g}) = \det(D g D^T) = (\det D)^2 \det g$$

$$\Rightarrow d\tilde{V} = dV$$

$$\Rightarrow d\tilde{V} = \sqrt{\det \tilde{g}} d\tilde{q}_1 \dots d\tilde{q}_n = |\det D|^{-1} \sqrt{\det g} d\tilde{q}_1 \dots d\tilde{q}_n = \sqrt{\det g} dq_1 \dots dq_n$$

2.  $h \rightarrow h'h$

$$\det g \rightarrow \det \tilde{g}$$

$$\tilde{g}_{ik} = \left\langle \frac{\partial(h'h)}{\partial q^i}, \frac{\partial(h'h)}{\partial q^k} \right\rangle$$

$$= \left\langle \frac{\partial(h'h)}{\partial q^i}, \frac{\partial(h'h)}{\partial q^k} \right\rangle$$

$$h' \text{ is fixed} = \left\langle \frac{\partial h'}{\partial q^i} h + h' \frac{\partial h}{\partial q^i}, \frac{\partial h'}{\partial q^k} h + h' \frac{\partial h}{\partial q^k} \right\rangle \quad \checkmark$$

$$= \left\langle h' \frac{\partial h^*}{\partial g^i}, h' \frac{\partial h}{\partial g^k} \right\rangle$$

unitarity

$$= \left\langle \frac{\partial h^*}{\partial g^i}, \frac{\partial h}{\partial g^k} \right\rangle = g_{ik}$$

$$h \rightarrow h h'$$

$$g_{ik} \rightarrow \tilde{g}_{ik} = \left\langle \frac{\partial (h h')}{\partial g^i}, \frac{\partial (h h')}{\partial g^k} \right\rangle$$

$$= \left\langle \frac{\partial h}{\partial g^i} h', \frac{\partial h}{\partial g^k} h' \right\rangle$$

$$= \left\langle \frac{\partial h}{\partial g^i}, \frac{\partial h}{\partial g^k} \right\rangle$$

$$= g_{ik}$$

H.4  $g_1, g_2 \in G$ , an abelian group

$$g_1 g_2 = g_2 g_1$$

$$\begin{aligned} D(g_1) D(g_2) &= D(g_1 g_2) \quad \forall g_1, g_2 \\ &= D(g_2 g_1) \end{aligned}$$

$$\Rightarrow D(g_1) D(g_2) = D(g_2) D(g_1) \quad \forall g_1, g_2$$

There are two possibilities  $\rightarrow D(g_i) = \lambda \mathbb{1}$ ,  $\Rightarrow$  1-D

$$1) \underline{D: G \rightarrow \mathbb{R}}, \text{ that is } D(g) \in \mathbb{R}$$

2)  $D(g)$  is reducible, because of Schur's lemma

$\Rightarrow$  All irreducible repr. abelian group must be one-dimensional.