H2.

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1. Permutations of n objects:

There are n possibilities to choose first element. After this, only $n-1$. And so on, untill every one is ordered.

$\Rightarrow \quad \# \sigma = n!$

S_n is a group:

- There is always an one-to-one map between elements and itself
- Inverse also exists for all $\sigma \in S_n$, since we can order the elements "inversely".

$$\sigma = \begin{pmatrix} 1 & \cdots & n \\ \sigma(1) & \cdots & \sigma(n) \end{pmatrix} , \quad \sigma^{-1} = \begin{pmatrix} \sigma(1) & \cdots & \sigma(n) \\ 1 & \cdots & n \end{pmatrix}$$

- Closure is also true, since σ is a bijective mapping.

2. $$\sigma = \begin{pmatrix} 1 & \cdots & n \\ \sigma(1) & \cdots & \sigma(n) \end{pmatrix}$$

$$\Rightarrow \quad \sigma = \sigma(1) \circ \sigma(2) \cdots \sigma(n)$$

3. $$\sum_{l=1}^{n} n_l l < n$$

$$\sum_{l=1}^{n} n_l l = n_1 + 2n_2 + \cdots + n\, n_n < n$$

$$\Rightarrow \quad n_1 + \cdots + n(n_n - 1) < 0$$

$$\Rightarrow \quad n_n > 1, \quad \text{or} \quad n_n \geq 2$$

Since $\sigma \in S_n$, n_n cannot be bigger than 1

$$\sum n_l l = n_1 + 2n_2 + \cdots + n\, n_n > n$$

$$\Rightarrow \quad n_1 + \cdots + n(n_n - 1) > 0$$

H3.

1. To show $d\tilde{V} = dV$

$$\sqrt{\det f'}\sqrt{\det\tilde{g}}\, d\tilde{q}^1 \dots d\tilde{q}^{n^2} = \sqrt{\det g}\, dq^1 \cdots dq^{n^2} \quad , \quad \tilde{q}^i = f^i(q) \in \mathbb{R}$$

$$\Rightarrow \quad \tilde{g}_{ik} = \left\langle \frac{\partial h}{\partial \tilde{q}^i} , \frac{\partial h}{\partial \tilde{q}^k} \right\rangle$$

$$= \left\langle \frac{\partial h}{\partial q^\mu} \frac{\partial q^\mu}{\partial \tilde{q}^i} , \frac{\partial h}{\partial q^\nu} \frac{\partial q^\nu}{\partial \tilde{q}^k} \right\rangle$$

$$= \left\langle \frac{\partial h}{\partial q^\mu} \left(\frac{\partial f^i(q)}{\partial q^\mu}\right)^{-1} , \frac{\partial h}{\partial q^\nu} \left(\frac{\partial f^k(q)}{\partial q^\nu}\right)^{-1} \right\rangle$$

$$= \mathrm{Re}\left(tr\left[\frac{\partial h^*}{\partial q^\mu} \frac{\partial q^\mu}{\partial f^i(q)} \frac{\partial h}{\partial q^\nu} \frac{\partial q^\nu}{\partial f^k(q)} \right] \right)$$

$$= \frac{\partial q^\mu}{\partial f^i(q)} \frac{\partial q^\nu}{\partial f^k(q)} \left\langle \frac{\partial h}{\partial q^\mu} , \frac{\partial h}{\partial q^\nu} \right\rangle$$

$$= \underbrace{\frac{\partial q^\mu}{\partial f^i(q)} \frac{\partial q^\nu}{\partial f^k(q)}}_{\text{cancels out } \sqrt{\det \tilde{g}}} g_{\mu\nu}$$

$$\Rightarrow \quad d\tilde{V} = dV$$

2. $h \to h'h$

$$\det g \longrightarrow \det \tilde{g}$$

$$\tilde{g}_{ik} = \left\langle \frac{\partial(h'h)}{\partial q^i} , \frac{\partial(h'h)}{\partial q^k} \right\rangle$$

$$= \left\langle \frac{\partial(h'h)}{\partial q^i} , \frac{\partial(h'h)}{\partial q^k} \right\rangle$$

h' is fixed $$= \left\langle \cancel{\frac{\partial h'}{\partial q^i}} h + h' \frac{\partial h}{\partial q^i} , \cancel{\frac{\partial h'}{\partial q^k}} h + h' \frac{\partial h}{\partial q^k} \right\rangle$$

$$= \langle h' \frac{\partial h^*}{\partial q^i} \quad , h' \frac{\partial h}{\partial q^k} \rangle$$

unitarity

$$= \langle \frac{\partial h^*}{\partial q^i} , \frac{\partial h}{\partial q^k} \rangle = g_{ik}$$

$$h \to h h'$$

$$g_{ik} \to \tilde{g}_{ik} = \langle \frac{\partial (h h')}{\partial q^i} , \frac{\partial (h h')}{\partial q^k} \rangle$$

$$= \langle \frac{\partial h}{\partial q^i} h' , \frac{\partial h}{\partial q^k} h' \rangle$$

$$= \langle \frac{\partial h}{\partial q^i} , \frac{\partial h}{\partial q^k} \rangle$$

$$= g_{ik}$$

H.4 $\quad g_1, g_2 \in G$, an abelian group

$$g_1 g_2 = g_2 g_1$$

$$D(g_1) D(g_2) = D(g_1 g_2) \qquad \forall g_1, g_2$$

$$= D(g_2 g_1)$$

$$\Rightarrow D(g_1) D(g_2) = D(g_2) D(g_1) \qquad \forall g_1, g_2$$

There are two possibilities

1) $D: G \to \mathbb{R}$, that is $D(g) \in \mathbb{R}$

2) $D(g)$ is reducible, because of Schur's lemma

$\Rightarrow$ All irreducible repre. abelian group must be one-dimensional.