H18.

Chenhuan Wang

1. $L = \{\Lambda : \Lambda^T \eta \Lambda = \eta\}$

$$(\tilde{\Lambda}_{p'}(h) \cdot \tilde{\Lambda}_{p'}(h'))$$

$$= ([\underline{h} - 2(\underline{p}'\cdot\underline{h})\underline{p}'] \cdot [\underline{h}' - 2(\underline{p}'\cdot\underline{h}')\underline{p}'])$$

$$= (\underline{h}\cdot\underline{h}) + 4(\underline{p}'\cdot\underline{h})\cancel{(\underline{p}'\cdot\underline{h}')}\underbrace{\underline{p}'\cdot\underline{p}'}_{=1} - 2(\underline{p}'\cdot\underline{h})\cancel{(\underline{p}'\cdot\underline{h}')} - 2\cancel{(\underline{p}'\cdot\underline{h}')}(\underline{p}'\cdot\underline{h})$$

$$= (\underline{h}\cdot\underline{h})$$

$\rightarrow \Lambda_{p'} \in L$

$\rightarrow \hat{\Lambda}_{p'} = \Lambda_{p'} T \in L$

$\rightarrow \det(\eta) = \det(\Lambda_{p'}^T \eta \Lambda_{p'})$

$-1 = (\det \Lambda_{p'})^2 \, -1$

$\det \Lambda_{p'} = \pm 1 \quad \rightarrow \quad \text{if } \det \Lambda_{p'} = -1 , \ \det \tilde{\Lambda}_{p'} = +1$

Further $\quad \Lambda_{p'}(\underline{h}) = \underline{h} - 2(\underline{p}'\cdot\underline{h})\underline{p}'$

$$\Lambda_{p'}(\underline{h})^\mu = \underline{h}^\mu - 2(\underline{p}'\cdot\underline{h})\underline{p}'^\mu$$

$$= \underline{h}^\mu - 2\eta_{\alpha\beta}\underline{p}'^\alpha \underline{h}^\beta \underline{p}'^\mu$$

$$= \sum_{a=0}^{3} \delta^{\mu a}\underline{h}^a - 2\eta_{\alpha\beta}\underline{p}'^\alpha \underline{p}'^\mu \delta^{\beta a}\underline{h}^a$$

$$= \sum_{a=0}^{3} \left(\delta^{\mu a} - 2\eta^{\alpha\beta}\underline{p}'_\alpha \underline{p}'_\mu \delta^{\beta a}\right)\underline{h}^a$$

$\longrightarrow$ somehow $\det \Lambda_{p'} = -1$

2. $\breve{\Lambda}_{\underline{p}'}(\underline{e}_0) = \Lambda_{\underline{p}'}(-\underline{e}_0) = -\underline{e}_0 + 2(\underline{p}'\cdot\underline{e}_0)\underline{p}' \overset{!}{=} \underline{p}$

$$\rightarrow (\underline{p}'\cdot\underline{e}_0)\underline{p}' = (\underline{p}+\underline{e}_0)\tfrac{1}{2}$$

$$p'^0\ \underline{p}' = (\underline{p}+\underline{e}_0)/2$$

0-th component: $\quad (p'^0)^2 = 1 + |\vec{p}'|^2 = (p'^0+1)/2$

$$\Leftrightarrow x = p'^0, \quad x^2 - \tfrac{1}{2}x - \tfrac{1}{2} = 0$$

$$(x-\tfrac{1}{2})^2 = \tfrac{1}{2} + \tfrac{1}{4} = \tfrac{3}{4}$$

$$\rightarrow p'^0 = \pm\tfrac{\sqrt{3}}{2} + \tfrac{1}{2}$$

$$p'^0 > 0, \quad p'^0 = \frac{\sqrt{3}+1}{2}$$

$$|\vec{p}'|^2 = \left(\frac{\sqrt{3}+1}{2}\right)^2 - 1 = \frac{3+1+2\sqrt{3}}{4} - 1 = \frac{\sqrt{3}}{2}$$

1-th component $\quad p'^0\, p'^1 = \tfrac{1}{2}p^1$

$$\rightarrow p'^1 = \tfrac{1}{2}p^1 \cdot \frac{2}{\sqrt{3}+1} = \frac{p^1}{\sqrt{3}+1}$$

$$\rightarrow \vec{p}' = \frac{1}{\sqrt{3}+1}\vec{p}$$

and p must be chose according to $\left|\frac{1}{\sqrt{3}+1}\vec{p}\right|^2 = \frac{3}{4}$

3. $\forall g \in L_+, \quad g = g_0 \cdot \breve{\Lambda}_{\underline{p}'}, \quad g_0 \in SO(3)$?

$g_0 \in SO(3) \Rightarrow \det(g_0) = 1 \Rightarrow g \in L_+$

$g = g_0' \breve{\Lambda}_{p'} = g_0 \breve{\Lambda}_{\underline{k}}, \quad (\underline{p}'\cdot\underline{p}') = 1, \ p'^0 > 0$, same for $\underline{k}$

$$(g_0'\breve{\Lambda}_{p'})^T \eta (g_0 \breve{\Lambda}_k) = \eta$$

$\text{LHS} = \breve{\Lambda}_{p'}^T g_0'^T \eta g_0 \breve{\Lambda}_k$

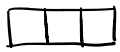
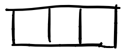
$\text{RHS} = \eta = \breve{\Lambda}^T_{p'} \eta \breve{\Lambda}_k$ $\quad \rightarrow \eta = \lambda \mathbb{1}$?

$$\rightarrow g_0'^T \eta g_0 = \eta \qquad \text{Schur's lemma}, \ \eta \neq 0$$

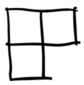
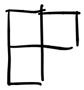
$$g_0'^{-1} \eta g_0 = \eta \qquad \Rightarrow g_0 = g_0'$$

$$\eta g_0 = g_0' \eta \qquad \rightarrow \breve{\Lambda}_{\underline{p}'} = \breve{\Lambda}_{\underline{k}} \Rightarrow \text{unique}$$

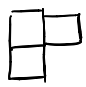
H.19

1. spin $\frac{3}{2}$  $\otimes$  $SU(3)$

$$\begin{aligned}\dim &= Y_S^{U(3)}(1,1,1) \\ &= K_S^{U(3)}(1,1,1) = \sum_{i\le j\le k=1}^{3} (1)^3 = 10, \qquad I = \tfrac{3}{2}, \quad I_3 = -\tfrac{3}{2}, -1, -\tfrac{1}{2}, 0, \tfrac{1}{2}, 1, \tfrac{3}{2}\end{aligned}$$

spin $\frac{1}{2}$  $\otimes$ 

$$\begin{aligned}\dim &= Y_M^{U(3)}(1,1,1) \\ &= K_S^{U(3)}(1,1,1) = 8, \qquad I = 1, \quad I_3 = -1, -\tfrac{1}{2}, 0, \tfrac{1}{2}, 1\end{aligned}$$

2. $L = 1$, $\pi = -$ $\rightarrow$  for orbit part

4. $\rho(\tilde{g}_1 \tilde{g}_2) h = \sigma^{-1}(\tilde{g}_1 \tilde{g}_2 \sigma(\underline{h}) \underbrace{(\tilde{g}_1 \tilde{g}_2)^*}_{= \tilde{g}_2^* \tilde{g}_1^*})$

$$\rho(\tilde{g}_1)\rho(\tilde{g}_2) h = \sigma^{-1}(\tilde{g}_1 \underbrace{\sigma(\rho(\tilde{g}_2) h)}\, \tilde{g}_1^*)$$

$$= \sigma^{-1}(\tilde{g}_1 \quad \tilde{g}_2 \sigma(\underline{h}) \tilde{g}_2^* \tilde{g}_1^*)$$

$$\rightarrow \quad \rho(\tilde{g}_1 \tilde{g}_2) h = \rho(\tilde{g}_1)\rho(\tilde{g}_2) h$$

$(\rho(\tilde{g})\underline{h} \cdot \rho(\tilde{g})\underline{h}) = (\underline{h} \cdot \underline{h})$ since $\rho : SL(\mathbb{C}^2) \rightarrow L_+$

$\rightarrow$ is unitary rep.

5. $\epsilon\, \overline{\sigma(\underline{h})}\, \epsilon^{-1} = \sigma(P\underline{h})$

$$LHS = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \underbrace{\overline{(\sigma^\mu \underline{h}_\mu)}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$= \sigma^0 \underline{h}_0 + \sigma^1 \underline{h}_1 - \sigma^2 \underline{h}_2 + \sigma_3 \underline{h}_3$$

$$\epsilon \sigma^0 \epsilon^{-1} = \mathbb{1} = \sigma^0$$

$$\epsilon \sigma^1 \epsilon^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

$$= -\sigma^1$$

$$\epsilon \sigma^2 \epsilon^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\sigma^2$$

$$\epsilon \sigma^3 \epsilon^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix} = -\sigma^3$$

$$\rightarrow LHS = \sigma(P\underline{h})$$

H.20

1. fundamental reps.

$$\ydiagram{1} = \frac{\begin{ytableau} n \end{ytableau}}{\begin{ytableau} 1 \end{ytableau}} = n$$

anti-fundamental reps:

$$n-1 \left\{ \begin{array}{c} \ydiagram{1,1} \\ \vdots \\ \ydiagram{1} \end{array} \right. = \begin{array}{c} \begin{ytableau} n \\ n-1 \end{ytableau} \\ \vdots \\ \begin{ytableau} 2 \end{ytableau} \end{array} \Bigg/ \begin{array}{c} \begin{ytableau} 1 \end{ytableau} \\ \vdots \\ \begin{ytableau} 1 \end{ytableau} \end{array} = n!$$

adjoint reps.

2. $SU(2)$, $n=2$

$$\ydiagram{1} \otimes \ydiagram{1} \otimes \ydiagram{1} = \left(\ydiagram{1,1} \oplus \ydiagram{2} \right) \otimes \ydiagram{1}$$

$$= \ydiagram{2,1} \oplus \ydiagram{3}$$

$$\simeq \frac{2\cdot3\cdot1}{3\cdot1\cdot1} \oplus \frac{2\cdot3\cdot4}{1\cdot2\cdot3} = 2 \oplus 4 \quad \rightarrow 6 \text{ baryons of } u,d$$

$SU(3)$

$$\ydiagram{1} \otimes \ydiagram{1} \otimes \ydiagram{1} = \left(\ydiagram{1,1} \oplus \ydiagram{2} \right) \otimes \ydiagram{1}$$

$$= \ydiagram{1,1,1} \oplus \ydiagram{2,1} \oplus \ydiagram{2,1} \oplus \ydiagram{3}$$

$$\simeq 1 \oplus \frac{3\cdot4\cdot2}{3\cdot1\cdot1} \oplus \frac{3\cdot4\cdot2}{3\cdot1\cdot1} \oplus \frac{3\cdot4\cdot5}{1\cdot2\cdot3}$$

$$= 1 \oplus 8 \oplus 8 \oplus 10 \quad \rightarrow 27 \text{ baryons of } u,d,s$$