Lab Report
S261: Optical Astronomy and Gravitational Lensing

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1 Theoretical Background

1.1 Cosmic Expansion

The general expression for a spatially homogeneous and isotropic universe can be written as [**manual**]

$$ds^2 = c^2dt^2 - a^2(t)[dw^2 + f_K^2(w)(d\theta + sin^2\theta d\psi^2)] \tag{1}$$

Where, t is cosmic time and a(t) is the cosmic scale factor which describe the isotropic expansion. Finally, the K denotes the space-time curvature is given by,

$$f_K(w) = \left\{ \begin{array}{cc} \frac{1}{\sqrt{K}} sin(\sqrt{K}w) & K > o \\ w & K = 0 \\ \frac{1}{\sqrt{-K} sinh(\sqrt{-K}w)} & K < 0 \end{array} \right\} \tag{2}$$

Where, K= 0 represents a flat geometry.
The redshift of a source is given by,

$$z = \frac{\lambda_{obj} - \lambda_{em}}{\lambda_{em}} \tag{3}$$

Where, λ_{obs} and λ_{em} are, respectively, the wavelengths at time of observation and emission. Equation 3 is directly related to the scale factor as,

$$1 + z = \frac{1}{a(t_{em})} \tag{4}$$

This means that a source at redshift z = 1 is observed at a time when the Universe was half of its current size (a = 1/2).

Due to the expansion of the Universe, a set of comoving observers sees the recession of surrounding objects. The corresponding velocity is,

$$v = \dot{a}x = \frac{\dot{a}}{a}r = H(t)r \tag{5}$$

where r = ax, and $H(t) = \frac{\dot{a}}{a}$ is the Hubble parameter, a measure of the cosmic expansion rate. The local Hubble law according to today's result are given by the following formula,

$$v_{esc} = H_0 D \tag{6}$$

Where, H_0 is Hubble constant and D is the distance between object and observer.

1.2 Distances

Accordingly, one defines the angular diameter distance as exactly this ratio,

$$D_{ang}(z) = 2R/\delta = a(z) f_K(w) \tag{7}$$

Where, 'R' is the radius of the distant object, 'δ' is the angular diameter, and 'z' is the cosmological redshift. If we consider an observer at redshift z_1 gives the angular diameter of another object at redshift z_2, so equation 7 becomes

$$D_{ang}(z_1, z_2) = a(z_2) f_K[w(z_2) = w(z_1)] \tag{8}$$

Another distance measure relates the observed flux, S, of a source to its luminosity, L. For a known luminosity, the distance to the source can be determined as,

$$D_{lum}(z) = \sqrt{\frac{L}{4\pi S}} \tag{9}$$

1.3 Quasars

1.4 Gravitational lensing

1.4.1 Lens Equation

For a given source, the lens equation is given by,

$$\beta = \theta - \alpha(\theta) \tag{10}$$

Where, θ be the apparent angular position of the source in the sky, β is its true position, and α is the scaled deflection angle.

The angles can be related to the physical distances as,

$$\eta = \frac{D_s}{D_d}\xi - D_{ds}\hat{\alpha}(\xi) \tag{11}$$

With, $D_d\theta = \xi$, $\eta = D_s\beta$, and $\hat{\alpha}$ is the true deflection angle which is related to the scaled deflection angle via:

$$\hat{\alpha}(\xi) = \frac{4G}{c^2} \int d^2\xi' \sum(\xi') \frac{\xi - \xi'}{\mid \xi - \xi' \mid^2} \tag{12}$$

Where, $\sum(\xi')$ is the surface mass density and $\mid \xi - \xi' \mid$ is the impact parameter for $\sum(\xi')$. Now, we can define a dimensionless surface mas density with convergence as,

$$\kappa(\theta) = \frac{\sum(D_d\theta)}{\sum_{cr}} \tag{13}$$

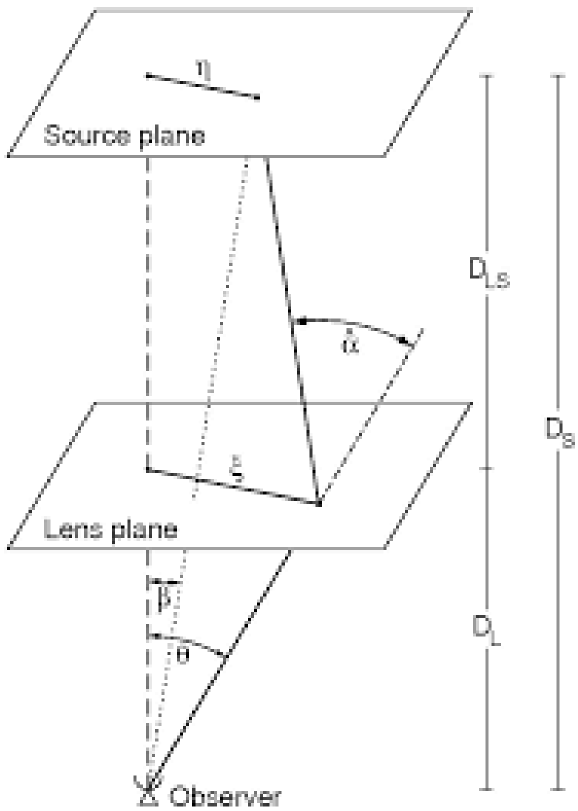

Figure 1: Schematic diagram of the gravitational lensing system [**wiki**].

with the critical surface mass density,

$$\Sigma cr = \frac{c^2}{4\pi G} \frac{D_s}{D_d D_{ds}} \tag{14}$$

Now, on rewriting the expression for scaled deflection angle,

$$\alpha(\theta) = \frac{1}{\pi} \int d^2\theta'_{\kappa}(\theta') \frac{\theta - \theta'}{\mid \theta - \theta' \mid^2} \tag{15}$$

Along with this we also have to define a deflection potential which is to gives information about the mass distribution of the lens is as follow,

$$\psi(\theta) = \frac{1}{\pi} \int d^2\theta'_{\kappa}(\theta') In \mid \theta - \theta' \mid \tag{16}$$

The use of this quantity is motivated because it encloses all information of the mass distribution of the lens. By means of the deflection potential the following relations can be derived:

$$\alpha(\theta) = \triangledown \psi(\theta) \tag{17}$$

For a better understanding further reading of the references is advised. From the deflection potential a further scalar function, the Fermat potential, can be derived:

$$\tau(\theta; \beta) = \frac{1}{2}(\beta - \theta)^2 - \psi(\theta) \tag{18}$$

Finally to find the magnification of the images, which is the ratio of a lens to un-lensed flux, is given by

$$\mu = (detA)^{-1} \tag{19}$$

Where, A is the Jacobian matrix of lens mapping

$$A_{ij} = \frac{\partial \beta_i}{\partial \theta_j} \tag{20}$$

1.4.2 The SIS (Singular Isothermal Sphere)

A simple model to describe the mass distribution of a galaxy acting as a lens is the singular isothermal sphere (SIS):

$$\rho(r) = \frac{\sigma_v^2}{2\pi G r^2} \tag{21}$$

Where σ_v^2 is the velocity dispersion. Integration along the line of sight yields the surface mass density

$$\Sigma(\xi) = \frac{\sigma_v^2}{2G\xi} \tag{22}$$

A characteristic angular scale of an axisymmetric lens is given by the Einstein radius θ_E , defined as the angle inside which the mean of the convergence is unity. As a consequence, the projected mass inside θ_E can be written as,

$$M(\theta \leq \theta_E) = \pi\theta_E^2 D_d^2 \Sigma_{cr} \tag{23}$$

For an SIS the Einstein radius reads

$$\theta_E = 4\pi\left(\frac{\sigma_v}{c}\right)^2 \frac{D_{ds}}{D_s} \tag{24}$$

2 Preparatory Tasks

P.3.1 Calculation of the deflection potential, $\psi(\theta)$ and the scaled deflection angle of an SIS lens.

From equation 16 the deflection potential, which gives the information about the mass distribution of the lens, is define as,

$$\psi(\theta) = \frac{1}{\pi}\int d^2\theta'\kappa(\theta')In \mid \theta - \theta' \mid \tag{25}$$

In the case of axial symmetry of SIS lens equation 16 simplifies to (Given on the question)

$$\psi(\theta) = 2\int_0^\theta d\theta'\theta'\kappa(\theta')ln\left(\frac{\theta}{\theta'}\right) \tag{26}$$

By substituting the values of $\Sigma(D_d\theta)$, where $D_d\theta = \xi$ from equation 22 and Σ_{cr} from equation 14 on equation 13, then by plugging the new expression of $\kappa(\theta)$ equation 26 becomes

$$\psi(\theta) = \frac{4\pi}{c^2}\frac{D_{ds}}{D_s}\sigma_v^2 \int_0^{\theta} d\theta' ln\left(\frac{\theta}{\theta'}\right) \tag{27}$$

By integrating,

$$= \frac{4\pi}{c^2}\frac{D_{ds}}{D_s}\sigma_v^2 [\theta' ln(\frac{\theta}{\theta'}) + \theta'] \tag{28}$$

$$= \theta_E(\theta ln\theta - \theta ln\theta + \theta) \tag{29}$$

Therefore,

$$\psi(\theta) = \theta_E \theta \tag{30}$$

From equation 17 scaled deflection angle is defined as

$$\alpha(\theta) = \bigtriangledown\psi(\theta) \tag{31}$$

From equations: 30 and 31,

$$\alpha(\theta) = \bigtriangledown\theta_E\theta = \theta_E\hat{\theta} \tag{32}$$

P.3.2: Solving the lens equation and finding the separation between images

The lens equation is given by,

$$\beta = \theta - \alpha(\theta) \tag{33}$$

By using the expression for scaled deflection angle for SIS from 32

$$\beta = \theta - \theta_E\hat{\theta} \tag{34}$$

or,

$$\alpha(\theta) = \beta + \theta_E\hat{\theta} \tag{35}$$

for $\hat{\theta} > 0$,

$$\theta_A = \beta + \theta_E \tag{36}$$

for $\hat{\theta} < 0$,

$$\theta_B = \beta - \theta_E \tag{37}$$

Thus, the separation of these images is given by,

$$\Delta\theta = \theta_A - \theta_B = \beta + \theta_E - (\beta - \theta_E) = 2\theta_E \tag{38}$$

P.3.3: Magnification ratio of the two images of SIS lens.

The magnification of a gravitational lens is given by,

$$\mu = (det\mathrm{A})^{-1} \tag{39}$$

P.3.4: Time delay derivation for SIS lens as a function of θ_A and θ_B.

The time delay is given by[**manual**]

$$\mathrm{c}\Delta\mathrm{t}(\beta) = (1+\mathrm{z}_d)\frac{\mathrm{D}_d D_s}{D_{ds}}[\tau(\theta_A;\beta) - \tau(\theta_B;\beta)] \tag{40}$$

Also, from 18 the Format's potential is defined as,

$$\tau(\theta;\beta) = \frac{1}{2}(\beta-\alpha)^2 - \psi(\theta) \tag{41}$$

So,

$$\tau(\theta_A;\beta) = \frac{1}{2}(\beta-\theta_A)^2 - \psi(\theta_A) \tag{42}$$

By plugging the values of β and $\psi(\theta_A)$,

$$= \frac{1}{2}(\theta_A - \theta_E - \theta_A)^2 - \theta_E\theta_A \tag{43}$$

Therefore,

$$\tau(\theta_A;\beta) = \frac{1}{2}\theta_E^2 - \theta_E\theta_A \tag{44}$$

Similarly,

$$\tau(\theta_B;\beta) = \frac{1}{2}(\beta-\theta_B)^2 - \psi(\theta_B) \tag{45}$$

$$= \frac{1}{2}(\theta_B + \theta_E - \theta_B)^2 - \theta_E\theta_B \tag{46}$$

$$\tau(\theta_B;\beta) = \frac{1}{2}\theta_E^2 - \theta_E\theta_B \tag{47}$$

By substituting these values equation 40 becomes,

$$\mathrm{c}\Delta\mathrm{t}(\beta) = (1+\mathrm{z}_d)\frac{\mathrm{D}_d D_s}{D_{ds}}(\frac{1}{2}\theta_E^2 - \theta_E\theta_A - \frac{1}{2}\theta_E^2 + \theta_E\theta_B) \tag{48}$$

Therefore,

$$\Delta\mathrm{t}(\beta) = \frac{(1+\mathrm{z}_d)}{\mathrm{c}}\frac{\mathrm{D}_d D_s}{D_{ds}}\theta_E(\theta_B - \theta_A) \tag{49}$$

Thus, the time delay is proportional to the Einstein radius.

P.3.5: Minimum Dispersion Estimator

The minimum dispersion estimator is a simple, efficient and well tested method to estimate time delay from observed light curve[**manual**] It helps to fine the difference between the curve at various delay times. Since it assume that light curves have the same shape but they are separated by time.

P3.6: The approximation of dispersion function near the minimum by a parabola.

The dispersion function near the minimum can be find by Taylor expanding of the functional function. i.e.

$$d^2(\lambda) = D^2(\lambda_0) + \frac{\mathrm{d}D^2(\lambda_0)}{\mathrm{d}\lambda}(\lambda - \lambda_0) + \frac{\mathrm{d}^2 D^2(\lambda_0)}{\mathrm{d}\lambda^2}(\lambda - \lambda_0)^2 + 0 \tag{50}$$

Where, D^2 is the dispersion function and λ is the time shift. Here high order terms are negligible compared to first and second order terms.

The dispersion function is minimum at $\lambda = \lambda_0$. Now the second term of equation 50 goes to zero. i.e.

$$d^2(\lambda) = D^2(\lambda_0) + \frac{\mathrm{d}^2 D^2(\lambda_0)}{\mathrm{d}\lambda^2}(\lambda - \lambda_0)^2 \tag{51}$$

it is parabolic in shape.