4.9 Chenhuan Weng

a) $\mathcal{L} = \sum_{i=1}^{3} \frac{1}{2}(\partial_\mu \phi_i)(\partial^\mu \phi_i) - \frac{1}{2} m^2 \left(\sum_{i=1}^{3} \phi_i^2 \right) - \frac{\lambda}{8} \left(\sum_{i=1}^{3} \phi_i^3 \right)^2$

$\mathcal{H}_{int} = \frac{\lambda}{8} \left(\sum \phi_i^2 \right)^2$

$\phi_1(x) = \int \frac{d^3k}{(2\pi)^3 \sqrt{2\omega_k}} \left(e^{-ikx} a_{1,k} + e^{+ik\cdot x} a_{1,k}^{+} \right)$

$\phi_2(x) = \int \frac{d^3k}{(2\pi)^3 \sqrt{2\omega_k}} \left(e^{-ikx} a_{2,k} + e^{+ik\cdot x} a_{2,k}^{+} \right)$

$\phi_3(x) = \int \frac{d^3k}{(2\pi)^3 \sqrt{2\omega_k}} \left(e^{-ikx} a_{3,k} + e^{+ik\cdot x} a_{3,k}^{+} \right)$

$i \neq j \rightarrow \langle 0 | T \phi_i(x) \phi_j(y) | 0 \rangle$

$\propto \langle P_{\phi_i} | P_{\phi_j} \rangle = 0$, orthogonality condition.

$i = j \rightarrow \langle 0 | T \phi_i(x) \phi_i(y) | 0 \rangle = D_F(x-y)$

$\Rightarrow \langle 0 | T \phi_i(x) \phi_i(y) | 0 \rangle = \delta_{ij} D_F(x-y)$

Feynman rules in position space:

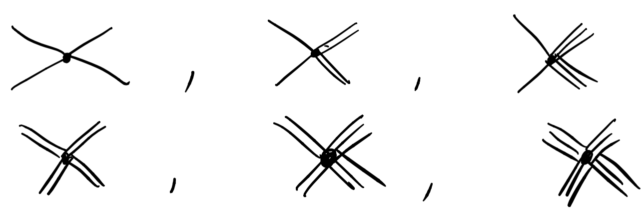
- Propagator

 $= \phi_1(x) \phi_1(y) = D^1_F(x-y)$

 $= \phi_2(x) \phi_2(y) = D^2_F(x-y)$

 $= \phi_3(x) \phi_3(y) = D^3_F(x-y)$

- Vertices



$= -3i\lambda \int d^4x$

3 comes from $\frac{4!}{8} = 3$

- external lines

$x \xleftarrow{p}$ $e^{-ip\cdot x}$ same for ϕ_2 and ϕ_3

$x \xrightarrow{p}$ $e^{+ip\cdot x}$

- Symmetry factor

b) $\mathcal{L}_0 \,, \quad \mathcal{L}_1 = -g\phi\psi\chi$

$$
\begin{aligned}
S_{fi} &= \delta_{fi} + i(2\pi)^4 \delta^{(4)}(P_f - P_i)\, \mathcal{M}_{fi} \\
&= \langle f \,|\, T \exp(-i \textstyle\int d^4x\, \mathcal{H}_{int}(x)) \,|\, i \rangle \\
&= \langle P_3, P_4 \,|\, T \exp(-i \textstyle\int d^4x\, \mathcal{H}_{int}(x)) \,|\, P_1, P_2 \rangle \\
&= \langle P_3, P_4 \,|\, T \Big[1 - i \textstyle\int d^4x\, (-g\phi(x)\psi(x)\chi(x)) \\
&\quad - \tfrac{1}{2} \textstyle\int d^4x \int d^4y\; g^2\, \phi(x)\psi(x)\chi(x)\, \phi(y)\psi(y)\chi(y) + \mathcal{O}(g^3) \Big] \,|\, P_1, P_2 \rangle
\end{aligned}
$$

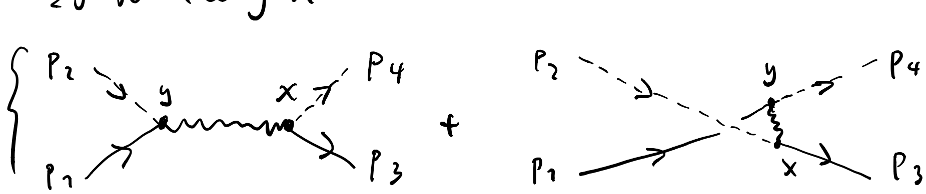
$$S^{(0)} = 0$$

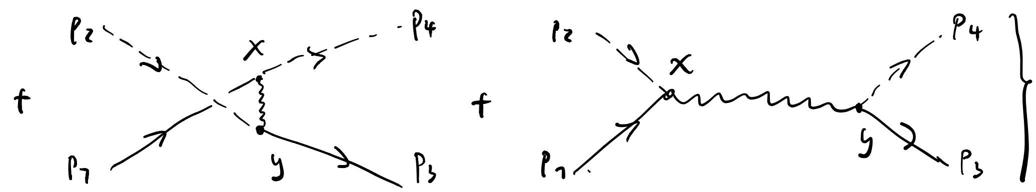
$$S^{(1)} = ig \int d^4x \;\; \langle \underset{\phi}{\underset{\uparrow}{P_3}}, \underset{\psi}{\underset{\uparrow}{P_4}} \,|\; T\,(\phi(x)\,\psi(x)\;\chi(x)) \;|\, \underset{\phi}{\underset{\uparrow}{P_1}}, \underset{\psi}{\underset{\uparrow}{P_2}} \rangle = 0$$

$$S^{(2)} = -\frac{1}{2} g^2 \int d^4x\, d^4y \;\langle P_3, P_4 \,|\, T(\phi(x)\psi(x)\chi(x)\;\phi(y)\psi(y)\chi(y)) \,|\, P_1, P_2 \rangle$$

$$
\begin{aligned}
\overset{\text{Wick}}{=} -\frac{1}{2} g^2 \int d^4x\, d^4y \Big\{ & \langle P_3, P_4 \,|\, \phi(x)\psi(x)\chi(x)\,\phi(y)\psi(y)\chi(y) \,|\, P_1, P_2 \rangle \\
+ & \langle P_3, P_4 \,|\, \phi(x)\psi(x)\chi(x)\,\phi(y)\psi(y)\chi(y) \,|\, P_1, P_2 \rangle \\
+ & \langle P_3, P_4 \,|\, \phi(x)\psi(x)\chi(x)\,\phi(y)\psi(y)\chi(y) \,|\, P_1, P_2 \rangle \\
+ & \langle P_3, P_4 \,|\, \phi(x)\psi(x)\chi(x)\,\phi(y)\psi(y)\chi(y) \,|\, P_1, P_2 \rangle \Big\}
\end{aligned}
$$

$$= -\frac{1}{2} g^2 \int d^4x\, d^4y\; X$$





The scattering amplitude associated with diagram on sheet :

$$= - g^2 \int d^4x \, d^4y \quad e^{-ip_2 y} \, e^{-ip_1 y} \, D_F^x(y-x) \, e^{ip_3 x} \, e^{ip_4 x}$$

$$= - g^2 \int d^4x \, d^4y \, d^4p \; e^{-i(p_1+p_2)y + i(p_3+p_4)x}$$

$$\frac{1}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\varepsilon} \cdot e^{-ip(x-y)}$$

$$= - g^2 \int d^4y \, d^4p \;\; \delta^{(4)}(-p + p_3 + p_4) \, e^{-i(p_1+p_2)y} \frac{i}{p^2 - m^2 + i\varepsilon} \; e^{+ipy}$$

$$= - g^2 (2\pi)^4 \int d^4p \; \frac{i}{p^2 + m^2 + i\varepsilon} \; \delta^{(4)}(-p + p_3 + p_4) \, \delta^{(4)}(-p + p_1 + p_2)$$

$$= - g^2 (2\pi)^4 \; \delta^{(4)}(p_1 + p_2 - p_3 - p_4) \; \frac{i}{(p_1+p_2)^2 - m^2 + i\varepsilon}$$

$$\stackrel{!}{=} i(2\pi)^4 \; \delta^{(4)}(p_f - p_i) \; \mathcal{M}$$

$$\Rightarrow \quad \mathcal{M} = \frac{-g^2}{(p_1+p_2)^2 - M^2} = \frac{-g^2}{s - M^2}$$