H.2 Chenhuan Wang 20/20

a) $$Q^+ = i\int d^3x\,[(\partial^0\phi^+)^+\phi^+ - \phi(\partial^0\phi)^+]$$
$$= i\int d^3x\,(\partial^0\phi\cdot\phi^+ - \phi\,\partial^0\phi^+)$$
$$= -i\int d^3x\,(\partial^0\phi^+\phi - \phi^+\partial^0\phi)$$

(b) $$\partial_0 Q \propto \int d^3x\;\partial_0[(\partial^0\phi^+)\phi - \phi^+(\partial^0\phi)]$$
$$= \int d^3x\,(\partial_0\partial^0\phi^+\cdot\phi + \cancel{\partial^0\phi^+\cdot\partial_0\phi} - \cancel{\partial_0\phi^+\cdot\partial^0\phi} - \phi^+\partial_0\partial^0\phi)$$
$$= \int d^3x\,(\partial_0^2\phi^+\cdot\phi - \phi^+\partial_0^2\phi)$$

1^{st} term $= \phi\,\partial_0^2\phi^+$
$$= \int \frac{d^3k\,d^3k'}{\sqrt{2\omega_k\cdot 2\omega_{k'}}}\frac{1}{(2\pi)^6}(e^{-i(\omega_k t-\underset{\sim}{k}\cdot\underset{\sim}{x})}b_{\underset{\sim}{k}} + e^{+i(\omega_k t-\underset{\sim}{k}\cdot\underset{\sim}{x})}c_k^+)\times$$
$$\underbrace{\partial_0^2(e^{+i(\omega_k t-\underset{\sim}{k}'\cdot\underset{\sim}{x})}b_{\underset{\sim}{k}}^+ + e^{-i(\omega_k t-\underset{\sim}{k}\cdot\underset{\sim}{x})}c_k)}$$

(Define $\alpha := \omega_k t - \underset{\sim}{k}\cdot\underset{\sim}{x}$
$\alpha' := \omega_{k'} t - \underset{\sim}{k}'\cdot\underset{\sim}{x}$)

$$\left[\begin{aligned} &= i\omega_{k'}\,\partial_0(e^{i\alpha'}b_k^+ - e^{-i\alpha'}c_k) \\ &= -\omega_{k'}^2(e^{i\alpha'}b_k^+ + e^{-i\alpha'}c_k)\end{aligned}\right]$$

$$= \int \frac{d^3k\,d^3k'}{\sqrt{2\omega_k\cdot 2\omega_{k'}}}\frac{-\omega_{k'}^2}{(2\pi)^6}(e^{-i\alpha}b_k + e^{i\alpha}c_k^+)(e^{i\alpha'}b_{k'}^+ + e^{-i\alpha'}c_{k'})$$
$$= \int \frac{d^3k\,d^3k'}{\sqrt{2\omega_k\cdot 2\omega_{k'}}}\frac{-\omega_{k'}^2}{(2\pi)^6}\Big(e^{i(\alpha'-\alpha)}b_k b_{k'}^+ + e^{-i(\alpha+\alpha')}b_k c_{k'} + e^{i(\alpha+\alpha')}c_k^+ b_{k'}^+$$
$$+ e^{i(\alpha-\alpha')}c_k^+ c_{k'}\Big)$$

with integration over $\underset{\sim}{x}$ and $\underset{\sim}{k}'$
$$= \int \frac{d^3k}{2\omega_k}\frac{-\omega_k^2}{(2\pi)^3}\Big(b_k b_k^+ + e^{-2i\omega_k t}b_k c_{-k} + e^{2i\omega_k t}c_k^+ b_{-k}^+ + c_k^+ c_k\Big)$$

2^{nd} term $= \phi^+\partial_0^2\phi$
$$= \int\frac{d^3k\,d^3k'}{\sqrt{2\omega_k\cdot 2\omega_{k'}}}\frac{1}{(2\pi)^6}(e^{-i\alpha}c_k + e^{i\alpha}b_k^+)\,\partial_0^2(e^{-i\alpha'}b_k + e^{i\alpha'}c_k^+)$$
$$= \int\frac{d^3k\,d^3k'}{\sqrt{2\omega_k\cdot 2\omega_{k'}}}\frac{-\omega_{k'}}{(2\pi)^6}(e^{-i\alpha}c_k + e^{i\alpha}b_k^+)(e^{-i\alpha'}b_{k'} + e^{i\alpha'}c_{k'}^+)$$

integration over $\underline{x}$ and $\underline{k}'$

$$= \int \frac{d^3k}{2\omega_k} \frac{-\omega_k^2}{(2\pi)^3} \left(e^{-2i\omega_k t} c_k b_{-k} + c_k c_k^{\dagger} + b_k^{\dagger} b_k + e^{2i\omega_k t} b_k^{\dagger} c_{-k}^{\dagger} \right)$$

$$\Rightarrow \partial_0 Q = -i \int d^3x \left(\partial_0^2 \phi^{\dagger} \cdot \phi - \phi^{\dagger} \partial_0^2 \phi \right)$$

$$= +i \int \frac{d^3k \; \omega_k}{2(2\pi)^3} \left([b_k, b_k^{\dagger}] + [c_k^{\dagger}, c_k] \right)$$

$$= i \int \frac{d^3k \; \omega_k}{2(2\pi)^3} \left((2\pi)^3 \delta^{(3)} \cancel{(\underline{k}-\underline{k})} - (2\pi)^3 \delta^{(3)} \cancel{(\underline{k}-\underline{k})} \right)$$

$$= 0$$

c) $$\partial^0 \phi^+ \phi = i\omega_k \int \frac{d^3k\, d^3k'}{\sqrt{2\omega_k \cdot 2\omega_{k'}}} \frac{1}{(2\pi)^6} \left(-e^{-i\alpha} c_k + e^{i\alpha} b_k^+\right)\left(e^{-i\alpha'} b_{k'} + e^{i\alpha'} c_{k'}^+\right)$$

$$= \int \frac{d^3k\, d^3k'}{\sqrt{2\omega_k \cdot 2\omega_{k'}}} \frac{i\omega_k}{(2\pi)^6} \Big(-e^{-i(\alpha+\alpha')} c_k b_{k'} - e^{i(\alpha'-\alpha)} c_k c_{k'}^+ + e^{i(\alpha-\alpha')} b_k^+ b_{k'} + e^{i(\alpha+\alpha')} b_k^+ c_{k'}^+ \Big)$$

integrating over $\underset{\sim}{x}$ and $\underset{\sim}{k}$

$$= \int \frac{d^3k}{2\cancel{\omega_k}} \frac{i\cancel{\omega_k}}{(2\pi)^3} \left(-e^{-2i\omega_k t} c_k b_{-k} - c_k c_k^+ + b_k^+ b_k + e^{2i\omega_k t} b_k^+ c_{-k}^+\right)$$

$$\partial^0 \phi\, \phi^+ = i\omega_k \int \frac{d^3k\, d^3k'}{\sqrt{2\omega_k \cdot 2\omega_{k'}}} \frac{1}{(2\pi)^6} \left(-e^{-i\alpha} b_k + e^{i\alpha} c_k^+\right)\left(e^{-i\alpha'} c_{k'} + e^{i\alpha'} b_{k'}^+\right)$$

integrating

$$= \int \frac{d^3k}{2\cancel{\omega_k}} \frac{i\cancel{\omega_k}}{(2\pi)^3} \left(-e^{-2i\omega_k t} b_k c_{-k} - b_k b_k^+ + c_k^+ c_k + e^{2i\omega_k t} c_k^+ b_{-k}^+\right)$$

$$\Rightarrow \quad Q = -i \int \frac{i\, d^3k}{2(2\pi)^3} \Big[\cancel{-e^{-2i\omega_k t} c_k b_{-k}} - c_k c_k^+ + b_k^+ b_k + \cancel{e^{2i\omega_k t} b_k^+ c_{-k}^+} + \cancel{e^{-2i\omega_k t} b_k c_{-k}} + b_k b_k^+ - c_k^+ c_k - \cancel{e^{2i\omega_k t} c_k^+ b_{-k}^+} \Big]$$

since $[b_k, c_k] = 0$

$$\Rightarrow \quad :Q: = \int \frac{d^3k}{(2\pi)^3} \Big(\underbrace{b_k^+ b_k}_{\substack{\text{positively charged} \\ \text{particle: } b_k}} - \underbrace{c_k^+ c_k}_{\substack{\text{negatively charged} \\ \text{particle: } c_k}} \Big)$$