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A.1

a)
- S is a Poincaré scalar
- $\delta S = 0$ if ε is a constant
- $|\varepsilon| << 1$

$$\delta S = \int d^4x \, [\, \varepsilon(x)\, \partial_\mu j^\mu(x) + j^\mu(x)\, \partial^\mu \varepsilon(x) \,]$$

b)
$$\delta S = \int \left\{ \frac{\partial \mathcal{L}}{\partial \phi_k} \varepsilon(x)\, \delta\phi_k(x) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_k)} \partial_\mu [\varepsilon(x)\, \delta\phi_k(x)] \right\} d^4x$$

$$= \int \left\{ \varepsilon(x) \underbrace{\left[\frac{\partial \mathcal{L}}{\partial \phi_k} \delta\phi_k(x) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_k)} \partial_\mu \delta\phi_k(x) \right]}_{\text{since } \delta S = 0 \text{ if } \varepsilon = \text{const}} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_k)} \delta\phi_k(x)\, \partial_\mu \varepsilon(x) \right\} d^4x$$

$$\Rightarrow \delta S = \int \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_k)} \delta\phi_k(x)\, \partial_\mu \varepsilon(x)\, d^4x$$

$$\Rightarrow \quad j^\mu(x) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_k)} \delta\phi_k$$

c)
$$\delta S = 0 = \int j^\mu(x)\, \partial_\mu \varepsilon(x)\, d^4x = -\int \varepsilon(x)\, \partial_\mu j^\mu\, d^4x$$

$\varepsilon(x)$ is now arbitary $\Rightarrow \partial_\mu j^\mu(x) = 0$

$$\partial_0 Q = \partial_0 \int j^0(x)\, d^3x = \int \partial_0 j^0(x)\, d^3x = -\int \partial_k j^k(x)\, d^3x$$

$$= \int_{\partial \mathbb{R}^3} \vec{j}\, d^3x = 0$$

A.2 $$\mathcal{L} = \partial_\mu \phi^*(x) \partial^\mu \phi(x) - m^2 \phi^*(x)\phi(x)$$

a)

$\phi(x)$: $$\partial_\omega \frac{\partial \mathcal{L}}{\partial(\partial_\omega \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} = \partial_\omega \left(\partial_\mu \phi^* g^{\mu\nu} \frac{\partial(\partial_\nu \phi)}{\partial(\partial_\omega \phi)} \right) + m^2 \phi^* \frac{\partial \phi}{\partial \phi}$$

$$= (\partial_\omega \partial^\omega + m^2)\phi^* = 0$$

$\phi^*(x)$: $$(\partial_\omega \partial^\omega + m^2)\phi = 0$$

b) $|\Lambda| << 1$ ← not losing any generality because of group property

$$\phi(x) \to \phi(x) - i\Lambda\phi(x)$$
$$\phi^*(x) \to \phi^*(x) + i\Lambda\phi^*(x)$$

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} \delta\phi^* = i\Lambda(\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*)$$

$\delta \to \partial^\mu$

$$Q = \int d^3x\, j^0(x)$$

$$= i\int d^3x \left(\phi^* \frac{\partial \phi}{\partial t} - \phi \frac{\partial \phi^*}{\partial t} \right)$$

- overall factor for j^μ symmetry, related to gauge coupling when symmetry is gauged

c) $$\phi(x) \to \phi(x) - i\Lambda(x)\phi(x)$$

$$A_\mu(x) \to A_\mu(x) + \frac{1}{e}\partial_\mu \Lambda(x)$$

$$D_\mu \phi = (\partial_\mu + ieA_\mu)\phi$$

$$D_\mu \phi \to [\partial_\mu + ieA_\mu + i(\partial_\mu \Lambda)][\phi - i\Lambda\phi]$$

$$= (\partial_\mu + ieA_\mu)\phi - i\Lambda(\partial_\mu + ieA_\mu)\phi$$

$$= D_\mu \phi - i\Lambda D_\mu \phi$$

same as $\phi(x)$'s transformation!

d) $|\Lambda| << 1$

$D_\mu \phi$ is multiplied by a factor $1 - i\Lambda$

Repeatedly $\lim_{N\to\infty} (1 - \frac{i\Lambda}{N})^N = e^{i\Lambda}$ (with redefinition of Λ)

$\to \mathcal{L}$ is invariant under local gauge

e) $F^{\mu\nu}$ is already gauge invariant itself

$$F^{\mu\nu} \longmapsto \partial^\mu (A^\nu + \frac{1}{e}(\partial^\nu \Lambda)) - \partial^\nu (A^\mu + (\frac{1}{e}\partial^\mu \Lambda))$$

$$= \partial^\mu A^\nu - \partial^\nu A^\mu + \frac{1}{e}\partial^\mu \partial^\nu \Lambda - \frac{1}{e}\partial^\mu \partial^\nu \Lambda$$

$$= F^{\mu\nu}$$

(f) $\mathcal{L}_M(x) = M^2 A^\mu(x) A_\mu(x)$

$$\delta \mathcal{L}_M = M^2 (\delta A_\mu) A^\mu + M^2 A_\mu (\delta A^\mu)$$

$$= 2 M^2 A^\mu (\delta A_\mu) = \frac{2M^2}{e} A^\mu \partial_\mu \Lambda \neq 0$$

(space-time is not transformed, internal symmetry

$\delta \mathcal{L} = 0 \Leftarrow \delta S = 0$)