Quantum Field Theory

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6 Radiative corrections

6.1 Optical theorem

We have seen in Advanced Quantum Theory that tree diagrams are in general real. So there is no imaginary parts. Need to restore perturbatively in higher-order corrections. Then the optical theorem is valid again.

S-matrix is unitary: $S^\dagger S = \mathbb{1}$ with $S = \mathbb{1} + iT$. Thus

$$-i(T - T^\dagger) = T^\dagger T$$

We take matrix element for $k_1 k_2 \to p_1 p_2$ scattering. On RHS, insert a complete set of states,

$$\langle p_1 p_2 | T^\dagger T | k_1 k_2 \rangle = \sum_n \prod_{i=1}^n \int \frac{\mathrm{d}^3 q_i}{(2\pi)^3 2E_i} \langle p_1 p_2 | T^\dagger | q_1 \ldots q_n \rangle \langle q_1 \ldots q_n | T | k_1 k_2 \rangle$$

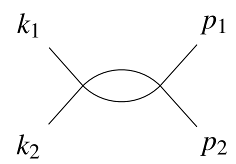
Reduce $T_{fi} = (2\pi)^4 \delta^{(4)}(p_f - p_i) M_{fi}$ and omitting overall $(2\pi)^4 \delta^{(4)}(p_f - p_i)$

$$\begin{aligned} -i\left[\mathcal{M}(k_1 k_2 \to p_1 p_2) - \mathcal{M}^*(p_1 p_2 \to k_1 k_2)\right] \\ = \underbrace{\sum \prod_{i=1}^n \int \frac{\mathrm{d}^3 q_i}{(2\pi)^3 2E_i}}_{\text{invariant phase-space volume element}} \mathcal{M}^*(p_1 p_2 \to q_1 \ldots q_n) \mathcal{M}(k_1 k_2 \to q_1 \ldots q_n)(2\pi)^4 \delta^{(4)}(k_1 + k_2 - \sum_i q_i) \end{aligned}$$

So optical theorem, for forward scattering ($p_1 = k_1, p_2 = k_2$) reads (see 4.5.1)

$$\begin{aligned} \operatorname{Im} \mathcal{M}(k_1 k_2 \to k_1 k_2) &= 2F\sigma_{\text{tot}}(k_1 k_2 \to \text{anything}) \\ 2\sqrt{s}\,|f_i^{\text{CMS}}| &= \lambda^{\frac{1}{2}}(s, m_1^2, m_2^2) \end{aligned}$$

Optical theorem for Feynman diagrams Consider a specific diagram contributing to the imaginary part, e.g. in ϕ^4-theory.



$$p_s = k_1 + k_2 = p_1 + p_2,\ p_s^2 = s$$

$$i\mathcal{M}(s) = \frac{\lambda^2}{2} \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{1}{[(p_s/2 - q)^2 - M^2 + i\epsilon]\,[(p_s/2 + q)^2 - M^2 + i\epsilon]} \tag{6.1.1}$$

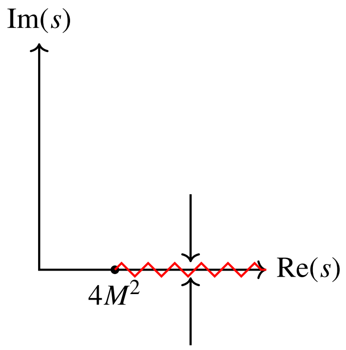
From optical theorem, $\operatorname{Im} \mathcal{M}(s < 4M^2) = 0$, so $\mathcal{M}(s < 4M^2) \in \mathbb{R}$, (since the scattering is not physical, the cross section must vanish) when regarding $\mathcal{M}(s)$ as an analytic function of s beyond what physical S-matrix element allow.

Schwarz reflection principle If (in some region) analytic function $\mathcal{M}(s)$ is real at least for a finite, non-vanishing interval $\in \mathbb{R}$, then

$$\mathcal{M}(s^*) = \mathcal{M}^*(s) \tag{6.1.2}$$

Hence

$$\mathcal{M}(s+i\epsilon) - \mathcal{M}(s-i\epsilon) \equiv \operatorname{disc}\mathcal{M}(s) = \mathcal{M}(s+i\epsilon) - \mathcal{M}^*(s+i\epsilon) = 2i\,\operatorname{Im}\mathcal{M}(s+i\epsilon)$$



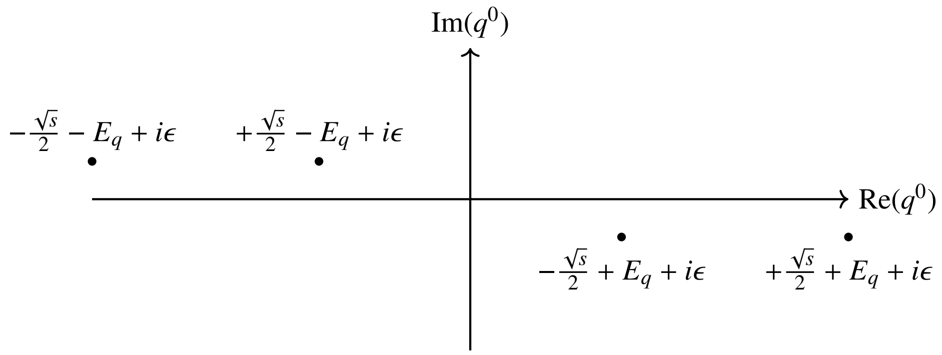
Onset of imaginary part for $s \leq 4M^2$ necessarily leads to a "branch cut", a non-trivial discontinuity in the complex energy plane. The branch cut is equivalent to $\sqrt{4M^2 - s}$. Function has discontinuity, a cut, on real axis.

How can we calculate the discontinuity (= imaginary part) of the above diagram? Use centre-of-mass system $p_s = (\sqrt{s}, \mathbf{0})$. Poles from propagators

$$\frac{s}{4} \mp \sqrt{s}q^0 + q^2 - M^2 + i\epsilon = 0$$
$$\Leftrightarrow (q^0)^2 \pm \sqrt{s}q^0 + \frac{s}{4} - |\boldsymbol{q}|^2 - M^2 + i\epsilon = 0$$

First propagator $\quad q^0 = +\frac{\sqrt{s}}{2} \pm (\sqrt{M^2 + |\boldsymbol{q}|^2} - i\epsilon) = +\frac{\sqrt{s}}{2} \pm (E_q - i\epsilon)$

Second propagator $\quad q^0 = -\frac{\sqrt{s}}{2} \pm (E_q - i\epsilon)$



If we close the contour of the q_0 integration in the lower half plane, we only pick up the 2 residues at $\mp\frac{\sqrt{s}}{2} + E_q - i\epsilon$. As E_q is positive, only $-\frac{\sqrt{s}}{2} + E_q - i\epsilon$ from second propagator contributes to discontinuity.

So pinching up the residue equivalent to replacement under q^0 integration

$$\frac{1}{(p_s/2+q)^2-M^2+i\epsilon} \longmapsto \underbrace{-2\pi i}_{\text{orientation of contour}} \quad \delta((p_s/2+q)^2-M^2)$$

Determine the residue of the rest at the pole at $-\frac{\sqrt{s}}{2}+E_q-i\epsilon$

$$M(s) \longmapsto -\frac{\lambda^2}{2}\int\frac{\mathrm{d}^3 q}{(2\pi)^3}\frac{1}{2E_q\sqrt{s}(\sqrt{s}-2E_q)}$$

With no angular dependence and using substitution (note the limits of integral also change)$\mathrm{d}^3 q \to 4\pi|\boldsymbol{q}|^2\mathrm{d}|\boldsymbol{q}| = 4\pi|\boldsymbol{q}|E_q\mathrm{d}E_q$

$$= -\frac{\lambda^2}{8\pi^2}\int_M^\infty \frac{\mathrm{d}E_q\sqrt{E_q^2-M^2}}{\sqrt{s}(\sqrt{s}-2E_q)} \tag{6.1.3}$$

It has pole at $E_q = \frac{\sqrt{s}}{2}$. The second pole in 6.1.1 at $\frac{\sqrt{s}}{2}+E_q-i\epsilon$ would produce a pole in 6.1.3 for $E_q = -\frac{\sqrt{s}}{2}$, outside the integration range $M \le E_q < \infty$.

- for $\sqrt{s} < 2M$, 6.1.3 is manifestly real.
- for $\sqrt{s} > 2M$, the pole at $E_q = \frac{\sqrt{s}}{2}$ in 6.1.3 contributes <u>differently</u> depending on $\sqrt{s} \pm i\epsilon$; difference yields discontinuity.

Use

$$\frac{1}{\sqrt{s}-2E_q\pm i\epsilon} = \underbrace{\frac{P}{\sqrt{s}-2E_q}}_{\text{real}}\underbrace{\mp i\pi\delta(\sqrt{s}-2E_q)}_{\text{yields discontinuity}}$$

So for calculation of the discontinuity, have replacement

$$\frac{1}{(p_s/2-q)^2-M^2+i\epsilon} \longmapsto -2\pi i\delta((p_s/2-q)^2-M^2)$$

for other propagator too!

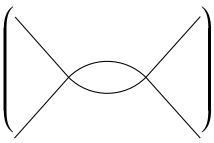
Cuthosky rules (1960) replace cut propagator according to

$$\frac{1}{p^2-M^2+i\epsilon} \longmapsto -2\pi i\delta(p^2-M^2) \tag{6.1.4}$$

to calculate discontinuity across the cut!

Calculation completed:

disc



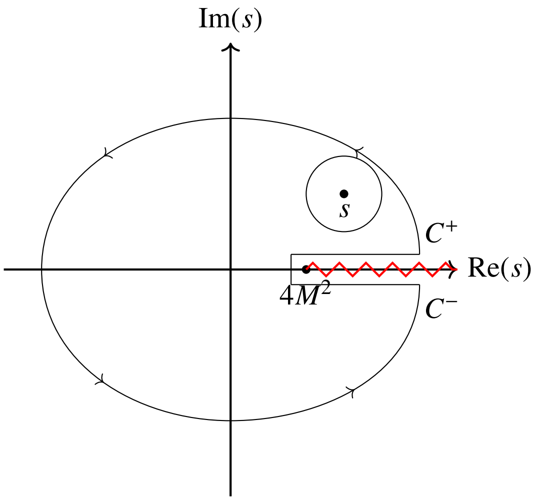
$$= i\frac{\lambda^2}{2}\int\frac{\mathrm{d}^4 q}{(2\pi)^4}2\pi\delta(q^2-M^2)2\pi\delta((p_s-q)^2-M^2)$$

using $d^4q = dq^0 dq|q|^2 d\Omega_q$ and $(p_s - q)^2 - M^2 = s - 2\sqrt{s}q^0$

$$
\begin{aligned}
&= \frac{\lambda^2}{2}\frac{i}{4\pi^2}\int \frac{|q|^2 d|q| d\Omega_q}{2q^0}\delta(s - 2\sqrt{s}q^0) \\
&= \frac{\lambda^2}{2}\frac{i}{8\pi^2}\int \sqrt{(q^0)^0 - M^2} dq^0 d\Omega_q \delta(s - 2\sqrt{s}q^0) \\
&= \frac{\lambda^2}{2}\frac{i}{8\pi^2}\frac{\sqrt{s/4 - M^2}}{2\sqrt{s}}\int d\Omega_q \\
&= \frac{\lambda^2}{2}\frac{i}{8\pi}\sqrt{1 - \frac{4M^2}{s}} \\
\mathrm{Im}\mathcal{M} &= \frac{\lambda^2}{4}\frac{1}{8\pi}\sqrt{1 - \frac{4M^2}{s}}
\end{aligned}
$$

Note $\sigma = \frac{\lambda^2}{32\pi}$ and $2F = s\sqrt{1 - \frac{4M^2}{s}}$. Thus optical theorem is still valid.

We can do more. Construct the complete $\mathcal{M}(s)$ from Im $\mathcal{M}(s)$ through a dispersion relation!



Use Cauchy's theorem:

$$
\mathcal{M}(s) = \frac{1}{2\pi i}\oint \frac{\mathcal{M}(z)dz}{z - s} \tag{6.1.5}
$$

dropping the large circle

$$
\begin{aligned}
&\longmapsto \frac{1}{2\pi i}\int_{C_+ + C_-} \frac{\mathcal{M}(z)dz}{z - s} \\
&= \frac{1}{2\pi i}\left[\int_{4M^2}^{\infty} \frac{M(z + i\epsilon)dz}{z - s} - \int_{4M^2}^{\infty} \frac{M(z - i\epsilon)dz}{z - s}\right] \\
&= \frac{1}{2\pi i}\int_{4M^2}^{\infty} \frac{\mathrm{disc}\mathcal{M}(z)dz}{z - s} \\
&= \frac{1}{\pi}\int_{4M^2}^{\infty} \frac{\mathrm{Im}\,\mathcal{M}(z)dz}{z - s}
\end{aligned} \tag{6.1.6}
$$

Repeat the exercise for $\frac{\mathcal{M}(s)-\mathcal{M}(0)}{s}$ (no pole introduced!).

$$\operatorname{Im}\left(\frac{\mathcal{M}(s)-\mathcal{M}(0)}{s}\right) = \frac{\operatorname{Im}\mathcal{M}(s)}{s}$$
$$\mathcal{M}(s)-\mathcal{M}(0) = \frac{s}{\pi}\int_{4M^2}^{\infty}\frac{\operatorname{Im}\mathcal{M}(z)\mathrm{d}z}{z(z-s)}$$
$$= \frac{\lambda^2}{2}\frac{s}{(4\pi)^2}\int_{4M^2}^{\infty}\frac{\mathrm{d}z}{z(z-s)}\sqrt{1-\frac{4M^2}{z}}$$

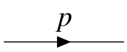
using $\sigma = \sqrt{1-\frac{4M^2}{s}}$ and $\zeta = \sqrt{1-\frac{4M^2}{z}}$

$$= \frac{\lambda^2}{2}\frac{1}{8\pi^2}\int_0^1\frac{\zeta^2}{\zeta^2-\sigma^2}\mathrm{d}\zeta$$
$$= \frac{\lambda^2}{2}\begin{cases}\frac{1}{8\pi^2}\left(1-\frac{\sigma}{2}\log\frac{\sigma+1}{\sigma-1}\right) & s<0 \Leftrightarrow \sigma>1\\ \frac{1}{8\pi^2}\left(1-\sqrt{-\sigma^2}\arctan\frac{1}{\sqrt{-\sigma^2}}\right) & 0<s<4M^2, \sigma^2<0\\ \frac{1}{8\pi^2}\left(1-\frac{\sigma}{2}\log\frac{1+\sigma}{1-\sigma}+\frac{i\sigma}{16\pi}\right) & s>M^2, 0<\sigma<1\end{cases}$$

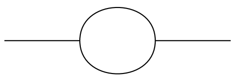
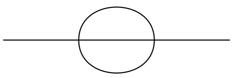
Note We are going to calculate this diagram again, noticing that $\int\frac{\mathrm{d}^4q}{(q^2\dots)(q^2\dots)}$ is logarithmically divergent! The above representation demonstrates that this divergence resides in $M(0)$!

6.2 Field-strength renormalization

What is structure of the propagator $\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle$ at higher orders? At lower order



$$= \frac{i}{p^2-M^2+i\epsilon}$$

Beyond this the propagator is not a simple pole. In ϕ^3-theory  branch cuts are at $p^2 \leq 4M^2$. In ϕ^4-theory  branch cuts are at $p^2 \leq 9M^2$. To induce cuts in the analytic structure.

Insert complete set of intermediate states ($x^0 > y^0$)

$$\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle = \sum_\lambda\int\frac{\mathrm{d}^3p}{(2\pi)^3 2E_p(\lambda)}\langle\Omega|\phi(x)|\lambda_{\boldsymbol{p}}\rangle\langle\lambda_{\boldsymbol{p}}|\phi(y)|\Omega\rangle$$

with

λ multi-particle state

λ_0 "rest frame", i.e. $\hat{\boldsymbol{P}}|\lambda_0\rangle = 0$

$\lambda_{\boldsymbol{p}}$ boosted to momentum $\boldsymbol{p}$

Call energy of $\lambda_0 = m_\lambda$. From single particle to multi particle $E_{\boldsymbol{p}}(\lambda) = \sqrt{m_\lambda^2 + |\boldsymbol{p}|^2}$.

$$\begin{aligned}\langle\Omega|\phi(x)|\lambda_{\boldsymbol{p}}\rangle &= \langle\Omega|e^{i\hat{P}x}\phi(0)e^{-i\hat{P}x}|\lambda_{\boldsymbol{p}}\rangle \\ &= \langle\Omega|\phi(0)|\lambda_{\boldsymbol{p}}\rangle\, e^{-ipx}\Big|_{p^0=E_{\boldsymbol{p}}}\end{aligned}$$

Ω and $\phi(0)$ are invariant under momentum boost

$$= \langle\Omega|\phi(0)|\lambda_0\rangle\, e^{-ipx}\Big|_{p^0=E_{\boldsymbol{p}}}$$

$$\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle = \sum_\lambda \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2E_p(\lambda)} e^{-ip(x-y)} |\,\langle\Omega|\phi(0)|\lambda_0\rangle\,|^2 \tag{6.2.1}$$

$$= \sum_\lambda \underbrace{\int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i}{p^2 - m_\lambda^2 + i\epsilon} e^{-ip(x-y)}}_{D_F(x-y;m_\lambda^2)\text{ when combined with }y^0>x^0} |\,\langle\Omega|\phi(0)|\lambda_0\rangle\,|^2 \tag{6.2.2}$$

$$\tag{6.2.3}$$

Formally write this as

$$\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle = \int_0^\infty \frac{\mathrm{d}s}{2\pi}\rho(s)D_F(x-y;s) \tag{6.2.4}$$

with $\rho(s)$ the spectral density function.

$$\rho(s) := \sum_\lambda (2\pi)\delta(s - m_\lambda^2)\,|\langle\Omega|\phi(0)|\lambda_0\rangle|^2 \tag{6.2.5}$$

A typical spectral function looks like

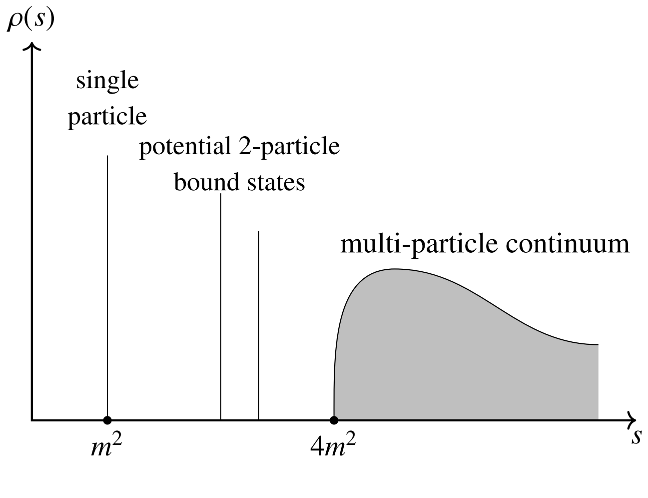


Figure 6.1: typical spectral function

Single particle contribution

$$\rho(s) = 2\pi\delta(s - m^2)Z + (\text{contributions} \geq 4m^2) \tag{6.2.6}$$

with $Z = |\langle\Omega|\phi(0)|\lambda_0\rangle|^2$ the field-strength renormalization factor.
Fourier transforming two-point function

$$\begin{aligned}&\int \mathrm{d}^4 x e^{ipx} \langle\Omega|T\phi(x)\phi(0)|\Omega\rangle \\ =&\int_0^\infty \frac{\mathrm{d}s}{2\pi}\rho(s)\frac{i}{p^2 - s + i\epsilon} \\ =&\frac{iZ}{p^2 - m^2 + i\epsilon} + \int_{\sim 4m^2}^\infty \frac{\mathrm{d}s}{2\pi}\rho(s)\frac{i}{p^2 - s + i\epsilon}\end{aligned}$$

Comparing to free theory: $\langle 0|\phi(0)|\boldsymbol{p}\rangle = 1$ hence $Z = 1$.

6.3 LSZ reduction formula†

Reminder A complete set of intermediate states

$$\mathbb{1} = |\Omega\rangle\langle\Omega| + \sum_\lambda \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2E_p(\lambda)} |\lambda_{\boldsymbol{p}}\rangle\langle\lambda_{\boldsymbol{p}}| \tag{6.3.1}$$

with

- λ multi-particle state
- λ_0 "rest frame" state, i.e. $\hat{\boldsymbol{\mathcal{P}}}|\lambda_0\rangle = 0$. Energy of λ_0: $m_\lambda \leftarrow E_{\boldsymbol{p}}(\lambda) = \sqrt{m_\lambda^2 + \boldsymbol{p}_\lambda^2}$
- $\lambda_{\boldsymbol{p}}$ state boosted to momentum $\boldsymbol{p}$

using the translation operators

$$\langle\Omega|\phi(x)|\lambda_{\boldsymbol{p}}\rangle = \langle\Omega|e^{i\hat{P}\cdot x}\phi(0)e^{-i\hat{P}\cdot x}|\lambda_{\boldsymbol{p}}\rangle$$

since Ω and $\phi(0)$ are invariant under momentum boost

$$= \langle\Omega|\phi(0)|\lambda_0\rangle\, e^{-ip\cdot x}|_{p^0 = E_{\boldsymbol{p}}}$$

We claim the Fourier transform of $\langle\Omega|T\phi(x_1)\dots\phi(x_n)\phi(y_1)\dots\phi(y_n)|\Omega\rangle$ contains poles in all external momenta. The residue is the S-matrix element $\langle p_1 \dots p_n|S|k_1 \dots k_m\rangle$ multiplied by $\sqrt{Z}$ for each external leg.

Fourier transform with respect to the first coordinated x_1 and let $x_2^0, \dots, x_n^0 \in [T_-, T_+]$ divide

$$\begin{aligned}&\int \mathrm{d}x_1^0 = \int_{T_+}^\infty + \int_{T_-}^{T_+} + \int_{-\infty}^{T_-} \\ \Rightarrow &\int_{T_+}^\infty \mathrm{d}x_1^0 \int \mathrm{d}^3 x_1 e^{iP_1 \dot{x}_1} \langle\Omega|\phi(x_1)\phi(x_2)\dots\phi(x_n)|\Omega\rangle \\ = &\int_{T_+}^\infty \mathrm{d}x_1^0 \int \mathrm{d}^3 x_1 e^{iP_1 \dots x_1} \sum_\lambda \int \frac{\mathrm{d}^3 q}{(2\pi)^3 2E_{\boldsymbol{q}}(\lambda)} \langle\Omega|\phi(x_1)|\lambda_{\boldsymbol{q}}\rangle\langle\lambda_{\boldsymbol{q}}|T\phi(x_2)\dots\phi(x_n)|\Omega\rangle\end{aligned}$$

†see also Peskin ans Schröder, chapter 7.2; Martin Mojzis, QFT I, page 110-113

use $\langle\Omega|\phi(x_1)|\lambda_{\boldsymbol{q}}\rangle = \langle\Omega|\phi(0)|\lambda_0\rangle\, e^{-iq\cdot x_1}|_{q^0=E_{\boldsymbol{q}}}$ and integrate over $\boldsymbol{x}$.

$$= \sum_\lambda \int_{T_+}^\infty \mathrm{d}x_1^0 \int \frac{\mathrm{d}^3 q}{(2\pi)^3 2E_{\boldsymbol{q}}(\lambda)} e^{i(p_1^0-q^0+i\epsilon)x_1^0}(2\pi)^3\delta^{(3)}(\boldsymbol{p_1}-\boldsymbol{q})\,\langle\Omega|\phi(0)|\lambda_0\rangle\,\langle\lambda_{\boldsymbol{q}}|T\phi(x_2)\dots\phi(x_n)|\Omega\rangle$$

Integrate over $\boldsymbol{q}$ and x^0

$$= \sum_\lambda \frac{1}{2E_{\boldsymbol{p}}(\lambda)}\frac{ie^{i(p_1^0-E_{\boldsymbol{p}_1}(\lambda))T_+}}{p_1^0-E_{\boldsymbol{p}_1}(\lambda)+i\epsilon}\,\langle\Omega|\phi(0)|\lambda_0\rangle\,\langle\lambda_{\boldsymbol{q}}|T\phi(x_2)\dots\phi(x_n)|\Omega\rangle$$

We know from the previous section

- only single-particle states λ_0 will produce a pole
- multi-particle produce "milder" singularities, like continuous cuts
- for single-particle state of mass m, the above has precisely the pole of $\frac{1}{p_1^2-m^2+i\epsilon}$ with residue $\langle\Omega|\phi(0)|\boldsymbol{p}_1\rangle = \sqrt{Z} = |\langle\Omega|\phi(0)|\lambda_0\rangle|$

hence

$$\int \mathrm{d}^4x_1 e^{ip_1\cdot x_1}\,\langle\Omega|T(\phi(x_1)\phi(x_2)\dots\phi(x_n))|\Omega\rangle \tag{6.3.2}$$

$$\overset{p_1^0\to+E_{\boldsymbol{p}}}{=} \frac{i\sqrt{Z}}{p_1^2-m^2+i\epsilon}\,{}_{\text{out}}\langle\boldsymbol{p}_1|T(\phi(x_2)\dots\phi(x_n))|\Omega\rangle + \text{(less singular stuff)} \tag{6.3.3}$$

What about the other integration regions in x_1^0?

- Integral $\int_{T_-}^{T_+}\mathrm{d}x_1^0$ is bounded, hence yields analytic, non-singular function
- $$\int_{-\infty}^{T_1}\mathrm{d}x_1^0\int \mathrm{d}^3x_1 e^{ip_1\cdot x_1}\,\langle\Omega|T(\phi(x_2)\dots\phi(x_n))\phi(x_1)|\Omega\rangle$$
  has pole for $p_1^0 \to -E_{\boldsymbol{p}_1}$
  $$= \dots = \frac{i\sqrt{Z}}{p_1^2-m^2+i\epsilon}\,\langle\Omega|T(\phi(x_1)\dots\phi(x_n))|-\boldsymbol{p}_1\rangle_{\text{in}} + \dots$$
  it has pole for an in- instead of an out-state

How do we go from here. Fourier-transform with respect to the second coordinate x_2, with the same assumption on T-ordering as before

$$\int \mathrm{d}^4x_2 e^{ip_2\cdot x_2}\,{}_{\text{out}}\langle\boldsymbol{p}_1|\phi(x_2)T(\phi(x_3)\dots\phi(x_n))|\Omega\rangle$$
$$= \sum_\lambda \int \mathrm{d}^4x_2 e^{ip_2\cdot x_2}\int\frac{\mathrm{d}^3q}{(2\pi)^3 2E_{\boldsymbol{q}}(\lambda)}\,\langle\boldsymbol{p}_1|\phi(x_2)|\lambda_{\boldsymbol{q}}\rangle\,\langle\lambda_{\boldsymbol{q}}|T(\phi(x_3)\dots\phi(x_n))|\Omega\rangle \tag{6.3.4}$$

We want to find the poles in p_2. But from which intermediate states?

- $|\lambda_{\boldsymbol{q}}\rangle = |\Omega\rangle$ yields $\propto \int \mathrm{d}^4x_2 e^{ip_2\cdot x_2}\,\langle\boldsymbol{p}_1|\phi(x_2)|\Omega\rangle\,\frac{\mathrm{d}^3q}{2E_{\boldsymbol{q}}} \propto \int \mathrm{d}^4x_2 e^{i(p_2+p_1)\cdot x_2}\frac{\mathrm{d}^3q}{2E_{\boldsymbol{q}}}$
  No singularity in p_2 (no isolated $\frac{1}{2E_{\boldsymbol{p}_2}}$-term).

- $|\lambda_{\boldsymbol{q}}\rangle = |\boldsymbol{q}\rangle \longmapsto \langle \boldsymbol{p}_1|\phi(x_2)|\boldsymbol{q}\rangle\, e^{ip_2\cdot x_2} = \langle \boldsymbol{p}_1|\phi(0)|\boldsymbol{q}\rangle\, e^{i(p_1+p_2-q)\cdot x_2}$
  Upon $\int \frac{\mathrm{d}^3 q}{2E_{\boldsymbol{q}}}$ cut in $p_1 + p_2$ at best
- $|\lambda_{\boldsymbol{q}}\rangle = |\boldsymbol{q}_1, \boldsymbol{q}_2\rangle$ Crucial assumption that by using wave packets, we can define asymptotically (for $t \to 0$) "non-interacting" single-particle states, such that

$$\sum_\lambda \int \frac{\mathrm{d}^3 q}{(2\pi)^3 2E_{\boldsymbol{q}}(\lambda)} \mapsto \int \frac{\mathrm{d}^3 q_1}{(2\pi)^3 2E_{\boldsymbol{q}_1}} \int \frac{\mathrm{d}^3 q_2}{(2\pi)^3 2E_{\boldsymbol{q}_2}} + \text{(higher states)}$$

So we have

$$\int \mathrm{d}^4 x_2 e^{ip_2\cdot x_2} \langle \boldsymbol{p}_1|\phi(x_2) T(\phi(x_3)\dots\phi(x_n))|\Omega\rangle$$

$$\longmapsto \dots + \int \mathrm{d}^4 x_2 e^{ip_2\cdot x_2} \int \frac{\mathrm{d}^3 q_1}{(2\pi)^3 2E_{\boldsymbol{q}_1}} \int \frac{\mathrm{d}^3 q_2}{(2\pi)^3 2E_{\boldsymbol{q}_2}} \langle p_1 |\phi(x_2)| \boldsymbol{q}_1, \boldsymbol{q}_2\rangle \langle \boldsymbol{q}_1 \boldsymbol{q}_2 | T(\phi(x_2) \cdots \phi(x_n))| s\rangle$$

Auxiliary calculation

$$\langle p_1 |\phi(x_2)| q_1, q_2\rangle = \langle p_1|\phi(0)|q_1, q_2\rangle\, e^{i(p_1-q_1-q_1)\cdot x_2}$$

use commutation relation and $\langle p_1|\, a_{\boldsymbol{q}_1}^{+(\text{asymp})} = \langle\Omega\rangle \sqrt{2E_{\boldsymbol{q}_1}} (2\pi)^3 \delta^{(3)} (\boldsymbol{p}_1 - \boldsymbol{q}_1)$

$$\begin{aligned} &= \sqrt{2E_{\boldsymbol{q}_1}} \left\langle \boldsymbol{p}_1 \left| \phi(0) a_{\boldsymbol{q}_1}^{+(\text{asymp})} \right| \boldsymbol{q}_2 \right\rangle e^{i(p_1-q_1-q_2)\cdot x_2} \\ &= 2E_{\boldsymbol{q}_1} (2\pi)^3 \delta^{(3)}(\boldsymbol{p}_1 - \boldsymbol{q}_1) \langle \Omega|\phi(0)|\boldsymbol{p}_2\rangle\, e^{i(p_1-q_1-q_2)\cdot x_2} \end{aligned}$$

Note $[\phi(0), a_{\boldsymbol{q}_2}^{+(\text{asymp})}]$ yields no singular piece in p_2 (no $\frac{1}{2E_{\boldsymbol{p}_2}}$).
Then

$$\int \mathrm{d}^4 x_2 e^{ip_2\cdot x_2} \int \frac{\mathrm{d}^3 q_1}{(2\pi)^3 2E_{\boldsymbol{q}_1}} \int \frac{\mathrm{d}^3 q_2}{(2\pi)^3 2E_{\boldsymbol{q}_2}} \langle p_1 |\phi(x_2)| \boldsymbol{q}_1 \boldsymbol{q}_2\rangle \langle \boldsymbol{q}_1 \boldsymbol{q}_2 | T(\phi(x_3) \dots \phi(x_n) | s\rangle$$

$$= \text{(non-singular-pieces)} + \int \mathrm{d}^4 x_2 \int \frac{\mathrm{d}^3 q_2}{((2\pi)^3)\, 2E_{\boldsymbol{q}_2}} e^{i(p_2-q_2)\cdot x_2} \langle\Omega|\phi(0)|\boldsymbol{p}_2\rangle \langle \boldsymbol{p}_1 \boldsymbol{q}_2 | T(\phi(x_3) \cdot \phi(x_n))| \Omega\rangle$$

x_2 integration turns into delta distribution and then gets integrated out; $\langle\Omega|\phi(0)|\boldsymbol{p}_2\rangle$ turns into $\sqrt{Z}$

$$= \dots + \frac{i\sqrt{z}}{p_2^2 - m^2 + i\epsilon}{}_{\text{out}} \langle \boldsymbol{p}_1 \boldsymbol{p}_2 | T(\phi(x_3) \dots \phi(x_n) | \Omega\rangle$$

Combine them together

$$\prod_{i=1}^n \int d^4 x_i e^{ip_i x_i} \langle \Omega | T(\phi(x_1) \phi(x_2) \phi(x_3) \dots \phi(x_n))| \Omega\rangle$$

$$= \frac{i\sqrt{Z}}{p_1^2 - m^2 + i\epsilon} \frac{i\sqrt{Z}}{p_2^2 - m^2 + i\epsilon}{}_{\text{out}} \langle \boldsymbol{p}_1 \boldsymbol{p}_2 | T(\phi(x_3) \dots \phi(x_n))| \Omega\rangle + \text{(non-pole terms)} \tag{6.3.5}$$

Repeat these steps and remember that the singular piece in the propagator (two-point function) was $\frac{iZ}{p^2-m^2+i\epsilon}$. Write ${}_{\text{out}}\langle \boldsymbol{p}_1 \boldsymbol{p}_2 \dots \boldsymbol{p}_r | \boldsymbol{p}_{r+1} \dots \boldsymbol{p}_n\rangle_{\text{in}} = {}_{\text{(in)}}\langle \boldsymbol{p}_1 \boldsymbol{p}_2 \dots \boldsymbol{p}_r | S | \boldsymbol{p}_{r+1} \dots \boldsymbol{p}_n\rangle_{\text{(in)}}$

$$\langle \boldsymbol{p}_1 \boldsymbol{p}_2 \dots \boldsymbol{p}_r | S | \boldsymbol{p}_{r+1} \dots \boldsymbol{p}_n\rangle = \prod_{k=1}^n \sqrt{Z} \lim_{p_k^2 \mapsto m_k^2} \left(\frac{iZ}{p_k^2 - m^2 + i_\epsilon} \right)^{-1} G(p_1, \dots, p_n)$$

$$\text{where } G(p_1, \dots, p_n) = \prod_{i=1}^r \int \mathrm{d}^4 x_i e^{ip_i x_i} \prod_{j=r+1}^n \int \mathrm{d}^4 x_j e^{-ip_j\cdot x_j} \langle \Omega | T\phi(x_1) \dots \phi(x_n)| \Omega\rangle \tag{6.3.6}$$

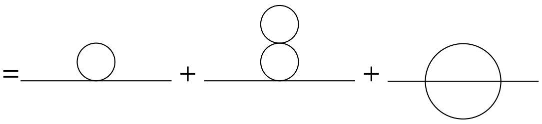
The S-matrix element is the on-shell limit (which takes care of the non-pole pieces) of the momentum-space vacuum correlation function, multiplied by the inverse propagator (with the dressed mass m) and a factor $\sqrt{Z}$ (wave-function renormalization) for each external leg.

Remark A more rigorous proof (see e.g. Itzykson and Zuber, Chapter.5.1-3) is based on careful definition of in-/out-state operator etc. (see also Schwatz, Chapter 6.1).

6.4 The propagator (again)[‡]

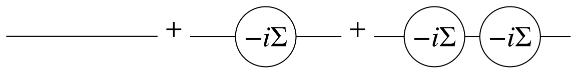
How do we calculate the propagator and the wave-function renormalization factor Z in perturbation theory, using Feynman diagrams? Call mass parameter in $\mathcal{L} = \frac{1}{2}(\partial_\mu\phi_0)^2 - \frac{m_0^2}{2}\phi_0^2 - \frac{\lambda_0}{4!}\phi_0^4$ the *bare mass*.

One-particle-irreducibles (1PIs) in ϕ^4-theory are the diagrams that cannot be disconnected by cutting internal lines. Their contributions are

$-i\Sigma(p^2) =$  $+ \dots$

Then the complete propagator using $D_F^0(p^2) = \frac{i}{p^2 - m_0^2 + i\epsilon}$ is

$$D_F(p^2) = \int \mathrm{d}^4 x e^{ipx} \langle 0|T\phi(x)\phi(0)|0\rangle \tag{6.4.1}$$

$=$  $+ \dots$

$$= D_F^0(p^2) + D_F^0(p^2)\left(-i\Sigma(p^2)\right) D_F^0(p^2) + \dots$$

it is cleary a geometric series

$$= \frac{D_F^0(p^2)}{1 + i\Sigma(p^2) D_F^0(p^2)}$$

$$= \frac{i}{p^2 - m_0^2 - \Sigma(p^2)} \tag{6.4.2}$$

The pole of propagator does not occur at m_0^2 anymore. It will be shifted by $\Sigma \sim O(\lambda)$!

Expansion of divergent integrals [§] Notice that the integral in 6.1.1 $\propto \int \frac{\mathrm{d}^4 q}{q^4}$. If we differentiate it with respect to q, the integral becomes convergent. This holds true for integral of general loop diagrams (although more than one differentiation might be needed). Thus we can expand this kind of integral into convergent and divergent term(s).

Expand

$$\Sigma(p^2) = \Sigma(m^2) + (p^2 - m^2)\Sigma'(m^2) + (p^2 - m^2)\tilde{\Sigma}(p^2) \tag{6.4.3}$$

where $\Sigma(m^2)$ is quadratically and $\Sigma'(m^2)$ logarithmically divergent. $\tilde{\Sigma}$ represents a correction (to first order Taylor expansion) and it satisfies $\tilde{\Sigma}(m^2) = \tilde{\Sigma}'(m^2) = 0$.

[‡] see also Peskin and Schröder, Chapter 10.2

[§] see also Cheng and Li, Chapter 2.1

Mass and field renormalization The mass m by the condition

$$m^2 = m_0^2 + \Sigma(m^2) \tag{6.4.4}$$

This is indeed physical mass, since the expression for propagator in 6.4.2 has a pole at $p^2 = m^2$.

Then the propagator

$$D_F(p^2) = \frac{i}{p^2 - m_0^2 - \Sigma(p^2)} = \frac{i}{p^2 - m^2 - (p^2 - m^2)(\Sigma'(m^2) + \tilde{\Sigma}(p^2))}$$

using 6.4.3

$$\begin{aligned} &= \frac{i}{(p^2 - m^2)(1 - \Sigma'(m^2) - \tilde{\Sigma}(p^2))} \\ &= \frac{iZ}{p^2 - m^2} \cdot \frac{1}{1 - Z\tilde{\Sigma}(p^2)} \\ &= \frac{iZ}{p^2 - m^2} + (\text{regular at } p^2 = m^2) \end{aligned} \tag{6.4.5}$$

with $Z = \left(1 - \Sigma'(m^2)\right)^{-1}$. This expression is to be compared with 6.3.1.

Starting point Lagrangian is $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_0)^2 - \frac{m_0^2}{2}\phi_0^2 - \frac{\lambda_0}{4!}\phi_0^4$. To remove Z from numerator in the propagator and instead put $\sqrt{Z}$ onto the couplings at each end. Since each internal vertex has 4 lines (remember the vertex carries the coupling constant)

$$\lambda_0 \longmapsto \lambda_1 = Z^2 \lambda_0 \tag{6.4.6}$$

In Σ and $\tilde{\Sigma}$, there are 2 external lines without $\sqrt{Z}$, so

$$\Sigma(p^2, \lambda_0, \text{old } D_F) = \frac{1}{Z}\Sigma_1(p^2, \lambda_1, \text{new } D'_F) \tag{6.4.7}$$

(same expression for $\tilde{\Sigma}$).

Thus we get the new propagator

$$D'_F(p^2) = \frac{i}{p^2 - m^2} \cdot \frac{1}{1 - \tilde{\Sigma}_1(p^2)} \tag{6.4.8}$$

where $\tilde{\Sigma}_1(m^2) = 0$.

Define the renomalized field

$$Z^{-\frac{1}{2}}\phi_0 = \phi \tag{6.4.9}$$

then D'_F is the Fourier transform of $\langle 0|T\phi(x)\phi(y)|0\rangle$

Rewrite the Lagrangian as

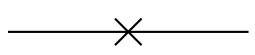
$$\mathcal{L} = \frac{1}{2}\left((\partial_\mu \phi)^2 - m^2\phi^2\right) \boxed{-\frac{\lambda_1}{4!}\phi^4 \underbrace{-\frac{1}{2}\delta m^2 \phi^2 + \frac{1}{2}(Z-1)\left((\partial_\mu \phi)^2 - m^2\phi^2\right)}_{\text{counter-terms}}} \tag{6.4.10}$$

where $\delta m^2 = -Z(m^2 + m_0^2) = -Z\Sigma(m^2) = -\Sigma_1(m^2)$. Everything inside the $\boxed{\text{box}}$ can be considered as interaction.

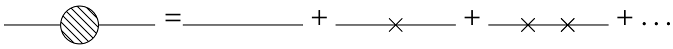
It may look weird given the kinetic/mass-like terms, but there is no contradiction. Consider just $\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2$. The mass-term ≡ "interaction". A massless propagator

 $= \dfrac{i}{p^2}$

and interaction

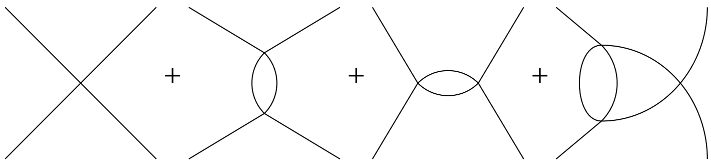
 $= -im^2$

The resummed propagator is then



$$
\begin{aligned}
&= \frac{i}{p^2}\left(1 + \frac{i}{p^2}(-im^2) + \dots\right) \\
&= \frac{i}{p^2}\left(1 - \frac{i}{p^2}(-im^2)\right)^{-1} = \frac{i}{p^2 - m^2}
\end{aligned}
$$

Actually this is not all. We will also have to further renormalize λ_1

 $+ \dots = \lambda_1 \{1 + L + \Gamma(s,t,u)\}$

L is value of the sum of all 1PI vertex contributions at the same kinematic point and Γ defined by $\Gamma(s = t = u = \frac{4}{3}M^2) = 0$ (for instance, $P_i^2 = M^2$ and $P_iP_j = -\frac{M^2}{3}$ with $i \neq j$).

Define

$$Z_\lambda := (1 + L)^{-1} \tag{6.4.11}$$

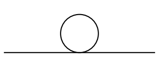
and the renormalized coupling is

$$\lambda = Z_\lambda^{-1}\lambda_1 = Z_\lambda^{-1}Z^2\lambda_0 \tag{6.4.12}$$

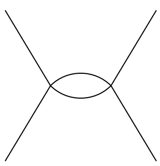
Write Lagrangian in terms of renormalized λ and add another counter-term $-\frac{(Z_\lambda - 1)}{4!}\lambda\phi^4$.

Note that all counter-terms we have introduced are of the same form as the original Lagrangian: $(\partial\phi)^2$, ϕ^2 and ϕ^4. There is no need to introduce new structure or new coupling parameters. It is property of a renormalizable theory.

6.5 Divergent graphs and dimensional regularization

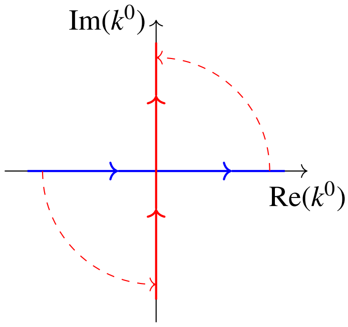


$$M_2 = \frac{\lambda}{2}\frac{1}{i}\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2 - M^2 + i\epsilon}$$



$$M_4 = \frac{\lambda^2}{2}\frac{1}{i}\int \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2 - M^2 + i\epsilon)((k-p)^2 - M^2 + i\epsilon)}$$

Wick rotation Poles of M_2 in the complex k^0-plane at $k^0 = \pm\sqrt{\boldsymbol{k}^2 + M^2 - i\epsilon}$. The position of the poles allow us to rotate the integration path to go $-i\infty \mapsto +i\infty$ instead. So no singularities are hit!

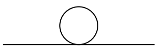


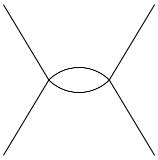
Define a Euclidean momentum $k^0 = ik_E^0$, $\boldsymbol{k} = \boldsymbol{k}_E$

$$\frac{1}{i}\int \frac{\mathrm{d}k^0\mathrm{d}^3k}{(2\pi)^4}\frac{1}{(k_0)^2 - \boldsymbol{k}^2 - M^2 + i\epsilon} \longmapsto -\int \frac{\mathrm{d}^4k_E}{(2\pi)^4}\frac{1}{k_E^2 + M^2 - i\epsilon}$$

Now we are far from singularities, $i\epsilon$ can thus be ignored.

This form allows us to see

 $\sim \int \frac{\mathrm{d}kk^3}{k^2}$ is quadratically divergent

 $\sim \int \frac{\mathrm{d}kk^3}{k^4}$ is logarithmically divergent

Hope of renormalization program is that all such divergences can be absorbed into bare/unrenormalized couplings to produce physical/renormalized/observable parameters.

There are different methods to regularize divergent loop integrals in order to keep track of divergences

1. momentum (Λ) cutoff: study the limit $\Lambda \longmapsto \infty$ in the end

2. Pauli-Villars: subtract propagator(s) with heavy mass(es)*

$$\frac{1}{k^2} \longmapsto \frac{1}{k^2} - \frac{1}{k^2 - M_{\mathrm{PV}}^2},\ M_{\mathrm{PV}} \longmapsto \infty$$

3. dimensional regularization: work in d dimension instead of 4, 1 time-like, $d-1$ space-like. For small d integral converge, consider $d \longmapsto 4$ in the end. The divergences appear as poles in $\frac{1}{d-4}$.

Main advantage of dimensional regularization is that all symmetries are preserved (massless photons etc.). Downside is that it is somewhat unphysical and unintuitive.

*for details see Ryder, Chapter 9.2

Feynman parameters * Combine multiple propagators into one (to some power)

$$\frac{1}{A_1 \dots A_n} = \int_0^1 \mathrm{d}x_1 \dots \mathrm{d}x_n \delta\left(\sum_i^n x_i - 1\right) \frac{(n-1)!}{(x_1 A_1 + \dots + x_n A_n)^n} \tag{6.5.1}$$

using

$$\frac{1}{A_i} = \int_0^\infty \mathrm{d}\alpha_i e^{-\alpha_i A_i}$$
$$\int \mathrm{d}\alpha_1 \dots \mathrm{d}\alpha_n e^{-\sum_i \alpha_i A_i} = \int_0^1 \mathrm{d}x_1 \dots \mathrm{d}x_n \delta\left(\sum_i^n x_i - 1\right) \int_0^\infty \mathrm{d}t\, t^{n-1} e^{-t \sum_i x_i A_i}$$

Special case

$$\frac{1}{AB} = \int_0^1 \mathrm{d}x \frac{1}{[xA + (1-x)B]^2} \tag{6.5.2}$$

With $A = (k-p)^2 - M^2$ and $B = k^2 - M^2$

$$xA + (1-x)B = k^2 - xp(2k-p) - M^2 = (k-p)^2 - (M^2 - x(1-x)p^2)$$

Thus after shifting the integration variable $k \longmapsto k + xp$ and with $\Delta(x) := M^2 - x(1-x)p^2$

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{(k^2 - M^2)((k-p)^2 - M^2)} = \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{[k^2 - \Delta(x)]^2}$$

Dimensional regularization formula *

$$\frac{1}{i} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{(k^2 - \Delta)^n} = \frac{(-1)^n}{(4\pi)^{d/2}} \frac{\Gamma(n - d/2)}{\Gamma(n)} \frac{1}{\Delta^{n-d/2}} \tag{6.5.3}$$

Γ-function has following definition and properties

- $\Gamma(n+1) = \int_0^\infty \mathrm{d}x x^n e^{-x}$
- $\Gamma(n+1) = n!$ for $n \in \mathbb{N}$, $n\Gamma(n) = \Gamma(n+1)$
- $\Gamma(n)$ has poles for negative integers $n = 0, -1, -2, \dots$

Proof by induction

- $n = 1$: introduce Schwinger parameter α and $i\epsilon$ part enforces convergence.

$$\frac{1}{i} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{k^2 - \Delta + i\epsilon} = -\int_0^\infty \mathrm{d}\alpha \int \frac{\mathrm{d}^d k}{(2\pi)^d} e^{i\alpha(k^2 - \Delta + i\epsilon)}$$

using Wick rotation

$$= -i \int_0^\infty \mathrm{d}\alpha \int \frac{\mathrm{d}^d k_E}{(2\pi)^d} e^{-i\alpha(k_E^2 + \Delta - i\epsilon)}$$

*see also Peskin and Schröder, Chapter 6.3; Ryder, Chapter 9.2
*see also Peskin and Schröder, Chapter 7.5

Gaussian integral in higher dimension; in general $\int \exp\left(-\frac{1}{2}x\cdot A\cdot x + J\cdot x\right)d^n x = \sqrt{\frac{(2\pi)^n}{\det A}}\exp\left(\frac{1}{2}J\cdot A^{-1}\cdot J\right)$

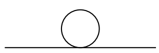
$$\begin{aligned}
&= \frac{-i}{(2\pi)^d}\int_0^\infty \mathrm{d}\alpha \sqrt{\frac{\pi}{i\alpha}}^{d} e^{-i\alpha\Delta} \\
&= \frac{-i}{(4\pi)^{d/2}}\int_0^\infty \mathrm{d}\alpha (i\alpha)^{-d/2}e^{-i\alpha\Delta} \\
&= -\frac{1}{(4\pi)^{d/2}}\frac{1}{\Delta^{1-d/2}}\int_0^\infty \mathrm{d}x x^{-d/2}e^{-x} \\
&= \frac{(-1)}{(4\pi)^{d/2}}\frac{1}{\Delta^{1-d/2}}\Gamma(1-d/2)
\end{aligned}$$

- Induction $n \to n+1$

$$\begin{aligned}
\frac{1}{i}\int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{(k^2-\Delta)^{n+1}} &= \frac{1}{n}\frac{\partial}{\partial\Delta}\frac{1}{i}\int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{(k^2-\Delta)^n} \\
&= \frac{1}{n}\frac{\partial}{\partial\Delta}\left(\frac{(-1)^n}{(4\pi)^{d/2}}\frac{\Gamma(n-d/2)}{\Gamma(n)}\frac{1}{\Delta^{n-d/2}}\right) \\
&= \frac{(-1)^{n+1}}{(4\pi)^{d/2}}\frac{\Gamma(n-d/2)}{n\Gamma(n)}\left(n-\frac{d}{2}\right)\frac{1}{\Delta^{n+1-d/2}} \\
&= \frac{(-1)^{n+1}}{(4\pi)^{d/2}}\frac{\Gamma(n+1-d/2)}{\Gamma(n+1)}\frac{1}{\Delta^{n+1-d/2}} \qquad \square
\end{aligned}$$

There is another change in d dimensions. Since $S = \int \mathrm{d}^d x\mathcal{L}$ is dimensionless (keep in mind we are working in natural units), $[\mathcal{L}] = M^d$. So $\mathcal{L}_{\mathrm{KG}} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{M^2}{2}\phi^2$ suggests now $[\phi] = M^{d/2-1}$ and in Dirac theory $[\psi] = M^{\frac{d-1}{2}}$. So in order to keep $[\lambda] = M^0 = 1$, $\mathcal{L}_{\phi^4} = -\mu^{4-d}\frac{\lambda}{4!}\phi^4$ with μ an arbitrary mass parameter $[\mu] = M^1$.

With dimensional regularization



$$\begin{aligned}
&= \frac{\mu^{4-d}\lambda}{2}\frac{1}{i}\int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{k^2-M^2+i\epsilon} \\
&= \frac{\lambda}{2}\mu^{4-d}\left(-\frac{1}{(4\pi)^{d/2}}\right)M^{d-2}\Gamma(1-d/2)
\end{aligned}$$

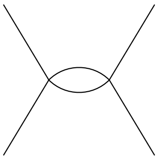
$$\boxed{\begin{array}{ll}
 & \text{Laurent expansion} \\
\Gamma(z) & = \frac{1}{z} - \gamma_E + O(z), \quad z\to 0 \\
\Gamma(z-1) & = \frac{1}{z-1}\Gamma(z), \quad z \to 0 \\
 & = -\left(1+z+O(z^2)\right)\Gamma(z) \\
 & = -\frac{1}{z} + \gamma_E - 1 + O(z) \\
\gamma_E & = 0.5772\ldots
\end{array}}$$

$$= -\frac{\lambda}{2}\frac{M^2}{8\pi^2}\left(\frac{M^2}{4\pi\mu^2}\right)^{\frac{d-4}{2}}\left[\frac{1}{d-4}+\frac{1}{2}(\gamma_E-1)+O(d-4)\right]$$

Taylor expansion around $\epsilon = 0$, $a^\epsilon = 1 + \epsilon\ln a$

$$= -\frac{\lambda}{2}\frac{M^2}{8\pi^2}\left\{\frac{1}{d-4}+\frac{1}{2}\left[\gamma_E - 1 - \ln(4\pi)\right] + \ln\frac{M}{\mu} + O(d-4)\right\}$$

with $\Delta(x) = M^2 - x(1-x)p^2$



$$= \frac{\mu^{2(4-d)}\lambda^2}{2}\frac{1}{i}\int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{[k^2-\Delta(x)]^2}$$

$$= \frac{\lambda^2}{2}\mu^{2(4-d)}\int_0^1 \mathrm{d}x \frac{1}{(4\pi)^{d/2}}\frac{\Gamma(2-d/2)}{2}\frac{1}{\Delta(x)^{2-d/2}}$$
$$= \frac{\lambda^2}{2}\frac{\mu^{4-d}}{(4\pi)^2}\left\{-2\left[\frac{1}{d-4}+\frac{1}{2}(\gamma_E - \ln 4\pi) + \ln\left(\frac{M}{\mu}\right)\right] - \int_0^1 \mathrm{d}x \ln\left(\frac{\Delta(x)}{M^2}\right)\right\}$$
$$\int_0^1 \mathrm{d}x \ln\left(\frac{\Delta(x)}{M^2}\right) = \int_0^1 \mathrm{d}x \ln\frac{M^2 - x(1-x)p^2}{M^2}$$
$$= \int_0^1 \mathrm{d}x \ln\left[\left(\frac{\sigma+1}{2}-x\right)\left(x+\frac{\sigma-1}{2}\right)\right] - \ln\frac{\sigma^2-1}{4}, \quad \sigma = \sqrt{1-\frac{4M^2}{p^2}}$$
$$= \sigma \ln\frac{\sigma+1}{\sigma-1} - 2$$

Valid for $p^2 < 0$, rest by analytic continuation

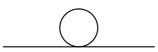
Compare $M(s) - M(0)$ calculated based on Cutkosky and dispersion integral. Easier

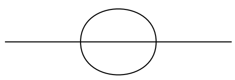
$$M(0) = \frac{1}{i}\int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{(k^2-M^2)^2}$$
$$= \frac{\partial}{\partial M^2}\frac{1}{i}\int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{k^2-M^2}$$
$$= \frac{\partial}{\partial M^2}\left\{-\frac{M^2}{8\pi^2}\left[\frac{1}{d-4}+\frac{1}{2}(\gamma_E - 1 - \ln 4\pi) + \frac{1}{2}\ln\frac{M^2}{\mu^2}\right]\right\}$$

1 gets cancalled by the derivative of ln

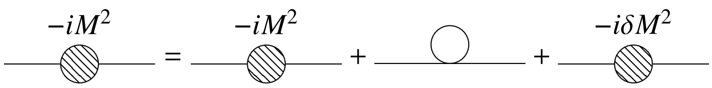
$$= -\frac{1}{8\pi^2}\left[\frac{1}{d-4}+\frac{1}{2}\left(\gamma_E - \ln 4\pi + \frac{1}{2}\ln\frac{M^2}{\mu^2}\right)\right]$$

Lets summarise the renormalization of ϕ^4 at one loop

-  is independent of p^2! Hence $\Sigma(p^2)$ at $O(\lambda)$ only renormalises the mass, there is no wave-function renormalization $Z(\sim \frac{\partial\Sigma}{\partial p^2}|_{p^2=M^2}) \quad \to Z = 1 + O(\lambda^2)$

  This does change at $O(\lambda^2)$  $\to Z \neq 1$

- Mass renormalization



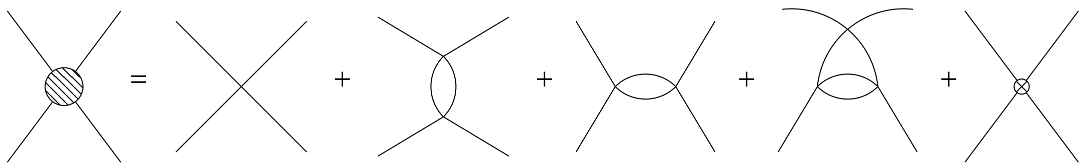
$$\delta M^2 = M_0^2 - M^2$$

then

$$M^2 = M^2 + \frac{\lambda M^2}{16\pi^2}\left[\frac{1}{d-4} + \frac{1}{2}\left(\gamma_E - 1 - \log 4\pi + \log\frac{M}{\mu}\right)\right] - M^2 + M_0^2$$
$$\delta M^2 = \frac{\lambda M^2}{16\pi^2}\left[\frac{1}{d-4} + \frac{1}{2}\left(\gamma_E - 1 - \log 4\pi + \log\frac{M}{\mu}\right) + O(\lambda, (d-4))\right]$$

Physical mass M^2_{phy} cannot be dependent on μ, meaning $\lambda\mu^{4-d}M^2 = \lambda_0 M_0^2 + O(\lambda^2)$ and λ_0 and M_0 are independent of μ.

- Coupling constant renormalization. Lets choose renormalization point for λ at $s = t = u = 0$ for simplicity:



$$= -i\lambda\mu^{4-d} + i\,(M(s) + M(t) + M(u)) - i(Z_\lambda - 1)\lambda\mu^{4-d} + O(\lambda^3)$$

with $Z = 1$

$$\lambda_0 = \lambda\mu^{4-d}Z_\lambda = \lambda\mu^{4-d}\Bigl\{\underbrace{1 - \frac{3}{\lambda}16\pi^2\Bigl[\frac{1}{d-4}}_{Z_\lambda^{MS}\text{ minimal subtraction}} + \frac{1}{2}\Bigl(\gamma_E - \log 4\pi + \log\frac{M}{\mu}\Bigr)\Bigr] + O(\lambda^2)\Bigr\}$$
$$= \lambda\mu^{4-d}Z_\lambda = \lambda\mu^{4-d}\Bigl\{\underbrace{1 - \frac{3}{\lambda}16\pi^2\Bigl[\frac{1}{d-4} + \frac{1}{2}\Bigl(\gamma_E - \log 4\pi}_{Z_\lambda^{\overline{MS}}\text{ modified minimal subtraction}} + \log\frac{M}{\mu}\Bigr)\Bigr] + O(\lambda^2)\Bigr\}$$

these two Z are mass-indepent

$$= \lambda\mu^{4-d}Z_\lambda = \lambda\mu^{4-d}\Bigl\{\underbrace{1 - \frac{3}{\lambda}16\pi^2\Bigl[\frac{1}{d-4} + \frac{1}{2}\Bigl(\gamma_E - \log 4\pi + \log\frac{M}{\mu}\Bigr)\Bigr]}_{Z_\lambda\text{mass-dependent}} + O(\lambda^2)\Bigr\}$$

6.6 Superficial degree of divergence

How do we know that we are done renormalising the theory with

- wave function
- mass
- coupling

Can't there be more divergences?
Want to analyse superficial degree of divergence D of an arbitrary loop diagram with

- d dimension
- L number of loops
- I number of internal propagators

- E number of external lines
- V number of vertices

Matrix element of an arbitrary diagram generically

$$\sim \lambda^V \int \frac{\mathrm{d}^d k_1 \mathrm{d}^d k_2 \ldots \mathrm{d}^d k_L}{(k_{i_1}^2 - M^2)\ldots(k_{i_I}^2 - M^2)}$$

So clearly

$$D = dL - 2I \tag{6.6.1}$$

$D \geq 0$ divergent ($D = 0$ logarithmically divergent) and $D < 0$ convergent.
Express L and I in terms of V and E

$$\begin{aligned} L &= \text{number of undetermined integration momenta} \\ &= \text{number of internal propagators} - \text{number of momentum conservations at vertices} \\ &\quad + 1 \text{ (because of overall momentum conservation)} \\ L &= I - V + 1 \end{aligned} \tag{6.6.2}$$

One vertex is linked to 4 legs. Internal lines are attached to 2 vertices and external line to 1.

$$4V = 2I + E \tag{6.6.3}$$

solve 6.6.2 and 6.6.3 for L and I

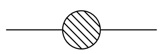
$$D = d + (d-4)V - \left(\frac{d}{2} - 1\right)E \tag{6.6.4}$$

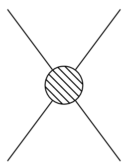
in physical 4 dimension

$$D = 4 - E \tag{6.6.5}$$

Remarks

- for $d = 4$, D is independent of V, only dependent on E.
- only a few small E produce $D \geq 0$, here in ϕ^4

$E = 2$ 

$E = 4$ 

- distinguish theories of different d
  - $d < 4$: D decreases with V, only finite number of diagrams (not n-point functions) diverges. **super-renormalizable**
  - $d = 4$: D is independent of V, only a finite number of amplitudes diverges, but at each order in perturbation theory. **renormalizable**

  - $d > 4$: D grows with V, even amplitude becomes divergent at some order in perturbation theory. **non-renormalizable**

- alternative characterisation in terms of mass dimension of coupling constant

$$\mathcal{L}_{\phi^4} = -\mu^{4-d}\frac{\lambda}{4!}\phi^4 = -\frac{\tilde{\lambda}}{4!}\phi^4$$

  so $[\tilde{\lambda}] = 4 - d$ in d dimension; hence
  - $[\tilde{\lambda}] > 0$ super-renormalizable
  - $[\tilde{\lambda}] = 0$ renormalizable
  - $[\tilde{\lambda}] < 0$ non-renormalizable

- why is this "superficial"? There can always be divergent sub-graphs! These sub-graphs are regularised and renormalised by the treatment of the "primitive divergences" we have already seen before.

Conclusion for ϕ^4 the only primitive divergences are $E = 2$ and $E = 4$ (and $E = 0$ the vacuum graphs) and we renormalise the theory by

$$\begin{aligned}
M_0^2 &= M^2\left\{1 + c_m^{(1)}\frac{\lambda}{d-4} + c_m^{(2)}\frac{\lambda^2}{(d-4)^2} + \ldots\right\}\\
\lambda_0 &= \lambda\left\{1 + c_\lambda^{(1)}\frac{\lambda}{d-4} + c_\lambda^{(2)}\frac{\lambda^2}{(d-4)^2} + \ldots\right\}\\
Z &= 1 + c_z^{(2)}\frac{\lambda^2}{(d-4)^2} + \ldots
\end{aligned}$$

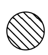
6.7 Sketch of renormalization of QED

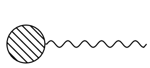
Superficial degree of divergence with E_γ external photons and E_e external electrons.

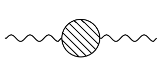
$$\begin{aligned}
D &= d + V\left(\frac{d-4}{2}\right) - E_e\left(\frac{d-1}{2}\right) - E_\gamma\left(\frac{d-2}{2}\right)\\
&= 4 - \frac{3}{2}E_e - E_\gamma
\end{aligned} \tag{6.7.1}$$

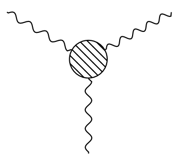
Again the superficial degree of divergence D is independent of V, i.e. QED is renormalizable (e is dimensionless as well).

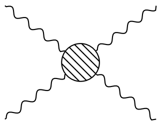
We have following divergent diagrams

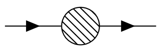
(a)  $D = 4$

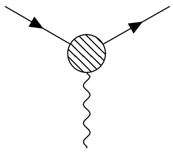
(b)  $D = 3$

(c)  $D = 2$

(d)  $D = 1$

(e)  $D = 0$

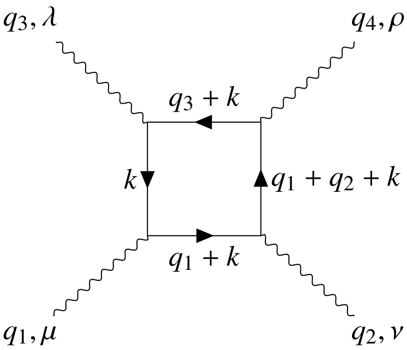
(f)  $D = 1$

(g)  $D = 1$

Still need to show that all divergences can be absorbed in the renormalization of the parameters of the theory.

$$
\begin{aligned}
e_0 &\longmapsto e \\
m_0 &\longmapsto m \\
\psi &\longmapsto Z_2^{-1/2}\psi \\
A_\mu &\longmapsto Z_3^{-1/2}A_\mu \\
Z_1 &\equiv \text{ vertex correction (diagram (g) as } Z_\lambda \text{ in } \phi^4)
\end{aligned}
$$

We now focus on individual divergent graphs. Diagram (a), vacuum diagram, is to be ignored. Diagrams (b) and (d) are not possible, since QED is C-invariant, $A_\mu \longmapsto -A_\mu$. It can be generalized into Furry's theorem: correlation functions of odd number of photons vanish. Diagram (e) is worrisome! Divergence would require counter-terms $\sim (F_{\mu\nu}F^{\mu\nu})^2, (F_{\mu\nu}\tilde{F}^{\mu\nu})^2$, dimension 8 operator. $[g_{A^4}] = M^{-4}$, i.e. non-renormalizable! Here gauge-invariance is to rescue.



$$= \frac{e^4}{i}\int \frac{\mathrm{d}^d k}{(2\pi)^4} \frac{\gamma_\mu(\not{k} + m)\gamma_\lambda(\not{q}_3 + \not{k} + m)\gamma_\rho(\not{q}_1 + \not{q}_2 + \not{k} + m)\gamma_\nu(\not{q}_1 + \not{k} + m)}{(k^2 - m^2)[(q_3 + k)^2 - m^2][(q_1 + q_2 + k)^2 - m^2][(q_1 + k)^2 - m^2]}$$

$$
\begin{aligned}
&= \frac{e^4}{i}\int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{\gamma_\mu \not{k}\gamma_\lambda \not{k}\gamma_\rho \not{k}\gamma_\nu \not{k}}{k^8} + \text{convergent terms} \\
&= e^4 I(g_{\mu\lambda}g_{\rho\nu} + g_{\mu\rho}g_{\lambda\nu} + g_{\mu\nu}g_{\lambda\rho}) + \text{ finite}
\end{aligned}
$$

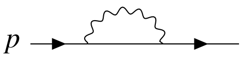
because of Ward Identity $q_1^\mu(\dots) = 0$

$$= I(q_{1\lambda}g_{\rho\nu} + q_{1\rho}g_{\lambda\nu} + q_{1\nu}g_{\lambda\rho}) + \text{ finite } = 0$$

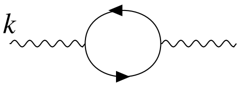
thus digram (e) is finite.

As a result the only primitively divergent graphs are diagram (c) photon self energy, (f) electron self energy, (g) vertex graph!

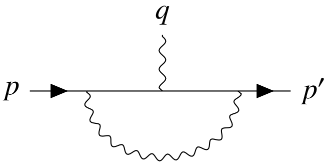
We will discuss these in detail next term. The results at one loop are



$$= -i\Sigma(p) = \frac{-ie^2}{8\pi^2(d-4)}(\not{p} - 4m) + \text{finite}$$



$$= -i\Pi_{\mu\nu}(k) = \frac{ie^2}{6\pi^2(d-4)}(g_{\mu\nu}k^2 - k_\mu k_\nu)\left[\frac{1}{d-4} + (\text{finite, const.}) - \frac{k^2}{10m^2} + \dots\right]$$



$$= ie\mu^{2-d/2}\Lambda_\mu; \quad \Lambda_\mu = \frac{-e^2}{8\pi^2(d-4)}\gamma_\mu + \text{ finite}$$

"finite" contains $\frac{\alpha}{2\pi}\frac{i\sigma_{\mu\nu}q^\nu}{2m}$ anomolous magnetic moment

We have three renormalization factors

- vertex

$$Z_1 = 1 + \frac{e^2}{8\pi^2(d-4)} \tag{6.7.2}$$

- electron wave-function

$$Z_2 = 1 + \frac{e^2}{8\pi^2(d-4)} \tag{6.7.3}$$

- photon wave-function

$$Z_3 = 1 + \frac{e^2}{6\pi^2(d-4)} \tag{6.7.4}$$

Mass renormalization

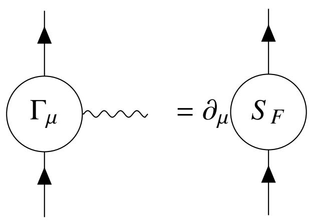
$$m_0 = Z_2^{-1}(m + \delta m) = m\left(1 + \frac{3e^2}{8\pi^2(d-4)}\right) \tag{6.7.5}$$

Coupling renormalization

$$e_0 = \mu^{2-d/2}Z_1Z_2^{-1}Z_3^{-1/2}e = \mu^{2-d/2}Z_3^{-1/2}e \tag{6.7.6}$$

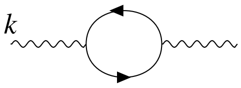
Remarks

- $Z_1 = Z_2$ is a fundamental consequence of the Ward Identity (consequence of gauge-invariance)



$$\Gamma_\mu(p.0, p) = \frac{\partial}{\partial p^\mu} iS_F^{-1}$$

Charge renormalization only depends on photon's self energy (vacuum polarisation). This is an essential reason for many particles (e. μ, p, π, ...) having the same charge!

- Vacuum polarisation  does not generate a photon mass term!

- The k^2-dependent (finite) correction in vacuum polarisation does remain after renormalization

$$D'_{\mu\nu}(k) = -ig_{\mu\nu}\left(\frac{1}{k^2} - \frac{e^2}{60\pi^2 m^2} + O(k^2)\right)$$

Fourier transformation yields potential between two charges.

$$V(r) = \frac{e^2}{4\pi r} + \frac{e^4}{60\pi^2 m^2}\delta^{(3)}(\boldsymbol{r}) \tag{6.7.7}$$

This shift, *Lamb shift*, S-levels in hydrogen atom.

6.8 The (idea of the) renormalization group

Consider N-point vacuum correlation function in momentum space $G_N(p_1, \ldots, p_N; \lambda, M, \mu)$. Its relation to bare Green's function is given by

$$G_N(p_1, \ldots, p_N; \lambda, M, \mu) = Z^{-N/2} G_N^0(p_1, \ldots, p_N; \lambda_0, M_0) \tag{6.8.1}$$

using the relation $Z^{-1/2}\phi_0 = \phi$

Since G_N^0 is independent of μ

$$\left[\mu\frac{\partial}{\partial\mu} + \mu\frac{\partial\lambda}{\partial\mu}\frac{\partial}{\partial\lambda} + \mu\frac{\partial M}{\partial\mu}\frac{\partial}{\partial M} + \frac{N}{2}\mu\frac{\partial}{\partial\mu}\ln Z\right] G_N(p; \lambda, M, \mu) = 0$$

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(\lambda)\frac{\partial}{\partial\lambda} + M\gamma_M(\lambda)\frac{\partial}{\partial M} + \frac{N}{2}\gamma(\lambda)\right] G_N(p; \lambda, M, \mu) = 0 \tag{6.8.2}$$

with $\beta(\lambda) = \mu\frac{\partial}{\partial\mu}\lambda$, $\gamma(\lambda) = \mu\frac{\partial}{\partial\mu}\ln Z$ and $M\gamma_M(\lambda) = \mu\frac{\partial}{\partial\mu}M$

(6.8.3)

This is the *renormalization group equation* (Callan-Symanzik equation)!

What is the mass dimension of G_N?

$$[\phi] = M \Rightarrow [G(x_1, \ldots, x_N)] = M^N$$
$$\Rightarrow [G_N(p_1, \ldots, p_N)] = M^{4-3N} = M^{D_N}$$

We need to recover D_N by counting

- the power of momenta
- the power of masses
- the power of μ

or

$$\left[t\frac{\partial}{\partial t} + M\frac{\partial}{\partial M} + \mu\frac{\partial}{\partial \mu} - D_N\right] G_N\left(\{t_p\}; \lambda, M, \mu\right) = 0 \tag{6.8.4}$$

Eliminating $\mu\frac{\partial}{\partial\mu}$ between 6.8.2 and 6.8.4 leads to

$$\left[-t\frac{\partial}{\partial t} + \beta\frac{\partial}{\partial \lambda} + M(\gamma_M - 1)\frac{\partial}{\partial M} + (D_N + \frac{N}{2}\gamma)\right] G_N\left(\{t_p\}; \lambda, M.\mu\right) = 0 \tag{6.8.5}$$

Expresses the effect of scaling momenta in G_N by a factor of t. $\frac{N}{2}\gamma$ is the <u>anomalous dimension</u> which gets added to the "engineering dimension" D_N. This equation suggests that an overall change in momentum scale t can be compensated by

- changing the coupling λ
- rescaling the mass M
- an overall factor

hence

$$G_N\left(\{t_p\}; \lambda, M, \mu\right) = f(t) G_N\left(\{p\}; \lambda(t), M(t), \mu\right)$$

A little algebra and comparing to 6.8.5 leads to

$$\begin{aligned}
t\frac{\partial \lambda(t)}{\partial t} &= \beta(\lambda) \\
t\frac{\partial M(t)}{\partial t} &= M(\gamma_M(\lambda) - 1) \\
t\frac{\partial f(t)}{\partial t} &= D_N + \frac{N}{2}\gamma \\
\Rightarrow f(t) &= t^{D_N} \exp\left\{\frac{N}{2}\int_1^t \frac{\gamma(\lambda(s))}{s} \mathrm{d}s\right\}
\end{aligned}$$

Solution in terms of <u>running mass</u> $M(t)$ and <u>running coupling</u> $\lambda(t)$! Discuss running coupling $\lambda(t)$ (determined by "beta function" $\beta(\lambda)$). In ϕ^4-theory

$$\lambda(\mu) = \mu^{4-d}\lambda_0 + \frac{3\lambda_0^2}{16\pi^2}\mu^{2(d-4)}\left[\frac{1}{d-4} + \dots\right] + O(\lambda_0^3, d-4) \tag{6.8.6}$$

with $\lambda^2 = \lambda_0^2\mu^{2(d-4)} + O(\lambda_0^3)$ and $\mu\frac{\partial}{\partial\mu}\lambda_0 = 0$

$$\beta(\lambda) = \lim_{d\to 4}\mu\frac{\partial}{\partial\mu}\lambda = \frac{3\lambda^2}{16\pi^2} + O(\lambda^3) > 0 \tag{6.8.7}$$

$\lambda(t)$ grows with t; approaching large(r) momenta

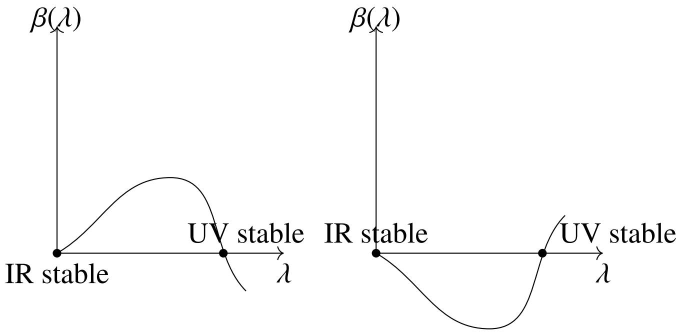
$$\tag{6.8.8}$$

Solution to $\frac{\mathrm{d}\lambda}{\mathrm{d}\ln\mu} = \frac{3\lambda^2}{16\pi^2}$ is (with $\lambda(\mu_0)$ as an integration constant)

$$\lambda(\mu) = \frac{\lambda(\mu_0)}{1 - \frac{3}{16\pi^2}\lambda(\mu_0)\ln(\mu/\mu_0)} \tag{6.8.9}$$

In different theories, sign of β (coupling) can be different, coupling can decrease with energy (e.g. Yang-Mills, QCD, …)

Interesting asymptotic (IR/UV) behaviour of couplings depending on form of $\beta(\lambda) = \mu\frac{\partial}{\partial\mu}\lambda(\mu)$



Non-perturbative zeros $\beta(\lambda_{NP}) = 0$ and perturbative $\beta(0) = 0$ are fixed points for of RG flow. The second picture has *asymptotic freedom* for $t \longmapsto \infty$, $\lambda(t) \longmapsto 0$ is a (UV-stable) fixed point; we believe QCD to have this property.