A.5 Gamma matrices

a)
$$\mathbb{1}_d \mathbb{1}_d = \mathbb{1}_d$$
$$\gamma^\mu \gamma^\mu = \frac{1}{2}(\gamma^\mu\gamma^\mu + \gamma^\mu\gamma^\mu) = \frac{1}{2}\{\gamma^\mu, \gamma^\mu\} = g^{\mu\mu}\mathbb{1}_d$$
$$\gamma^5\gamma^5 = \mathbb{1}_d$$
$$\gamma^5\gamma^\mu\gamma^5\gamma^\mu = -(\gamma^5)^2\gamma^\mu\gamma^\mu = -g^{\mu\mu}\mathbb{1}_d$$
$$\gamma^\mu\gamma^\nu\gamma^\mu\gamma^\nu = -\gamma^\mu\gamma^\mu\gamma^\nu\gamma^\nu = -g^{\mu\mu}g^{\nu\nu}\mathbb{1}_d$$

b)
$$c A^0 A^1 A^2 A^3, \quad c \in \mathbb{Z}, \qquad A^\mu \in \{\mathbb{1}_d, \gamma^\mu\}$$ ← any four elements from this set
$$\gamma^0\gamma^1\gamma^2\gamma^3 = -i\gamma^5$$
$$\gamma^0\gamma^1\gamma^3 = -i\gamma^5\gamma^2 \qquad , \quad (\gamma^2)^2 = -1$$

c) (3), (4) => closure

(3) => inverse

Γ^1 is the identity

matrix multiplication is associative

d) suppose two anticommuting matrices
$$\mathrm{tr}[AB] = \frac{1}{2}(\mathrm{tr}[AB] + \mathrm{tr}[BA]) = \frac{1}{2}\mathrm{tr}[\{A,B\}] = 0$$
→ rewrite the $\Gamma^A, A \neq 1$ as products of anticommutating matrices
$$\gamma^\mu = \gamma^\mu\mathbb{1}_d = \gamma^\mu g_{\nu\nu}\gamma^\nu\gamma^\nu = (\gamma^\mu\gamma^\nu)(g_{\nu\nu}\gamma^\nu) = -(\gamma^\nu\gamma^\mu)(g_{\nu\nu}\gamma^\nu)$$
$$= -(g_{\nu\nu}\gamma^\nu)(\gamma^\mu\gamma^\nu)$$
$$\gamma^5 = i\gamma^0(\gamma^1\gamma^2\gamma^3) = -(\gamma^1\gamma^2\gamma^3)\, i\gamma^0$$

e)
$$\mathrm{tr}[\Gamma^A\Gamma^B] \propto \alpha^{AB}\,\mathrm{tr}[\Gamma^{c(A,B)}] = 0$$

f)
$$0 = \sum_{A=1}^{16}\lambda^A\Gamma^A, \quad \lambda^A \in \mathbb{C} \qquad | \cdot \Gamma^B$$
$$0 = \sum_{A=1}^{16}\lambda^A\Gamma^A\Gamma^B = \lambda^B\alpha^{BB}\mathbb{1}_d + \sum_{A\neq B}\lambda^A\Gamma^B\Gamma^A$$

take the trace of both sides

$$0 = \lambda^B \alpha^{BB} d \quad \Rightarrow \quad \lambda^B = 0$$

$\Rightarrow$ linear independent

g) 16 l.i. Γ,

$d=4$, $d^2 = 16$, they form a basis

h) consider the kernel of h; $\ker(h) = \{ v \in F^N : hv = 0 \}$

If $v \in \ker(h)$, one has

$$h \rho(g) v = \sigma(g) h v = \sigma(g) 0 = 0$$

i.e. $\rho(g) v \in \ker(h)$

Hence $\ker(h)$ is an invariant subspace w.r.t. ρ

$\Rightarrow$ either $\ker(h) = F^N$ where $h=0$ or $\ker(h) = \{0\}$

In the latter case, h is invertible $\blacksquare$

i) $S := \sum_{A=1}^{16} \Gamma'^A M (\Gamma^A)^{-1}$, $M \in \mathrm{mat}_{\mathbb{C}}(d', d)$, $\Gamma'^B S = S \Gamma^B$

$$\Gamma'^B \Gamma'^A = \alpha^{BA} \Gamma'^{C(B,A)}$$

$$\Gamma^B \Gamma^A = \alpha^{BA} \Gamma^{C(B,A)} \qquad | (\Gamma^{C(B,A)})^{-1} \cdot \,, \ (\Gamma^A)^{-1}$$

$$\Rightarrow (\Gamma^{C(B,A)})^{-1} \Gamma^B = \alpha^{BA} (\Gamma^A)^{-1}$$

For fixed B, one has for $A \neq D \Rightarrow C(B,A) \neq C(B,D)$

$$\left(\begin{array}{l} \Gamma^B \Gamma^A = \alpha^{BA} \Gamma^{C(B,A)} = \alpha^{BA} \Gamma^{C(B,D)} = \frac{\alpha^{BA}}{\alpha^{BD}} \Gamma^B \Gamma^D \\ \text{multiply by } (\Gamma^B)^{-1} \Rightarrow \Gamma^A \propto \Gamma^D \end{array} \right)$$

$$\Gamma'^B S = \sum_{A=1}^{16} \Gamma'^B \Gamma'^A M (\Gamma^A)^{-1} = \sum_{A=1}^{16} \Gamma'^{C(B,A)} M \alpha^{BA} (\Gamma^A)^{-1}$$

$$= \sum_{A=1}^{16} \Gamma'^{C(B,A)} M (\Gamma^{C(B,A)})^{-1} \Gamma^B = S \Gamma^B$$

j) $d=4$, Γ matrices are irrep. of the group G

k) $0 = (i\not{\partial} - m)\psi$,

$$0 = S(i\not{\partial} - m)\psi = S(i\gamma^\mu \partial_\mu - m) S^{-1} S \psi$$

$$= (i\gamma'^\mu \partial_\mu - m) S\psi$$

$\Rightarrow$ Dirac eq. holds for all equivalent rep. of γ^μ

l) $S\gamma^\mu S^+ = S \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix} S^+ = \frac{1}{2} \begin{pmatrix} \sigma^\mu + \bar\sigma^\mu & \sigma^\mu - \bar\sigma^\mu \\ \bar\sigma^\mu + \sigma^\mu & -\sigma^\mu - \bar\sigma^\mu \end{pmatrix} = \gamma'^\mu$

m) $S \begin{pmatrix} \varphi \\ \eta \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \eta + \varphi \\ \eta - \varphi \end{pmatrix}$, $S u_{\frac{1}{2}}(\vec{0}) = \sqrt{m} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ ...

$m = 0$, or $\beta \approx 1$ $\Rightarrow$ use chiral rep.

$(\gamma^\mu)^+ = \gamma^0 \gamma^\mu \gamma^0$, $\quad S(\Lambda) = \exp(-\frac{i}{4} \omega_{\mu\nu} \sigma^{\mu\nu})$, $\psi'(x') = S(\Lambda)\psi(x)$

$\sigma_{\mu\nu}^+ = \gamma^0 \sigma_{\mu\nu} \gamma^0$, $\quad S(\Lambda)^+ = \gamma^0 \exp(+\frac{i}{4} \omega_{\mu\nu} \sigma^{\mu\nu}) \gamma^0$

Dirac — Chiral

γ^0 diagonal — γ^0 diagonal

$e^A = \sum \frac{A^n}{n!}$, $\quad (e^A)^+ = e^{A^+}$, $\quad u e^A u^{-1} = e^{uAu^{-1}}$

$(e^A)^{-1} = e^{-A}$, $\quad \det e^A = e^{\mathrm{tr} A}$

$e^A e^B = e^{A + B + \frac{1}{2}[A,B]} + \ldots$