A.9

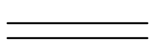
a) $\mathcal{L} = \frac{1}{2}(\partial_\mu \Phi)^2 - \frac{1}{2}M^2\Phi^2 + \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\mu}{2}\Phi\phi^2$

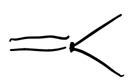
$\mathcal{H}_{int}(x) = \frac{\mu}{2}\Phi(x)\,\phi^2(x)$

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}}\left(e^{-ik\cdot x}a_k + e^{+ik\cdot x}a_k^\dagger\right), \quad k^0 = \omega_k = \sqrt{\vec{k}^2+m^2}$$

$$\Phi(x) = \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}}\left(e^{-ik\cdot x}b_k + e^{+ik\cdot x}b_k^\dagger\right), \quad k^0 = E_k = \sqrt{\vec{k}^2+M^2}$$

 $= \overbracket{\phi(x)\,\phi(y)} = D_F^m(x-y) = \int \frac{d^4q}{(2\pi)^4}\frac{i}{p^2-m^2+i\varepsilon}e^{-iq(x-y)}$

 $= \overbracket{\Phi(x)\,\Phi(y)} = D_F^M(x-y)$

 $= (-i\mu)\int d^4x$ — interaction can happen at any space-time

$\frac{1}{2}$ takes into account permutations of 2ϕ fields.

at the contraction with the incoming particle.

$$\langle 0|\phi(x)\,|P_\phi\rangle = \langle 0|\overbracket{\phi(x)\,|P_\phi}\rangle$$

$$= \langle 0|\phi(x)\sqrt{2\omega_p}\,a_{\vec{p}}^\dagger|0\rangle$$

$$= \int\frac{d^3q}{(2\pi)^3}\sqrt{\frac{2\omega_p}{2\omega_q}}\langle 0|\left(e^{-iqx}a_{\vec{q}} + e^{+iqx}a_{\vec{q}}^\dagger\right)a_{\vec{p}}^\dagger|0\rangle$$

$$= \int\frac{d^3q}{(2\pi)^3}\sqrt{\frac{\omega_p}{\omega_q}}\,e^{-iqx}(2\pi)^3\delta^{(3)}(\vec{q}-\vec{p})$$

$$= e^{-ip\cdot x}$$

$$\langle 0|\Phi(x)\,|P_\Phi\rangle = \langle 0|\overbracket{\Phi(x)\,|P_\Phi}\rangle = e^{-ip\cdot x}$$

b) $S = \langle \vec{p}_1\,\vec{p}_2|\,T\exp\left(-i\int d^4x\,\mathcal{H}_{int}(x)\right)|\vec{p}_A\rangle$

$S = 1 + iT, \quad T = (2\pi)^4\delta^{(4)}(p_1+p_2-p_A)\,\mathcal{M}$

$$S^{(0)} = \langle \vec{p}_1 \vec{p}_2 | \vec{p}_A \rangle = \sqrt{8\omega_{p_1}\omega_{p_2}E_{p_A}}\ \langle 0 | a_{\vec{p}_1} a_{\vec{p}_2}\, b^{\dagger}_{\vec{p}_A} | 0 \rangle$$

$$\propto \langle 0 | b_{\vec{p}_A}\, a_{\vec{p}_1} a_{\vec{p}_2} | 0 \rangle = 0$$

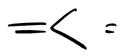
$$S^{(1)} = \int d^4x\, \langle \vec{p}_1 \vec{p}_2 | T\left[-i\frac{\mu}{2}\, \phi(x)\phi(x)\Phi(x)\right] | \vec{p}_A \rangle$$

$$= \frac{-i\mu}{2} \int d^4x\, \langle \vec{p}_1 \vec{p}_2 | \phi(x)\, \phi(x)\, \Phi(x) | \vec{p}_A \rangle$$

$$+ \frac{-i\mu}{2} \int d^4x\, \langle \vec{p}_1 \vec{p}_2 | \phi(x)\, \phi(x)\, \Phi(x) | \vec{p}_A \rangle$$

$$= -i\mu \int d^4x\ e^{ix(p_1+p_2-p_A)} = -i\mu\,(2\pi)^4\, \delta^{(4)}(p_1+p_2-p_A)$$

$$\mathcal{M} = -i\mu$$

c) $S^{(1)} =$  $= \underbrace{-i\mu \int d^4x}_{\text{vertex}}\ \underbrace{e^{-ix(p_A-p_1-p_2)}}_{\text{external lines}} = -i\mu\,(2\pi)^4\,\delta^{(4)}(p_1+p_2-p_A)$

d) $$d\Gamma = \frac{1}{2}\ \frac{1}{2M}\ \underbrace{\frac{d^3p_1}{(2\pi)^3 2\omega_{p_1}}\ \frac{d^3p_2}{(2\pi)^3 2\omega_{p_2}}}\ \mu^2 (2\pi)^4\, \delta^{(4)}(p_A - p_1 - p_2)$$

two identical particle

incoming flux

phase space factor of final state

$$\Gamma = \frac{1}{4M} \int \frac{d^3p_1}{(2\pi)^3 2\omega_{p_1}}\ \frac{d^3p_2}{(2\pi)^3 2\omega_{p_2}}\ \mu^2 (2\pi)^4\, \delta^{(3)}(\underbrace{\vec{p}_A}_{0} - \vec{p}_1 - \vec{p}_2)\, \delta(\underbrace{E_A}_{M} - \omega_{p_1} - \omega_{p_2})$$

$\omega_p = \omega_{-p}$

$$= \frac{1}{4M} \int \frac{d^3p_1}{(2\pi)^3}\ \frac{\mu^2}{4\omega_{p_1}^2}\ (2\pi)\, \delta(M - 2\omega_{p_1})$$

$$= \frac{1}{4M}\ \frac{\mu^2}{4(2\pi)^2} \int_0^\infty d|\vec{p}|\ |\vec{p}|^2 \int_0^{2\pi} d\phi \int_{-1}^{1} d\cos\theta\ \frac{1}{|p|^2+m^2}\ \delta\left(M - 2\sqrt{|p|^2+m^2}\right)$$

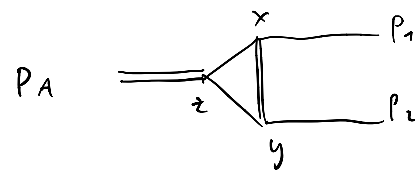
$$= \frac{1}{16\pi}\ \frac{\mu^2}{M} \int_0^\infty d|\vec{p}|\ \frac{|\vec{p}|^2}{\vec{p}^2+m^2}\ \delta\left(M - 2\sqrt{\vec{p}^2+m^2}\right)$$

$$= \frac{1}{16\pi}\ \frac{\mu^2}{M} \int_m^\infty d\omega\ \frac{\sqrt{\omega^2-m^2}}{\omega}\ \delta(M - 2\omega)$$

$$= \frac{1}{32\pi} \frac{\mu^2}{M} \sqrt{1-\frac{4m^2}{M^2}}$$

e) $$S^{(3)} = \frac{1}{3!}\left(\frac{-i\mu}{2}\right)^3 \int d^4x\, d^4y\, d^4z$$

$$\langle p_1 p_2 | T[\Phi(x)\phi(x)\phi(x)\,\Phi(y)\phi(y)\phi(y)\,\Phi(z)\phi(z)\phi(z)] | p_A\rangle$$



1: $\Phi(z)\,|p_A\rangle$: 1

2: $\langle p_1|\,\phi(x)$, $\langle p_2|\,\phi(y)$ $\quad 2\cdot 2 = 4$

3. $\phi(x)\phi(z)$ and $\phi(y)\phi(z)$: 2

4. interchange of vertices : $3!$

$$= \frac{1}{3!}\, 3!\, 4\cdot 2 \left(\frac{-i\mu}{2}\right) \int d^4x\, d^4y\, d^4z\, \underbrace{\sqrt{2E_{p_A}}\; \rho(z)\, b^+_{p_A}}_{e^{-ip_A z}} \times$$

$$\underbrace{\sqrt{2\omega_{p_1}}\, a_{p_1}\phi(x)}\; \underbrace{\sqrt{2\omega_{p_2}}\, a_{p_2}\phi(y)}\; \Phi(x)\Phi(y)\; \phi(x)\phi(z)\; \phi(y)\phi(z)$$

$$\int \frac{d^4Q}{(2\pi)^4} \frac{i e^{-iQ(x-y)}}{Q^2-m^2+i\varepsilon} = (-i\mu)^3 \int d^4x\, d^4y\, d^4z\, \frac{d^4q_1}{(2\pi)^4}\frac{d^4q_2}{(2\pi)^4} \times$$

$$\frac{d^4Q}{(2\pi)^4}\, e^{-iz(p_A-p_1-p_2)}\, e^{+ix(p_1-q_1-Q)}\, e^{+iy(p_2-q_2+Q)} \times$$

$$\frac{i}{p_1^2+m^2+i\varepsilon}\; \frac{i}{p_2^2-m^2+i\varepsilon}\; \frac{i}{Q^2-\mu^2+i\varepsilon}$$