Quantum Field Theory

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Contents

1 Classical field theory

1.1 Field theory in continuum

Euler-Lagrange-equation

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \tag{1.1.1}$$

momentum density

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}(\boldsymbol{x})} \tag{1.1.2}$$

Hamiltonian density

$$\mathcal{H}(\phi(\boldsymbol{x}), \pi(\boldsymbol{x})) = \pi(\boldsymbol{x})\dot{\phi}(\boldsymbol{x}) - \mathcal{L}(\phi, \partial_\mu \phi) \tag{1.1.3}$$

1.2 Noether Theorem

If a Lagrangian field theory has an infinitisimal symmetry, then there is an associated current j^μ, which is conserved.

$$\partial_\mu j^\mu = 0 \tag{1.2.1}$$

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta\phi - X^\mu \tag{1.2.2}$$

Energy-momentum tensor (stress-energy tensor)

Asymmetric version

$$\Theta^\mu_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu_\nu \mathcal{L} \tag{1.2.3}$$

General version

$$T^{\mu\nu} = \Theta^{\mu\nu} + \partial_\lambda f^{\mu\nu\lambda} \tag{1.2.4}$$

with $f^{\lambda\mu\nu} = -f^{\mu\lambda\nu}$ or $\partial_\mu \partial \nu f^{\lambda\mu\nu} = 0$

2 Klein-Gordon theory

(Real) Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 \tag{2.0.1}$$

Quantization

$$\begin{aligned} [\phi(\boldsymbol{x}), \phi(\boldsymbol{x}')] = [\pi(\boldsymbol{x}), \pi(\boldsymbol{x}')] = 0 \\ [\phi(\boldsymbol{x}), \pi(\boldsymbol{x}')] = i\delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}') \end{aligned} \tag{2.0.2}$$

Decomposition into Fourier modes

$$\phi(\boldsymbol{x}) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(a_{\boldsymbol{p}}e^{i\boldsymbol{p}\cdot\boldsymbol{x}} + a^\dagger_{\boldsymbol{p}}e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\right) \tag{2.0.3}$$

$$\pi(\boldsymbol{x}) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3}(-i)\sqrt{\frac{E_p}{2}}\left(a_{\boldsymbol{p}}e^{i\boldsymbol{p}\cdot\boldsymbol{x}} - a^\dagger_{\boldsymbol{p}}e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\right) \tag{2.0.4}$$

thus the commutation relations for ladder operators:

$$\left[a_{\boldsymbol{p}}, a_{\boldsymbol{p}'}\right] = \left[a^\dagger_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{p}'}\right] = 0 \tag{2.0.5}$$

$$\left[a_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{p}'}\right] = (2\pi)^3\delta^{(3)}(\boldsymbol{p} - \boldsymbol{p}') \tag{2.0.6}$$

Hamiltonian in terms of ladder operator

$$H = \int \frac{\mathrm{d}^3 p}{(2\pi)^3}E_p\left(a_{\boldsymbol{p}}a^\dagger_{\boldsymbol{p}} + \frac{1}{2}\left[a_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{p}}\right]\right) \tag{2.0.7}$$

Normalisation (Lorentz-invariant)

$$\langle p|q\rangle = 2E_p(2\pi)^3\delta^{(3)}(\boldsymbol{p} - \boldsymbol{q}) \tag{2.0.8}$$

2.1 Heisenberg-picture fields

Heisenberg-picture

$$|\psi_H\rangle = e^{iHt}|\psi_s(t)\rangle \tag{2.1.1}$$

$$O_H(t) = e^{iHt}O_S e^{-iHt} \tag{2.1.2}$$

Field operator

$$\phi(x) = \phi(\boldsymbol{x}, t) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(a_p e^{ipx} + a^\dagger_p e^{-ipx}\right) \tag{2.1.3}$$

2.2 Commutations and propogators

Commutations

$$[\phi(x), \phi(y)] = D(x-y) - D(y-x) \begin{cases} = 0 & \text{if } (x-y) \text{ is space-like} \\ \neq 0 & \text{otherwise} \end{cases} \tag{2.2.1}$$

$$D(x-y) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip(x-y)} \tag{2.2.2}$$

Propogator

$$\langle 0|\phi(x)\phi(y)|0\rangle = D(x-y) \tag{2.2.3}$$

Feynman propagator

$$\begin{aligned} D_F(x-y) &= \langle 0|T\phi(x)\phi(y)|0\rangle \\ &= \Theta(x^0 - y^0)D(x-y) + \Theta(y^0 - x^0)D(y-x) \end{aligned} \tag{2.2.4}$$

$$D_F(x-y) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip(x-y)} \tag{2.2.5}$$

3 Quantization of the Dirac field

3.1 Dirac equation

$$\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi(x) = 0 \tag{3.1.1}$$

Standard representation (Dirac's)

$$\gamma_0 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix} \quad \boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix} \tag{3.1.2}$$

Lorentz transformation

$$\Lambda = \exp\left(\frac{1}{2}\omega_{\mu\nu}M^{\mu\nu}\right) \tag{3.1.3}$$

with ω set of parameters and M the generator of Lie algebra.

Spinor representation

$$S^{\rho\sigma} = \frac{1}{4}\left[\gamma^{\rho}, \gamma^{\sigma}\right] = \frac{1}{2i}\sigma^{\rho\sigma} \tag{3.1.4}$$

Spinor transformation

$$S(\Lambda) = \exp\left(\frac{1}{2}\omega_{\mu\nu}S^{\mu\nu}\right) \tag{3.1.5}$$

$$\psi'_a(x) = S_{ab}(\Lambda)\psi_b(\Lambda^{-1}x) \tag{3.1.6}$$

Adjoint spinor

$$\bar{\psi} = \psi^{\dagger}\gamma^{0} \tag{3.1.7}$$

Fifth gamma matrix

$$\gamma^5 := i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{3.1.8}$$

$$\left\{\gamma^{\mu}, \gamma^5\right\} = 0 \tag{3.1.9}$$

$$(\gamma^5)^2 = \mathbb{1}_4 \tag{3.1.10}$$

Plane wave solutions

$$\psi(x) = \begin{cases} u(p)e^{-ipx} & \text{positive frequency} \\ v(p)e^{ipx} & \text{negative frequency} \end{cases} \tag{3.1.11}$$

$$(\not{p} - m)u(p) = 0 \qquad u_s(p) = \sqrt{E_p + m}\begin{pmatrix} \chi_s \\ \frac{\boldsymbol{u}\cdot\boldsymbol{p}}{E_p+m}\chi_s \end{pmatrix} \tag{3.1.12}$$

$$(\not{p} + m)v(p) = 0 \qquad v_s(p) = \sqrt{E_p + m}\begin{pmatrix} \frac{\boldsymbol{u}\cdot\boldsymbol{p}}{E_p+m}\tilde{\chi}_s \\ \tilde{\chi}_s \end{pmatrix} \tag{3.1.13}$$

with

$$\chi_{\frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad x_{-\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$s = \pm\frac{1}{2} \quad \tilde{\chi}_s = \chi_{-s}$$

Orthogonality of spinor

$$\bar{u}_s(p)u_{s'}(p) = -\bar{v}_s(p)v_{s'}(p) = 2m\delta_{ss'} \tag{3.1.14}$$
$$\bar{u}_s(p)v_{s'}(p) = 0 \tag{3.1.15}$$

Spin sums

$$\sum_s u_s(p)\bar{u}_s(p) = \not{p} + m \tag{3.1.16}$$
$$\sum_s v_s(p)\bar{v}_s(p) = \not{p} - m \tag{3.1.17}$$

3.2 Dirac Lagrangian and quantization

$$\mathcal{L} = \bar{\psi}(x)(i\not{\partial} - m)\psi(x) \tag{3.2.1}$$

Quantization

$$\left\{\psi_a(\boldsymbol{x}), \psi_b^\dagger(\boldsymbol{x}')\right\} = \delta_{ab}\delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}') \tag{3.2.2}$$
$$\{\psi_a(\boldsymbol{x}), \psi_b(\boldsymbol{x}')\} = \left\{\psi_a^\dagger(\boldsymbol{x}), \psi_b^\dagger(\boldsymbol{x}')\right\} = 0 \tag{3.2.3}$$

Field operators

$$\psi(\boldsymbol{x}) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3\sqrt{2E_p}} \sum_s (a_{\boldsymbol{p}}^s u_s(\boldsymbol{p})e^{i\boldsymbol{p}\cdot\boldsymbol{x}} + b_{\boldsymbol{p}}^{s\,\dagger} v_s(\boldsymbol{p})e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}) \tag{3.2.4}$$

thus the anticommutations of ladder operators:

$$\left\{a_{\boldsymbol{p}}^s, a_{\boldsymbol{p}'}^{s'\,\dagger}\right\} = \left\{b_{\boldsymbol{p}}^s, b_{\boldsymbol{p}'}^{s'\,\dagger}\right\} = (2\pi)^3\delta_{ss'}\delta^{(3)}(\boldsymbol{p} - \boldsymbol{p}')$$
$$\{a, a\} = \left\{a^\dagger, a^\dagger\right\} = \cdots = 0$$

Hamiltonian in terms of Fourier modes (with normal ordering)

$$H = \int \frac{d^3p}{(2\pi)^3} \sum_s E_p (a_{\boldsymbol{p}}^{s\dagger} a_{\boldsymbol{p}}^s - b_{\boldsymbol{p}}^{s\dagger} b_{\boldsymbol{p}}^s) \tag{3.2.5}$$

3.3 Particles and antiparticles

$$Q = e \int d^3x \psi^\dagger(x)\psi(x) \tag{3.3.1}$$

$$: Q : = e \int \frac{d^3p}{(2\pi)^3} \sum_s (a_{\boldsymbol{p}}^{s\dagger} a_{\boldsymbol{p}}^s - b_{\boldsymbol{p}}^{s\dagger} b_{\boldsymbol{p}}^s) \tag{3.3.2}$$

3.4 Dirac propagator and anticommutators

$$\begin{aligned} S_{ab}(x-y) &= \{\psi_a(x), \bar{\psi}_b(y)\} \\ &= (i\partial\!\!\!/ + m)\left[D(x-y) - D(y-x)\right] \end{aligned} \tag{3.4.1}$$

Time ordering of Dirac fields

$$T(\phi_a(x)\bar{\psi}_b(y)) = \Theta(x^0 - y^0)\psi_a(x)\bar{\psi}_b(y) - \Theta(y^0 - x^0)\bar{\psi}_b(y)\psi_a(x) \tag{3.4.2}$$

Feynman propogator for the Dirac field

$$S_F(x-y) = \langle 0|T\psi(x)\bar{\psi}(y)|0\rangle = \int \frac{d^4p}{(2\pi)^4} \frac{i(p\!\!\!/ + m)}{p^2 - m^2 + i\epsilon} e^{-ip\cdot(x-y)} \tag{3.4.3}$$

3.5 Discrete symmetries of the Dirac Field

	orientation perserving	orientation not perserving
(ortho)chronous	$\mathcal{L}_+^\uparrow$	$\mathcal{L}_-^\uparrow = \mathcal{P}\mathcal{L}_+^\uparrow$
non-orthochronous	$\mathcal{L}_-^\downarrow = \mathcal{T}\mathcal{L}_+^\uparrow$	$\mathcal{L}_+^\downarrow = \mathcal{P}\mathcal{T}\mathcal{L}_+^\uparrow$

4 Interacting Quantum Field Theory

4.1 Introduction and examples

Theories discussed so far are Klein-Gordon theory with spin 0

$$\mathcal{L}_{KG} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 \tag{4.1.1}$$

and Dirac theory spin $\frac{1}{2}$

$$\mathcal{L}_D = \bar{\psi}(i\partial\!\!\!/ - m)\psi \tag{4.1.2}$$

There is also $\mathcal{L}_{EM} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ for a massless vector filed. Its quantisation gives photon.

One thing they have in common is they are all quadratic in the fields. As a result:

- linear field equations
- exact quantisation
- multi-particle states without scattering or interaction
- linear Fourier decomposition, no momentum changes

To have an interacting theory with scattering, need higher powers in the field in the Lagrangians. A few examples are following

Scalar ϕ^4 theory

$$\mathcal{L} = \mathcal{L}_{KG} + \frac{\lambda}{4!}\phi^4$$

need positive sign $\lambda > 0$ for a stable theory, otherwise classical energy can be arbitrarily negative.

Equation of motions

$$(\partial^2 + m^2)\phi = -\frac{\lambda}{3!}\phi^3$$

is non-linear, cannot be solved by Fourier decomposition.

Yukawa-theory

$$\mathcal{L} = \mathcal{L}_{KG} + \mathcal{L}_D - g\bar{\psi}\psi\phi$$

is originally developed as a theory for nuclear forces with ψ nucleon, ϕ pion. In the Standard Model it is similar to interactions in Higgs mechanism.

Quantum Electrodynamics (QED)

$$\mathcal{L} = \mathcal{L}_{EM} + \mathcal{L}_D - eA_\mu \bar{\psi}\gamma^\mu \psi$$

describes electrons, their antiparticles positrons and photons.

Yang-Mills theory generalises $\mathcal{L}_{EM}$ with terms like A^4 or $A^2\partial A$

Scalar QED describes pions and photons

$$\begin{aligned}\mathcal{L} &= \mathcal{L}_{EM} + D_\mu\phi D^\mu\phi^* - m^2|\phi|^2 \\ &= \mathcal{L}_{EM} + \partial_\mu\phi\partial^\mu\phi^* - m^2\phi\phi^* + ieA_\mu(\phi\partial^\mu\phi^* - \phi^*\partial^\mu\phi) + e^2A_\mu A^\mu\phi\phi^*\end{aligned}$$

Remarks

- Interaction terms in $H_{\text{int}} = \int \mathrm{d}^3x\mathcal{H}_{\text{int}} = -\int \mathrm{d}^3x\mathcal{L}_{\text{int}}$ always involves products of fields at the same point $\boldsymbol{x}$. It ensures causality, no "instant at a distance".
- There are no derivative interactions. These may complicate quantisation as

$$\pi(\boldsymbol{x}) = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi(\boldsymbol{x}))}$$

- Why are we taking the examples above? There must be zillions of theories (Lagrangians)?
  We have the criterion of **renormalizability**. Note the mass dimensions of fields;

$$[S] = 1 \text{ so } [\mathcal{L}] = [M]^4 \;\Rightarrow [\phi] = [M],\ [\psi] = [M]^{\frac{3}{2}},\ [A_\mu] = [M]$$

  So in all the interaction terms indicated above, the coupling constant λ, e, g are all **dimensionless**! Can add $-\frac{\mu}{3!}\phi^3$ to the ϕ^4 theory. This leads to $[\mu] = [M]$ and all these generate renormalizable interactions.
  All higher interaction terms require coupling constants of **negative** mass dimension, e.g. $G\bar{\psi}\psi\bar{\psi}\psi$ and then $[G] = [M]^{-2}$. These are non-renormalizable and create trouble when performing higher-order calculation in perturbation theory. (with energy cut-off; corrections $G\Lambda^2$, $\Lambda \to \infty$)
- We haven't quantised the photon yet. The reason is that its is a vector field, i.e. 4 degrees of freedom, but photon has just 2 physical polarisation states. It is linked to gauge symmetry and complicates quantisation somewhat.

4.2 The interaction picture

Consider the ϕ^4 theory,

$$\mathcal{L}_{int} = -\frac{\lambda}{4!}\phi(x)^4 \tag{4.2.1}$$

Hamiltonian $H = H_0 + H_{int}$ with

$$H_0 = \int \mathrm{d}^3x \left\{\frac{1}{2}\pi^2(x) + \frac{1}{2}(\boldsymbol{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2\right\} \tag{4.2.2}$$

$$H_{\text{int}} = -\int \mathrm{d}^3x\mathcal{L}_{\text{int}} = \frac{\lambda}{4!}\int \mathrm{d}^3x\phi^4 \tag{4.2.3}$$

Interaction picture means that *operators* evolve in time using H_0 (only), in particular

$$\phi_I(t,\boldsymbol{x}) = e^{iH_0t}\phi(\boldsymbol{x})e^{-iH_0t} \tag{4.2.4}$$

Time-dependence of the free field obeys classical equation of motion $\left(\partial^2 + m^2\right)\phi_I(t,\boldsymbol{x}) = 0$. Solution in terms of Fourier modes as before:

$$\phi_I(t,\boldsymbol{x}) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3\sqrt{2E_p}}(a^I_{\boldsymbol{p}}e^{-ipx} + a^{I\dagger}_{\boldsymbol{p}}e^{+ipx}) \tag{4.2.5}$$

as in the free theory with standard commutation relations $[a^I_{\boldsymbol{p}}, a^{I\dagger}_{\boldsymbol{p}}] = (2\pi)^3\delta^{(3)}(\boldsymbol{p}-\boldsymbol{p}')$. The state satisfying $a^I_p\,|0\rangle = 0$ is the vacuum of the free, non-interacting theory.

Relation between interaction and Schrödinger picture states:

$$|\psi_I(t)\rangle = e^{iH_0t}|\psi_S(t)\rangle \tag{4.2.6}$$

Schrödinger equation becomes:

$$\begin{aligned}
i\frac{\partial}{\partial t}\,|\psi_S\rangle &= (H_0 + H_{\text{int}})\,|\psi_S\rangle \\
\text{LHS} &= i\frac{\partial}{\partial t}\left(e^{-iH_0t}\,|\phi_I\rangle\right) = H_0e^{-iH_0t}\,|\phi_I\rangle + e^{-iH_0t}i\frac{\partial}{\partial t}\,|\phi_I\rangle \\
\text{RHS} &= (H_0 + H_{\text{int}})\,e^{-iH_0t}\,|\phi_I\rangle \\
\Rightarrow i\frac{\partial}{\partial t}\,|\phi_I\rangle &= e^{iH_0t}H_{\text{int}}e^{-iH_0t} = H_I(t)\,|\phi_I\rangle
\end{aligned} \tag{4.2.7}$$

with H_I interaction Hamiltonian in the interaction picture. Clearly

$$H_I = \frac{\lambda}{4!}\int \mathrm{d}^3x\phi_I^4(x)$$

What is the solution of 4.2.7 for the time evolution of $|\phi_I(t)\rangle$? Define time-evolution operator in the interaction picture.

$$|\phi_I(t)\rangle = U(t,t_0)\,|\phi_I(t_0)\rangle \tag{4.2.8}$$

$$\text{with } U(t,t_0) = e^{iH_0(t-t_0)}e^{-iH(t-t_0)} \tag{4.2.9}$$

With 4.2.7 and 4.2.8:

$$i\frac{\partial}{\partial t}U(t,t_0) = H_I(t)U(t,t_0) \tag{4.2.10}$$

To solve with boundary conditions $U(t_0,t_0) = \mathbb{1}$. The formal solution is then:

$$U(t,t_0) = 1 - i\int_{t_0}^{t}\mathrm{d}t'\,H_I(t')U(t',t_0)$$

Substitute back in and we get:

$$U(t,t_0) = 1 - i\int_{t_0}^{t}\mathrm{d}t'\,H_I(t') + (-i)^2\int_{t_0}^{t}\mathrm{d}t'\int_{t_0}^{t'}\mathrm{d}t''\,H_I(t')H_I(t'') + \dots$$

H_I inside the integral is automatically time-ordered. Ranges of integration is not.

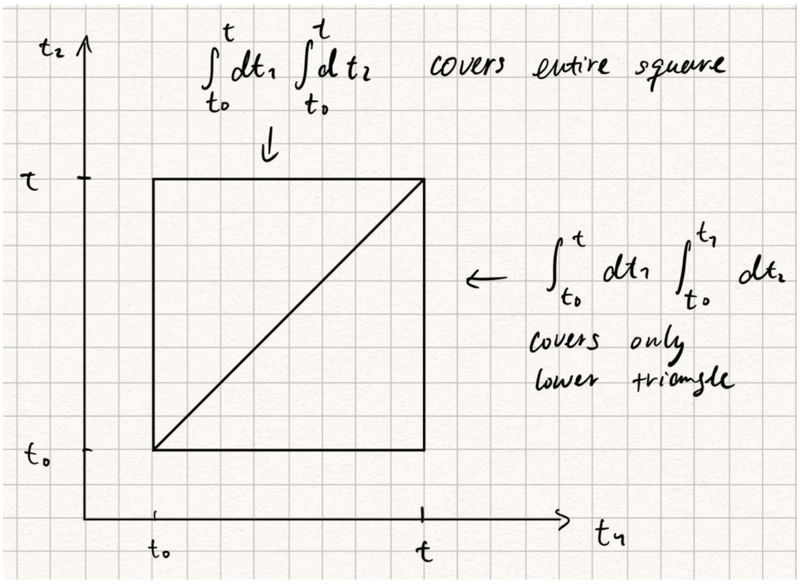


Figure 4.1: Time ordering

Upper triangle has the wrong time order. We are going to "repair" it by hand.

$$
\begin{aligned}
U(t, t_0) &= 1 - i \int_{t_0}^{t} \mathrm{d}t' H_I(t') + \frac{(-i)^2}{2} \int_{t_0}^{t} \mathrm{d}t' \int_{t_0}^{t} \mathrm{d}t'' T(H_I(t')H_I(t'')) + \dots \\
&= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{t_0}^{t} \mathrm{d}t_1 \cdots \int_{t_0}^{t} \mathrm{d}t_n T(H_I(t_1) \dots H_I(t_n)) \\
&= T \exp\left\{-i \int_{t_0}^{t} \mathrm{d}t' H_I(t')\right\}
\end{aligned}
\tag{4.2.11}
$$

It is interesting for scattering to transition into asymptotic state for $t \to \infty$

$$
\begin{aligned}
S &= \lim_{t\to\infty} U(t, -t) = T \exp\left\{-i \int_{-\infty}^{\infty} \mathrm{d}t H_I(t)\right\} \\
&\overset{\phi^4}{=} T \exp\left\{-i \int \mathrm{d}^4 x \frac{\lambda}{4!} \phi_I^4(x)\right\}
\end{aligned}
\tag{4.2.12}
$$

Both U and S are formally unitary

Composition law for time evolution operator

$$
U(t_2, t_0) = U(t_2, t_1)U(t, t_0) = U(t_2, t_1)U(t_0, t_1)^\dagger \tag{4.2.13}
$$

4.2.1 Scattering amplitudes and the S-matrix

Take $|i\rangle$ the initial (multi-particle) state and $|f\rangle$ the final (multi-particle) state. Time evolution of $|i\rangle$ then is

$$
\lim_{t\to\infty} U(t, -t) |i\rangle = S |i\rangle
$$

Probability that $|i\rangle$ evolves into $|f\rangle$ is proportional to the squared "**S-matrix element**"

$$
|\langle f,\ t \to \infty | i, t \to -\infty\rangle|^2 = |\langle f|S|i\rangle|^2 = \left|S_{fi}\right|^2 \tag{4.2.14}
$$

The non-trivial part of the S-matrix is the T-matrix:

$$S_{fi} := \delta_{fi} + iT_{fi} \tag{4.2.15}$$

Use momentum conservation (from translation invariance) to define matrix element

$$S_{fi} = \delta_{fi} + i(2\pi)^4\delta^{(4)}(p_f - p_i)M_{fi} \tag{4.2.16}$$

M_{fi} measures "genuine scattering" from $|i\rangle$ to $|f\rangle$.

How are we going to calculate correlation functions in the interacting theory:

$$\langle\Omega|\, T\phi(x)\phi(y)|\Omega\rangle$$

or more generally $\langle\Omega|T\phi(x_1)\phi(x_2)\ldots|\Omega\rangle$? Here $|\Omega\rangle$ is the vacuum/ground state of the interacting theory $\mathcal{H}$ (in contrast to $|0\rangle$ is the ground state of $\mathcal{H}_0$) and $\phi(x)$ the Heisenberg operators.

Ignore $|\Omega\rangle \neq |0\rangle$ for the moment saying that we want to study the time evolution from the vacuum at $t \to -\infty$ to $t \to +\infty$. So rewriting in terms $\phi_I(x)$, assuming $x^0 > y^0$ for now:

$$\langle 0|U(\infty, x^0)\phi_I(x^0)U(x^0, y^0)\phi_I(y^0)U(y^0, -\infty)|0\rangle = \langle 0|T(\phi_I(x)\phi_I(y)S)|0\rangle \tag{4.2.17}$$

still holds if $x^0 < y^0$ because of T.

Now $|\Omega\rangle \neq |0\rangle$: this can be taken care of by dividing out the time evolution of the (free) vacuum $\langle 0|S|0\rangle$, so

$$\begin{aligned}\langle\Omega|T(\phi(x)\phi(y))|\Omega\rangle &= \frac{\langle 0|T(\phi_I(x)\phi_I(y)S)|0\rangle}{\langle 0|S|0\rangle} \\ &\overset{\phi^4}{=} \frac{\langle 0|T\phi_I(x)\phi_I(y)\exp\left\{-i\int \mathrm{d}^4x'\,\frac{\lambda}{4!}\phi^4(x')\right\}|0\rangle}{\langle 0|T\exp\left\{-i\int \mathrm{d}^4x'\,\frac{\lambda}{4!}\phi^4(x')\right\}|0\rangle}\end{aligned} \tag{4.2.18}$$

Proof can be found in Peskin. It will also be illustrated practically later ("vacuum bubbles").

Perturbation theory is viable when λ (or some other coupling) is "small" and then expands $U(t, t_0)$ or S in powers of λ.

4.3 Wick's theorem

From now on drop the subscript for interaction picture fields $\phi_I(x) \to \phi(x)$ for convenience.

Want to calculate stuff like $\langle 0|T\phi(x_1)\ldots\phi(x_n)S|0\rangle$ in perturbation theory, e.g. at order λ^n.

$$\frac{1}{n!}\left(-i\frac{\lambda}{4!}\right)^n \int \mathrm{d}^4y_1\ldots\mathrm{d}^4y_n\,\langle 0|T\phi(x_1)\ldots\phi(x_n)\phi^4(y_1)\ldots\phi^4(y_n)|0\rangle \tag{4.3.1}$$

We know $\langle 0|T\phi(x_1)\phi(x_2)|0\rangle$ is the Feynman propagator!

Recall **normal ordering** with $\phi(x) = \phi^+(x) + \phi^-(x)$

$$:\phi^+\phi^-:=:\phi^-\phi^+:=\phi^-\phi^+ \tag{4.3.2}$$

where

$$\phi^+ = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\boldsymbol{p}}}}a_{\boldsymbol{p}}e^{-ip\cdot x} \tag{4.3.3}$$

$$\phi^- = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\boldsymbol{p}}}}a^\dagger_{\boldsymbol{p}}e^{+ip\cdot x} \tag{4.3.4}$$

Wick's theorem expresses time-ordered products in terms of normal-ordered ones. Then it is easy to take vacuum expectation values, as $\langle 0| : \phi(x_1)\dots\phi(x_n) : |0\rangle = 0$

Take two fields and $x^0 > y^0$:

$$\begin{aligned} T\phi(x)\phi(y) = \phi(x)\phi(y) &= \left(\phi^+(x) + \phi^-(x)\right)\left(\phi^+(y) + \phi^-(y)\right) \\ &= \phi^+(x)\phi^+(y) + \phi^-(x)\phi^-(y) + \phi^-(x)\phi^+(y) + \phi^-(x)\phi^+(y) + [\phi^+(x), \phi^-(y)] \\ &=: \phi(x)\phi(y) : + [\phi^+(x), \phi^-(y)] \end{aligned}$$

Particularly for $y^0 > x^0$:

$$T\phi(x)\phi(y) =: \phi(x)\phi(y) : + [\phi^+(y), \phi^-(x)]$$

Thus altogether:

$$T\phi(x)\phi(y) =: \phi(x)\phi(y) : + D_f(x-y) \tag{4.3.5}$$

as $\Theta(x^0 - y^0)[\phi^+(x), \phi^-(y)] + \Theta(y^0 - x^0)[\phi^+(y), \phi^-(x)] = D_F(x-y)$.

Worth noting that $D_F(x-y)$ is still a c-number, not operator (yet). Thus it can be pulled out of any matrix element or expectation value.

We now define "contraction":

$$\contraction{}{\phi}{(x_1)}{\phi}\phi(x_1)\phi(x_2) = D_F(x_1 - x_2) \tag{4.3.6}$$

Thus we can remove the fields from the product leaving only the propagators:

$$T\phi(x)\phi(y) =: \phi(x)\phi(y) : + \contraction{}{\phi}{(x)}{\phi}\phi(x)\phi(y) \tag{4.3.7}$$

General form of **Wick's theorem** for arbitrary number of fields

$$T\phi(x_1)\dots\phi(x_n) =: \phi(x_1)\dots\phi(x_n) : + : (\text{sum over all possible contractions}) : \tag{4.3.8}$$

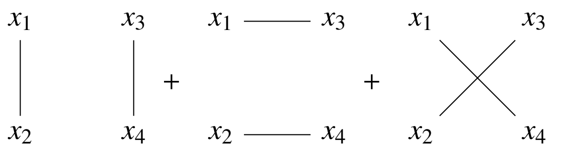
Example with four fields:

$$\begin{aligned} T(\phi_1\phi_2\phi_3\phi_4) =&: \phi_1\phi_2\phi_3\phi_4 : \\ &+ \contraction{}{\phi_1}{}{\phi_2}\phi_1\phi_2 : \phi_3\phi_4 : + \contraction{}{\phi_1}{}{\phi_3}\phi_1\phi_3 : \phi_2\phi_4 : + \contraction{}{\phi_1}{}{\phi_4}\phi_1\phi_4 : \phi_2\phi_3 : + \contraction{}{\phi_2}{}{\phi_3}\phi_2\phi_3 : \phi_1\phi_4 : + \contraction{}{\phi_2}{}{\phi_4}\phi_2\phi_4 : \phi_1\phi_3 : + \contraction{}{\phi_3}{}{\phi_4}\phi_3\phi_4 : \phi_1\phi_2 : \\ &+ \contraction{}{\phi_1}{}{\phi_2}\contraction{\phi_1\phi_2}{\phi_3}{}{\phi_4}\phi_1\phi_2\phi_3\phi_4 + \contraction{}{\phi_1}{}{\phi_3}\contraction{\phi_1\phi_3}{\phi_2}{}{\phi_4}\phi_1\phi_3\phi_2\phi_4 + \contraction{}{\phi_1}{}{\phi_4}\contraction{\phi_1\phi_4}{\phi_2}{}{\phi_3}\phi_1\phi_4\phi_2\phi_3 \end{aligned}$$

Thus

$$\langle 0|T(\phi_1\phi_2\phi_3\phi_4)|0\rangle = D_F(x_1 - x_2)D_F(x_3 - x_4) + D_F(x_1 - x_3)D_F(x_2 - x_4) + D_F(x_1 - x_4)D_F(x_2 - x_3)$$

which can be visually represented as



Proof of the general theorem by *induction* in the number of fields (see exercise). The idea is to suppose it is true for $\phi_2\dots\phi_m$, $x_1^0 > x_{k>1}^0$. Then

$$\begin{aligned} T\phi_1\phi_2\dots\phi_m &= (\phi_1^+ + \phi_1^-)T\phi_2\dots\phi_m \\ &= (\phi_1^+ + \phi_1^-)[: \phi_2\dots\phi_m : + : \text{contractions} :] \end{aligned}$$

ϕ_1^- can stay as it is part of (: $\phi_1\phi_2\dots\phi_m$:). But ϕ_1^+ needs to be commuted past all ϕ_1^- operators, giving rise to additional contractions $\contraction{}{\phi_1}{}{\phi_2}\phi_1\phi_2$.

Consequences

- $n = 2k+1,\ k \in \mathbb{N}$
$$\langle 0|T\phi_1 \dots \phi_m|0\rangle = 0$$
- $n = 2k,\ k \in \mathbb{N}$
$$\langle 0|T\phi_1 \dots \phi_m|0\rangle = \sum_{\text{pairing of fields}} D_F(x_{i_1} - x_{i_2}) \dots D_F(x_{i_{m-1}} - x_{i_m})$$

4.3.1 Wick's theorem and the S-Matrix

Apply Wick's theorem to correlation functions $\langle 0|T(\phi_1 \dots \phi_m)S|0\rangle$ n-th term in the perturbative expansion of S with $\phi(x_1) := \phi_1$.

$$\frac{1}{n!}\left(\frac{-i\lambda}{4!}\right)^n \int \mathrm{d}^4y_1 \dots \mathrm{d}^4y_n \, \langle 0|T(\phi_1 \dots \phi_m \phi^4(y_1) \dots \phi^4(y_n))|0\rangle$$

Example with $m = 4,\ n = 1$

$$\begin{aligned}
&-\frac{i\lambda}{4!}\int \mathrm{d}^4x \, \langle 0|T\phi_1\phi_2\phi_3\phi_4\phi^4(x)|0\rangle \\
&= -\frac{i\lambda}{4!}\int \mathrm{d}^4x \, \langle 0|\phi_1\phi_2\phi_3\phi_4\phi(x)\phi(x)\phi(x)\phi(x)|0\rangle + 23 \text{ permutations} \\
&\quad -\frac{i\lambda}{4!}\int \mathrm{d}^4x \, \langle 0|\phi_1\phi_2\phi_3\phi_4\phi(x)\phi(x)\phi(x)\phi(x)|0\rangle + 11 \text{ permutations} + 5 \text{ similar} \\
&\quad -\frac{i\lambda}{4!}\int \mathrm{d}^4x \, \langle 0|\phi_1\phi_2\phi_3\phi_4\phi(x)\phi(x)\phi(x)\phi(x)|0\rangle + 2 \text{ permutations} + 2 \text{ similar} \\
&= -i\lambda \int \mathrm{d}^4x D_F(x_1 - x)D_F(x_2 - x)D_F(x_3 - x)D_F(x_4 - x) \\
&\quad -\frac{i\lambda}{2}D_F(x_1 - x_2)\int \mathrm{d}^4x D_F(x_3 - x)D_F(x_4 - x)D_F(x - x) + 5 \text{ similar} \\
&\quad -\frac{i\lambda}{8}D_F(x_1 - x_2)D_F(x_3 - x_4)\int \mathrm{d}^4x D_F(x - x) + 2 \text{ similar}
\end{aligned}$$

Permutation means permutation of $\phi(x)$ and similar means exchanging external states ϕ_i, $i \in \{1, 2, 3, 4\}$ without changing the shape of diagram. Note that the pre-factors of the integrals are called *symmetry factor*. The number of permutations of first diagram is equivalent to arrangement of 4 elements

$$\frac{4!}{4!} = 1$$

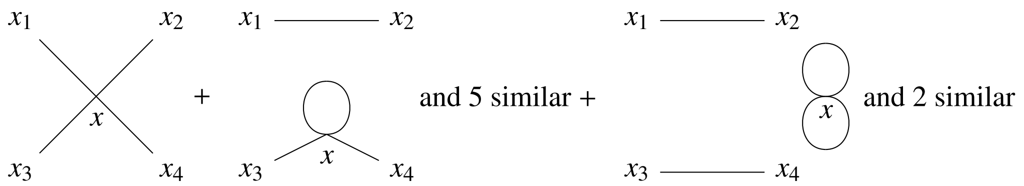
The permutations of second digram are to link two internal points to external ones and the rest external points can link to either of internal one.

$$\binom{4}{2} \cdot 2 \cdot \frac{1}{4!} = \frac{1}{2}$$

There are only three ways to permute a vacuum bubble

$$\frac{3}{4!} = \frac{1}{8}$$

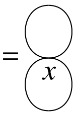
To be represented in Feynman diagrams:



In fact $D_F(x-x) = D_F(0)$ diverges!

Example with $m = 0$, $n = 1$ vacuum diagram

$$
\begin{aligned}
&-\frac{i\lambda}{4!}\int \mathrm{d}^4x \,\langle 0|T\phi^4(x)|0\rangle \\
&= -\frac{i\lambda}{8}[D_F(0)]^2 \int \mathrm{d}^4x \\
&=
\end{aligned}
$$



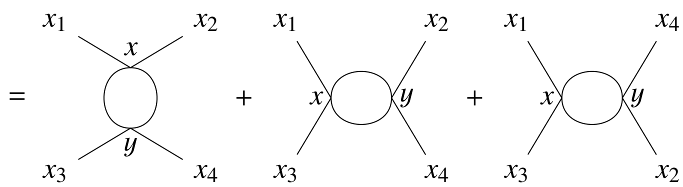
Example: 2nd order S-matrix term

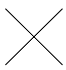
$$
\frac{1}{2!}\left(\frac{-i\lambda}{4!}\right)^4 \int \mathrm{d}^4x\mathrm{d}^4y \,\langle 0|T\phi_1\phi_2\phi_3\phi_4\phi^4(x)\phi^4(y)|0\rangle
$$

It has many contractions and some of the fully connected ones are of the type there are

(4×3)[choose $\phi(x)$] $\times$ (4×3)[choose $\phi(y)$] $\times$ 2[x-y-cont.] $\times$ 2(x-y-symm.) + 2 similar, exchanging external points

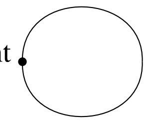
$$
= \frac{(-i\lambda)^2}{2}\int \mathrm{d}^4x\mathrm{d}^4y D_F(x_1-x)D_F(x_2-x)D_F(x_3-y)D_F(x_4-y)[D_F(x-y)]^2 + 2 \text{ similar}
$$

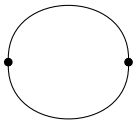


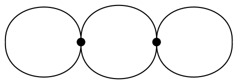
Symmetry factors A lot of the contractions eliminate the factors $\frac{1}{n!}\left(\frac{1}{4!}\right)^n$ in the denominators; the $\frac{1}{4!}$ was chosen to yield  $\sim -i\lambda$

See examples above. Sometimes, factors are not completely cancelled and thus procedure gets "reversed". Divide diagrams by *symmetry factor* ("missing factors").

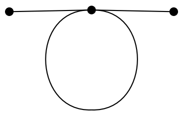
Where does it come from?

- Factor 2 from the line that starts and ends at the same point 

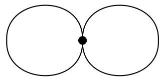
- Factor $j!$ for j lines linking the same 2 points 

- Factor $k!$ for k equivalent vertices 

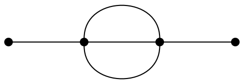
When in doubt, can always go back to Wick's theorem and count the contractions explicitly.
Examples:



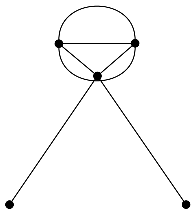
$S = 2$



$S = 8$



$S = 3! = 6$



$S = 2 \cdot 3! = 12$

Summary of Feynman rules

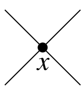
$$\langle 0|T\phi_1 \ldots \phi_m \exp\left(-\frac{i\lambda}{4!}\int \mathrm{d}^4x\phi^4(x)\right)|0\rangle$$
$$= \text{sum of all diagrams with m external points;}$$

usually organised by number of internal points (i.e. power of λ).
Each diagram is built of

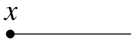
- propagators
- vertices (n)
- external points (m)

Feynman rules in position space Analytic expression obtained by combining

- For each propagator 

$= D_F(x-y)$
- For each vertex 

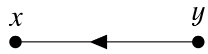
$= -i\lambda \int \mathrm{d}^4x$

- For each external point 

$= 1$
- Divide diagram by its symmetry factor S

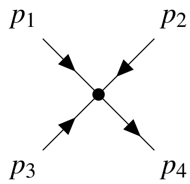
Since the propagator $D_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m^2+i\epsilon} e^{-ip(x-y)}$. It is actually simpler to express these in momentum space instead.

The way to do it is to asign a momentum p to each propagator. (direction arbitrary)



- assign e^{ipy} to y-vertex (arrow out)
- assign e^{-ipx} to x-vertex (arrow in)
- $\frac{i}{p^2-m^2+i\epsilon}$ to the line and the integration $\int \frac{d^4p}{(2\pi)^4}$

At vertex x:

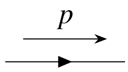


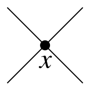
$$= -i\lambda \int d^4x e^{-i(p_1+p_2+p_3)x+ip_4x}$$

$$= -i\lambda (2\pi)^4 \delta^{(4)}(p_1+p_2+p_3-p_4)$$

This imposes momentum conservation at vertex. $\delta^{(4)}$-functions make some of the momentum integrals trivial, always with $(2\pi)^4$ cancelled appropriately.

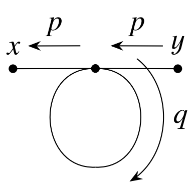
Momentum space Feynman rules

- Propagator 

$= \frac{i}{p^2-m^2+i\epsilon}$
- Vertex (position integrated out) 

$= -i\lambda$
- External points $\begin{cases} e^{-ipx} & \text{incoming} \\ e^{+ipx} & \text{outgoing} \end{cases}$
- Impose momentum conservation at each vertex
- Integrate over each undetermined momentum $\int \frac{d^4p}{(2\pi)^4}$
- Divided by symmetry factor

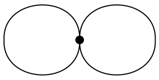
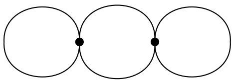
e.g.:



$$= (-i\lambda) \int \frac{d^4p}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} \left(\frac{i}{p^2-m^2+i\epsilon}\right)^2 \frac{i}{q^2-m^2+i\epsilon} e^{-ip(x-y)}$$

Vacuum diagrams Disconnected pieces in Feynman diagrams are pretty bad. Not only $D_F(0) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m^2+i\epsilon}$ is divergent (that will be taken care of later), it also contains an integral $\int d^4x \,\text{const}$. Thus divergent once more!

Typical diagram contributing to 2-point function. one piece connected to x and y, plus disconnected pieces.

Call disconnected pieces $V_i \in$ { , , ... }. Points are connected internally, but not to external points.

V_i can occur n_i-times, then

$$[\text{diagram}] = [\text{connected pieces}] \times \prod_i \frac{1}{n!} (V_i)^{n_i}$$

The factorial is the symmetry factor of n_i disconnected copies of V_i.

Then

$$\begin{aligned}
&\langle 0|T\phi_1 \dots \phi_n S|0\rangle \\
&= \sum_{\text{connected}} \sum_{\text{all}\{n_i\}} [\text{connected}] \times \prod_i \frac{1}{n_i!}(V_i)^{n_i} \\
&= \left(\sum [\text{connected}]\right) \times \sum_{\text{all}\{n_i\}} \left(\prod_i \frac{1}{n_i!}(V_i)^{n_i}\right) \\
&= \left(\sum [\text{connected}]\right) \times \prod_i \left(\sum_{n_i} \frac{1}{n_i!}(V_i)^{n_i}\right) \\
&= \left(\sum [\text{connected}]\right) \times \exp\left(\sum_i V_i\right)
\end{aligned}$$

Thus

$$\begin{aligned}
\text{sum of ALL diagrams} = &(\text{sum of all CONNECTED diagrams}) \\
&\times \exp(\text{sum of all DISCONNECTED diagrams})
\end{aligned}$$

Obvious from the above:

$$\langle 0|S|0\rangle = \langle 0|T\{\exp\left(-\frac{i\lambda}{4!}\int d^4x\phi^4(x)\right)\}|0\rangle = \exp(\text{sum of all vacuum bubbles})$$

Conclusion from the (unproven) formula for n-point correlation functions in the true, interacting vacuum:

$$\begin{aligned}
\langle \Omega|T\phi_1 \dots \phi_m|\Omega\rangle &= \frac{\langle 0|T\phi_1 \dots \phi_m S|0\rangle}{\langle 0|S|0\rangle} && (4.3.9) \\
&= \sum (\text{connected diagrams with m external points}) && (4.3.10)
\end{aligned}$$

Here "connected" means connected to any external point. External points do not have to linked to each other.

4.4 S-matrix elements and Feynman diagrams

What is the correlation function in interacting vacuum $\langle\Omega|T\phi_1\dots\phi_m|\Omega\rangle$ good for? For scattering, shouldn't we rather look at $\langle p_1\dots p_m|S|p_Ap_B\rangle$ with the perturbative expansion of S as before?

Decompose the S-matrix

$$S_{fi}=\delta_{fi}+iT_{fi}=\delta_{fi}+i(2\pi)^4\delta^{(4)}(p_f-p_i)M_{fi} \tag{4.4.1}$$

M_{fi} is the invariant matrix element, used to calculate cross section etc..

Zeroth term in the expansion of S

$$\begin{aligned}\langle p_1p_2|p_Ap_B\rangle &= \sqrt{2E_12E_22E_A2E_B}\,\langle 0|a_1a_2a_A^+a_B^+|0\rangle\\ &= 2E_A2E_B(2\pi)^6\left(\delta^{(3)}(\boldsymbol{p}_A-\boldsymbol{p}_1)\delta^{(3)}(\boldsymbol{p}_B-\boldsymbol{p}_2)+\delta^{(3)}(\boldsymbol{p}_A-\boldsymbol{p}_2)\delta^{(3)}(\boldsymbol{p}_B-\boldsymbol{p}_1)\right)\end{aligned}$$

This actually is "no scattering", part of the $\mathbb{1}$ in the S-matrix.

First term is

$$\begin{aligned}&\langle p_1p_2|T\left(-\frac{i\lambda}{4!}\int \mathrm{d}^4x\phi^4(x)\right)|p_Ap_B\rangle\\ &\overset{\text{wick}}{=}\langle p_1p_2|:\left(-\frac{i\lambda}{4!}\int \mathrm{d}^4x\phi^4(x)+\text{contractions}\right):|p_Ap_B\rangle\end{aligned}$$

However now the expectation value of a normal-ordered expression doesn't vanish!

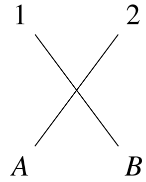
$$\begin{aligned}\phi^+(x)\,|\boldsymbol{p}\rangle &= \int\frac{\mathrm{d}^3k}{(2\pi)^3\sqrt{2E_k}}a_{\boldsymbol{k}}e^{-ikx}\sqrt{2E_p}a_{\boldsymbol{p}}^+\,|0\rangle\\ &= \int\frac{\mathrm{d}^3k}{\sqrt{2E_k}}e^{-ikx}\sqrt{2E_p}\delta^{(3)}(\boldsymbol{k}-\boldsymbol{p})\,|0\rangle\\ &= e^{-ipx}\,|0\rangle\end{aligned}$$

So in general, need two field operators to annihilate the in-state and two field operators to create the out-states. We have new type of Feynman diagram to deal with external states. Define contractions of field operators with external states according to

$$\begin{aligned}\overbrace{\phi(x)\,|\boldsymbol{p}\rangle} &= e^{-ipx}\,|0\rangle\\ \overbrace{\langle\boldsymbol{p}|\,\phi(x)} &= e^{+ipx}\,|0\rangle\end{aligned}$$

How does this work for $p_Ap_B\to p_1p_2$ in ϕ^4 at first order? It contains three types of terms: $\phi\phi\phi\phi$, $\overbrace{\phi\phi}\phi\phi$ and $\overbrace{\phi\phi}\,\overbrace{\phi\phi}$ (fields without contraction are contracted with external states).

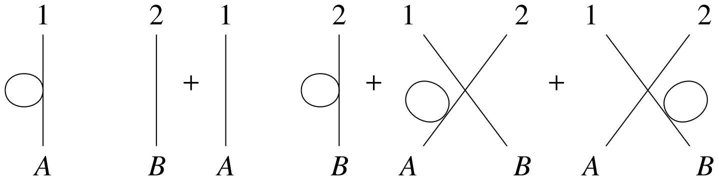
1. $\phi\phi\phi\phi$ allows full contractions with all external states. There is 4! possibilities



$$= 4!\frac{-i\lambda}{4!}\int \mathrm{d}^4x e^{-i(p_A+p_B-p_1-p_2)x}=-i\lambda\underbrace{(2\pi)^4\delta^{(4)}(p_A+p_B-p_1-p_2)}_{\text{Pre-factor in definition of }i\mathcal{M}}$$

$i\mathcal{M}$ receives a contribution $-i\lambda$!

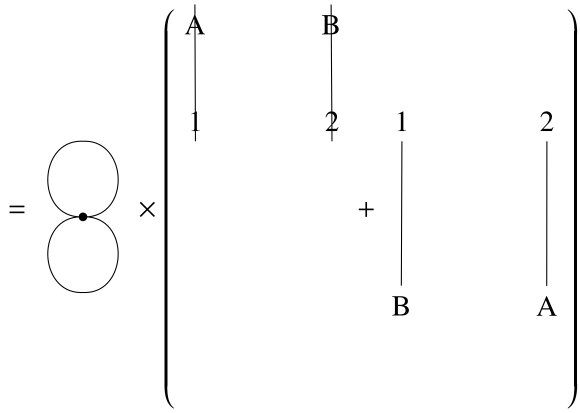
2. $\phi\phi\phi\phi$ leaves two operators to connect to external particles. Momentum conservation at each vertex. Still trivial!



Only fully connected Feynman diagrams contribute to $iT/i\mathcal{M}$!

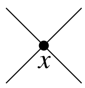
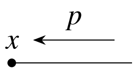
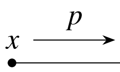
3.

$$-\frac{i\lambda}{4!}\int \mathrm{d}^4x\,\langle p_1p_2|\phi\phi\phi\phi|p_Ap_B\rangle$$



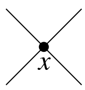
4.4.1 Feynman rules (with external lines)

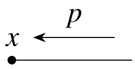
Position space calculate iT by summing overall fully connected diagrams with

- propagator  $= D_F(x-y)$
- vertex  $= -i\lambda\int \mathrm{d}^4x$
- external lines "in"  $= e^{-ip\cdot x}$;  $= e^{ip\cdot x}$
- divide diagram by its symmetry factor S

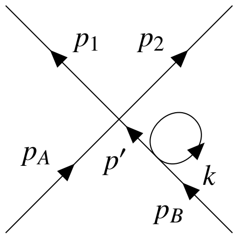
Momentum space We have seen it before. Now (with external lines) all positions are integrated over. Result is a function of external momenta only. Integrating out all momentum-conserving δ-distribution yields overall momentum conservation: $(2\pi)^4\delta^{(4)}(P_f-P_i)$

Momentum space Feynman rules for calculating iM:

- internal propagator  $= \frac{i}{p^2-M^2+i\epsilon}$
- vertex  $= -i\lambda$

- external lines ("in" or "out")  $= 1$
- impose 4-momentum conservation at each vertex
- integrate over all undetermined momenta $\int \frac{d^4 p}{(2\pi)^4}$
- divide diagram by its symmetry factor S

There is still trouble in there. Consider the next-to-leading contribution to the scattering amplitude

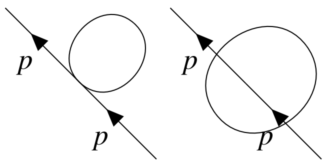


$$=\frac{1}{2}\int \frac{d^4 p'}{(2\pi)^4}\frac{i}{p'^2 - m^2}\int \frac{d^4 k}{(2\pi)^4}\frac{i}{k^2 - m^2}$$
$$\times(-i\lambda)(2\pi)^4\delta^{(4)}(p_A + p' - p_1 - p_2)\cdot(-i\lambda)(2\pi)^4\delta^{(4)}(p_B - p')$$

This contains the internal propagator $\frac{i}{P_B^2 - m^2 + i\epsilon}$, but all the external particle are on their mass-shell, i.e.

$$P_A^2 = P_B^2 = P_1^2 = P_2^2 = m^2 \;\Rightarrow\; \frac{i}{P_B^2 - m^2} = \frac{i}{0}$$

In addition to having fully connected diagrams, also need to confine ourselves to amputated diagrams: disregard all these diagrams with loops attached to external legs.



These diagrams represent the transition from the free to the interacting asymptotic states.

Lehmann-Symanzik-Zimmermann (LSZ) reduction formula Proof on relation between correlation functions and S-matrix elements will be provided later.

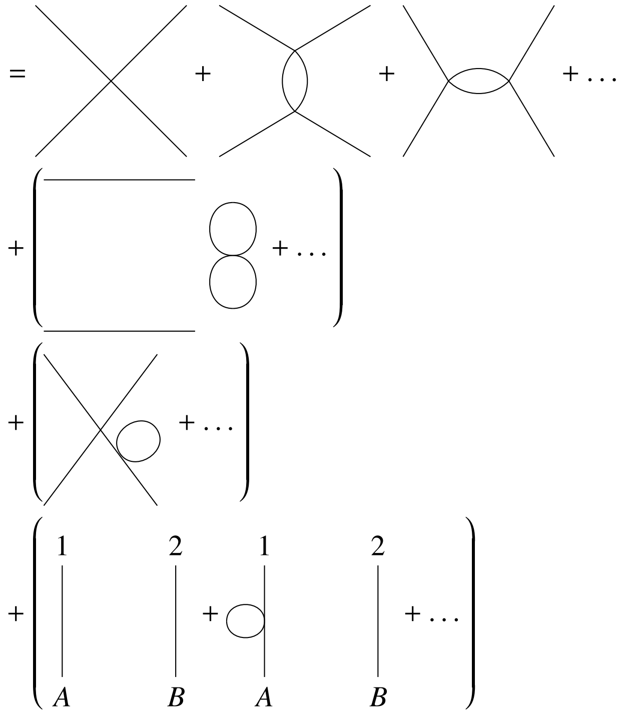
$$\begin{aligned}&\prod_{i=1}^{n}\int d^4 x_i e^{ip_i\cdot x_i}\prod_{j=1}^{m}\int d^4 y_j e^{-ik_j\cdot y_i}\langle\Omega|T\phi(x_1)\ldots\phi(x_n)\phi(y_1)\ldots\phi(y_m)|\Omega\rangle\\ &=(\text{disconnected stuff}) + \underbrace{\prod_{i=1}^{n}\frac{i\sqrt{Z}}{p_i^2 - m^2 + i\epsilon}\prod_{j=1}^{m}\frac{i\sqrt{Z}}{{}^2_j - m^2 + i\epsilon}}_{\text{remove poles from external legs}}\langle p_1\ldots p_n|S|k_i\ldots k_m\rangle\end{aligned} \tag{4.4.2}$$

z is the wave-function renormalization factor.

Then amend Feynman rules above

consider only fully connected, amputated diagrams

$\langle p_1 p_2 | iT | p_A p_B \rangle =$

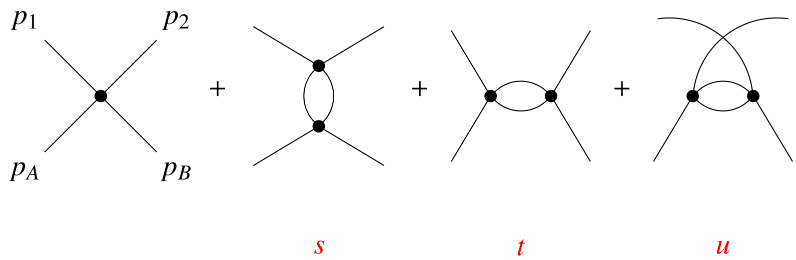


yields $|0\rangle \to |\Omega\rangle$

yields $|p\rangle_{\text{free}} \to |p\rangle_{\text{int}}$

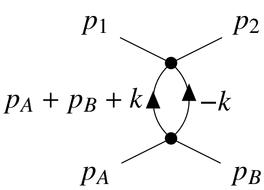
yields $\mathbb{1}$ in S-matrix

All allowed scattering diagrams $2 \to 2$ in ϕ^4 up to $\mathcal{O}(\lambda^2)$:



Define the Lorentz-invariant quantities, *Mandelstam variables*:

$$s = (p_A + p_B)^2\,, \quad t = (p_A - p_1)^2\,, \quad u = (p_A - p_2)^2 \tag{4.4.3}$$



$$= \frac{1}{2}(-i\lambda)^2 \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} \frac{i}{(p_A + p_B + k)^2 - m^2 + i\epsilon} =: \frac{1}{2}(-i\lambda)^2 iJ(s)$$

Then the complete invariant amplitude is

$$M = -\lambda - \frac{\lambda^2}{2}\left(J(s) + J(t) + J(u)\right) \tag{4.4.4}$$

4.5 Scattering cross section

This section is based on Itzykson & Zuber, Chapter 5.1.
The aim is to relate (differential) cross section to reduced/invariant matrix element M_{fi}. First we describe the initial states not as momentum eigenstates $|p_A p_B\rangle$, but as wave packets.

$$|i\rangle = \int \frac{\mathrm{d}^3 k_A}{(2\pi)^3 2k_A^0} \frac{\mathrm{d}^3 k_B}{(2\pi)^3 2k_B^0} f(k_A) g(k_B) \, |k_A k_B\rangle$$

with $f(k_A)$, $g(k_B)$ strongly peaked at $k_A \approx p_A$, $k_B \approx p_B$.
We can write the transition amplitude to the final state $|f\rangle \propto |p_1 p_2\rangle$ (note: normalisation not the same)

$$\begin{aligned} A_{fi} &= \int \frac{\mathrm{d}^3 k_A}{(2\pi)^3 2k_A^0} \frac{\mathrm{d}^3 k_B}{(2\pi)^3 2k_B^0} f(k_A) g(k_B) \, \langle f | iT | k_A k_B \rangle \\ &= \int \frac{\mathrm{d}^3 k_A}{(2\pi)^3 2k_A^0} \frac{\mathrm{d}^3 k_B}{(2\pi)^3 2k_B^0} f(k_A) g(k_B) (2\pi)^4 \delta^{(4)}(\underbrace{p_f}_{=p_1+p_2} - k_A - k_B) iM(f, k_A, k_B) \end{aligned}$$

Thus the transition probability:

$$\begin{aligned} \omega_{fi} = (2\pi)^8 &\int \frac{\mathrm{d}^3 k_A}{(2\pi)^3 2k_A^0} \frac{\mathrm{d}^3 k_B}{(2\pi)^3 2k_B^0} \frac{\mathrm{d}^3 q_A}{(2\pi)^3 2q_A^0} \frac{\mathrm{d}^3 q_B}{(2\pi)^3 2q_B^0} f(k_A) g(k_B) f(q_A)^* g(q_B)^* \\ \times &\underbrace{\delta^{(4)}(p_f - k_A - k_B) \delta^{(4)}(p_f - q_A - q_B)}_{=\delta^{(4)}(q_A + q_B - k_A - k_B)\delta^{(4)}(p_f - p_A - p_B)} \underbrace{M(f, k_A, k_B) M^*(f, q_A, q_B)}_{\approx |M(f, p_A, p_B)|^2} \end{aligned}$$

Using the fourier representation of delta function $\delta^{(4)}(q_A + q_B - k_A - k_B) = (2\pi)^{-4} \int \mathrm{d}^4 x e^{i(k_A + k_B - q_A - q_B)\cdot x}$

$$\begin{aligned} = \int \mathrm{d}^4 x &\underbrace{\int \frac{\mathrm{d}^3 k_A}{(2\pi)^3 2k_A^0} \frac{\mathrm{d}^3 q_A}{(2\pi)^3 2q_A^0} e^{i(k_A - q_A)\cdot x} f(k_A) f^*(q_A)}_{:=|\tilde{f}(x)|^2} \\ \times &\underbrace{\int \frac{\mathrm{d}^3 k_B}{(2\pi)^3 2k_B^0} \frac{\mathrm{d}^3 q_B}{(2\pi)^3 2q_B^0} e^{i(k_B - q_B)\cdot x} g(k_B) g^*(q_B)}_{:=|\tilde{g}(x)|^2} (2\pi)^4 \delta^{(4)}(p_f - p_A - p_B) \cdot |M(f, p_A, p_B)|^2 \end{aligned}$$

Using Fourier transformation $\tilde{g}(x) := \int \frac{\mathrm{d}^3 q}{(2\pi)^3 2q^0} e^{iq\cdot x} g(q)$

$$= \int \mathrm{d}^4 x |\tilde{f}(x)|^2 |\tilde{g}(x)|^2 (2\pi)^4 \delta^{(4)}(p_f - p_A - p_B) \cdot |M(f, p_A, p_B)|^2$$

note that $M(f, p_A, p_B)$ and $M(p_1, p_2, p_A, p_B)$ have different normalisation.
We now consider transition probability per unit volume per unit time:

$$\frac{\mathrm{d}\omega_{fi}}{\mathrm{d}V \mathrm{d}t} = (\text{incident flux}) \cdot (\text{target density}) \cdot \mathrm{d}\sigma$$

with $\mathrm{d}\sigma$ the infinitesimal cross section for scattering into final state $\langle f|$.
Product (incident flux) · (target density) denotes overlap of wave function. Necessary condition!

Covariant renormalization of states $\langle \boldsymbol{p}|\boldsymbol{q}\rangle \sim 2p^0\delta^3(\boldsymbol{p}-\boldsymbol{q})$ means the number of particles per unit volume is $2p_A^0|\tilde{f}(x)|^2$ and $2p_B^0|\tilde{g}(x)|^2$, respectively.

Assume

$$\bullet \xrightarrow{p_A} \bullet\ \boldsymbol{p}_B = 0$$

in target rest frame. Then $2p_B^0 = 2m_B$ and target density $= 2m_B|\tilde{g}(x)|^2$

Incident flux $= |\boldsymbol{v}_A| \cdot 2p_A^0|\tilde{f}(x)|^2 = 2|\boldsymbol{p}_A||\tilde{f}(x)|^2$ since $|\boldsymbol{v}_A| = |\boldsymbol{p}_A|/p_A^0$. Then

$$\mathrm{d}\sigma = (2\pi)^4\delta^{(4)}(p_f - p_A - p_B)\frac{1}{4m_B|\boldsymbol{p_A}|}|M(f, p_A, p_B)|^2$$

for $A + B \to 1 + 2$ processes

$$= \int_\Delta \frac{\mathrm{d}^3p_1}{(2\pi)^3 2p_1^0}\frac{\mathrm{d}^3p_2}{(2\pi)^3 2p_2^0}(2\pi)^4\delta^{(4)}(p_1 + p_2 - p_A - p_B)\frac{1}{4m_B|\boldsymbol{p_A}|}|M(p_1, p_2, p_A, p_B)|^2$$

with Δ energy-momentum resolution of 4-momentum of final state $|f\rangle$.

Covariant form of

$$m_B \cdot |\boldsymbol{p}_A| = m_B\sqrt{(p_A^0)^2 - m_A^2} = \sqrt{(p_A \cdot p_B)^2 - m_A^2 m_B^2} =: F \tag{4.5.1}$$

This is scattering into arbitrary final state subject to 4-momentum conservation: $p_A + p_B = p_1 + p_2$.

Consider now <u>differential</u> cross section for scattering into a particular infinitesimal solid angle $\mathrm{d}\Omega$, hence specific momentum $\mathrm{d}p_1, \mathrm{d}p_2$ variations:

$$\begin{aligned}
\mathrm{d}\sigma &= \frac{1}{4F}\prod_f \frac{\mathrm{d}^3p_f}{(2\pi)^3 2p_f^0}(2\pi)^4\delta^{(4)}(p_A + p_B - \sum_f p_f)|M|^2 \\
&\overset{f=1,2}{=} \frac{1}{4F}\frac{\mathrm{d}^3p_1}{(2\pi)^3 2p_1^0}\frac{\mathrm{d}^3p_2}{(2\pi)^3 2p_2^0}(2\pi)^4\delta^{(4)}(p_i - p_f)|M|^2 \\
&= \frac{1}{64\pi^2 F}\frac{\mathrm{d}^3p_1}{E_1}\frac{\mathrm{d}^3p_2}{E_2}\delta^{(4)}(p_1 + p_2 - p_i)|M|^2
\end{aligned}$$

$$\boxed{\begin{aligned}
&\int \tfrac{\mathrm{d}^3p_1}{E_1}\tfrac{\mathrm{d}^3p_2}{E_2}\delta^{(4)}(p_1 + p_2 - p_i) \\
&\overset{\text{CMS}}{=} \int \mathrm{d}|\boldsymbol{p}_1|\mathrm{d}\Omega_1 \tfrac{|\boldsymbol{p}_1|^2}{E_1E_2}\delta(E_1 + E_2 - E_i) \\
&= \int \mathrm{d}(E_1 + E_2)\tfrac{\mathrm{d}|\boldsymbol{p}_1|}{\mathrm{d}(E_1+E_2)}\mathrm{d}\Omega_1\tfrac{|\boldsymbol{p}_1|^2}{E_1E_2}\delta(E_1 + E_2 - E_i) \\
&= \tfrac{|\boldsymbol{p}_1|^2}{E_1E_2}\left(\tfrac{|\boldsymbol{p}_1|}{E_1} + \tfrac{|\boldsymbol{p}_1|}{E_2}\right)^{-1}\mathrm{d}\Omega_1 \\
&= \tfrac{|\boldsymbol{p}_1|\mathrm{d}\Omega_1}{E_1+E_2} = \tfrac{|\boldsymbol{p}_1|\mathrm{d}\Omega_1}{E_i}
\end{aligned}}$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2}\frac{|\boldsymbol{p}_1|}{F \cdot E_i}|M|^2 \tag{4.5.2}$$

Rewrite all kinematic factors in terms of $s = (p_A + p_B)^2 = (p_1 + p_2)^2$. Define the function

$$\lambda(x, y, z) := x^2 + y^2 + z^2 - 2(xy + xz + yz) \tag{4.5.3}$$

then

$$F = \sqrt{(p_A \cdot p_B)^2 - m_A^2 m_B^2} = \frac{1}{2}\lambda^{\frac{1}{2}}(s, m_A^2, m_B^2) = \sqrt{s}|\boldsymbol{p}_i|$$

$$\boxed{\begin{aligned} \lambda(s, m_A^2, m_B^2) &= s^2 - 2s(m_A^2 + m_B^2) - (m_A^2 - m_B^2)^2 = (s - (m_A + m_B)^2)(s - (m_A - m_B)^2) \\ &= (2p_A \cdot p_B - 2m_A \cdot m_B) \cdot (2p_A \cdot p_B + 2m_A \cdot m_B) = 4\left[(p_A p_B)^2 - m_A^2 m_B^2\right] \\ &\qquad \boxed{\begin{aligned} &p_A = (c\sqrt{s}, \boldsymbol{p}_i),\ c \in [0,1] && \to m_A^2 = c^2 s - |\boldsymbol{p}_i|^2 \\ &p_B = ((1-c)\sqrt{s}, -\boldsymbol{p}_i) && \to m_B^2 = (1-c)^2 s - |\boldsymbol{p}_i|^2 \end{aligned}} \\ &= 4\left[\left((c(1-c)s + p_i^2\right)^2 + (c^2 s - p_i^2)((1-c)^2 s - p_i^2)\right] = 4s|\boldsymbol{p}_i|^2 \end{aligned}}$$

$$|\boldsymbol{p}_f| = \sqrt{E_{1,2}^2 - m_{1,2}^2} = \frac{1}{2\sqrt{s}}\lambda^{\frac{1}{2}}(s, m_1^2, m_2^2)$$
$$E_i = \sqrt{s}$$

$$\frac{d\sigma}{d\Omega_{CMS}} = \frac{1}{64\pi^2 s}\frac{|\boldsymbol{p}_f|}{|\boldsymbol{p}_i|}|M|^2 = \frac{1}{64\pi^2 s}\sqrt{\frac{\lambda(s, m_1^2, m_2^2)}{\lambda(s, m_A^2, m_A^2)}}|M|^2 \tag{4.5.4}$$

Decay rate instead of cross section means no "incident flux" to divide by, only "target density"

$$d\Gamma = \frac{1}{2m_A}\prod_f \frac{d^3 p_f}{(2\pi)^3 2p_f^0}(2\pi)^4\delta^{(4)}(p_A - \sum_f p_f)|M|^2 \tag{4.5.5}$$

Particles with spin (unpolarized): sum over outgoing or average over initial spins

$$|M|^2 \to \frac{1}{(2s_A + 1)(2s_B + 1)}\sum_{s_i, s_f}|M_{fi}|^2 \tag{4.5.6}$$

Symmetry factor $|M|^2 \to \frac{1}{s}|M|^2$ with $s = \prod_i k_i!$ if there are k_i identical particles of species i in the final states.

If 1 and 2 are identical, then factor $\frac{1}{s} = \frac{1}{2}$ on the right hand side.

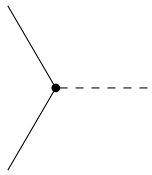
4.6 Feynman rules for fermions

Consider the simplest interacting theory with fermions, Yukawa-theory. We will treat QED later.

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{M^2}{2}\phi^2 + \bar{\psi}(i\partial\!\!\!/ - m)\psi - g\bar{\psi}\psi\phi \tag{4.6.1}$$

Feynman rules will involve:

- scalar x •- - - - - - -• y $= D_F(x-y) = \int \frac{d^4}{(2\pi)^2}\frac{i}{p^2 - M^2 + i\epsilon}e^{-ip(x-y)}$
- fermions x, α •——◀——• y, β $= S_F(x-y)_{\alpha\beta} = \int \frac{d^4}{(2\pi)^4}\frac{i(p\!\!\!/ + m)}{p^2 - m^2 + i\epsilon}e^{-ip(x-y)}$

- vertices  $= -ig \int \mathrm{d}^4 x$

What previous steps need reconsideration due to the anti-commutating fermion operators? Interaction Hamiltonian $\sim \bar{\psi}\psi\phi$ and in general compose of even number of fermion fields (spin conservation and fermion number conservation). Thus there is no problem with time-ordered exponential in definition of S-matrix. (Time ordering always takes two or even number of fields.)

Remember the relation

$$T(\psi_\alpha(x)\bar{\psi}_\beta(x)) = -\bar{\psi}_\beta(x)\psi_\alpha(x) \text{ when } y^0 > x^0 \tag{4.6.2}$$

Similarly in normal product:

$$:\psi^+\psi^-: = -\psi^-\psi^+ \tag{4.6.3}$$

Then Wick's theorem is formally the same as before

$$T(\psi_\alpha(x)\bar{\psi}_\beta(x)) =: \psi_\alpha(x)\bar{\psi}_\beta(x) : + \overbracket{\psi_\alpha(x)\bar{\psi}_\beta}(x)$$

note by definition $\overbracket{\psi\psi} = \overbracket{\bar{\psi}\bar{\psi}} = 0$

Thus contractions inside normal-ordered products would be

$$:\psi_1\psi_2\bar{\psi}_3\bar{\psi}_4: = -\overbracket{\psi_1\bar{\psi}_3} : \psi_2\bar{\psi}_4 : = -S_F(x_1 - x_3) : \psi_2\bar{\psi}_4 :$$

because of the additional operator exchange.

We will want to consider fermion-(anti-)fermion scattering. Leading contribution at $O(g^2)$:

$$\frac{1}{2!}(-ig)^2 \int \mathrm{d}^4x \mathrm{d}^4y \, \langle p', k' | T\bar{\phi}(x)\phi(x)\phi(x)\bar{\phi}(y)\phi(y)\phi(y) | p, k \rangle$$

Contractions with initial-/final-state fermions?

$$\begin{aligned} \phi^+(x)\,|p, s\rangle &= \int \frac{\mathrm{d}^3k}{(2\pi)^3\sqrt{2E_k}} \sum_r a_k^r u_r(k) e^{-ik\cdot x} \sqrt{2E_p} a_p^{s\dagger}\,|0\rangle \\ &= e^{-ip\cdot x} u_s(p)\,|0\rangle \end{aligned}$$

So define

$$\begin{aligned} \overbracket{\psi(x)\,|p}, s\rangle &= e^{-ip\cdot x} u_s(p) \\ \langle \overbracket{p, s|\,\bar{\psi}}(x) &= e^{ip\cdot x} \bar{u}_s(p) \end{aligned} \tag{4.6.4}$$

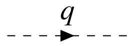
Note, though, for anti-fermion states $|p', s'\rangle$:

$$\begin{aligned} \overbracket{\bar{\psi}(x)\,|p}, s\rangle &= e^{-ip'\cdot x} \bar{v}_{s'}(p') \\ \langle \overbracket{p', s'|\,\psi}(x) &= e^{ip'\cdot x} v_{s'}(p') \end{aligned} \tag{4.6.5}$$

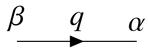
In short $\overbracket{\psi|}\,\rangle$ contracts with a fermion, $\langle\overbracket{|\psi}$ with an anti-fermion; vice verse for $\bar{\psi}$.

Momentum space Feynman rule for iM

- internal propagators

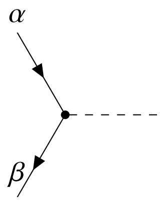


$$= \frac{i}{q^2 - M^2 + i\epsilon}$$

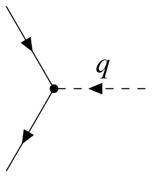


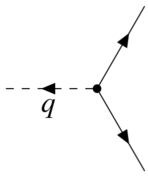
$$= \frac{i(\not{p} + m)_{\alpha\beta}}{p^2 - m^2 + i\epsilon}$$

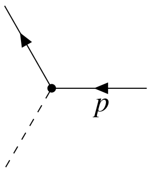
- vertex

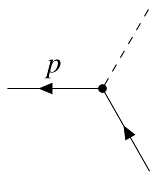


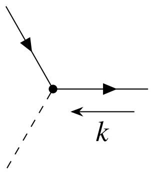
$$= -ig \int \mathrm{d}^4 x = ig\delta_{\beta\alpha}$$

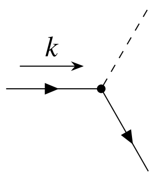
- external lines
  - "in" and "out" scalar 

$=$ 

$= 1$
  - "in"-fermion 

$= u_s(p)$ and "out" fermion 

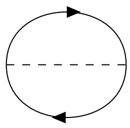
$= \bar{u}_s(p)$
  - "in"-anti-fermion 

$= \bar{v}_s(k)$ and "out" anti-fermion 

$= v_s(k)$

- impose energy-momentum conservation at each vertex
- integrate over undetermined (loop) momenta
- include an overall sign for the diagram

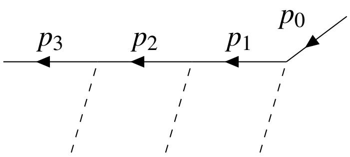
Note

- Arrows on the fermion lines by convention denote fermion (or charge) flow. They must flow consistently through the diagram. (≡ fermion number conservation) (Only potential confusion: external anti-fermion lines)
- No symmetry factors (except vacuum bubbles  $\frac{1}{s} = \frac{1}{2}$). $\bar{\psi}\psi\phi$ allows for unambiguous contractions.

- Dirac indices are summed over at each vertex

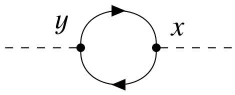
$$\mathcal{L}_{\text{int}} \approx \bar{\psi}_\alpha(x)\psi_\alpha(x)\phi(x)$$

$(\not{p} + m)$ terms in propagator are matrix-multiplied contracted with external spinors, e.g.



$$\sim \bar{u}_\alpha(p_3)\frac{i(\not{p}+m)_{\alpha\beta}}{p_2^2 - m^2 + i\epsilon}\frac{i(\not{p}+m)_{\beta\gamma}}{p_1^2 - m^2 + i\epsilon}u_\gamma(p_0)$$

- closed fermion loop



$$\sim \bar{\psi}_\alpha(x)\psi_\alpha(x)\bar{\psi}_\beta(y)\psi_\beta(y) = -\psi_\alpha(x)\bar{\psi}_\beta(y)\psi_\alpha(x)\bar{\psi}_\beta(y)$$

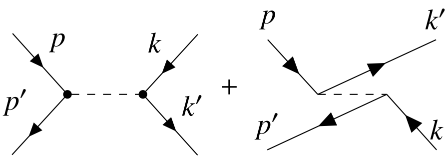
$$= -S_F(y-x)_{\beta\alpha}S_F(x-y)_{\alpha\beta} = -\mathrm{Tr}\left(S_F(y-x)S_F(x-y)\right)$$

It always (also with more propagators/couplings) involves an overall (-1) and a trace $\mathrm{Tr}(\dots)$.

Examples

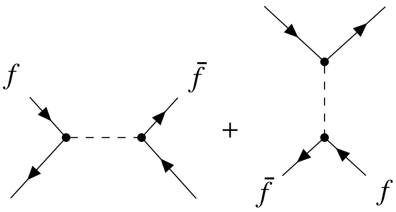
- fermion-fermion scattering to lowest order $O(g^2)$

$iM =$



$$= (-ig)^2\left\{\bar{u}(p')u(p)\frac{i}{\underbrace{(p'-p)^2}_{t} - M^2 + i\epsilon}\bar{u}(k')u(k) - \bar{u}(p')u(k)\frac{i}{\underbrace{(p'-k)^2}_{u} - M^2 + i\epsilon}\bar{u}(k')u(p)\right\}$$

- fermion-anti-fermion scattering



These are tree diagrams. Thus there is no undetermined momenta to integrate.

5 Quantum Electrodynamics (QED)

5.1 Classical Electrodynamics and Maxwell's equations

We have the gauge potential $A^\mu = (A^0, \boldsymbol{A}) = (\phi, \boldsymbol{A})$ (or $A_\mu = (A^0, -\boldsymbol{A}) = (\phi, -\boldsymbol{A})$) and the field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

Then

- electric field is
  $E_i = F_{0i} = \partial_0 A_i - \partial_i A_0 \to \boldsymbol{E} = -\dot{\boldsymbol{A}} - \boldsymbol{\nabla}\phi$
- magnetic field is
  $B^i = -\frac{1}{2}\epsilon^{ijk} F_{jk} \to \boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$

Lagrangian density $\mathcal{L}_{EM} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{2}(\boldsymbol{E} \cdot \boldsymbol{E} - \boldsymbol{B} \cdot \boldsymbol{B})$. The field equation $\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} \right) - \frac{\partial \mathcal{L}}{\partial A_\nu} = 0$ leads to

$$\partial_\mu F^{\mu\nu} = 0 \tag{5.1.1}$$

It is half of Maxwell's equations (in vacuum).

The other half is Bianchi identities following from the definition of $F_{\mu\nu}$:

$$\begin{aligned} &\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0 \;\Leftrightarrow\; \epsilon^{\sigma\lambda\mu\nu} \partial_\lambda F_{\mu\nu} = 0 \\ \text{or} \quad &\partial_\lambda \tilde{F}^{\sigma\lambda} = 0, \; \tilde{F}^{\sigma\lambda} = \frac{1}{2} \epsilon^{\sigma\lambda\mu\nu} F_{\mu\nu} \end{aligned}$$

In terms of $\boldsymbol{E}$ and $\boldsymbol{B}$:

$$\begin{aligned} &\boldsymbol{\nabla} \cdot \boldsymbol{E} = 0, \; \dot{\boldsymbol{E}} = \boldsymbol{\nabla} \times \boldsymbol{B} && \text{dynamical equations} \\ &\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0, \; \dot{\boldsymbol{B}} = -\boldsymbol{\nabla} \times \boldsymbol{E} && \text{Bianchi identities} \end{aligned}$$

Remarks

- Lagrangian density does not depend on $\dot{A}_0$, since A_0 is not really dynamical.

  $$\boldsymbol{\nabla} \cdot \boldsymbol{E} = 0 \to \boldsymbol{\nabla}^2 A_0 + \boldsymbol{\nabla} \cdot \dot{\boldsymbol{A}} = 0$$

  Solve this Poisson equation for $A_0(\boldsymbol{x}, t) = \frac{1}{4\pi} \int \mathrm{d}^3 y \frac{\boldsymbol{\nabla} \cdot \dot{\boldsymbol{A}}(\boldsymbol{y}, t)}{|\boldsymbol{y} - \boldsymbol{x}|}$. Thus A_0 is given in terms of the other components of A.

- Gauge invariance ensures field strength tensor invariant under the transformation $A_\mu \longmapsto A_\mu - \partial_\mu \mathrm{X}$ due to commuting derivatives. This leads to gauge invariance of Maxwell equations.
  Choose X to satisfy $\partial_\mu \partial^\mu \mathrm{X} = \partial^2 \mathrm{X} = \partial_\mu A^\mu$ allows us to demand the condition (Lorenz condition)

  $$\partial_\mu A^\mu = 0 \tag{5.1.2}$$

  such that A_μ belongs to the "Lorenz gauge" and reduces the degrees of freedom from 4 to 3.

- Further freedom is eliminated by adding any X with $\partial^2 X = 0$, e.g. $\partial_t X = A_0$. Then we get the Coulomb or radiation gauge

$$A_0 = 0,\ \nabla \cdot \boldsymbol{A} = 0 \tag{5.1.3}$$

Note: vice versa imposing $\nabla \cdot \boldsymbol{A} = 0$ first, yields $A_0 = 0$.
In Coulomb gauge:

$$\begin{aligned}
&\boldsymbol{E} = -\dot{\boldsymbol{A}},\ \boldsymbol{B} = \nabla \times \boldsymbol{A},\ \nabla \times \boldsymbol{A} = 0 \\
&-\ddot{\boldsymbol{A}} = \dot{\boldsymbol{E}} \overset{\text{Maxwell}}{=} \nabla \times \boldsymbol{B} = \nabla \times (\nabla \times \boldsymbol{A}) = \nabla(\underbrace{\nabla \cdot \boldsymbol{A}}_{=0}) - \nabla^2 \boldsymbol{A} \\
&\Rightarrow \partial^2 \boldsymbol{A} = 0
\end{aligned}$$

This wave equation is massless KG equation for each spatial component.

Then the solutions are obvious: $\boldsymbol{A} = \boldsymbol{\epsilon} e^{-ik\cdot x}$ with $k^2 = 0$ and $\boldsymbol{\epsilon} \cdot \boldsymbol{k} = 0$. The polarization vector $\boldsymbol{\epsilon}$ is transverse to $\boldsymbol{k}$.

Can write the lagrangian in Coulmb gauge

$$\mathcal{L}_{\text{EM}} = \frac{1}{2}\dot{\boldsymbol{A}}\dot{\boldsymbol{A}} - \frac{1}{2}\boldsymbol{B} \cdot \boldsymbol{B}$$

Then the conjugate momentum to $\boldsymbol{A}$ is $\boldsymbol{\Pi} = \frac{\partial \mathcal{L}}{\partial \dot{A}} = \dot{\boldsymbol{A}} = -\boldsymbol{E}$. It has only 3 components, there is no conjugate momentum to A_0!. Because of Coulomb gauge $\boldsymbol{\Pi}$ is subject to the constraint $\nabla \cdot \boldsymbol{\Pi} = 0$

Hamiltonian

$$H_{\text{EM}} = \int \mathrm{d}^3 x \left(\frac{1}{2}\boldsymbol{\Pi} \cdot \boldsymbol{\Pi} + \frac{1}{2}\boldsymbol{B} \cdot \boldsymbol{B} \right)$$

5.2 Quantizing the Maxwell field

We would like to impose canonical commutation relations, à la

$$\begin{aligned}
&[A_i(\boldsymbol{x}), A_j(\boldsymbol{y})] = [\Pi_i(\boldsymbol{x}), \Pi_j(\boldsymbol{y})] = 0 \\
&[A_i(\boldsymbol{x}), \Pi_j(\boldsymbol{y})] = i\delta_{ij}\delta^{(3)}(\boldsymbol{x} - \boldsymbol{y})
\end{aligned}$$

However this cannot be true. Take either derivative of the last equation and it needs to vanish due to $\nabla \cdot \boldsymbol{A} = \nabla \cdot \boldsymbol{\Pi} = 0$. But

$$[\partial^i A_i(\boldsymbol{x}), \Pi_k(\boldsymbol{y})] = i\delta_{ij}\partial^i \delta^{(3)}(\boldsymbol{x} - \boldsymbol{y})$$

here the derivative is taken with respect to $\boldsymbol{x}$, i.e. $\partial^i = \frac{\partial}{\partial x_i}$.

Replace δ_{ij} by Δ_{ij}

$$\begin{aligned}
[\partial^i A_i(\boldsymbol{x}), \Pi_j(\boldsymbol{y})] &= i\Delta_{ij}\partial^i \frac{1}{(2\pi)^3}\int \mathrm{d}^3 k e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})} \\
&= -\frac{1}{(2\pi)^3}\int \mathrm{d}^3 k (k^i \Delta_{ij}) e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})} \overset{!}{=} 0
\end{aligned}$$

It works for $\Delta_{ij} = \delta_{ij} - \frac{k_i k_j}{\boldsymbol{k}^2}$ in momentum space or $\Delta_{ij} = \delta_{ij} - \nabla^{-2}\partial_i \partial_j$ in position space.

$$[A_i(\boldsymbol{x}), \Pi_j(\boldsymbol{y})] = i\left(\delta_{ij} - \boldsymbol{\nabla}^{-2}\partial_i\partial_j\right)\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}) \tag{5.2.1}$$

As before we have the mode expansion

$$\begin{aligned}
\boldsymbol{A}(\boldsymbol{x}) &= \int \frac{\mathrm{d}^3k}{(2\pi)^3\sqrt{2|\boldsymbol{k}|}}\left(\boldsymbol{a}_{\boldsymbol{k}}e^{i\boldsymbol{k}\cdot\boldsymbol{x}} + \boldsymbol{a}^\dagger_{\boldsymbol{k}}e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\right)\\
\boldsymbol{\Pi}(\boldsymbol{x}) &= \int \frac{\mathrm{d}^3k}{(2\pi)^3}(-i)\sqrt{\frac{|\boldsymbol{k}|}{2}}\left(\boldsymbol{a}_{\boldsymbol{k}}e^{i\boldsymbol{k}\cdot\boldsymbol{x}} - \boldsymbol{a}^\dagger_{\boldsymbol{k}}e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\right)
\end{aligned}$$

with $\boldsymbol{k}\cdot\boldsymbol{a}_{\boldsymbol{k}} = \boldsymbol{k}\cdot\boldsymbol{a}^\dagger_{\boldsymbol{k}} = 0$.

Introduce two orthogonal polarization vectors $\boldsymbol{\epsilon}^{(1)}(\boldsymbol{k})$ and $\boldsymbol{\epsilon}^{(2)}(\boldsymbol{k})$ for each $\boldsymbol{k}$.

$$\begin{aligned}
\boldsymbol{a}_{\boldsymbol{k}} &= a^{(1)}_{\boldsymbol{k}}\boldsymbol{\epsilon}^{(1)} + a^{(2)}_{\boldsymbol{k}}\boldsymbol{\epsilon}^{(2)} = \sum_{\lambda=1}^{2} a^{(\lambda)}_{\boldsymbol{k}}\boldsymbol{\epsilon}^{(\lambda)}(\boldsymbol{k})\\
\text{with}\quad & \boldsymbol{k}\cdot\boldsymbol{\epsilon}^{(1)}(\boldsymbol{k}) = \boldsymbol{k}\cdot\boldsymbol{\epsilon}^{(2)}(\boldsymbol{k}) = 0,\ \boldsymbol{\epsilon}^{(\lambda)}\cdot\boldsymbol{\epsilon}^{(\lambda;)} = \delta_{\lambda\lambda'}
\end{aligned}$$

Creation and annihilation operator have the standard commutation relations

$$[a^{(\lambda)}_{\boldsymbol{k}}, a^{(\lambda')\dagger}_{\boldsymbol{k}'}] = (2\pi)^3\delta_{\lambda\lambda'}\delta^{(3)}(\boldsymbol{k}-\boldsymbol{k}') \tag{5.2.2}$$

and all other commutators vanish. Geometrically, still possible to write including the unphysical longitudinal components:

$$\begin{aligned}
[\boldsymbol{a}_{\boldsymbol{k}}, \boldsymbol{a}_{\boldsymbol{l}}] &= [\boldsymbol{a}^\dagger_{\boldsymbol{k}}, \boldsymbol{a}^\dagger_{\boldsymbol{l}}] = 0\\
[a^i_{\boldsymbol{k}}, a^{j\dagger}_{\boldsymbol{l}}] &= (2\pi)^3\left(\delta^{ij} - \frac{k^ik^j}{\boldsymbol{k}^2}\right)\delta^{(3)}(\boldsymbol{k}-\boldsymbol{l})
\end{aligned}$$

$a^{(\lambda)}_{\boldsymbol{k}}$ and $a^{(\lambda)\dagger}_{\boldsymbol{k}}$ create and destroy photons of momentum $\boldsymbol{k}$, energy $|\boldsymbol{k}|$ and (electric) polarization along $\boldsymbol{\epsilon}^{(\lambda)}(\boldsymbol{k})$.

Next steps are analogous to KG theory.

Hamiltonian

$$H = \frac{1}{2}\int \mathrm{d}^3x\left(\boldsymbol{E}^2 + \boldsymbol{B}^2\right) = \frac{1}{2}\int \mathrm{d}^3x\left(\dot{\boldsymbol{A}}^2 + (\boldsymbol{\nabla}\times\boldsymbol{A})\cdot(\boldsymbol{\nabla}\times\boldsymbol{A})\right)$$

using identity $\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C}) = \boldsymbol{B}\cdot(\boldsymbol{C}\times\boldsymbol{A})$

$$= \frac{1}{2}\int \mathrm{d}^3x\left(\dot{\boldsymbol{A}}^2 + \boldsymbol{A}\cdot\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{A})\right)$$

using the identity $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{A} - \boldsymbol{\nabla}^2\boldsymbol{A})$

$$= \frac{1}{2}\int \mathrm{d}^3x\left(\dot{\boldsymbol{A}}^2 - \boldsymbol{A}\cdot\boldsymbol{\nabla}^2\boldsymbol{A} + \boldsymbol{A}\cdot\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{A})\right)$$

using coulomb gauge condition

$$= \frac{1}{2}\int \mathrm{d}^3x\left(\dot{\boldsymbol{A}}^2 - \boldsymbol{A}\cdot\nabla^2\boldsymbol{A}\right)$$

the first term vanishes and use normal ordering

$$= \int \frac{\mathrm{d}^3k}{(2\pi)^3}|\boldsymbol{k}|\,\boldsymbol{a}^\dagger_{\boldsymbol{k}}\cdot\boldsymbol{a}_{\boldsymbol{k}} = \sum_{\lambda=1}^{2}\int \frac{\mathrm{d}^3k}{(2\pi)^3}|\boldsymbol{k}|\,a^{(\lambda\dagger)}_{\boldsymbol{k}}a^{\lambda}_{\boldsymbol{k}}$$

Heisenberg field

$$\boldsymbol{A}(\boldsymbol{x},t) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\sqrt{2|\boldsymbol{k}|}}\left(\boldsymbol{a}_{\boldsymbol{k}}e^{-ik\cdot x} + \boldsymbol{a}_{\boldsymbol{k}}^{\dagger}e^{ik\cdot x}\right)$$

Photon propagator

$$\langle 0|TA_i(x)A_j(y)|0\rangle =: D_{ij}^{\mathrm{tr}}(x-y) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{i}{k^2+i\epsilon}\left(\delta_{ij} - \frac{k_ik_j}{|\boldsymbol{k}|^2}\right)e^{-ik\cdot(x-y)} \tag{5.2.3}$$

tr stands for transverse: photon polarization perpendicular to its momentum. This is NOT the final version of the photon propagator!

5.3 Inclusion of matter - QED

$$\mathcal{L}_{\mathrm{QED}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\not{D} - m)\psi \tag{5.3.1}$$

where $D_\mu = \partial_\mu + ieA_\mu$ is the (gauge) covariante derivative

$$= \mathcal{L}_{\mathrm{EM}} + \mathcal{L}_D - e\underbrace{\bar{\psi}\gamma^\mu\psi}_{j^\mu}A_\mu \tag{5.3.2}$$

Field equations would be

$$\partial_\mu F^{\mu\nu} = ej^\nu \qquad (i\not{D} - m)\psi = 0$$

where ej^ν is the electromagnetic 4-current.

Gauge invariance under the transformation

$$\begin{cases}\psi(x) \longmapsto \psi'(x) = e^{ie\chi(x)}\psi \\ A_\mu(x) \longmapsto A'_\mu(x) = A_\mu(x) - \partial_\mu\chi(x)\end{cases}$$

To check the consistence: cavariant derivative transforms like $D_\mu \longmapsto D'_\mu\psi'(x) = e^{ie\chi(x)}D_\mu\psi(x)$. Since the adjoint spinor transforms like $\bar{\psi}(x) \longmapsto \bar{\psi}'(x) = \bar{\psi}(x)e^{-ie\chi(x)}$, the Lagrangian and field equations are gauge invariant.

Again we choose Coulomb gauge $\boldsymbol{\nabla}\cdot\boldsymbol{A} = 0$, then equation for A^0:

$$\begin{aligned}\partial_i F^{i0} &= ej^0 \\ \Rightarrow -\boldsymbol{\nabla}^2 A^0 &= ej^0 = e\bar{\psi}\gamma^0\psi \\ &= e\bar{\psi}\gamma^0\psi = e\psi^\dagger\psi \\ &= e\rho(x) \\ A^0(\boldsymbol{x},t) &= e\int \mathrm{d}^3y \frac{\rho(\boldsymbol{y},t)}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\end{aligned} \tag{5.3.3}$$

We want to derive the interaction Hamiltonian. Note

$$\int \mathrm{d}^3x \frac{1}{2}\boldsymbol{E}^2 = \int \mathrm{d}^3x\frac{1}{2}\left(\dot{\boldsymbol{A}} + \boldsymbol{\nabla}A^0\right)^2$$

cross terms vanish after integration by parts due to $\boldsymbol{\nabla} \cdot \dot{\boldsymbol{A}} = 0$

$$= \int \mathrm{d}^3 x \frac{1}{2} \left(\dot{\boldsymbol{A}}^2 + (\boldsymbol{\nabla} A^0)^2 \right)$$

$$= \int \mathrm{d}^3 x \frac{1}{2} \left(\dot{\boldsymbol{A}}^0 - A^0 \boldsymbol{\nabla}^2 A^0 \right)$$

$-e j^0 = -e\rho$

$$= \int \mathrm{d}^3 x \frac{1}{2} \dot{\boldsymbol{A}}^2 + \underbrace{\frac{e^2}{2} \int \mathrm{d}^3 x \mathrm{d}^3 y \frac{\rho(\boldsymbol{x})\rho(\boldsymbol{y})}{4\pi |\boldsymbol{x} - \boldsymbol{y}|}}_{= \frac{e^2}{2} j^0 A_0}$$

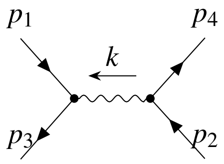
Combined Hamiltonian

$$\begin{aligned} H = \int \mathrm{d}^3 x \left\{ \frac{1}{2} \boldsymbol{\Pi} \cdot \boldsymbol{\Pi} + \frac{1}{2} \boldsymbol{B} \cdot \boldsymbol{B} + i \bar{\psi} \boldsymbol{\gamma} \cdot \boldsymbol{\nabla} \psi + m \bar{\psi} \psi \right\} & \quad \text{free photon and fermion} \\ + \frac{e^2}{2} \int \mathrm{d}^3 x \mathrm{d}^3 y \frac{\rho(\boldsymbol{x})\rho(\boldsymbol{y})}{4\pi |\boldsymbol{x} - \boldsymbol{y}|} - e \int \mathrm{d}^3 x \boldsymbol{j} \cdot \boldsymbol{A} & \quad \text{interactions} \end{aligned}$$

where $\rho = \psi^\dagger \psi = \bar{\psi} \gamma^0 \psi$, $\boldsymbol{j} = \bar{\psi} \boldsymbol{\gamma} \psi$ for 2 types of interactions.

5.4 Lorentz-invariant propagator

Consider $e^- e^-$ scattering at $O(e^2)$



$$k = p_3 - p_1 = p_2 - p_4$$

We expect this to involve

- spinors for external fermions
- $-ie\gamma^\mu$
- Photon propagator $D_{\mu\nu}(x - y)$

What we have found in Coulomb gauge is actually

- vertices $ie\gamma^i$, transverse propagator $D^{\mathrm{tr}}_{\mu\nu}(x - y)$
- vertices $\pm ie\gamma^0$, instantaneous Coumlomb interaction $\frac{1}{4\pi |\boldsymbol{x} - \boldsymbol{y}|} \delta(x^0 - y^0)$

Effectively combine these propagators terms into $D^{\mathrm{Coul}}_{\mu\nu}(x - y)$, where the $D^{\mathrm{Coul}}_{00}(x - y) = \frac{1}{4\pi |\boldsymbol{x} - \boldsymbol{y}|} \delta(x^0 - y^0)$. This component in momentum space is simply

$$\int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{e^{i \boldsymbol{k} \cdot \boldsymbol{r}}}{|\boldsymbol{k}|^2} = \frac{1}{4\pi |\boldsymbol{r}|}$$

Therefore Coumlomb propagator in momentum space:

$$D_{\mu\nu}^{\text{Coul}}(k) = \begin{cases} \frac{i}{k^2+i\epsilon}\left(\delta_{ij} - \frac{k_i k_j}{|\boldsymbol{k}|^2}\right) & \mu = i, \nu = j \\ \frac{i}{|\boldsymbol{k}|^2} & \mu = \nu = 0 \\ 0 & \text{otherwise} \end{cases}$$

Consider contraction to scattering amplitude from vertex at x:

$$\sim e\bar{u}(p_3)\gamma^\mu u(p_1)e^{i(p_3-p_1)x}$$

current conservation $\partial_\mu j^\mu = 0$ written in momentum space

$$\underbrace{(p_3 - p_1)_\mu}_{k_\mu}\bar{u}(p_3)\gamma^\mu u(p_1) = 0$$

so in the complete diagram $D_{\mu\nu}^{\text{Coul}}$ occurs in a form

$$\begin{aligned} &a^\mu D_{\mu\nu}^{\text{Coul}}(k)b^\nu \\ =&a^0\frac{i}{|\boldsymbol{k}|^2}b^0 + a^i\left[\frac{i}{k^2+i\epsilon}\left(\delta_{ij} - \frac{k_i k_j}{|\boldsymbol{k}|^2}\right)\right]b^j \end{aligned}$$

where $k^\mu a^\mu = 0, k_\mu a^\mu = 0$

$$\begin{aligned} =&i\left[\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{k^2}\underbrace{-\frac{k_0^2 a_0 b_0}{k^2|\boldsymbol{k}|^2} + \frac{a_0 b_0}{|\boldsymbol{k}|^2}}_{=\frac{-k_0^2a_0b_0+a_0b_0(k_0^2-|\boldsymbol{k}|^2)}{k^2|\boldsymbol{k}|}^2}\right] \\ =&\frac{i}{k^2}(\boldsymbol{a}\cdot\boldsymbol{b} - a_0b_0) = -\frac{i}{k}a_\mu b^\mu \end{aligned}$$

Conclusion in this diagram (and in fact, in general), we may replace the $D_{\mu\nu}^{\text{Coul}}(k)$ by the manifestly Lorentz covariant propagator

$$D_{\mu\nu}(k) = -\frac{ig_{\mu\nu}}{k+i\epsilon} \tag{5.4.1}$$

This can be generalised to

$$D_{\mu\nu}(k) = -\frac{i}{k^2+i\epsilon}\left(g_{\mu\nu} - (1-\alpha)\frac{k_\mu k_\nu}{k^2}\right) \tag{5.4.2}$$

as, by current consevation, additional term doesn't contribute.

Feynman gauge $\alpha = 1$; Landau gauge $\alpha = 0$.

Remark one can also try to quantise photons in a manifestly covariant way, imposing Lorentz gauge $\partial_\mu A^\mu = 0$

$$[A_\mu(\boldsymbol{x}), \Pi_\nu(\boldsymbol{y})] = ig_{\mu\nu}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})$$

This is trouble since $\Pi^0 = \frac{\partial\mathcal{L}}{\partial\dot{A}_0} = 0$. This cannot hold!

We thus change the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\alpha}(\partial_\mu A^\mu)^2$$

with "gauge fixing term". The equation of motion from this is

$$\partial^2 A^\mu - \left(1 - \frac{1}{\alpha}\right)\partial^\mu(\partial_\lambda A^\lambda) = 0$$

e.g. $\alpha = 1$ is the Feynman gauge.

With this Lagrangian we can the 0th component of conjugate momentum

$$\Pi^0 = -\frac{1}{\alpha}\partial_\mu A^\mu$$

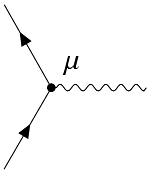
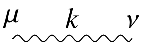
but this seems as bad as before!

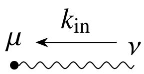
We cannot impose Coulomb gauge condition $\partial_\mu A^\mu = 0$ as an operator identity. Instead demand a weaker condition $\langle \text{out}|\partial_\mu A^\mu|\text{in}\rangle = 0$ for all physical states.

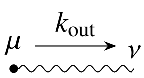
This in turn tells us which states are actually physical. The 4 polarisation states consist of physical, timelike(scalar) and longitudinal states. The negative-norm states cancel each ther out (Gopta-Bleuler formalism).

Feynman rules for QED diagrams constucted from electron (positron) and photon; rules for fermions are valid as before.

In addition

- vertex  $= -ie\gamma^\mu$;
- photon propagator  $= -\frac{ig_{\mu\nu}}{k^2+i\epsilon}$
- external photons

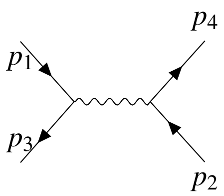
 $= \epsilon_\mu$

 $= \epsilon^*_\nu$

ϵ_μ polarisation vector of in/out photon and ϵ^*_μ for out photon required for complex (circular) polarisation.

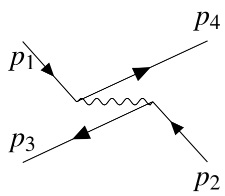
5.5 QED process at tree level

Example $e^- e^- \to e^- e^-$



$$\sim \frac{1}{(p_1 - p_3)^2} = \frac{1}{t}$$

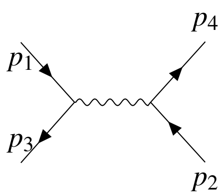
t-channel



$$\sim \frac{1}{(p_1 - p_4)^2} = \frac{1}{u}$$

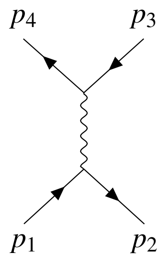
u-channel

Example $e^- e^+ \to e^- e^+$



$$\sim \frac{1}{(p_1 - p_3)^2} = \frac{1}{t}$$

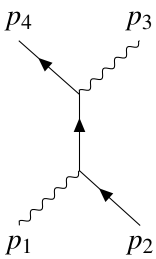
t-channel



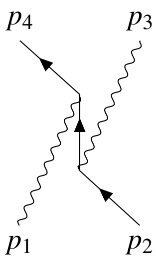
$$\sim \frac{1}{(p_1 + p_2)^2} = \frac{1}{s}$$

s-channel

Compton scattering $\gamma e^- \to \gamma e^-$

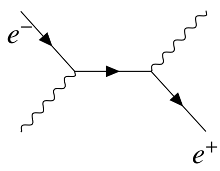


s-channel

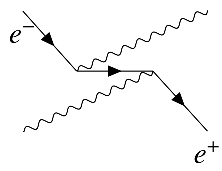


u-channel

Example $e^+ e^- \to \gamma\gamma$



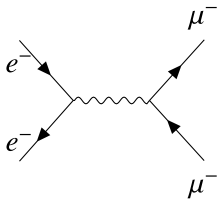
t-channel

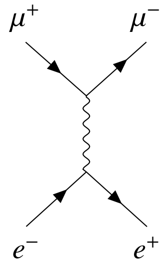


u-channel

These are important for lifetime of positronium.

All these amplitudes are $O(e^2)$, $\alpha = \frac{e^2}{4\pi} = \frac{1}{137.036}$ the fine structure constant.
Muons $\mu^\pm$, like electrons, just ca. 200 times heavier.

$e^-\mu^- \to e^-\mu^- \Rightarrow$  Just one diagram

$e^+e^- \to \mu^+\mu^- \Rightarrow$  Just one diagram

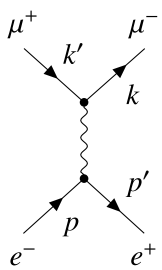
$\mu^\pm$ decay into $e^\pm$ and neutrinos in weak interactions.

For tree level diagrams the photon propagator does not need to have the $i\epsilon$ in the denominator, since we will never be able to see a singularity/pole.

5.5.1 Some hints and tricks for cross section calculations

See application in exercises!

Example $e^+e^- \to \mu^+\mu^-$



$$\begin{aligned} iM &= \bar{v}_e^{s'}(-ie\gamma^\mu)u_e^s(p)\left.\frac{-ig_{\mu\nu}}{s}\right|_{s=q^2}\bar{u}_\mu^r(k)(-ie\gamma^\nu)v_\mu^{r'}(k') \\ &= \frac{ie^2}{s}\left(\bar{v}_e(p')\gamma^\mu u_e(p)\right)\left(\bar{u}_\mu(k)\gamma_\mu v_\mu(k')\right) \end{aligned}$$

See section 4.5, $|M|^2$ is needed for cross section. M^* involves things like

$$\begin{aligned} (\bar{v}\gamma^\mu u)^* &= (\bar{v}\gamma^\mu u)^\dagger = u^\dagger\gamma^{\mu\dagger}\gamma_0^\dagger v \\ &= u^\dagger\gamma_0\gamma^\mu\gamma_0\gamma_0 v = \bar{u}\gamma^\mu v \end{aligned}$$

So

$$|M|^2 = \frac{e^4}{s^2}\left[\bar{v}(p')\gamma^\mu u(p)\bar{u}(p)\gamma^\nu v(p')\right]_{e^\pm}\cdot\left[\bar{u}(k)\gamma_\mu v(p)\bar{v}(k')\gamma_\nu u(k)\right]_{\mu^\pm}$$

Unpolarized scattering= $\frac{1}{4}\sum_{r,s,r',s'}|M|^2$.

Now $\bar{v}\gamma^\mu u$, $\bar{u}\gamma^\nu v$ etc. are scalars in Dirac/spinor space:

$$\begin{aligned}
&\sum_{s,s'} \bar{v}_{s'} p' \gamma^\mu u_s(p) \bar{u}_s(p) \gamma^\nu v_{s'}(p') \\
\text{(taking trace of scalar)} \ &= \sum_{s,s'} \mathrm{Tr}\left(\bar{v}_{s'} p' \gamma^\mu u_s(p) \bar{u}_s(p) \gamma^\nu v_{s'}(p')\right) \\
&= \sum_{s,s'} \mathrm{Tr}\left(v_{s'}(p') \bar{v}_{s'}(p') \gamma^\mu u_s(p) \bar{u}_s(p) \gamma^\nu\right)
\end{aligned}$$

using spin sums

$$= \mathrm{Tr}\left((\not{p}' - m)\gamma^\mu(\not{p} + m)\gamma^\nu\right)$$

Trace technology

- remember $\mathrm{Tr}\gamma_\mu = 0$
- $\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4g_{\mu\nu}$
- $\mathrm{Tr}(\text{odd number of } \gamma) = 0$
- $\mathrm{Tr}(\gamma_\mu\gamma_\nu\gamma_\alpha\gamma_\beta) = 4\left(g_{\mu\nu}g_{\alpha\beta} + g_{\mu\beta}g_{\nu\alpha} - g_{\mu\alpha}g_{\nu\beta}\right)$
- more rules involving γ_5 (weak interactions!)

So

$$\mathrm{Tr}\left((\not{p}' - m)\gamma^\mu(\not{p} + m)\gamma^\nu\right) = 4\left(p'^\mu p^\nu + p'^\nu p^\mu - g^{\mu\nu}(p \cdot p' + m^2)\right)$$

Mandelstam variables with 4 equal masses, center-of-mass system (CMS):

$$p = (E, \boldsymbol{p}), \quad p' = (E, -\boldsymbol{p}), \quad k = (E, \boldsymbol{k}), \quad \theta = \measuredangle(\boldsymbol{p}, \boldsymbol{k})$$

$$s \overset{\text{CMS}}{=} (p + p')^2 = 4E^2 \tag{5.5.1}$$

$$t = (p - k)^2 = -(\boldsymbol{p} - \boldsymbol{k})^2 = -2|\boldsymbol{p}|^2(1 - \cos\theta) \tag{5.5.2}$$

$$u = (p' - k)^2 = -2|\boldsymbol{p}|^2(1 + \cos\theta) \tag{5.5.3}$$

$$|\boldsymbol{p}|^2 = E^2 - m^2 = \frac{s}{4} - m^2 \tag{5.5.4}$$

Only 2 Mandelstam variables are independent.

$$\begin{aligned}
s + t + u &= (p_1 + p_2)^2 + (p_1 - p_3)^2 + (p_1 - p_4)^2 \\
&= p_1^2 + p_2^2 + p_3^2 + p_4^2 + 2p_1 \underbrace{(p_1 + p_2 - p_3 - p_4)}_{=0} \\
&= \sum_i m_i^2 = \text{const}
\end{aligned}$$

photon polarisation sums Analogy to fermion spin sums before, Feynman rules for external photons involve $\epsilon^{(}*)_\mu$; e.g. Compton amplitude of the form

$$M \sim \epsilon^*_\mu(p_3)\epsilon_\nu(p_1)T^{\mu\nu}$$

Thus

$$\sum_{\text{spin, pol.}} |M|^2 = \sum_{\text{spin, pol.}} \epsilon^*_\mu(p_3)\epsilon_\alpha(p_3)\epsilon^*_\beta(p_1)\epsilon_\nu(p_1)T^{\mu\nu}T^{\alpha\beta^*}$$

How can we simplify $\sum_{pol}\epsilon^*_\mu(k)\epsilon_\nu(k)$? Again we have only 2 physical polarisation states, but want to do it in a covariant form.

Assume a simpler process (than Compton) with a single external photon, $\epsilon^*_\mu(k)M^\mu$. Choose

$$k^\mu = (k,0,0,k), \quad \epsilon^\mu_{(1)} = (0,1,0,0), \quad \epsilon^\mu_{(2)} = (0,0,1,0)$$

so $\sum_{\text{pol}} |\epsilon^*_\mu(k)M^\mu|^2 = |M_1|^2 + |M_2|^2$

Remember that photon coupled source j^μ, current sonvervation $\partial_\mu j^\mu = 0$. We will see (next term) this holds in general as Ward identity

$$k_\mu M^\mu = 0 \tag{5.5.5}$$

In exercises, show $p_{3\mu}T^{\mu\nu} = 0 = p_{1\nu}T^{\mu\nu}$ for Compton

Here $kM^0 - kM^3 = 0 \Rightarrow M^0 = M^3$ and we can rewrite

$$\sum_{pol} \epsilon^*_\mu\epsilon_\nu M^\mu M^{*\nu} = |M_1|^2 + |M_2|^2 + \underbrace{|M_3|^2 - |M_0|^2}_{=0} = -g_{\mu\nu}M^\mu M^{*\nu}$$

so effectively

$$\sum_{pol} \epsilon^*_\mu(k)\epsilon_\nu(k) = -g_{\mu\nu} \tag{5.5.6}$$

side remark

- KG propagator $\frac{i}{p^2-M^2+i\epsilon}$
- Dirac propagator $\frac{i(\not{p}+m)}{p^2-m^2+i\epsilon} = \frac{i\sum_s u_s(p)\bar{u}_s(p)}{p^2-m^2+i\epsilon}$
- Photon propagator $\frac{-ig_{\mu\nu}}{p^2+i\epsilon} = \frac{i\sum_{pol}\epsilon^*_\mu(p)\epsilon_\nu(p)}{p^2+i\epsilon}$

6 Radiative corrections

6.1 Optical theorem

We have seen in Advanced Quantum Theory that tree diagrams are in general real. So there is no imaginary parts. Need to restore perturbatively in higher-order corrections. Then the optical theorem is valid again.

S-matrix is unitary: $S^\dagger S = \mathbb{1}$ with $S = \mathbb{1} + iT$. Thus

$$-i(T - T^\dagger) = T^\dagger T$$

We take matrix element for $k_1 k_2 \to p_1 p_2$ scattering. On RHS, insert a complete set of states,

$$\langle p_1 p_2 | T^\dagger T | k_1 k_2 \rangle = \sum_n \prod_{i=1}^n \int \frac{\mathrm{d}^3 q_i}{(2\pi)^3 2E_i} \langle p_1 p_2 | T^\dagger | q_1 \dots q_n \rangle \langle q_1 \dots q_n | T | k_1 k_2 \rangle$$

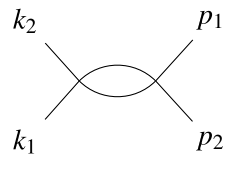
Reduce $T_{fi} = (2\pi)^4 \delta^{(4)}(p_f - p_i) M_{fi}$ and omitting overall $(2\pi)^4 \delta^{(4)}(p_f - p_i)$

$$\begin{aligned} -i\left[\mathcal{M}(k_1 k_2 \to p_1 p_2) - \mathcal{M}^*(p_1 p_2 \to k_1 k_2)\right] & \\ = \underbrace{\sum \prod_{i=1}^n \int \frac{\mathrm{d}^3 q_i}{(2\pi)^3 2E_i}}_{\text{invariant phase-space volume element}} & \mathcal{M}^*(p_1 p_2 \to q_1 \dots q_n) \mathcal{M}(k_1 k_2 \to q_1 \dots q_n) (2\pi)^4 \delta^{(4)}(k_1 + k_2 - \sum_i q_i) \end{aligned}$$

So optical theorem, for forward scattering ($p_1 = k_1, p_2 = k_2$) reads (see 4.5.1)

$$\begin{aligned} \operatorname{Im} \mathcal{M}(k_1 k_2 \to k_1 k_2) &= 2F\sigma_{\text{tot}}(k_1 k_2 \to \text{anything}) \\ 2\sqrt{s}\,|f_i^{\text{CMS}}| &= \lambda^{\frac{1}{2}}(s, m_1^2, m_2^2) \end{aligned}$$

Optical theorem for Feynman diagrams Consider a specific diagram contributing to the imaginary part, e.g. in ϕ^4-theory.



$$p_s = k_1 + k_2 = p_1 + p_2, p_s^2 = s$$

$$i\mathcal{M}(s) = \frac{\lambda^2}{2} \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{1}{[(p_s/2 - q)^2 - M^2 + i\epsilon]\,[(p_s/2 + q)^2 - M^2 + i\epsilon]} \tag{6.1.1}$$

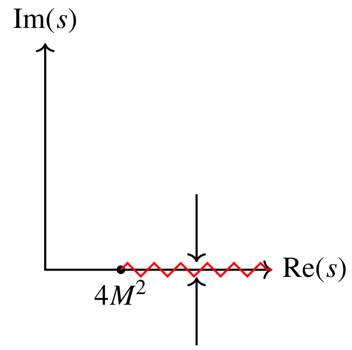
From optical theorem, $\operatorname{Im} \mathcal{M}(s < 4M^2) = 0$, so $\mathcal{M}(s < 4M^2) \in \mathbb{R}$, (since the scattering is not physical, the cross section must vanish) when regarding $\mathcal{M}(s)$ as an analytic function of s beyond what physical S-matrix element allow.

Schwarz reflection principle If (in some region) analytic function $\mathcal{M}(s)$ is real at least for a finite, non-vanishing interval $\in \mathbb{R}$, then

$$\mathcal{M}(s^*) = \mathcal{M}^*(s) \tag{6.1.2}$$

Hence

$$\mathcal{M}(s+i\epsilon) - \mathcal{M}(s-i\epsilon) \equiv \mathrm{disc}\mathcal{M}(s) = \mathcal{M}(s+i\epsilon) - \mathcal{M}^*(s+i\epsilon) = 2i\,\mathrm{Im}\,\mathcal{M}(s+i\epsilon)$$



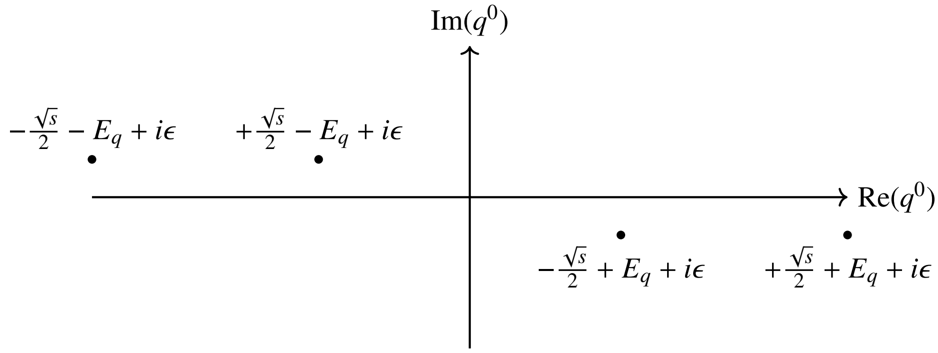
Onset of imaginary part for $s \leq 4M^2$ necessarily leads to a "branch cut", a non-trivial discontinuity in the complex energy plane. The branch cut is equivalent to $\sqrt{4M^2 - s}$. Function has discontinuity, a cut, on real axis.

How can we calculate the discontinuity (= imaginary part) of the above diagram? Use centre-of-mass system $p_s = (\sqrt{s}, \mathbf{0})$. Poles from propagators

$$\begin{aligned} &\frac{s}{4} \mp \sqrt{s}q^0 + q^2 - M^2 + i\epsilon = 0 \\ \Leftrightarrow\ &(q^0)^2 \pm \sqrt{s}q^0 + \frac{s}{4} - |\boldsymbol{q}|^2 - M^2 + i\epsilon = 0 \end{aligned}$$

First propagator $\quad q^0 = +\frac{\sqrt{s}}{2} \pm (\sqrt{M^2 + |\boldsymbol{q}|^2} - i\epsilon) = +\frac{\sqrt{s}}{2} \pm (E_q - i\epsilon)$

Second propagator $\quad q^0 = -\frac{\sqrt{s}}{2} \pm (E_q - i\epsilon)$



If we close the contour of the q_0 integration in the lower half plane, we only pick up the 2 residues at $\mp\frac{\sqrt{s}}{2} + E_q - i\epsilon$. As E_q is positive, only $-\frac{\sqrt{s}}{2} + E_q - i\epsilon$ from second propagator contributes to discontinuity.

So pinching up the residue equivalent to replacement under q^0 integration

$$\frac{1}{(p_s/2+q)^2 - M^2 + i\epsilon} \longmapsto \underbrace{-2\pi i}_{\text{orientation of contour}} \quad \delta((p_s/2+q)^2 - M^2)$$

Determine the residue of the rest at the pole at $-\frac{\sqrt{s}}{2} + E_q - i\epsilon$

$$M(s) \longmapsto -\frac{\lambda^2}{2}\int \frac{\mathrm{d}^3 q}{(2\pi)^3}\frac{1}{2E_q\sqrt{s}(\sqrt{s}-2E_q)}$$

With no angular dependence and using substitution (note the limits of integral also change)$\mathrm{d}^3 q \to 4\pi|\boldsymbol{q}|^2\mathrm{d}|\boldsymbol{q}| = 4\pi|\boldsymbol{q}|E_q\mathrm{d}E_q$

$$= -\frac{\lambda^2}{8\pi^2}\int_M^\infty \frac{\mathrm{d}E_q\sqrt{E_q^2 - M^2}}{\sqrt{s}(\sqrt{s}-2E_q)} \tag{6.1.3}$$

It has pole at $E_q = \frac{\sqrt{s}}{2}$. The second pole in 6.1.1 at $\frac{\sqrt{s}}{2} + E_q - i\epsilon$ would produce a pole in 6.1.3 for $E_q = -\frac{\sqrt{s}}{2}$, outside the integration range $M \leq E_q < \infty$.

- for $\sqrt{s} < 2M$, 6.1.3 is manifestly real.
- for $\sqrt{s} > 2M$, the pole at $E_q = \frac{\sqrt{s}}{2}$ in 6.1.3 contributes differently depending on $\sqrt{s} \pm i\epsilon$; difference yields discontinuity.

Use

$$\frac{1}{\sqrt{s}-2E_q \pm i\epsilon} = \underbrace{\frac{P}{\sqrt{s}-2E_q}}_{\text{real}} \underbrace{\mp i\pi\delta(\sqrt{s}-2E_q)}_{\text{yields discontinuity}}$$

So for calculation of the discontinuity, have replacement

$$\frac{1}{(p_s/2-q)^2 - M^2 + i\epsilon} \longmapsto -2\pi i\delta((p_s/2-q)^2 - M^2)$$

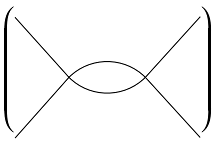
for other propagator too!

Cuthosky rules (1960) replace cut propagator according to

$$\frac{1}{p^2 - M^2 + i\epsilon} \longmapsto -2\pi i\delta(p^2 - M^2) \tag{6.1.4}$$

to calculate discontinuity across the cut!

Calculation completed:

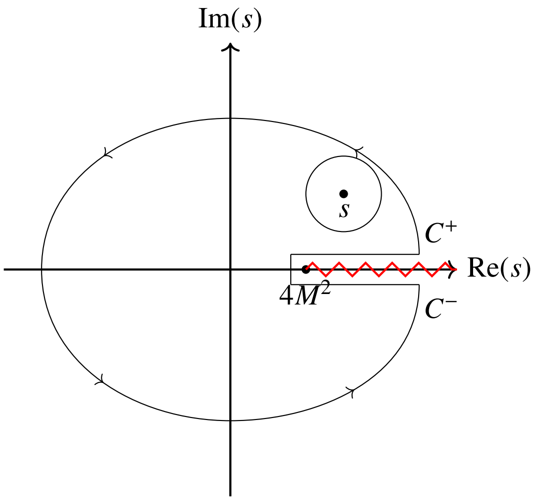
disc  $= i\frac{\lambda^2}{2}\int \frac{\mathrm{d}^4 q}{(2\pi)^4} 2\pi\delta(q^2 - M^2)2\pi\delta((p_s-q)^2 - M^2)$

using $\mathrm{d}^4q = \mathrm{d}q^0\mathrm{d}|q||q|^2\mathrm{d}\Omega_q$ and $(p_s - q)^2 - M^2 = s - 2\sqrt{s}q^0$

$$
\begin{aligned}
&= \frac{\lambda^2}{2}\frac{i}{4\pi^2}\int \frac{|q|^2\mathrm{d}|q|\mathrm{d}\Omega_q}{2q^0}\delta(s - 2\sqrt{s}q^0) \\
&= \frac{\lambda^2}{2}\frac{i}{8\pi^2}\int \sqrt{(q^0)^0 - M^2}\mathrm{d}q^0\mathrm{d}\Omega_q\delta(s - 2\sqrt{s}q^0) \\
&= \frac{\lambda^2}{2}\frac{i}{8\pi^2}\frac{\sqrt{s/4 - M^2}}{2\sqrt{s}}\int \mathrm{d}\Omega_q \\
&= \frac{\lambda^2}{2}\frac{i}{8\pi}\sqrt{1 - \frac{4M^2}{s}} \\
\mathrm{Im}\mathcal{M} &= \frac{\lambda^2}{4}\frac{1}{8\pi}\sqrt{1 - \frac{4M^2}{s}}
\end{aligned}
$$

Note $\sigma = \frac{\lambda^2}{32\pi}$ and $2F = s\sqrt{1 - \frac{4M^2}{s}}$. Thus optical theorem is still valid.

We can do more. Construct the complete $\mathcal{M}(s)$ from Im $\mathcal{M}(s)$ through a dispersion relation!



Use Cauchy's theorem:

$$
\mathcal{M}(s) = \frac{1}{2\pi i}\oint \frac{\mathcal{M}(z)\mathrm{d}z}{z - s} \tag{6.1.5}
$$

dropping the large circle

$$
\begin{aligned}
&\longmapsto \frac{1}{2\pi i}\int_{C_+ + C_-} \frac{\mathcal{M}(z)\mathrm{d}z}{z - s} \\
&= \frac{1}{2\pi i}\left[\int_{4M^2}^{\infty} \frac{M(z + i\epsilon)\mathrm{d}z}{z - s} - \int_{4M^2}^{\infty} \frac{M(z - i\epsilon)\mathrm{d}z}{z - s}\right] \\
&= \frac{1}{2\pi i}\int_{4M^2}^{\infty} \frac{\mathrm{disc}\mathcal{M}(z)\mathrm{d}z}{z - s} \\
&= \frac{1}{\pi}\int_{4M^2}^{\infty} \frac{\mathrm{Im}\,\mathcal{M}(z)\mathrm{d}z}{z - s}
\end{aligned} \tag{6.1.6}
$$

Repeat the exercise for $\frac{\mathcal{M}(s)-\mathcal{M}(0)}{s}$ (no pole introduced!).

$$\begin{aligned}
\operatorname{Im}\left(\frac{\mathcal{M}(s)-\mathcal{M}(0)}{s}\right) &= \frac{\operatorname{Im}\mathcal{M}(s)}{s} \\
\mathcal{M}(s)-\mathcal{M}(0) &= \frac{s}{\pi}\int_{4M^2}^{\infty}\frac{\operatorname{Im}\mathcal{M}(z)\mathrm{d}z}{z(z-s)} \\
&= \frac{\lambda^2}{2}\frac{s}{(4\pi)^2}\int_{4M^2}^{\infty}\frac{\mathrm{d}z}{z(z-s)}\sqrt{1-\frac{4M^2}{z}}
\end{aligned}$$

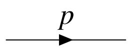
using $\sigma = \sqrt{1-\frac{4M^2}{s}}$ and $\zeta = \sqrt{1-\frac{4M^2}{z}}$

$$\begin{aligned}
&= \frac{\lambda^2}{2}\frac{1}{8\pi^2}\int_0^1\frac{\zeta^2}{\zeta^2-\sigma^2}\mathrm{d}\zeta \\
&= \frac{\lambda^2}{2}\begin{cases}
\frac{1}{8\pi^2}\left(1-\frac{\sigma}{2}\log\frac{\sigma+1}{\sigma-1}\right) & s<0 \Leftrightarrow \sigma>1 \\
\frac{1}{8\pi^2}\left(1-\sqrt{-\sigma^2}\arctan\frac{1}{\sqrt{-\sigma^2}}\right) & 0<s<4M^2, \sigma^2<0 \\
\frac{1}{8\pi^2}\left(1-\frac{\sigma}{2}\log\frac{1+\sigma}{1-\sigma}+\frac{i\sigma}{16\pi}\right) & s>M^2, 0<\sigma<1
\end{cases}
\end{aligned}$$

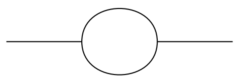
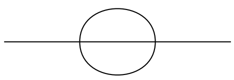
Note We are going to calculate this diagram again, noticing that $\int\frac{\mathrm{d}^4q}{(q^2\dots)(q^2\dots)}$ is logarithmically divergent! The above representation demonstrates that this divergence resides in $M(0)$!

6.2 Field-strength renormalization

What is structure of the propagator $\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle$ at higher orders? At lower order



$$= \frac{i}{p^2-M^2+i\epsilon}$$

Beyond this the propagator is not a simple pole. In ϕ^3-theory  branch cuts are at $p^2 \le 4M^2$. In ϕ^4-theory  branch cuts are at $p^2 \le 9M^2$. To induce cuts in the analytic structure.

Insert complete set of intermediate states ($x^0 > y^0$)

$$\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle = \sum_\lambda\int\frac{\mathrm{d}^3p}{(2\pi)^3 2E_p(\lambda)}\langle\Omega|\phi(x)|\lambda_{\boldsymbol{p}}\rangle\langle\lambda_{\boldsymbol{p}}|\phi(y)|\Omega\rangle$$

with

λ multi-particle state

λ_0 "rest frame", i.e. $\hat{\boldsymbol{P}}|\lambda_0\rangle = 0$

$\lambda_{\boldsymbol{p}}$ boosted to momentum $\boldsymbol{p}$

Call energy of $\lambda_0 = m_\lambda$. From single particle to multi particle $E_{\boldsymbol{p}}(\lambda) = \sqrt{m_\lambda^2 + |\boldsymbol{p}|^2}$.

$$\begin{aligned}\langle\Omega|\phi(x)|\lambda_{\boldsymbol{p}}\rangle &= \langle\Omega|e^{i\hat{P}x}\phi(0)e^{-i\hat{P}x}|\lambda_{\boldsymbol{p}}\rangle \\ &= \langle\Omega|\phi(0)|\lambda_{\boldsymbol{p}}\rangle\, e^{-ipx}\Big|_{p^0=E_{\boldsymbol{p}}}\end{aligned}$$

Ω and $\phi(0)$ are invariant under momentum boost

$$= \langle\Omega|\phi(0)|\lambda_0\rangle\, e^{-ipx}\Big|_{p^0=E_{\boldsymbol{p}}}$$

$$\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle = \sum_\lambda \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2E_p(\lambda)} e^{-ip(x-y)} |\,\langle\Omega|\phi(0)|\lambda_0\rangle\,|^2 \tag{6.2.1}$$

$$= \sum_\lambda \underbrace{\int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i}{p^2 - m_\lambda^2 + i\epsilon} e^{-ip(x-y)}}_{D_F(x-y;m_\lambda^2)\text{ when combined with }y^0>x^0} |\,\langle\Omega|\phi(0)|\lambda_0\rangle\,|^2 \tag{6.2.2}$$

$$\tag{6.2.3}$$

Formally write this as

$$\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle = \int_0^\infty \frac{\mathrm{d}s}{2\pi}\rho(s) D_F(x-y;s) \tag{6.2.4}$$

with $\rho(s)$ the spectral density function.

$$\rho(s) := \sum_\lambda (2\pi)\delta(s - m_\lambda^2)\, |\langle\Omega|\phi(0)|\lambda_0\rangle|^2 \tag{6.2.5}$$

A typical spectral function looks like

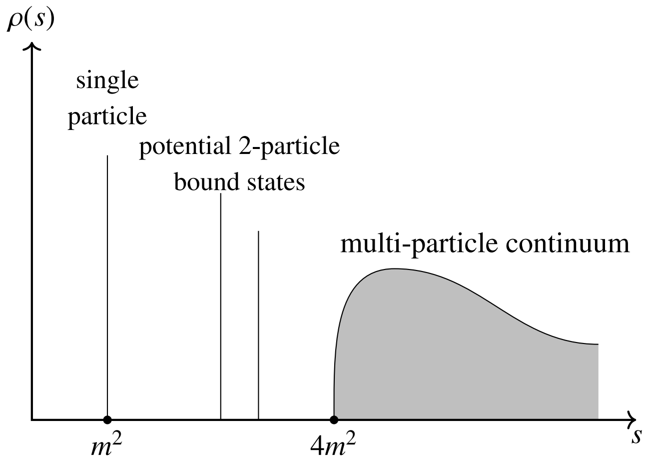


Figure 6.1: typical spectral function

Single particle contribution

$$\rho(s) = 2\pi\delta(s - m^2)Z + (\text{contributions} \geq 4m^2) \tag{6.2.6}$$

with $Z = |\langle\Omega|\phi(0)|\lambda_0\rangle|^2$ the field-strength renormalization factor.

Fourier transforming two-point function

$$\begin{aligned}&\int \mathrm{d}^4 x e^{ipx} \langle\Omega|T\phi(x)\phi(0)|\Omega\rangle\\ =&\int_0^\infty \frac{\mathrm{d}s}{2\pi}\rho(s)\frac{i}{p^2-s+i\epsilon}\\ =&\frac{iZ}{p^2-m^2+i\epsilon}+\int_{\sim 4m^2}^\infty \frac{\mathrm{d}s}{2\pi}\rho(s)\frac{i}{p^2-s+i\epsilon}\end{aligned}$$

Comparing to free theory: $\langle 0|\phi(0)|\boldsymbol{p}\rangle = 1$ hence $Z = 1$.

6.3 LSZ reduction formula*

Reminder A complete set of intermediate states

$$\mathbb{1} = |\Omega\rangle\langle\Omega| + \sum_\lambda \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2E_p(\lambda)} |\lambda_{\boldsymbol{p}}\rangle\langle\lambda_{\boldsymbol{p}}| \tag{6.3.1}$$

with

- λ multi-particle state
- λ_0 "rest frame" state, i.e. $\hat{\boldsymbol{\mathcal{P}}}|\lambda_0\rangle = 0$. Energy of λ_0: $m_\lambda \leftarrow E_{\boldsymbol{p}}(\lambda) = \sqrt{m_\lambda^2 + \boldsymbol{p}_\lambda^2}$
- $\lambda_{\boldsymbol{p}}$ state boosted to momentum $\boldsymbol{p}$

using the translation operators

$$\langle\Omega|\phi(x)|\lambda_{\boldsymbol{p}}\rangle = \langle\Omega|e^{i\hat{P}\cdot x}\phi(0)e^{-i\hat{P}\cdot x}|\lambda_{\boldsymbol{p}}\rangle$$

since Ω and $\phi(0)$ are invariant under momentum boost

$$= \langle\Omega|\phi(0)|\lambda_0\rangle\, e^{-ip\cdot x}|_{p^0=E_{\boldsymbol{p}}}$$

We claim the Fourier transform of $\langle\Omega|T\phi(x_1)\ldots\phi(x_n)\phi(y_1)\ldots\phi(y_n)|\Omega\rangle$ contains poles in all external momenta. The residue is the S-matrix element $\langle p_1\ldots p_n|S|k_1\ldots k_m\rangle$ multiplied by $\sqrt{Z}$ for each external leg.

Fourier transform with respect to the first coordinated x_1 and let $x_2^0,\ldots,x_n^0 \in [T_-,T_+]$ divide

$$\begin{aligned}&\int \mathrm{d}x_1^0 = \int_{T_+}^\infty + \int_{T_-}^{T_+} + \int_{-\infty}^{T_-}\\ \Rightarrow&\int_{T_+}^\infty \mathrm{d}x_1^0 \int \mathrm{d}^3 x_1 e^{iP_1\dot{x}_1} \langle\Omega|\phi(x_1)\phi(x_2)\ldots\phi(x_n)|\Omega\rangle\\ =&\int_{T_+}^\infty \mathrm{d}x_1^0 \int \mathrm{d}^3 x_1 e^{iP_1\ldots x_1} \sum_\lambda \int \frac{\mathrm{d}^3 q}{(2\pi)^3 2E_{\boldsymbol{q}}(\lambda)} \langle\Omega|\phi(x_1)|\lambda_{\boldsymbol{q}}\rangle\langle\lambda_{\boldsymbol{q}}|T\phi(x_2)\ldots\phi(x_n)|\Omega\rangle\end{aligned}$$

*see also Peskin ans Schröder, chapter 7.2; Martin Mojzis, QFT I, page 110-113 use $\langle\Omega|\phi(x_1)|\lambda_{\boldsymbol{q}}\rangle = \langle\Omega|\phi(0)|\lambda_0\rangle\, e^{-iq\cdot x_1}|_{q^0=E_{\boldsymbol{q}}}$ and integrate over $\boldsymbol{x}$.

$$= \sum_\lambda \int_{T_+}^\infty \mathrm{d}x_1^0 \int \frac{\mathrm{d}^3 q}{(2\pi)^3 2E_{\boldsymbol{q}}(\lambda)} e^{i(p_1^0-q^0+i\epsilon)x_1^0}(2\pi)^3\delta^{(3)}(\boldsymbol{p_1}-\boldsymbol{q})\,\langle\Omega|\phi(0)|\lambda_0\rangle\,\langle\lambda_{\boldsymbol{q}}|T\phi(x_2)\dots\phi(x_n)|\Omega\rangle$$

Integrate over $\boldsymbol{q}$ and x^0

$$= \sum_\lambda \frac{1}{2E_{\boldsymbol{p}}(\lambda)} \frac{ie^{i(p_1^0-E_{\boldsymbol{p}_1}(\lambda))T_+}}{p_1^0-E_{\boldsymbol{p}_1}(\lambda)+i\epsilon}\,\langle\Omega|\phi(0)|\lambda_0\rangle\,\langle\lambda_{\boldsymbol{q}}|T\phi(x_2)\dots\phi(x_n)|\Omega\rangle$$

We know from the previous section

- only single-particle states λ_0 will produce a pole
- multi-particle produce "milder" singularities, like continuous cuts
- for single-particle state of mass m, the above has precisely the pole of $\frac{1}{p_1^2-m^2+i\epsilon}$ with residue $\langle\Omega|\phi(0)|\boldsymbol{p}_1\rangle = \sqrt{Z} = |\langle\Omega|\phi(0)|\lambda_0\rangle|$

hence

$$\int \mathrm{d}^4x_1 e^{ip_1\cdot x_1}\,\langle\Omega|T(\phi(x_1)\phi(x_2)\dots\phi(x_n))|\Omega\rangle \tag{6.3.2}$$

$$\overset{p_1^0\to+E_{\boldsymbol{p}}}{=} \frac{i\sqrt{Z}}{p_1^2-m^2+i\epsilon}\,{}_{\text{out}}\langle\boldsymbol{p}_1|T(\phi(x_2)\dots\phi(x_n))|\Omega\rangle + (\text{less singular stuff}) \tag{6.3.3}$$

What about the other integration regions in x_1^0?

- Integral $\int_{T_-}^{T_+}\mathrm{d}x_1^0$ is bounded, hence yields analytic, non-singular function
- $$\int_{-\infty}^{T_1}\mathrm{d}x_1^0\int\mathrm{d}^3x_1 e^{ip_1\cdot x_1}\,\langle\Omega|T(\phi(x_2)\dots\phi(x_n))\phi(x_1)|\Omega\rangle$$
  has pole for $p_1^0\to -E_{\boldsymbol{p}_1}$
  $$= \dots = \frac{i\sqrt{Z}}{p_1^2-m^2+i\epsilon}\,\langle\Omega|T(\phi(x_1)\dots\phi(x_n))|-\boldsymbol{p}_1\rangle_{\text{in}} + \dots$$
  it has pole for an in- instead of an out-state

How do we go from here. Fourier-transform with respect to the second coordinate x_2, with the same assumption on T-ordering as before

$$\int \mathrm{d}^4x_2 e^{ip_2\cdot x_2}\,{}_{\text{out}}\langle\boldsymbol{p}_1|\phi(x_2)T(\phi(x_3)\dots\phi(x_n))|\Omega\rangle$$
$$= \sum_\lambda \int \mathrm{d}^4x_2 e^{ip_2\cdot x_2}\int\frac{\mathrm{d}^3q}{(2\pi)^3 2E_{\boldsymbol{q}}(\lambda)}\,\langle\boldsymbol{p}_1|\phi(x_2)|\lambda_{\boldsymbol{q}}\rangle\,\langle\lambda_{\boldsymbol{q}}|T(\phi(x_3)\dots\phi(x_n))|\Omega\rangle \tag{6.3.4}$$

We want to find the poles in p_2. But from which intermediate states?

- $|\lambda_{\boldsymbol{q}}\rangle = |\Omega\rangle$ yields $\propto \int\mathrm{d}^4x_2 e^{ip_2\cdot x_2}\,\langle\boldsymbol{p}_1|\phi(x_2)|\Omega\rangle\,\frac{\mathrm{d}^3q}{2E_{\boldsymbol{q}}} \propto \int\mathrm{d}^4x_2 e^{i(p_2+p_1)\cdot x_2}\frac{\mathrm{d}^3q}{2E_{\boldsymbol{q}}}$
  No singularity in p_2 (no isolated $\frac{1}{2E_{\boldsymbol{p}_2}}$-term).

- $|\lambda_{\boldsymbol{q}}\rangle = |\boldsymbol{q}\rangle \longmapsto \langle \boldsymbol{p}_1|\phi(x_2)|\boldsymbol{q}\rangle\, e^{ip_2\cdot x_2} = \langle \boldsymbol{p}_1|\phi(0)|\boldsymbol{q}\rangle\, e^{i(p_1+p_2-q)\cdot x_2}$
  Upon $\int \frac{\mathrm{d}^3 q}{2E_{\boldsymbol{q}}}$ cut in $p_1 + p_2$ at best
- $|\lambda_{\boldsymbol{q}}\rangle = |\boldsymbol{q}_1, \boldsymbol{q}_2\rangle$ Crucial assumption that by using wave packets, we can define asymptotically (for $t \to 0$) "non-interacting" single-particle states, such that

$$\sum_\lambda \int \frac{\mathrm{d}^3 q}{(2\pi)^3 2E_{\boldsymbol{q}}(\lambda)} \mapsto \int \frac{\mathrm{d}^3 q_1}{(2\pi)^3 2E_{\boldsymbol{q}_1}} \int \frac{\mathrm{d}^3 q_2}{(2\pi)^3 2E_{\boldsymbol{q}_2}} + \text{(higher states)}$$

So we have

$$\int \mathrm{d}^4 x_2 e^{ip_2\cdot x_2} \langle \boldsymbol{p}_1|\phi(x_2)T(\phi(x_3)\dots\phi(x_n))|\Omega\rangle$$

$$\longmapsto \dots + \int \mathrm{d}^4 x_2 e^{ip_2\cdot x_2} \int \frac{\mathrm{d}^3 q_1}{(2\pi)^3 2E_{\boldsymbol{q}_1}} \int \frac{\mathrm{d}^3 q_2}{(2\pi)^3 2E_{\boldsymbol{q}_2}} \langle p_1 \left|\phi\left(x_2\right)\right| \boldsymbol{q}_1, \boldsymbol{q}_2\rangle \langle \boldsymbol{q}_1\boldsymbol{q}_2 \left|T\left(\phi\left(x_2\right)\cdots\phi\left(x_n\right)\right)\right| s\rangle$$

Auxiliary calculation

$$\langle p_1 \left|\phi\left(x_2\right)\right| q_1, q_2\rangle = \langle p_1|\phi(0)|q_1, q_2\rangle\, e^{i(p_1-q_1-q_1)\cdot x_2}$$

use commutation relation and $\langle p_1|\, a^{+(\text{asymp})}_{\boldsymbol{q}_1} = \langle\Omega\rangle \sqrt{2E_{\boldsymbol{q}_1}}(2\pi)^3\delta^{(3)}\left(\boldsymbol{p}_1 - \boldsymbol{q}_1\right)$

$$\begin{aligned} &= \sqrt{2E_{\boldsymbol{q}_1}} \left\langle \boldsymbol{p}_1 \left|\phi(0) a^{+(\text{asymp})}_{\boldsymbol{q}_1}\right| \boldsymbol{q}_2 \right\rangle e^{i(p_1-q_1-q_2)\cdot x_2} \\ &= 2E_{\boldsymbol{q}_1}(2\pi)^3\delta^{(3)}(\boldsymbol{p}_1 - \boldsymbol{q}_1)\, \langle\Omega|\phi(0)|\boldsymbol{p}_2\rangle\, e^{i(p_1-q_1-q_2)\cdot x_2} \end{aligned}$$

Note $[\phi(0), a^{+(\text{asymp})}_{\boldsymbol{q}_2}]$ yields no singular piece in p_2 (no $\frac{1}{2E_{\boldsymbol{p}_2}}$).
Then

$$\int \mathrm{d}^4 x_2 e^{ip_2\cdot x_2} \int \frac{\mathrm{d}^3 q_1}{(2\pi)^3 2E_{\boldsymbol{q}_1}} \int \frac{\mathrm{d}^3 q_2}{(2\pi)^3 2E_{\boldsymbol{q}_2}} \langle p_1 \left|\phi\left(x_2\right)\right| \boldsymbol{q}_1\boldsymbol{q}_2\rangle \langle \boldsymbol{q}_1\boldsymbol{q}_2 \left|T\left(\phi\left(x_3\right)\dots\phi\left(x_n\right)\right|s\right\rangle$$

$$= \text{(non-singular-pieces)} + \int \mathrm{d}^4 x_2 \int \frac{\mathrm{d}^3 q_2}{\left((2\pi)^3\right) 2E_{\boldsymbol{q}_2}} e^{i(p_2-q_2)\cdot x_2} \langle\Omega|\phi(0)|\boldsymbol{p}_2\rangle \langle \boldsymbol{p}_1\boldsymbol{q}_2 \left|T\left(\phi\left(x_3\right)\cdot\phi\left(x_n\right)\right)\right|\Omega\rangle$$

x_2 integration turns into delta distribution and then gets integrated out; $\langle\Omega|\phi(0)|\boldsymbol{p}_2\rangle$ turns into $\sqrt{Z}$

$$= \dots + \frac{i\sqrt{z}}{p_2^2 - m^2 + i\epsilon}{}_{\text{out}} \langle \boldsymbol{p}_1\boldsymbol{p}_2 \left|T\left(\phi\left(x_3\right)\dots\phi\left(x_n\right)\right|\Omega\right\rangle$$

Combine them together

$$\begin{aligned} &\prod_{i=1}^{n} \int d^4 x_i e^{ip_i x_i} \langle\Omega \left|T\left(\phi\left(x_1\right)\phi\left(x_2\right)\phi\left(x_3\right)\dots\phi\left(x_n\right)\right)\right|\Omega\rangle \\ &= \frac{i\sqrt{Z}}{p_1^2 - m^2 + i\epsilon} \frac{i\sqrt{Z}}{p_2^2 - m^2 + i\epsilon}{}_{\text{out}} \langle \boldsymbol{p}_1\boldsymbol{p}_2 \left|T\left(\phi\left(x_3\right)\dots\phi\left(x_n\right)\right)\right|\Omega\rangle + \text{(non-pole terms)} \end{aligned} \tag{6.3.5}$$

Repeat these steps and remember that the singular piece in the propagator (two-point function) was $\frac{iZ}{p^2-m^2+i\epsilon}$. Write ${}_{\text{out}}\langle \boldsymbol{p}_1\boldsymbol{p}_2\dots\boldsymbol{p}_r|\boldsymbol{p}_{r+1}\dots\boldsymbol{p}_n\rangle_{\text{in}} = {}_{(\text{in})}\langle \boldsymbol{p}_1\boldsymbol{p}_2\dots\boldsymbol{p}_r|S\,|\boldsymbol{p}_{r+1}\dots\boldsymbol{p}_n\rangle_{(\text{in})}$

$$\langle \boldsymbol{p}_1\boldsymbol{p}_2\dots\boldsymbol{p}_r|S\,|\boldsymbol{p}_{r+1}\dots\boldsymbol{p}_n\rangle = \prod_{k=1}^{n} \sqrt{Z} \lim_{p_k^2 \mapsto m_k^2} \left(\frac{iZ}{p_k^2 - m^2 + i_\epsilon}\right)^{-1} G\left(p_1, \dots, p_n\right)$$

$$\text{where } G\left(p_1, \dots, p_n\right) = \prod_{i=1}^{r} \int \mathrm{d}^4 x_i e^{ip_i x_i} \prod_{j=r+1}^{n} \int \mathrm{d}^4 x_j e^{-ip_j\cdot x_j} \langle\Omega \left|T\phi\left(x_1\right)\dots\phi\left(x_n\right)\right|\Omega\rangle \tag{6.3.6}$$

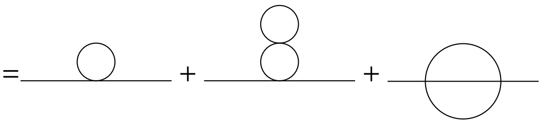
The S-matrix element is the on-shell limit (which takes care of the non-pole pieces) of the momentum-space vacuum correlation function, multiplied by the inverse propagator (with the dressed mass m) and a factor $\sqrt{Z}$ (wave-function renormalization) for each external leg.

Remark A more rigorous proof (see e.g. Itzykson and Zuber, Chapter.5.1-3) is based on careful definition of in-/out-state operator etc. (see also Schwatz, Chapter 6.1).

6.4 The propagator (again)*

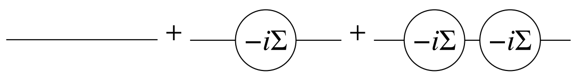
How do we calculate the propagator and the wave-function renormalization factor Z in perturbation theory, using Feynman diagrams? Call mass parameter in $\mathcal{L} = \frac{1}{2}(\partial_\mu\phi_0)^2 - \frac{m_0^2}{2}\phi_0^2 - \frac{\lambda_0}{4!}\phi_0^4$ the *bare mass*.

One-particle-irreducibles (1PIs) in ϕ^4-theory are the diagrams that cannot be disconnected by cutting internal lines. Their contributions are

$-i\Sigma(p^2) =$  $+ \dots$

Then the complete propagator using $D_F^0(p^2) = \frac{i}{p^2 - m_0^2 + i\epsilon}$ is

$$D_F(p^2) = \int \mathrm{d}^4x e^{ipx} \langle 0|T\phi(x)\phi(0)|0\rangle \tag{6.4.1}$$

$=$  $+ \dots$

$$= D_F^0(p^2) + D_F^0(p^2)\left(-i\Sigma(p^2)\right)D_F^0(p^2) + \dots$$

it is cleary a geometric series

$$= \frac{D_F^0(p^2)}{1 + i\Sigma(p^2)D_F^0(p^2)}$$
$$= \frac{i}{p^2 - m_0^2 - \Sigma(p^2)} \tag{6.4.2}$$

The pole of propagator does not occur at m_0^2 anymore. It will be shifted by $\Sigma \sim O(\lambda)$!

Expansion of divergent integrals † Notice that the integral in 6.1.1 $\propto \int \frac{\mathrm{d}^4q}{q^4}$. If we differentiate it with respect to q, the integral becomes convergent. This holds true for integral of general loop diagrams (although more than one differentiation might be needed). Thus we can expand this kind of integral into convergent and divergent term(s).

Expand

$$\Sigma(p^2) = \Sigma(m^2) + (p^2 - m^2)\Sigma'(m^2) + (p^2 - m^2)\tilde{\Sigma}(p^2) \tag{6.4.3}$$

where $\Sigma(m^2)$ is quadratically and $\Sigma'(m^2)$ logarithmically divergent. $\tilde{\Sigma}$ represents a correction (to first order Taylor expansion) and it satisfies $\tilde{\Sigma}(m^2) = \tilde{\Sigma}'(m^2) = 0$.

*see also Peskin and Schröder, Chapter 10.2
†see also Cheng and Li, Chapter 2.1

Mass and field renormalization The mass m by the condition

$$m^2 = m_0^2 + \Sigma(m^2) \tag{6.4.4}$$

This is indeed physical mass, since the expression for propagator in 6.4.2 has a pole at $p^2 = m^2$.

Then the propagator

$$D_F(p^2) = \frac{i}{p^2 - m_0^2 - \Sigma(p^2)} = \frac{i}{p^2 - m^2 - (p^2 - m^2)(\Sigma'(m^2) + \tilde{\Sigma}(p^2))}$$

using 6.4.3

$$\begin{aligned}
&= \frac{i}{(p^2 - m^2)(1 - \Sigma'(m^2) - \tilde{\Sigma}(p^2))} \\
&= \frac{iZ}{p^2 - m^2} \cdot \frac{1}{1 - Z\tilde{\Sigma}(p^2)} \\
&= \frac{iZ}{p^2 - m^2} + (\text{regular at } p^2 = m^2)
\end{aligned} \tag{6.4.5}$$

with $Z = \left(1 - \Sigma'(m^2)\right)^{-1}$. This expression is to be compared with 6.3.1.

Starting point Lagrangian is $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_0)^2 - \frac{m_0^2}{2}\phi_0^2 - \frac{\lambda_0}{4!}\phi_0^4$. To remove Z from numerator in the propagator and instead put $\sqrt{Z}$ onto the couplings at each end. Since each internal vertex has 4 lines (remember the vertex carries the coupling constant)

$$\lambda_0 \longmapsto \lambda_1 = Z^2 \lambda_0 \tag{6.4.6}$$

In Σ and $\tilde{\Sigma}$, there are 2 external lines without $\sqrt{Z}$, so

$$\Sigma(p^2, \lambda_0, \text{old } D_F) = \frac{1}{Z}\Sigma_1(p^2, \lambda_1, \text{new } D_F') \tag{6.4.7}$$

(same expression for $\tilde{\Sigma}$).

Thus we get the new propagator

$$D_F'(p^2) = \frac{i}{p^2 - m^2} \cdot \frac{1}{1 - \tilde{\Sigma}_1(p^2)} \tag{6.4.8}$$

where $\tilde{\Sigma}_1(m^2) = 0$.

Define the renomalized field

$$Z^{-\frac{1}{2}}\phi_0 = \phi \tag{6.4.9}$$

then D_F' is the Fourier transform of $\langle 0|T\phi(x)\phi(y)|0\rangle$

Rewrite the Lagrangian as

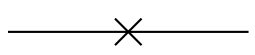
$$\mathcal{L} = \frac{1}{2}\left((\partial_\mu\phi)^2 - m^2\phi^2\right) \boxed{-\frac{\lambda_1}{4!}\phi^4 \underbrace{-\frac{1}{2}\delta m^2\phi^2 + \frac{1}{2}(Z-1)\left((\partial_\mu\phi)^2 - m^2\phi^2\right)}_{\text{counter-terms}}} \tag{6.4.10}$$

where $\delta m^2 = -Z(m^2 + m_0^2) = -Z\Sigma(m^2) = -\Sigma_1(m^2)$. Everything inside the $\boxed{\text{box}}$ can be considered as interaction.

It may look weird given the kinetic/mass-like terms, but there is no contradiction. Consider just $\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2$. The mass-term ≡ "interaction". A massless propagator

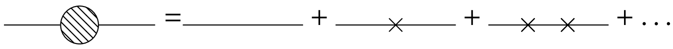
$$= \frac{i}{p^2}$$

and interaction



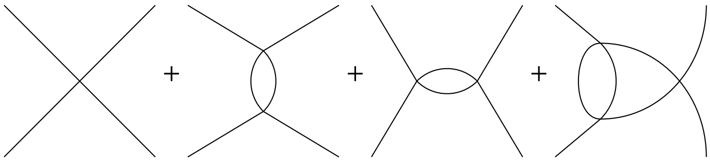
$$= -im^2$$

The resummed propagator is then



$$\begin{aligned} &= \frac{i}{p^2}\left(1 + \frac{i}{p^2}(-im^2) + \dots\right) \\ &= \frac{i}{p^2}\left(1 - \frac{i}{p^2}(-im^2)\right)^{-1} = \frac{i}{p^2 - m^2} \end{aligned}$$

Actually this is not all. We will also have to further renormalize λ_1



$$+ \dots = \lambda_1 \{1 + L + \Gamma(s, t, u)\}$$

L is value of the sum of all 1PI vertex contributions at the same kinematic point and Γ defined by $\Gamma(s = t = u = \frac{4}{3}M^2) = 0$ (for instance, $P_i^2 = M^2$ and $P_iP_j = -\frac{M^2}{3}$ with $i \neq j$).

Define

$$Z_\lambda := (1 + L)^{-1} \tag{6.4.11}$$

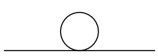
and the renormalized coupling is

$$\lambda = Z_\lambda^{-1}\lambda_1 = Z_\lambda^{-1}Z^2\lambda_0 \tag{6.4.12}$$

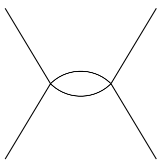
Write Lagrangian in terms of renormalized λ and add another counter-term $-\frac{(Z_\lambda - 1)}{4!}\lambda\phi^4$.

Note that all counter-terms we have introduced are of the same form as the original Lagrangian: $(\partial\phi)^2$, ϕ^2 and ϕ^4. There is no need to introduce new structure or new coupling parameters. It is property of a renormalizable theory.

6.5 Divergent graphs and dimensional regularization

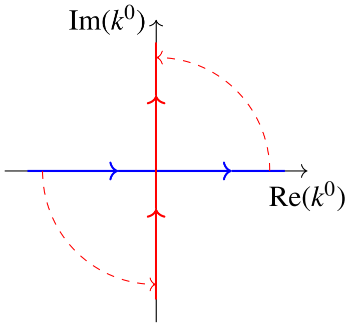


$$M_2 = \frac{\lambda}{2}\frac{1}{i}\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2 - M^2 + i\epsilon}$$



$$M_4 = \frac{\lambda^2}{2}\frac{1}{i}\int \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2 - M^2 + i\epsilon)((k-p)^2 - M^2 + i\epsilon)}$$

Wick rotation Poles of M_2 in the complex k^0-plane at $k^0 = \pm\sqrt{\boldsymbol{k}^2 + M^2 - i\epsilon}$. The position of the poles allow us to rotate the integration path to go $-i\infty \mapsto +i\infty$ instead. So no singularities are hit! If the singularities lie on the real axis (in the limit $\epsilon \to 0$), the integral becomes undefined.

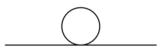


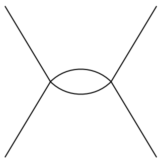
Define a Euclidean momentum $k^0 = ik_E^0$, $\boldsymbol{k} = \boldsymbol{k}_E$

$$\frac{1}{i}\int \frac{\mathrm{d}k^0\mathrm{d}^3k}{(2\pi)^4}\frac{1}{(k_0)^2 - \boldsymbol{k}^2 - M^2 + i\epsilon} \longmapsto -\int \frac{\mathrm{d}^4k_E}{(2\pi)^4}\frac{1}{k_E^2 + M^2 - i\epsilon}$$

Now we are far from singularities, $i\epsilon$ can thus be ignored.

This form allows us to see

 $\sim \int \frac{\mathrm{d}kk^3}{k^2}$ is quadratically divergent

 $\sim \int \frac{\mathrm{d}kk^3}{k^4}$ is logarithmically divergent

Hope of renormalization program is that all such divergences can be absorbed into bare/unrenormalized couplings to produce physical/renormalized/observable parameters.

There are different methods to regularize divergent loop integrals in order to keep track of divergences

1. momentum (Λ) cutoff: study the limit $\Lambda \longmapsto \infty$ in the end

2. Pauli-Villars: subtract propagator(s) with heavy mass(es)*

$$\frac{1}{k^2} \longmapsto \frac{1}{k^2} - \frac{1}{k^2 - M_{\mathrm{PV}}^2}, \; M_{\mathrm{PV}} \longmapsto \infty$$

3. dimensional regularization: work in d dimension instead of 4, 1 time-like, $d-1$ space-like. For small d integral converge, consider $d \longmapsto 4$ in the end. The divergences appear as poles in $\frac{1}{d-4}$.

Main advantage of dimensional regularization is that all symmetries are preserved (massless photons etc.). Downside is that it is somewhat unphysical and unintuitive.

*for details see Ryder, Chapter 9.2

Feynman parameters [*] Combine multiple propagators into one (to some power)

$$\frac{1}{A_1 \dots A_n} = \int_0^1 \mathrm{d}x_1 \dots \mathrm{d}x_n \delta\left(\sum_i^n x_i - 1\right) \frac{(n-1)!}{(x_1 A_1 + \dots + x_n A_n)^n} \tag{6.5.1}$$

using

$$\frac{1}{A_i} = \int_0^\infty \mathrm{d}\alpha_i e^{-\alpha_i A_i}$$

$$\int \mathrm{d}\alpha_1 \dots \mathrm{d}\alpha_n e^{-\sum_i \alpha_i A_i} = \int_0^1 \mathrm{d}x_1 \dots \mathrm{d}x_n \delta\left(\sum_i^n x_i - 1\right) \int_0^\infty \mathrm{d}t\, t^{n-1} e^{-t \sum_i x_i A_i}$$

Special case

$$\frac{1}{AB} = \int_0^1 \mathrm{d}x \frac{1}{[xA + (1-x)B]^2} \tag{6.5.2}$$

With $A = (k-p)^2 - M^2$ and $B = k^2 - M^2$

$$xA + (1-x)B = k^2 - xp(2k - p) - M^2 = (k-p)^2 - (M^2 - x(1-x)p^2)$$

Thus after shifting the integration variable $k \longmapsto k + xp$ and with $\Delta(x) := M^2 - x(1-x)p^2$

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{(k^2 - M^2)((k-p)^2 - M^2)} = \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{[k^2 - \Delta(x)]^2}$$

Dimensional regularization formula [†]

$$\frac{1}{i} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{(k^2 - \Delta)^n} = \frac{(-1)^n}{(4\pi)^{d/2}} \frac{\Gamma(n - d/2)}{\Gamma(n)} \frac{1}{\Delta^{n-d/2}} \tag{6.5.3}$$

Γ-function has following definition and properties

- $\Gamma(n+1) = \int_0^\infty \mathrm{d}x x^n e^{-x}$
- $\Gamma(n+1) = n!$ for $n \in \mathbb{N}$, $n\Gamma(n) = \Gamma(n+1)$
- $\Gamma(n)$ has poles for negative integers $n = 0, -1, -2, \dots$

Proof by induction

- $n = 1$: introduce Schwinger parameter α and $i\epsilon$ part enforces convergence.

$$\frac{1}{i} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{k^2 - \Delta + i\epsilon} = -\int_0^\infty \mathrm{d}\alpha \int \frac{\mathrm{d}^d k}{(2\pi)^d} e^{i\alpha(k^2 - \Delta + i\epsilon)}$$

using Wick rotation

$$= -i \int_0^\infty \mathrm{d}\alpha \int \frac{\mathrm{d}^d k_E}{(2\pi)^d} e^{-i\alpha(k_E^2 + \Delta - i\epsilon)}$$

[*] see also Peskin and Schröder, Chapter 6.3; Ryder, Chapter 9.2
[†] see also Peskin and Schröder, Chapter 7.5

Gaussian integral in higher dimension; in general $\int \exp\left(-\frac{1}{2}x\cdot A\cdot x + J\cdot x\right) d^n x = \sqrt{\frac{(2\pi)^n}{\det A}}\exp\left(\frac{1}{2}J\cdot A^{-1}\cdot J\right)$

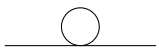
$$
\begin{aligned}
&= \frac{-i}{(2\pi)^d}\int_0^\infty \mathrm{d}\alpha \sqrt{\frac{\pi}{i\alpha}}^{\,d} e^{-i\alpha\Delta} \\
&= \frac{-i}{(4\pi)^{d/2}}\int_0^\infty \mathrm{d}\alpha (i\alpha)^{-d/2} e^{-i\alpha\Delta} \\
&= -\frac{1}{(4\pi)^{d/2}}\frac{1}{\Delta^{1-d/2}}\int_0^\infty \mathrm{d}x x^{-d/2} e^{-x} \\
&= \frac{(-1)}{(4\pi)^{d/2}}\frac{1}{\Delta^{1-d/2}}\Gamma(1-d/2)
\end{aligned}
$$

- Induction $n \to n+1$

$$
\begin{aligned}
\frac{1}{i}\int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{(k^2-\Delta)^{n+1}} &= \frac{1}{n}\frac{\partial}{\partial\Delta}\frac{1}{i}\int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{(k^2-\Delta)^n} \\
&= \frac{1}{n}\frac{\partial}{\partial\Delta}\left(\frac{(-1)^n}{(4\pi)^{d/2}}\frac{\Gamma(n-d/2)}{\Gamma(n)}\frac{1}{\Delta^{n-d/2}}\right) \\
&= \frac{(-1)^{n+1}}{(4\pi)^{d/2}}\frac{\Gamma(n-d/2)}{n\Gamma(n)}\left(n-\frac{d}{2}\right)\frac{1}{\Delta^{n+1-d/2}} \\
&= \frac{(-1)^{n+1}}{(4\pi)^{d/2}}\frac{\Gamma(n+1-d/2)}{\Gamma(n+1)}\frac{1}{\Delta^{n+1-d/2}} \qquad \square
\end{aligned}
$$

There is another change in d dimensions. Since $S = \int \mathrm{d}^d x\mathcal{L}$ is dimensionless (keep in mind we are working in natural units), $[\mathcal{L}] = M^d$. So $\mathcal{L}_{\mathrm{KG}} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{M^2}{2}\phi^2$ suggests now $[\phi] = M^{d/2-1}$ and in Dirac theory $[\psi] = M^{\frac{d-1}{2}}$. So in order to keep $[\lambda] = M^0 = 1$, $\mathcal{L}_{\phi^4} = -\mu^{4-d}\frac{\lambda}{4!}\phi^4$ with μ an arbitrary mass parameter $[\mu] = M^1$.

With dimensional regularization



$$
\begin{aligned}
&= \frac{\mu^{4-d}\lambda}{2}\frac{1}{i}\int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{k^2-M^2+i\epsilon} \\
&= \frac{\lambda}{2}\mu^{4-d}\left(-\frac{1}{(4\pi)^{d/2}}\right)M^{d-2}\Gamma(1-d/2)
\end{aligned}
$$

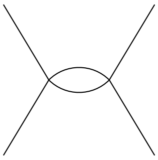
	Laurent expansion
$\Gamma(z)$	$= \frac{1}{z} - \gamma_E + O(z), \quad z \to 0$
$\Gamma(z-1)$	$= \frac{1}{z-1}\Gamma(z), \quad z \to 0$
	$= -\left(1 + z + O(z^2)\right)\Gamma(z)$
	$= -\frac{1}{z} + \gamma_E - 1 + O(z)$
γ_E	$= 0.5772\ldots$

$$
= -\frac{\lambda}{2}\frac{M^2}{8\pi^2}\left(\frac{M^2}{4\pi\mu^2}\right)^{\frac{d-4}{2}}\left[\frac{1}{d-4}+\frac{1}{2}(\gamma_E - 1) + O(d-4)\right]
$$

Taylor expansion around $\epsilon = 0$, $a^\epsilon = 1 + \epsilon\ln a$

$$
= -\frac{\lambda}{2}\frac{M^2}{8\pi^2}\left\{\frac{1}{d-4}+\frac{1}{2}\left[\gamma_E - 1 - \ln(4\pi)\right] + \ln\frac{M}{\mu} + O(d-4)\right\}
$$

with $\Delta(x) = M^2 - x(1-x)p^2$



$$= \frac{\mu^{2(4-d)}\lambda^2}{2}\frac{1}{i}\int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{[k^2-\Delta(x)]^2}$$

$$= \frac{\lambda^2}{2}\mu^{2(4-d)}\int_0^1 \mathrm{d}x \frac{1}{(4\pi)^{d/2}}\frac{\Gamma(2-d/2)}{2}\frac{1}{\Delta(x)^{2-d/2}}$$

$$= \frac{\lambda^2}{2}\frac{\mu^{4-d}}{(4\pi)^2}\left\{-2\left[\frac{1}{d-4}+\frac{1}{2}(\gamma_E - \ln 4\pi) + \ln\left(\frac{M}{\mu}\right)\right] - \int_0^1 \mathrm{d}x \ln\left(\frac{\Delta(x)}{M^2}\right)\right\}$$

$$\int_0^1 \mathrm{d}x \ln\left(\frac{\Delta(x)}{M^2}\right) = \int_0^1 \mathrm{d}x \ln \frac{M^2 - x(1-x)p^2}{M^2}$$

$$= \int_0^1 \mathrm{d}x \ln\left[\left(\frac{\sigma+1}{2} - x\right)\left(x + \frac{\sigma-1}{2}\right)\right] - \ln\frac{\sigma^2-1}{4}, \quad \sigma = \sqrt{1-\frac{4M^2}{p^2}}$$

$$= \sigma \ln\frac{\sigma+1}{\sigma-1} - 2$$

Valid for $p^2 < 0$, rest by analytic continuation

Compare $M(s) - M(0)$ calculated based on Cutkosky and dispersion integral. Easier

$$M(0) = \frac{1}{i}\int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{(k^2-M^2)^2}$$

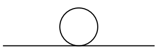
$$= \frac{\partial}{\partial M^2}\frac{1}{i}\int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{k^2-M^2}$$

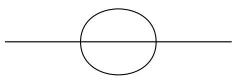
$$= \frac{\partial}{\partial M^2}\left\{-\frac{M^2}{8\pi^2}\left[\frac{1}{d-4}+\frac{1}{2}(\gamma_E - 1 - \ln 4\pi) + \frac{1}{2}\ln\frac{M^2}{\mu^2}\right]\right\}$$

1 gets cancalled by the derivative of ln

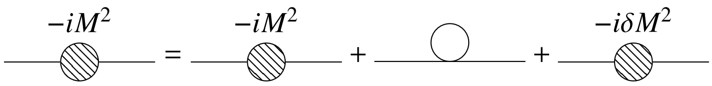
$$= -\frac{1}{8\pi^2}\left[\frac{1}{d-4}+\frac{1}{2}\left(\gamma_E - \ln 4\pi + \frac{1}{2}\ln\frac{M^2}{\mu^2}\right)\right]$$

Lets summarise the renormalization of ϕ^4 at one loop

-  is independent of p^2! Hence $\Sigma(p^2)$ at $O(\lambda)$ only renormalises the mass, there is no wave-function renormalization $Z(\sim \frac{\partial\Sigma}{\partial p^2}|_{p^2=M^2}) \quad \to Z = 1 + O(\lambda^2)$

  This does change at $O(\lambda^2)$  $\to Z \neq 1$

- Mass renormalization



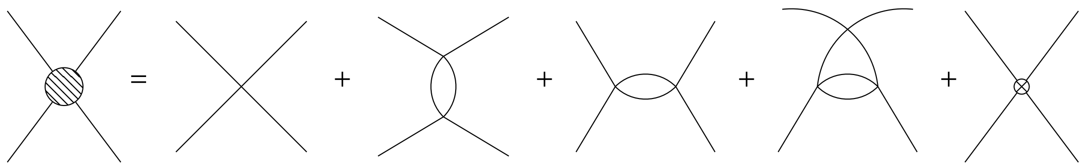
$$\delta M^2 = M_0^2 - M^2$$

then

$$M^2 = M^2 + \frac{\lambda M^2}{16\pi^2}\left[\frac{1}{d-4} + \frac{1}{2}\left(\gamma_E - 1 - \log 4\pi + \log\frac{M}{\mu}\right)\right] - M^2 + M_0^2$$
$$\delta M^2 = \frac{\lambda M^2}{16\pi^2}\left[\frac{1}{d-4} + \frac{1}{2}\left(\gamma_E - 1 - \log 4\pi + \log\frac{M}{\mu}\right) + O(\lambda, (d-4))\right]$$

Physical mass M^2_{phy} cannot be dependent on μ, meaning $\lambda\mu^{4-d}M^2 = \lambda_0 M_0^2 + O(\lambda^2)$ and λ_0 and M_0 are independent of μ.

- Coupling constant renormalization. Lets choose renormalization point for λ at $s = t = u = 0$ for simplicity:



$$= -i\lambda\mu^{4-d} + i\left(M(s) + M(t) + M(u)\right) - i(Z_\lambda - 1)\lambda\mu^{4-d} + O(\lambda^3)$$

with $Z = 1$

$$\lambda_0 = \lambda\mu^{4-d}Z_\lambda = \lambda\mu^{4-d}\Big\{\underbrace{1 - \frac{3}{\lambda}16\pi^2\Big[\frac{1}{d-4}}_{Z_\lambda^{MS}\text{ minimal subtraction}} + \frac{1}{2}\Big(\gamma_E - \log 4\pi + \log\frac{M}{\mu}\Big)\Big] + O(\lambda^2)\Big\}$$
$$= \lambda\mu^{4-d}Z_\lambda = \lambda\mu^{4-d}\Big\{\underbrace{1 - \frac{3}{\lambda}16\pi^2\Big[\frac{1}{d-4} + \frac{1}{2}\Big(\gamma_E - \log 4\pi}_{Z_\lambda^{\overline{MS}}\text{ modified minimal subtraction}} + \log\frac{M}{\mu}\Big)\Big] + O(\lambda^2)\Big\}$$

these two Z are mass-indepent

$$= \lambda\mu^{4-d}Z_\lambda = \lambda\mu^{4-d}\Big\{\underbrace{1 - \frac{3}{\lambda}16\pi^2\Big[\frac{1}{d-4} + \frac{1}{2}\Big(\gamma_E - \log 4\pi + \log\frac{M}{\mu}\Big)\Big]}_{Z_\lambda\text{mass-dependent}} + O(\lambda^2)\Big\}$$

6.6 Superficial degree of divergence

How do we know that we are done renormalising the theory with

- wave function
- mass
- coupling

Can't there be more divergences?
Want to analyse superficial degree of divergence D of an arbitrary loop diagram with

- d dimension
- L number of loops
- I number of internal propagators

- E number of external lines
- V number of vertices

Matrix element of an arbitrary diagram generically

$$\sim \lambda^V \int \frac{\mathrm{d}^d k_1 \mathrm{d}^d k_2 \dots \mathrm{d}^d k_L}{(k_{i_1}^2 - M^2) \dots (k_{i_I}^2 - M^2)}$$

So clearly

$$D = dL - 2I \tag{6.6.1}$$

$D \geq 0$ divergent ($D = 0$ logarithmically divergent) and $D < 0$ convergent.
Express L and I in terms of V and E

$$\begin{aligned} L &= \text{number of undetermined integration momenta} \\ &= \text{number of internal propagators} - \text{number of momentum conservations at vertices} \\ &\quad + 1 \text{ (because of overall momentum conservation)} \\ L &= I - V + 1 \end{aligned} \tag{6.6.2}$$

One vertex is linked to 4 legs. Internal lines are attached to 2 vertices and external line to 1.

$$4V = 2I + E \tag{6.6.3}$$

solve 6.6.2 and 6.6.3 for L and I

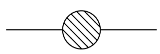
$$D = d + (d-4)V - \left(\frac{d}{2} - 1\right)E \tag{6.6.4}$$

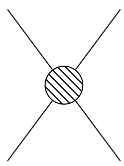
in physical 4 dimension

$$D = 4 - E \tag{6.6.5}$$

Remarks

- for $d = 4$, D is independent of V, only dependent on E.
- only a few small E produce $D \geq 0$, here in ϕ^4

$E = 2$ 

$E = 4$ 

- distinguish theories of different d
  - $d < 4$: D decreases with V, only finite number of diagrams (not n-point functions) diverges. **super-renormalizable**
  - $d = 4$: D is independent of V, only a finite number of amplitudes diverges, but at each order in perturbation theory. **renormalizable**

  - $d > 4$: D grows with V, even amplitude becomes divergent at some order in perturbation theory. **non-renormalizable**

- alternative characterisation in terms of mass dimension of coupling constant

$$\mathcal{L}_{\phi^4} = -\mu^{4-d}\frac{\lambda}{4!}\phi^4 = -\frac{\tilde{\lambda}}{4!}\phi^4$$

  so $[\tilde{\lambda}] = 4 - d$ in d dimension; hence
  - $[\tilde{\lambda}] > 0$ super-renormalizable
  - $[\tilde{\lambda}] = 0$ renormalizable
  - $[\tilde{\lambda}] < 0$ non-renormalizable

- why is this "superficial"? There can always be divergent sub-graphs! These sub-graphs are regularised and renormalised by the treatment of the "primitive divergences" we have already seen before.

Conclusion for ϕ^4 the only primitive divergences are $E = 2$ and $E = 4$ (and $E = 0$ the vacuum graphs) and we renormalise the theory by

$$\begin{aligned} M_0^2 &= M^2\left\{1 + c_m^{(1)}\frac{\lambda}{d-4} + c_m^{(2)}\frac{\lambda^2}{(d-4)^2} + \dots\right\} \\ \lambda_0 &= \lambda\left\{1 + c_\lambda^{(1)}\frac{\lambda}{d-4} + c_\lambda^{(2)}\frac{\lambda^2}{(d-4)^2} + \dots\right\} \\ Z &= 1 + c_z^{(2)}\frac{\lambda^2}{(d-4)^2} + \dots \end{aligned}$$

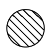
6.7 Sketch of renormalization of QED

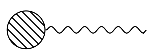
Superficial degree of divergence with E_γ external photons and E_e external electrons.

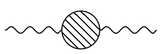
$$\begin{aligned} D &= d + V\left(\frac{d-4}{2}\right) - E_e\left(\frac{d-1}{2}\right) - E_\gamma\left(\frac{d-2}{2}\right) \\ &= 4 - \frac{3}{2}E_e - E_\gamma \end{aligned} \tag{6.7.1}$$

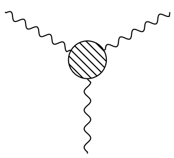
Again the superficial degree of divergence D is independent of V, i.e. QED is renormalizable (e is dimensionless as well).

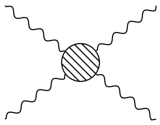
We have following divergent diagrams

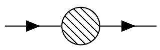
(a)  $D = 4$

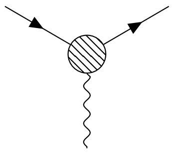
(b)  $D = 3$

(c)  $D = 2$

(d)  $D = 1$

(e)  $D = 0$

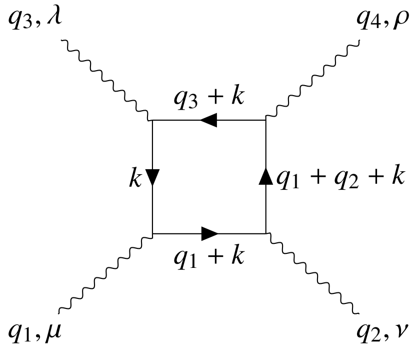
(f)  $D = 1$

(g)  $D = 1$

Still need to show that all divergences can be absorbed in the renormalization of the parameters of the theory.

$$\begin{aligned}
e_0 &\longmapsto e \\
m_0 &\longmapsto m \\
\psi &\longmapsto Z_2^{-1/2}\psi \\
A_\mu &\longmapsto Z_3^{-1/2}A_\mu \\
Z_1 &\equiv \text{ vertex correction (diagram (g) as } Z_\lambda \text{ in } \phi^4)
\end{aligned}$$

We now focus on individual divergent graphs. Diagram (a), vacuum diagram, is to be ignored. Diagrams (b) and (d) are not possible, since QED is C-invariant, $A_\mu \longmapsto -A_\mu$. It can be generalized into Furry's theorem: correlation functions of odd number of photons vanish. Diagram (e) is worrisome! Divergence would require counter-terms $\sim (F_{\mu\nu}F^{\mu\nu})^2, (F_{\mu\nu}\tilde{F}^{\mu\nu})^2$, dimension 8 operator. $[g_{A^4}] = M^{-4}$, i.e. non-renormalizable! Here gauge-invariance is to rescue.



$$= \frac{e^4}{i}\int \frac{\mathrm{d}^d k}{(2\pi)^4} \frac{\gamma_\mu(\not{k} + m)\gamma_\lambda(\not{q}_3 + \not{k} + m)\gamma_\rho(\not{q}_1 + \not{q}_2 + \not{k} + m)\gamma_\nu(\not{q}_1 + \not{k} + m)}{(k^2 - m^2)[(q_3 + k)^2 - m^2][(q_1 + q_2 + k)^2 - m^2][(q_1 + k)^2 - m^2]}$$

$$\begin{aligned}
&= \frac{e^4}{i}\int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{\gamma_\mu \not{k}\gamma_\lambda \not{k}\gamma_\rho \not{k}\gamma_\nu \not{k}}{k^8} + \text{convergent terms} \\
&= e^4 I(g_{\mu\lambda}g_{\rho\nu} + g_{\mu\rho}g_{\lambda\nu} + g_{\mu\nu}g_{\lambda\rho}) + \text{ finite}
\end{aligned}$$

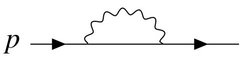
because of Ward Identity $q_1^\mu(\dots) = 0$

$$= I(q_{1\lambda}g_{\rho\nu} + q_{1\rho}g_{\lambda\nu} + q_{1\nu}g_{\lambda\rho}) + \text{ finite } = 0$$

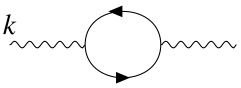
thus digram (e) is finite.

As a result the only primitively divergent graphs are diagram (c) photon self energy, (f) electron self energy, (g) vertex graph!

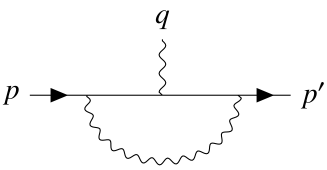
We will discuss these in detail next term. The results at one loop are



$$= -i\Sigma(p) = \frac{-ie^2}{8\pi^2(d-4)}(\not{p} - 4m) + \text{finite}$$



$$= -i\Pi_{\mu\nu}(k) = \frac{ie^2}{6\pi^2(d-4)}(g_{\mu\nu}k^2 - k_\mu k_\nu)\left[\frac{1}{d-4} + (\text{finite, const.}) - \frac{k^2}{10m^2} + \dots\right]$$



$$= ie\mu^{2-d/2}\Lambda_\mu; \quad \Lambda_\mu = \frac{-e^2}{8\pi^2(d-4)}\gamma_\mu + \text{ finite}$$

"finite" contains $\frac{\alpha}{2\pi}\frac{i\sigma_{\mu\nu}q^\nu}{2m}$ anomolous magnetic moment

We have three renormalization factors

- vertex

$$Z_1 = 1 + \frac{e^2}{8\pi^2(d-4)} \tag{6.7.2}$$

- electron wave-function

$$Z_2 = 1 + \frac{e^2}{8\pi^2(d-4)} \tag{6.7.3}$$

- photon wave-function

$$Z_3 = 1 + \frac{e^2}{6\pi^2(d-4)} \tag{6.7.4}$$

Mass renormalization

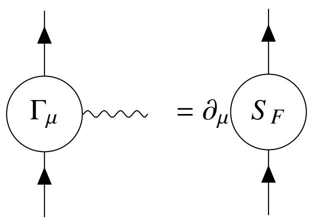
$$m_0 = Z_2^{-1}(m + \delta m) = m\left(1 + \frac{3e^2}{8\pi^2(d-4)}\right) \tag{6.7.5}$$

Coupling renormalization

$$e_0 = \mu^{2-d/2}Z_1 Z_2^{-1} Z_3^{-1/2} e = \mu^{2-d/2} Z_3^{-1/2} e \tag{6.7.6}$$

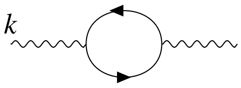
Remarks

- $Z_1 = Z_2$ is a fundamental consequence of the Ward Identity (consequence of gauge-invariance)



$$\Gamma_\mu(p.0, p) = \frac{\partial}{\partial p^\mu} i S_F^{-1}$$

Charge renormalization only depends on photon's self energy (vacuum polarisation). This is an essential reason for many particles (e. μ, p, π, ...) having the same charge!

- Vacuum polarisation  does not generate a photon mass term!

- The k^2-dependent (finite) correction in vacuum polarisation does remain after renormalization

$$D'_{\mu\nu}(k) = -ig_{\mu\nu}\left(\frac{1}{k^2} - \frac{e^2}{60\pi^2 m^2} + O(k^2)\right)$$

Fourier transformation yields potential between two charges.

$$V(r) = \frac{e^2}{4\pi r} + \frac{e^4}{60\pi^2 m^2}\delta^{(3)}(\boldsymbol{r}) \tag{6.7.7}$$

This shift, *Lamb shift*, S-levels in hydrogen atom.

6.8 The (idea of the) renormalization group

Consider N-point vacuum correlation function in momentum space $G_N(p_1, \ldots, p_N; \lambda, M, \mu)$. Its relation to bare Green's function is given by

$$G_N(p_1, \ldots, p_N; \lambda, M, \mu) = Z^{-N/2} G_N^0(p_1, \ldots, p_N; \lambda_0, M_0) \tag{6.8.1}$$

using the relation $Z^{-1/2}\phi_0 = \phi$

Since G_N^0 is independent of μ

$$\left[\mu\frac{\partial}{\partial\mu} + \mu\frac{\partial\lambda}{\partial\mu}\frac{\partial}{\partial\lambda} + \mu\frac{\partial M}{\partial\mu}\frac{\partial}{\partial M} + \frac{N}{2}\mu\frac{\partial}{\partial\mu}\ln Z\right] G_N(p; \lambda, M, \mu) = 0$$

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(\lambda)\frac{\partial}{\partial\lambda} + M\gamma_M(\lambda)\frac{\partial}{\partial M} + \frac{N}{2}\gamma(\lambda)\right] G_N(p; \lambda, M, \mu) = 0 \tag{6.8.2}$$

with $\beta(\lambda) = \mu\frac{\partial}{\partial\mu}\lambda$, $\gamma(\lambda) = \mu\frac{\partial}{\partial\mu}\ln Z$ and $M\gamma_M(\lambda) = \mu\frac{\partial}{\partial\mu}M$

(6.8.3)

This is the *renormalization group equation* (Callan-Symanzik equation)!

What is the mass dimension of G_N?

$$[\phi] = M \Rightarrow [G(x_1, \ldots, x_N)] = M^N$$
$$\Rightarrow [G_N(p_1, \ldots, p_N)] = M^{4-3N} = M^{D_N}$$

We need to recover D_N by counting

- the power of momenta
- the power of masses
- the power of μ

or

$$\left[t\frac{\partial}{\partial t} + M\frac{\partial}{\partial M} + \mu\frac{\partial}{\partial \mu} - D_N\right] G_N\left(\{t_p\}; \lambda, M, \mu\right) = 0 \tag{6.8.4}$$

Eliminating $\mu\frac{\partial}{\partial\mu}$ between 6.8.2 and 6.8.4 leads to

$$\left[-t\frac{\partial}{\partial t} + \beta\frac{\partial}{\partial \lambda} + M(\gamma_M - 1)\frac{\partial}{\partial M} + (D_N + \frac{N}{2}\gamma)\right] G_N\left(\{t_p\}; \lambda, M.\mu\right) = 0 \tag{6.8.5}$$

Expresses the effect of scaling momenta in G_N by a factor of t. $\frac{N}{2}\gamma$ is the anomalous dimension which gets added to the "engineering dimension" D_N. This equation suggests that an overall change in momentum scale t can be compensated by

- changing the coupling λ
- rescaling the mass M
- an overall factor

hence

$$G_N\left(\{t_p\}; \lambda, M, \mu\right) = f(t) G_N\left(\{p\}; \lambda(t), M(t), \mu\right)$$

A little algebra and comparing to 6.8.5 leads to

$$\begin{aligned}
t\frac{\partial\lambda(t)}{\partial t} &= \beta(\lambda) \\
t\frac{\partial M(t)}{\partial t} &= M(\gamma_M(\lambda) - 1) \\
t\frac{\partial f(t)}{\partial t} &= D_N + \frac{N}{2}\gamma \\
\Rightarrow f(t) &= t^{D_N}\exp\left\{\frac{N}{2}\int_1^t \frac{\gamma(\lambda(s))}{s}\mathrm{d}s\right\}
\end{aligned}$$

Solution in terms of running mass $M(t)$ and running coupling $\lambda(t)$! Discuss running coupling $\lambda(t)$ (determined by "beta function" $\beta(\lambda)$). In ϕ^4-theory

$$\lambda(\mu) = \mu^{4-d}\lambda_0 + \frac{3\lambda_0^2}{16\pi^2}\mu^{2(d-4)}\left[\frac{1}{d-4} + \dots\right] + O(\lambda_0^3, d-4) \tag{6.8.6}$$

with $\lambda^2 = \lambda_0^2\mu^{2(d-4)} + O(\lambda_0^3)$ and $\mu\frac{\partial}{\partial\mu}\lambda_0 = 0$

$$\beta(\lambda) = \lim_{d\to 4}\mu\frac{\partial}{\partial\mu}\lambda = \frac{3\lambda^2}{16\pi^2} + O(\lambda^3) > 0 \tag{6.8.7}$$

$\lambda(t)$ grows with t; approaching large(r) momenta

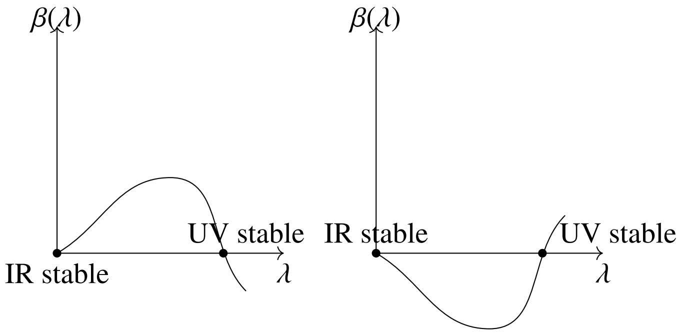
(6.8.8)

Solution to $\frac{\mathrm{d}\lambda}{\mathrm{d}\ln\mu} = \frac{3\lambda^2}{16\pi^2}$ is (with $\lambda(\mu_0)$ as an integration constant)

$$\lambda(\mu) = \frac{\lambda(\mu_0)}{1 - \frac{3}{16\pi^2}\lambda(\mu_0)\ln(\mu/\mu_0)} \tag{6.8.9}$$

In different theories, sign of β (coupling) can be different, coupling can decrease with energy (e.g. Yang-Mills, QCD, ...)

Interesting asymptotic (IR/UV) behaviour of couplings depending on form of $\beta(\lambda) = \mu\frac{\partial}{\partial\mu}\lambda(\mu)$



Non-perturbative zeros $\beta(\lambda_{NP}) = 0$ and perturbative $\beta(0) = 0$ are fixed points for of RG flow. The second picture has *asymptotic freedom* for $t \longmapsto \infty$, $\lambda(t) \longmapsto 0$ is a (UV-stable) fixed point; we believe QCD to have this property.