H.7

a) Lemma: $tr(\gamma^\mu) = 0$

since $\{\gamma^\mu, \gamma^\mu\} = 2g^{\mu\mu}\mathbb{1} \Rightarrow \gamma^\mu\gamma^\mu = g^{\mu\mu}\mathbb{1}$

$\Rightarrow tr(\gamma^\mu) = tr(\mathbb{1}\cdot\gamma^\mu) = \frac{1}{g^{\nu\nu}} tr(\gamma^\nu\gamma^\nu\gamma^\mu)$

$$\underset{\text{anticommtator}}{=} -\frac{1}{g^{\nu\nu}} tr(\gamma^\nu\gamma^\mu\gamma^\nu) \underset{\text{permutation}}{=} -\frac{1}{g^{\nu\nu}} tr(\gamma^\nu\gamma^\nu\gamma^\mu)$$

$\Rightarrow tr(\gamma^\mu) = 0$

$tr(\gamma^{\mu_1} \cdots \gamma^{\mu_{2n+1}})$, μ_k, $k = \{0, \ldots, 2n+1\}$ don't have to be distinctive

<u>Method 1:</u>

Using anticommtations we can rewrite the whole product into these two expressions

$$\gamma^{\mu_1} \cdots \gamma^{\mu_{2n+1}} = \begin{cases} (-1)^\xi \gamma^\nu, \\ (-1)^\xi \gamma^\alpha\gamma^\beta\gamma^\nu, \\ \alpha \neq \beta \neq \nu, \\ \alpha, \beta, \nu \in \{0,1,2,3\} \end{cases}$$

ξ is an intger number, come from anticommtations and $(\gamma^k)^2 = -\mathbb{1}$

(no summation is used)

$Tr((-1)^\xi\gamma^\nu) = (-1)^\xi Tr(\gamma^\nu) = 0$

$$Tr((-1)^\xi\gamma^\alpha\gamma^\beta\gamma^\nu) = (-1)^\xi Tr(\gamma^\alpha\gamma^\beta\gamma^\nu) = (-1)^\xi Tr(\gamma^5\gamma^5\gamma^\alpha\gamma^\beta\gamma^\nu)$$
$$= (-1)^\xi Tr(\gamma^5\gamma^\alpha\gamma^\beta\gamma^\nu\gamma^5)$$
$$\underset{\text{anticommtation}}{=} -(-1)^\xi Tr(\gamma^5\gamma^5\gamma^\alpha\gamma^\beta\gamma^\nu)$$

$\Rightarrow Tr(\gamma^{\mu_1} \cdots \gamma^{\mu_{2n+1}}) = 0$

<u>Method 2:</u> (better I suppose)

$tr(\gamma^{\mu_1} \cdots \gamma^{\mu_{2n+1}}) = tr(\gamma^5\gamma^5\gamma^{\mu_1} \cdots \gamma^{\mu_{2n}}\gamma^{\mu_{2n+1}})$

$$= tr(\gamma^5 \gamma^{\mu_1} \cdots \gamma^{\mu_{2n+1}} \gamma^5)$$

$$= (-1)^{2n+1} tr(\gamma^5 \gamma^5 \gamma^{\mu_1} \cdots \gamma^{\mu_{2n+1}})$$

$\{\gamma^5, \gamma^\mu\} = 0$

$$= - tr(\gamma^{\mu_1} \cdots \gamma^{\mu_{2n+1}}) = 0$$

$$tr(\gamma^{\mu_1} \cdots \gamma^{\mu_{2n+1}} \gamma^5)$$

$$= tr(\gamma^5 \gamma^{\mu_1} \cdots \gamma^{\mu_{2n+1}})$$

$$= (-1)^{2n+1} tr(\gamma^{\mu_1} \cdots \gamma^{\mu_{2n+1}} \gamma^5)$$

$$= 0$$

b) $$tr(\gamma^5) = tr(\gamma^5 \gamma^0 \gamma^0)$$

$$= - tr(\gamma^0 \gamma^5 \gamma^0)$$

$$= - tr(\gamma^5 \gamma^0 \gamma^0)$$

$$= 0$$

$$tr(\gamma^5 \gamma^\mu \gamma^\nu) = (-1)^\xi tr(\gamma^5 \gamma^\alpha \gamma^\alpha \gamma^\mu \gamma^\nu) ,$$

$$\left(\begin{array}{l} \alpha \neq \mu , \ \alpha = \nu \\ \xi = 0 \ \text{if} \ \alpha = 0 \\ \xi = 1 \ \text{if} \ \alpha = 1,2,3 \end{array} \right.$$

$$= (-1)^{\xi+1} tr(\gamma^\alpha \gamma^5 \gamma^\mu \gamma^\nu \gamma^\alpha)$$

$$= (-1)^{\xi+1} tr(\gamma^5 \gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\alpha)$$

$$= - tr(\gamma^5 \gamma^\mu \gamma^\nu)$$

$$= 0$$

c) $$tr(\gamma^{\mu_1} \cdots \gamma^{\mu_{2n}})$$

$$= tr(\gamma^{\mu_1} \gamma^{\mu_2} \cdots \gamma^{\mu_k} \cdots \gamma^{\mu_{2n}})$$

$$= tr\left[(-\gamma^{\mu_2} \gamma^{\mu_1} + 2 g^{\mu_1 \mu_2} \mathbb{1}) \gamma^{\mu_3} \cdots \gamma^{\mu_k} \cdots \gamma^{\mu_{2n}}\right]$$

$$= -\mathrm{tr}[\gamma^{\mu_2}\gamma^{\mu_1}\gamma^{\mu_3}\dots\gamma^{\mu_k}\dots\gamma^{\mu_{2n}}] + 2g^{\mu_1\mu_2}\,\mathrm{tr}[\gamma^{\mu_3}\dots\gamma^{\mu_k}\dots\gamma^{\mu_{2n}}]$$

$$\vdots$$

$$= (-1)^{2n-1}\,\mathrm{tr}[\gamma^{\mu_2}\dots\gamma^{\mu_k}\dots\gamma^{\mu_{2n}}\gamma^{\mu_1}]$$

$$+ 2\sum_{k=2}^{2n}(-1)^k g^{\mu_1\mu_k}\,\mathrm{tr}[\gamma^{\mu_2}\dots\gamma^{\mu_{k-1}}\gamma^{\mu_{k+1}}\dots\gamma^{\mu_{2n}}]$$

1st. term is $\mathrm{tr}[\gamma^{\mu_1}\dots\gamma^{\mu_{2n}}]$

$$\Rightarrow \quad \mathrm{tr}[\gamma^{\mu_1}\dots\gamma^{\mu_{2n}}]$$

$$= \sum_{k=2}^{2n}(-1)^k g^{\mu_1\mu_k}\,\mathrm{tr}[\gamma^{\mu_2}\dots\gamma^{\mu_{k-1}}\gamma^{\mu_{k+1}}\dots\gamma^{\mu_{2n}}]$$

d) $\mathrm{tr}[\gamma^\mu\gamma^\nu] = g^{\mu\nu}\,\mathrm{tr}[\mathbb{1}]$, $n=1$

$$= 4g^{\mu\nu}$$

e) $\mathrm{tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] = \sum_{k=2}^{4}(-1)^k g^{\mu\mu_k}\,\mathrm{tr}[\dots]$, $n=4$

$$= g^{\mu\nu}\,\mathrm{tr}[\gamma^\rho\gamma^\sigma] - g^{\mu\rho}\,\mathrm{tr}[\gamma^\nu\gamma^\sigma] + g^{\mu\sigma}\,\mathrm{tr}[\gamma^\nu\gamma^\rho]$$

$$= 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})$$

f) $n=1$, $\mathrm{tr}[\gamma^\mu\gamma^\nu] = 4g^{\mu\nu}$

$n=2$, $\mathrm{tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})$

Assume (9) is true $\forall n$

$$\mathrm{tr}[\gamma^{\mu_1}\dots\gamma^{\mu_{2n}}] = 4\sum_{p\in P}\mathrm{sgn}(p)\prod_{i=1}^{n} g^{\mu_{\alpha i}}g^{\mu_{\beta i}}$$

$$\mathrm{tr}[\gamma^{\mu_1}\dots\gamma^{\mu_{2n}}\gamma^{\mu_{2n+1}}\gamma^{\mu_{2n+2}}]$$

$$\underset{(8)}{=} \sum_{k=2}^{2n} g^{\mu_1\mu_k}(-1)^k\,\mathrm{Tr}[\gamma^{\mu_2}\dots\gamma^{\mu_{k-1}}\gamma^{\mu_{k+1}}\dots\gamma^{2n+2}]$$

$$= \sum_{k=2}^{2n} g^{\mu_1\mu_k}(-1)^k \cdot 4\sum_{p\in P}\mathrm{sgn}(p)\prod_{i=1}^{n} g^{\mu_{\alpha i}}g^{\mu_{\beta i}}$$

$$= 4 \sum_{p \in P} \mathrm{sgn}(p) \prod_{i=1}^{n+1} g^{\mu_{\alpha_i}} g^{\mu_{\beta_i}}$$

$\Rightarrow$ proven

(g) if μ, ν, ρ, σ not permutation

$$\Rightarrow \quad tr[\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^5] \propto tr[\gamma^\alpha \gamma^\beta \gamma^5] = 0$$

$$tr[\gamma^0 \gamma^1 \gamma^2 \gamma^3 \gamma^5] = -i\, tr[(\gamma^5)^2] = -i\, 4 = -4i$$

$$\text{if } tr[\gamma^0 \underbrace{\gamma^2 \gamma^1}_{\text{anti commute}} \gamma^3 \gamma^5] = (-1) \cdot (-4i)$$

$$\Rightarrow \quad tr[\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^5] = -4i\, \epsilon^{\mu\nu\rho\sigma}$$