

A.1

- a) • S is a Poincaré scalar
 • $\delta S = 0$ if ε is a constant
 • $|\varepsilon| \ll 1$

$$\delta S = \int d^4x [\varepsilon(x) \partial_\mu j^\mu(x) + j^\mu(x) \partial_\mu \varepsilon(x)]$$

$$\begin{aligned} \text{b) } \delta S &= \int \left\{ \frac{\partial \mathcal{L}}{\partial \phi_k} \varepsilon(x) \delta \phi_k(x) + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_k)} \partial_\mu [\varepsilon(x) \delta \phi_k(x)] \right\} d^4x \\ &= \int \left\{ \varepsilon(x) \left[\frac{\partial \mathcal{L}}{\partial \phi_k} \delta \phi_k(x) + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_k)} \partial_\mu \delta \phi_k(x) \right] + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_k)} \delta \phi_k(x) \partial_\mu \varepsilon(x) \right\} d^4x \\ &\quad \underbrace{\hspace{10em}}_{\text{since } \delta S = 0 \text{ if } \varepsilon = \text{const}} \end{aligned}$$

$$\Rightarrow \delta S = \int \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_k)} \delta \phi_k(x) \partial_\mu \varepsilon(x) d^4x$$

$$\Rightarrow j^\mu(x) = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_k)} \delta \phi_k$$

$$\text{c) } \delta S = 0 = \int j^\mu(x) \partial_\mu \varepsilon(x) d^4x = - \int \varepsilon(x) \partial_\mu j^\mu d^4x$$

$$\varepsilon(x) \text{ is now arbitrary} \Rightarrow \partial_\mu j^\mu(x) = 0$$

$$\begin{aligned} \partial_0 Q &= \partial_0 \int j^0(x) d^3x = \int \partial_0 j^0(x) d^3x = - \int \partial_k j^k(x) d^3x \\ &= \int_{\partial \mathbb{R}^3} \vec{j} \cdot d^3x = 0 \end{aligned}$$

A.2 $\mathcal{L} = \partial_\mu \phi^*(x) \partial^\mu \phi(x) - m^2 \phi^*(x) \phi(x)$

a) $\phi(x): \partial_\omega \frac{\partial \mathcal{L}}{\partial(\partial_\omega \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} = \partial_\omega \left(\partial_\mu \phi^* g^{\mu\nu} \frac{\partial(\partial_\nu \phi)}{\partial(\partial_\omega \phi)} \right) + m^2 \phi^* \frac{\partial \phi}{\partial \phi}$
 $= (\partial_\omega \partial^\omega + m^2) \phi^* = 0$

$\phi^*(x): (\partial_\omega \partial^\omega + m^2) \phi = 0$

b) $|\Lambda| \ll 1 \quad \leftarrow \text{not losing any generality because of group property}$

$\phi(x) \rightarrow \phi(x) - i\Lambda \phi(x)$

$\phi^*(x) \rightarrow \phi^*(x) + i\Lambda \phi^*(x)$

$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} \delta \phi^* = i(\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*)$
 $\delta \rightarrow \partial^\mu$

$Q = \int d^3x j^0(x)$

$= i \int d^3x \left(\phi^* \frac{\partial \phi}{\partial t} - \phi \frac{\partial \phi^*}{\partial t} \right)$

• overall factor for j^μ symmetry, related to gauge coupling when symmetry is gauged

c) $\phi(x) \rightarrow \phi(x) - i\Lambda(x) \phi(x)$

$A_\mu(x) \rightarrow A_\mu(x) + \frac{1}{e} \partial_\mu \Lambda(x)$

$D_\mu \phi = (\partial_\mu + ieA_\mu) \phi$

$D_\mu \phi \rightarrow [\partial_\mu + ieA_\mu + i(\partial_\mu \Lambda)] [\phi - i\Lambda \phi]$

$= (\partial_\mu + ieA_\mu) \phi - i\Lambda (\partial_\mu + ieA_\mu) \phi$

$= D_\mu \phi - i\Lambda D_\mu \phi$

same as $\phi(x)$'s transformation!

d) $|\Lambda| \ll 1$

$D_\mu \phi$ is multiplied by a factor $1 - i\Lambda$

Repeatedly $\lim_{N \rightarrow \infty} (1 - \frac{i\lambda}{N})^N = e^{i\lambda}$ (with redefinition of λ)

$\rightarrow \mathcal{L}$ is invariant under local gauge

e) $F^{\mu\nu}$ is already gauge invariant itself

$$\begin{aligned} F^{\mu\nu} &\mapsto \partial^\mu (A^\nu + \frac{1}{e}(\partial^\nu \lambda)) - \partial^\nu (A^\mu + \frac{1}{e}(\partial^\mu \lambda)) \\ &= \partial^\mu A^\nu - \partial^\nu A^\mu + \frac{1}{e} \partial^\mu \partial^\nu \lambda - \frac{1}{e} \partial^\nu \partial^\mu \lambda \\ &= F^{\mu\nu} \end{aligned}$$

(f) $\mathcal{L}_M(x) = M^2 A^\mu(x) A_\mu(x)$

$$\begin{aligned} \delta \mathcal{L}_M &= M^2 (\delta A_\mu) A^\mu + M^2 A_\mu (\delta A^\mu) \\ &= 2 M^2 A^\mu (\delta A_\mu) = \frac{2M^2}{e} A^\mu \partial_\mu \lambda \neq 0 \end{aligned}$$

(space-time is not transformed, internal symmetry)

$$\delta \mathcal{L} = 0 \Leftrightarrow \delta S = 0$$