Quantum Field Theory

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1 Interacting Quantum Field Theory

4.1 Introduction and examples

Theories discussed so far are Klein-Gordon theory with spin 0

$$\mathcal{L}_{KG} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 \tag{4.1.1}$$

and Dirac theory spin $\frac{1}{2}$

$$\mathcal{L}_D = \bar{\psi}(i\not{\partial} - m)\psi \tag{4.1.2}$$

There is also $\mathcal{L}_{EM} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ for a massless vector filed. Its quantisation gives photon.

One thing they have in common is they are all quadratic in the fields. As a result:

- linear field equations
- exact quantisation
- multi-particle states without scattering or interaction
- linear Fourier decomposition, no momentum changes

To have an interacting theory with scattering, need higher powers in the field in the Lagrangians. A few examples are following

Scalar ϕ^4 theory

$$\mathcal{L} = \mathcal{L}_{KG} + \frac{\lambda}{4!}\phi^4$$

need positive sign $\lambda > 0$ for a stable theory, otherwise classical energy can be arbitrarily negative.

Equation of motions

$$(\partial^2 + m^2)\phi = -\frac{\lambda}{3!}\phi^3$$

is non-linear, cannot be solved by Fourier decomposition.

Yukawa-theory

$$\mathcal{L} = \mathcal{L}_{KG} + \mathcal{L}_D - g\bar{\psi}\psi\phi$$

is originally developed as a theory for nuclear forces with ψ nucleon, ϕ pion. In the Standard Model it is similar to interactions in Higgs mechanism.

Quantum Electrodynamics (QED)

$$\mathcal{L} = \mathcal{L}_{EM} + \mathcal{L}_D - eA_\mu\bar{\psi}\gamma^\mu\psi$$

describes electrons, their antiparticles positrons and photons.

Yang-Mills theory generalises $\mathcal{L}_{EM}$ with terms like A^4 or $A^2\partial A$

Scalar QED describes pions and photons

$$\begin{aligned}\mathcal{L} &= \mathcal{L}_{EM} + D_\mu\phi D^\mu\phi^* - m^2|\phi|^2 \\ &= \mathcal{L}_{EM} + \partial_\mu\phi\partial^\mu\phi^* - m^2\phi\phi^* + ieA_\mu(\phi\partial^\mu\phi^* - \phi^*\partial^\mu\phi) + e^2A_\mu A^\mu\phi\phi^*\end{aligned}$$

Remarks

- Interaction terms in $H_{\text{int}} = \int d^3x\mathcal{H}_{\text{int}} = -\int d^3x\mathcal{L}_{\text{int}}$ always involves products of fields at the same point $\boldsymbol{x}$. It ensures causality, no "instant at a distance".
- There are no derivative interactions. These may complicate quantisation as

$$\pi(\boldsymbol{x}) = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi(\boldsymbol{x}))}$$

- Why are we taking the examples above? There must be zillions of theories (Lagrangians)?
We have the criterion of **renormalizability**. Note the mass dimensions of fields;

$$[S] = 1 \text{ so } [\mathcal{L}] = [M]^4 \Rightarrow [\phi] = [M],\ [\psi] = [M]^{\frac{3}{2}},\ [A_\mu] = [M]$$

So in all the interaction terms indicated above, the coupling constant λ, e, g are all **dimensionless**!
Can add $-\frac{\mu}{3!}\phi^3$ to the ϕ^4 theory. This leads to $[\mu] = [M]$ and all these generate renormalizable interactions.
All higher interaction terms require coupling constants of **negative** mass dimension, e.g. $G\bar{\psi}\psi\bar{\psi}\psi$ and then $[G] = [M]^{-2}$. These are non-renormalizable and create trouble when performing higher-order calculation in perturbation theory. (with energy cut-off; corrections $G\Lambda^2$, $\Lambda \to \infty$)

- We haven't quantised the photon yet. The reason is that its is a vector field, i.e. 4 degrees of freedom, but photon has just 2 physical polarisation states. It is linked to gauge symmetry and complicates quantisation somewhat.

4.2 The interaction picture

Consider the ϕ^4 theory,

$$\mathcal{L}_{int} = -\frac{\lambda}{4!}\phi(x)^4 \tag{4.2.1}$$

Hamiltonian $H = H_0 + H_{int}$ with

$$H_0 = \int d^3x \left\{\frac{1}{2}\pi^2(x) + \frac{1}{2}(\boldsymbol{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2\right\} \tag{4.2.2}$$

$$H_{\text{int}} = -\int d^3x\mathcal{L}_{\text{int}} = \frac{\lambda}{4!}\int d^3x\phi^4 \tag{4.2.3}$$

Interaction picture means that *operators* evolve in time using H_0 (only), in particular

$$\phi_I(t, \boldsymbol{x}) = e^{iH_0 t}\phi(\boldsymbol{x})e^{-iH_0 t} \tag{4.2.4}$$

Time-dependence of the free field obeys classical equation of motion $\left(\partial^2 + m^2\right)\phi_I(t, \boldsymbol{x}) = 0$. Solution in terms of Fourier modes as before:

$$\phi_I(t, \boldsymbol{x}) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3 \sqrt{2E_p}}(a^I_{\boldsymbol{p}} e^{-ipx} + a^{I\,\dagger}_{\boldsymbol{p}} e^{+ipx}) \tag{4.2.5}$$

as in the free theory with standard commutation relations $[a^I_{\boldsymbol{p}}, a^{I\,\dagger}_{\boldsymbol{p}}] = (2\pi)^3\delta^{(3)}(\boldsymbol{p}-\boldsymbol{p}')$. The state satisfying $a^I_p\,|0\rangle = 0$ is the vacuum of the free, non-interacting theory.

Relation between interaction and Schrödinger picture states:

$$|\psi_I(t)\rangle = e^{iH_0 t}|\psi_S(t)\rangle \tag{4.2.6}$$

Schrödinger equation becomes:

$$\begin{aligned}
i\frac{\partial}{\partial t}\,|\psi_S\rangle &= (H_0 + H_{\text{int}})\,|\psi_S\rangle \\
\text{LHS} &= i\frac{\partial}{\partial t}\left(e^{-iH_0 t}\,|\phi_I\rangle\right) = H_0 e^{-iH_0 t}\,|\phi_I\rangle + e^{-iH_0 t} i\frac{\partial}{\partial t}\,|\phi_I\rangle \\
\text{RHS} &= (H_0 + H_{\text{int}})\,e^{-iH_0 t}\,|\phi_I\rangle \\
\Rightarrow i\frac{\partial}{\partial t}\,|\phi_I\rangle &= e^{iH_0 t}H_{\text{int}}e^{-iH_0 t} = H_I(t)\,|\phi_I\rangle
\end{aligned} \tag{4.2.7}$$

with H_I interaction Hamiltonian in the interaction picture. Clearly

$$H_I = \frac{\lambda}{4!}\int \mathrm{d}^3 x \phi_I^4(x)$$

What is the solution of **??** for the time evolution of $|\phi_I(t)\rangle$? Define time-evolution operator in the interaction picture.

$$|\phi_I(t)\rangle = U(t, t_0)\,|\phi_I(t_0)\rangle \tag{4.2.8}$$

$$\text{with } U(t, t_0) = e^{iH_0(t-t_0)}e^{-iH(t-t_0)} \tag{4.2.9}$$

With **??** and **??**:

$$i\frac{\partial}{\partial t}U(t, t_0) = H_I(t)U(t, t_0) \tag{4.2.10}$$

To solve with boundary conditions $U(t_0, t_0) = \mathbb{1}$. The formal solution is then:

$$U(t, t_0) = 1 - i\int_{t_0}^{t} \mathrm{d}t' H_I(t')U(t', t_0)$$

Substitute back in and we get:

$$U(t, t_0) = 1 - i\int_{t_0}^{t} \mathrm{d}t' H_I(t') + (-i)^2\int_{t_0}^{t} \mathrm{d}t' \int_{t_0}^{t'} \mathrm{d}t'' H_I(t')H_I(t'') + \dots$$

H_I inside the integral is automatically time-ordered. Ranges of integration is not.

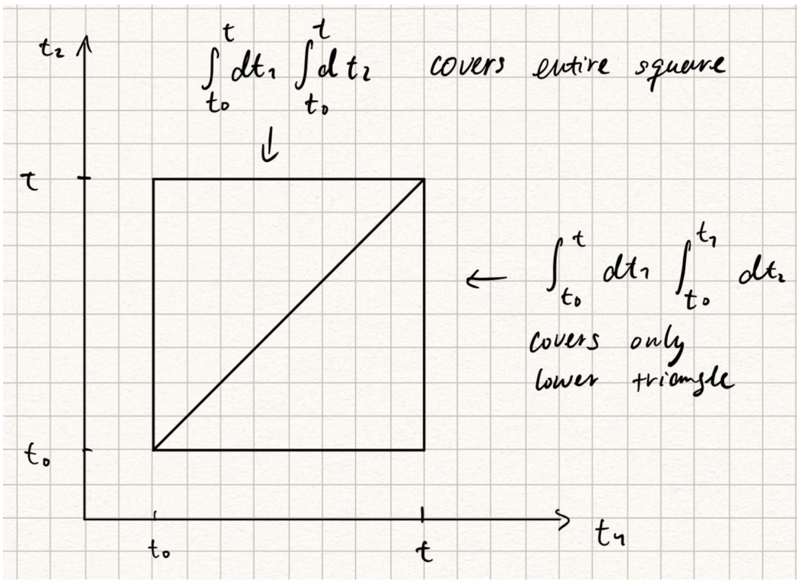


Figure 4.1: Time ordering

Upper triangle has the wrong time order. We are going to "repair" it by hand.

$$\begin{aligned}
U(t,t_0) &= 1 - i\int_{t_0}^{t}\mathrm{d}t' H_I(t') + \frac{(-i)^2}{2}\int_{t_0}^{t}\mathrm{d}t'\int_{t_0}^{t}\mathrm{d}t'' T(H_I(t')H_I(t'')) + \ldots \\
&= \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int_{t_0}^{t}\mathrm{d}t_1 \cdots \int_{t_0}^{t}\mathrm{d}t_n T(H_I(t_1)\ldots H_I(t_n)) \\
&= T\exp\left\{-i\int_{t_0}^{t}\mathrm{d}t' H_I(t')\right\} \qquad (4.2.11)
\end{aligned}$$

It is interesting for scattering to transition into asymptotic state for $t \to \infty$

$$\begin{aligned}
S &= \lim_{t\to\infty} U(t,-t) = T\exp\left\{-i\int_{-\infty}^{\infty}\mathrm{d}t H_I(t)\right\} \qquad (4.2.12) \\
&\overset{\phi^4}{=} T\exp\left\{-i\int \mathrm{d}^4x \frac{\lambda}{4!}\phi_I^4(x)\right\}
\end{aligned}$$

Both U and S are formally unitary

Composition law for time evolution operator

$$U(t_2,t_0) = U(t_2,t_1)U(t,t_0) = U(t_2,t_1)U(t_0,t_1)^\dagger \qquad (4.2.13)$$

4.2.1 Scattering amplitudes and the S-matrix

Take $|i\rangle$ the initial (multi-particle) state and $|f\rangle$ the final (multi-particle) state. Time evolution of $|i\rangle$ then is

$$\lim_{t\to\infty} U(t,-t)\,|i\rangle = S\,|i\rangle$$

Probability that $|i\rangle$ evolves into $|f\rangle$ is proportional to the squared "**S-matrix element**"

$$|\langle f,\ t\to\infty | i, t\to -\infty\rangle|^2 = |\langle f|S|i\rangle|^2 = \left|S_{fi}\right|^2 \qquad (4.2.14)$$

The non-trivial part of the S-matrix is the T-matrix:

$$S_{fi} := \delta_{fi} + iT_{fi} \tag{4.2.15}$$

Use momentum conservation (from translation invariance) to define matrix element

$$S_{fi} = \delta_{fi} + i(2\pi)^4\delta^{(4)}(p_f - p_i)M_{fi} \tag{4.2.16}$$

M_{fi} measures "genuine scattering" from $|i\rangle$ to $|f\rangle$.

How are we going to calculate correlation functions in the interacting theory:

$$\langle\Omega|\, T\phi(x)\phi(y)|\Omega\rangle$$

or more generally $\langle\Omega|T\phi(x_1)\phi(x_2)\ldots|\Omega\rangle$? Here $|\Omega\rangle$ is the vacuum/ground state of the interacting theory $\mathcal{H}$ (in contrast to $|0\rangle$ is the ground state of $\mathcal{H}_0$) and $\phi(x)$ the Heisenberg operators.

Ignore $|\Omega\rangle \neq |0\rangle$ for the moment saying that we want to study the time evolution from the vacuum at $t \to -\infty$ to $t \to +\infty$. So rewriting in terms $\phi_I(x)$, assuming $x^0 > y^0$ for now:

$$\langle 0|U(\infty, x^0)\phi_I(x^0)U(x^0, y^0)\phi_I(y^0)U(y^0, -\infty)|0\rangle = \langle 0|T(\phi_I(x)\phi_I(y)S)|0\rangle \tag{4.2.17}$$

still holds if $x^0 < y^0$ because of T.

Now $|\Omega\rangle \neq |0\rangle$: this can be taken care of by dividing out the time evolution of the (free) vacuum $\langle 0|S|0\rangle$, so

$$\begin{aligned}\langle\Omega|T(\phi(x)\phi(y))|\Omega\rangle &= \frac{\langle 0|T(\phi_I(x)\phi_I(y)S)|0\rangle}{\langle 0|S|0\rangle} \\ &\overset{\phi^4}{=} \frac{\langle 0|T\phi_I(x)\phi_I(y)\exp\Big\{-i\int \mathrm{d}^4x'\,\frac{\lambda}{4!}\phi^4(x')\Big\}|0\rangle}{\langle 0|T\exp\Big\{-i\int \mathrm{d}^4x'\,\frac{\lambda}{4!}\phi^4(x')\Big\}|0\rangle}\end{aligned} \tag{4.2.18}$$

Proof can be found in Peskin. It will also be illustrated practically later ("vacuum bubbles").

Perturbation theory is viable when λ (or some other coupling) is "small" and then expands $U(t, t_0)$ or S in powers of λ.

4.3 Wick's theorem

From now on drop the subscript for interaction picture fields $\phi_I(x) \to \phi(x)$ for convenience.

Want to calculate stuff like $\langle 0|T\phi(x_1)\ldots\phi(x_n)S|0\rangle$ in perturbation theory, e.g. at order λ^n.

$$\frac{1}{n!}\left(-i\frac{\lambda}{4!}\right)^n \int \mathrm{d}^4y_1\ldots\mathrm{d}^4y_n\,\langle 0|T\phi(x_1)\ldots\phi(x_n)\phi^4(y_1)\ldots\phi^4(y_n)|0\rangle \tag{4.3.1}$$

We know $\langle 0|T\phi(x_1)\phi(x_2)|0\rangle$ is the Feynman propagator!

Recall **normal ordering** with $\phi(x) = \phi^+(x) + \phi^-(x)$

$$:\phi^+\phi^- :=: \phi^-\phi^+ := \phi^-\phi^+ \tag{4.3.2}$$

where

$$\phi^+ = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\boldsymbol{p}}}}a_{\boldsymbol{p}}e^{-ip\cdot x} \tag{4.3.3}$$

$$\phi^- = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\boldsymbol{p}}}}a^\dagger_{\boldsymbol{p}}e^{+ip\cdot x} \tag{4.3.4}$$

Wick's theorem expresses time-ordered products in terms of normal-ordered ones. Then it is easy to take vacuum expectation values, as $\langle 0| : \phi(x_1)\ldots\phi(x_n) : |0\rangle = 0$

Take two fields and $x^0 > y^0$:

$$\begin{aligned} T\phi(x)\phi(y) = \phi(x)\phi(y) &= \left(\phi^+(x) + \phi^-(x)\right)\left(\phi^+(y) + \phi^-(y)\right) \\ &= \phi^+(x)\phi^+(y) + \phi^-(x)\phi^-(y) + \phi^-(x)\phi^+(y) + \phi^-(x)\phi^+(y) + [\phi^+(x), \phi^-(y)] \\ &=: \phi(x)\phi(y) : + [\phi^+(x), \phi^-(y)] \end{aligned}$$

Particularly for $y^0 > x^0$:

$$T\phi(x)\phi(y) =: \phi(x)\phi(y) : + [\phi^+(y), \phi^-(x)]$$

Thus altogether:

$$T\phi(x)\phi(y) =: \phi(x)\phi(y) : + D_f(x-y) \tag{4.3.5}$$

as $\Theta(x^0 - y^0)[\phi^+(x), \phi^-(y)] + \Theta(y^0 - x^0)[\phi^+(y), \phi^-(x)] = D_F(x-y)$.

Worth noting that $D_F(x-y)$ is still a c-number, not operator (yet). Thus it can be pulled out of any matrix element or expectation value.

We now define "contraction":

$$\wick{\c{\phi}(x_1)\c{\phi}(x_2)} = D_F(x_1 - x_2) \tag{4.3.6}$$

Thus we can remove the fields from the product leaving only the propagators:

$$T\phi(x)\phi(y) =: \phi(x)\phi(y) : + \wick{\c{\phi}(x)\c{\phi}(y)} \tag{4.3.7}$$

General form of **Wick's theorem** for arbitrary number of fields

$$T\phi(x_1)\ldots\phi(x_n) =: \phi(x_1)\ldots\phi(x_n) : + : \text{(sum over all possible contractions)} : \tag{4.3.8}$$

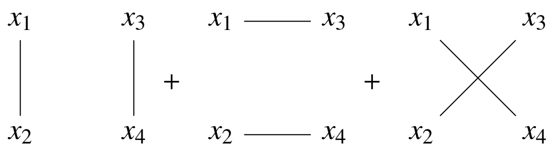
Example with four fields:

$$\begin{aligned} T(\phi_1\phi_2\phi_3\phi_4) =&: \phi_1\phi_2\phi_3\phi_4 : \\ &+ \wick{\c{\phi_1}\c{\phi_2}} : \phi_3\phi_4 : + \wick{\c{\phi_1}\c{\phi_3}} : \phi_2\phi_4 : + \wick{\c{\phi_1}\c{\phi_4}} : \phi_2\phi_3 : + \wick{\c{\phi_2}\c{\phi_3}} : \phi_1\phi_4 : + \wick{\c{\phi_2}\c{\phi_4}} : \phi_1\phi_3 : + \wick{\c{\phi_3}\c{\phi_4}} : \phi_1\phi_2 : \\ &+ \wick{\c{\phi_1}\c{\phi_2}}\wick{\c{\phi_3}\c{\phi_4}} + \wick{\c{\phi_1}\c{\phi_3}}\wick{\c{\phi_2}\c{\phi_4}} + \wick{\c{\phi_1}\c{\phi_4}}\wick{\c{\phi_2}\c{\phi_3}} \end{aligned}$$

Thus

$$\langle 0| T\left(\phi_1\phi_2\phi_3\phi_4\right)|0\rangle = D_F(x_1 - x_2)D_F(x_3 - x_4) + D_F(x_1 - x_3)D_F(x_2 - x_4) + D_F(x_1 - x_4)D_F(x_2 - x_3)$$

which can be visually represented as



Proof of the general theorem by *induction* in the number of fields (see exercise). The idea is to suppose it is true for $\phi_2\ldots\phi_m$, $x_1^0 > x_{k>1}^0$. Then

$$\begin{aligned} T\phi_1\phi_2\ldots\phi_m &= (\phi_1^+ + \phi_1^-)T\phi_2\ldots\phi_m \\ &= (\phi_1^+ + \phi_1^-)[: \phi_2\ldots\phi_m : + : \text{contractions} :] \end{aligned}$$

ϕ_1^- can stay as it is part of (: $\phi_1\phi_2\ldots\phi_m$:). But ϕ_1^+ needs to be commuted past all ϕ_1^- operators, giving rise to additional contractions $\wick{\c{\phi_1}\c{\phi_2}}$.

Consequences

- $n = 2k+1,\ k \in \mathbb{N}$
$$\langle 0|T\phi_1 \dots \phi_m|0\rangle = 0$$
- $n = 2k,\ k \in \mathbb{N}$
$$\langle 0|T\phi_1 \dots \phi_m|0\rangle = \sum_{\text{pairing of fields}} D_F(x_{i_1} - x_{i_2}) \dots D_F(x_{i_{m-1}} - x_{i_m})$$

4.3.1 Wick's theorem and the S-Matrix

Apply Wick's theorem to correlation functions $\langle 0|T(\phi_1 \dots \phi_m)S|0\rangle$ n-th term in the perturbative expansion of S with $\phi(x_1) := \phi_1$.

$$\frac{1}{n!}\left(\frac{-i\lambda}{4!}\right)^n \int \mathrm{d}^4y_1 \dots \mathrm{d}^4y_n \,\langle 0|T(\phi_1 \dots \phi_m\phi^4(y_1)\dots\phi^4(y_n))|0\rangle$$

Example with $m = 4,\ n = 1$

$$\begin{aligned}
&-\frac{i\lambda}{4!}\int \mathrm{d}^4x\,\langle 0|T\phi_1\phi_2\phi_3\phi_4\phi^4(x)|0\rangle \\
&= -\frac{i\lambda}{4!}\int \mathrm{d}^4x\,\langle 0|\phi_1\phi_2\phi_3\phi_4\phi(x)\phi(x)\phi(x)\phi(x)|0\rangle + 23 \text{ permutations} \\
&\quad -\frac{i\lambda}{4!}\int \mathrm{d}^4x\,\langle 0|\phi_1\phi_2\phi_3\phi_4\phi(x)\phi(x)\phi(x)\phi(x)|0\rangle + 11 \text{ permutations} + 5 \text{ similar} \\
&\quad -\frac{i\lambda}{4!}\int \mathrm{d}^4x\,\langle 0|\phi_1\phi_2\phi_3\phi_4\phi(x)\phi(x)\phi(x)\phi(x)|0\rangle + 2 \text{ permutations} + 2 \text{ similar} \\
&= -i\lambda\int \mathrm{d}^4x D_F(x_1 - x)D_F(x_2 - x)D_F(x_3 - x)D_F(x_4 - x) \\
&\quad -\frac{i\lambda}{2}D_F(x_1 - x_2)\int \mathrm{d}^4x D_F(x_3 - x)D_F(x_4 - x)D_F(x - x) + 5 \text{ similar} \\
&\quad -\frac{i\lambda}{8}D_F(x_1 - x_2)D_F(x_3 - x_4)\int \mathrm{d}^4x D_F(x - x) + 2 \text{ similar}
\end{aligned}$$

Permutation means permutation of $\phi(x)$ and similar means exchanging external states $\phi_i,\ i \in \{1,2,3,4\}$ without changing the shape of diagram. Note that the pre-factors of the integrals are called *symmetry factor*. The number of permutations of first diagram is equivalent to arrangement of 4 elements

$$\frac{4!}{4!} = 1$$

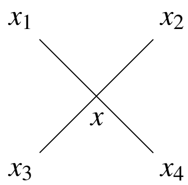
The permutations of second digram are to link two internal points to external ones and the rest external points can link to either of internal one.

$$\binom{4}{2} \cdot 2 \cdot \frac{1}{4!} = \frac{1}{2}$$

There are only three ways to permute a vacuum bubble

$$\frac{3}{4!} = \frac{1}{8}$$

To be represented in Feynman diagrams:



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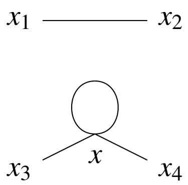
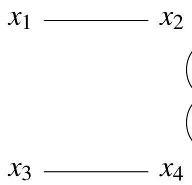
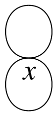


and 5 similar +


and 2 similar

In fact $D_F(x-x) = D_F(0)$ diverges!

Example with $m = 0$, $n = 1$ vacuum diagram

$$-\frac{i\lambda}{4!}\int \mathrm{d}^4x \,\langle 0|T\phi^4(x)|0\rangle$$
$$= -\frac{i\lambda}{8}[D_F(0)]^2 \int \mathrm{d}^4x$$
$$=$$


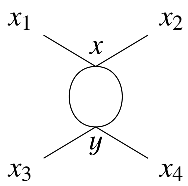
Example: 2nd order S-matrix term

$$\frac{1}{2!}\left(\frac{-i\lambda}{4!}\right)^4 \int \mathrm{d}^4x\mathrm{d}^4y \,\langle 0|T\phi_1\phi_2\phi_3\phi_4\phi^4(x)\phi^4(y)|0\rangle$$

It has many contractions and some of the fully connected ones are of the type there are

(4×3)[choose $\phi(x)$] $\times$ (4×3)[choose $\phi(y)$] $\times$ 2[x-y-cont.] $\times$ 2(x-y-symm.) + 2 similar, exchanging external points

$$= \frac{(-i\lambda)^2}{2}\int \mathrm{d}^4x\mathrm{d}^4y D_F(x_1-x)D_F(x_2-x)D_F(x_3-y)D_F(x_4-y)[D_F(x-y)]^2 + 2\text{ similar}$$

=


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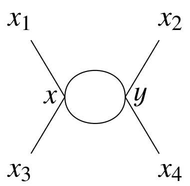

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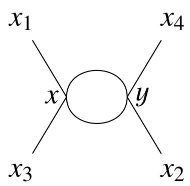
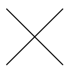
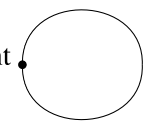
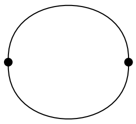


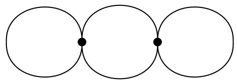
Symmetry factors A lot of the contractions eliminate the factors $\frac{1}{n!}\left(\frac{1}{4!}\right)^n$ in the denominators; the $\frac{1}{4!}$ was chosen to yield  $\sim -i\lambda$

See examples above. Sometimes, factors are not completely cancelled and thus procedure gets "reversed". Divide diagrams by *symmetry factor* ("missing factors").

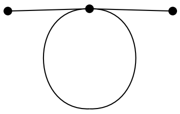
Where does it come from?

- Factor 2 from the line that starts and ends at the same point 

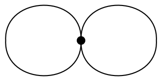
- Factor $j!$ for j lines linking the same 2 points 

- Factor $k!$ for k equivalent vertices 

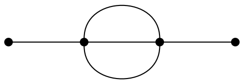
When in doubt, can always go back to Wick's theorem and count the contractions explicitly.
Examples:



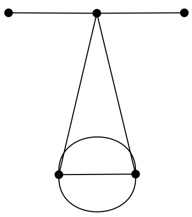
$S = 2$



$S = 8$



$S = 3! = 6$



$S = 2 \cdot 3! = 12$

Summary of Feynman rules

$$\langle 0|T\phi_1 \ldots \phi_m \exp\left(-\frac{i\lambda}{4!}\int d^4x \phi^4(x)\right)|0\rangle$$
$$= \text{sum of all diagrams with m external points;}$$

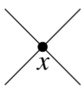
usually organised by number of internal points (i.e. power of λ).

Each diagram is built of

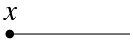
- propagators
- vertices (n)
- external points (m)

Feynman rules in position space Analytic expression obtained by combining

- For each propagator 

$= D_F(x - y)$
- For each vertex 

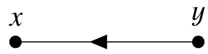
$= -i\lambda \int d^4x$

- For each external point 

$= 1$
- Divide diagram by its symmetry factor S

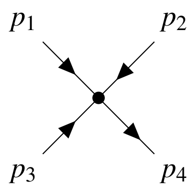
Since the propagator $D_F(x-y) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip(x-y)}$. It is actually simpler to express these in momentum space instead.

The way to do it is to asign a momentum p to each propagator. (direction arbitrary)



- assign e^{ipy} to y-vertex (arrow out)
- assign e^{-ipx} to x-vertex (arrow in)
- $\frac{i}{p^2 - m^2 + i\epsilon}$ to the line and the integration $\int \frac{\mathrm{d}^4 p}{(2\pi)^4}$

At vertex x:

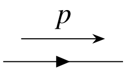


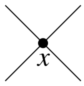
$$= -i\lambda \int \mathrm{d}^4 x e^{-i(p_1+p_2+p_3)x + ip_4 x}$$

$$= -i\lambda (2\pi)^4 \delta^{(4)}(p_1 + p_2 + p_3 - p_4)$$

This imposes momentum conservation at vertex. $\delta^{(4)}$-functions make some of the momentum integrals trivial, always with $(2\pi)^4$ cancelled appropriately.

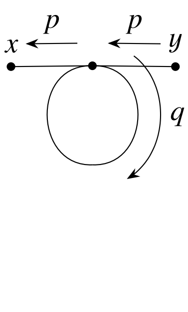
Momentum space Feynman rules

- Propagator 

$= \frac{i}{p^2 - m^2 + i\epsilon}$
- Vertex (position integrated out) 

$= -i\lambda$
- External points $\begin{cases} e^{-ipx} & \text{incoming} \\ e^{+ipx} & \text{outgoing} \end{cases}$
- Impose momentum conservation at each vertex
- Integrate over each undetermined momentum $\int \frac{\mathrm{d}^4 p}{(2\pi)^4}$
- Divided by symmetry factor

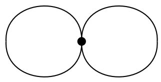
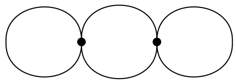
e.g.:



$$= (-i\lambda) \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{\mathrm{d}^4 q}{(2\pi)^4} \left(\frac{i}{p^2 - m^2 + i\epsilon} \right)^2 \frac{i}{q^2 - m^2 + i\epsilon} e^{-ip(x-y)}$$

Vacuum diagrams Disconnected pieces in Feynman diagrams are pretty bad. Not only $D_F(0) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon}$ is divergent (that will be taken care of later), it also contains an integral $\int d^4x$ const. Thus divergent once more!

Typical diagram contributing to 2-point function. one piece connected to x and y, plus disconnected pieces.

Call disconnected pieces $V_i \in \Big\{$  ,  , $\dots \Big\}$. Points are connected internally, but not to external points.

V_i can occur n_i-times, then

$$[\text{diagram}] = [\text{connected pieces}] \times \prod_i \frac{1}{n!} (V_i)^{n_i}$$

The factorial is the symmetry factor of n_i disconnected copies of V_i.

Then

$$\begin{aligned}
&\langle 0|T\phi_1 \dots \phi_n S|0\rangle \\
&= \sum_{\text{connected}} \sum_{\text{all}\{n_i\}} [\text{connected}] \times \prod_i \frac{1}{n_i!} (V_i)^{n_i} \\
&= \Big(\sum [\text{connected}]\Big) \times \sum_{\text{all}\{n_i\}} \left(\prod_i \frac{1}{n_i!} (V_i)^{n_i}\right) \\
&= \Big(\sum [\text{connected}]\Big) \times \prod_i \left(\sum_{n_i} \frac{1}{n_i!} (V_i)^{n_i}\right) \\
&= \Big(\sum [\text{connected}]\Big) \times \exp\left(\sum_i V_i\right)
\end{aligned}$$

Thus

$$\begin{aligned}
\text{sum of ALL diagrams} = &\ (\text{sum of all CONNECTED diagrams}) \\
&\times \exp(\text{sum of all DISCONNECTED diagrams})
\end{aligned}$$

Obvious from the above:

$$\langle 0|S|0\rangle = \langle 0|T\{\exp\left(-\frac{i\lambda}{4!} \int d^4x \phi^4(x)\right)\}|0\rangle = \exp(\text{sum of all vacuum bubbles})$$

Conclusion from the (unproven) formula for n-point correlation functions in the true, interacting vacuum:

$$\langle \Omega|T\phi_1 \dots \phi_m|\Omega\rangle = \frac{\langle 0|T\phi_1 \dots \phi_m S|0\rangle}{\langle 0|S|0\rangle} \tag{4.3.9}$$

$$= \sum (\text{connected diagrams with m external points}) \tag{4.3.10}$$

Here "connected" means connected to any external point. External points do not have to linked to each other.

4.4 S-matrix elements and Feynman diagrams

What is the correlation function in interacting vacuum $\langle\Omega|T\phi_1\dots\phi_m|\Omega\rangle$ good for? For scattering, shouldn't we rather look at $\langle p_1\dots p_m|S|p_A p_B\rangle$ with the perturbative expansion of S as before?

Decompose the S-matrix

$$S_{fi} = \delta_{fi} + iT_{fi} = \delta_{fi} + i(2\pi)^4\delta^{(4)}(p_f - p_i)M_{fi} \tag{4.4.1}$$

M_{fi} is the invariant matrix element, used to calculate cross section etc..

Zeroth term in the expansion of S

$$\begin{aligned}\langle p_1p_2|p_Ap_B\rangle &= \sqrt{2E_1 2E_2 2E_A 2E_B}\,\langle 0|a_1a_2a_A^+a_B^+|0\rangle\\ &= 2E_A2E_B(2\pi)^6\left(\delta^{(3)}(\boldsymbol{p}_A-\boldsymbol{p}_1)\delta^{(3)}(\boldsymbol{p}_B-\boldsymbol{p}_2)+\delta^{(3)}(\boldsymbol{p}_A-\boldsymbol{p}_2)\delta^{(3)}(\boldsymbol{p}_B-\boldsymbol{p}_1)\right)\end{aligned}$$

This actually is "no scattering", part of the $\mathbb{1}$ in the S-matrix.

First term is

$$\begin{aligned}&\langle p_1p_2|T\left(-\frac{i\lambda}{4!}\int \mathrm{d}^4x\phi^4(x)\right)|p_Ap_B\rangle\\ &\overset{\text{wick}}{=}\langle p_1p_2|:\left(-\frac{i\lambda}{4!}\int \mathrm{d}^4x\phi^4(x)+\text{contractions}\right):|p_Ap_B\rangle\end{aligned}$$

However now the expectation value of a normal-ordered expression doesn't vanish!

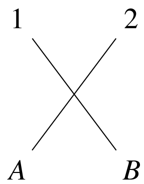
$$\begin{aligned}\phi^+(x)\,|\boldsymbol{p}\rangle &= \int\frac{\mathrm{d}^3k}{(2\pi)^3\sqrt{2E_k}}a_{\boldsymbol{k}}e^{-ikx}\sqrt{2E_p}a^+_{\boldsymbol{p}}\,|0\rangle\\ &= \int\frac{\mathrm{d}^3k}{\sqrt{2E_k}}e^{-ikx}\sqrt{2E_p}\delta^{(3)}(\boldsymbol{k}-\boldsymbol{p})\,|0\rangle\\ &= e^{-ipx}\,|0\rangle\end{aligned}$$

So in general, need two field operators to annihilate the in-state and two field operators to create the out-states. We have new type of Feynman diagram to deal with external states. Define contractions of field operators with external states according to

$$\begin{aligned}\phi(x)\,|\boldsymbol{p}\rangle &= e^{-ipx}\,|0\rangle\\ \langle\boldsymbol{p}|\,\phi(x) &= e^{+ipx}\,|0\rangle\end{aligned}$$

How does this work for $p_Ap_B\to p_1p_2$ in ϕ^4 at first order? It contains three types of terms: $\phi\phi\phi\phi$, $\phi\phi\phi\phi$ and $\phi\phi\phi\phi$ (fields without contraction are contracted with external states).

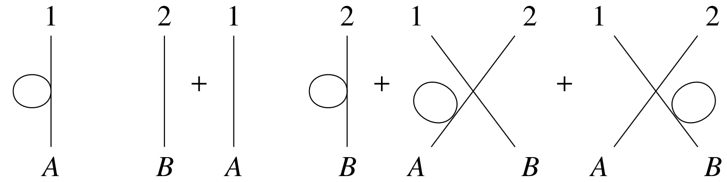
1. $\phi\phi\phi\phi$ allows full contractions with all external states. There is 4! possibilities



$$= 4!\frac{-i\lambda}{4!}\int \mathrm{d}^4x e^{-i(p_A+p_B-p_1-p_2)x} = -i\lambda\,\underbrace{(2\pi)^4\delta^{(4)}(p_A+p_B-p_1-p_2)}_{\text{Pre-factor in definition of } i\mathcal{M}}$$

$i\mathcal{M}$ receives a contribution $-i\lambda$!

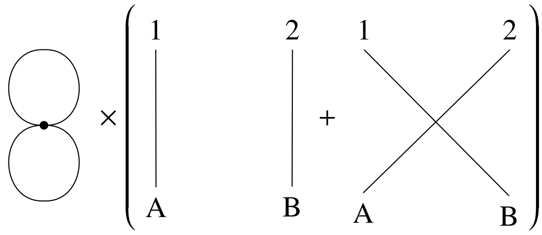
2. $\phi\phi\phi\phi$ leaves two operators to connect to external particles. Momentum conservation at each vertex. Still trivial!



Only fully connected Feynman diagrams contribute to $iT/i\mathcal{M}$!

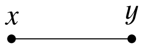
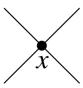
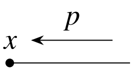
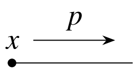
3.

$$-\frac{i\lambda}{4!}\int \mathrm{d}^4x\,\langle p_1p_2|\phi\phi\phi\phi|p_Ap_B\rangle$$

$=$ 

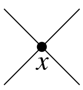
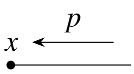
4.4.1 Feynman rules (with external lines)

Position space calculate iT by summing overall fully connected diagrams with

- propagator  $= D_F(x-y)$
- vertex  $= -i\lambda\int \mathrm{d}^4x$
- external lines "in"  $= e^{-ip\cdot x}$;  $= e^{ip\cdot x}$
- divide diagram by its symmetry factor S

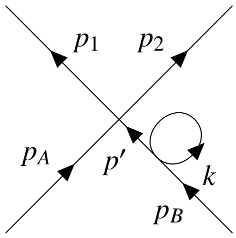
Momentum space We have seen it before. Now (with external lines) all positions are integrated over. Result is a function of external momenta only. Integrating out all momentum-conserving δ-distribution yields overall momentum conservation: $(2\pi)^4\delta^{(4)}(P_f - P_i)$

Momentum space Feynman rules for calculating iM:

- internal propagator  $= \frac{i}{p^2-M^2+i\epsilon}$
- vertex  $= -i\lambda$
- external lines ("in" or "out")  $= 1$
- impose 4-momentum conservation at each vertex

- integrate over all undetermined momenta $\int \frac{\mathrm{d}^4 p}{(2\pi)^4}$
- divide diagram by its symmetry factor S

There is still trouble in there. Consider the next-to-leading contribution to the scattering amplitude

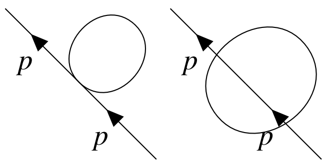


$$=\frac{1}{2}\int \frac{\mathrm{d}^4 p'}{(2\pi)^4}\frac{i}{p'^2 - m^2}\int \frac{\mathrm{d}^4 k}{(2\pi)^4}\frac{i}{k^2 - m^2}$$
$$\times(-i\lambda)(2\pi)^4\delta^{(4)}(p_A + p' - p_1 - p_2)\cdot(-i\lambda)(2\pi)^4\delta^{(4)}(p_B - p')$$

This contains the internal propagator $\frac{i}{P_B^2 - m^2 + i\epsilon}$, but all the external particle are on their mass-shell, i.e.

$$P_A^2 = P_B^2 = P_1^2 = P_2^2 = m^2 \;\Rightarrow\; \frac{i}{P_B^2 - m^2} = \frac{i}{0}$$

In addition to having fully connected diagrams, also need to confine ourselves to amputated diagrams: disregard all these diagrams with loops attached to external legs.



These diagrams represent the transition from the free to the interacting asymptotic states.

Lehmann-Symanzik-Zimmermann (LSZ) reduction formula Proof on relation between correlation functions and S-matrix elements will be provided later.

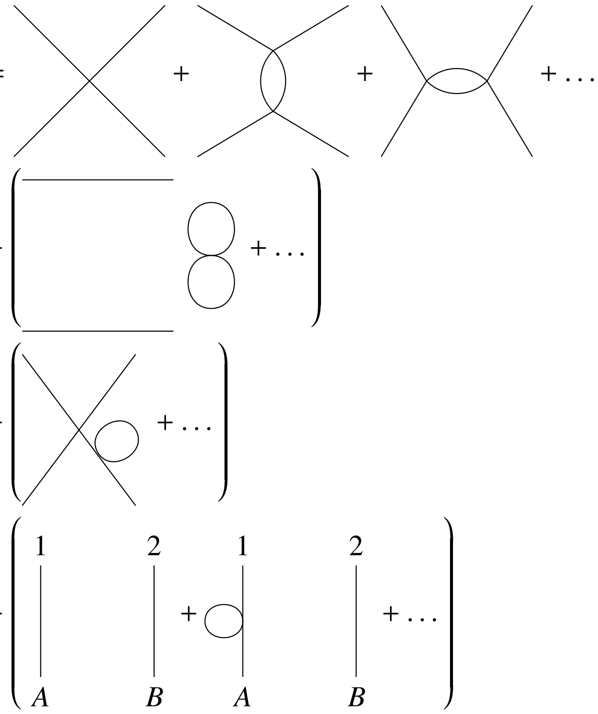
$$\begin{aligned}&\prod_{i=1}^{n}\int \mathrm{d}^4 x_i e^{ip_i\cdot x_i}\prod_{j=1}^{m}\int \mathrm{d}^4 y_j e^{-ik_j\cdot y_i}\langle\Omega|T\phi(x_1)\ldots\phi(x_n)\phi(y_1)\ldots\phi(y_m)|\Omega\rangle\\ &=(\text{disconnected stuff}) + \underbrace{\prod_{i=1}^{n}\frac{i\sqrt{Z}}{p_i^2 - m^2 + i\epsilon}\prod_{j=1}^{m}\frac{i\sqrt{Z}}{{}^2_j - m^2 + i\epsilon}}_{\text{remove poles from external legs}}\langle p_1\ldots p_n|S|k_i\ldots k_m\rangle\end{aligned} \tag{4.4.2}$$

z is the wave-function renormalization factor.

Then amend Feynman rules above

consider only fully connected, amputated diagrams

$\langle p_1 p_2 | iT | p_A p_B \rangle =$

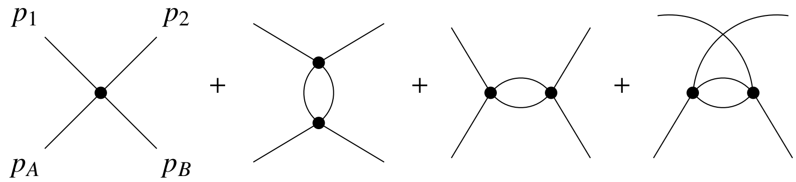


yields $|0\rangle \to |\Omega\rangle$

yields $|p\rangle_{\text{free}} \to |p\rangle_{\text{int}}$

yields $\mathbb{1}$ in S-matrix

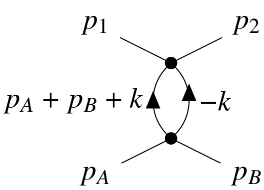
All allowed scattering diagrams $2 \to 2$ in ϕ^4 up to $\mathcal{O}(\lambda^2)$:



s t u

Define the Lorentz-invariant quantities, *Mandelstam variables*:

$$s = (p_A + p_B)^2\,, \quad t = (p_A - p_1)^2\,, \quad u = (p_A - p_2)^2 \tag{4.4.3}$$



$$= \frac{1}{2}(-i\lambda)^2 \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} \frac{i}{(p_A + p_B + k)^2 - m^2 + i\epsilon} =: \frac{1}{2}(-i\lambda)^2 iJ(s)$$

Then the complete invariant amplitude is

$$M = -\lambda - \frac{\lambda^2}{2}\left(J(s) + J(t) + J(u)\right) \tag{4.4.4}$$

4.5 Scattering cross section

This section is based on Itzykson & Zuber, Chapter 5.1.
The aim is to relate (differential) cross section to reduced/invariant matrix element M_{fi}. First we describe the initial states not as momentum eigenstates $|p_A p_B\rangle$, but as wave packets.

$$|i\rangle = \int \frac{\mathrm{d}^3 k_A}{(2\pi)^3 2k_A^0} \frac{\mathrm{d}^3 k_B}{(2\pi)^3 2k_B^0} f(k_A) g(k_B) \, |k_A k_B\rangle$$

with $f(k_A)$, $g(k_B)$ strongly peaked at $k_A \approx p_A$, $k_B \approx p_B$.
We can write the transition amplitude to the final state $|f\rangle \propto |p_1 p_2\rangle$ (note: normalisation not the same)

$$\begin{aligned} A_{fi} &= \int \frac{\mathrm{d}^3 k_A}{(2\pi)^3 2k_A^0} \frac{\mathrm{d}^3 k_B}{(2\pi)^3 2k_B^0} f(k_A) g(k_B) \, \langle f | iT | k_A k_B \rangle \\ &= \int \frac{\mathrm{d}^3 k_A}{(2\pi)^3 2k_A^0} \frac{\mathrm{d}^3 k_B}{(2\pi)^3 2k_B^0} f(k_A) g(k_B) (2\pi)^4 \delta^{(4)}(\underbrace{p_f}_{=p_1+p_2} - k_A - k_B) iM(f, k_A, k_B) \end{aligned}$$

Thus the <u>transition probability</u>:

$$\begin{aligned} \omega_{fi} = (2\pi)^8 &\int \frac{\mathrm{d}^3 k_A}{(2\pi)^3 2k_A^0} \frac{\mathrm{d}^3 k_B}{(2\pi)^3 2k_B^0} \frac{\mathrm{d}^3 q_A}{(2\pi)^3 2q_A^0} \frac{\mathrm{d}^3 q_B}{(2\pi)^3 2q_B^0} f(k_A) g(k_B) f(q_A)^* g(q_B)^* \\ \times \quad &\underbrace{\delta^{(4)}(p_f - k_A - k_B) \delta^{(4)}(p_f - q_A - q_B)}_{=\delta^{(4)}(q_A + q_B - k_A - k_B)\delta^{(4)}(p_f - p_A - p_B)} \underbrace{M(f, k_A, k_B) M^*(f, q_A, q_B)}_{\approx |M(f, p_A, p_B)|^2} \end{aligned}$$

Using the fourier representation of delta function $\delta^{(4)}(q_A + q_B - k_A - k_B) = (2\pi)^{-4} \int \mathrm{d}^4 x e^{i(k_A + k_B - q_A - q_B)\cdot x}$

$$\begin{aligned} = \int \mathrm{d}^4 x &\underbrace{\int \frac{\mathrm{d}^3 k_A}{(2\pi)^3 2k_A^0} \frac{\mathrm{d}^3 q_A}{(2\pi)^3 2q_A^0} e^{i(k_A - q_A)\cdot x} f(k_A) f^*(q_A)}_{:=|\tilde{f}(x)|^2} \\ \times \quad &\underbrace{\int \frac{\mathrm{d}^3 k_B}{(2\pi)^3 2k_B^0} \frac{\mathrm{d}^3 q_B}{(2\pi)^3 2q_B^0} e^{i(k_B - q_B)\cdot x} g(k_B) g^*(q_B)}_{:=|\tilde{g}(x)|^2} (2\pi)^4 \delta^{(4)}(p_f - p_A - p_B) \cdot |M(f, p_A, p_B)|^2 \end{aligned}$$

Using Fourier transformation $\tilde{g}(x) := \int \frac{\mathrm{d}^3 q}{(2\pi)^3 2q^0} e^{iq\cdot x} g(q)$

$$= \int \mathrm{d}^4 x |\tilde{f}(x)|^2 |\tilde{g}(x)|^2 (2\pi)^4 \delta^{(4)}(p_f - p_A - p_B) \cdot |M(f, p_A, p_B)|^2$$

note that $M(f, p_A, p_B)$ and $M(p_1, p_2, p_A, p_B)$ have different normalisation.
We now consider transition probability per unit volume per unit time:

$$\frac{\mathrm{d}\omega_{fi}}{\mathrm{d}V \mathrm{d}t} = (\text{incident flux}) \cdot (\text{target density}) \cdot \mathrm{d}\sigma$$

with $\mathrm{d}\sigma$ the infinitesimal cross section for scattering into final state $\langle f|$.
Product (incident flux) $\cdot$ (target density) denotes overlap of wave function. Necessary condition!

Covariant renormalization of states $\langle \boldsymbol{p}|\boldsymbol{q}\rangle \sim 2p^0\delta^3(\boldsymbol{p}-\boldsymbol{q})$ means the number of particles per unit volume is $2p_A^0|\tilde{f}(x)|^2$ and $2p_B^0|\tilde{g}(x)|^2$, respectively.

Assume

$$\bullet \xrightarrow{\;p_A\;} \bullet\ \boldsymbol{p}_B = 0$$

in target rest frame. Then $2p_B^0 = 2m_B$ and target density $= 2m_B|\tilde{g}(x)|^2$

Incident flux $= |\boldsymbol{v}_A|\cdot 2p_A^0|\tilde{f}(x)|^2 = 2|\boldsymbol{p}_A||\tilde{f}(x)|^2$ since $|\boldsymbol{v}_A| = |\boldsymbol{p}_A|/p_A^0$. Then

$$\mathrm{d}\sigma = (2\pi)^4\delta^{(4)}(p_f - p_A - p_B)\frac{1}{4m_B|\boldsymbol{p_A}|}|M(f, p_A, p_B)|^2$$

for $A + B \to 1 + 2$ processes

$$= \int_\Delta \frac{\mathrm{d}^3p_1}{(2\pi)^3 2p_1^0}\frac{\mathrm{d}^3p_2}{(2\pi)^3 2p_2^0}(2\pi)^4\delta^{(4)}(p_1 + p_2 - p_A - p_B)\frac{1}{4m_B|\boldsymbol{p_A}|}|M(p_1, p_2, p_A, p_B)|^2$$

with Δ energy-momentum resolution of 4-momentum of final state $|f\rangle$.

Covariant form of

$$m_B\cdot|\boldsymbol{p}_A| = m_B\sqrt{(p_A^0)^2 - m_A^2} = \sqrt{(p_A\cdot p_B)^2 - m_A^2m_B^2} =: F \tag{4.5.1}$$

This is scattering into arbitrary final state subject to 4-momentum conservation: $p_A + p_B = p_1 + p_2$.

Consider now <u>differential</u> cross section for scattering into a particular infinitesimal solid angle $\mathrm{d}\Omega$, hence specific momentum $\mathrm{d}p_1$, $\mathrm{d}p_2$ variations:

$$\begin{aligned}
\mathrm{d}\sigma &= \frac{1}{4F}\prod_f \frac{\mathrm{d}^3p_f}{(2\pi)^3 2p_f^0}(2\pi)^4\delta^{(4)}(p_A + p_B - \sum_f p_f)|M|^2\\
&\overset{f=1,2}{=} \frac{1}{4F}\frac{\mathrm{d}^3p_1}{(2\pi)^3 2p_1^0}\frac{\mathrm{d}^3p_2}{(2\pi)^3 2p_2^0}(2\pi)^4\delta^{(4)}(p_i - p_f)|M|^2\\
&= \frac{1}{64\pi^2 F}\frac{\mathrm{d}^3p_1}{E_1}\frac{\mathrm{d}^3p_2}{E_2}\delta^{(4)}(p_1 + p_2 - p_i)|M|^2\\
&\quad \boxed{\begin{aligned}
&\textstyle\int \frac{\mathrm{d}^3p_1}{E_1}\frac{\mathrm{d}^3p_2}{E_2}\delta^{(4)}(p_1 + p_2 - p_i)\\
&\overset{\text{CMS}}{=} \textstyle\int \mathrm{d}|\boldsymbol{p}_1|\mathrm{d}\Omega_1\frac{|\boldsymbol{p}_1|^2}{E_1E_2}\delta(E_1 + E_2 - E_i)\\
&= \textstyle\int \mathrm{d}(E_1 + E_2)\frac{\mathrm{d}|\boldsymbol{p}_1|}{\mathrm{d}(E_1+E_2)}\mathrm{d}\Omega_1\frac{|\boldsymbol{p}_1|^2}{E_1E_2}\delta(E_1 + E_2 - E_i)\\
&= \textstyle\frac{|\boldsymbol{p}_1|^2}{E_1E_2}\left(\frac{|\boldsymbol{p}_1|}{E_1} + \frac{|\boldsymbol{p}_1|}{E_2}\right)^{-1}\mathrm{d}\Omega_1\\
&= \textstyle\frac{|\boldsymbol{p}_1|\mathrm{d}\Omega_1}{E_1+E_2} = \frac{|\boldsymbol{p}_1|\mathrm{d}\Omega_1}{E_i}
\end{aligned}}\\
\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} &= \frac{1}{64\pi^2}\frac{|\boldsymbol{p}_1|}{F\cdot E_i}|M|^2
\end{aligned} \tag{4.5.2}$$

Rewrite all kinematic factors in terms of $s = (p_A + p_B)^2 = (p_1 + p_2)^2$. Define the function

$$\lambda(x, y, z) := x^2 + y^2 + z^2 - 2(xy + xz + yz) \tag{4.5.3}$$

then

$$F = \sqrt{(p_A \cdot p_B)^2 - m_A^2 m_B^2} = \frac{1}{2}\lambda^{\frac{1}{2}}(s, m_A^2, m_B^2) = \sqrt{s}|\boldsymbol{p}_i|$$

$$\boxed{\begin{aligned} \lambda(s, m_A^2, m_B^2) &= s^2 - 2s(m_A^2 + m_B^2) - (m_A^2 - m_B^2)^2 = (s - (m_A + m_B)^2)(s - (m_A - m_B)^2) \\ &= (2p_A \cdot p_B - 2m_A \cdot m_B) \cdot (2p_A \cdot p_B + 2m_A \cdot m_B) = 4\left[(p_A p_B)^2 - m_A^2 m_B^2\right] \\ &\qquad \boxed{\begin{aligned} &p_A = (c\sqrt{s}, \boldsymbol{p}_i),\ c \in [0,1] && \to m_A^2 = c^2 s - |\boldsymbol{p}_i|^2 \\ &p_B = ((1-c)\sqrt{s}, -\boldsymbol{p}_i) && \to m_B^2 = (1-c)^2 s - |\boldsymbol{p}_i|^2 \end{aligned}} \\ &= 4\left[\left((c(1-c)s + p_i^2\right)^2 + (c^2 s - p_i^2)((1-c)^2 s - p_i^2)\right] = 4s|\boldsymbol{p}_i|^2 \end{aligned}}$$

$$\begin{aligned} |\boldsymbol{p}_f| &= \sqrt{E_{1,2}^2 - m_{1,2}^2} = \frac{1}{2\sqrt{s}}\lambda^{\frac{1}{2}}(s, m_1^2, m_2^2) \\ E_i &= \sqrt{s} \end{aligned}$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega_{CMS}} = \frac{1}{64\pi^2 s}\frac{|\boldsymbol{p}_f|}{|\boldsymbol{p}_i|}|M|^2 = \frac{1}{64\pi^2 s}\sqrt{\frac{\lambda(s, m_1^2, m_2^2)}{\lambda(s, m_A^2, m_A^2)}}|M|^2 \tag{4.5.4}$$

Decay rate instead of cross section means no "incident flux" to divide by, only "target density"

$$\mathrm{d}\Gamma = \frac{1}{2m_A}\prod_f \frac{\mathrm{d}^3 p_f}{(2\pi)^3 2p_f^0}(2\pi)^4 \delta^{(4)}(p_A - \sum_f p_f)|M|^2 \tag{4.5.5}$$

Particles with spin (unpolarized): sum over outgoing or average over initial spins

$$|M|^2 \to \frac{1}{(2s_A + 1)(2s_B + 1)}\sum_{s_i, s_f}|M_{fi}|^2 \tag{4.5.6}$$

Symmetry factor $|M|^2 \to \frac{1}{s}|M|^2$ with $s = \prod_i k_i!$ if there are k_i identical particles of species i in the final states.

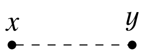
If 1 and 2 are identical, then factor $\frac{1}{s} = \frac{1}{2}$ on the right hand side.

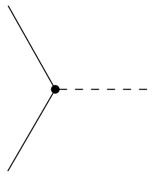
4.6 Feynman rules for fermions

Consider the simplest interacting theory with fermions, Yukawa-theory. We will treat QED later.

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{M^2}{2}\phi^2 + \bar{\psi}(i\partial\!\!\!/ - m)\psi - g\bar{\psi}\psi\phi \tag{4.6.1}$$

Feynman rules will involve:

- scalar  $= D_F(x-y) = \int \frac{\mathrm{d}^4}{(2\pi)^2}\frac{i}{p^2 - M^2 + i\epsilon}e^{-ip(x-y)}$
- fermions  $= S_F(x-y)_{\alpha\beta} = \int \frac{\mathrm{d}^4}{(2\pi)^4}\frac{i(p\!\!\!/ + m)}{p^2 - m^2 + i\epsilon}e^{-ip(x-y)}$

- vertices  $= -ig \int \mathrm{d}^4 x$

What previous steps need reconsideration due to the anti-commutating fermion operators? Interaction Hamiltonian $\sim \bar{\psi}\psi\phi$ and in general compose of even number of fermion fields (spin conservation and fermion number conservation). Thus there is no problem with time-ordered exponential in definition of S-matrix. (Time ordering always takes two or even number of fields.)

Remember the relation

$$T(\psi_\alpha(x)\bar{\psi}_\beta(x)) = -\bar{\psi}_\beta(x)\psi_\alpha(x) \text{ when } y^0 > x^0 \tag{4.6.2}$$

Similarly in normal product:

$$: \psi^+\psi^- := -\psi^-\psi^+ \tag{4.6.3}$$

Then Wick's theorem is formally the same as before

$$T(\psi_\alpha(x)\bar{\psi}_\beta(x)) =: \psi_\alpha(x)\bar{\psi}_\beta(x) : + \overbracket{\psi_\alpha(x)\bar{\psi}_\beta}(x)$$

note by definition $\overbracket{\psi\psi} = \overbracket{\bar{\psi}\bar{\psi}} = 0$

Thus contractions inside normal-ordered products would be

$$: \psi_1\psi_2\bar{\psi}_3\bar{\psi}_4 := -\overbracket{\psi_1\bar{\psi}_3} : \psi_2\bar{\psi}_4 := -S_F(x_1 - x_3) : \psi_2\bar{\psi}_4 :$$

because of the additional operator exchange.

We will want to consider fermion-(anti-)fermion scattering. Leading contribution at $O(g^2)$:

$$\frac{1}{2!}(-ig)^2 \int \mathrm{d}^4x\mathrm{d}^4y \, \langle p', k'|T\bar{\phi}(x)\phi(x)\phi(x)\bar{\phi}(y)\phi(y)|p, k\rangle$$

Contractions with initial-/final-state fermions?

$$\begin{aligned}\phi^+(x)\,|p, s\rangle &= \int \frac{\mathrm{d}^3k}{(2\pi)^3\sqrt{2E_k}} \sum_r a_k^r u_r(k) e^{-ik\cdot x} \sqrt{2E_p} a_p^{s\dagger}\,|0\rangle \\ &= e^{-ip\cdot x} u_s(p)\,|0\rangle\end{aligned}$$

So define

$$\begin{aligned}\overbracket{\psi(x)\,|p}, s\rangle &= e^{-ip\cdot x} u_s(p) \\ \langle p, \overbracket{s|\,\bar{\psi}}(x) &= e^{ip\cdot x} \bar{u}_s(p)\end{aligned} \tag{4.6.4}$$

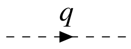
Note, though, for anti-fermion states $|p', s'\rangle$:

$$\begin{aligned}\overbracket{\bar{\psi}(x)\,|p}, s\rangle &= e^{-ip'\cdot x} \bar{v}_{s'}(p') \\ \langle p', \overbracket{s'|\,\psi}(x) &= e^{ip'\cdot x} v_{s'}(p')\end{aligned} \tag{4.6.5}$$

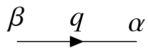
In short $\overbracket{\psi|}\,\rangle$ contracts with a fermion, $\langle\overbracket{|\psi}$ with an anti-fermion; vice verse for $\bar{\psi}$.

Momentum space Feynman rule for iM

- internal propagators

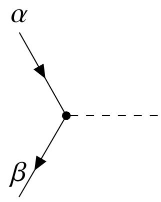


$$= \frac{i}{q^2 - M^2 + i\epsilon}$$

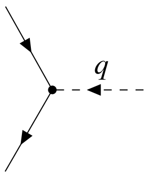
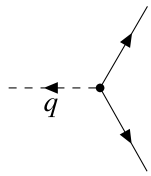
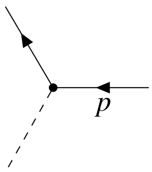
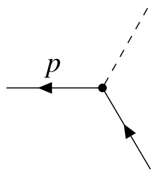
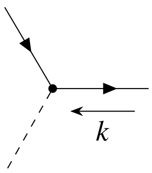
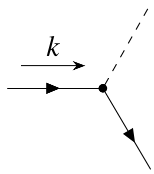


$$= \frac{i(\not{p} + m)_{\alpha\beta}}{p^2 - m^2 + i\epsilon}$$

- vertex



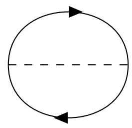
$$= -ig \int \mathrm{d}^4x = ig\delta_{\beta\alpha}$$

- external lines
  - "in" and "out" scalar  =  $= 1$
  - "in"-fermion  $= u_s(p)$ and "out" fermion  $= \bar{u}_s(p)$
  - "in"-anti-fermion  $= \bar{v}_s(k)$ and "out" anti-fermion  $= v_s(k)$

- impose energy-momentum conservation at each vertex
- integrate over undetermined (loop) momenta
- include an overall sign for the diagram

Note

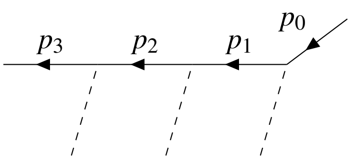
- Arrows on the fermion lines by convention denote fermion (or charge) flow. They must flow consistently through the diagram. ($\equiv$ fermion number conservation) (Only potential confusion: external anti-fermion lines)

- No symmetry factors (except vacuum bubbles  $\frac{1}{s} = \frac{1}{2}$). $\bar{\psi}\psi\phi$ allows for unambiguous contractions.

- Dirac indices are summed over at each vertex

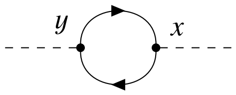
$$\mathcal{L}_{\text{int}} \approx \bar{\psi}_\alpha(x)\psi_\alpha(x)\phi(x)$$

$(\not{p} + m)$ terms in propagator are matrix-multiplied contracted with external spinors, e.g.



$$\sim \bar{u}_\alpha(p_3)\frac{i(\not{p}+m)_{\alpha\beta}}{p_2^2 - m^2 + i\epsilon}\frac{i(\not{p}+m)_{\beta\gamma}}{p_1^2 - m^2 + i\epsilon}u_\gamma(p_0)$$

- closed fermion loop



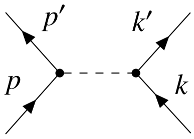
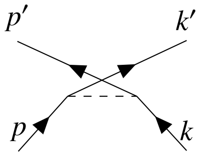
$$\sim \bar{\psi}_\alpha(x)\psi_\alpha(x)\bar{\psi}_\beta(y)\psi_\beta(y) = -\psi_\alpha(x)\bar{\psi}_\beta(y)\psi_\beta(y)\bar{\psi}_\alpha(x)\bar{\psi}_\beta(y)$$

$$= -S_F(y-x)_{\beta\alpha}S_F(x-y)_{\alpha\beta} = -\text{Tr}\left(S_F(y-x)S_F(x-y)\right)$$

It always (also with more propagators/couplings) involves an overall (-1) and a trace $\text{Tr}(\ldots)$.

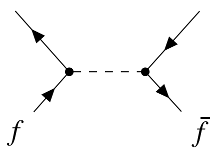
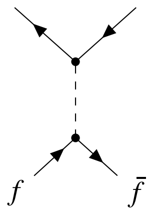
Examples

- fermion-fermion scattering to lowest order $O(g^2)$

$iM =$  $+$ 

$$= (-ig)^2\left\{\bar{u}(p')u(p)\frac{i}{\underbrace{(p'-p)^2}_{t} - M^2 + i\epsilon}\bar{u}(k')u(k) - \bar{u}(p')u(k)\frac{i}{\underbrace{(p'-k)^2}_{u} - M^2 + i\epsilon}\bar{u}(k')u(p)\right\}$$

- fermion-anti-fermion scattering

 $+$ 

These are tree diagrams. Thus there is no undetermined momenta to integrate.