H.7

a) Lemma: $tr(\gamma^\mu) = 0$

since $\{\gamma^\mu, \gamma^\mu\} = 2 g^{\mu\mu} \mathbb{1} \Rightarrow \gamma^\mu \gamma^\mu = g^{\mu\mu} \mathbb{1}$

$$\Rightarrow tr(\gamma^\mu) = tr(\mathbb{1} \cdot \gamma^\mu) = \frac{1}{g^{\nu\nu}} tr(\gamma^\nu \gamma^\nu \gamma^\mu)$$

$$\underset{\text{anticommutator}}{=} -\frac{1}{g^{\nu\nu}} tr(\gamma^\nu \gamma^\mu \gamma^\nu) \underset{\text{permutation}}{=} -\frac{1}{g^{\nu\nu}} tr(\gamma^\nu \gamma^\nu \gamma^\mu)$$

$$\Rightarrow tr(\gamma^\mu) = 0$$

$tr(\gamma^{\mu_1} \cdots \gamma^{\mu_{2n+1}})$, μ_k, $k = \{0, \ldots, 2n+1\}$ don't have to be distinctive

Method 1:

Using anticommutations we can rewrite the whole product into these two expressions

$$\gamma^{\mu_1} \cdots \gamma^{\mu_{2n+1}} = \begin{cases} (-1)^\xi \gamma^\nu, \\ (-1)^\xi \gamma^\alpha \gamma^\beta \gamma^\nu, \quad \alpha \neq \beta \neq \nu, \\ \quad \alpha, \beta, \nu \in \{0, 1, 2, 3\} \end{cases}$$

ξ is an intger number, come from anticommutations and $(\gamma^k)^2 = -\mathbb{1}$

(no summation is used)

$$Tr((-1)^\xi \gamma^\nu) = (-1)^\xi Tr(\gamma^\nu) = 0$$

$$Tr((-1)^\xi \gamma^\alpha \gamma^\beta \gamma^\nu) = (-1)^\xi Tr(\gamma^\alpha \gamma^\beta \gamma^\nu) = (-1)^\xi Tr(\gamma^5 \gamma^5 \gamma^\alpha \gamma^\beta \gamma^\nu)$$

$$= (-1)^\xi Tr(\gamma^5 \gamma^\alpha \gamma^\beta \gamma^\nu \gamma^5)$$

$$\underset{\text{anticommutation}}{=} -(-1)^\xi Tr(\gamma^5 \gamma^5 \gamma^\alpha \gamma^\beta \gamma^\nu)$$

$$\Rightarrow Tr(\gamma^{\mu_1} \cdots \gamma^{\mu_{2n+1}}) = 0$$

Method 2: (better I suppose)

$$tr(\gamma^{\mu_1} \cdots \gamma^{\mu_{2n+1}}) = tr(\gamma^5 \gamma^5 \gamma^{\mu_1} \cdots \gamma^{\mu_{2n}} \gamma^{\mu_{2n+1}})$$

$$= \mathrm{tr}(\gamma^5 \gamma^{\mu_1} \cdots \gamma^{\mu_{2n+1}} \gamma^5)$$

$$= (-1)^{2n+1} \mathrm{tr}(\gamma^5 \gamma^5 \gamma^{\mu_1} \cdots \gamma^{\mu_{2n+1}})$$

$\{\gamma^5, \gamma^\mu\} = 0$

$$= - \mathrm{tr}(\gamma^{\mu_1} \cdots \gamma^{\mu_{2n+1}}) = 0$$

$$\mathrm{tr}(\gamma^{\mu_1} \cdots \gamma^{\mu_{2n+1}} \gamma^5)$$

$$= \mathrm{tr}(\gamma^5 \gamma^{\mu_1} \cdots \gamma^{\mu_{2n+1}})$$

$$= (-1)^{2n+1} \mathrm{tr}(\gamma^{\mu_1} \cdots \gamma^{\mu_{2n+1}} \gamma^5)$$

$$= 0$$

b) $$\mathrm{tr}(\gamma^5) = \mathrm{tr}(\gamma^5 \gamma^0 \gamma^0)$$

$$= -\mathrm{tr}(\gamma^0 \gamma^5 \gamma^0)$$

$$= -\mathrm{tr}(\gamma^5 \gamma^0 \gamma^0)$$

$$= 0$$

$$\mathrm{tr}(\gamma^5 \gamma^\mu \gamma^\nu) = (-1)^\xi \mathrm{tr}(\gamma^5 \gamma^\alpha \gamma^\alpha \gamma^\mu \gamma^\nu),$$

$$\left(\begin{array}{l} \alpha \neq \mu, \ \alpha = \nu \\ \xi = 0 \ \text{if} \ \alpha = 0 \\ \xi = 1 \ \text{if} \ \alpha = 1, 2, 3 \end{array} \right.$$

$$= (-1)^{\xi+1} \mathrm{tr}(\gamma^\alpha \gamma^5 \gamma^\mu \gamma^\nu \gamma^\alpha)$$

$$= (-1)^{\xi+1} \mathrm{tr}(\gamma^5 \gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\alpha)$$

$$= - \mathrm{tr}(\gamma^5 \gamma^\mu \gamma^\nu)$$

$$= 0$$

c) $$\mathrm{tr}(\gamma^{\mu_1} \cdots \gamma^{\mu_{2n}})$$

$$= \mathrm{tr}(\gamma^{\mu_1} \gamma^{\mu_2} \cdots \gamma^{\mu_k} \cdots \gamma^{\mu_{2n}})$$

$$= \mathrm{tr}\left[(-\gamma^{\mu_2} \gamma^{\mu_1} + 2 g^{\mu_1 \mu_2} \mathbb{1}) \gamma^{\mu_3} \cdots \gamma^{\mu_k} \cdots \gamma^{\mu_{2n}}\right]$$

$$= - tr[\gamma^{\mu_2}\gamma^{\mu_1}\gamma^{\mu_3} \dots \gamma^{\mu_k} \dots \gamma^{\mu_{2n}}] + 2 g^{\mu_1\mu_2} tr[\gamma^{\mu_3} \dots \gamma^{\mu_k} \dots \gamma^{\mu_{2n}}]$$

$$\vdots$$

$$= (-1)^{2n-1} tr[\gamma^{\mu_2} \dots \gamma^{\mu_k} \dots \gamma^{\mu_{2n}}\gamma^{\mu_1}]$$

$$+ 2 \sum_{k=2}^{2n} (-1)^k g^{\mu_1\mu_k} tr[\gamma^{\mu_2} \dots \gamma^{\mu_{k-1}}\gamma^{\mu_{k+1}} \dots \gamma^{\mu_{2n}}]$$

1st. term is $tr[\gamma^{\mu_1} \dots \gamma^{\mu_{2n}}]$

$\Rightarrow$ $tr[\gamma^{\mu_1} \dots \gamma^{\mu_{2n}}]$

$$= \sum_{k=2}^{2n} (-1)^k g^{\mu_1\mu_k} tr[\gamma^{\mu_2} \dots \gamma^{\mu_{k-1}}\gamma^{\mu_{k+1}} \dots \gamma^{\mu_{2n}}]$$

d) $tr[\gamma^\mu\gamma^\nu] = g^{\mu\nu} tr[\mathbb{1}]$, $n=1$

$$= 4 g^{\mu\nu}$$

e) $tr[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] = \sum_{k=2}^{4} (-1)^k g^{\mu\mu_k} tr[\dots]$, $n=4$

$$= g^{\mu\nu} tr[\gamma^\rho\gamma^\sigma] - g^{\mu\rho} tr[\gamma^\nu\gamma^\sigma] + g^{\mu\sigma} tr[\gamma^\nu\gamma^\rho]$$

$$= 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})$$

f) $n=1$, $tr[\gamma^\mu\gamma^\nu] = 4g^{\mu\nu}$

$n=2$, $tr[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})$

Assume (9) is true $\forall n$

$$tr[\gamma^{\mu_1} \dots \gamma^{\mu_{2n}}] = 4 \sum_{p\in P} sgn(p) \prod_{i=1}^{n} g^{\mu_{\alpha i}} g^{\mu_{\beta i}}$$

$$tr[\gamma^{\mu_1} \dots \gamma^{\mu_{2n}}\gamma^{\mu_{2n+1}}\gamma^{\mu_{2n+2}}]$$

$$\underset{(8)}{=} \sum_{k=2}^{2n+2} g^{\mu_1\mu_k} (-1)^k \, Tr[\gamma^{\mu_2} \dots \gamma^{\mu_{k-1}}\gamma^{\mu_{k+1}} \dots \gamma^{2n+2}]$$

$$= \sum_{k=2}^{2n+2} g^{\mu_1\mu_k} (-1)^k \cdot 4 \sum_{p\in P} sgn(p) \prod_{i=1}^{n} g^{\mu_{\alpha i}\mu_{\beta i}}$$

$$= 4 \sum_{p \in P} \text{sgn}(p) \prod_{i=1}^{n+1} g^{\mu_{\alpha i} \mu_{\beta i}}$$

$\Rightarrow$ proven

(g) if μ, ν, ρ, σ not permutation

$$\Rightarrow \quad tr[\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^5] \propto tr[\gamma^\alpha \gamma^\beta \gamma^5] = 0$$

$$tr[\gamma^0 \gamma^1 \gamma^2 \gamma^3 \gamma^5] = -i\, tr[(\gamma^5)^2] = -i\, 4 = -4i$$

$$\text{if } tr[\gamma^0 \underbrace{\gamma^2 \gamma^1}_{\text{anti commute}} \gamma^3 \gamma^5] = (-1)\cdot(-4i)$$

$$\Rightarrow \quad tr[\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^5] = -4i\, \epsilon^{\mu\nu\rho\sigma}$$