H.4 $\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi$ 16/20

a) EL-eq. $\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\right) - \frac{\partial\mathcal{L}}{\partial\psi} = 0$

$$\partial_\mu(\bar{\psi}i\gamma^\mu) + m\bar{\psi} = 0$$

$$\bar{\psi}(i\gamma^\mu\overleftarrow{\partial}_\mu + m) = 0$$

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})}\right) - \frac{\partial\mathcal{L}}{\partial\bar{\psi}} = 0$$

thinking $\not{\partial}$ acting on ψ

$$(i\not{\partial} - m)\psi = 0$$

(hermitian conjugation:

$$\psi^\dagger(i\gamma^\mu\partial_\mu - m)^\dagger = 0$$

$$\psi^\dagger(-i\gamma^0\gamma^\mu\gamma^0\partial_\mu - m) = 0$$

$$\psi^\dagger(-i\gamma^0\gamma^\mu\partial_\mu - m\gamma^0) = 0$$

$$\rightarrow \bar{\psi}(i\gamma^\mu\overleftarrow{\partial}_\mu + m) = 0$$
)

b) $T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\partial^\nu\psi + (\partial^\nu\bar{\psi})\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})} - g^{\mu\nu}\mathcal{L}$

$$= \bar{\psi}i\gamma^\mu\partial^\nu\psi - g^{\mu\nu}\bar{\psi}(i\not{\partial} - m)\psi$$

if one uses EOM here, $T^{\mu\nu} = i\bar{\psi}\gamma^\mu\partial^\nu\psi$

(why are we allowed to use EOM here?)

PS: one can get symmetric $T^{\mu\nu}$ using symmetric $\mathcal{L}$ ✓

c) $(i\not{\partial} - m)\psi = 0$, ψ is a solution

$m=0$: $\quad i\not{\partial}\psi = 0$

$$i\not{\partial}\gamma^5\psi = i\gamma^\mu\gamma^5\partial_\mu\psi = -i\gamma^5\gamma^\mu\partial_\mu\psi = 0$$

$(\{\gamma^\mu, \gamma^5\} = 0)$ $\quad \Rightarrow \gamma^5\psi$ is also a solution

d) $P_{L/R} = \frac{1}{2}(\mathbb{1} \mp \gamma^5)$

i) $P_{L/R}^2 = \frac{1}{4}(\mathbb{1} \mp \gamma^5)^2 = \frac{1}{4}(\mathbb{1} + (\gamma^5)^2 \mp 2\mathbb{1}\gamma^5)$

$= \frac{1}{4}(2\mathbb{1} \mp 2\mathbb{1}\gamma^5) = \frac{1}{2}(\mathbb{1} \mp \gamma^5) = P_{L/R}$

$P_L P_R = P_R P_L = 0$

$P_L + P_R = \mathbb{1}$

ii) $\psi_{L/R} := P_{L/R}\psi$

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},$$

$$P_{L/R}\psi = \frac{1}{2}\begin{pmatrix} \mathbb{1} \pm \mathbb{1} & 0 \\ 0 & \mathbb{1} \mp \mathbb{1} \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \quad \text{(using chiral rep.)}$$

$$= \begin{cases} \begin{pmatrix} \psi_1 \\ 0 \end{pmatrix} = \psi_L, & L \\ \begin{pmatrix} 0 \\ \psi_2 \end{pmatrix} = \psi_R, & R \end{cases}$$

$\Rightarrow \psi = \psi_L + \psi_R$

$$\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi = (\bar{\psi}_L + \bar{\psi}_R)(i\not{\partial} - m)(\psi_L + \psi_R)$$

$$= \bar{\psi}_L(i\not{\partial} - m)\psi_L + \underbrace{\bar{\psi}_L(i\not{\partial} - m)\psi_R + \bar{\psi}_R(i\not{\partial} - m)\psi_L}_{\text{diagonal and } (1,0)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0} + \bar{\psi}_R(i\not{\partial} - m)\psi_R$$

$$= \bar{\psi}_L(i\not{\partial} - m)\psi_L + \bar{\psi}_R(i\not{\partial} - m)\psi_R = \mathcal{L}_L + \mathcal{L}_R$$

$\gamma^\mu P_{R/L} = P_{L/R}\gamma^\mu$, $\bar\psi P_{L/R} = \psi^\dagger\gamma^0 P_{L/R} = \psi^\dagger P_{R/L}\gamma^0 = (P_{R/L}\psi)^\dagger\gamma^0 = \bar\psi_{R/L}$

kinetic term $= i\bar\psi\not\partial(P_L+P_R)\psi = i\bar\psi\not\partial P_L\psi + i\bar\psi\not\partial P_R\psi \left(= i\bar\psi\not\partial P_L^2\psi + i\bar\psi\not\partial P_R^2\psi\right)$

$= i\bar\psi P_R\not\partial P_L\psi + i\bar\psi P_L\not\partial P_R\psi = i\bar\psi_L\not\partial\psi_L + i\bar\psi_R\not\partial\psi_R$

mass term $m\bar\psi\psi = m\bar\psi_L\psi_R + m\bar\psi_R\psi_L$

e) the trafo $\begin{Bmatrix}\psi\\ \psi^\dagger\end{Bmatrix} \longmapsto \begin{Bmatrix}\gamma^5\psi\\ \psi^\dagger\gamma^5\end{Bmatrix}$

i.e. $\begin{Bmatrix}\psi\\ \bar\psi\end{Bmatrix} = \begin{Bmatrix}\psi\\ \psi^\dagger\gamma^0\end{Bmatrix} \longrightarrow \begin{Bmatrix}\psi\\ \psi^\dagger\gamma^5\gamma^0\end{Bmatrix} = \begin{Bmatrix}\psi\\ -\bar\psi\gamma^5\end{Bmatrix}$

$\bar\psi\psi \to -\bar\psi\gamma^5\gamma^5\psi = -\bar\psi\psi$

$\bar\psi i\gamma^5\psi \longrightarrow -\bar\psi\gamma^5 i\gamma^5\gamma^5\psi = -\bar\psi i\gamma^5\psi$

$\bar\psi\gamma^\mu\psi \longrightarrow -\bar\psi\gamma^5\gamma^\mu\gamma^5\psi \underset{\{\gamma^5,\gamma^\mu\}=0}{=} +\bar\psi\gamma^\mu\psi$

$\bar\psi\gamma^\mu\gamma^5\psi \longrightarrow -\bar\psi\gamma^5\gamma^\mu\gamma^5\gamma^5\psi = +\bar\psi\gamma^\mu\gamma^5\psi$

$\bar\psi\sigma^{\mu\nu}\psi \longrightarrow -\bar\psi\gamma^5\sigma^{\mu\nu}\gamma^5\psi = \bar\psi\sigma^{\mu\nu}\psi$

$$\left(\begin{aligned}\sigma^{\mu\nu} &= \tfrac{i}{2}[\gamma^\mu,\gamma^\nu]\\ \{\sigma^{\mu\nu},\gamma^5\} &= \tfrac{i}{2}\{\gamma^\mu\gamma^\nu-\gamma^\nu\gamma^\mu,\gamma^5\} = 0\end{aligned}\right)$$

f) $\partial_\mu V^\mu(x) = \partial_\mu(\bar\psi\gamma^\mu\psi) = \partial_\mu\bar\psi\gamma^\mu\psi + \bar\psi\gamma^\mu\partial_\mu\psi$

$[(i\not\partial - m)\psi = 0;\ \bar\psi(i\overleftarrow{\not\partial}+m) = 0]$

$= im\bar\psi\psi + \bar\psi(-im)\psi = 0$

$\partial_\mu A^\mu(x) = \partial_\mu(\bar\psi\gamma^\mu\gamma^5\psi) = \partial_\mu\bar\psi\gamma^\mu\gamma^5\psi + \bar\psi\gamma^\mu\gamma^5\partial_\mu\psi$

$= \not\partial\bar\psi\gamma^5\psi - \bar\psi\gamma^5\not\partial\psi$

$= im\bar\psi\gamma^5\psi + i\bar\psi\gamma^5 m\psi$

$\partial_\mu A^\mu = 0$ iff $m = 0$

$\psi(x) \to e^{i\alpha}\psi(x)$, $\alpha\Delta\psi = i\alpha\psi$

$\alpha\Delta\bar\psi = -i\alpha\bar\psi$

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \Delta\psi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar\psi)} \Delta\bar\psi \underbrace{- X^\mu}_{=0}$$

$$= \bar\psi \cdot i\gamma^\mu \cdot i\alpha\psi$$

$$= -\alpha \bar\psi \gamma^\mu \psi \quad \propto \quad V(x)$$

g) $\mathcal{L} = \bar\psi (i\not\partial - m)\psi$

$$\rightarrow \hat{\mathcal{L}} = \mathcal{L} + \partial_\mu X^\mu(x)$$

$$= \frac{i}{2} \bar\psi(\gamma^\mu \overrightarrow{\partial_\mu} - m)\psi - \frac{i}{2}\bar\psi(\overleftarrow{\partial_\mu}\gamma^\mu + m)\psi$$

$$\partial_\mu X^\mu(x) = -\frac{i}{2}\bar\psi(\gamma^\mu \overrightarrow{\partial_\mu} - \cancel{m})\psi - \frac{i}{2}\bar\psi(\overleftarrow{\partial_\mu}\gamma^\mu + \cancel{m})\psi$$

$$= -\frac{i}{2}\bar\psi\gamma^\mu(\overrightarrow{\partial_\mu} + \overleftarrow{\partial_\mu})\psi = -\frac{i}{2}\partial_\mu(\bar\psi\gamma^\mu\psi)$$

the difference can be rewritten into total divergence, thus the EOM from $\hat{\mathcal{L}}$ is the same from $\mathcal{L}$

$$\Rightarrow \hat{T}^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu\psi)}\partial^\nu\psi + (\partial^\nu\bar\psi)\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar\psi)} \underbrace{- g^{\mu\nu}\mathcal{L}}_{=0}$$

$$= \frac{i}{2}\bar\psi\gamma^\mu(\partial^\nu\psi) - (\partial^\nu\bar\psi)\frac{i}{2}\gamma^\mu\psi$$

$$= \frac{i}{2}\bar\psi\gamma^\mu(\overrightarrow{\partial^\nu} - \overleftarrow{\partial^\nu})\psi$$

$$\mathcal{L} = \frac{i}{2}\bar\psi(\gamma^\mu\overrightarrow{\partial_\mu} - m)\psi - \frac{i}{2}\bar\psi(\overleftarrow{\partial_\mu}\gamma^\mu + m)\psi$$

$$\partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)} - \frac{\partial\mathcal{L}}{\partial\psi} = 0 \quad \Rightarrow \quad \bar\psi(i\overleftarrow{\not\partial} + m) = 0$$

w.r.t. $\bar\psi$ $\quad \Rightarrow \quad (i\not\partial - m)\psi = 0$