H.3 Lorentz group: two casimir operator, $p^2 = m^2$, $\omega^2 = -m^2 s(s+1)$ 18/20

rotation & boost $\to L_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu)$, $P_\mu = -i\partial_\mu$ are two forms of generators

a) $$G(x,y) = \langle 0 | \phi(x)\phi(y) | 0 \rangle$$
$$\overset{(1)}{=} \langle 0 | \underbrace{U^\dagger(\Lambda, a)}_{\text{unitary}} \phi(x)\phi(y) U(\Lambda,a) | 0 \rangle$$
$$= \langle 0 | U^{-1}\phi(x) U U^{-1} \phi(y) U | 0 \rangle$$
$$= \langle 0 | \phi(\Lambda x + a)\phi(\Lambda y + a) | 0 \rangle \quad ✓$$

b) Give a such boost and translation so that
$$\Lambda x + a \to 0 \quad , \quad \Lambda y + a \to \Lambda(y-x)$$
$$\Rightarrow G(x,y) = G(0, \Lambda(y-x)) \underset{\substack{(4),\\ \text{Lorentz invariance}}}{=} G(0, y-x) = D(y-x)$$

For $\Lambda = \mathbb{1}$, $G(x,y) = G(x+a, x+y)$ $\forall a \in \mathbb{R}^4$, $z_\pm \doteq \frac{x \pm y}{2}$

$G(x,y) = G(z_+ + z_-, z_+ - z_-) =: \tilde{G}(z_+, z_-)$

$\Rightarrow \tilde{G}(z_+, z_-) = \tilde{G}(z_+ + a, z_-)$ with $\Lambda = \mathbb{1}$,

Thus $0 = \frac{d}{da}\tilde{G}(z_+ + a, z_-)|_{a=0} = \frac{\partial}{\partial z_+}\tilde{G}(z_+, z_-) \Rightarrow \tilde{G}$ is indep. of $z_+ \Rightarrow G(x,y) = D(x,y)$

c) $$D(\Lambda(y-x)) = G(0, \Lambda(y-x)) = G(0, y-x) = D(y-x)$$

d) $$D(x-y) = \langle 0 | \phi(x)\phi(y) | 0 \rangle$$
$$= \langle 0 | \int \frac{d^3p \, d^3q}{(2\pi)^6 \sqrt{2E_p \cdot 2E_q}} (e^{-ipx} a_p + e^{ipx} a_p^\dagger)(e^{-iqy} a_q + e^{iqy} a_q^\dagger) | 0 \rangle$$

$$\left(\text{use} \quad \langle 0 | \underbrace{a_p^\dagger a_q}_{a_q a_p^\dagger} | 0 \rangle = (2\pi)^3 \delta^{(3)}(p-q) \right)$$

$$= \int \frac{d^3p \, d^3q}{(2\pi)^3 \sqrt{2E_p \cdot 2E_q}} e^{-ipx + iqy} \delta^{(3)}(p-q)$$
$$= \int \frac{d^3p}{(2\pi)^3 2E_p} e^{-ip(-y+x)} \quad ✓$$

(e) Arbitrary function f
$$I = \int \frac{d^4p}{(2\pi)^3} \Theta(p^0) \delta(p^2 - m^2) f(p)$$
$$= \frac{1}{(2\pi)^3} \int dE \, d^3p \, \Theta(p^0) \delta(E^2 - p^2 - m^2) f(p)$$

$$= \frac{1}{(2\pi)^3} \int dE\, d^3p\, \Theta(p^0) \sum_{\pm} \frac{\delta(E_p \mp (\underline{p}^2 + m^2))}{2E_p} f(p)$$

$$\left(\delta(g(x)) = \sum_{g(x_0)=0} \frac{\delta(x - x_0)}{\| g'(x_0) \|} \right)$$

$$= \frac{1}{(2\pi)^3} \int d^3p \frac{1}{2E_p} f(p) = \int \frac{d^3p}{(2\pi)^3 \cdot 2E_p} f(p)$$

$$\Rightarrow D(x-y) = \int \frac{d^4p}{(2\pi)^3} \Theta(p^0) \delta(p^2 - m^2) e^{-ip(x-y)}$$ ✓

f) This expression is obviously lorentz invariante,

$$d^4p \longrightarrow |\det(\Lambda)| d^4p = d^4p$$

$\Theta(p^0) \longrightarrow \Theta(p^0)$, since Λ ($SO_+(1,3)$) cannot change the sign of $p^0 = E_p$

$$p^2 = p_\mu p^\mu \longrightarrow p_\mu p^\mu$$

$$p \cdot (x-y) = p_\mu (x-y)^\mu \longrightarrow p_\mu (x-y)^\mu$$ ✓

g) x: space like $\Leftrightarrow x_\mu x^\mu = x^{0\,2} - |\underline{x}|^2 < 0$

Under lorentz-trafo $\tilde{\Lambda} \longrightarrow \tilde{x}_\mu \tilde{x}^\mu = x_\mu x^\mu = x^{0\,2} - |\underline{x}|^2 < 0$

where $\tilde{x} = (\tilde{x}^0, \tilde{x}^1, 0, 0)$.

Thus it is sufficient to prove using $\tilde{x}$.

$x' = \Lambda \tilde{x}$, so that $x'_\mu x'^\mu = -((x')^1)^2 = x^{0\,2} - |\underline{x}|^2 < 0$ ✗

since $\Lambda^T g \Lambda = g$

g) rotate : $x \longrightarrow \tilde{x} = (\tilde{x}^0, \tilde{x}^1, 0, 0)$, space-like $\tilde{x}^{0\,2} - \tilde{x}^{1\,2} < 0$

Lorentz boost in x^1-direction with rapidity α

$$\Rightarrow \tilde{x} \longrightarrow x' = \Lambda x, \quad \begin{cases} (x')^0 = \tilde{x}^0 \cosh(\alpha) - \tilde{x}^1 \sinh(\alpha) \overset{!}{=} 0 \\ (x')^1 = \tilde{x}^1 \cosh(\alpha) - \tilde{x}^0 \sinh(\alpha) \end{cases}$$

$\Rightarrow \frac{\tilde{x}^0}{\tilde{x}^1} = \tanh(\alpha)$, $\forall \alpha \in \mathbb{R}$, $\tanh(\alpha) \in (-1, 1)$, $(\tilde{x}^0) < (\tilde{x}^1)^2$

$\Rightarrow \frac{\tilde{x}^0}{\tilde{x}^1} \in (-1, 1)$, therefore $\exists$ solution

h) Even function : $f(x) = f(-x)$

$D(\Lambda(x-y)) = D(x-y); \quad \forall \Lambda, \ \Lambda x = (0,(x')^1,0,0)^T$

$$D(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip(x-y)}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-i(\Lambda p)_\mu (\Lambda(x-y))^\mu}$$

$$= \int \frac{d^3p'}{(2\pi)^3} \frac{1}{2E_{p'}} e^{+i(p')^1 \cdot (x'-y')^1}$$

$$= \underbrace{\int \frac{d^3p'}{(2\pi)^3} \frac{1}{2E_{p'}}}_{= \int \frac{d^3(-p')}{(2\pi)^3} \frac{1}{2E_{(-p')}}} e^{+i(-p')^1 \cdot (y'-x')^1}$$

$$= D(y-x)$$

i) $$(x^0, x^1, x^2, x^3) \xrightarrow{\text{rotation}} (x^0, (x^{1\,2}+x^{2\,2})^{1/2}, 0, x^3)$$

$$\xrightarrow{\text{rotation}} (x^0, (x^{1\,2}+x^{2\,2}+x^{3\,2})^{1/2}, 0, 0)$$

$$\xrightarrow{\text{boost}} \begin{pmatrix} \gamma & -\gamma\beta & & \\ -\gamma\beta & \gamma & & \\ & & 0 & \\ & & & 0 \end{pmatrix} \begin{pmatrix} x^0 \\ |\underline{x}| \\ 0 \\ 0 \end{pmatrix}$$

$$\beta = \frac{x^0}{|\underline{x}|}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-x^{0\,2}/|\underline{x}|^2}}$$

$\gamma \in \mathbb{R}$, iff $|\underline{x}|^2 > x^{0\,2}$

aka spacelike

$$= \gamma \begin{pmatrix} x^0 - \beta|\underline{x}| \\ -\beta x^0 + |\underline{x}| \\ 0 \\ 0 \end{pmatrix} = \gamma \begin{pmatrix} x^0 - x^0 \\ -x^{0\,2}/|\underline{x}| + |\underline{x}| \\ 0 \\ 0 \end{pmatrix}$$

$$= \frac{1}{\sqrt{1-x^{0\,2}/|\underline{x}|^2}} \begin{pmatrix} 0 \\ -x^{0\,2}+|\underline{x}|^2 \\ 0 \\ 0 \end{pmatrix} \frac{1}{|\underline{x}|}$$

$$= \frac{1}{\sqrt{|\underline{x}|^2 - x^{0\,2}}} \begin{pmatrix} 0 \\ -x^{0\,2}+|\underline{x}|^2 \\ 0 \\ 0 \end{pmatrix}$$

$$= (0, \sqrt{|\underline{x}|^2 - x^{0\,2}}, 0, 0)^T = (0, \sqrt{x^{1\,2}+x^{2\,2}+x^{3\,2}-x^{0\,2}}, 0, 0)^T$$

$$\xrightarrow{\text{rotation}} -(0, \sqrt{-x_\mu x^\mu}, 0, 0)^T$$

$$\xrightarrow{\text{go back}} -(x^0, x^1, x^2, x^3) \quad \checkmark$$

$$\Rightarrow \Lambda x = -x$$

i) $$x = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \xrightarrow{R} \begin{pmatrix} x^0 \\ (x')^1 \\ 0 \\ 0 \end{pmatrix} \xrightarrow{\Lambda_\rho} \begin{pmatrix} 0 \\ (x'')^1 \\ 0 \\ 0 \end{pmatrix} \xrightarrow{R_-} \begin{pmatrix} 0 \\ -(x'')^1 \\ 0 \\ 0 \end{pmatrix}$$

$$\xrightarrow{\Lambda_\rho^{-1}} \begin{pmatrix} -x^0 \\ -(x')^1 \\ 0 \\ 0 \end{pmatrix} \xrightarrow{R^{-1}} -x$$

j) $$D(\Lambda(x-y)) = D(x-y) = D(y-x) \Rightarrow \text{even}$$

k) $$[\phi(x), \phi(y)] = [\phi_1(x), \phi_2(y)] + [\phi_2(x), \phi_1(y)]$$

with $\phi(x) = \underbrace{\phi_1(x)}_{\text{term with } a_p} + \underbrace{\phi_2(x)}_{\text{term with } a^\dagger_p}$

$$\langle 0 | [\phi_1(x), \phi_2(y)] | 0 \rangle = \int \frac{d^3p \, d^3q}{(2\pi)^6 \sqrt{2E_p 2E_q}} e^{-ipx} a_p e^{iqy} a_q^\dagger (2\pi)^3 \delta^{(3)}(p-q)$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{ip(-x+y)} = D(y-x)$$

$$\langle 0 | [\phi_2(x), \phi_1(y)] | 0 \rangle = D(x-y)$$

So if $(x-y)$ is ~~timelike~~ space like, $D(x-y) = D(y-x)$

$$\Rightarrow \langle 0 | [\phi(x), \phi(y)] | 0 \rangle = 0$$