Quantum Field Theory

July 14, 2019

Contents

6 Radiative corrections

6.1 Optical theorem

We have seen in Advanced Quantum Theory that tree diagrams are in general real. So there is no imaginary parts. Need to restore perturbatively in higher-order corrections. Then the optical theorem is valid again.

S-matrix is unitary: $S^\dagger S = \mathbb{1}$ with $S = \mathbb{1} + iT$. Thus

$$-i(T - T^\dagger) = T^\dagger T$$

We take matrix element for $k_1 k_2 \to p_1 p_2$ scattering. On RHS, insert a complete set of states,

$$\langle p_1 p_2 | T^\dagger T | k_1 k_2 \rangle = \sum_n \prod_{i=1}^n \int \frac{\mathrm{d}^3 q_i}{(2\pi)^3 2E_i} \langle p_1 p_2 | T^\dagger | q_1 \dots q_n \rangle \langle q_1 \dots q_n | T | k_1 k_2 \rangle$$

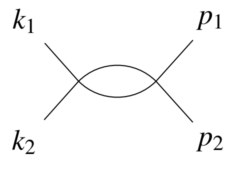
Reduce $T_{fi} = (2\pi)^4 \delta^{(4)}(p_f - p_i) M_{fi}$ and omitting overal $(2\pi)^4 \delta^{(4)}(p_f - p_i)$

$$\begin{aligned} -i\left[\mathcal{M}(k_1 k_2 \to p_1 p_2) - \mathcal{M}^*(p_1 p_2 \to k_1 k_2)\right] & \\ = \underbrace{\sum \prod_{i=1}^n \int \frac{\mathrm{d}^3 q_i}{(2\pi)^3 2E_i}}_{\text{invariant phase-space volume element}} & \mathcal{M}^*(p_1 p_2 \to q_1 \dots q_n) \mathcal{M}(k_1 k_2 \to q_1 \dots q_n) (2\pi)^4 \delta^{(4)}(k_1 + k_2 - \sum_i q_i) \end{aligned}$$

So optical theorem, for forward scattering ($p_1 = k_1, p_2 = k_2$) reads (see 4.5.1)

$$\begin{aligned} \operatorname{Im} \mathcal{M}(k_1 k_2 \to k_1 k_2) &= 2F \sigma_{\text{tot}}(k_1 k_2 \to \text{anything}) \\ 2\sqrt{s} |f_i^{\text{CMS}}| &= \lambda^{\frac{1}{2}}(s, m_1^2, m_2^2) \end{aligned}$$

Optical theorem for Feynman diagrams Consider a specific diagram contributing to the imaginary part, e.g. in ϕ^4-theory.



$$p_s = k_1 + k_2 = p_1 + p_2, p_s^2 = s$$

$$i\mathcal{M}(s) = \frac{\lambda^2}{2} \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{1}{[(p_s/2 - q)^2 - M^2 + i\epsilon][(p_s/2 + q)^2 - M^2 + i\epsilon]} \tag{6.1.1}$$

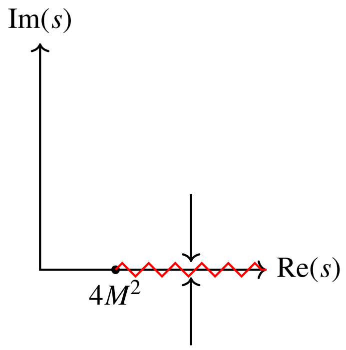
From optical theorem: $\operatorname{Im} \mathcal{M}(s < 4M^2) = 0$, so $\mathcal{M}(s < 4M^2) \in \mathbb{R}$, (Since it is physical case, the cross section must vanish) when regarding $\mathcal{M}(s)$ as an analytic function of s beyond what physical S-matrix element allow.

Schwarz reflection principle If (in some region) analytic function $\mathcal{M}(s)$ is real at least for a finite, nonvanishing interval $\in \mathbb{R}$, then

$$\mathcal{M}(s^*) = \mathcal{M}^*(s) \tag{6.1.2}$$

Hence

$$\mathcal{M}(s+i\epsilon) - \mathcal{M}(s-i\epsilon) \equiv \mathrm{disc}\mathcal{M}(s) = \mathcal{M}(s+i\epsilon) - \mathcal{M}^*(s+i\epsilon) = 2i\,\mathrm{Im}\,\mathcal{M}(s+i\epsilon)$$



Onset of imaginary part for $s \leq 4M^2$ necessarily leads to a "branch cut", a nontrivial discontinuity in the comlex energy plane. The branch cut is equivalent to $\sqrt{4M^2 - s}$. Function has discontinuity, a cut, on real axis.

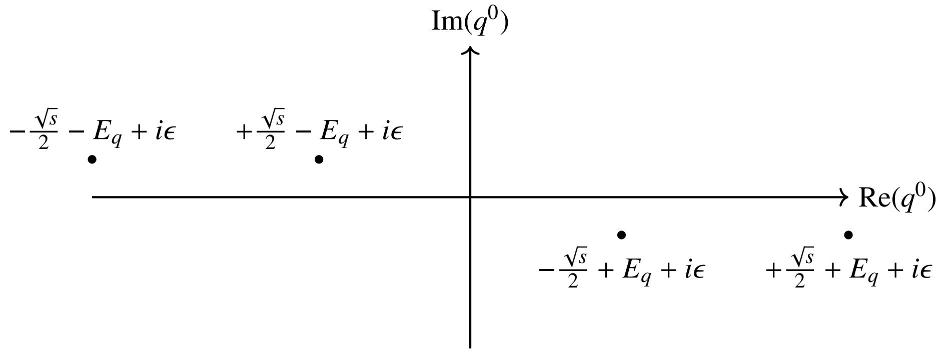
How can we calculate the discontinuity (= imaginary part) of the above diagram?

Use centre-of-mass system $p_s = (\sqrt{s}, \mathbf{0})$. Poles from propagators

$$\begin{aligned} &\frac{s}{4} \mp \sqrt{s}q^0 + q^2 - M^2 + i\epsilon = 0 \\ \Leftrightarrow &(q^0)^2 \pm \sqrt{s}q^0 + \frac{s}{4} - |\boldsymbol{q}|^2 - M^2 + i\epsilon = 0 \end{aligned}$$

first propagator $\qquad q^0 = +\frac{\sqrt{s}}{2} \pm (\sqrt{M^2 + |\boldsymbol{q}|^2} - i\epsilon) = +\frac{\sqrt{s}}{2} \pm (E_q - i\epsilon)$

second propagator $\qquad q^0 = -\frac{\sqrt{s}}{2} \pm (E_q - i\epsilon)$



If we close the contour of the q_0 integration in the lower half plane, we only pick up the 2 residues at $\mp\frac{\sqrt{s}}{2} + E_q - i\epsilon$. As E_q is positive, only $-\frac{\sqrt{s}}{2} + E_q - i\epsilon$ from second propagator contirbutes to discontinuity.

So pinching up the residue equivalent to replacement under q^0 integration

$$\frac{1}{(p_s/2+q)^2-M^2+i\epsilon} \longmapsto \underbrace{-2\pi i}_{\text{orientation of contour}} \quad \delta((p_s/2+q)^2-M^2)$$

Determine the residue of the rest at the pole at $-\frac{\sqrt{s}}{2}+E_q-i\epsilon$

$$M(s) \longmapsto -\frac{\lambda^2}{2}\int \frac{\mathrm{d}^3 q}{(2\pi)^3}\frac{1}{2E_q\sqrt{s}(\sqrt{s}-2E_q)}$$

With no angular dependence and using substitution (note the limits of integral also change)$\mathrm{d}^3 q \to 4\pi|\boldsymbol{q}|^2\mathrm{d}|\boldsymbol{q}| = 4\pi|\boldsymbol{q}|E_q\mathrm{d}E_q$

$$= -\frac{\lambda^2}{8\pi^2}\int_M^\infty \frac{\mathrm{d}E_q\sqrt{E_q^2-M^2}}{\sqrt{s}(\sqrt{s}-2E_q)} \tag{6.1.3}$$

It has pole at $E_q = \frac{\sqrt{s}}{2}$. The second pole in 6.1.1 at $\frac{\sqrt{s}}{2}+E_q-i\epsilon$ would produce a pole in 6.1.3 for $E_q = -\frac{\sqrt{s}}{2}$, outside the integration range $M \le E_q < \infty$.

- for $\sqrt{s} < 2M$, 6.1.3 is manifestly real.
- for $\sqrt{s} > 2M$, the pole at $E_q = \frac{\sqrt{s}}{2}$ in 6.1.3 contributes differently depending on $\sqrt{s} \pm i\epsilon$; difference yields discontinuity.

Use

$$\frac{1}{\sqrt{s}-2E_q \pm i\epsilon} = \underbrace{\frac{P}{\sqrt{s}-2E_q}}_{\text{real}} \underbrace{\mp i\pi\delta(\sqrt{s}-2E_q)}_{\text{yields discontinuity}}$$

So for calculation of the discontinuity, have replacement

$$\frac{1}{(p_s/2-q)^2-M^2+i\epsilon} \longmapsto -2\pi i\delta((p_s/2-q)^2-M^2)$$

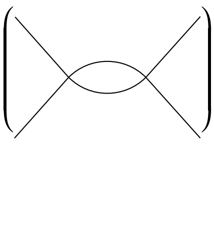
for other propagator too!

Cuthosky rules (1960) replace cut propagator according to

$$\frac{1}{p^2-M^2+i\epsilon} \longmapsto -2\pi i\delta(p^2-M^2) \tag{6.1.4}$$

to calculate discontinuity across the cut!

Calculateion completed:

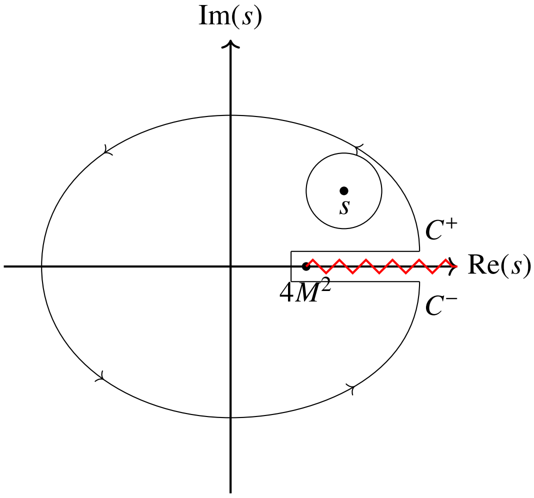
disc  $= i\frac{\lambda^2}{2}\int \frac{\mathrm{d}^4 q}{(2\pi)^4} 2\pi\delta(q^2-M^2)2\pi\delta((p_s-q)^2-M^2)$

using $\mathrm{d}^4q = \mathrm{d}q^0\mathrm{d}q|q|^2\mathrm{d}\Omega_q$ and $(p_s - q)^2 - M^2 = s - 2\sqrt{s}q^0$

$$
\begin{aligned}
&= \frac{\lambda^2}{2}\frac{i}{4\pi^2}\int \frac{|q|^2\mathrm{d}|q|\mathrm{d}\Omega_q}{2q^0}\delta(s - 2\sqrt{s}q^0) \\
&= \frac{\lambda^2}{2}\frac{i}{8\pi^2}\int \sqrt{(q^0)^0 - M^2}\mathrm{d}q^0\mathrm{d}\Omega_q\delta(s - 2\sqrt{s}q^0) \\
&= \frac{\lambda^2}{2}\frac{i}{8\pi^2}\frac{\sqrt{s/4 - M^2}}{2\sqrt{s}}\int \mathrm{d}\Omega_q \\
&= \frac{\lambda^2}{2}\frac{i}{8\pi}\sqrt{1 - \frac{4M^2}{s}} \\
\mathrm{Im}\mathcal{M} &= \frac{\lambda^2}{4}\frac{1}{8\pi}\sqrt{1 - \frac{4M^2}{s}}
\end{aligned}
$$

Note $\sigma = \frac{\lambda^2}{32\pi}$ and $2F = s\sqrt{1 - \frac{4M^2}{s}}$. Thus optical theorem is still valid.

We can do more. Construct the complete $\mathcal{M}(s)$ from Im $\mathcal{M}(s)$ through a <u>dispersion relation</u>!



Use Cauchy's theorem:

$$\mathcal{M}(s) = \frac{1}{2\pi i}\oint \frac{\mathcal{M}(z)\mathrm{d}z}{z - s} \tag{6.1.5}$$

dropping the large circle

$$
\begin{aligned}
&\longmapsto \frac{1}{2\pi i}\int_{C_+ + C_-}\frac{\mathcal{M}(z)\mathrm{d}z}{z - s} \\
&= \frac{1}{2\pi i}\left[\int_{4M^2}^{\infty}\frac{M(z + i\epsilon)\mathrm{d}z}{z - s} - \int_{4M^2}^{\infty}\frac{M(z - i\epsilon)\mathrm{d}z}{z - s}\right] \\
&= \frac{1}{2\pi i}\int_{4M^2}^{\infty}\frac{\mathrm{disc}\mathcal{M}(z)\mathrm{d}z}{z - s} \\
&= \frac{1}{\pi}\int_{4M^2}^{\infty}\frac{\mathrm{Im}\,\mathcal{M}(z)\mathrm{d}z}{z - s}
\end{aligned} \tag{6.1.6}
$$

Repeat the exercise for $\frac{\mathcal{M}(s)-\mathcal{M}(0)}{s}$ (no pole introduced!).

$$\begin{aligned}
\mathrm{Im}\left(\frac{\mathcal{M}(s)-\mathcal{M}(0)}{s}\right) &= \frac{\mathrm{Im}\,\mathcal{M}(s)}{s} \\
\mathcal{M}(s)-\mathcal{M}(0) &= \frac{s}{\pi}\int_{4M^2}^{\infty}\frac{\mathrm{Im}\,\mathcal{M}(z)\mathrm{d}z}{z(z-s)} \\
&= \frac{\lambda^2}{2}\frac{s}{(4\pi)^2}\int_{4M^2}^{\infty}\frac{\mathrm{d}z}{z(z-s)}\sqrt{1-\frac{4M^2}{z}}
\end{aligned}$$

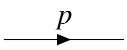
using $\sigma = \sqrt{1-\frac{4M^2}{s}}$ and $\zeta = \sqrt{1-\frac{4M^2}{z}}$

$$\begin{aligned}
&= \frac{\lambda^2}{2}\frac{1}{8\pi^2}\int_0^1\frac{\zeta^2}{\zeta^2-\sigma^2}\mathrm{d}\zeta \\
&= \frac{\lambda^2}{2}\begin{cases}\frac{1}{8\pi^2}\left(1-\frac{\sigma}{2}\log\frac{\sigma+1}{\sigma-1}\right) & s<0 \Leftrightarrow \sigma>1 \\ \frac{1}{8\pi^2}\left(1-\sqrt{-\sigma^2}\arctan\frac{1}{\sqrt{-\sigma^2}}\right) & 0<s<4M^2, \sigma^2<0 \\ \frac{1}{8\pi^2}\left(1-\frac{\sigma}{2}\log\frac{1+\sigma}{1-\sigma}+\frac{i\sigma}{16\pi}\right) & s>M^2, 0<\sigma<1\end{cases}
\end{aligned}$$

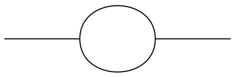
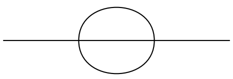
Note: we are going to calculate this diagram again, noticing that $\int\frac{\mathrm{d}^4q}{(q^2\ldots)(q^2\ldots)}$ is logarithmically divergent!. The above representation demonstrates that this divergence resides in $M(0)$!

6.2 Field-strength renomrlization

What is structure of the propagator $\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle$ at higher orders? At lower order



$$= \frac{i}{p^2-M^2+i\epsilon}$$

Beyond this the propagator is not a simple pole. In ϕ^3-theory  branch cuts are at $p^2\leq 4M^2$. In ϕ^4-theory  branch cuts are at $p^2\leq 9M^2$. To induce cuts in the analytic structure.

Insert complete set of intermediate states ($x^0>y^0$)

$$\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle = \sum_\lambda\int\frac{\mathrm{d}^3p}{(2\pi)^3 2E_p(\lambda)}\langle\Omega|\phi(x)|\lambda_{\boldsymbol{p}}\rangle\langle\lambda_{\boldsymbol{p}}|\phi(y)|\Omega\rangle$$

with

λ multiparticle state

λ_0 "rest frame", i.e. $\hat{\boldsymbol{P}}|\lambda_0\rangle = 0$

$\lambda_{\boldsymbol{p}}$ boosted to momentum $\boldsymbol{p}$

Call energy of $\lambda_0 = m_\lambda$. From single particle to multi particle $E_{\boldsymbol{p}}(\lambda) = \sqrt{m_\lambda^2+|\boldsymbol{p}|^2}$.

$$\begin{aligned}\langle\Omega|\phi(x)|\lambda_{\boldsymbol{p}}\rangle &= \langle\Omega|e^{i\hat{P}x}\phi(0)e^{-i\hat{P}x}|\lambda_{\boldsymbol{p}}\rangle \\ &= \langle\Omega|\phi(0)|\lambda_{\boldsymbol{p}}\rangle\, e^{-ipx}\Big|_{p^0=E_{\boldsymbol{p}}}\end{aligned}$$

Ω and $\phi(0)$ are invariant under momentum boost

$$= \langle\Omega|\phi(0)|\lambda_0\rangle\, e^{-ipx}\Big|_{p^0=E_{\boldsymbol{p}}}$$

$$\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle = \sum_\lambda \int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_p(\lambda)} e^{-ip(x-y)} |\langle\Omega|\phi(0)|\lambda_0\rangle|^2 \tag{6.2.1}$$

$$= \sum_\lambda \underbrace{\int \frac{\mathrm{d}^4p}{(2\pi)^4} \frac{i}{p^2 - m_\lambda^2 + i\epsilon} e^{-ip(x-y)}}_{D_F(x-y;m_\lambda^2)\text{ when combined with }y^0>x^0} |\langle\Omega|\phi(0)|\lambda_0\rangle|^2 \tag{6.2.2}$$

$$\tag{6.2.3}$$

Formally write this as

$$\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle = \int_0^\infty \frac{\mathrm{d}s}{2\pi}\rho(s)D_F(x-y;s) \tag{6.2.4}$$

with $\rho(s)$ the spectral density function.

$$\rho(s) := \sum_\lambda (2\pi)\delta(s - m_\lambda^2)\, |\langle\Omega|\phi(0)|\lambda_0\rangle|^2 \tag{6.2.5}$$

A typical spectral function looks like

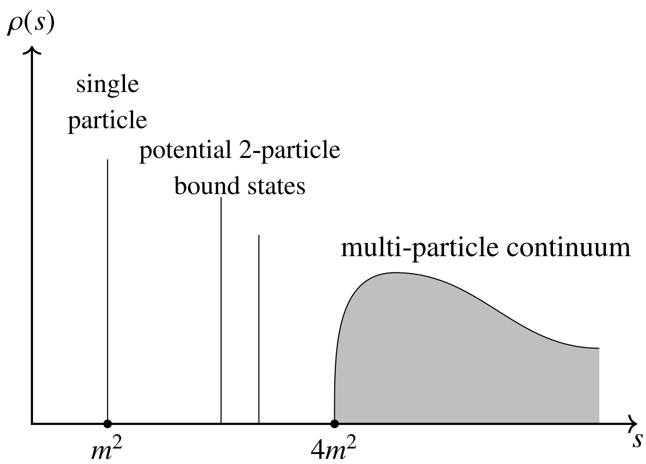


Figure 6.1: typical spectral function

Single particle contribution

$$\rho(s) = 2\pi\delta(s - m^2)Z + (\text{contributions} \geq 4m^2) \tag{6.2.6}$$

with $Z = |\langle\Omega|\phi(0)|\lambda_0\rangle|^2$ the field-strength renomrlization factor.

Fourier transforming two-point function

$$\begin{aligned}
&\int \mathrm{d}^4 x e^{ipx} \langle\Omega|T\phi(x)\phi(0)|\Omega\rangle \\
=&\int_0^\infty \frac{\mathrm{d}s}{2\pi}\rho(s)\frac{i}{p^2 - s + i\epsilon} \\
=&\frac{iZ}{p^2 - m^2 + i\epsilon} + \int_{\sim 4m^2}^\infty \frac{\mathrm{d}s}{2\pi}\rho(s)\frac{i}{p^2 - s + i\epsilon}
\end{aligned}$$

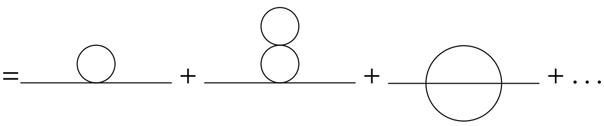
Comparing to free theory: $\langle 0|\phi(0)|\boldsymbol{p}\rangle = 1$ hence $Z = 1$.

6.3 LSZ reduction formula

6.4 The propagator (again)†

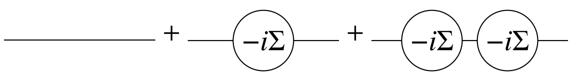
How do we calculate the propagtor and the wave-function renormalization factor Z in perturbation theory, using Feynman diagrams? Call mass parameter in $\mathcal{L} = \frac{1}{2}(\partial_\mu\phi_0)^2 - \frac{m_0^2}{2}\phi_0^2 - \frac{\lambda_0}{4!}\phi_0^4$ the *bare mass*.

1-particle-irreducibles (1PIs) in ϕ^4-theory are the diagrams that cannot be disconnected by cutting internal lines. Their contributions are

$-i\Sigma(p^2) =$ 

Then the complete propagator using $D_F^0(p^2) = \frac{i}{p^2 - m_0^2 + i\epsilon}$ is

$$\begin{aligned}
D_F(p^2) &= \int \mathrm{d}^4 x e^{ipx} \langle 0|T\phi(x)\phi(0)|0\rangle \qquad (6.4.1)
\end{aligned}$$

$=$ 

$$= D_F^0(p^2) + D_F^0(p^2)\left(-i\Sigma(p^2)\right)D_F^0(p^2) + D_F^0(p^2)\left(-i\Sigma(p^2)\right)D_F^0(p^2)\left(-i\Sigma(p^2)\right)D_F^0(p^2)$$

it is cleary a geometric series

$$\begin{aligned}
&= \frac{D_F^0(p^2)}{1 + i\Sigma(p^2)D_F^0(p^2)} \\
&= \frac{i}{p^2 - m_0^2 - \Sigma(p^2)} \qquad (6.4.2)
\end{aligned}$$

The pole of propagator does not occur at m_0^2 anynore. It will be shifted by $\Sigma \sim O(\lambda)$!

Expansion of divergent integrals * Notice that the integral in 6.1.1 $\propto \int \frac{\mathrm{d}^4 q}{q^4}$. If we differentiate it with respect to q, the integral becomes convergent. This holds true for integral of general loop diagrams (although more than one differentiation might be needed). Thus we can expand this kind of integral into convergent and divergent term(s).

†see also Peskin and Schröder, Chapter 10.2

*see also Cheng and Li, Chapter 2.1

Expand

$$\Sigma(p^2) = \Sigma(m^2) + (p^2 - m^2)\Sigma'(m^2) + (p^2 - m^2)\tilde{\Sigma}(p^2) \tag{6.4.3}$$

where $\Sigma(m^2)$ is quadratically and $\Sigma'(m^2)$ logarithmically divergent. $\tilde{\Sigma}$ represents a correction (to first order Taylor expansion) and it satisfies $\tilde{\Sigma}(m^2) = \tilde{\Sigma}'(m^2) = 0$.

Mass and field renomrlization The mass m by the condition

$$m^2 = m_0^2 + \Sigma(m^2) \tag{6.4.4}$$

This is indeed physical mass, since the expression for propagator in 6.4.2 has a pole at $p^2 = m^2$.

Then the propagator

$$D_F(p^2) = \frac{i}{p^2 - m_0^2 - \Sigma(p^2)} = \frac{i}{p^2 - m^2 - (p^2 - m^2)(\Sigma'(m^2) + \tilde{\Sigma}(p^2))}$$

using 6.4.3

$$\begin{aligned} &= \frac{i}{(p^2 - m^2)(1 - \Sigma'(m^2) - \tilde{\Sigma}(p^2))} \\ &= \frac{iZ}{p^2 - m^2} \cdot \frac{1}{1 - Z\tilde{\Sigma}(p^2)} \\ &= \frac{iZ}{p^2 - m^2} + (\text{regular at } p^2 = m^2) \end{aligned} \tag{6.4.5}$$

with $Z = \left(1 - \Sigma'(m^2)\right)^{-1}$. This expression is to be compared with 6.4.1.

Starting point Lagrangian is $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_0)^2 - \frac{m_0^2}{2}\phi_0^2 - \frac{\lambda_0}{4!}\phi_0^4$. To remove Z from numerator in the propagator and instead put $\sqrt{Z}$ onto the couplings at each end. Since each internal vertex has 4 lines (remember the vertex carries the coupling constant)

$$\lambda_0 \longmapsto \lambda_1 = Z^2 \lambda_0 \tag{6.4.6}$$

In Σ and $\tilde{\Sigma}$, there are 2 external lines without $\sqrt{Z}$, so

$$\Sigma(p^2, \lambda_0, \text{old } D_F) = \frac{1}{Z}\Sigma_1(p^2, \lambda_1, \text{new } D_F') \tag{6.4.7}$$

(same expression for $\tilde{\Sigma}$).

Thus we get the new propagator

$$D_F'(p^2) = \frac{i}{p^2 - m^2} \cdot \frac{1}{1 - \tilde{\Sigma}_1(p^2)} \tag{6.4.8}$$

where $\tilde{\Sigma}_1(m^2) = 0$.

Define the renomalized field

$$Z^{-\frac{1}{2}}\phi_0 = \phi \tag{6.4.9}$$

then D_F' is the Fourier transform of $\langle 0|T\phi(x)\phi(y)|0\rangle$

Rewrite the Lagrangian as

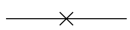
$$\mathcal{L} = \frac{1}{2}\left((\partial_\mu\phi)^2 - m^2\phi^2\right) \boxed{-\frac{\lambda_1}{4!}\phi^4 \underbrace{-\frac{1}{2}\delta m^2\phi^2 + \frac{1}{2}(Z-1)\left((\partial_\mu\phi)^2 - m^2\phi^2\right)}_{\text{counter-terms}}} \tag{6.4.10}$$

where $\delta m^2 = -Z(m^2 + m_0^2) = -Z\Sigma(m^2) = -\Sigma_1(m^2)$. Everythin inside the $\boxed{\text{box}}$ can be considered as "interaction".

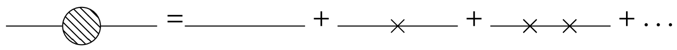
It may look weird given the kinetic/mass-like terms, but no contradiction. Consider just $\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2$. The mass-term ≡ "interaction". A massless propagator

 $= \dfrac{i}{p^2}$

and interaction

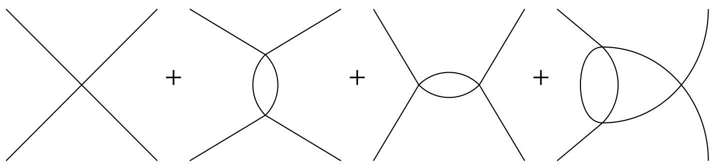
 $= -im^2$

The resummed propagator is then



$$= \frac{i}{p^2}\left(1 + \frac{i}{p^2}(-im^2) + \dots\right)$$
$$= \frac{i}{p^2}\left(1 - \frac{i}{p^2}(-im^2)\right)^{-1} = \frac{i}{p^2 - m^2}$$

Actually this is not all. We will also have to further renomalize λ_1

 $+ \dots = \lambda_1\{1 + L + \Gamma(s,t,u)\}$

L is value of the sum of all 1PI vertex contributions at the same kinematical point and Γ defined by $\Gamma(s = t = u = \frac{4}{3}M^2) = 0$ (for instance, $P_i^2 = M^2$ and $P_iP_j = -\frac{M^2}{3}$ with $i \neq j$).

Define

$$Z_\lambda := (1 + L)^{-1} \tag{6.4.11}$$

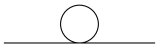
and the renormalized coupling is

$$\lambda = Z_\lambda^{-1}\lambda_1 = Z_\lambda^{-1}Z^2\lambda_0 \tag{6.4.12}$$

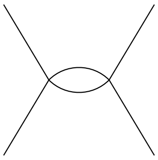
Write Lagrangian in terms of renormalized λ and add another counterterm $-\frac{(Z_\lambda - 1)}{4!}\lambda\phi^4$.

Note that all counter-terms we have introduced are of the same form as the original Lagrangian: $(\partial\phi)^2$, ϕ^2 and ϕ^4. There is no need to introduce new structure or new coupling parameters. It is property of a renormalizable theory.

6.5 Divergent graphs and dimensional regularization

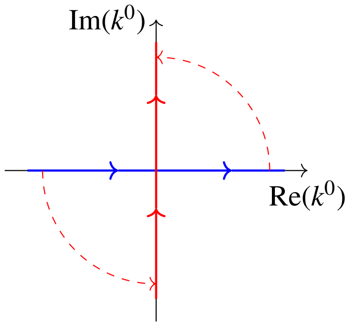


$$M_2 = \frac{\lambda}{2}\frac{1}{i}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{k^2 - M^2 + i\epsilon}$$



$$M_4 = \frac{\lambda^2}{2}\frac{1}{i}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{(k^2 - M^2 + i\epsilon)((k-p)^2 - M^2 + i\epsilon)}$$

Wick rotation Poles of M_2 in the complex k^0-plane at $k^0 = \pm\sqrt{\boldsymbol{k}^2 + M^2 - i\epsilon}$. The position of the poles allow us to rotate the integration path to go $-i\infty \mapsto +i\infty$ instead. So no sigularities are hit!

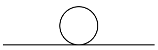


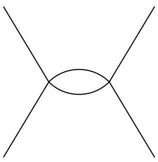
Define a Euclidean momentum $k^0 = ik^0_E$, $\boldsymbol{k} = \boldsymbol{k}_E$

$$\frac{1}{i}\int \frac{\mathrm{d}k^0\mathrm{d}^3k}{(2\pi)^4}\frac{1}{(k_0)^2 - \boldsymbol{k}^2 - M^2 + i\epsilon} \longmapsto -\int \frac{\mathrm{d}^4k_E}{(2\pi)^4}\frac{1}{k_E^2 + M^2 - i\epsilon}$$

Now we are far from singularities, $i\epsilon$ can thus be ignored.

This form allows us to see

 $\sim \int \frac{\mathrm{d}k k^3}{k^2}$ is quadratically divergent

 $\sim \int \frac{\mathrm{d}k k^3}{k^4}$ is logarithmically divergent

Hope of renormalization program is that all such divergences can be absorbed into bare/unrenormalized couplings to produce physical/renormalized/observable parameters.

There are different methods to regularize divergent loop integrals in order to keep track of divergences

1. momentum (Λ) cutoff: study the limit $\Lambda \longmapsto \infty$ in the end
2. Pauli-Villars: subtract propagator(s) with heavy mass(es)*

$$\frac{1}{k^2} \longmapsto \frac{1}{k^2} - \frac{1}{k^2 - M_{\mathrm{PV}}^2},\ M_{\mathrm{PV}} \longmapsto \infty$$

*for details see Ryder, Chapter 9.2

3. dimensional regularization: work in d dimension instead of 4, 1 time-like, $d-1$ space-like. For small d integral converge, consider $d \longmapsto 4$ in the end. The divergences appear as poles in $\frac{1}{d-4}$.

Main advantage of dimensional regularization is that all symmetries are perserved (massless photons etc.). Downside is that it is somewhat unphysical and unintuitive.

Feynman parameters [†] Combine multiple propagators into one (to some power)

$$\frac{1}{A_1 \dots A_n} = \int_0^1 \mathrm{d}x_1 \dots \mathrm{d}x_n \delta(\sum_i^n x_i - 1) \frac{(n-1)!}{(x_1 A_1 + \cdots + x_n A_n)^n} \tag{6.5.1}$$

using

$$\frac{1}{A_i} = \int_0^\infty \mathrm{d}\alpha_i e^{-\alpha_i A_i}$$

$$\int \mathrm{d}\alpha_1 \dots \mathrm{d}\alpha_n e^{-\sum_i \alpha_i A_i} = \int_0^1 \mathrm{d}x_1 \dots \mathrm{d}x_n \delta\left(\sum_i^n x_i - 1\right) \int_0^\infty \mathrm{d}t\, t^{n-1} e^{-t \sum_i x_i A_i}$$

Special case

$$\frac{1}{AB} = \int_0^1 \mathrm{d}x \frac{1}{[xA + (1-x)B]^2} \tag{6.5.2}$$

With $A = (k-p)^2 - M^2$ and $B = k^2 - M^2$

$$xA + (1-x)B = k^2 - xp(2k-p) - M^2 = (k-p)^2 - (M^2 - x(1-x)p^2)$$

Thus after shifting the integration variable $k \longmapsto k + xp$ and with $\Delta(x) := M^2 - x(1-x)p^2$

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{(k^2 - M^2)((k-p)^2 - M^2)} = \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{[k^2 - \Delta(x)]^2}$$

Dimensional regularization formula [*]

$$\frac{1}{i} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{(k^2 - \Delta)^n} = \frac{(-1)^n}{(4\pi)^{d/2}} \frac{\Gamma(n - d/2)}{\Gamma(n)} \frac{1}{\Delta^{n-d/2}} \tag{6.5.3}$$

Γ-function has following definition and properties

- $\Gamma(n+1) = \int_0^\infty \mathrm{d}x x^n e^{-x}$
- $\Gamma(n+1) = n!$ for $n \in \mathbb{N}$, $n\Gamma(n) = \Gamma(n+1)$
- $\Gamma(n)$ has poles for negative integers $n = 0, -1, -2, \dots$

[†] see also Peskin and Schröder, Chapter 6.3; Ryder, Chapter 9.2
[*] see also Peskin and Schröder, Chapter 7.5

Proof by induction

- $n = 1$: introduce Schwinger parameter α and $i\epsilon$ part enforces convergence.

$$\frac{1}{i}\int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{k^2-\Delta+i\epsilon} = -\int_0^\infty \mathrm{d}\alpha \int \frac{\mathrm{d}^d k}{(2\pi)^d} e^{i\alpha(k^2-\Delta+i\epsilon)}$$

using Wick rotation

$$= -i\int_0^\infty \mathrm{d}\alpha \int \frac{\mathrm{d}^d k_E}{(2\pi)^d} e^{-i\alpha(k_E^2+\Delta-i\epsilon)}$$

Gaussian integral in higher dimension; in general $\int \exp\left(-\frac{1}{2}x\cdot A\cdot x + J\cdot x\right) d^n x = \sqrt{\frac{(2\pi)^n}{\det A}}\exp\left(\frac{1}{2}J\cdot A^{-1}\cdot J\right)$

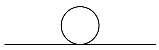
$$= \frac{-i}{(2\pi)^d}\int_0^\infty \mathrm{d}\alpha \sqrt{\frac{\pi}{i\alpha}}^d e^{-i\alpha\Delta}$$
$$= \frac{-i}{(4\pi)^{d/2}}\int_0^\infty \mathrm{d}\alpha (i\alpha)^{-d/2} e^{-i\alpha\Delta}$$
$$= -\frac{1}{(4\pi)^{d/2}}\frac{1}{\Delta^{1-d/2}}\int_0^\infty \mathrm{d}x x^{-d/2} e^{-x}$$
$$= \frac{(-1)}{(4\pi)^{d/2}}\frac{1}{\Delta^{1-d/2}}\Gamma(1-d/2)$$

- Induction $n \to n+1$

$$\frac{1}{i}\int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{(k^2-\Delta)^{n+1}} = \frac{1}{n}\frac{\partial}{\partial\Delta}\frac{1}{i}\int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{(k^2-\Delta)^n}$$
$$= \frac{1}{n}\frac{\partial}{\partial\Delta}\left(\frac{(-1)^n}{(4\pi)^{d/2}}\frac{\Gamma(n-d/2)}{\Gamma(n)}\frac{1}{\Delta^{n-d/2}}\right)$$
$$= \frac{(-1)^{n+1}}{(4\pi)^{d/2}}\frac{\Gamma(n-d/2)}{n\Gamma(n)}\left(n-\frac{d}{2}\right)\frac{1}{\Delta^{n+1-d/2}}$$
$$= \frac{(-1)^{n+1}}{(4\pi)^{d/2}}\frac{\Gamma(n+1-d/2)}{\Gamma(n+1)}\frac{1}{\Delta^{n+1-d/2}} \qquad \square$$

There is another change in d dimensions. Since $S = \int \mathrm{d}^d x\mathcal{L}$ is dimensionless (keep in mind we are working in natural units), $[\mathcal{L}] = M^d$. So $\mathcal{L}_{\mathrm{KG}} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{M^2}{2}\phi^2$ suggests now $[\phi] = M^{d/2-1}$ and in Dirac theory $[\psi] = M^{\frac{d-1}{2}}$. So in order to keep $[\lambda] = M^0 = 1$, $\mathcal{L}_{\phi^4} = -\mu^{4-d}\frac{\lambda}{4!}\phi^4$ with μ an arbitary mass parameter $[\mu] = M^1$.

With dimensional regularization

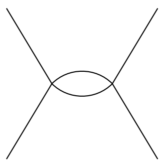


$$
\begin{aligned}
&= \frac{\mu^{4-d}\lambda}{2}\frac{1}{i}\int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{k^2-M^2+i\epsilon}\\
&= \frac{\lambda}{2}\mu^{4-d}\left(-\frac{1}{(4\pi)^{d/2}}\right)M^{d-2}\Gamma(1-d/2)
\end{aligned}
$$

$$
\boxed{
\begin{array}{ll}
 & \text{Laurent expansion}\\
\Gamma(z) & = \frac{1}{z} - \gamma_E + O(z), \quad z\to 0\\
\Gamma(z-1) & = \frac{1}{z-1}\Gamma(z), \quad z\to 0\\
 & = -\left(1+z+O(z^2)\right)\Gamma(z)\\
 & = -\frac{1}{z} + \gamma_E - 1 + O(z)\\
\gamma_E & = 0.5772\ldots
\end{array}
}
$$

$$
\begin{aligned}
&= -\frac{\lambda}{2}\frac{M^2}{8\pi^2}\left(\frac{M^2}{4\pi\mu^2}\right)^{\frac{d-4}{2}}\left[\frac{1}{d-4}+\frac{1}{2}(\gamma_E-1)+O(d-4)\right]\\
&= -\frac{\lambda}{2}\frac{M^2}{8\pi^2}\left\{\frac{1}{d-4}+\frac{1}{2}\left[\gamma_E-1-\ln(4\pi)\right]+\ln\frac{M}{\mu}+O(d-4)\right\}
\end{aligned}
$$

with $\Delta(x) = M^2 - x(1-x)p^2$



$$
\begin{aligned}
&= \frac{\mu^{2(4-d)}\lambda^2}{2}\frac{1}{i}\int_0^1\mathrm{d}x\int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{[k^2-\Delta(x)]^2}\\
&= \frac{\lambda^2}{2}\mu^{2(4-d)}\int_0^1\mathrm{d}x\frac{1}{(4\pi)^{d/2}}\frac{\Gamma(2-d/2)}{2}\frac{1}{\Delta(x)^{(2-d/2)}}\\
&= \frac{\lambda^2}{2}\frac{\mu^{4-d}}{(4\pi)^2}\left\{-2\left[\frac{1}{d-4}+\frac{1}{2}(\gamma_E-\ln(4\pi))+\ln\!\left(\frac{M}{\mu}\right)\right]-\int_0^1\mathrm{d}x\ln\!\left(\frac{\Delta(x)}{M^2}\right)\right\}
\end{aligned}
$$

$$
\begin{aligned}
\int_0^1\mathrm{d}x\ln\!\left(\frac{\Delta(x)}{M^2}\right) &= \int_0^1\mathrm{d}x\ln\frac{M^2-x(1-x)p^2}{M^2}\\
&= \int_0^1\mathrm{d}x\ln\left[\left(\frac{\sigma+1}{2}-x\right)\left(x+\frac{\sigma-1}{2}\right)\right]-\ln\frac{\sigma^2-1}{4}, \qquad \sigma=\sqrt{1-\frac{4M^2}{p^2}}\\
&= \sigma\ln\frac{\sigma+1}{\sigma-1}-2
\end{aligned}
$$

Valid for $p^2 < 0$, rest by analytic contination

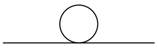
Compare $M(s) - M(0)$ calculated based on Cutkosky and dispersion integral. Easier

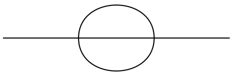
$$
\begin{aligned}
M(0) &= \frac{1}{i}\int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{(k^2-M^2)^2}\\
&= \frac{\partial}{\partial M^2}\frac{1}{i}\int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{k^2-M^2}\\
&= \frac{\partial}{\partial M^2}\left\{-\frac{M^2}{8\pi^2}\left[\frac{1}{d-4}+\frac{1}{2}(\gamma_E-1-\log 4\pi)+\frac{1}{2}\log\frac{M^2}{\mu^2}\right]\right\}
\end{aligned}
$$

1 gets cancalled by the derivative of log

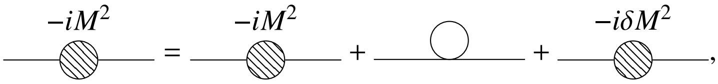
$$= -\frac{1}{8\pi^2}\left[\frac{1}{d-4} + \frac{1}{2}\left(\gamma_E - \log(4\pi) + \frac{1}{2}\log\frac{M^2}{\mu^2}\right)\right]$$

Lets summarise the renormalization of ϕ^4 at one loop

-  is <u>independent</u> of p^2! Hence $\Sigma(p^2)$ at $O(\lambda)$ only renormalises the <u>mass</u>, there is no wavefunction renormalisation $Z(\sim \frac{\partial\Sigma}{\partial p^2}|_{p^2=M^2}) \quad \to Z = 1 + O(\lambda^2)$

  This does change at $O(\lambda^2)$  $\to Z \neq 1$

- Mass renomalisation

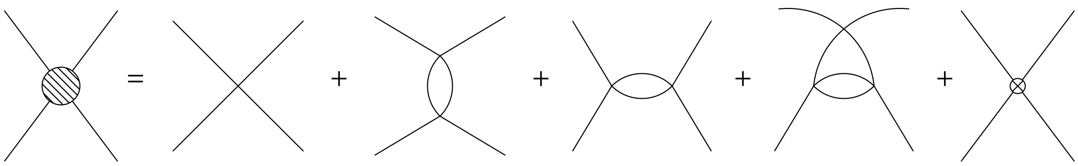
  $\delta M^2 = M_0^2 - M^2$

  Then

$$\begin{aligned}M^2 &= M^2 + \frac{\lambda M^2}{16\pi^2}\left[\frac{1}{d-4} + \frac{1}{2}\left(\gamma_E - 1 - \log 4\pi + \log\frac{M}{\mu}\right)\right] - M^2 + M^2\\ &= M_0^2 + \frac{\lambda M^2}{16\pi^2}\left[\frac{1}{d-4} + \frac{1}{2}(\gamma_E - 1 - \log 4\pi + \log\frac{M}{\mu}) + O(\lambda, (d-4))\right]\end{aligned}$$

  Physical mass M^2_{phy} cannot be dependent on μ, meaning $\lambda\mu^{4-d}M^2 = \lambda_0 M_0^2 + O(\lambda^2)$ and λ_0 and M_0 are independent of μ.

- Coupling constant renormlaisation. Lets choose renormlisation point for λ at $s = t = u = 0$ for simplicity:

$$= -i\lambda\mu^{4-d} + i\left(M(s) + M(t) + M(u)\right) + -i(Z_\lambda - 1)\lambda\mu^{4-d} + O(\lambda^3)$$

  with $Z = 1$

$$\lambda_0 = \lambda\mu^{4-d}Z_\lambda = \lambda\mu^{4-d}\left\{\underbrace{1 - \frac{3}{\lambda}16\pi^2[\frac{1}{d-4}}_{Z^MS_\lambda \text{ minimal subtraction}} + \frac{1}{2}(\gamma_E - \log 4\pi + \log\frac{M}{\mu})] + O(\lambda^2)\right\}$$

$$\lambda_0 = \lambda\mu^{4-d}Z_\lambda = \lambda\mu^{4-d}\left\{\underbrace{1 - \frac{3}{\lambda}16\pi^2[\frac{1}{d-4} + \frac{1}{2}(\gamma_E - \log 4\pi}_{Z_\lambda^{\bar{M}S} \text{ modified minimal subtraction}} + \log\frac{M}{\mu})] + O(\lambda^2)\right\}$$

  these two Z are mass-indepent

$$\lambda_0 = \lambda\mu^{4-d}Z_\lambda = \lambda\mu^{4-d}\left\{\underbrace{1 - \frac{3}{\lambda}16\pi^2[\frac{1}{d-4} + \frac{1}{2}(\gamma_E - \log 4\pi + \log\frac{M}{\mu})]}_{Z_\lambda \text{mass-dependent}} + O(\lambda^2)\right\}$$

6.6 Superficial defree of divergence

How do we know that we are done renormalising the theory with

- wave function
- mass
- coupling

Can't there be more divergences?

Want to analyse superficial degree of divergence D of an arbitary loop diagram with

- d dimension
- L number of loops
- I number of internal propagators
- E number of external lines
- V number of vertices

Matrix element of an arbitary diagram generically

$$\sim \lambda^V \int \frac{\mathrm{d}^d k_1 \mathrm{d}^d k_2 \dots \mathrm{d}^d k_L}{(k_{i_1}^2 - M^2)\dots(k_{i_I}^2 - M^2)}$$

So clearly

$$D = dL - 2I \tag{6.6.1}$$

$D \geq 0$ divergend ($D = 0$ logarithmically divergent) and $D < 0$ convergent.

Express L and I in terms of V and E

$$\begin{aligned} L &= \text{number of undetermined intergration momenta} \\ &= \text{number of internal propagators} - \text{number of momentum conservations at vertices} \\ &\quad + 1 \text{ (because of overall momentum conservation)} \\ L &= I - V + 1 \end{aligned} \tag{6.6.2}$$

One vertex is linked to 4 legs. Internal lines are attached to 2 vertices and external line to 1.

$$4V = 2I + E \tag{6.6.3}$$

solve 6.6.2 and 6.6.3 for L and I

$$D = d + (d-4)V - \left(\frac{d}{2} - 1\right)E \tag{6.6.4}$$

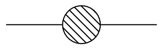
in physical 4 dimension

$$D = 4 - E \tag{6.6.5}$$

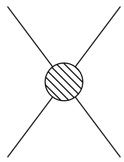
Remarks

- for $d = 4$, D is independent of V, only dependent on E.
- only a few small E produce $D \geq 0$, here in ϕ^4

$E = 2$



$E = 4$



- distinguish theories of different d
  - $d < 4$: D decreases with V, only finite number of digrams (not n-point functions) diverges. **super-renormalisable**
  - $d = 4$: D is independent of V, only a finite number of amplitudes diverges, but at each order in perturbation theory. **renormalisable**
  - $d > 4$: D frows with V, even ampitude becomes divergent at some prder in perturbation theory. **non-renormalisable**
- alternative characterisation in terms of mass dimension of coupling constant

$$\mathcal{L}_{\phi^4} = -\mu^{4-d}\frac{\lambda}{4!}\phi^4 = -\frac{\tilde{\lambda}}{4!}\phi^4$$

  so $[\tilde{\lambda}] = 4 - d$ in d dimension; hence
  - $[\tilde{\lambda}] > 0$ super-renormalisable
  - $[\tilde{\lambda}] = 0$ renormalisable
  - $[\tilde{\lambda}] < 0$ non-renormalisable
- why is this "superficial"? There can always be divergent subgraphs! These subgraphs are regularised and renormalised by the treatment of the "primitive divergences" we have already seen before.

Conclusion for ϕ^4 the only primitive divergences are $E = 2$ and $E = 4$ (and $E = 0$ the vacuum graphs) and we renormalise the theory by

$$M_0^2 = M^2\left\{1 + c_m^{(1)}\frac{\lambda}{d-4} + c_m^{(2)}\frac{\lambda^2}{(d-4)^2} + \ldots\right\}$$
$$\lambda_0 = \lambda\left\{1 + c_\lambda^{(1)}\frac{\lambda}{d-4} + c_\lambda^{(2)}\frac{\lambda^2}{(d-4)^2} + \ldots\right\}$$
$$Z = 1 + c_z^{(2)}\frac{\lambda^2}{(d-4)^2} + \ldots$$

6.7 Sketch of renormlisation of QED