Quantum Field Theory

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Contents

1 Classical field theory

1.1 Field theory in continuum

Euler-Lagrange-equation

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \tag{1.1.1}$$

momentum density

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}(\boldsymbol{x})} \tag{1.1.2}$$

Hamiltonian density

$$\mathcal{H}(\phi(\boldsymbol{x}), \pi(\boldsymbol{x})) = \pi(\boldsymbol{x})\dot{\phi}(\boldsymbol{x}) - \mathcal{L}(\phi, \partial_\mu \phi) \tag{1.1.3}$$

1.2 Noether Theorem

If a Lagrangian field theory has an infinitisimal symmetry, then there is an associated current j^μ, which is conserved.

$$\partial_\mu j^\mu = 0 \tag{1.2.1}$$

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta\phi - X^m u \tag{1.2.2}$$

Energy-momentum tensor (stress-energy tensor)
Asymmetric version

$$\Theta^\mu_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu_\nu \mathcal{L} \tag{1.2.3}$$

General version

$$T^{\mu\nu} = \Theta^{\mu\nu} + \partial_\lambda f^{\mu\nu\lambda} \tag{1.2.4}$$

with $f^{\lambda\mu\nu} = -f^{\mu\lambda\nu}$ or $\partial_\mu \partial \nu f^{\lambda\mu\nu} = 0$

2 Klein-Gordon theory

(Real) Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 \tag{2.0.1}$$

Quantization

$$\begin{aligned} [\phi(\boldsymbol{x}), \phi(\boldsymbol{x}')] = [\pi(\boldsymbol{x}), \pi(\boldsymbol{x}')] = 0 \\ [\phi(\boldsymbol{x}), \pi(\boldsymbol{x}')] = i\delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}') \end{aligned} \tag{2.0.2}$$

Decomposition into Fourier modes

$$\phi(\boldsymbol{x}) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(a_{\boldsymbol{p}}e^{i\boldsymbol{p}\cdot\boldsymbol{x}} + a_{\boldsymbol{p}}^\dagger e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\right) \tag{2.0.3}$$

$$\pi(\boldsymbol{x}) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3}(-i)\sqrt{\frac{E_p}{2}}\left(a_{\boldsymbol{p}}e^{i\boldsymbol{p}\cdot\boldsymbol{x}} - a_{\boldsymbol{p}}^\dagger e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\right) \tag{2.0.4}$$

thus the commutation relations for ladder operators:

$$\left[a_{\boldsymbol{p}}, a_{\boldsymbol{p}'}\right] = \left[a_{\boldsymbol{p}}^\dagger, a_{\boldsymbol{p}'}^\dagger\right] = 0 \tag{2.0.5}$$

$$\left[a_{\boldsymbol{p}}, a_{\boldsymbol{p}'}^\dagger\right] = (2\pi)^3\delta^{(3)}(\boldsymbol{p} - \boldsymbol{p}') \tag{2.0.6}$$

Hamiltonian in terms of ladder operator

$$H = \int \frac{\mathrm{d}^3 p}{(2\pi)^3}E_p\left(a_{\boldsymbol{p}}a_{\boldsymbol{p}}^\dagger + \frac{1}{2}\left[a_{\boldsymbol{p}}, a_{\boldsymbol{p}}^\dagger\right]\right) \tag{2.0.7}$$

Normlisation it's also lorentz-invariante

$$\langle p|q\rangle = 2E_p(2\pi)^3\delta^{(3)}(\boldsymbol{p} - \boldsymbol{q}) \tag{2.0.8}$$

2.1 Heisenberg-picture fields

Heisenberg-picture

$$|\psi_H\rangle = e^{iHt}|\psi_s(t)\rangle \tag{2.1.1}$$

$$O_H(t) = e^{iHt}O_S e^{-iHt} \tag{2.1.2}$$

Field operator

$$\phi(x) = \phi(\boldsymbol{x}, t) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(a_p e^{ipx} + a_p^\dagger e^{-ipx}\right) \tag{2.1.3}$$

2.2 Commutations and propogators

Commutations

$$[\phi(x), \phi(y)] = D(x-y) - D(y-x) \begin{cases} = 0 & \text{if } (x-y) \text{ is space-like} \\ \neq 0 & \text{otherwise} \end{cases} \tag{2.2.1}$$

$$D(x-y) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip(x-y)} \tag{2.2.2}$$

Propogator

$$\langle 0|\phi(x)\phi(y)|0\rangle = D(x-y) \tag{2.2.3}$$

Feynman propagator

$$\begin{aligned} D_F(x-y) &= \langle 0|T\phi(x)\phi(y)|0\rangle \\ &= \Theta(x^0 - y^0)D(x-y) + \Theta(y^0 - x^0)D(y-x) \end{aligned} \tag{2.2.4}$$

$$D_F(x-y) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip(x-y)} \tag{2.2.5}$$

3 Quantization of the Dirac field

3.1 Dirac equation

$$\left(i\gamma^{\mu}\partial_{\mu} - m\right)\phi(x) = 0 \tag{3.1.1}$$

Standard representation (Dirac's)

$$\gamma_0 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix} \quad \boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix} \tag{3.1.2}$$

Lorentz transformation

$$\Lambda = \exp\left(\frac{1}{2}\omega_{\mu\nu}M^{\mu\nu}\right) \tag{3.1.3}$$

with ω set of parameters and M the generator of Lie algebra.

Spinor representation

$$S^{\rho\sigma} = \frac{1}{4}\left[\gamma^{\rho}, \gamma^{\sigma}\right] = \frac{1}{2i}\sigma^{\rho\sigma} \tag{3.1.4}$$

$$\tag{3.1.5}$$

Spinor transformation

$$S(\Lambda) = \exp\left(\frac{1}{2}\omega_{\mu\nu}S^{\mu\nu}\right) \tag{3.1.6}$$

$$\psi'_a(x) = S_{ab}(\Lambda)\psi_b(\Lambda^{-1}x) \tag{3.1.7}$$

adjoint spinor

$$\bar{\psi} = \psi^{\dagger}\gamma^0 \tag{3.1.8}$$

Fifth gamma matrix

$$\gamma^5 := i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{3.1.9}$$

$$\left\{\gamma^{\mu}, \gamma^5\right\} = 0 \tag{3.1.10}$$

$$(\gamma^5)^2 = \mathbb{1}_4 \tag{3.1.11}$$

Plane wavesolutions

$$\psi(x) = \begin{cases} u(p)e^{-ipx} & \text{positive frequency} \\ v(p)e^{ipx} & \text{negative frequency} \end{cases} \tag{3.1.12}$$

$$u_s(p) = \sqrt{E_p + m}\begin{pmatrix} \chi_s \\ \frac{\boldsymbol{u}\cdot\boldsymbol{p}}{E_p+m}\chi_s \end{pmatrix} e^{-ipx} v_s(p) = \sqrt{E_p + m}\begin{pmatrix} \frac{\boldsymbol{u}\cdot\boldsymbol{p}}{E_p+m}\tilde{\chi}_s \\ \tilde{\chi}_s \end{pmatrix} e^{ipx} \tag{3.1.13}$$

with

$$\chi_{\frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad x_{-\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$s = \pm\frac{1}{2} \quad \tilde{\chi}_s = \chi_{-s}$$

Orthogonality of spinor

$$\bar{u}_s(p)u_{s'}(p) = -\bar{v}_s(p)v_{s'}(p) = 2m\delta_{ss'} \tag{3.1.14}$$
$$\bar{u}_s(p)v_{s'}(p) = 0 \tag{3.1.15}$$

Spin sums

$$\sum_s u_s(p)\bar{u}_s(p) = \not{p} + m \tag{3.1.16}$$
$$\sum_s v_s(p)\bar{v}_s(p) = \not{p} - m \tag{3.1.17}$$

3.2 Dirac Lagrangian and quantization

$$\mathcal{L} = \bar{\psi}(x)(i\not{\partial} - m)\psi(x) \tag{3.2.1}$$

Quantization

$$\left\{\psi_a(\boldsymbol{x}), \psi_b^\dagger(\boldsymbol{x}')\right\} = \delta_{ab}\delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}') \tag{3.2.2}$$
$$\{\psi_a(\boldsymbol{x}), \psi_b(\boldsymbol{x}')\} = \left\{\psi_a^\dagger(\boldsymbol{x}), \psi_b^\dagger(\boldsymbol{x}')\right\} = 0 \tag{3.2.3}$$

Field operators

$$\psi(\boldsymbol{x}) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3\sqrt{2E_p}} \sum_s (a_{\boldsymbol{p}}^s u_s(\boldsymbol{p}) e^{i\boldsymbol{p}\cdot\boldsymbol{x}} + b_{\boldsymbol{p}}^{s\,\dagger} v_s(\boldsymbol{p}) e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}) \tag{3.2.4}$$

thus the anticommutations of ladder operators:

$$\left\{a_{\boldsymbol{p}}^s, a_{\boldsymbol{p}'}^{s'\,\dagger}\right\} = \left\{b_{\boldsymbol{p}}^s, b_{\boldsymbol{p}'}^{s'\,\dagger}\right\} = (2\pi)^3\delta_{ss'}\delta^{(3)}(\boldsymbol{p} - \boldsymbol{p}')$$
$$\{a, a\} = \left\{a^\dagger, a^\dagger\right\} = \cdots = 0$$

Hamiltonian in terms of Fourier modes (with normal ordering)

$$H = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \sum_s E_p (a_{\boldsymbol{p}}^{s\,\dagger} a_{\boldsymbol{p}}^s - b_{\boldsymbol{p}}^{s\,\dagger} b_{\boldsymbol{p}}^s) \tag{3.2.5}$$

3.3 Particles and antiparticles

$$Q = e \int \mathrm{d}^3 x \psi^\dagger(x)\psi(x) \tag{3.3.1}$$

$$: Q : = e \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \sum_s (a_{\boldsymbol{p}}^{s\,\dagger} a_{\boldsymbol{p}}^s - b_{\boldsymbol{p}}^{s\,\dagger} b_{\boldsymbol{p}}^s) \tag{3.3.2}$$

3.4 Dirac propagator and anticommutators

$$\begin{aligned} S_{ab}(x-y) &= \{\psi_a(x), \bar{\psi}_b(y)\} \\ &= (i\not{\partial} + m)\left[D(x-y) - D(y-x)\right] \end{aligned} \tag{3.4.1}$$

Time ordering of Dirac fields

$$T(\phi_a(x)\bar{\psi}_b(y)) = \Theta(x^0 - y^0)\psi_a(x)\bar{\psi}_b(y) - \Theta(y^0 - x^0)\bar{\psi}_b(y)\psi_a(x) \tag{3.4.2}$$

Feynman propogator for the Dirac field

$$S_F(x-y) = \langle 0|T\psi(x)\bar{\psi}(y)|0\rangle = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon} e^{-ip\cdot(x-y)} \tag{3.4.3}$$

3.5 Discrete symmetries of the Dirac Field

	orientation perserving	orientation not perserving
(ortho)chronous	$\mathcal{L}_+^\uparrow$	$\mathcal{L}_-^\uparrow = \mathcal{P}\mathcal{L}_+^\uparrow$
non-orthochronous	$\mathcal{L}_-^\downarrow = \mathcal{T}\mathcal{L}_+^\uparrow$	$\mathcal{L}_+^\downarrow = \mathcal{P}\mathcal{T}\mathcal{L}_+^\uparrow$

4 Interacting QFT

4.1 Introduction and examples

Theories discussed so far are Klein-Gordon theory (spin 0)

$$\mathcal{L}_{KG} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2$$

and Dirac theory (spin $\frac{1}{2}$)

$$\mathcal{L}_D = \bar{\psi}(i\partial\!\!\!/ - m)\psi$$

There is also $\mathcal{L}_{EM} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ for a massless vector filed. Its quantiasation gives photon

One thing they have in common is quadratic in the fields. As result:

- linear field equations
- exact quantisation
- multi-particle states without scattering or interaction
- linear fourier decompositions , no mementum changes

To have an interacting theory with scattering, need higher powers in the field in the Lagrangians. A few examples are following

scalar ϕ^4 theory

$$\mathcal{L} = \mathcal{L}_{KG} + \frac{\lambda}{4!}\phi^4$$

need positive sign $\lambda > 0$ for a stable theory, otherwise classical energy can be arbitarily negative.

Equation of motions

$$(\partial^2 + m^2)\phi = -\frac{\lambda}{3!}\phi^3$$

is nonlinear, cannot be solved by Fourier decomposition.

Yukawa-theory

$$\mathcal{L} = \mathcal{L}_{KG} + \mathcal{L}_D - g\bar{\psi}\psi\phi$$

It is originally developed as a theory for nuclear forces with ψ nucleon, ϕ pion. In the Standard Model it is similar to interactions in Higgs mechanism.

Quantum Electrodynamics (QED)

$$\mathcal{L} = \mathcal{L}_{EM} + \mathcal{L}_D - eA_\mu \bar{\psi}\gamma^\mu \psi$$

descreibes electrons, their antiparticles positrons and photons.

Yang-Mills theory generalises $\mathcal{L}_{EM}$ with terms like A^4 or $A^2\partial A$

Scalar QED descreibes pions and photons

$$\begin{aligned}\mathcal{L} &= \mathcal{L}_{EM} + D_\mu\phi D^\mu\phi^* - m^2|\phi|^2 \\ &= \mathcal{L}_{EM} + \partial_\mu\phi\partial^\mu\phi^* - m^2\phi\phi^* + ieA_\mu(\phi\partial^\mu\phi^* - \phi^*\partial^\mu\phi) + e^2A_\mu A^\mu\phi\phi^*\end{aligned}$$

Remarks

1. Interaction terms in $H_{\text{int}} = \int d^3\mathcal{H}_{\text{int}} = -\int d^3x\mathcal{L}_{\text{int}}$ always involves products of fields at the same point $\boldsymbol{x}$. It ensures causality, no "instant at a distance".
2. There are no derivative interactions. These may complicate quantisation as
$$\pi(\boldsymbol{x}) = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi(\boldsymbol{x}))}$$
3. Why are we taking the examples above? There must be zillions of theories (Lagrangians)?
We have the criterion of **renormalizability**. Note the mass dimensions of fields;
$$[S] = 1 \text{ so } [\mathcal{L}] = [M]^4 \Rightarrow [\phi] = [M],\ [\psi] = [M]^{\frac{3}{2}},\ [A_\mu] = [M]$$
So in all the interaction terms indicated above, the coupling constant λ, e, g are all **dimensionless**!
Can add $-\frac{\mu}{3!}\phi^3$ to the ϕ^4 theory. This leads to $[\mu] = [M]$ and all these generate renormalisable interactions.
All higher interaction terms require coupling constants of **negative** mass dimension, e.g. $G\bar{\psi}\psi\bar{\psi}\psi$ and then $[G] = [M]^{-2}$. These are nonrenormalisable and create trouble when performing higher-order calculation in perturbation theory. (with energy cutoff; corrections $G\Lambda^2$, $\Lambda \to \infty$)
4. we haven't quantised the photon yet. The reason is that its is a vector field, i.e. 4 degrees of freedom, but photon has just 2 physical polarisaion states. It is linked to gauge symmetry and complicates quantisation somewhat.

4.2 The interaction picture

Consider the ϕ^4 theory,

$$\mathcal{L}_{int} = -\frac{\lambda}{4!}\phi(x)^4 \tag{4.2.1}$$

Hamiltonian $H = H_0 + H_{int}$ with

$$H_0 = \int d^3x \left\{\frac{1}{2}\pi^2(x) + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\right\} \tag{4.2.2}$$

$$H_{int} = -\int d^3x\mathcal{L}_{int} = \frac{\lambda}{4!}\int d^3x\phi^4 \tag{4.2.3}$$

Interaction picture means that operators evolve in time using H_0 (only), in particular

$$\phi_I(t, \boldsymbol{x}) = e^{iH_0 t}\phi(\boldsymbol{x})e^{-iH_0 t} \tag{4.2.4}$$

Time-dependence of the free field, obeys classical equation of motion $\left(\partial^2 + m^2\right)\phi_I(t, \boldsymbol{x}) = 0$. Solution in terms if fourier modes as before:

$$\phi_I = \int \frac{\mathrm{d}^3 p}{(2\pi)^3 \sqrt{2E_p}}(a^I_{\boldsymbol{p}} e^{-ipx} + a^{I\,\dagger}_{\boldsymbol{p}} e^{+ipx}) \tag{4.2.5}$$

as in the free theory with standard commutation relations $[a^I_{\boldsymbol{p}}, a^{I\,\dagger}_{\boldsymbol{p}}] = (2\pi)^3\delta^{(3)}(\boldsymbol{p} - \boldsymbol{p}')$. The state satisfing $a^I_p\,|0\rangle = 0$ is the vacuum of the free, noninteratcting theory.

Relation between interaction and Schrödinger picure states:

$$|\phi_I(t)\rangle = e^{iH_0 t}|\psi_S(t)\rangle \tag{4.2.6}$$

Schrödinger equation becomes:

$$\begin{aligned} i\frac{\partial}{\partial t}\,|\psi_S\rangle &= (H_0 + H_{\text{int}})\,|\psi_S\rangle \\ \text{LHS} &= i\frac{\partial}{\partial t}(e^{-iH_0 t}\,|\phi_I\rangle) = H_0 e^{-iH_0 t}\,|\phi_I\rangle + e^{-iH_0 t} i\frac{\partial}{\partial t}\,|\phi_I\rangle \\ \text{RHS} &= (H_0 + H_{\text{int}})\,e^{-iH_0 t}\,|\phi_I\rangle \\ \Rightarrow i\frac{\partial}{\partial t}\,|\phi_I\rangle &= e^{iH_0 t} H_{int} e^{-iH_0 t} = H_I(t)\,|\phi_I\rangle \end{aligned} \tag{4.2.7}$$

with H_I interaction Hamiltonian in the interaction picture. Clearly

$$H_I = \frac{\lambda}{4!}\int \mathrm{d}^3 x \phi_I^4(x)$$

What is the solution of **??** for the time evolution of $|\phi_I(t)\rangle$? Define time-evolution operator in the interaction picture.

$$|\phi_I(t)\rangle = U(t, t_0)\,|\phi_I(t_0)\rangle \tag{4.2.8}$$

$$\text{where } U(t, t_0) = e^{iH_0(t-t_0)}e^{-iH(t-t_0)} \tag{4.2.9}$$

With **??** and **??**:

$$i\frac{\partial}{\partial t}U(t, t_0) = H_I(t)U(t, t_0) \tag{4.2.10}$$

To solve with boundary conditions: $U(t_0, t_0) = \mathbb{1}$. The formal solution:

$$U(t, t_0) = 1 - i\int_{t_0}^{t} \mathrm{d}t' H_I(t')U(t', t_0)$$

Substitute back in and we get:

$$U(t, t_0) = 1 - i\int_{t_0}^{t} \mathrm{d}t' H_I(t') + (-i)^2\int_{t_0}^{t} \mathrm{d}t' \int_{t_0}^{t'} \mathrm{d}t'' H_I(t')H_I(t'') + \dots \tag{4.2.11}$$

Ranges of integration: H_I in the product is automatically time-ordered.

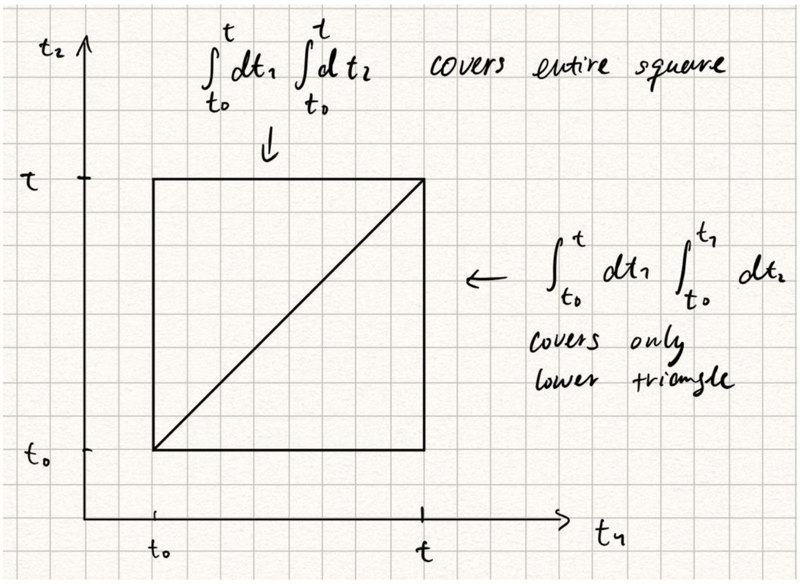


Figure 4.1: Time ordering

Upper triangle has the wrong time order. We are going to "repair" it by hand.

$$\begin{aligned}
U(t,t_0) &= 1 - i\int_{t_0}^{t} \mathrm{d}t' H_I(t') + \frac{(-i)^2}{2}\int_{t_0}^{t}\mathrm{d}t' \int_{t_0}^{t'} \mathrm{d}t'' T(H_I(t')H_I(t'')) + \dots \\
&= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{t_0}^{t} \mathrm{d}t_1 \cdots \int_{t_0}^{t} \mathrm{d}t_n T(H_I(t_1)\dots H_I(t_n)) \\
&= T\exp\left\{-i\int_{t_0}^{t}\mathrm{d}t' H_I(t')\right\}
\end{aligned} \tag{4.2.12}$$

It is interesing for scattering to transition into asymptotic state for $t \to \infty$

$$\begin{aligned}
S &= \lim_{t\to\infty} U(t,-t) = T\exp\left\{-i\int_{-\infty}^{\infty}\mathrm{d}t H_I(t)\right\} \\
&\overset{\phi^4}{=} T\exp\left\{-i\int \mathrm{d}^4x \frac{\lambda}{4!}\phi_I^4(x)\right\}
\end{aligned} \tag{4.2.13}$$

Both U and S are formally unitary

Composition law for time evolution operator:

$$U(t_2,t_0) = U(t_2,t_1)U(t,t_0) = U(t_2,t_1)U(t_0,t_1)^\dagger \tag{4.2.14}$$

4.2.1 Scattering amplitudes and the S-matrix

Take $|i\rangle$ the initial (multi-particle) state and $|f\rangle$ the final (multi-particle) state. Time evolution of $|i\rangle$ then is

$$\lim(t\to\infty)U(t,-\infty)|i\rangle = S\,|i\rangle$$

Probability that $|i\rangle$ evolves into $|f\rangle$ is proporional to the squared "**S-matrix element**"

$$|\langle f,\ t\to\infty|i, t\to-\infty\rangle|^2 = |\langle f|S|i\rangle|^2 = \left|S_{fi}\right|^2 \tag{4.2.15}$$

The nontrivial part of the S-matrix is the T-matrix:

$$S_{fi} := \delta_{fi} + iT_{fi} \tag{4.2.16}$$

Use momentum conservation (from translation invariance) to define matrix element

$$S_{fi} = \delta_{fi} + i(2\pi)^4\delta^{(4)}(p_f - p_i)M_{fi} \tag{4.2.17}$$

M_{fi} measures "genuine scattering" from $|i\rangle$ to $|f\rangle$.

How are we going to calculate correlation functions in the interacting theory:

$$\langle\Omega|\, T\phi(x)\phi(y)|\Omega\rangle \tag{4.2.18}$$

or more generally $\langle\Omega|T\phi(x_1)\phi(x_2)\dots|\Omega\rangle$, where $|\Omega\rangle$ is the vaccum/ground state of the interacting theory and $\phi(x)$ the Heisenberg operators.

Ignore $|\Omega\rangle \neq |0\rangle$ for the moment other than saying: we want to study the time evolution from the vacuum at $t \to -\infty$ to $t \to +\infty$. So rewriting in terms $\phi_I(x)$, assuming $x^0 > y^0$ for now:

$$\langle 0|U(\infty, x^0)\phi_I(x^0)U(x^0, y^0)\phi_I(y^0)U(y^0, -\infty)|0\rangle = \langle 0|T(\phi_I(x)\phi_I(y)S)|0\rangle \tag{4.2.19}$$

still holds if $x^0 < y^0$ because of T.

Now $|\Omega\rangle \neq |0\rangle$: this can be taken care of by dividing out the time evolution of the (free) vacuum $\langle 0|S|0\rangle$, so

$$\begin{aligned}\langle\Omega|T(\phi(x)\phi(y))|\Omega\rangle& \\ &= \frac{\langle 0|T(\phi_I(x)\phi_I(y)S)|0\rangle}{\langle 0|S|0\rangle} \\ &\overset{\phi^4}{=} \frac{\langle 0|T\phi_I(x)\phi_I(y)\exp\left\{-i\int \mathrm{d}^4x'\,\frac{\lambda}{4!}\phi^4(x')\right\}|0\rangle}{\langle 0|T\exp\left\{-i\int \mathrm{d}^4x'\,\frac{\lambda}{4!}\phi^4(x')\right\}|0\rangle}\end{aligned} \tag{4.2.20}$$

Proof can be found in Peskin. It will also be illustrated parctically later ("vacuum bubbles").

Perturbation theory is viable when λ (or some other coupling) is "small" and then expands $U(t, t_0)$ or S in powers of λ.

4.3 Wick's theorem

From now on drop the subscript for interaction pictire fields $\phi_I(x) \to \phi(x)$.

Want to calculate stuff like $\langle 0|T\phi(x_1)\dots\phi(x_n)S|0\rangle$ in perturbation theory; so e.g. at order λ^n. So

$$\frac{1}{n!}\left(-i\frac{\lambda}{4!}\right)^n \int \mathrm{d}^4y_1\dots\mathrm{d}^4y_n\,\langle 0|T\phi(x_1)\dots\phi(x_n)\phi^4(y_1)\dots\phi^4(y_n)|0\rangle \tag{4.3.1}$$

is tough!

We know $\langle 0|T\phi(x_1)\phi(x_2)|0\rangle$ is the Feynman propagator!

Recall **normal ordering** with $\phi(x) = \phi^+(x) + \phi^-(x)$

$$:\phi^+\phi^- :=: \phi^-\phi^+ := \phi^-\phi^+ \tag{4.3.2}$$

Wick's therem expresses time-ordered products in terms of normal-ordered ones. Then it is easy to take vacuum expectation values, as $\langle 0| : \phi(x_1)\dots\phi(x_n) : |0\rangle = 0$

Take two fields and $x^0 > y^0$:

$$\begin{aligned} T\phi(x)\phi(y) &= \phi(x)\phi(y) = \left(\phi^+(x) + \phi^-(x)\right)\left(\phi^+(y) + \phi^-(y)\right) \\ &= \phi^+(x)\phi^+(y) + \phi^-(x)\phi^-(y) + \phi^-(x)\phi^+(y) + \phi^-(x)\phi^+(y) + [\phi^+(x), \phi^-(y)] \\ &=: \phi(x)\phi(y) : +[\phi^+(x), \phi^-(y)] \end{aligned}$$

Particularly for $y^0 > x^0$:

$$T\phi(x)\phi(y) =: \phi(x)\phi(y) : +[\phi^+(y), \phi^-(x)]$$

Thus altogether:

$$T\phi(x)\phi(y) =: \phi(x)\phi(y) : +D_f(x-y) \tag{4.3.3}$$

as $\Theta(x^0 - y^0)[\phi^+(x), \phi^-(y)] + \Theta(y^0 - x^0)[\phi^+(y), \phi^-(x)] = D_F(x-y)$.

Worth noting that $D_F(x-y)$ is still a c-number, not operator (yet). Thus it can be pulled out of any matrix element or expectation value.

We now define "contraction":

$$\wick{\c\phi(x_1)\c\phi(x_2)} = D_F(x_1 - x_2) \tag{4.3.4}$$

Thus we can remove the fields from the product leaving only the propagators:

$$T\phi(x)\phi(y) =: \phi(x)\phi(y) : +\wick{\c\phi(x)\c\phi(y)} \tag{4.3.5}$$

General form of **Wick's theorem** for arbitary number of fields

$$T\phi(x_1)\ldots\phi(x_n) =: \phi(x_1)\ldots\phi(x_n) : + : \text{(sum over all possible contractions)} : \tag{4.3.6}$$

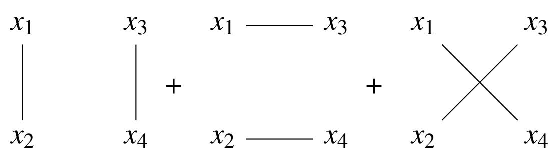
Example with four fields:

$$\begin{aligned} T(\phi_1\phi_2\phi_3\phi_4) =&: \phi_1\phi_2\phi_3\phi_4 : \\ &+ \wick{\c\phi_1\c\phi_2} : \phi_3\phi_4 : + \wick{\c\phi_1\c\phi_3} : \phi_2\phi_4 : + \wick{\c\phi_1\c\phi_4} : \phi_2\phi_3 : + \wick{\c\phi_2\c\phi_3} : \phi_1\phi_4 : + \wick{\c\phi_2\c\phi_4} : \phi_1\phi_3 : + \wick{\c\phi_3\c\phi_4} : \phi_1\phi_2 : \\ &+ \wick{\c\phi_1\c\phi_2}\wick{\c\phi_3\c\phi_4} + \wick{\c\phi_1\c\phi_3}\wick{\c\phi_2\c\phi_4} + \wick{\c\phi_1\c\phi_4}\wick{\c\phi_2\c\phi_3} \end{aligned}$$

Thus

$$\langle 0|T\left(\phi_1\phi_2\phi_3\phi_4\right)|0\rangle = D_F(x_1 - x_2)D_F(x_3 - x_4) + D_F(x_1 - x_3)D_F(x_2 - x_4) + D_F(x_1 - x_4)D_F(x_2 - x_3)$$

which can be visually represented as



Proof of the general theorem by *induction* in the number of fields (see exercise). The idea is to suppose it is true for $\phi_2\ldots\phi_m$, $x_1^0 > x_{k>1}^0$. Then

$$\begin{aligned} T\phi_1\phi_2\ldots\phi_m &= (\phi_1^+ + \phi_1^-)T\phi_2\ldots\phi_m \\ &= (\phi_1^+ + \phi_1^-)[: \phi_2\ldots\phi_m : + : \text{contractions} :] \end{aligned}$$

ϕ_1^- can stay as it is part of (: $\phi_1\phi_2\ldots\phi_m$:). But ϕ_1^+ needs to be comuted past all ϕ_1^- operators, giving rise to additional contractions $\wick{\c\phi_1\c\phi_2}$.

Consequences

- $n = 2k+1,\ k \in \mathbb{N}$
$$\langle 0|T\phi_1 \dots \phi_m|0\rangle = 0$$
- $n = 2k,\ k \in \mathbb{N}$
$$\langle 0|T\phi_1 \dots \phi_m|0\rangle = \sum_{\text{pairing of fields}} D_F(x_{i_1} - x_{i_2}) \dots D_F(x_{i_{m-1}} - x_{i_m})$$

4.3.1 Wick's theorem and the S-Matrix

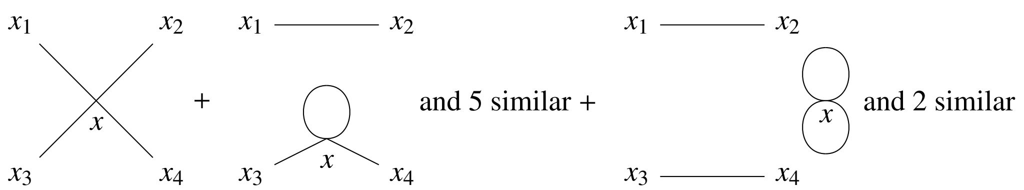
Apply Wick's theorem to correlation functions $\langle 0|T\{\phi_1 \dots \phi_m\}S|0\rangle$ n-th term in the perturbative expansion of S with $\phi(x_1) := \phi_1$.

$$\frac{1}{n!}\left(\frac{-i\lambda}{4!}\right)^n \int \mathrm{d}^4 y_1 \dots \mathrm{d}^4 y_n \,\langle 0|T\{\phi_1 \dots \phi_m \phi^4(y_1) \dots \phi^4(y_n)\}|0\rangle$$

Example with $m = 4,\ n = 1$

$$\begin{aligned}
&-\frac{i\lambda}{4!}\int \mathrm{d}^4 x \,\langle 0|T\phi_1\phi_2\phi_3\phi_4\phi^4(x)|0\rangle \\
&= -\frac{i\lambda}{4!}\int \mathrm{d}^4 x \,\langle 0|\phi_1\phi_2\phi_3\phi_4\phi(x)\phi(x)\phi(x)\phi(x)|0\rangle + 23 \text{ permutations} \\
&\quad -\frac{i\lambda}{4!}\int \mathrm{d}^4 x \,\langle 0|\phi_1\phi_2\phi_3\phi_4\phi(x)\phi(x)\phi(x)\phi(x)|0\rangle + 11 \text{ permutations} + 5 \text{ similar} \\
&\quad -\frac{i\lambda}{4!}\int \mathrm{d}^4 x \,\langle 0|\phi_1\phi_2\phi_3\phi_4\phi(x)\phi(x)\phi(x)\phi(x)|0\rangle + 2 \text{ permutations} + 2 \text{ similar} \\
&= -i\lambda \int \mathrm{d}^4 x D_F(x_1 - x)D_F(x_2 - x)D_F(x_3 - x)D_F(x_4 - x) \\
&\quad -\frac{i\lambda}{2} D_F(x_1 - x_2)\int \mathrm{d}^4 x D_F(x_3 - x)D_F(x_4 - x)D_F(x - x) + 5 \text{ similar} \\
&\quad -\frac{i\lambda}{8} D_F(x_1 - x_2)D_F(x_3 - x_4)\int \mathrm{d}^4 x D_F(x - x) + 2 \text{ similar}
\end{aligned}$$

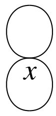
Permutation means permutation of $\phi(x)$ and similar means exchanging $\phi_i,\ i \in 1,2,3,4$ without changing the diagram. Represented in Feynman diagrams:



In fact $D_F(x - x) = D_F(0)$ diverges!

Example with $m = 0$, $n = 1$ vacuum diagram

$$-\frac{i\lambda}{4!}\int d^4x\,\langle 0|T\phi^4(x)|0\rangle$$
$$= -\frac{i\lambda}{8}[D_F(0)]^2\int d^4x$$
$$=$$



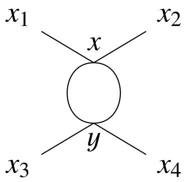
Example: 2nd order S-matrix term

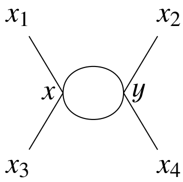
$$\frac{1}{2!}\left(\frac{-i\lambda}{4!}\right)^4\int d^4x d^4y\,\langle 0|T\phi_1\phi_2\phi_3\phi_4\phi^4(x)\phi^4(y)|0\rangle$$

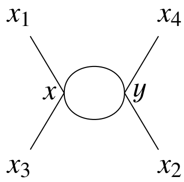
It has many contractions and some of the fully connected ones are of the type there are

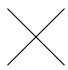
(4×3)[choose$\phi(x)$] $\times$ (4×3)[choose$\phi(y)$] $\times$ 2[x-y-cont.] $\times$ 2(x-y-symm.) + 2 similar, exchanging external points

$$= \frac{(-i\lambda)^2}{2}\int d^4x d^4y D_F(x_1-x)D_F(x_2-x)D_F(x_3-y)D_F(x_4-y)[D_F(x-y)]^2 + 2\text{ similar}$$

$=$


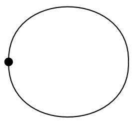
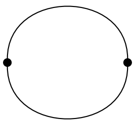
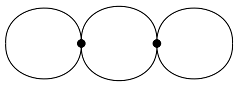
$+$


$+$


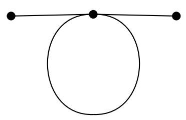
Symmetry factors A lot of the contractions eliminate the factors $\frac{1}{n!}\left(\frac{1}{4!}\right)^4$ in the denominators; the $\frac{1}{4!}$ was chosen as to yield  $\sim -i\lambda$

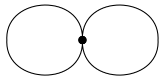
See examples above. Sometimes, factors are not completely cancelled and thus procedure gets "reversed". Divide diagrams by *symmetry factor* $\hat{=}$ the "missing factors".

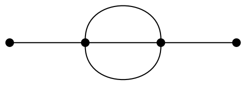
Where does it come from?

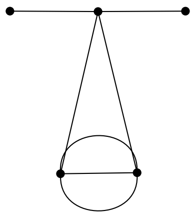
- factor 2 from the line that starts and ends at the same point. 
- two (or more) lines linking the same 2 points. 
- 2 vertices can be equivalent. 

When in doubt, can always go back to Wick's theorem and count the contractions explicitely.
Examples:


$S = 2$


$S = 8$


$S = 3! = 6$


$S = 3! = 6$

Summary of Feynman rules

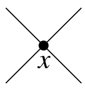
$$\langle 0|T\phi_1 \dots \phi_m \exp\left(-\frac{i\lambda}{4!}\int \mathrm{d}^4x\phi^4(x)\right)|0\rangle$$
$$= \text{sum of all diagrams with m external points;}$$

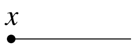
usually organised by number of internal points(i.e. power of λ).
Each diagram built cut of

- propagators
- vertices (n)
- external points (m)

Feynman rules in position space Analytic expression obtained by combining

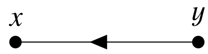
- for each propagator 

$= D_F(x-y)$
- for each vertex 

$= -i\lambda\int \mathrm{d}^4x$
- for each external point 

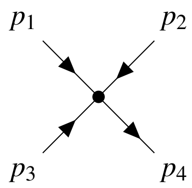
$= 1$
- divide diagram by its symmetry factor S

Since the propagator $D_F(x-y) = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\epsilon}e^{-ip(x-y)}$. It is actually simpler to express these in momentum space instead.
The way to do it is to sssign a momentum p to each propagator. (direction arbitary)



- assign e^{ipy} to y-vertex (arrow out)
- assign e^{-ipx} to x-vertex (arrow in)
- $\frac{i}{p^2-m^2+i\epsilon}$ to the line and the integration $\int \frac{\mathrm{d}^4 p}{(2\pi)^4}$

At vertex x:

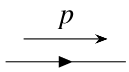
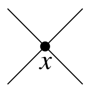


$$= -i\lambda \int \mathrm{d}^4 x e^{-i(p_1+p_2+p_3)x+ip_4x}$$

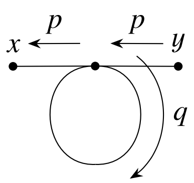
$$= -i\lambda(2\pi)^4 \delta^{(4)}(p_1 + p_2 + p_3 - p_4)$$

This imposes momentum conservation at vetex. $\delta^{(4)}$-functions make some of the momentum integrals trivial, always with $(2\pi)^4$ cancelled appropriately.

Momentum space Feynman rules

- propagator  $= \frac{i}{p^2-m^2+i\epsilon}$
- vertex (position integrated out)  $= -i\lambda$
- external points $\begin{cases} e^{-ipx} & \text{incoming} \\ e^{+ipx} & \text{outgoing} \end{cases}$
- impose momentum conservation at each vertex
- integrate over each undetermined momentum $\int \frac{\mathrm{d}^4 p}{(2\pi)^4}$
- divide by symmetry factor

e.g.:

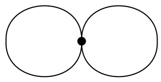
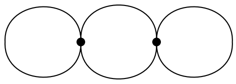


$$= (-i\lambda) \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{\mathrm{d}^4 1}{(2\pi)^4} \left(\frac{i}{p^2 - m^2 + i\epsilon}\right)^2 \frac{i}{q^2 - m^2 + i\epsilon} e^{-ip(x-y)}$$

Vacuum diagrams

Disconnected pieces in Feynman diagrams are pretty bad. Not only $D_F(0) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon}$ is divergent (that will be taken care of later), it also contains an integral $\int \mathrm{d}^4 x\,\text{const.}$ thus divergent once more!

Typical diagram contributing to 2-point function. one piece connected to x and y, plus disconnected pieces.

Call disconnected pieces $V_i \in \Big\{$  ,  , $\ldots\Big\}$. Points are connected internally, but not to external points.

V_i can occur n_i-times, then

$$[\text{diagram}] = [\text{connected pieces}] \times \prod_i \frac{1}{n!} (V_i)^{n_i}$$

The factorial is the symmetry factor of n_i disconnected copies of V_i.

Then

$$\begin{aligned}
&\langle 0|T\phi_1 \ldots \phi_n S|0\rangle \\
&= \sum_{\text{connected}} \sum_{\text{all}\{n_i\}} [\text{connected}] \times \prod_i \frac{1}{n_i!} (V_i)^{n_i} \\
&= \left(\sum_{\text{connected}} [\text{connected}] \right) \times \sum_{\text{all}\{n_i\}} \left(\prod (i) \frac{1}{n_i!} (V_i)^{n_i} \right) \\
&= \prod_i \left(\sum_{n_i} \frac{1}{n_i!} (V_i)^{n_i} \right) \\
&= \exp\left(\sum_i V_i \right)
\end{aligned}$$

Thus

$$\begin{aligned}
\text{sum of ALL diagrams} &= (\text{sum of all CONNECTED diagrams}) && (4.3.7)\\
&\quad \times \exp(\text{sum of all DISCONNECTED diagrams}) && (4.3.8)
\end{aligned}$$

Obvious from the above:

$$\langle 0|S|0\rangle = \langle 0|T\{\exp\left(-\frac{i\lambda}{4!} \int \mathrm{d}^4 x \phi^4(x) \right)\}|0\rangle = \exp(\text{sum of all vacuum bubbles})$$

Conclusion from the (unproven) formula for n-point correlation functions in the true, interacting vacuum:

$$\begin{aligned}
\langle \Omega|T\phi_1 \ldots \phi_m|\Omega\rangle &= \frac{\langle 0|T\phi_1 \ldots \phi_m S|0\rangle}{\langle 0|S|0\rangle} && (4.3.9)\\
&= \sum (\text{connected diagrams with m external points}) && (4.3.10)
\end{aligned}$$

Here: "connected" means connected to any external point. External points do not have to linked to each other.

4.4 S-matrix elements and Feynman diagrams

What is the correlation function in interacting vacuum $\langle\Omega|T\phi_1\dots\phi_m|\Omega\rangle$ good for? For scattering, shouldn't we rather look at $\langle p_1\dots p_m|S|p_Ap_B\rangle$ with the perturbative expansion of S as before?

Decomposition

$$S_{fi} = \delta_{fi} + iT_{fi} = \delta_{fi} + i(2\pi)^4\delta^{(4)}(p_f - p_i)M_{fi} \tag{4.4.1}$$

M_{fi} is the invariant matrix element, used to calculate cross section etc..

Zeroth term in the expansion of S

$$\begin{aligned}\langle p_1p_2|p_Ap_B\rangle &= \sqrt{2E_12E_22E_A2E_B}\,\langle 0|a_1a_2a_A^+a_B^+|0\rangle\\ &= 2E_A2E_B(2\pi)^6\left\{\delta^{(3)}(\boldsymbol{p}_A-\boldsymbol{p}_1)\delta^{(3)}(\boldsymbol{p}_B-\boldsymbol{p}_2)+\delta^{(3)}(\boldsymbol{p}_A-\boldsymbol{p}_2)\delta^{(3)}(\boldsymbol{p}_B-\boldsymbol{p}_1)\right\}\end{aligned}$$

This actually is "no scattering", part of the $\mathbb{1}$ in the S-matrix.

First term is

$$\begin{aligned}&\langle p_1p_2|T\left(-\frac{i\lambda}{4!}\int \mathrm{d}^4x\phi^4(x)\right)|p_Ap_B\rangle\\ &\overset{\text{wick}}{=}\langle p_1p_2|:\left(-\frac{i\lambda}{4!}\int \mathrm{d}^4x\phi^4(x)+\text{contractions}\right):|p_Ap_B\rangle\end{aligned}$$

However now the expectation value of a normal-ordered expression doesn't vanish!

$$\begin{aligned}\phi^+(x)\,|\boldsymbol{p}\rangle &= \int\frac{\mathrm{d}^3k}{(2\pi)^3\sqrt{2E_k}}a_{\boldsymbol{k}}e^{-ikx}\sqrt{2E_p}a_{\boldsymbol{p}}^+\,|0\rangle\\ &= \int\frac{\mathrm{d}^3k}{(2\pi)^3\sqrt{2E_k}}e^{-ikx}\sqrt{2E_p}\delta^{(3)}(\boldsymbol{k}-\boldsymbol{p})\,|0\rangle\\ &= e^{-ipx}\,|0\rangle\end{aligned}$$

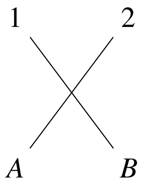
So in general, need two field operators to annihilate the in-state and m fields operators to create the out-states.

New type of Feynman diagram to deal with external states. Define contractions of field operators with external states according to

$$\begin{aligned}\phi(x)\,|\boldsymbol{p}\rangle &= e^{-ipx}\,|0\rangle\\ \langle\boldsymbol{p}|\,\phi(x) &= e^{+ipx}\,|0\rangle\end{aligned}$$

How does this work for $p_Ap_B\to p_1p_2$ in ϕ^4 at $O(\lambda)$? The above contains 3 types of terms: $:\phi\phi\phi\phi:$, $\phi\phi:\phi\phi:$ and $\phi\phi\phi\phi$.

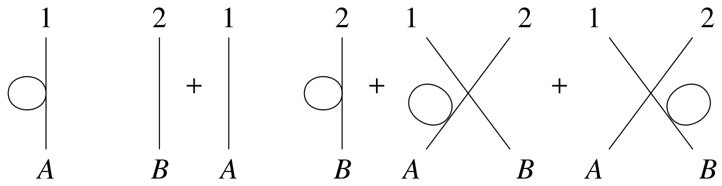
1. $:\phi\phi\phi\phi:$ allows for contractions with all external states. There is 4! possiblities



$$= 4!\frac{-i\lambda}{4!}\int \mathrm{d}^4xe^{-i(p_A+p_B-p_1-p_2)x} = -i\lambda\underbrace{(2\pi)^4\delta^{(4)}(p_A+p_B-p_1-p_2)}_{\text{Prefactor in definition of } i\mathcal{M}}$$

$i\mathcal{M}$ receives a contributuon $-i\lambda$!

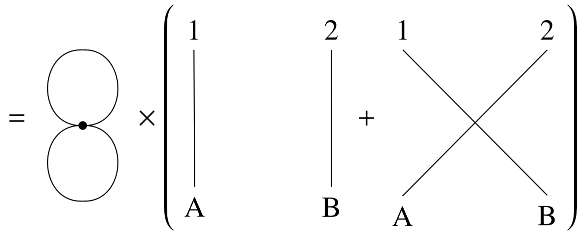
2. $\phi\phi\phi\phi$ leaves 2 operators to connect to external particles. Momentum conservation at each vertex. Still trivial!



Only fully connected Feynman diagrams contribute to $iT/i\mathcal{M}$!

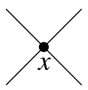
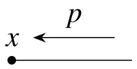
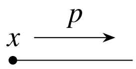
3.

$$-\frac{i\lambda}{4!}\int \mathrm{d}^4x\,\langle p_1 p_2|\phi\phi\phi\phi|p_A p_B\rangle$$



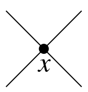
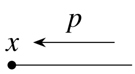
4.4.1 Feynman rules (with external lines)

position space calculate iT by summing overall fully connected diagrams with

- propagator  $= D_F(x-y)$
- vertex  $= -i\lambda\int \mathrm{d}^4x$
- external lines "in"  $= e^{-ip\cdot x}$;  $= e^{ip\cdot x}$
- divide diagram by its symmetry factor $\frac{1}{S}$

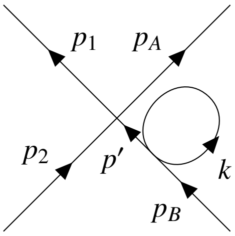
momentum space We have seen it before. Now (with external lines) all positions are integrated over. Result is a function of external momenta only. Integrating out all momentum-conserving δ-distribution yields overall momentum conservation: $(2\pi)^4\delta^{(4)}(P_f - P_i)$

Momentum space Feynman rules for calculating iM:

- internal propagator  $= \frac{i}{p^2-M^2+i\epsilon}$
- vertex  $= -i\lambda\int \mathrm{d}^4x$
- external lines ("in" or "out")  $= 1$
- impose 4-momentum conservation at each vertex

- integrate over all undetermined momenta $\int \frac{\mathrm{d}^4 p}{(2\pi)^4}$
- divide diagram by its symmetry factor $\frac{1}{S}$

There is still trouble in there. Consider the next-to-leading contribution to the scattering amplitude

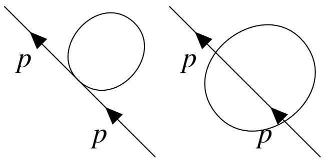


$$
\begin{aligned}
&= \frac{1}{2} \int \frac{\mathrm{d}^4 p'}{(2\pi)^4} \frac{i}{p'^2 - m^2} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{i}{k^2 - m^2} \times \\
&(-i\lambda)(2\pi)^4 \delta^{(4)}(p_A + p' - p_1 - p_2) \cdot (-i\lambda)(2\pi)^4 \delta^{(4)}(p_B - p')
\end{aligned}
$$

This contains the internal propagator $\frac{i}{P_B^2 - m^2 + i\epsilon}$, but all the external particle are on their mass-shell, i.e.

$$
P_A^2 = P_B^2 = P_1^2 = P_2^2 = m^2 \;\Rightarrow\; \frac{i}{P_B^2 + m^2} = \frac{i}{0}
$$

In Addition to having fully connected diagrams, also need to confine ourselves to amputated diagrams: disregard all these diagrams with loops attached to external legs.



These diagrams represent the transition from the free to the interacting asymptotic states.

Lehmann-Symanzik-Zimmermann (LSZ) reduction formula

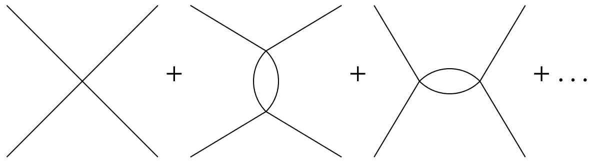
Proof on relation between correlation functions and S-matrix elements will be provided later.

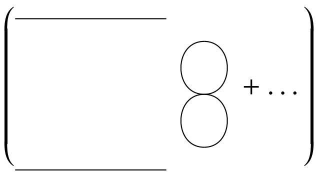
$$
\begin{aligned}
&\prod_{i=1}^{n} \int \mathrm{d}^4 x_i e^{ip_i \cdot x_i} \prod_{j=1}^{m} \int \mathrm{d}^4 y_j e^{-ik_j \cdot y_i} \langle \Omega | T\phi(x_1) \ldots \phi(x_n) \phi(y_1) \ldots \phi(y_m) | \Omega \rangle \\
&\overset{\text{LSZ}}{=} (\text{disconnected stuff}) + \underbrace{\prod_{i=1}^{n} \frac{\sqrt{z} i}{p_i^2 - m^2 + i\epsilon} \prod_{j=1}^{m} \frac{\sqrt{z} i}{k_j^2 - m^2 + i\epsilon}}_{\text{remove poles from external legs}} \langle p_1 \ldots p_n | S | k_i \ldots k_m \rangle
\end{aligned} \tag{4.4.2}
$$

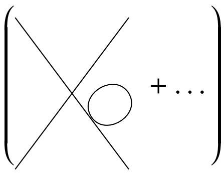
z is the wave-function renormalization factor.

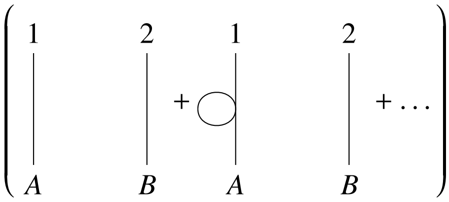
Then amend feynman rules above

consider only fully connected, amputated diagrams

$\langle p_1 p_2 | iT | p_A p_B \rangle =$ 

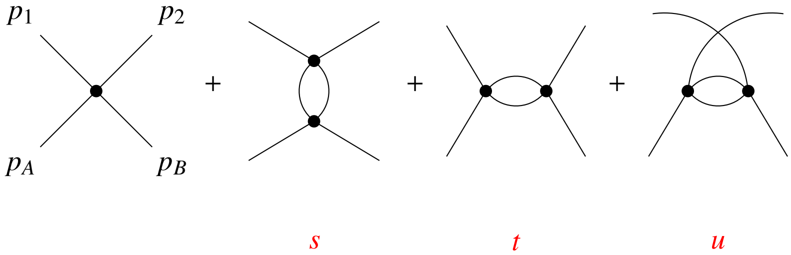
$+$  yields $|0\rangle \to |\Omega\rangle$

$+$  yields $|p\rangle_{\text{free}} \to |p\rangle_{\text{int}}$

$+$ 

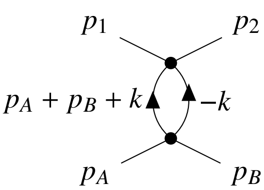
yields $\mathbb{1}$ in S-matrix

All allowed scattering diagrams $2 \to 2$ in ϕ^4 up to $\mathcal{O}(\lambda^2)$:



Define the Lorentz-invariant quantities, *Mandelstam variables*:

$$s = (p_A + p_B)^2\,, \quad t = (p_A - p_1)^2\,, \quad u = (p_A - p_2)^2 \tag{4.4.3}$$



$$= \frac{1}{2}(-i\lambda)^2 \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} \frac{i}{(p_A + p_B + k)^2 - m^2 + i\epsilon} \overset{(p_A+p_B)^2=s}{=:} \frac{1}{2}(-i\lambda)^2 iJ(s)$$

Then the complete invariant amplitude is

$$M = -\lambda - \frac{\lambda^2}{2}\left(J(s) + J(t) + J(u)\right) \tag{4.4.4}$$

4.5 Scattering cross section

(Itzykson & Zuber, Chapter 5.1)
The aim is to relate (differential) cross section to reduced/invariant matrix element M_{fi}. First we describe the initial states not as momentum eigenstates $|p_A p_B\rangle$, but as wave packets.

$$|i\rangle = \int \frac{\mathrm{d}^3 k_A}{(2\pi)^3 2k_A^0} \frac{\mathrm{d}^3 k_B}{(2\pi)^3 2k_B^0} f(k_A) g(k_B) \, |k_A k_B\rangle$$

with $f(k_A)$, $g(k_B)$ strongly peaked at $k_A \approx p_A$, $k_B \approx p_B$.

We can write the transition amplitude to the final state $|f\rangle \propto |p_1 p_2\rangle$ (note: normalisation not the same)

$$\begin{aligned} A_{fi} &= \int \frac{\mathrm{d}^3 k_A}{(2\pi)^3 2k_A^0} \frac{\mathrm{d}^3 k_B}{(2\pi)^3 2k_B^0} f(k_A) g(k_B) \langle f | iT | k_A k_B \rangle \\ &= \int \frac{\mathrm{d}^3 k_A}{(2\pi)^3 2k_A^0} \frac{\mathrm{d}^3 k_B}{(2\pi)^3 2k_B^0} f(k_A) g(k_B) (2\pi)^4 \delta^{(4)}(\underbrace{p_f}_{=p_1+p_2} - k_A - k_B) iM(f, k_A, k_B) \end{aligned}$$

Thus the transition probablity:

$$\begin{aligned} \omega_{fi} = (2\pi)^8 \int & \frac{\mathrm{d}^3 k_A}{(2\pi)^3 2k_A^0} \frac{\mathrm{d}^3 k_B}{(2\pi)^3 2k_B^0} \frac{\mathrm{d}^3 q_A}{(2\pi)^3 2q_A^0} \frac{\mathrm{d}^3 q_B}{(2\pi)^3 2q_B^0} f(k_A) g(k_B) f(q_A)^* g(q_B)^* \times \\ & \underbrace{\delta^{(4)}(p_f - k_A - k_B) \delta^{(4)}(p_f - q_A - q_B)}_{=\delta^{(4)}(q_A+q_B-k_A-k_B)\delta^{(4)}(p_f-p_A-p_B)} \underbrace{M(f, k_A, k_B) M^*(f, q_A, q_B)}_{\approx |M(f,p_A,p_B)|^2} \\ & \left[\delta^{(4)}(q_A + q_B - k_A - k_B) = (2\pi)^{-4} \int \mathrm{d}^4 x e^{i(k_A+k_B-q_A-q_B)\cdot x} \right] \\ = & \underbrace{\int \mathrm{d}^4 x \int \frac{\mathrm{d}^3 k_A}{(2\pi)^3 2k_A^0} \frac{\mathrm{d}^3 q_A}{(2\pi)^3 2q_A^0} e^{i(k_A-q_A)\cdot x} f(k_A) f^*(q_A)}_{:=|\tilde{f}(x)|^2} \times \\ & \underbrace{\int \frac{\mathrm{d}^3 k_B}{(2\pi)^3 2k_B^0} \frac{\mathrm{d}^3 q_B}{(2\pi)^3 2q_B^0} e^{i(k_B-q_B)\cdot x} g(k_B) g^*(q_B)}_{:=|\tilde{g}(x)|^2} (2\pi)^4 \delta^{(4)}(p_f - p_A - p_B) \cdot |M(f, p_A, p_B)|^2 \\ & \left[\text{using Fourier transformation } \tilde{g}(x) := \int \frac{\mathrm{d}^3 q}{(2\pi)^3 2q^0} e^{iq\cdot x} g(q) \right] \\ = & \int \mathrm{d}^4 x |\tilde{f}(x)|^2 |\tilde{g}(x)|^2 (2\pi)^4 \delta^{(4)}(p_f - p_A - p_B) \cdot |M(f, p_A, p_B)|^2 \end{aligned}$$

note that $M(f, p_A, p_B)$ and $M(p_1, p_2, p_A, p_B)$ have different normalisation.

We now consider transition probabilty per unit volume per unit time:

$$\frac{\mathrm{d}\omega_{fi}}{\mathrm{d}V \mathrm{d}t} = (\text{incident flux}) \cdot (\text{target density}) \cdot \mathrm{d}\sigma$$

with $\mathrm{d}\sigma$ the infinitismal cross section for scattering into final state $\langle f|$.

Product (incident flux) · (target density) denotes overlap of wave function. Necessary condition!

Covariant renormalization of states $\langle \boldsymbol{p} | \boldsymbol{q} \rangle \sim 2p^0 \delta^3(\boldsymbol{p} - \boldsymbol{q})$ means the number of particles per unit volume is $2p_A^0 |\tilde{f}(x)|^2$ and $2p_B^0 |\tilde{g}(x)|^2$, respectively.

Assume $\bullet \xrightarrow{p_A} \bullet\ \boldsymbol{p_B} = 0$ in target rest frame. Then $p_B^0 = 2m_B$.
Incident flux $= |\boldsymbol{v}_A| \cdot 2p_A^0 |\tilde{f}(x)|^2 = 2|\boldsymbol{p}_A||\tilde{f}(x)|^2$ since $|\boldsymbol{v}_A| = |\boldsymbol{p}_A|/p_A^0$. Then

$$\begin{aligned} \mathrm{d}\sigma &= (2\pi)^4 \delta^{(4)}(p_f - p_A - p_B) \frac{1}{4m_B|\boldsymbol{p_A}|} |M(f, p_A, p_B)|^2 \\ (\text{for } A + B \to 1 + 2) &= \int_\Delta \frac{\mathrm{d}^3 p_1}{(2\pi)^3 2p_1^0} \frac{\mathrm{d}^3 p_2}{(2\pi)^3 2p_2^0} (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_A - p_B) \frac{1}{4m_B|\boldsymbol{p_A}|} |M(p_1, p_2, p_A, p_B)|^2 \end{aligned}$$

with Δ energy-momentum resolution of 4-momentum of final state $|f\rangle$.
Covariant form of

$$m_B \cdot |\boldsymbol{p}_A| = m_B \sqrt{(p_A^0)^0 - m_A^2} = \sqrt{(p_A \cdot p_B)^2 - m_A^2 m_B^2} =: F \tag{4.5.1}$$

This is scattering into arbitary final state subject to 4-momentum conservation: $p_A + p_B = p_1 + p_2$.
Consider now differential cross section for scattering into a particular infinitismal solid angle $\mathrm{d}\Omega$, hence specific momentum $\mathrm{d}p_1$, $\mathrm{d}p_2$ variations:

$$\begin{aligned} \mathrm{d}\sigma &= \frac{1}{4F} \prod_f \frac{\mathrm{d}^3 p_f}{(2\pi)^3 2p_f^0} (2\pi)^4 \delta^{(4)}(p_A + p_B - \sum_f p_f) |M|^2 \\ &\overset{f=1,2}{=} \frac{1}{4F} \frac{\mathrm{d}^3 p_1}{(2\pi)^3 2p_1^0} \frac{\mathrm{d}^3 p_2}{(2\pi)^3 2p_2^0} (2\pi)^4 \delta^{(4)}(p_i - p_f) |M|^2 \\ &= \frac{1}{64\pi^2 F} \frac{\mathrm{d}^3 p_1}{E_1} \frac{\mathrm{d}^3 p_2}{E_2} \delta^{(4)}(p_1 + p_2 - p_i) |M|^2 \\ &\quad \left[\begin{aligned} &\int \tfrac{\mathrm{d}^3 p_1}{E_1} \tfrac{\mathrm{d}^3 p_2}{E_2} \delta^{(4)}(p_1 + p_2 - p_i) \\ &\overset{\text{CMS}}{=} \int \mathrm{d}|\boldsymbol{p}_1| \mathrm{d}\Omega_1 \tfrac{|\boldsymbol{p}_1|^2}{E_1 E_2} \delta(E_1 + E_2 - E_i) \\ &= \int \mathrm{d}(E_1 + E_2) \tfrac{\mathrm{d}|\boldsymbol{p}_1|}{\mathrm{d}(E_1 + E_2)} \mathrm{d}\Omega_1 \tfrac{|\boldsymbol{p}_1|^2}{E_1 E_2} \delta(E_1 + E_2 - E_i) \\ &= \tfrac{|\boldsymbol{p}_1|^2}{E_1 E_2} \left(\tfrac{|\boldsymbol{p}_1|}{E_1} + \tfrac{|\boldsymbol{p}_1|}{E_2} \right)^{-1} \mathrm{d}\Omega_1 \\ &= \tfrac{|\boldsymbol{p}_1| \mathrm{d}\Omega_1}{E_1 + E_2} = \tfrac{|\boldsymbol{p}_1| \mathrm{d}\Omega_1}{E_i} \end{aligned} \right] \\ \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} &= \frac{1}{64\pi^2} \frac{|\boldsymbol{p}_1|}{F \cdot E_i} |M|^2 \end{aligned} \tag{4.5.2}$$

Rewrite all kinematical factors in terms of $s = (p_A + p_B)^2 = (p_1 + p_2)^2$. Define the function

$$\lambda(x, y, z) := x^2 + y^2 + z^2 - 2(xy + xz + yz) \tag{4.5.3}$$

then

$$\begin{aligned} F &= \sqrt{(p_A \cdot p_B)^2 - m_A^2 m_B^2} = \frac{1}{2} \lambda^{\frac{1}{2}}(s, m_A^2, m_B^2) = \sqrt{s} |\boldsymbol{p}_i| \\ &\left[\begin{aligned} \lambda(s, m_A^2, m_B^2) &= s^2 - 2s(m_A^2 + m_B^2) - (m_A^2 - m_B^2)^2 = (s - (m_A + m_B)^2)(s - (m_A - m_B)^2) \\ &= (2p_A \cdot p_B - 2m_A \cdot m_B) \cdot (2p_A \cdot p_B + 2m_A \cdot m_B) = 4\left[(p_A p_B)^2 - m_A^2 m_B^2 \right] \\ &\quad \left[\begin{array}{ll} p_A = (c\sqrt{s}, \boldsymbol{p}_i),\ c \in [0, 1] & \to m_A^2 = c^2 s - |\boldsymbol{p}_i|^2 \\ p_B = ((1 - c)\sqrt{s}, -\boldsymbol{p}_i) & \to m_B^2 = (1 - c)^2 s - |\boldsymbol{p}_i|^2 \end{array} \right] \\ &= 4\left[(\tfrac{s}{4} + |\boldsymbol{p}_i|)^2 - (\tfrac{s}{4} - |\boldsymbol{p}_i|)^2 \right] = 4s|\boldsymbol{p}_i|^2 \end{aligned} \right] \\ |\boldsymbol{p}_f| &= \sqrt{E_{1,2}^2 - m_{1,2}^2} = \frac{1}{2\sqrt{s}} \lambda^{\frac{1}{2}}(s, m_1^2, m_2^2) \\ E_i &= \sqrt{s} \end{aligned}$$

$$\frac{d\sigma}{d\Omega_{CMS}} = \frac{1}{64\pi^2 s}\frac{|\boldsymbol{p}_f|}{|\boldsymbol{p}_i|}|M|^2 = \frac{1}{64\pi^2 s}\sqrt{\frac{\lambda(s, m_1^2, m_2^2)}{\lambda(s, m_A^2, m_A^2)}}|M|^2 \tag{4.5.4}$$

Decay rate instead of cross section means no "incident flux" to divide by, only "target density"

$$d\Gamma = \frac{1}{2m_A}\prod_f \frac{d^3 p_f}{(2\pi)^3 2p_f^0}(2\pi)^4\delta^{(4)}(p_A - \sum_f p_f)|M|^2 \tag{4.5.5}$$

Particles with spin (unpolarized): sum over outgoing or average over initial spins

$$|M|^2 \to \frac{1}{(2s_A+1)(2s_B+1)}\sum s_i, s_f |M_{fi}|^2 \tag{4.5.6}$$

Symmetry factor $|M|^2 \to \frac{1}{s}|M|^2$ with $s = \prod_i k_i!$ if there are k_i identical particles of species i in the final states.

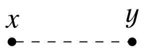
If 1 & 2 are identical, then facotr $\frac{1}{s} = \frac{1}{2}$ on the right hand side.

4.6 Feynman rules for fermions

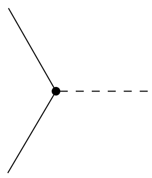
Consider the simplest interacting theory with fermions, Yukawa-theory. We will treat QED later.

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{M^2}{2}\phi^2 + \bar{\psi}(i\partial\!\!\!/ - m)\psi - g\bar{\psi}\psi\phi \tag{4.6.1}$$

Feynman rules will involve:

- scalar 

$= D_F(x-y) = \int \frac{d^4}{(2\pi)^2}\frac{i}{p^2 - M^2 + i\epsilon}e^{-ip(x-y)}$
- fermions 

$= S_F(x-y)_{\alpha\beta} = \int \frac{d^4}{(2\pi)^4}\frac{i(p\!\!\!/ + m)}{p^2 - m^2 + i\epsilon}e^{-ip(x-y)}$
- vertices  $= -ig\int d^4x$

What previous steps need reconsideration due tot he anticommutating fermion operators? Interaction Hamiltonina $\sim\ \bar{\psi}\psi\phi$ and in general compose of even number of fermion fields (spin conservation and fermion number conservation). Thus there is no problem with time-ordered exponential in definition of S-matrix. (Time ordering always takes two or even number of fields.)

Remember the relation

$$T(\psi_\alpha(x)\bar{\psi}_\beta(x)) = -\bar{\psi}_\beta(x)\psi_\alpha(x) \leftarrow y^0 > x^0 \tag{4.6.2}$$

Similarly in normal product:

$$:\psi^+\psi^- = -\psi^-\psi^+: \tag{4.6.3}$$

Then Wick's theorem is formally the same as before

$$T(\psi_\alpha(x)\bar{\psi}_\beta(x)) =: \psi_\alpha(x)\bar{\psi}_\beta(x) : + \overbracket{\psi_\alpha(x)\bar{\psi}_\beta(x)}$$

note by definition $\overbracket{\psi\psi} = \overbracket{\bar{\psi}\bar{\psi}} = 0$

Thus contractions inside normal-ordered products would be

$$: \psi_1\psi_2\bar{\psi}_3\bar{\psi}_4 := -\overbracket{\psi_1\bar{\psi}_3} : \psi_2\bar{\psi}_4 := -S_F(x_1 - x_3) : \psi_2\bar{\psi}_4 :$$

because of the additional operator exchange.

We will want to consider fermion-(anti-)fermion scattering. Leading contribution at $O(g^2)$:

$$\frac{1}{2!}(-ig)^2 \int \mathrm{d}^4x\mathrm{d}^4y \, \langle p', k'| T\bar{\phi}(x)\phi(x)\phi(x)\bar{\phi}(y)\phi(y)|p, k\rangle$$

Contractions with initial-/final-state fermions?

$$\begin{aligned}\phi^+(x)\,|p, s\rangle &= \int \frac{\mathrm{d}^3k}{(2\pi)^3\sqrt{2E_k}} \sum_r a_k^r u_r(k) e^{-ik\cdot x}\sqrt{(2E_p)} a_p^{s\dagger}\,|0\rangle \\ &= e^{-ip\cdot x} u_s(p)\,|0\rangle\end{aligned}$$

So define

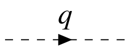
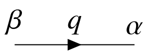
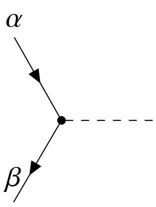
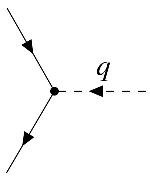
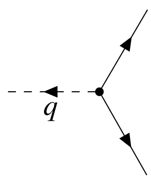
$$\begin{aligned}\overbracket{\psi(x)\,|p,} s\rangle &= e^{-ip\cdot x} u_s(p) \\ \langle \overbracket{p, s|\,\bar{\psi}}(x) &= e^{ip\cdot x} \bar{u}_s(p)\end{aligned} \tag{4.6.4}$$

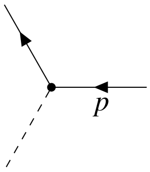
note, though, for <u>antifermion states</u> $|p', s'\rangle$:

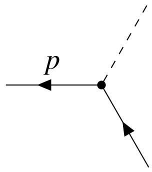
$$\begin{aligned}\overbracket{\bar{\psi}(x)\,|p, s}\rangle &= e^{-ip'\cdot x} \bar{v}_{s'}(p') \\ \langle p', \overbracket{s'|\,\psi}(x) &= e^{ip'\cdot x} v_{s'}(p')\end{aligned} \tag{4.6.5}$$

In short $\overbracket{\psi|}\,\rangle$ contractes with a fermion, $\langle\overbracket{|\psi}$ with an antifermion; vice verse for $\bar{\psi}$.

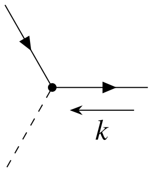
Momentum space feynman rule for iM

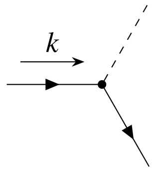
- internal propagators  $= \frac{i}{q^2 - M^2 + i\epsilon}$;  $= \frac{i}{q^2 - M^2 + i\epsilon} = \frac{i(\not{p}+m)_{\alpha\beta}}{p^2 - m^2 + i\epsilon}$
- vertex  $= -ig\int \mathrm{d}^4x = ig\delta_{\beta\alpha}$
- external lines:
  - "in" and "out" scalar  =  $= 1$

– "in"-fermion 

$= u_s(p)$ and "out" fermion 

$= \bar{u}_s(p)$

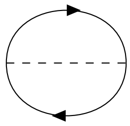
– "in"-antifermion 

$= \bar{v}_s(k)$ and "out" antifermion 

$= v_s(k)$

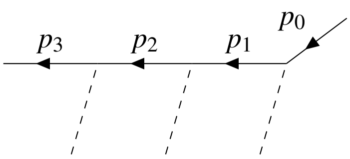
- impose energy-momentum conservation at each vertex
- integrate over undetermined (loop) momenta
- include an overall sign for the diagram

note

- Arrowas on the fermion lines by convention denote fermion (or charge) flow. They must flow consistently through the diagram. ($\equiv$ fermion number conservation) (Only potential confusion: external antifermion lines)
- No symmetry factors (except vacuum bubbles  $\frac{1}{s} = \frac{1}{2}$). $\bar{\psi}\psi\phi$ allows for unambiguous contractions.
- Dirac indices are summed over at each vertex

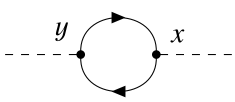
$$\mathcal{L}_{\text{int}} \approx \bar{\psi}_\alpha(x)\psi_\alpha(x)\phi(x)$$

$(\not{p} + m)$ terms in propagator are matrix-multiplied contracted with external spinors, e.g.



$$\sim \bar{u}_\alpha(p_3)\frac{i(\not{p}+m)_{\alpha\beta}}{p_2^2 - m^2 + i\epsilon}\frac{i(\not{p}+m)_{\beta\gamma}}{p_1^2 - m^2 + i\epsilon}u_\gamma(p_0)$$

- closed fermion loop



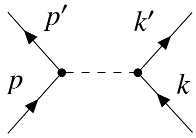
$$\sim \bar{\psi}_\alpha(x)\psi_\alpha(x)\bar{\psi}_\beta(y)\psi_\beta(y) = -\psi_\alpha(x)\bar{\psi}_\beta(y)\psi_\alpha(x)\bar{\psi}_\beta(y)$$

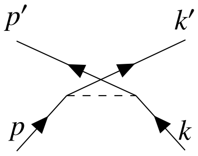
$$= -S_F(y-x)_{\beta\alpha}S_F(x-y)_{\alpha\beta} = -\mathrm{Tr}\,(S_F(y-x)S_F(x-y))$$

It always (also with more propagators/couplings) involves an overal (-1) and a trace $\mathrm{Tr}(\dots)$.

Examples

- fermion-fermion scattering to lowest order $O(g^2)$

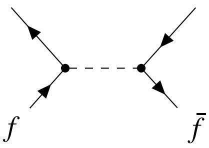
$iM =$ 

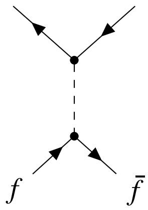
$+$ 

$$= (-ig)^2 \left\{ \bar{u}(p')u(p)\frac{i}{\underbrace{(p'-p)^2}_{t} - M^2 + i\epsilon}\bar{u}(k')u(k) - \bar{u}(p')u(k)\frac{i}{\underbrace{(p'-k)^2} u - M^2 + i\epsilon}\bar{u}(k')u(p) \right\}$$

Mandelstam variables $t = (p'-p)^2$ and $u = (p'-k)^2$

- fermion-antifermion scattering



$+$ 

These are tree diagrams. Thus there is no undetermined momenta to integrate.

5 Quantum Electrodynamics (QED)

5.1 Classical Electrodynamics and Maxwell's equations

We have the gauge potential $A^\mu = (A^0, \boldsymbol{A}) = (\phi, \boldsymbol{A})$ & $A_\mu = (A^0, -\boldsymbol{A}) = (\phi, -\boldsymbol{A})$ and the field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

Then

- electric field $E_i = F_{0i} = \partial_0 A_i - \partial_i A_0 \to \boldsymbol{E} = -\dot{\boldsymbol{A}} - \boldsymbol{\nabla}\phi$
- magnetic field $B^i = -\frac{1}{2}\epsilon^{ijk} F_{jk} \to \boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$

Lagrangian density $\mathcal{L}_{EM} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{2}(\boldsymbol{E} \cdot \boldsymbol{E} - \boldsymbol{B} \cdot \boldsymbol{B})$. The field equation $\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} \right) - \frac{\partial \mathcal{L}}{\partial A_\nu} = 0$ leads to

$$\partial_\mu F^{\mu\nu} = 0 \tag{5.1.1}$$

it is half of Maxwell's equations (in vacuum).

The other half are Bianchi identities following from the definition of $F_{\mu\nu}$:

$$\begin{aligned} \partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0 \Leftrightarrow \epsilon^{\sigma\lambda\mu\nu} \partial_\lambda F_{\mu\nu} = 0 \\ \text{or } \partial_\lambda \tilde{F}^{\sigma\lambda} = 0, \ \tilde{F}^{\sigma\lambda} = \frac{1}{2} \epsilon^{\sigma\lambda\mu\nu} F_{\mu\nu} \end{aligned}$$

In terms of $\boldsymbol{E}$ and $\boldsymbol{B}$:

$$\begin{aligned} \boldsymbol{\nabla} \cdot \boldsymbol{E} = 0, \ \dot{\boldsymbol{E}} = \boldsymbol{\nabla} \times \boldsymbol{B} \quad & \text{dynamical equations} \\ \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0, \ \dot{\boldsymbol{B}} = -\boldsymbol{\nabla} \times \boldsymbol{E} \quad & \text{Bianchi identities} \end{aligned}$$

Remarks

- Lagrangian density does not depend on $\dot{A}_0$, since A_0 is not really dynamical.

  $$\boldsymbol{\nabla} \cdot \boldsymbol{E} = 0 \to \boldsymbol{\nabla}^2 A_0 + \boldsymbol{\nabla} \cdot \dot{\boldsymbol{A}} = 0$$

  Solve this Poisson equation for $A_0(\boldsymbol{x}, t) = \frac{1}{4\pi} \int \mathrm{d}^3 y \frac{\boldsymbol{\nabla} \cdot \dot{\boldsymbol{A}}(\boldsymbol{y}, t)}{|\boldsymbol{y} - \boldsymbol{x}|}$. Thus A_0 is given in terms of the other components of A.

- gauge invariance: field strength tensor invariant under the transformation $A_\mu \longmapsto A_\mu - \partial_\mu \mathrm{X}$ due to commuting derivatives. This leads to gauge invariance of Maxwell equations.
  Choose X to satisfy $\partial_\mu \partial^\mu \mathrm{X} = \partial^2 \mathrm{X} = \partial_\mu A^\mu$ allows us to demand the condition (Lorenz condition)

  $$\partial_\mu A^\mu = 0 \tag{5.1.2}$$

  such that A_μ belongs to the "Lorenz gauge" and reduces the degrees of freedom from 4 to 3.

– Further freedom is eliminated by adding any X with $\partial^2 X = 0$, e.g. $\frac{\partial}{\partial t} X = A_0$. Then we get the Coulomb or radiation gauge

$$A_0 = 0,\ \boldsymbol{\nabla} \cdot \boldsymbol{A} = 0 \tag{5.1.3}$$

Note: vice versa imposing $\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0$ first, yields $A_0 = 0$ (using Lorenz condition?).
In Coulomb gauge:

$$\begin{aligned}
&\boldsymbol{E} = -\dot{\boldsymbol{A}}.\ \boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A},\ \boldsymbol{\nabla} \times \boldsymbol{A} = 0 \\
&-\ddot{\boldsymbol{A}} = \dot{\boldsymbol{E}} \overset{\text{Maxwell}}{=} \boldsymbol{\nabla} \times \boldsymbol{B} = \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{A}) = \boldsymbol{\nabla}(\underbrace{\boldsymbol{\nabla} \cdot \boldsymbol{A}}_{=0}) - \boldsymbol{\nabla}^2 \boldsymbol{A} \\
&\Rightarrow \partial^2 \boldsymbol{A} = 0
\end{aligned}$$

This wave equation is massless KG equation for each spatial component.

Then the solutions are obvious: $\boldsymbol{A} = \boldsymbol{\epsilon} e^{-ik\cdot x}$ with $k^2 = 0$ and $\boldsymbol{\epsilon} \cdot \boldsymbol{k} = 0$. The polarization vector $\boldsymbol{\epsilon}$ is transverse to $\boldsymbol{k}$.

Can write the lagrangian in Coulmb gauge

$$\mathcal{L}_{\text{EM}} = \frac{1}{2}\dot{\boldsymbol{A}}\dot{\boldsymbol{A}} - \frac{1}{2}\boldsymbol{B} \cdot \boldsymbol{B}$$

Then the conjugate momentum to $\boldsymbol{A}$ is $\boldsymbol{\Pi} = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\boldsymbol{A}} = -\boldsymbol{E}$. (only 3 components, there is no conjugate momentum to A_0!) $\boldsymbol{\Pi}$ is subject to the constraint $\boldsymbol{\nabla} \cdot \boldsymbol{\Pi} = 0$

Hamiltonian

$$H_{\text{EM}} = \int \mathrm{d}^3 x \left(\frac{1}{2} \boldsymbol{\Pi} \cdot \boldsymbol{\Pi} + \frac{1}{2} \boldsymbol{B} \cdot \boldsymbol{B} \right)$$

5.2 Quantizing the Maxwell field

We would like to impose canonical commutation relations, à la

$$\begin{aligned}
&[A_i(\boldsymbol{x}), A_j(\boldsymbol{y})] = [\Pi_i(\boldsymbol{x}), \Pi_j(\boldsymbol{y})] = 0 \\
&[A_i(\boldsymbol{x}), \Pi_j(\boldsymbol{y})] = i\delta_{ij}\delta^{(3)}(\boldsymbol{x} - \boldsymbol{y})
\end{aligned}$$

However this cannot be true. Take either derivative of the last equation and it needs to vanish deu to $\boldsymbol{\nabla} \cdot \boldsymbol{A} = \boldsymbol{\nabla} \cdot \boldsymbol{\Pi} = 0$. But

$$[\partial^i A_i(\boldsymbol{x}), \Pi_k(\boldsymbol{y})] = i\delta_{ij}\partial^i \delta^{(3)}(\boldsymbol{x} - \boldsymbol{y})$$

here the derivative is takev with respect to $\boldsymbol{x}$, i.e. $\partial^i = \frac{\partial}{\partial x_i}$.

Replace δ_{ij} by Δ_{ij}

$$\begin{aligned}
[\partial^i A_i(\boldsymbol{x}), \Pi_j(\boldsymbol{y})] &= i\Delta_{ij}\partial^i \frac{1}{(2\pi)^3} \int \mathrm{d}^3 k e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})} \\
&= -\frac{1}{(2\pi)^3} \int \mathrm{d}^3 k (k^i \Delta_{ij}) e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})} \overset{!}{=} 0
\end{aligned}$$

it works for $\Delta_{ij} = \delta_{ij} - \frac{k_i k_j}{\boldsymbol{k}^2}$ in momentum space or $\Delta_{ij} = \delta_{ij} - \boldsymbol{\nabla}^{-2}\partial_i \partial_j$ in position space.

$$[A_i(\boldsymbol{x}), \Pi_j(\boldsymbol{y})] = i\left(\delta_{ij} - \boldsymbol{\nabla}^{-2}\partial_i\partial_j\right)\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}) \tag{5.2.1}$$

As before we have the mode expansion

$$\boldsymbol{A}(\boldsymbol{x}) = \int \frac{\mathrm{d}^3k}{(2\pi)^3\sqrt{2|\boldsymbol{k}|}}\left(\boldsymbol{a_k}e^{i\boldsymbol{k}\cdot\boldsymbol{x}} + \boldsymbol{a}_{\boldsymbol{k}}^{\dagger}e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\right)$$
$$\boldsymbol{\Pi}(\boldsymbol{x}) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}(-i)\sqrt{\frac{|\boldsymbol{k}|}{2}}\left(\boldsymbol{a_k}e^{i\boldsymbol{k}\cdot\boldsymbol{x}} - \boldsymbol{a}_{\boldsymbol{k}}^{\dagger}e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\right)$$

with $\boldsymbol{k}\cdot\boldsymbol{a_k} = \boldsymbol{k}\cdot\boldsymbol{a}_{\boldsymbol{k}}^{\dagger} = 0$.

Introduce 2 orthogonal polarization vectors $\boldsymbol{\epsilon}^{(1)}(\boldsymbol{k})$ and $\boldsymbol{\epsilon}^{(2)}(\boldsymbol{k})$ for each $\boldsymbol{k}$.

$$\boldsymbol{a_k} = a_{\boldsymbol{k}}^{(1)}\boldsymbol{\epsilon}^{(1)} + a_{\boldsymbol{k}}^{(2)}\boldsymbol{\epsilon}^{(2)} = \sum_{\lambda=1}^{2} a_{\boldsymbol{k}}^{(\lambda)}\boldsymbol{\epsilon}^{(\lambda)}(\boldsymbol{k})$$
$$\text{with } \boldsymbol{k}\cdot\boldsymbol{\epsilon}^{(1)}(\boldsymbol{k}) = \boldsymbol{k}\cdot\boldsymbol{\epsilon}^{(2)}(\boldsymbol{k}) = 0,\ \boldsymbol{\epsilon}^{(\lambda)}\cdot\boldsymbol{\epsilon}^{(\lambda;)} = \delta_{\lambda\lambda'}$$

Creation and annihilation operator have the standard commutation relations

$$[a_{\boldsymbol{k}}^{(\lambda)}, a_{\boldsymbol{k}'}^{(\lambda')\dagger}] = (2\pi)^3\delta_{\lambda\lambda'}\delta^{(3)}(\boldsymbol{k}-\boldsymbol{k}') \tag{5.2.2}$$

and all other commutators vanish. Geometrically, still possible to write

$$[\boldsymbol{a_k}, \boldsymbol{a_l}] = [\boldsymbol{a}_{\boldsymbol{k}}^{\dagger}, \boldsymbol{a}_{\boldsymbol{l}}^{\dagger}] = 0$$
$$[a_{\boldsymbol{k}}^i, a_{\boldsymbol{l}}^{j\dagger}] = (2\pi)^3\left(\delta^{ij} - \frac{k^ik^j}{\boldsymbol{k}^2}\right)\delta^{(3)}(\boldsymbol{k}-\boldsymbol{l})$$

$a_{\boldsymbol{k}}^{(\lambda)}$ and $a_{\boldsymbol{k}}^{(\lambda)\dagger}$ create and destroy photons of momentum $\boldsymbol{k}$, energy $|\boldsymbol{k}|$ and (electric) polarization along $\boldsymbol{\epsilon}^{(\lambda)}(\boldsymbol{k})$.

Next steps are analogout to KG theory.

Hamiltonian

$$H = \frac{1}{2}\int \mathrm{d}^3x\left(\boldsymbol{E}^2 + \boldsymbol{B}^2\right) = \frac{1}{2}\int \mathrm{d}^3x\left(\dot{\boldsymbol{A}}^2 + (\boldsymbol{\nabla}\times\boldsymbol{A})\cdot(\boldsymbol{\nabla}\times\boldsymbol{A})\right)$$

using identity $\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C}) = \boldsymbol{B}\cdot(\boldsymbol{C}\times\boldsymbol{A})$

$$= \frac{1}{2}\int \mathrm{d}^3x\left(\dot{\boldsymbol{A}}^2 + \boldsymbol{A}\cdot\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{A})\right)$$

using the identity $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{A} - \boldsymbol{\nabla}^2\boldsymbol{A})$

$$= \frac{1}{2}\int \mathrm{d}^3x\left(\dot{\boldsymbol{A}}^2 - \boldsymbol{A}\cdot\boldsymbol{\nabla}^2\boldsymbol{A} + \boldsymbol{A}\cdot\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{A})\right)$$

using coulomb gauge condition

$$= \frac{1}{2}\int \mathrm{d}^3x\left(\dot{\boldsymbol{A}}^2 - \boldsymbol{A}\cdot\nabla^2\boldsymbol{A}\right)$$

the first term vanishes and use normal ordering

$$= \int \frac{\mathrm{d}^3k}{(2\pi)^3}|\boldsymbol{k}|\,\boldsymbol{a}_{\boldsymbol{k}}^{\dagger}\cdot\boldsymbol{a_k} = \sum_{\lambda=1}^{2}\int \frac{\mathrm{d}^3k}{(2\pi)^3}|\boldsymbol{k}|\,a_{\boldsymbol{k}}^{(\lambda\dagger)}a_{\boldsymbol{k}}^{\lambda}$$

Heisenberg field

$$\boldsymbol{A}(\boldsymbol{x},t)=\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\sqrt{2|\boldsymbol{k}|}}\left(\boldsymbol{a}_{\boldsymbol{k}}e^{-ik\cdot x}+\boldsymbol{a}_{\boldsymbol{k}}^{\dagger}e^{ik\cdot x}\right)$$

Photon propagator

$$\langle 0|TA_i(x)A_j(y)|0\rangle =: D_{ij}^{\mathrm{tr}}(x-y)=\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{i}{k^2+i\epsilon}\left(\delta_{ij}-\frac{k_ik_j}{|\boldsymbol{k}|^2}\right)e^{-ik\cdot(x-y)} \tag{5.2.3}$$

tr stands for transverse: photon polarization perpendicular to its momentum. This is NOT the final version of the photon propagator!

5.3 Inclusion of matter - QED

$$\mathcal{L}_{\mathrm{QED}}=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\bar{\psi}(i\not{D}-m)\psi \tag{5.3.1}$$

where $D_\mu=\partial_\mu+ieA_\mu$ is the (gauge) covariante derivative

$$=\mathcal{L}_{\mathrm{EM}}+\mathcal{L}_D-e\underbrace{\bar{\psi}\gamma^\mu\psi}_{j^\mu}A_\mu \tag{5.3.2}$$

Field equations would be

$$\partial_\mu F^{\mu\nu}=ej^\nu \qquad (i\not{D}-m)\psi=0$$

where ej^ν is the electromagnetic 4-current.

Gauge invariance under the transformation

$$\begin{cases}\psi(x)\longmapsto\psi'(x)=e^{ie\chi(x)}\psi\\ A_\mu(x)\longmapsto A'_\mu(x)=A_\mu(x)-\partial_\mu\chi(x)\end{cases}$$

To check the consistence: cavariant derivative transforms like $D_\mu\longmapsto D'_\mu\psi'(x)=e^{ie\chi(x)}D_\mu\psi(x)$. Since the adjoint spinor transforms like $\bar{\psi}(x)\longmapsto\bar{\psi}'(x)=\bar{\psi}(x)e^{-ie\chi(x)}$, the Lagrangian and field equations are gauge invariant.

Again we choose Coulomb gauge $\boldsymbol{\nabla}\cdot\boldsymbol{A}=0$, then equation for A^0:

$$\begin{aligned}\partial_iF^{i0}&=ej^0\\ \Rightarrow -\boldsymbol{\nabla}^2A^0&=ej^0=e\bar{\psi}\gamma^0\psi\\ &=e\bar{\psi}\gamma^0\psi=e\psi^\dagger\psi\\ &=e\rho(x)\end{aligned}$$

$$A^0(\boldsymbol{x},t)=e\int\mathrm{d}^3y\frac{\rho(\boldsymbol{y},t)}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} \tag{5.3.3}$$