A.6

a) $\langle \psi, U\chi \rangle \underset{(3)}{=} \langle U^\dagger \psi, \chi \rangle^* \underset{(3)}{=} \langle \psi, (U^\dagger)^\dagger \chi \rangle \quad \Rightarrow \quad U = (U^\dagger)^\dagger$

b) $\langle \psi, \chi \rangle = \langle U\psi, U\chi \rangle^* = \langle \psi, U^\dagger U \chi \rangle \quad \Rightarrow \quad U^\dagger U = \mathbb{1}$

c) Assume the opposite

$\langle \psi, \chi \rangle = \langle U^\dagger U \psi, U^\dagger U \chi \rangle \neq \langle U\psi, U\chi \rangle^* = \langle \psi, \chi \rangle$

$\Rightarrow$ $U^\dagger$ must be antiunitary

d) $\langle \psi, U^\dagger(a\chi + b\xi) \rangle = \langle U\psi, a\chi + b\xi \rangle^*$

$= (a\langle U\psi, \chi \rangle + b\langle U\psi, \xi \rangle)^*$

$= a^* \langle U\psi, \chi \rangle^* + b^* \langle U\psi, \xi \rangle^*$

$= a^* \langle \psi, U^\dagger \chi \rangle^* + b^* \langle \psi, U^\dagger \xi \rangle^*$

$= \langle \psi, a^* U^\dagger \chi + b^* U^\dagger \xi \rangle$

e) $\langle UV\psi, UV\chi \rangle = \langle V\psi, V\chi \rangle^* = \langle \psi, \chi \rangle$

f) $\langle \psi, (UV)^\dagger \chi \rangle = \langle UV\psi, \chi \rangle^* = \langle V\psi, U^\dagger \chi \rangle^* = \langle \psi, V^\dagger U^\dagger \chi \rangle$

A.7

a) $$T\,T\,\psi(t,\underline{x})\,T^{-1}T^{-1} = T\,\eta\,M\,\psi(-t,\underline{x})\,T^{-1}$$
$$= \eta^* M^*\, T\,\psi(-t,\underline{x})\,T^{-1}$$
$$= \eta^* M^*\, \eta\, M\,\psi(t,\underline{x})$$
$$= |-\gamma^1\gamma^3|^2\,\psi(t,\underline{x})$$
$$= -\psi(t,\underline{x})$$

b) $$T^2\, a_s^+(\underline{p})\,(T^{-1})^2 = T\,\eta^*(-1)^{\frac{1}{2}+s}\, a^+_{-s}(-\underline{p})\,T^{-1}$$
$$= (-1)^{\frac{1}{2}+s+\frac{1}{2}-s}\; a^+_s(\underline{p})$$
$$= -\,a_s^+(\underline{p})$$

c) Assume the opposite : $\quad T|\psi_1\rangle = e^{i\alpha}|\psi_1\rangle$

$$T^2|\psi_1\rangle = e^{-i\alpha}\,T|\psi_1\rangle = |\psi_1\rangle \neq -|\psi_1\rangle$$

for fermions

d) $(\gamma^\mu)^+ = g^{\mu\mu}\gamma^\mu = \gamma^0\gamma^\mu\gamma^0$, $\quad M = -\gamma^0\gamma^3$, $\quad M^+ = \gamma^1\gamma^3$, $\quad M^+M = 1$

γ^μ are real for $\mu \neq 2$, imaginary for $\mu = 2$

$$T\,\bar\psi(t,\underline{x})\,T^{-1} = T\,\psi^+(t,\underline{x})\,T^{-1}\gamma^0 = (T\,\psi\,T^{-1})^+\gamma^0$$
$$= (\eta\,M\,\psi(t,\vec{x}))^+\gamma^0$$
$$= \psi^+(t,\vec{x})\,M^+\eta^*\gamma^0$$
$$= \eta^*\,\psi^+(-t,\vec{x})\,\gamma^0 M^+$$
$$= \eta^*\,\bar\psi(-t,\vec{x})\,M^+$$

$$T\,\bar\psi(t,\underline{x})\,\psi(t,\underline{x})\,T^{-1}$$
$$= T\,\bar\psi(t,\underline{x})\,T^{-1}\,T\,\psi(t,\underline{x})\,T^{-1}$$
$$= \bar\psi(-t,\vec{x})\,M^+M\,\psi(-t,\underline{x})$$
$$= \bar\psi(-t,\underline{x})\,\psi(-t,\underline{x})$$

$$T\bar{\psi}(t,\underline{x})\gamma^\mu\psi(t,\underline{x})T^{-1}$$

$$= a(\mu)\, T\bar{\psi}(t,\underline{x})T^{-1}\gamma^\mu T\psi(t,\underline{x})T^{-1},$$

$$\begin{cases} a(\mu) = -1, & \mu = 2 \\ a(\mu) = 1, & \mu \neq 2 \end{cases}$$

$$= a(\mu)\,\bar{\psi}(-t,\underline{x})\, M^\dagger\gamma^\mu M\,\psi(-t,\underline{x})$$

$$= g^{\mu\mu}\,\bar{\psi}(-t,\underline{x})\gamma^\mu \underbrace{M^\dagger M}_{=\mathbb{1}}\psi(-t,\underline{x})$$

e) $A^2 = \mathbb{1}$, $A^\dagger = A$.

$$C\bar{\psi}C^{-1} = C\psi^\dagger\gamma^0 C^{-1} = (C\psi C^{-1})^\dagger\gamma^0 = (\xi A\psi^*)^\dagger\gamma^0$$

$$= \psi^T A^\dagger \xi^*\gamma^0 = -\xi^*\psi^T\gamma^0 A = -\xi^*\bar{\psi}^* A$$

For any matrix B, we have

$$\psi^T(x) B\psi^*(x) = \psi_\alpha(x)\psi_\beta^* B_{\alpha\beta}$$

$$= [-\psi^*_\beta(x)\psi_\alpha(x) + \{\psi_\alpha(x), \psi^\dagger_\beta(x)\}]B_{\alpha\beta}$$

$$= -\psi^*_\beta B_{\alpha\beta}\psi_\alpha + \delta_{\alpha\beta}B_{\alpha\beta}\delta^{(3)}(\vec{0})$$

$$= -\psi^\dagger(x)B^T\psi(x) + tr(B)\,\delta^{(3)}(\vec{0})$$

$$(\gamma^\mu)^T = ((\gamma^\mu)^\dagger)^* = g^{\mu\mu}(\gamma^\mu)^* = a(\mu)\,g^{\mu\mu}\gamma^\mu$$

$$C\bar{\psi}B\psi C^{-1} = C\bar{\psi}BC^{-1}C\psi C^{-1}$$

$$= C\bar{\psi}C^{-1}BC\psi C^{-1}$$

$$= -\psi^T\gamma^0 ABA\psi^*$$

$$= \psi^\dagger(\gamma^0 ABA)^T\psi - tr(\gamma^0 ABA)\,\delta^{(3)}(\vec{0})$$

f) scalar bilinear, $B = \mathbb{1}$,

$$C\bar{\psi}\psi C^{-1} = \psi^\dagger(\gamma^0 \underset{\substack{\shortparallel \\ \mathbb{1}}}{A^2})^T\psi - \underbrace{tr(\gamma^0 A^2)}_{=0}\delta^{(3)}(\vec{0})$$

$$= \psi^\dagger\gamma^0\psi = \bar{\psi}\psi$$

vector bilinear, $B = \gamma^\mu$

$$C \bar{\psi} \gamma^\mu \psi C^{-1} = \psi^\dagger (\gamma^0 A \gamma^\mu A)^T \psi - tr(\gamma^0 A \gamma^\mu A) \delta^{(3)}(\vec{0})$$

$$= -a(\mu) \psi^\dagger (\gamma^0 \gamma^\mu A^2)^T \psi + a(\mu) tr(\gamma^0 \gamma^\mu A^2) \delta^{(3)}(\vec{0})$$

$$= -a(\mu) \psi^\dagger (\gamma^0 \gamma^\mu)^T \psi + a(\mu) tr(\gamma^0 \gamma^\mu) \delta^{(3)}(\vec{0})$$

$$= -a(\mu)^2 g^{\mu\mu} \psi^\dagger \gamma^\mu \gamma^0 \psi + 4 g^{0\mu} \delta^{(3)}(\vec{0})$$

$$= -\bar{\psi} \gamma^\mu \psi + \underline{4 \delta_{0\mu} \delta^{(3)}(\vec{0})}$$