H1. Chenhuan Wang 20/20

a) $$\mathcal{L}_{EM}(x) = -\frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x)$$
$$= -\frac{1}{4}(\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x))(\partial^\mu A^\nu(x) - \partial^\nu A^\mu(x))$$

We have $A_\mu(x)$ as field:

EL-eq.: $$\partial_\lambda \frac{\partial \mathcal{L}}{\partial(\partial_\lambda A_\sigma)} - \frac{\partial \mathcal{L}}{\partial A_\sigma} = 0$$

$$-\frac{1}{4}\partial_\lambda \frac{\partial}{\partial(\partial_\lambda A_\sigma)}\left[(\partial_\mu A_\nu - \partial_\nu A_\mu) g^{\alpha\mu} g^{\beta\nu}(\partial_\alpha A_\beta - \partial_\beta A_\alpha)\right] = 0$$

$$-\frac{1}{4}\partial_\lambda g^{\alpha\mu} g^{\beta\nu}\left[(\delta_{\lambda\mu}\delta_{\sigma\nu} - \delta_{\lambda\nu}\delta_{\sigma\mu})F_{\alpha\beta} + F_{\mu\nu}(\delta_{\lambda\alpha}\delta_{\sigma\beta} - \delta_{\lambda\beta}\delta_{\sigma\alpha})\right] = 0$$

$$\Rightarrow -\frac{1}{4}\partial_\lambda\left[(g^{\alpha\lambda}g^{\beta\sigma} - g^{\alpha\sigma}g^{\beta\lambda})F_{\alpha\beta} + F_{\mu\nu}(g^{\lambda\mu}g^{\sigma\nu} - g^{\sigma\mu}g^{\lambda\nu})\right] = 0$$

$$-\frac{1}{4}\partial_\lambda(F^{\lambda\sigma} - F^{\sigma\lambda} + F^{\lambda\sigma} - F^{\sigma\lambda}) = 0$$

$$\underset{\text{skew-symm.}}{\Rightarrow} \quad \partial_\lambda F^{\lambda\sigma} = 0 \quad \Leftrightarrow \quad \partial^2 A_\nu(x) - \partial_\nu(\partial A) = 0$$

b) $$\partial_\mu F^{\mu\nu}(x) = 0$$

$$\partial_\mu \tilde{F}^{\mu\nu} = \frac{1}{2}\partial_\mu \epsilon^{\mu\nu\lambda\xi} F_{\lambda\xi}(x)$$
$$= \frac{1}{2}\epsilon^{\mu\nu\lambda\xi}\partial_\mu(\partial_\lambda A_\xi(x) - \partial_\xi A_\lambda(x))$$
$$= \frac{1}{2}\left(\epsilon^{\mu\nu\lambda\xi}\partial_\mu\partial_\lambda A_\xi(x) - \epsilon^{\mu\nu\lambda\xi}\partial_\mu\partial_\xi A_\lambda(x)\right)$$

$\lambda \longleftrightarrow \xi$

$$= \frac{1}{2}(\epsilon^{\mu\nu\lambda\xi}\partial_\mu\partial_\lambda A_\xi - \epsilon^{\mu\nu\xi\lambda}\partial_\mu\partial_\lambda A_\xi)$$

$$= \underbrace{\epsilon^{\mu\nu\lambda\xi}}_{\substack{\text{antisymmetric}\\ \text{in } \mu,\lambda}} \underbrace{\partial_\mu\partial_\lambda}_{\text{symmetric in } \mu,\lambda} A_\xi$$

$$= 0$$

(c) $\mathcal{L}_{tot}(x) = [D_\mu \phi(x)]^* D^\mu \phi(x) - m^2 \phi^*(x)\phi(x) - \frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x)$

$D_\mu = \partial_\mu + ieA_\mu \quad , \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$$\partial_\mu \frac{\partial \mathcal{L}_{tot}}{\partial(\partial_\mu A_\nu)} - \frac{\partial \mathcal{L}_{tot}}{\partial(A_\nu)} = 0$$

$$\Rightarrow -\partial_\mu F^{\mu\nu} - \frac{\partial}{\partial(A_\nu)}\left\{[(\partial_\mu + ieA_\mu)\phi]^* (\partial^\mu + ieA^\mu)\phi\right\} = 0$$

from (a)

$$-\partial_\mu F^{\mu\nu} - ie[\delta^\nu_\mu \phi^* (D^\mu \phi) + (D_\mu\phi)^* g^{\mu\nu}\phi] = 0$$

$$\Rightarrow \partial_\mu F^{\mu\nu} = -ie\,[\phi^*(D^\nu\phi) + (D^\nu\phi)^*\phi]$$

$$= -eJ^\mu$$

with $J^\mu = i[\phi^*(D^\nu\phi) + (D^\nu\phi)^*\phi]$

$$\partial_\mu J^\mu = -\frac{1}{e}\underbrace{\partial_\mu \partial_\alpha}_{\text{symm.}}\underbrace{F^{\alpha\mu}}_{\text{skew-symm.}} = 0$$

d) $$\partial_\mu \frac{\partial \mathcal{L}_{tot}}{\partial(\partial_\mu \phi)} - \frac{\partial \mathcal{L}_{tot}}{\partial \phi} = 0$$

$$\Rightarrow \partial_\mu (D^\mu\phi)^* - (D_\mu\phi)^* ieA^\mu + m^2\phi^* = 0$$

$$(\partial_\mu - ieA_\mu)(D^\mu\phi)^* + m^2\phi^* = 0$$

$$D^*_\mu D^{*\mu}\phi^* + m^2\phi^* = 0$$

w.r.t. ϕ^*: $\quad D_\mu D^\mu \phi + m^2\phi = 0$

compare to KG: $\quad \partial_\mu \longrightarrow D_\mu$

$$\partial_\mu J^\mu(x) = i\,\partial_\mu\left(\phi^* D^\mu \phi - \phi (D^\mu\phi)^*\right)$$

$$= i\left[\partial_\mu \phi^* \cdot D^\mu \phi + \phi^* \underbrace{\partial_\mu D^\mu \phi}_{= -(ieA_\mu D^\mu + m^2)\phi} - \partial_\mu\phi \cdot (D^\mu\phi)^* - \phi \underbrace{\partial_\mu (D^\mu \phi)^*}_{= (ieA_\mu D^\mu - m^2)\phi^*}\right]$$

$$\propto \partial_\mu \phi^* \cdot D^\mu \phi - \partial_\mu \phi (D^\mu \phi)^* - \phi^* ieA_\mu D^\mu \phi - \phi\, ieA_\mu D^\mu \phi^*$$

$$= (D_\mu \phi)^* D^\mu \phi - D_\mu \phi (D^\mu \phi)^*$$

$$= 0$$

e) "Proca field"

$$\mathcal{L}(x) = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{m^2}{2} B_\mu B^\mu$$

$$F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$$

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu B_\nu)} - \frac{\partial \mathcal{L}}{\partial (B_\nu)} = 0$$

$$\underset{\text{from b)}}{\Rightarrow} \partial_\mu F^{\mu\nu} + \frac{m^2}{2} \cdot \underbrace{\frac{\partial}{\partial B_\nu}\left(B_\mu g^{\mu\alpha} B_\alpha\right)} = 0$$

$$= \delta_\mu^\nu g^{\mu\alpha} B_\alpha + B_\mu g^{\mu\alpha} \delta_\alpha^\nu$$

$$= B^\nu + B^\nu = 2B^\nu$$

$$\Rightarrow \partial_\mu F^{\mu\nu} + m^2 B^\nu = 0$$

f) $$m^2 \partial_\nu B^\nu + \underbrace{\partial_\mu \partial_\nu}_{\text{symm.}} \underbrace{F^{\mu\nu}}_{\text{skew-symm}} = 0 \qquad \Rightarrow \partial_\nu B^\nu = 0$$

Field equation(s): $$\partial_\mu (\partial^\mu B^\nu - \partial^\nu B^\mu) + m^2 B^\nu = 0$$

$$\partial^2 B^\nu + m^2 B^\nu = 0$$

(Klein - Gordon for each component!)