Quantum Field Theory

ST 2017

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1st Examination

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1 Short questions **20p**

(a) Derive the Euler–Lagrange equations for the Dirac and photon fields from the following Lagrangian

$$\mathcal{L} = \bar{\psi}\,(i\not{D} - m)\,\psi - \frac{1}{4}\epsilon^{\mu\nu\alpha\beta}\,F_{\alpha\beta}\,F_{\mu\nu}\,,$$

with $i\not{D}\psi = \left(i\not{\partial} - e\not{A}\right)\psi$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. **4p**

(b) Consider a Yukawa theory in 6 dimensions with

$$\mathcal{L} = \bar{\psi}\,(i\not{\partial} - m)\,\psi + \frac{1}{2}\left(\partial_\mu\phi\,\partial^\mu\phi - M^2\phi^2\right) - g\bar{\psi}i\gamma^5\psi\phi\,.$$

Is this theory super-renormalizable, renormalizable, or non-renormalizable? **3p**

(c) Write down the Fourier decomposition of a massive fermion field $\psi(x)$ in the Heisenberg picture. Calculate $\langle 0|\psi(x)|\mathbf{p}, r\rangle$, where $|\mathbf{p}, r\rangle$ is a free particle with spin r and three momentum $\mathbf{p}$. **3p**

(d) What are the propagators and vertex Feynman rules for the Lagrangian density

$$\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi + \tfrac{1}{2}\partial_\mu\pi\partial^\mu\pi - \sqrt{\lambda}\,(\phi^*\phi)^2 + \lambda\,(\phi^*\phi)\,\pi^2 - g\,(\phi^*\phi)\,\pi \quad ?$$ **4p**

<u>*Hint:*</u> *No proof is required. Be careful about the prefactors.*

(e) Consider ϕ^4 theory with $\mathcal{L}_{\text{int}} = -\frac{\lambda}{4!}\phi^4$. Give the symmetry factors of the two diagrams

(i) (ii)

In addition prove them using Wick contractions. **6p**

2 Proton–antiproton annihilation **20p**

Consider a Yukawa theory given by

$$\mathcal{L} = \bar{\psi}\,(i\partial\!\!\!/ - m)\,\psi + \bar{\chi}\,(i\partial\!\!\!/ - \mu)\,\chi + \frac{1}{2}\left(\partial^\mu\phi\partial_\mu\phi - M^2\phi^2\right) - g\bar{\psi}i\gamma^5\psi\phi - \bar{g}\bar{\chi}i\gamma^5\chi\phi\,.$$

where $\psi(x)$ denotes the proton field, $\chi(x)$ the neutron field and $\phi(x)$ a real scalar field. The coupling constants are given by g and $\bar{g}$. Furthermore, note $\mu > m$.
We now assign the spins $s, \bar{s}, r, \bar{r}$ and the momenta $p, \bar{p}, k, \bar{k}$ such that the process is given by

$$\psi^{(s)}(p) + \bar{\psi}^{(\bar{s})}(\bar{p}) \to \chi^{(r)}(k) + \bar{\chi}^{(\bar{r})}(\bar{k})\,.$$

It is useful to define the Mandelstam variables $s = (p+\bar{p})^2$, $t = (p-k)^2$ and $u = (p-\bar{k})^2$.

(a) What are the Feynman rules for the interaction vertices? **1p**
Hint: Be careful about the factor i.

(b) Which Feynman diagram(s) contribute to the process proton–antiproton to neutron–antineutron to the lowest order? Draw them, label the lines with momenta and spins, and determine the invariant matrix element $i\mathcal{M}$. **2p**

(c) Calculate $\overline{|\mathcal{M}|^2}$ by averaging over the initial and summing over the final spin states. Show that it can be written in terms of Mandelstam variables as

$$\overline{|\mathcal{M}|^2} = \left(\frac{g\bar{g}}{s - M^2}\right)^2 \cdot s^2\,.$$ **10p**

Hint: Use the relation $\left(\gamma^5\right)^\dagger = \gamma^5$.

(d) In the center-of-mass system (CMS) the differential cross section is given by

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\frac{|\mathbf{p}_f|}{|\mathbf{p}_i|}\overline{|\mathcal{M}|^2}\,,$$

where $\mathbf{p}_i$ ($\mathbf{p}_f$) are the CMS particle three momenta of the initial (final) state. Express the differential cross section in terms of the Mandelstam variables. What happens for $s < 4\mu^2$? **2p**

(e) Consider now proton–neutron scattering

$$\psi^{(s)}(p) + \chi^{(\bar{r})}(\bar{k}) \to \psi^{(\bar{s})}(\bar{p}) + \chi^{(r)}(k)\,.$$

We define our new Mandelstam variables to be $\tilde{s} = (p+\bar{k})^2$, $\tilde{t} = (p-\bar{p})^2$, and $\tilde{u} = (p-k)^2$. Calculate $\overline{|\mathcal{M}|^2}$ for this process and express it in terms of the new Mandelstam variables.
5p

3 Wave function renormalization **20p**

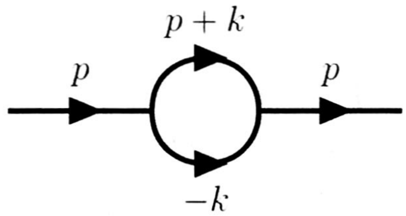
In the following we are going to study the self energy in ϕ^3-theory with $\mathcal{L}_{\text{int}} = -\frac{\lambda}{3!}\phi^3$ (in 4 dimensions) at next-to-leading order and determine its contribution to the wave function renormalization.

(a) Briefly show that the two-point function is given by

$$\int \mathrm{d}^4x\, e^{ipx} \langle \Omega | T\phi(x)\phi(0) | \Omega \rangle = \frac{i}{p^2 - m_0^2 - \Sigma(p^2)} \, .$$

where $-i\Sigma(p^2)$ denotes the sum of all one-particle-irreducible diagrams and m_0 the bare mass. Show that the correction to the wave function renormalization is given by $\frac{\mathrm{d}\Sigma(p^2)}{\mathrm{d}p^2}\Big|_{p^2=m^2}$. **6p**

(b) Consider the following second-order contribution to $\Sigma(p^2)$ denoted by $\Sigma_2(p^2)$:



Write down the amplitude corresponding to the diagram in d dimensions, combine the propagators using Feynman parameters, and simplify as far as possible. **6p**
Hint:

$$\frac{1}{AB} = \int_0^1 \mathrm{d}x \, \frac{1}{[xA + (1-x)B]^2}$$

(c) Perform a Wick rotation of the self energy contribution. What can you deduce about the degree of divergence of the loop integral for $\Sigma_2(p^2)$ and $\frac{\mathrm{d}\Sigma_2(p^2)}{\mathrm{d}p^2}$ for $d \to 4$? **4p**
Hint: Perform the differentiation $\frac{\mathrm{d}}{\mathrm{d}p^2}$ under the integral.

(d) We now want to calculate the contribution to the wave function renormalization. Perform the Euclidean integral. The remaining integral over the Feynman parameter evaluates to a real number and should not be carried out. **4p**
Hints:
1. $$\int \frac{\mathrm{d}^d l_E}{(2\pi)^d} \frac{1}{(l_E^2 + \Delta)^n} = \frac{1}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma\left(n - \frac{d}{2}\right)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}}$$
2. $\Gamma(x+1) = x \cdot \Gamma(x)$ and $\Gamma(1) = 1$
3. Note that $m^2 - m_0^2 = \mathcal{O}(\lambda^2)$.

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2nd Examination

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1 Short questions **20p**

(a) **Derive the Euler-Lagrange equations for the Dirac field** from the following Lagrangian:

$$\mathcal{L} = \bar{\psi}\left(i\not{D} - m\right)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} ,$$

with $\not{D}\psi = \left(\not{\partial} - ie\not{A}\right)\psi$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.
If $m = 0$ the Lagrangian is invariant under the global transformation $\psi \to e^{i\alpha\gamma^5}\psi$, $\bar{\psi} \to \psi^\dagger e^{-i\alpha\gamma^5}\gamma^0$. **Deduce the Noether current.** **4p**

(b) Consider the following interaction terms:

$$\mathcal{L}_{\text{int}} = c_1\left(\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}\right)^2 + c_2\left(\bar{\psi}i\sigma^{\mu\nu}\psi\right)F_{\mu\nu} + c_3\left(\bar{\psi}i\gamma^5\not{A}\psi\right)\phi^2 + c_4\left(\phi(\partial^\mu\phi)A_\mu\right) .$$

with the Klein-Gordon field ϕ, Dirac field ψ, electromagnetic field A_μ, its corresponding field strength tensor $F_{\mu\nu}$, its dual $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$, and $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$.
Determine the mass dimension of the coupling constants c_i, $i = 1, \ldots, 4$, in $d = 4$ dimensions and deduce whether the individual interaction terms are super-renormalizable, renormalizable, or non-renormalizable. **5p**

(c) Why do the creation and annihilation operators of Dirac particles fulfill anticommutation instead of commutation relations? What is the result of $\left(a_{\mathbf{p}}^{(s)}\right)^\dagger \left(a_{\mathbf{q}}^{(r)}\right)^\dagger |0\rangle$? **2p**

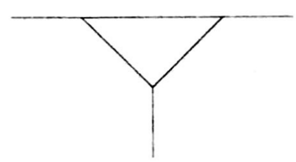
(d) Show that the superficial degree of divergence for an arbitrary loop-diagram in ϕ^4 theory can be written as

$$D = d + (d-4)\cdot V - \left(\frac{d}{2} - 1\right)\cdot E .$$

where d denotes the space-time-dimension. V stands for the number of vertices and E for the number of external lines.
Discuss the renormalizability of the theory for $d < 4$, $d = 4$, and $d > 4$. **6p**

<u>*Hint:*</u> *You may want to use the number of loops L and the number of internal lines I in the course of the derivation.*

(e) Consider $\mathcal{L}_{\text{int}} = -\frac{\lambda}{3!}\phi^3$. What is the symmetry factor of the following diagram?



Proof your result using Wick contractions. **3p**

2 Electron–positron annihilation **20p**

We consider electron–positron annihilation into a muon–antimuon pair. The interaction Lagrangian for this process is then given by:

$$\mathcal{L}_{\text{int}} = -e\,\bar{\psi}\,\gamma^\mu A_\mu\,\psi - e\,\bar{\xi}\,\gamma^\mu A_\mu\,\xi\,,$$

where ψ is the electron field, ξ the muon field, and $e = |e|$ is the elementary charge.
We assign momenta p, p', k, k' and spins s, s', r, r' in the following way:

$$\psi^{(s)}(p) + \bar{\psi}^{(s')}(p') \to \xi^{(r)}(k) + \bar{\xi}^{(r')}(k')\,.$$

Assume that we are in an energy region where we can neglect the electron mass m_e ($m_e \approx 0$) but <u>not</u> the muon mass m_μ.
It is useful to define the Mandelstam variables $s = (p+p')^2$, $t = (p-k)^2$, and $u = (p-k')^2$.

(a) Show

1) $s = 2pp' = 2m_\mu^2 + 2kk'$
2) $t = m_\mu^2 - 2pk = m_\mu^2 - 2p'k'$
3) $u = m_\mu^2 - 2pk' = m_\mu^2 - 2p'k$
4) $s + t + u = 2m_\mu^2$ **2p**

(b) Which diagram contributes to the leading order? Draw it, label the lines with momenta and spins, and determine the invariant matrix element $\mathcal{M}$. **3p**

(c) Calculate $\overline{|\mathcal{M}|^2}$ by averaging $|\mathcal{M}|^2$ over the initial spins and summing over the final spin states. Simplify $\overline{|\mathcal{M}|^2}$ as far as possible (using $m_e = 0$) and verify that it reduces to

$$\overline{|\mathcal{M}|^2} = \frac{2e^4}{s^2}\left(t^2 + u^2 + 4m_\mu^2 s - 2m_\mu^4\right)\,.$$ **12p**

<u>*Hint:*</u> $\quad \mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})$

(d) Assume now to be in the ultrarelativistic case in which $m_\mu \approx 0$. In the center-of-mass system (CMS) the differential cross section is given by

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\cdot\overline{|\mathcal{M}|^2}\,.$$

Express the cross section in terms of the Mandelstam variable s and the angle Θ between the incoming electron ($\mathbf{p}$) and the outgoing muon three momentum ($\mathbf{k}$). **3p**

3 The electron self energy **20p**

In the following we are going to study the electron self energy in QED at next-to-leading order.

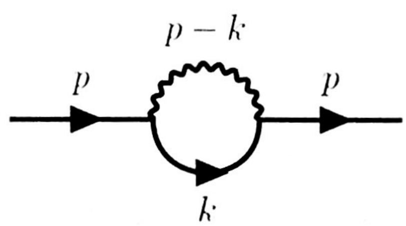
(a) Briefly show that the two-point function is given by

$$\int \mathrm{d}^4x\, e^{ipx} \langle \Omega | T\psi(x)\bar{\psi}(0) | \Omega \rangle = \frac{i}{\not{p} - m_0 - \Sigma(\not{p})} .$$

where $-i\Sigma(\not{p})$ denotes the sum of all one-particle-irreducible diagrams and m_0 the bare electron mass. What is the correction to the physical electron mass m? **4p**

<u>*Hint:*</u> *Use the relation* $\frac{i(\not{p}+m_0)}{p^2-m_0^2} = \frac{i}{\not{p}-m_0}$. *Furthermore use that the geometric sum for a matrix valued operator A is given by* $\sum_{n=0}^{\infty} A^n = [1 - A]^{-1}$.

(b) Show that the second-order self energy contribution $\Sigma_2(\not{p})$ is defined by

$-i\Sigma_2(\not{p}) =$ 

$$= -e^2 \int\limits_0^1 \mathrm{d}x \int \frac{\mathrm{d}^d q}{(2\pi)^d} \gamma^\mu (\not{p}x + m_0) \gamma_\mu \left[q^2 - \Delta(p^2) + i\epsilon\right]^{-2} .$$

with $\Delta(p^2) = (1-x)(m_0^2 - xp^2)$. **6p**

<u>*Hint:*</u> *Use the relation* $\frac{1}{AB} = \int_0^1 \mathrm{d}x \frac{1}{[xA + (1-x)B]^2}$.

(c) Show that in d dimensions the Dirac structure reduces to

$$\gamma^\mu (\not{p}x + m_0) \gamma_\mu = (2-d)x\not{p} + d\, m_0 .$$ **2p**

(d) What is the mass dimension of the coupling constant e in d dimensions? Rewrite it as $e = \mu^\alpha e_0$, where e_0 has the same mass dimension as e in four dimensions. What is α? **2p**

(e) Perform a Wick rotation in order to evaluate the momentum integral. **2p**

<u>*Hint:*</u> *Use the relation* $\int \frac{\mathrm{d}^d q_E}{(2\pi)^d} \frac{1}{(q_E^2 + \Delta)^n} = \frac{1}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma\left(n - \frac{d}{2}\right)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}}$

(f) Identify and calculate the divergent part of the mass renormalization in the variable $\epsilon = 4 - d$. **4p**

<u>*Hints:*</u> *Use the relations:*

1) $\Gamma(2) = 1$ *and* $\Gamma(x) = \frac{1}{x} - \Gamma_E + \mathcal{O}(x^2)$

2) $a^\epsilon = 1 + \epsilon \ln a$

3) $m_0 = m + \mathcal{O}(e^2)$