A.12

a) S-matrix $\hat{S}$.

to show $\langle p', n' | \hat{S} | p, n \rangle = \delta^{(4)}(p'-p) S_{n'n}(p)$

The theory is Poincaré-invariant, hence invariant under translations. This implies total 4-momentum is conserved.

b) We want to show $\left(\hat{S}^{+} \hat{S} = \mathbb{1} \Rightarrow S^{+} S = \mathbb{1} \right)$

$$\delta^{(4)}(p'-p) \delta_{nn'}$$

$$= \langle p'n' | pn \rangle = \langle p'n' | \hat{S}^{+} \hat{S} | p n \rangle$$

$$= \int d^4 q \sum_m \langle p' n' | \hat{S}^{+} | q \, m \rangle \langle q \, m | \hat{S} | p \, n \rangle$$

$$= \int d^4 q \sum_m \delta^{(4)}(p'-q) S^{*}_{mn'}(p') \delta^{(4)}(p-q) S_{mn}(p)$$

$$= \sum_m \delta^{(4)}(p'-p) S^{*}_{mn'}(p) S_{mn}(p)$$

$$\Rightarrow \quad \delta_{nn'} = \sum_m S^{*}_{mn'}(p) S_{mn}(p)$$

In matrix form: $\mathbb{1} = S^{+} S$

c) Eq. (2) has a pole at $E = E_R$

Since $\Gamma/M \ll 1$ and $|E - M| < \Gamma$, one has $M \gg \Gamma$ as well as $E \gg \Gamma$. Hence at the energies of interest the pole is close to the real axis, therefore it dominates the S-matrix. Moreover, B and R can be assumed to be constant, for we are interested in the small energy range $E \in (M-\Gamma, M+\Gamma)$, where the only significant energy dependence arises due to the nearby pole $\dfrac{R}{E - M + i\frac{\Gamma}{2}}$ is the resonant part of the S-matrix, where B is the non-resonant background.

d) $\mathbb{1} = S^\dagger(E)\, S(E)$

$$= \left[B + \frac{R}{E-M+\frac{i}{2}\Gamma} \right]^\dagger \left[B + \frac{R}{E-M+\frac{i}{2}\Gamma} \right]$$

$$= B^\dagger B + \frac{B^\dagger R}{E-M+\frac{i}{2}\Gamma} + \frac{R^\dagger B}{E-M-\frac{i}{2}\Gamma} + \frac{R^\dagger R}{(E-M)^2+\Gamma^2/4}$$

$$= B^\dagger B + \frac{B^\dagger R\,[E-M-\frac{i}{2}\Gamma] + R^\dagger B\,[E-M+\frac{i}{2}\Gamma] + R^\dagger R}{(E-M)^2 + \frac{1}{4}\Gamma^2}$$

energy-independent

$\Rightarrow$ $\underline{\mathbb{1} = B^\dagger B}$,

$$0 = B^\dagger R[E-M-\tfrac{i}{2}\Gamma] + R^\dagger B[E-M+\tfrac{i}{2}\Gamma] + R^\dagger R$$

$$= [B^\dagger R + R^\dagger B]E + R^\dagger R - B^\dagger R(M+\tfrac{i}{2}\Gamma) - R^\dagger B(M-\tfrac{i}{2}\Gamma)$$

This relation should hold for any value of E

$\Rightarrow$ $\underline{0 = B^\dagger R + R^\dagger B}$ (I)

$0 = R^\dagger R - B^\dagger R(M+\frac{i}{2}\Gamma) - R^\dagger B(M-\frac{i}{2}\Gamma)$ (II)

(I) in (II) $\Rightarrow$ $\underline{0 = R^\dagger R - i\Gamma B^\dagger R}$

e) $A := \frac{i}{\Gamma} R B^\dagger$

$0 = A^\dagger - A \quad \Leftrightarrow \quad 0 = BR^\dagger + RB^\dagger \quad | \; B^\dagger\cdot$

$\Leftrightarrow \quad 0 = R^\dagger + B^\dagger R B^\dagger \quad | \cdot B$

$\Leftrightarrow \quad 0 = R^\dagger B + B^\dagger R$

$\Leftrightarrow$ true

$\frac{i}{\Gamma} RB^\dagger = A = A^2 = A^\dagger A = \frac{1}{\Gamma^2} BR^\dagger RB^\dagger \longrightarrow$ a projector!

$\Leftrightarrow \quad 0 = BR^\dagger RB^\dagger - i\Gamma RB^\dagger \quad | \; B^\dagger\cdot \;, \; \cdot B$

$\Leftrightarrow \quad 0 = R^\dagger R - i\Gamma B^\dagger R$

$\Leftrightarrow$ true

f) $S(E) = B + \frac{R}{E-M+\frac{i}{2}\Gamma} = B - i\frac{\Gamma A B}{E-M+\frac{i}{2}\Gamma} = \left[1 - i\frac{\Gamma}{E-M+\frac{i}{2}\Gamma}\sum_r |r\rangle\langle r|\right]B$

g) $$S(E)\,|r\rangle = \left[1 - i\frac{\Gamma}{E-M+\frac{i}{2}\Gamma}\sum_s |s\rangle\langle s|\right]|r\rangle$$

$$= \left[|r\rangle - i\frac{\Gamma}{E-M+\frac{i}{2}\Gamma}\sum_s |s\rangle \underbrace{\langle s|r\rangle}_{\delta_{rs}}\right] = \lambda(E)\,|r\rangle$$

with $\lambda(E) = 1 - i\frac{\Gamma}{E-M+\frac{i}{2}\Gamma}$ (independent of $|r\rangle$)

h) $\lambda(E) = e^{2i\delta(E)}$

$$\tan(x) = \frac{\sin(x)}{\cos(x)} = -i\frac{e^{ix}-e^{-ix}}{e^{ix}+e^{-ix}} = -i\frac{e^{2ix}-1}{e^{2ix}+1}$$

$$\Rightarrow \tan(\delta(E)) = -i\frac{\lambda(E)-1}{\lambda(E)+1} = \frac{\Gamma}{2}\frac{1}{M-E}$$

$$\lim_{E \gg M} \tan(\delta(E)) = +\infty, \qquad \lim_{E \ll M} \tan(\delta(E)) = -\infty$$

$\Rightarrow$ Starting at energies slightly below M, the phase increases by π around $E=M$

i) $$(6) \to \quad S_{n'n}(E) = B_{n'n} - i\frac{\Gamma}{E-M+\frac{i}{2}\Gamma}\sum_r\sum_m \langle n'|r\rangle\langle r|m\rangle B_{mn}$$

$$\left(\sum_r \langle n'|r\rangle\langle r|B|n\rangle = \sum_r\sum_m \langle n'|r\rangle\langle r|m\rangle\langle m|B|n\rangle\right)$$

not a complete set of states

Since $n \to n'$ with $n \neq n'$ are forbidden, $S_{n'n} \propto \delta_{n'n}$.

Hence $B_{n'n} = b\,\delta_{n'n}$, $b = \text{const.}$

$$\Rightarrow \quad S_{n'n}(E) = b\left[\delta_{n'n} - i\frac{\Gamma}{E-M+\frac{i}{2}\Gamma}\sum_r \langle n'|r\rangle\langle r|n\rangle\right]$$

For all $\langle r|n\rangle \neq 0$ one needs to have $\langle n'|r\rangle = 0$ for $n \neq n'$.

such that $S_{n'n} \sim \delta_{n'n}$. For such an r the normalization of the states reads $\delta_{rs} = \langle r|s\rangle = \sum_{n'}\langle r|n'\rangle\langle n'|s\rangle = \langle r|n\rangle\langle n|s\rangle$

so $\langle n|s\rangle = 0$ for $n \neq s$

Hence only one term survives in the sum in $S_{n'n}$:

$$S_{n'n} = b\,\delta_{n'n}\left[1 - i\frac{\Gamma}{E - M + \frac{i}{2}\Gamma}\right] = b\,\delta_{n'n}\,\lambda(E)$$

$$= b\,\delta_{n'n}\,e^{2i\delta(E)}$$

B is unitary $\Rightarrow 1 = \sum_n (B^+)_{nn'} B_{n'n} = \sum_{n'} B^*_{n'n} B_{n'n} = |b|^2$

$$\Rightarrow b = e^{2i\delta_B}, \quad \delta_B \in \mathbb{R}$$

$$\Rightarrow S_{nn'}(E) = \delta_{nn'}\,e^{2i(\delta_B + \delta(E))}$$