A.8 $\qquad \phi^+ \propto a_p \quad , \quad \phi^- \propto a_p^+$

a) both sides of equation (3) are invariant under arbitary permutation of $x_1, \dots, x_n$ $\Rightarrow$ to relabel the coordinates if necessary

b) $$\phi^+(x_{n+1})\,\phi^-(x_1)\dots\phi^-(x_m)\,\phi^+(x_{m+1})\dots\phi^+(x_n)$$
$$= \phi^-(x_1)\phi^+(x_{n+1})\dots\phi^-(x_m)\phi^+(x_{m+1})\dots\phi^+(x_n)$$
$$+\phi^-(x_2)\dots\phi^-(x_m)\,\phi^+(x_{m+1})\dots\phi^+(x_n)\,\underbrace{\phi^+(x_{n+1})\,\phi^-(x_1)}$$

a c-number now!

$$= \phi^-(x_1)\dots\phi^-(x_m)\,\phi^+(x_{n+1})\,\phi^+(x_{m+1})\dots\phi^+(x_n)$$
$$+\ \phi^-(x_2)\dots\phi^-(x_m)\,\phi^+(x_{m+1})\dots\phi^+(x_m)\,\phi^+(x_{n+1})\,\phi^-(x_1)$$
$$+\ \dots$$

c)

i) $:\phi(x_1)\dots\phi(x_n):$

$$= :(\phi^+(x_1)+\phi^-(x_1))\dots(\phi^+(x_n)+\phi^-(x_n)):$$
$$= \phi^+(x_1)\phi^+(x_2)\dots\phi^+(x_n) + \sum_{i=1}^{n}\phi^-(x_i)\,\phi^+(x_2)\dots\phi^+(x_n)$$

$\binom{n}{0}$ $\qquad \binom{n}{2}$ terms $\qquad \binom{n}{1} = n$ terms $\qquad \binom{n}{n} = 1$

$$+\sum_{i,j=1}^{n}\phi^-(x_i)\,\phi^-(x_j)\,\phi^+(x_3)\dots\phi^+(x_n) + \dots + \phi^-(x_1)\dots\phi^-(x_n)$$

$$\sum_{k=0}^{n}\binom{n}{k} = 2^n \quad , \quad (x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k ,$$

setting $x = y = 1$

ii) $$\phi^+(x_{n+1}) :\phi(x_1)\dots\phi(x_n):$$
$$= \phi^+(x_{n+1})\,\phi^+(x_1)\dots\phi^+(x_n) + \phi^+(x_{n+1})\sum_{i=0}^{n}\phi^-(x_i)\dots\phi^+(x_n) + \dots +$$
$$+\quad \phi^+(x_{n+1})\,\phi^-(x_1)\dots\phi^-(x_n)$$

manual normal ordering $\overset{(4)}{=} \phi^+(x_{n+1})\phi^+(x_1)\dots\phi^+(x_n) + \sum_{i=1}^{n} \phi^-(x_i)\,\phi^+(x_{n+1})\phi^+(x_1)\dots\phi^+(x_n) + \dots +$

$+\phi^-(x_1)\dots\phi^-(x_n)\phi^+(x_{n+1})$

$+\phi^+(x_1)\dots\phi^+(x_n)\overbrace{\phi^+(x_{n+1})\phi^-(x_i)} + \phi^-(x_j)\phi^+(x_3)\dots\phi^+(x_n)\overbrace{\phi^+(x_{n+1})\phi^-(x_i)}$

↑ $\binom{n}{1}$ term

↑ $\binom{n}{2}\times 2$ term ↑ eq. (4)

$\binom{n}{n}\times n$ terms ↓

$+ \dots + \phi^-(x_2)\dots\phi^-(x_n)\,\phi^+(x_{n+1})\phi^-(x_i)$

iii) consider the structure $\phi^-(x_1)\dots\phi^-(x_k)\phi^+(k+1)\dots\phi^+(x_n)\phi^+(x_{n+1})\phi^-(x_1)$

there are $(n-1)$ different structures (with different number of positive frequency fields). In each structure, there are $\binom{n-1}{k-1}$ terms (interchanging $x_2 \dots x_k$ with $x_2 \dots x_n$)

$\Rightarrow$ in total $\sum_{k=1}^{n}\binom{n-1}{k-1} = 2^{n-1}$

From the normal ordering $: \cancel{\phi(x_1)} \dots \phi(x_n):$, we get exactly 2^{n-1} terms. Therefore

$\phi^+(x_{n+1}) :\phi(x_1)\dots\phi(x_n):$

$= :\phi^+(x_{n+1})\phi(x_1)\dots\phi(x_n): + \sum_{i=1}^{n} \phi^+(x_{n+1})\phi^-(x_i) :\phi(x_1)\dots\cancel{\phi(x_i)}\dots\phi(x_n):$

d) Also works for $\phi(x_{n+1})$?

$\phi^-(x_{n+1}) :\phi(x_1)\dots\phi(x_n): = :\phi^-(x_{n+1})\phi(x_1)\dots\phi(x_n):$

already in normal order

$\Rightarrow$ (8)

e) consider $x^0_{n+1} > x^0_i \quad \forall\, i \in 1, \dots, n$

$T[\phi(x_{n+1})\phi(x_1)\dots\phi(x_n)]$

$= \phi(x_{n+1})\, T[\phi(x_1)\dots\phi(x_n)]$

$= \phi(x_{n+1}) :\phi(x_1)\dots\phi(x_n): + \sum_{i,j} \overbrace{\phi(x_i)\phi(x_j)}\,\phi(x_{n+1}) :\cancel{\phi(x_1)\phi(x_j)}\dots\phi(x_n):$

$\leftarrow \binom{n}{2}$ terms

$$+\sum_{i,j,m,n} \phi(x_i)\phi(x_j)\phi(x_m)\phi(x_n) : \cdots \phi(x_n) : + \cdots$$

$\binom{n}{2}\binom{n-2}{2}/2!$ terms

odd n

$$+\begin{cases} \overbrace{\phi(x_1)\phi(x_2)}\cdots\overbrace{\phi(x_{n-2})\phi(x_{n-1})}\phi(x_{n+1})\phi(x_n), & \binom{n}{2}\binom{n-2}{2}\cdots\binom{3}{2}\Big/\left(\frac{n-1}{2}\right)! \\ \overbrace{\phi(x_1)\phi(x_2)}\cdots\overbrace{\phi(x_{n-1})\phi(x_n)}\phi(x_{n+1}), & \binom{n}{2}\binom{n-2}{2}\cdots\binom{2}{2}\Big/\left(\frac{n}{2}\right)! \end{cases}$$

even n

now consider all contractions with k contractions

$$\phi(x_{n+1})\overbrace{\phi(x_1)\phi(x_2)}\cdots\overbrace{\phi(x_{2k-1})\phi(x_{2k})} : \phi(x_{2k+1})\cdots\phi(x_n):$$

$$= \overbrace{\phi(x_1)\phi(x_2)}\cdots\overbrace{\phi(x_{m-1})\phi(x_{2k})} : \phi(x_{n+1})\phi(x_{2k+1})\cdots\phi(x_n):$$

$$+\sum_{i=2k+1}^{n} \phi(x_1)\phi(x_2)\cdots\phi(x_{2k-1})\phi(x_m)\overbrace{\phi(x_{n+1})\phi(x_i)} : \phi(x_{2k+1})\cdots \cancel{\phi(x_i)}$$

$$\cdots\phi(x_n):$$

There are $(n-2k)$ additional terms with $k+1$ contractions.

For a fixed number of k contractions

$$\begin{cases} k: & \binom{n}{2}\binom{n-2}{2}\cdots\binom{n-2k+2}{2}/k! \\ k-1: & \binom{n}{2}\binom{n-2}{2}\cdots\left(\frac{n-2k+4}{2}\right)\frac{n-2(k-1)}{(k-1)!} \end{cases}$$

$$\binom{n}{2}\binom{n-2}{2}\cdots\binom{n-2k+2}{2}\frac{1}{k!} + \binom{n}{2}\cdots\left(\frac{n-2k+4}{2}\right)\frac{n-2k+2}{(k-1)!}$$

$$= \frac{n!}{2^k(n-2k)!\,k!} + \frac{(n-2k+2)\,n!}{2^{k-1}(n-2k+2)!\,(k-1)!} = \frac{n!}{2^k(n-2k)!\,k!}\left(1+\frac{2k}{n-2k+1}\right)$$

$$= \frac{(n+1)!}{2^k(n+1-2k)!\,k!}$$

1 term

$$\phi(x_{n+1})\,T[\phi_1\cdots\phi_n] = :\phi(x_{n+1})\cdots\phi(x_1): + \cdots$$