Quantum Field Theory
ST 2017

B. Kubis, S. Ropertz

1st Examination

---

1 Short questions **20p**

(a) Derive the Euler–Lagrange equations for the Dirac and photon fields from the following Lagrangian

$$\mathcal{L} = \bar{\psi}\,(i\not{D} - m)\,\psi - \frac{1}{4}\,\epsilon^{\mu\nu\alpha\beta}\,F_{\alpha\beta}\,F_{\mu\nu}\,,$$

with $i\not{D}\psi = (i\not{\partial} - e\not{A})\,\psi$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. **4p**

(b) Consider a Yukawa theory in 6 dimensions with

$$\mathcal{L} = \bar{\psi}\,(i\not{\partial} - m)\,\psi + \frac{1}{2}\left(\partial_\mu\phi\,\partial^\mu\phi - M^2\phi^2\right) - g\bar{\psi}i\gamma^5\psi\phi\,.$$

Is this theory super-renormalizable, renormalizable, or non-renormalizable? **3p**

(c) Write down the Fourier decomposition of a massive fermion field $\psi(x)$ in the Heisenberg picture. Calculate $\langle 0|\psi(x)|\mathbf{p},r\rangle$, where $|\mathbf{p},r\rangle$ is a free particle with spin r and three momentum $\mathbf{p}$. **3p**

(d) What are the propagators and vertex Feynman rules for the Lagrangian density

$$\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi + \tfrac{1}{2}\partial_\mu\pi\partial^\mu\pi - \sqrt{\lambda}\,(\phi^*\phi)^2 + \lambda\,(\phi^*\phi)\,\pi^2 - g\,(\phi^*\phi)\,\pi \quad ?$$ **4p**

<u>*Hint*</u>*: No proof is required. Be careful about the prefactors.*

(e) Consider ϕ^4 theory with $\mathcal{L}_{\text{int}} = -\frac{\lambda}{4!}\phi^4$. Give the symmetry factors of the two diagrams

(i) (ii)

In addition prove them using Wick contractions. **6p**

2 Proton–antiproton annihilation **20p**

Consider a Yukawa theory given by

$$\mathcal{L} = \bar{\psi}\,(i\partial\!\!\!/ - m)\,\psi + \bar{\chi}\,(i\partial\!\!\!/ - \mu)\,\chi + \frac{1}{2}\left(\partial^\mu\phi\partial_\mu\phi - M^2\phi^2\right) - g\bar{\psi}i\gamma^5\psi\phi - \bar{g}\bar{\chi}i\gamma^5\chi\phi\,.$$

where $\psi(x)$ denotes the proton field, $\chi(x)$ the neutron field and $\phi(x)$ a real scalar field. The coupling constants are given by g and $\bar{g}$. Furthermore, note $\mu > m$.
We now assign the spins $s, \bar{s}, r, \bar{r}$ and the momenta $p, \bar{p}, k, \bar{k}$ such that the process is given by

$$\psi^{(s)}(p) + \bar{\psi}^{(\bar{s})}(\bar{p}) \to \chi^{(r)}(k) + \bar{\chi}^{(\bar{r})}(\bar{k})\,.$$

It is useful to define the Mandelstam variables $s = (p+\bar{p})^2$, $t = (p-k)^2$ and $u = (p-\bar{k})^2$.

(a) What are the Feynman rules for the interaction vertices? **1p**
<u>*Hint*</u>*: Be careful about the factor i.*

(b) Which Feynman diagram(s) contribute to the process proton–antiproton to neutron–antineutron to the lowest order? Draw them, label the lines with momenta and spins, and determine the invariant matrix element $i\mathcal{M}$. **2p**

(c) Calculate $\overline{|\mathcal{M}|^2}$ by averaging over the initial and summing over the final spin states. Show that it can be written in terms of Mandelstam variables as

$$\overline{|\mathcal{M}|^2} = \left(\frac{g\bar{g}}{s - M^2}\right)^2 \cdot s^2\,.$$ **10p**

<u>*Hint*</u>*: Use the relation $\left(\gamma^5\right)^\dagger = \gamma^5$.*

(d) In the center-of-mass system (CMS) the differential cross section is given by

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\frac{|\mathbf{p}_f|}{|\mathbf{p}_i|}\overline{|\mathcal{M}|^2}\,,$$

where $\mathbf{p}_i$ ($\mathbf{p}_f$) are the CMS particle three momenta of the initial (final) state. Express the differential cross section in terms of the Mandelstam variables. What happens for $s < 4\mu^2$? **2p**

(e) Consider now proton–neutron scattering

$$\psi^{(s)}(p) + \chi^{(\bar{r})}(\bar{k}) \to \psi^{(\bar{s})}(\bar{p}) + \chi^{(r)}(k)\,.$$

We define our new Mandelstam variables to be $\tilde{s} = (p+\bar{k})^2$, $\tilde{t} = (p-\bar{p})^2$, and $\tilde{u} = (p-k)^2$. Calculate $\overline{|\mathcal{M}|^2}$ for this process and express it in terms of the new Mandelstam variables.

5p

3 Wave function renormalization **20p**

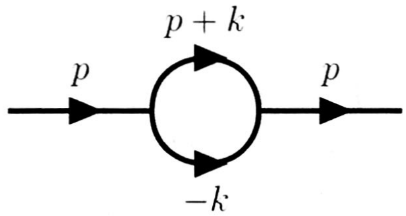
In the following we are going to study the self energy in ϕ^3-theory with $\mathcal{L}_{\text{int}} = -\frac{\lambda}{3!}\phi^3$ (in 4 dimensions) at next-to-leading order and determine its contribution to the wave function renormalization.

(a) Briefly show that the two-point function is given by

$$\int \mathrm{d}^4x \, e^{ipx} \langle \Omega | T\phi(x)\phi(0) | \Omega \rangle = \frac{i}{p^2 - m_0^2 - \Sigma(p^2)} ,$$

where $-i\Sigma(p^2)$ denotes the sum of all one-particle-irreducible diagrams and m_0 the bare mass. Show that the correction to the wave function renormalization is given by $\left.\frac{\mathrm{d}\Sigma(p^2)}{\mathrm{d}p^2}\right|_{p^2=m^2}$. **6p**

(b) Consider the following second-order contribution to $\Sigma(p^2)$ denoted by $\Sigma_2(p^2)$:



Write down the amplitude corresponding to the diagram in d dimensions, combine the propagators using Feynman parameters, and simplify as far as possible. **6p**
<u>*Hint:*</u>

$$\frac{1}{AB} = \int_0^1 \mathrm{d}x \frac{1}{[xA + (1-x)B]^2}$$

(c) Perform a Wick rotation of the self energy contribution. What can you deduce about the degree of divergence of the loop integral for $\Sigma_2(p^2)$ and $\frac{\mathrm{d}\Sigma_2(p^2)}{\mathrm{d}p^2}$ for $d \to 4$? **4p**
<u>*Hint:*</u> *Perform the differentiation $\frac{\mathrm{d}}{\mathrm{d}p^2}$ under the integral.*

(d) We now want to calculate the contribution to the wave function renormalization. Perform the Euclidean integral. The remaining integral over the Feynman parameter evaluates to a real number and should not be carried out. **4p**
<u>*Hints:*</u>

1. $$\int \frac{\mathrm{d}^d l_E}{(2\pi)^d} \frac{1}{(l_E^2 + \Delta)^n} = \frac{1}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma\left(n - \frac{d}{2}\right)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}}$$
2. $\Gamma(x+1) = x \cdot \Gamma(x)$ and $\Gamma(1) = 1$
3. Note that $m^2 - m_0^2 = \mathcal{O}(\lambda^2)$.

Quantum Field Theory
ST 2017

B. Kubis, S. Ropertz

2$^{\text{nd}}$ Examination

---

1 Short questions **20p**

(a) **Derive the Euler-Lagrange equations for the Dirac field** from the following Lagrangian:

$$\mathcal{L} = \bar{\psi}\left(i\not{D} - m\right)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,,$$

with $\not{D}\psi = \left(\not{\partial} - ie\not{A}\right)\psi$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.
If $m = 0$ the Lagrangian is invariant under the global transformation $\psi \to e^{i\alpha\gamma^5}\psi$, $\bar{\psi} \to \psi^\dagger e^{-i\alpha\gamma^5}\gamma^0$. **Deduce the Noether current.** **4p**

(b) Consider the following interaction terms:

$$\mathcal{L}_{\text{int}} = c_1\left(\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}\right)^2 + c_2\left(\bar{\psi}i\sigma^{\mu\nu}\psi\right)F_{\mu\nu} + c_3\left(\bar{\psi}i\gamma^5\not{A}\psi\right)\phi^2 + c_4\left(\phi(\partial^\mu\phi)A_\mu\right)\,,$$

with the Klein-Gordon field ϕ, Dirac field ψ, electromagnetic field A_μ, its corresponding field strength tensor $F_{\mu\nu}$, its dual $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$, and $\sigma^{\mu\nu} = \frac{i}{2}\left[\gamma^\mu, \gamma^\nu\right]$.
Determine the mass dimension of the coupling constants c_i, $i = 1,\ldots,4$, in $d = 4$ dimensions and deduce whether the individual interaction terms are super-renormalizable, renormalizable, or non-renormalizable. **5p**

(c) Why do the creation and annihilation operators of Dirac particles fulfill anticommutation instead of commutation relations? What is the result of $\left(a_{\mathbf{p}}^{(s)}\right)^\dagger\left(a_{\mathbf{q}}^{(r)}\right)^\dagger|0\rangle$? **2p**

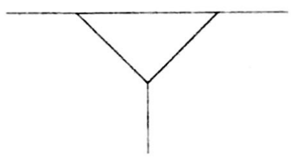
(d) Show that the superficial degree of divergence for an arbitrary loop-diagram in ϕ^4 theory can be written as

$$D = d + (d-4)\cdot V - \left(\frac{d}{2} - 1\right)\cdot E\,,$$

where d denotes the space-time-dimension, V stands for the number of vertices and E for the number of external lines.
Discuss the renormalizability of the theory for $d < 4$, $d = 4$, and $d > 4$. **6p**

<u>*Hint:*</u> *You may want to use the number of loops L and the number of internal lines I in the course of the derivation.*

(e) Consider $\mathcal{L}_{\text{int}} = -\dfrac{\lambda}{3!}\phi^3$. What is the symmetry factor of the following diagram?



Proof your result using Wick contractions. **3p**

2 Electron–positron annihilation **20p**

We consider electron–positron annihilation into a muon–antimuon pair. The interaction Lagrangian for this process is then given by:

$$\mathcal{L}_{\rm int} = -e\,\bar{\psi}\,\gamma^{\mu}\,A_{\mu}\,\psi - e\,\bar{\xi}\,\gamma^{\mu}\,A_{\mu}\,\xi\,,$$

where ψ is the electron field, ξ the muon field, and $e = |e|$ is the elementary charge. We assign momenta p, p', k, k' and spins s, s', r, r' in the following way:

$$\psi^{(s)}(p) + \bar{\psi}^{(s')}(p') \to \xi^{(r)}(k) + \bar{\xi}^{(r')}(k')\,.$$

Assume that we are in an energy region where we can neglect the electron mass m_e ($m_e \approx 0$) but <u>not</u> the muon mass m_μ.
It is useful to define the Mandelstam variables $s = (p+p')^2$, $t = (p-k)^2$, and $u = (p-k')^2$.

(a) Show

1) $s = 2pp' = 2m_\mu^2 + 2kk'$
2) $t = m_\mu^2 - 2pk = m_\mu^2 - 2p'k'$
3) $u = m_\mu^2 - 2pk' = m_\mu^2 - 2p'k$
4) $s + t + u = 2m_\mu^2$

2p

(b) Which diagram contributes to the leading order? Draw it, label the lines with momenta and spins, and determine the invariant matrix element $\mathcal{M}$. **3p**

(c) Calculate $\overline{|\mathcal{M}|^2}$ by averaging $|\mathcal{M}|^2$ over the initial spins and summing over the final spin states. Simplify $\overline{|\mathcal{M}|^2}$ as far as possible (using $m_e = 0$) and verify that it reduces to

$$\overline{|\mathcal{M}|^2} = \frac{2e^4}{s^2}\left(t^2 + u^2 + 4m_\mu^2 s - 2m_\mu^4\right)\,.$$ **12p**

<u>*Hint:*</u> $\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})$

(d) Assume now to be in the ultrarelativistic case in which $m_\mu \approx 0$. In the center-of-mass system (CMS) the differential cross section is given by

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2 s}\cdot\overline{|\mathcal{M}|^2}\,.$$

Express the cross section in terms of the Mandelstam variable s and the angle Θ between the incoming electron ($\mathbf{p}$) and the outgoing muon three momentum ($\mathbf{k}$). **3p**

3 The electron self energy **20p**

In the following we are going to study the electron self energy in QED at next-to-leading order.

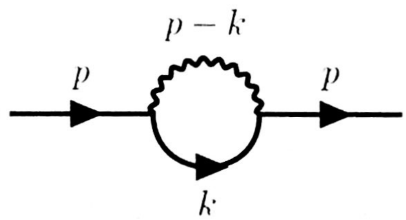
(a) Briefly show that the two-point function is given by

$$\int \mathrm{d}^4x\, e^{ipx} \langle \Omega | T\psi(x)\bar{\psi}(0) | \Omega \rangle = \frac{i}{\not{p} - m_0 - \Sigma(\not{p})}\,,$$

where $-i\Sigma(\not{p})$ denotes the sum of all one-particle-irreducible diagrams and m_0 the bare electron mass. What is the correction to the physical electron mass m? **4p**

<u>*Hint*:</u> *Use the relation* $\frac{i(\not{p}+m_0)}{p^2-m_0^2} = \frac{i}{\not{p}-m_0}$. *Furthermore use that the geometric sum for a matrix valued operator A is given by* $\sum\limits_{n=0}^{\infty} A^n = [1-A]^{-1}$.

(b) Show that the second-order self energy contribution $\Sigma_2(\not{p})$ is defined by

$-i\Sigma_2(\not{p}) =$ 

$$= -e^2 \int\limits_0^1 \mathrm{d}x \int \frac{\mathrm{d}^dq}{(2\pi)^d}\, \gamma^\mu \left(\not{p}x + m_0\right) \gamma_\mu \left[q^2 - \Delta(p^2) + i\epsilon\right]^{-2}.$$

with $\Delta(p^2) = (1-x)(m_0^2 - xp^2)$. **6p**

<u>*Hint:*</u> *Use the relation* $\frac{1}{AB} = \int_0^1 \mathrm{d}x \frac{1}{[xA + (1-x)B]^2}$.

(c) Show that in d dimensions the Dirac structure reduces to

$$\gamma^\mu \left(\not{p}x + m_0\right)\gamma_\mu = (2-d)x\not{p} + d\,m_0\,.$$ **2p**

(d) What is the mass dimension of the coupling constant e in d dimensions? Rewrite it as $e = \mu^\alpha e_0$, where e_0 has the same mass dimension as e in four dimensions. What is α? **2p**

(e) Perform a Wick rotation in order to evaluate the momentum integral. **2p**

<u>*Hint*:</u> *Use the relation* $\int \frac{\mathrm{d}^dq_E}{(2\pi)^d} \frac{1}{(q_E^2+\Delta)^n} = \frac{1}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma\left(n-\frac{d}{2}\right)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}}$

(f) Identify and calculate the divergent part of the mass renormalization in the variable $\epsilon = 4-d$. **4p**

<u>*Hints*:</u> *Use the relations:*

1) $\Gamma(2) = 1$ *and* $\Gamma(x) = \frac{1}{x} - \Gamma_E + \mathcal{O}(x^2)$
2) $a^\epsilon = 1 + \epsilon \ln a$
3) $m_0 = m + \mathcal{O}(e^2)$