Quantum Field Theory

July 16, 2019

Contents

3 Quantization of the Dirac field

3.1 Dirac equation

$$\left(i\gamma^\mu\partial_\mu - m\right)\psi(x) = 0 \tag{3.1.1}$$

Standard representation (Dirac's)

$$\gamma_0 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix} \quad \boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix} \tag{3.1.2}$$

Lorentz transformation

$$\Lambda = \exp\left(\frac{1}{2}\omega_{\mu\nu}M^{\mu\nu}\right) \tag{3.1.3}$$

with ω set of parameters and M the generator of Lie algebra.

Spinor representation

$$S^{\rho\sigma} = \frac{1}{4}\left[\gamma^\rho, \gamma^\sigma\right] = \frac{1}{2i}\sigma^{\rho\sigma} \tag{3.1.4}$$

$$\tag{3.1.5}$$

Spinor transformation

$$S(\Lambda) = \exp\left(\frac{1}{2}\omega_{\mu\nu}S^{\mu\nu}\right) \tag{3.1.6}$$

$$\psi'_a(x) = S_{ab}(\Lambda)\psi_b(\Lambda^{-1}x) \tag{3.1.7}$$

adjoint spinor

$$\bar{\psi} = \psi^\dagger\gamma^0 \tag{3.1.8}$$

Fifth gamma matrix

$$\gamma^5 := i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{3.1.9}$$

$$\left\{\gamma^\mu, \gamma^5\right\} = 0 \tag{3.1.10}$$

$$(\gamma^5)^2 = \mathbb{1}_4 \tag{3.1.11}$$

Plane wave solutions

$$\psi(x) = \begin{cases} u(p)e^{-ipx} & \text{positive frequency} \\ v(p)e^{ipx} & \text{negative frequency} \end{cases} \tag{3.1.12}$$

$$(\not{p} - m)u(p) = 0 \qquad u_s(p) = \sqrt{E_p + m}\begin{pmatrix} \chi_s \\ \frac{\boldsymbol{u}\cdot\boldsymbol{p}}{E_p+m}\chi_s \end{pmatrix} \tag{3.1.13}$$

$$(\not{p} + m)v(p) = 0 \qquad v_s(p) = \sqrt{E_p + m}\begin{pmatrix} \frac{\boldsymbol{u}\cdot\boldsymbol{p}}{E_p+m}\tilde{\chi}_s \\ \tilde{\chi}_s \end{pmatrix} \tag{3.1.14}$$

with

$$\chi_{\frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad x_{-\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$s = \pm\frac{1}{2} \quad \tilde{\chi}_s = \chi_{-s}$$

Orthogonality of spinor

$$\bar{u}_s(p)u_{s'}(p) = -\bar{v}_s(p)v_{s'}(p) = 2m\delta_{ss'} \tag{3.1.15}$$
$$\bar{u}_s(p)v_{s'}(p) = 0 \tag{3.1.16}$$

Spin sums

$$\sum_s u_s(p)\bar{u}_s(p) = \not{p} + m \tag{3.1.17}$$
$$\sum_s v_s(p)\bar{v}_s(p) = \not{p} - m \tag{3.1.18}$$

3.2 Dirac Lagrangian and quantization

$$\mathcal{L} = \bar{\psi}(x)(i\not{\partial} - m)\psi(x) \tag{3.2.1}$$

Quantization

$$\left\{\psi_a(\boldsymbol{x}), \psi_b^\dagger(\boldsymbol{x}')\right\} = \delta_{ab}\delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}') \tag{3.2.2}$$
$$\{\psi_a(\boldsymbol{x}), \psi_b(\boldsymbol{x}')\} = \left\{\psi_a^\dagger(\boldsymbol{x}), \psi_b^\dagger(\boldsymbol{x}')\right\} = 0 \tag{3.2.3}$$

Field operators

$$\psi(\boldsymbol{x}) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3\sqrt{2E_p}} \sum_s (a_{\boldsymbol{p}}^s u_s(\boldsymbol{p})e^{i\boldsymbol{p}\cdot\boldsymbol{x}} + b_{\boldsymbol{p}}^{s\,\dagger} v_s(\boldsymbol{p})e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}) \tag{3.2.4}$$

thus the anticommutations of ladder operators:

$$\left\{a_{\boldsymbol{p}}^s, a_{\boldsymbol{p}'}^{s'\,\dagger}\right\} = \left\{b_{\boldsymbol{p}}^s, b_{\boldsymbol{p}'}^{s'\,\dagger}\right\} = (2\pi)^3\delta_{ss'}\delta^{(3)}(\boldsymbol{p} - \boldsymbol{p}')$$
$$\{a, a\} = \left\{a^\dagger, a^\dagger\right\} = \cdots = 0$$

Hamiltonian in terms of Fourier modes (with normal ordering)

$$H = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \sum_s E_p (a_{\boldsymbol{p}}^{s\,\dagger} a_{\boldsymbol{p}}^s - b_{\boldsymbol{p}}^{s\,\dagger} b_{\boldsymbol{p}}^s) \tag{3.2.5}$$

3.3 Particles and antiparticles

$$Q = e \int \mathrm{d}^3 x \psi^\dagger(x)\psi(x) \tag{3.3.1}$$

$$: Q : = e \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \sum_s (a_{\boldsymbol{p}}^{s\,\dagger} a_{\boldsymbol{p}}^s - b_{\boldsymbol{p}}^{s\,\dagger} b_{\boldsymbol{p}}^s) \tag{3.3.2}$$

3.4 Dirac propagator and anticommutators

$$\begin{aligned} S_{ab}(x-y) &= \{\psi_a(x), \bar{\psi}_b(y)\} \\ &= (i\partial\!\!\!/ + m)\,[D(x-y) - D(y-x)] \end{aligned} \tag{3.4.1}$$

Time ordering of Dirac fields

$$T(\phi_a(x)\bar{\psi}_b(y)) = \Theta(x^0 - y^0)\psi_a(x)\bar{\psi}_b(y) - \Theta(y^0 - x^0)\bar{\psi}_b(y)\psi_a(x) \tag{3.4.2}$$

Feynman propogator for the Dirac field

$$S_F(x-y) = \langle 0|T\psi(x)\bar{\psi}(y)|0\rangle = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i(p\!\!\!/ + m)}{p^2 - m^2 + i\epsilon} e^{-ip\cdot(x-y)} \tag{3.4.3}$$

3.5 Discrete symmetries of the Dirac Field

	orientation perserving	orientation not perserving
(ortho)chronous	$\mathcal{L}_+^\uparrow$	$\mathcal{L}_-^\uparrow = \mathcal{P}\mathcal{L}_+^\uparrow$
non-orthochronous	$\mathcal{L}_-^\downarrow = \mathcal{T}\mathcal{L}_+^\uparrow$	$\mathcal{L}_+^\downarrow = \mathcal{P}\mathcal{T}\mathcal{L}_+^\uparrow$