A.17

a) $$\pi^{\lambda} = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_\lambda)} = -\frac{1}{2}\frac{\partial F_{\mu\nu}}{\partial(\partial_0 A_\lambda)} F^{\mu\nu} = -\frac{1}{2}(F^{0\lambda} - F^{\lambda 0}) = -F^{0\lambda}$$

$$\pi^0 = -F^{00} = 0$$

canonical quantitation: $[A^\alpha(t,\vec{x}), \pi^\beta(t,\vec{y})] = i\,\delta^{\alpha\beta}\,\delta^{(3)}(\vec{x}-\vec{y})$

but $[A^0(t,\vec{x}), \pi^0(t,\vec{y})] = 0 \neq i\,\delta^{(3)}(\vec{x}-\vec{y})$

b) $$\mathcal{L}'' = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \underbrace{-\frac{1}{2}(\partial_\mu A^\mu)^2}_{\text{gauge fixing term}}, \qquad \mathcal{L}' = -\frac{1}{2}(\partial^\mu A^\nu)(\partial_\mu A_\nu)$$

$$= -\frac{1}{2}(\partial^\mu A^\nu)(\partial_\mu A_\nu) + \frac{1}{2}[(\partial^\mu A^\nu)(\partial_\nu A_\mu) - (\partial^\mu A_\mu)(\partial^\nu A_\nu)]$$

$$= -\frac{1}{2}(\partial^\mu A^\nu)(\partial_\mu A_\nu) + \frac{1}{2}\partial^\mu[A_\nu(\partial^\nu A_\mu) - A_\mu(\partial^\nu A_\nu)]$$

the two-time differentiation terms cancel each other

$$= \mathcal{L}' + \partial^\mu[\quad]_\mu$$

c) $$\frac{\partial \mathcal{L}'}{\partial(\partial_\alpha A_\beta)} = -\frac{1}{2}\frac{\partial[(\partial^\mu A^\nu)(\partial_\mu A_\nu)]}{\partial(\partial_\alpha A_\beta)} = -\partial^\alpha A^\beta,$$

$\pi^\mu = -\partial^0 A^\mu = -\dot{A}_\mu$, in general non-vanishing

$$\partial_\alpha \frac{\partial \mathcal{L}'}{\partial(\partial_\alpha A_\beta)} - \frac{\partial \mathcal{L}'}{\partial A_\beta} = -\partial_\alpha\partial^\alpha A^\beta = -\partial^2 A^\beta = 0 \quad \text{on shell}$$

for Lorentz gauge $(\partial_\mu A^\mu = 0)$ $\Rightarrow$ e.o.m.

d) $\pi^\mu = -\partial^0 A^\mu = -\dot{A}^\mu$,

$$\pi_\nu(\vec{y},t) = i\sum_{r=0}^{3}\int \frac{d^3q}{(2\pi)^3}\sqrt{\frac{E_{\vec{q}}}{2}}\,[\varepsilon_\nu^{(r)}(\vec{q})\,a_r(\vec{q})\,e^{-iqy} - (\varepsilon_\nu^{(r)}(\vec{q}))^* a_r^\dagger(\vec{q})\,e^{iqy}]$$

$$[A_\mu(\vec{x},t), \pi_\nu(\vec{y},t)]$$
$$= -i\sum_{s,r=0}^{3}\int \frac{d^3k}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_k}}\sqrt{\frac{E_{\vec{q}}}{2}}\{\varepsilon_\mu^{(s)}(\vec{k})\,(\varepsilon_\nu^{(r)}(\vec{q}))^*\,e^{-ikx+iqy}[a_s(\vec{k}), a_r^\dagger(\vec{q})] +$$
$$+ (\varepsilon_\mu^{(s)}(\vec{k}))^*\,\varepsilon_\nu^{(r)}(\vec{q})\,e^{ikx-iqy}[a_r(\vec{q}), a_s^\dagger(\vec{k})]\}$$

$$= -i \sum_{r,s=0}^{3} \int \frac{d^3k}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \sqrt{\frac{E_{\vec{q}}}{4E_{\vec{k}}}} (2\pi)^3 \delta^{(3)}(\vec{q}-\vec{k}) [\varepsilon_\mu^{(s)}(\vec{k}) (\varepsilon_\nu^{(r)}(\vec{q}))^* e^{-ik\cdot x + iq y} (-g_{rs})$$

$$+ (\varepsilon_\mu^{(s)}(\vec{k}))^* \varepsilon_\nu^{(r)}(\vec{q}) (-g_{rs}) e^{ikx - iqy}]$$

$$= -\frac{i}{2} \int \frac{d^3k}{(2\pi)^3} (\sum_{r,s=0}^{3} (-\varepsilon_\mu^{(s)}(k) g_{\nu s} (\varepsilon_\nu^{(r)}(k))^*) e^{-ik(x-y)} + \sum_{r,s=0}^{3} (-(\varepsilon_\mu^{(s)}(k))^* g_{\nu s} \times$$

$$\varepsilon_\nu^{(k)}(k)) e^{ik(x-y)})$$

$$= i g_{\mu\nu} \delta^{(3)}(\vec{x}-\vec{y})$$

e) $$\langle 1 | 1 \rangle = \int \frac{d^3k}{(2\pi)^3 \sqrt{2E_{\vec{k}}}} \frac{d^3q}{(2\pi)^3 \sqrt{2E_{\vec{q}}}} f^*(\vec{k}) f(\vec{q}) \underbrace{\langle 0 | a_0(\vec{k}) a_0^\dagger(\vec{k}) | 0 \rangle}_{= \langle 0 | [a_0(\vec{k}), a_0^\dagger(\vec{k})] | 0 \rangle}$$

$$= -g^{00} \int \frac{d^3k}{(2\pi)^3} \frac{1}{2E_{\vec{k}}} |f(\vec{k})|^2 < 0$$

f) $[A_\mu(x), A_\nu(y)]$

$$= \sum_{r,s=0}^{3} \int \frac{d^3k}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \frac{1}{2\sqrt{E_{\vec{k}} E_{\vec{q}}}} \varepsilon_\mu^{(r)} \varepsilon_\nu^{(s)*} e^{-ikx + iqy} [a_r(\vec{k}), a_s^\dagger(\vec{q})] - h.c.$$

$$= (\quad \cdots \quad) (-g_{rs}) (2\pi)^3 \delta^{(3)}(\vec{k}-\vec{q}) - h.c.$$

$$= \sum_{r,s=0}^{3} \int \frac{d^3k}{(2\pi)^3} \frac{1}{2E_{\vec{k}}} \varepsilon_\mu^{(r)}(\vec{k}) \varepsilon_\nu^{(s)*}(\vec{k}) e^{-ik(x-y)} (-g_{rs}) - h.c.$$

$$= -g_{\mu\nu} \int \frac{d^3k}{(2\pi)^3} \frac{1}{2E_{\vec{k}}} e^{-ik\cdot(x-y)} - h.c.$$

$$= -g_{\mu\nu} [D(x-y) - D(y-x)]$$

g) $$0 = [\underset{\overset{\shortparallel}{0}}{\partial_x^\mu A_\mu(x)}, A_\nu(y)] = \partial_x^\mu [A_\mu(x), A_\nu(y)]$$

$$= -\partial_{x,\nu} (D(x-y) - D(y-x)) \neq 0 \quad \text{in general}$$

commutator inconsistent with lorenz gauge,

change the gauge condition to $\partial^\mu A_\mu^{(+)} | \psi \rangle = 0$

$$\langle \psi' | \partial^\mu A_\mu | \psi \rangle = \langle \psi' | \partial^\mu A_\mu^{(+)} + \partial^\mu A_\mu^{(-)} | \psi \rangle = \langle \psi' | \partial^\mu A_\mu^{(-)} | \psi \rangle$$

$$= \left(\langle \Psi' | \partial^\mu A_\mu^{(+)} | \Psi \rangle \right)^* = 0$$

physical Hilbert space is the null space of $\partial^\mu A_\mu^{(+)}$

<u>Gupta-Bleuler quantization</u>

h) $$\partial^\mu A_\mu^{(+)} |\Psi\rangle = -i \sum_{r=0}^{3} \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{k}}}} e^{-ik\cdot x} \left(k\cdot \varepsilon^{(r)}(\vec{k})\, a_r(\vec{k}) \right) |\Psi\rangle$$

$$= -i \int \frac{d^3k}{(2\pi)^3} \sqrt{\frac{E_{\vec{k}}}{2}} \left[a_0(\vec{k}) - a_3(\vec{k}) \right] |\Psi\rangle = 0 \quad \begin{cases} k\cdot\varepsilon^{(0)} = -k\cdot\varepsilon^{(3)} = E_{\vec{k}} \\ k\cdot\varepsilon^{(1)} = k\cdot\varepsilon^{(2)} = 0 \end{cases}$$

$$\left[a_0(\vec{k}) - a_3(\vec{k}) \right] |\Psi\rangle = 0$$

i) $\langle \Psi | :H: | \Psi \rangle$

$$= \int \frac{d^3k}{(2\pi)^3} E_{\vec{k}} \langle \Psi' | \sum_{i=1}^{2} a_i^\dagger(\vec{k}) a_i(\vec{k}) + a_3^\dagger(\vec{k}) a_3(\vec{k}) - a_0^\dagger(\vec{k}) a_0(\vec{k}) | \Psi \rangle$$

$$= \int \frac{d^3k}{(2\pi)^3} E_k \langle \Psi' | \sum_{i=1}^{2} a_i^\dagger(\vec{k}) a_i(\vec{k}) | \Psi \rangle + \int \frac{d^3k}{(2\pi)^3} E_k \langle \Psi' | a_3^\dagger(\vec{k}) \left(a_3(\vec{k}) - a_0(\vec{k}) \right)$$

$$+ \left(a_3^\dagger(\vec{k}) - a_0^\dagger(\vec{k}) \right) a_0(\vec{k}) | \Psi \rangle$$

$$= \int \frac{d^3k}{(2\pi)^3} E_k \langle \Psi' | \sum_{i=1}^{2} a_i^\dagger(\vec{k}) a_i(\vec{k}) | \Psi \rangle$$

only two a_i contribute to observables.

j) $\langle 0 | A_\mu(x) A_\nu(y) | 0 \rangle = -g_{\mu\nu} D(x-y)$

$$\langle 0 | T\{ A_\mu(x) A_\nu(y) \} | 0 \rangle$$

$$= \theta(x^0 - y^0) \langle 0 | A_\mu(x) A_\nu(y) | 0 \rangle + \theta(y^0 - x^0) \langle 0 | A_\nu(y) A_\mu(x) | 0 \rangle$$

$$= -g_{\mu\nu} \left[\theta(z^0) D(z) + \theta(-z^0) D(-z) \right] \Big|_{z = x-y}$$

$$\theta(a) = \frac{i}{2\pi} \int_{-\infty}^{+\infty} ds \frac{1}{s + i\varepsilon} e^{-isa}$$

$$\Rightarrow \theta(z^0) D(z) = i \int \frac{ds}{2\pi} \frac{d^3k}{(2\pi)^3} \frac{1}{s + i\varepsilon} e^{-i(s+k^0) z^0 + i\vec{k}\cdot\vec{z}}$$

$$= i\int \frac{d^4q}{(2\pi)^4}\,\frac{e^{-iq\cdot z}}{2|\vec{q}|(q^0-|\vec{q}|+i\varepsilon)}$$

where $q^0 = s+k^0$, $\vec{q}=\vec{k}$, $E_k = k^0 = |\vec{k}| = |\vec{q}|$

$$\Theta(-z_0)D(-z) = i\int \frac{d^4q}{(2\pi)^4}\,\frac{e^{-iq\cdot z}}{2|\vec{q}|(-q_0-|\vec{q}|+i\varepsilon)}$$

$$\Rightarrow \langle 0|T\{A_\mu(x)A_\nu(y)\}|0\rangle$$

$$= -ig_{\mu\nu}\int \frac{d^4q}{(2\pi)^4}\, e^{-iq\cdot z}\,\frac{1}{2|\vec{q}|}\left[\frac{1}{q^0-|\vec{q}|+i\varepsilon}+\frac{1}{-q^0-|\vec{q}|+i\varepsilon}\right]$$

$$= -ig_{\mu\nu}\int \frac{d^4q}{(2\pi)^4}\, e^{-iq\cdot z}\,\frac{1}{q^2+i\varepsilon}$$

CPT theorem: $(CPT)\,\mathcal{H}(x)\,(CPT)^{-1} = \mathcal{H}(-x)$

Micro causality: $[\phi(x), \phi(y)] = 0$

spinor $[\bar\psi(x)\psi(x), \bar\psi(y)\psi(y)] = 0, \quad (x-y)^2 < 0$