H.4 $\mathcal{L} = \bar\psi (i\not\partial - m)\psi$ 16/20

a) EL-eq. $\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\right) - \frac{\partial \mathcal{L}}{\partial \psi} = 0$

$$\partial_\mu (\bar\psi i \gamma^\mu) + m\bar\psi = 0$$

$$\bar\psi (i\gamma^\mu \overleftarrow{\partial}_\mu + m) = 0$$

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar\psi)}\right) - \frac{\partial \mathcal{L}}{\partial \bar\psi} = 0$$

thinking $\not\partial$ acting on $\psi$

$$(i\not\partial - m)\psi = 0$$

(hermitian conjugation:

$$\psi^\dagger (i\gamma^\mu \partial_\mu - m)^\dagger = 0$$

$$\psi^\dagger (-i\gamma^0\gamma^\mu\gamma^0 \partial_\mu - m) = 0$$

$$\psi^\dagger (-i\gamma^0\gamma^\mu \partial_\mu - m\gamma^0) = 0$$

$$\rightarrow \bar\psi (i\gamma^\mu \overleftarrow{\partial}_\mu + m) = 0$$ )

b) $T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \partial^\nu \psi + (\partial^\nu \bar\psi) \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar\psi)} - g^{\mu\nu} \mathcal{L}$

$$= \bar\psi i\gamma^\mu \partial^\nu \psi - g^{\mu\nu} \bar\psi (i\not\partial - m)\psi$$

if one uses EOM here, $T^{\mu\nu} = i\bar\psi \gamma^\mu \partial^\nu \psi$

( why are we allowed to use EOM here? )

PS: one can get symmetric $T^{\mu\nu}$ using symmetric $\mathcal{L}$ ✓

c) $(i\not\partial - m)\psi = 0$, $\psi$ is a solution

$m=0$: $i\not\partial\psi = 0$

$$i\not\partial\gamma^5\psi = i\gamma^\mu\gamma^5\partial_\mu\psi = -i\gamma^5\gamma^\mu\partial_\mu\psi = 0$$

$(\{\gamma^\mu, \gamma^5\} = 0)$ $\Rightarrow$ $\gamma^5\psi$ is also a solution

d) $P_{L/R} = \frac{1}{2}(\mathbb{1} \mp \gamma^5)$

i) $P_{L/R}^2 = \frac{1}{4}(\mathbb{1} \mp \gamma^5)^2 = \frac{1}{4}(\mathbb{1} + (\gamma^5)^2 \mp 2\mathbb{1}\gamma^5)$

$= \frac{1}{4}(2\mathbb{1} \mp 2\mathbb{1}\gamma^5) = \frac{1}{2}(\mathbb{1} \mp \gamma^5) = P_{L/R}$

✓ $P_L P_R = P_R P_L = 0$

$P_L + P_R = \mathbb{1}$

ii) $\psi_{L/R} := P_{L/R}\psi$

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},$$

$$P_{L/R}\psi = \frac{1}{2}\begin{pmatrix} \mathbb{1} \pm \mathbb{1} & 0 \\ 0 & \mathbb{1} \mp \mathbb{1} \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \quad \text{(using chiral rep.)}$$

$$= \begin{cases} \begin{pmatrix} \psi_1 \\ 0 \end{pmatrix} = \psi_L, & L \\ \begin{pmatrix} 0 \\ \psi_2 \end{pmatrix} = \psi_R, & R \end{cases}$$

$\Rightarrow \psi = \psi_L + \psi_R$

$$\mathcal{L} = \bar\psi(i\not\partial - m)\psi = (\bar\psi_L + \bar\psi_R)(i\not\partial - m)(\psi_L + \psi_R)$$

$$= \bar\psi_L(i\not\partial - m)\psi_L + \underbrace{\bar\psi_L(i\not\partial - m)\psi_R + \bar\psi_R(i\not\partial - m)\psi_L}_{\substack{\text{diagonal and} \\ (1,0)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0}} + \bar\psi_R(i\not\partial - m)\psi_R$$

$$= \bar\psi_L(i\not\partial - m)\psi_L + \bar\psi_R(i\not\partial - m)\psi_R = \mathcal{L}_L + \mathcal{L}_R$$ ✓

$$\gamma^\mu P_{R/L} = P_{L/R}\gamma^\mu \,, \quad \bar\psi P_{L/R} = \psi^\dagger\gamma^0 P_{L/R} = \psi^\dagger P_{R/L}\gamma^0 = (P_{R/L}\psi)^\dagger\gamma^0 = \bar\psi_{R/L}$$

kinetic term $= i\bar\psi\not\partial(P_L+P_R)\psi = i\bar\psi\underbrace{\not\partial}_{(P_L+P_R)} P_L\psi + i\bar\psi\underbrace{\not\partial}_{(P_L+P_R)} P_R\psi$ | and use $P_L P_R = P_R P_L = 0$

$$= i\bar\psi P_R \not\partial P_L \psi + i\bar\psi P_L \not\partial P_R\psi = i\bar\psi_L\not\partial\psi_L + i\bar\psi_R\not\partial\psi_R$$

mass term $m\bar\psi\psi = m\bar\psi_L\psi_R + m\bar\psi_R\psi_L$ ← Dirac mass term is not chiral symmetric !

e) the trafo $\begin{Bmatrix}\psi\\ \psi^\dagger\end{Bmatrix} \longrightarrow \begin{Bmatrix}\gamma^5\psi\\ \psi^\dagger\gamma^5\end{Bmatrix}$ If one wants to preserve CS, no dirac mass, Mass from Higgs

i.e. $\begin{Bmatrix}\psi\\ \bar\psi\end{Bmatrix} = \begin{Bmatrix}\psi\\ \psi^\dagger\gamma^0\end{Bmatrix} \longrightarrow \begin{Bmatrix}\psi\\ \psi^\dagger\gamma^5\gamma^0\end{Bmatrix} = \begin{Bmatrix}\psi\\ -\bar\psi\gamma^5\end{Bmatrix}$ ↑ Standard Model

$$\bar\psi\psi \to -\bar\psi\gamma^5\gamma^5\psi = -\bar\psi\psi$$

$$\bar\psi i\gamma^5\psi \longrightarrow -\bar\psi\gamma^5 i\gamma^5\gamma^5\psi = -\bar\psi i\gamma^5\psi$$

$$\bar\psi\gamma^\mu\psi \longrightarrow -\bar\psi\gamma^5\gamma^\mu\gamma^5\psi = +\bar\psi\gamma^\mu\psi$$

$$\{\gamma^5,\gamma^\mu\} = 0$$

$$\bar\psi\gamma^\mu\gamma^5\psi \longrightarrow -\bar\psi\gamma^5\gamma^\mu\gamma^5\gamma^5\psi = +\bar\psi\gamma^\mu\gamma^5\psi$$

$$\bar\psi\sigma^{\mu\nu}\psi \longrightarrow -\bar\psi\gamma^5\sigma^{\mu\nu}\gamma^5\psi = \bar\psi\sigma^{\mu\nu}\psi$$

$$\left(\begin{array}{c} \sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu,\gamma^\nu] \\ \{\sigma^{\mu\nu},\gamma^5\} = \frac{i}{2}\{\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu, \gamma^5\} = 0 \end{array}\right)$$

f) $\partial_\mu V^\mu(x) = \partial_\mu(\bar\psi\gamma^\mu\psi) = \partial_\mu\bar\psi\gamma^\mu\psi + \bar\psi\gamma^\mu\partial_\mu\psi$

$$\left[\, (i\not\partial - m)\psi = 0;\ \bar\psi(i\overleftarrow{\not\partial} + m) = 0 \,\right]$$

$$= im\bar\psi\psi + \bar\psi(-im)\psi = 0$$

$$\partial_\mu A^\mu(x) = \partial_\mu(\bar\psi\gamma^\mu\gamma^5\psi) = \partial_\mu\bar\psi\gamma^\mu\gamma^5\psi + \bar\psi\gamma^\mu\gamma^5\partial_\mu\psi$$

$$= \not\partial\bar\psi\gamma^5\psi - \bar\psi\gamma^5\not\partial\psi$$

$$= im\bar\psi\gamma^5\psi + i\bar\psi\gamma^5 m\psi$$

$$\partial_\mu A^\mu = 0 \quad \text{iff} \quad m = 0$$

$$\psi(x) \to e^{i\alpha}\psi(x)\,, \qquad \alpha\Delta\psi = i\alpha\psi$$

$$\alpha\Delta\bar\psi = -i\alpha\bar\psi$$

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \Delta\psi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar\psi)} \Delta\bar\psi \underbrace{- X^\mu}_{=0}$$

$$= \bar\psi \cdot i\gamma^\mu \cdot i\alpha\psi$$

$$= -\alpha \bar\psi \gamma^\mu \psi \quad \propto \quad V(x)$$

g) $\mathcal{L} = \bar\psi(i\partial\!\!\!/ - m)\psi$

$$\to \hat{\mathcal{L}} = \mathcal{L} + \partial_\mu X^\mu(x)$$

$$= \frac{i}{2}\bar\psi(\gamma^\mu \overrightarrow{\partial}_\mu - m)\psi - \frac{i}{2}\bar\psi(\overleftarrow{\partial}_\mu \gamma^\mu + m)\psi$$

$$\partial_\mu X^\mu(x) = -\frac{i}{2}\bar\psi(\gamma^\mu \overrightarrow{\partial}_\mu - \cancel{m})\psi - \frac{i}{2}\bar\psi(\overleftarrow{\partial}_\mu \gamma^\mu + \cancel{m})\psi$$

$$= -\frac{i}{2}\bar\psi\gamma^\mu(\overrightarrow{\partial}_\mu + \overleftarrow{\partial}_\mu)\psi = -\frac{i}{2}\partial_\mu(\bar\psi\gamma^\mu\psi)$$

the difference can be rewritten into total divergence, thus the EOM from $\hat{\mathcal{L}}$ is the same from $\mathcal{L}$

$$\Rightarrow \hat{T}^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\partial^\nu\psi + (\partial^\nu\bar\psi)\frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar\psi)} \underbrace{- g^{\mu\nu}\mathcal{L}}_{=0}$$

$$= \frac{i}{2}\bar\psi\gamma^\mu(\partial^\nu\psi) - (\partial^\nu\bar\psi)\frac{i}{2}\gamma^\mu\psi$$

$$= \frac{i}{2}\bar\psi\gamma^\mu(\overrightarrow{\partial}^\nu - \overleftarrow{\partial}^\nu)\psi$$

$$\mathcal{L} = \frac{i}{2}\bar\psi(\gamma^\mu \overrightarrow{\partial}_\mu - m)\psi - \frac{i}{2}\bar\psi(\overleftarrow{\partial}_\mu \gamma^\mu + m)\psi$$

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} - \frac{\partial \mathcal{L}}{\partial \psi} = 0 \quad \Rightarrow \quad \bar\psi(i\overleftarrow{\partial\!\!\!/} + m) = 0$$

w.r.t. $\bar\psi$ $\qquad \Rightarrow \quad (i\partial\!\!\!/ - m)\psi = 0$