H.4 $\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi$

a) EL-eq. $\partial_\mu \left( \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \right) - \frac{\partial \mathcal{L}}{\partial \psi} = 0$

$$\partial_\mu (\bar{\psi} i \gamma^\mu) + m\bar{\psi} = 0$$

$$\bar{\psi}(i\gamma^\mu \overleftarrow{\partial}_\mu + m) = 0$$

$$\partial_\mu \left( \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} \right) - \frac{\partial \mathcal{L}}{\partial \bar{\psi}} = 0$$

thinking $\partial\!\!\!/$ acting on $\psi$

$$(i\partial\!\!\!/ - m)\psi = 0$$

(hermitian conjugation:

$$\psi^\dagger (i\gamma^\mu \partial_\mu - m)^\dagger = 0$$

$$\psi^\dagger (-i\gamma^0 \gamma^\mu \gamma^0 \partial_\mu - m) = 0$$

$$\psi^\dagger (-i\gamma^0 \gamma^\mu \partial_\mu - m\gamma^0) = 0$$

$$\rightarrow \bar{\psi}(i\gamma^\mu \overleftarrow{\partial}_\mu + m) = 0 \text{ )}$$

b) $T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \partial^\nu \psi + (\partial^\nu \bar{\psi}) \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} - g^{\mu\nu}\mathcal{L}$

$$= \bar{\psi} i \gamma^\mu \partial^\nu \psi - g^{\mu\nu} \bar{\psi}(i\partial\!\!\!/ - m)\psi$$

if one uses EOM here, $T^{\mu\nu} = i\bar{\psi}\gamma^\mu \partial^\nu \psi$

(why are we allowed to use EOM here?)

PS: one can get symmetric $T^{\mu\nu}$ using symmetric $\mathcal{L}$

c) $(i\not{\partial}-m)\psi=0$, $\psi$ is a solution

$m=0$: $i\not{\partial}\psi=0$

$$i\not{\partial}\gamma^5\psi = i\gamma^\mu\gamma^5\partial_\mu\psi = -i\gamma^5\gamma^\mu\partial_\mu\psi = 0$$

$(\{\gamma^\mu,\gamma^5\}=0)$ $\Rightarrow$ $\gamma^5\psi$ is also a solution

d) $P_{L/R}=\frac{1}{2}(\mathbb{1}\mp\gamma^5)$

i) $$P_{L/R}^2 = \frac{1}{4}(\mathbb{1}\mp\gamma^5)^2 = \frac{1}{4}(\mathbb{1}+(\gamma^5)^2\mp 2\mathbb{1}\gamma^5)$$

$$= \frac{1}{4}(2\mathbb{1}\mp 2\mathbb{1}\gamma^5) = \frac{1}{2}(\mathbb{1}\mp\gamma^5) = P_{L/R}$$

ii) $\psi_{L/R} := P_{L/R}\psi$

$$\psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix},$$

$$P_{L/R}\psi = \frac{1}{2}\begin{pmatrix}\mathbb{1}\pm\mathbb{1} & 0\\ 0 & \mathbb{1}\mp\mathbb{1}\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \quad \text{(using chiral rep.)}$$

$$= \begin{cases}\begin{pmatrix}\psi_1\\ 0\end{pmatrix}=\psi_L, & L\\ \begin{pmatrix}0\\ \psi_2\end{pmatrix}=\psi_R, & R\end{cases}$$

$\Rightarrow \psi = \psi_L+\psi_R$

$$\mathcal{L} = \bar\psi(i\not{\partial}-m)\psi = (\bar\psi_L+\bar\psi_R)(i\not{\partial}-m)(\psi_L+\psi_R)$$

$$= \bar\psi_L(i\not{\partial}-m)\psi_L + \underbrace{\bar\psi_L(i\not{\partial}-m)\psi_R + \bar\psi_R(i\not{\partial}-m)\psi_L}_{\text{diagonal and } (1,0)\begin{pmatrix}0\\1\end{pmatrix}=0} + \bar\psi_R(i\not{\partial}-m)\psi_R$$

$$= \bar{\psi}_L (i\not{\partial} - m)\psi_L + \bar{\psi}_R (i\not{\partial} - m)\psi_R = \mathcal{L}_L + \mathcal{L}_R$$

e) the trafo $\begin{Bmatrix} \psi \\ \psi^\dagger \end{Bmatrix} \longmapsto \begin{Bmatrix} \gamma^5 \psi \\ \psi^\dagger \gamma^5 \end{Bmatrix}$

i.e. $\begin{Bmatrix} \psi \\ \bar{\psi} \end{Bmatrix} = \begin{Bmatrix} \psi \\ \psi^\dagger \gamma^0 \end{Bmatrix} \longrightarrow \begin{Bmatrix} \psi \\ \psi^\dagger \gamma^5 \gamma^0 \end{Bmatrix} = \begin{Bmatrix} \psi \\ -\bar{\psi} \gamma^5 \end{Bmatrix}$

$$\bar{\psi}\psi \longrightarrow -\bar{\psi}\gamma^5\gamma^5\psi = -\bar{\psi}\psi$$

$$\bar{\psi} i\gamma^5 \psi \longrightarrow -\bar{\psi}\gamma^5 i\gamma^5 \gamma^5 \psi = -\bar{\psi} i\gamma^5 \psi$$

$$\bar{\psi}\gamma^\mu\psi \longrightarrow -\bar{\psi}\gamma^5\gamma^\mu\gamma^5\psi \underset{\{\gamma^5,\gamma^\mu\}=0}{=} +\bar{\psi}\gamma^\mu\psi$$

$$\bar{\psi}\gamma^\mu\gamma^5\psi \longrightarrow -\bar{\psi}\gamma^5\gamma^\mu\gamma^5\gamma^5\psi = +\bar{\psi}\gamma^\mu\gamma^5\psi$$

$$\bar{\psi}\sigma^{\mu\nu}\psi \longrightarrow -\bar{\psi}\gamma^5\sigma^{\mu\nu}\gamma^5\psi = \bar{\psi}\sigma^{\mu\nu}\psi$$

$$\left( \begin{array}{l} \sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu] \\ \{\sigma^{\mu\nu}, \gamma^5\} = \frac{i}{2}\{\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu, \gamma^5\} = 0 \end{array} \right)$$

f) $\partial_\mu V^\mu(x) = \partial_\mu(\bar{\psi}\gamma^\mu\psi) = \partial_\mu\bar{\psi}\gamma^\mu\psi + \bar{\psi}\gamma^\mu\partial_\mu\psi$

$$[\,(i\not{\partial} - m)\psi = 0;\ \bar{\psi}(i\overleftarrow{\not{\partial}} + m) = 0\,]$$

$$= im\bar{\psi}\psi + \bar{\psi}(-im)\psi = 0$$

$$\partial_\mu A^\mu(x) = \partial_\mu(\bar{\psi}\gamma^\mu\gamma^5\psi) = \partial_\mu\bar{\psi}\gamma^\mu\gamma^5\psi + \bar{\psi}\gamma^\mu\gamma^5\partial_\mu\psi$$

$$= \not{\partial}\bar{\psi}\gamma^5\psi - \bar{\psi}\gamma^5\not{\partial}\psi$$

$$= im\bar{\psi}\gamma^5\psi + i\bar{\psi}\gamma^5 m\psi$$

$\partial_\mu A^\mu = 0$ iff $m = 0$

$\psi(x) \longrightarrow e^{i\alpha}\psi(x)$, $\quad \alpha\Delta\psi = i\alpha\psi$

$$\alpha\Delta\bar{\psi} = -i\alpha\bar{\psi}$$

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \Delta\psi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} \Delta\bar{\psi} \underbrace{- X^\mu}_{=0}$$

$$= \bar{\psi} \cdot i\gamma^\mu \cdot i\alpha\psi$$

$$= -\alpha \bar{\psi}\gamma^\mu\psi \quad \propto \quad V(x)$$

g) $\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi$

$$\rightarrow \hat{\mathcal{L}} = \mathcal{L} + \partial_\mu X^\mu(x)$$

$$= \frac{i}{2}\bar{\psi}(\gamma^\mu \overrightarrow{\partial}_\mu - m)\psi - \frac{i}{2}\bar{\psi}(\overleftarrow{\partial}_\mu \gamma^\mu + m)\psi$$

$$\partial_\mu X^\mu(x) = -\frac{i}{2}\bar{\psi}(\gamma^\mu \overrightarrow{\partial}_\mu \cancel{-m})\psi - \frac{i}{2}\bar{\psi}(\overleftarrow{\partial}_\mu \gamma^\mu \cancel{+m})\psi$$

$$= -\frac{i}{2}\bar{\psi}\gamma^\mu(\overrightarrow{\partial}_\mu + \overleftarrow{\partial}_\mu)\psi = -\frac{i}{2}\partial_\mu(\bar{\psi}\gamma^\mu\psi)$$

the difference can be rewritten into total divergence, thus the EOM from $\hat{\mathcal{L}}$ is the same from $\mathcal{L}$

$$\Rightarrow \hat{T}^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\partial^\nu\psi + (\partial^\nu\bar{\psi})\frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} \underbrace{- g^{\mu\nu}\mathcal{L}}_{=0}$$

$$= \frac{i}{2}\bar{\psi}\gamma^\mu(\partial^\nu\psi) - (\partial^\nu\bar{\psi})\frac{i}{2}\gamma^\mu\psi$$

$$= \frac{i}{2}\bar{\psi}\gamma^\mu(\overrightarrow{\partial}^\nu - \overleftarrow{\partial}^\nu)\psi$$

$$\mathcal{L} = \frac{i}{2}\bar{\psi}(\gamma^\mu \overrightarrow{\partial}_\mu - m)\psi - \frac{i}{2}\bar{\psi}(\overleftarrow{\partial}_\mu \gamma^\mu + m)\psi$$

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} - \frac{\partial \mathcal{L}}{\partial \psi} = 0 \quad \Rightarrow \quad \bar{\psi}(i\overleftarrow{\not{\partial}} + m) = 0$$

w.r.t. $\bar{\psi}$ $\quad \Rightarrow \quad (i\not{\partial} - m)\psi = 0$