Quantum Field Theory

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1 Quantum Electrodynamics (QED)

5.1 Classical Electrodynamics and Maxwell's equations

We have the gauge potential $A^\mu = (A^0, \boldsymbol{A}) = (\phi, \boldsymbol{A})$ (or $A_\mu = (A^0, -\boldsymbol{A}) = (\phi, -\boldsymbol{A})$) and the field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

Then

- electric field is
  $E_i = F_{0i} = \partial_0 A_i - \partial_i A_0 \to \boldsymbol{E} = -\dot{\boldsymbol{A}} - \boldsymbol{\nabla}\phi$
- magnetic field is
  $B^i = -\frac{1}{2}\epsilon^{ijk} F_{jk} \to \boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$

Lagrangian density $\mathcal{L}_{EM} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{2}(\boldsymbol{E} \cdot \boldsymbol{E} - \boldsymbol{B} \cdot \boldsymbol{B})$. The field equation $\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} \right) - \frac{\partial \mathcal{L}}{\partial A_\nu} = 0$ leads to

$$\partial_\mu F^{\mu\nu} = 0 \tag{5.1.1}$$

It is half of Maxwell's equations (in vacuum).

The other half is Bianchi identities following from the definition of $F_{\mu\nu}$:

$$\begin{aligned} &\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0 \;\Leftrightarrow\; \epsilon^{\sigma\lambda\mu\nu} \partial_\lambda F_{\mu\nu} = 0 \\ \text{or} \quad &\partial_\lambda \tilde{F}^{\sigma\lambda} = 0, \; \tilde{F}^{\sigma\lambda} = \frac{1}{2} \epsilon^{\sigma\lambda\mu\nu} F_{\mu\nu} \end{aligned}$$

In terms of $\boldsymbol{E}$ and $\boldsymbol{B}$:

$$\begin{aligned} &\boldsymbol{\nabla} \cdot \boldsymbol{E} = 0, \; \dot{\boldsymbol{E}} = \boldsymbol{\nabla} \times \boldsymbol{B} && \text{dynamical equations} \\ &\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0, \; \dot{\boldsymbol{B}} = -\boldsymbol{\nabla} \times \boldsymbol{E} && \text{Bianchi identities} \end{aligned}$$

Remarks

- Lagrangian density does not depend on $\dot{A}_0$, since A_0 is not really dynamical.

  $$\boldsymbol{\nabla} \cdot \boldsymbol{E} = 0 \to \boldsymbol{\nabla}^2 A_0 + \boldsymbol{\nabla} \cdot \dot{\boldsymbol{A}} = 0$$

  Solve this Poisson equation for $A_0(\boldsymbol{x}, t) = \frac{1}{4\pi} \int \mathrm{d}^3 y \frac{\boldsymbol{\nabla} \cdot \dot{\boldsymbol{A}}(\boldsymbol{y}, t)}{|\boldsymbol{y} - \boldsymbol{x}|}$. Thus A_0 is given in terms of the other components of A.

- Gauge invariance ensures field strength tensor invariant under the transformation $A_\mu \longmapsto A_\mu - \partial_\mu \mathrm{X}$ due to commuting derivatives. This leads to gauge invariance of Maxwell equations.
  Choose X to satisfy $\partial_\mu \partial^\mu \mathrm{X} = \partial^2 \mathrm{X} = \partial_\mu A^\mu$ allows us to demand the condition (Lorenz condition)

  $$\partial_\mu A^\mu = 0 \tag{5.1.2}$$

  such that A_μ belongs to the "Lorenz gauge" and reduces the degrees of freedom from 4 to 3.

- Further freedom is eliminated by adding any X with $\partial^2 X = 0$, e.g. $\partial_t X = A_0$. Then we get the Coulomb or radiation gauge

$$A_0 = 0, \ \nabla \cdot \boldsymbol{A} = 0 \tag{5.1.3}$$

Note: vice versa imposing $\nabla \cdot \boldsymbol{A} = 0$ first, yields $A_0 = 0$.
In Coulomb gauge:

$$\begin{aligned}
&\boldsymbol{E} = -\dot{\boldsymbol{A}}, \ \boldsymbol{B} = \nabla \times \boldsymbol{A}, \ \nabla \times \boldsymbol{A} = 0 \\
&-\ddot{\boldsymbol{A}} = \dot{\boldsymbol{E}} \overset{\text{Maxwell}}{=} \nabla \times \boldsymbol{B} = \nabla \times (\nabla \times \boldsymbol{A}) = \nabla(\underbrace{\nabla \cdot \boldsymbol{A}}_{=0}) - \nabla^2 \boldsymbol{A} \\
&\Rightarrow \partial^2 \boldsymbol{A} = 0
\end{aligned}$$

This wave equation is massless KG equation for each spatial component.

Then the solutions are obvious: $\boldsymbol{A} = \boldsymbol{\epsilon} e^{-ik\cdot x}$ with $k^2 = 0$ and $\boldsymbol{\epsilon} \cdot \boldsymbol{k} = 0$. The polarization vector $\boldsymbol{\epsilon}$ is transverse to $\boldsymbol{k}$.

Can write the lagrangian in Coulmb gauge

$$\mathcal{L}_{\text{EM}} = \frac{1}{2}\dot{\boldsymbol{A}}\dot{\boldsymbol{A}} - \frac{1}{2}\boldsymbol{B} \cdot \boldsymbol{B}$$

Then the conjugate momentum to $\boldsymbol{A}$ is $\boldsymbol{\Pi} = \frac{\partial \mathcal{L}}{\partial \dot{A}} = \dot{\boldsymbol{A}} = -\boldsymbol{E}$. It has only 3 components, there is no conjugate momentum to A_0!. Because of Coulomb gauge $\boldsymbol{\Pi}$ is subject to the constraint $\nabla \cdot \boldsymbol{\Pi} = 0$

Hamiltonian

$$H_{\text{EM}} = \int \mathrm{d}^3 x \left(\frac{1}{2}\boldsymbol{\Pi} \cdot \boldsymbol{\Pi} + \frac{1}{2}\boldsymbol{B} \cdot \boldsymbol{B} \right)$$

5.2 Quantizing the Maxwell field

We would like to impose canonical commutation relations, à la

$$\begin{aligned}
&[A_i(\boldsymbol{x}), A_j(\boldsymbol{y})] = [\Pi_i(\boldsymbol{x}), \Pi_j(\boldsymbol{y})] = 0 \\
&[A_i(\boldsymbol{x}), \Pi_j(\boldsymbol{y})] = i\delta_{ij}\delta^{(3)}(\boldsymbol{x} - \boldsymbol{y})
\end{aligned}$$

However this cannot be true. Take either derivative of the last equation and it needs to vanish due to $\nabla \cdot \boldsymbol{A} = \nabla \cdot \boldsymbol{\Pi} = 0$. But

$$[\partial^i A_i(\boldsymbol{x}), \Pi_k(\boldsymbol{y})] = i\delta_{ij}\partial^i \delta^{(3)}(\boldsymbol{x} - \boldsymbol{y})$$

here the derivative is taken with respect to $\boldsymbol{x}$, i.e. $\partial^i = \frac{\partial}{\partial x_i}$.

Replace δ_{ij} by Δ_{ij}

$$\begin{aligned}
[\partial^i A_i(\boldsymbol{x}), \Pi_j(\boldsymbol{y})] &= i\Delta_{ij}\partial^i \frac{1}{(2\pi)^3} \int \mathrm{d}^3 k e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})} \\
&= -\frac{1}{(2\pi)^3} \int \mathrm{d}^3 k (k^i \Delta_{ij}) e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})} \overset{!}{=} 0
\end{aligned}$$

It works for $\Delta_{ij} = \delta_{ij} - \frac{k_i k_j}{\boldsymbol{k}^2}$ in momentum space or $\Delta_{ij} = \delta_{ij} - \nabla^{-2}\partial_i\partial_j$ in position space.

$$[A_i(\boldsymbol{x}), \Pi_j(\boldsymbol{y})] = i\left(\delta_{ij} - \boldsymbol{\nabla}^{-2}\partial_i\partial_j\right)\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}) \tag{5.2.1}$$

As before we have the mode expansion

$$\begin{aligned}\boldsymbol{A}(\boldsymbol{x}) &= \int \frac{\mathrm{d}^3k}{(2\pi)^3\sqrt{2|\boldsymbol{k}|}}\left(\boldsymbol{a_k}e^{i\boldsymbol{k}\cdot\boldsymbol{x}} + \boldsymbol{a}_{\boldsymbol{k}}^\dagger e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\right) \\ \boldsymbol{\Pi}(\boldsymbol{x}) &= \int \frac{\mathrm{d}^3k}{(2\pi)^3}(-i)\sqrt{\frac{|\boldsymbol{k}|}{2}}\left(\boldsymbol{a_k}e^{i\boldsymbol{k}\cdot\boldsymbol{x}} - \boldsymbol{a}_{\boldsymbol{k}}^\dagger e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\right)\end{aligned}$$

with $\boldsymbol{k}\cdot\boldsymbol{a_k} = \boldsymbol{k}\cdot\boldsymbol{a}_{\boldsymbol{k}}^\dagger = 0$.

Introduce two orthogonal polarization vectors $\boldsymbol{\epsilon}^{(1)}(\boldsymbol{k})$ and $\boldsymbol{\epsilon}^{(2)}(\boldsymbol{k})$ for each $\boldsymbol{k}$.

$$\begin{aligned}\boldsymbol{a_k} &= a_{\boldsymbol{k}}^{(1)}\boldsymbol{\epsilon}^{(1)} + a_{\boldsymbol{k}}^{(2)}\boldsymbol{\epsilon}^{(2)} = \sum_{\lambda=1}^{2} a_{\boldsymbol{k}}^{(\lambda)}\boldsymbol{\epsilon}^{(\lambda)}(\boldsymbol{k}) \\ \text{with}\quad & \boldsymbol{k}\cdot\boldsymbol{\epsilon}^{(1)}(\boldsymbol{k}) = \boldsymbol{k}\cdot\boldsymbol{\epsilon}^{(2)}(\boldsymbol{k}) = 0,\ \boldsymbol{\epsilon}^{(\lambda)}\cdot\boldsymbol{\epsilon}^{(\lambda;)} = \delta_{\lambda\lambda'}\end{aligned}$$

Creation and annihilation operator have the standard commutation relations

$$[a_{\boldsymbol{k}}^{(\lambda)}, a_{\boldsymbol{k}'}^{(\lambda')\dagger}] = (2\pi)^3\delta_{\lambda\lambda'}\delta^{(3)}(\boldsymbol{k}-\boldsymbol{k}') \tag{5.2.2}$$

and all other commutators vanish. Geometrically, still possible to write including the unphysical longitudinal components:

$$\begin{aligned}[\boldsymbol{a_k}, \boldsymbol{a_l}] &= [\boldsymbol{a}_{\boldsymbol{k}}^\dagger, \boldsymbol{a}_{\boldsymbol{l}}^\dagger] = 0 \\ [a_{\boldsymbol{k}}^i, a_{\boldsymbol{l}}^{j\dagger}] &= (2\pi)^3\left(\delta^{ij} - \frac{k^ik^j}{\boldsymbol{k}^2}\right)\delta^{(3)}(\boldsymbol{k}-\boldsymbol{l})\end{aligned}$$

$a_{\boldsymbol{k}}^{(\lambda)}$ and $a_{\boldsymbol{k}}^{(\lambda)\dagger}$ create and destroy photons of momentum $\boldsymbol{k}$, energy $|\boldsymbol{k}|$ and (electric) polarization along $\boldsymbol{\epsilon}^{(\lambda)}(\boldsymbol{k})$.

Next steps are analogous to KG theory.

Hamiltonian

$$H = \frac{1}{2}\int \mathrm{d}^3x\left(\boldsymbol{E}^2 + \boldsymbol{B}^2\right) = \frac{1}{2}\int \mathrm{d}^3x\left(\dot{\boldsymbol{A}}^2 + (\boldsymbol{\nabla}\times\boldsymbol{A})\cdot(\boldsymbol{\nabla}\times\boldsymbol{A})\right)$$

using identity $\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C}) = \boldsymbol{B}\cdot(\boldsymbol{C}\times\boldsymbol{A})$

$$= \frac{1}{2}\int \mathrm{d}^3x\left(\dot{\boldsymbol{A}}^2 + \boldsymbol{A}\cdot\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{A})\right)$$

using the identity $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{A} - \boldsymbol{\nabla}^2\boldsymbol{A})$

$$= \frac{1}{2}\int \mathrm{d}^3x\left(\dot{\boldsymbol{A}}^2 - \boldsymbol{A}\cdot\boldsymbol{\nabla}^2\boldsymbol{A} + \boldsymbol{A}\cdot\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{A})\right)$$

using coulomb gauge condition

$$= \frac{1}{2}\int \mathrm{d}^3x\left(\dot{\boldsymbol{A}}^2 - \boldsymbol{A}\cdot\nabla^2\boldsymbol{A}\right)$$

the first term vanishes and use normal ordering

$$= \int \frac{\mathrm{d}^3k}{(2\pi)^3}|\boldsymbol{k}|\,\boldsymbol{a}_{\boldsymbol{k}}^\dagger\cdot\boldsymbol{a_k} = \sum_{\lambda=1}^{2}\int \frac{\mathrm{d}^3k}{(2\pi)^3}|\boldsymbol{k}|\,a_{\boldsymbol{k}}^{(\lambda\dagger)}a_{\boldsymbol{k}}^{\lambda}$$

Heisenberg field

$$\boldsymbol{A}(\boldsymbol{x},t) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\sqrt{2|\boldsymbol{k}|}}\left(\boldsymbol{a}_{\boldsymbol{k}}e^{-ik\cdot x} + \boldsymbol{a}_{\boldsymbol{k}}^\dagger e^{ik\cdot x}\right)$$

Photon propagator

$$\langle 0|TA_i(x)A_j(y)|0\rangle =: D_{ij}^{\mathrm{tr}}(x-y) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{i}{k^2+i\epsilon}\left(\delta_{ij} - \frac{k_ik_j}{|\boldsymbol{k}|^2}\right)e^{-ik\cdot(x-y)} \tag{5.2.3}$$

tr stands for transverse: photon polarization perpendicular to its momentum. This is NOT the final version of the photon propagator!

5.3 Inclusion of matter - QED

$$\mathcal{L}_{\mathrm{QED}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\not{D} - m)\psi \tag{5.3.1}$$

where $D_\mu = \partial_\mu + ieA_\mu$ is the (gauge) covariante derivative

$$= \mathcal{L}_{\mathrm{EM}} + \mathcal{L}_D - e\underbrace{\bar{\psi}\gamma^\mu\psi}_{j^\mu}A_\mu \tag{5.3.2}$$

Field equations would be

$$\partial_\mu F^{\mu\nu} = ej^\nu \qquad (i\not{D} - m)\psi = 0$$

where ej^ν is the electromagnetic 4-current.

Gauge invariance under the transformation

$$\begin{cases} \psi(x) \longmapsto \psi'(x) = e^{ie\chi(x)}\psi \\ A_\mu(x) \longmapsto A'_\mu(x) = A_\mu(x) - \partial_\mu\chi(x) \end{cases}$$

To check the consistence: cavariant derivative transforms like $D_\mu \longmapsto D'_\mu\psi'(x) = e^{ie\chi(x)}D_\mu\psi(x)$. Since the adjoint spinor transforms like $\bar{\psi}(x) \longmapsto \bar{\psi}'(x) = \bar{\psi}(x)e^{-ie\chi(x)}$, the Lagrangian and field equations are gauge invariant.

Again we choose Coulomb gauge $\boldsymbol{\nabla}\cdot\boldsymbol{A} = 0$, then equation for A^0:

$$\begin{aligned} \partial_i F^{i0} &= ej^0 \\ \Rightarrow -\boldsymbol{\nabla}^2 A^0 &= ej^0 = e\bar{\psi}\gamma^0\psi \\ &= e\bar{\psi}\gamma^0\psi = e\psi^\dagger\psi \\ &= e\rho(x) \\ A^0(\boldsymbol{x},t) &= e\int \mathrm{d}^3y\frac{\rho(\boldsymbol{y},t)}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} \end{aligned} \tag{5.3.3}$$

We want to derive the interaction Hamiltonian. Note

$$\int \mathrm{d}^3x\frac{1}{2}\boldsymbol{E}^2 = \int \mathrm{d}^3x\frac{1}{2}\left(\dot{\boldsymbol{A}} + \boldsymbol{\nabla}A^0\right)^2$$

cross terms vanish after integration by parts due to $\boldsymbol{\nabla} \cdot \dot{\boldsymbol{A}} = 0$

$$= \int \mathrm{d}^3 x \frac{1}{2}\left(\dot{\boldsymbol{A}}^2 + (\boldsymbol{\nabla} A^0)^2\right)$$

$$= \int \mathrm{d}^3 x \frac{1}{2}\left(\dot{\boldsymbol{A}}^0 - A^0 \boldsymbol{\nabla}^2 A^0\right)$$

$-e j^0 = -e\rho$

$$= \int \mathrm{d}^3 x \frac{1}{2}\dot{\boldsymbol{A}}^2 + \underbrace{\frac{e^2}{2}\int \mathrm{d}^3 x \mathrm{d}^3 y \frac{\rho(\boldsymbol{x})\rho(\boldsymbol{y})}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}}_{=\frac{e^2}{2} j^0 A_0}$$

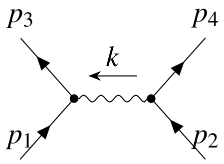
Combined Hamiltonian

$$\begin{aligned} H = \int \mathrm{d}^3 x &\left\{\frac{1}{2}\boldsymbol{\Pi}\cdot\boldsymbol{\Pi} + \frac{1}{2}\boldsymbol{B}\cdot\boldsymbol{B} + i\bar{\psi}\boldsymbol{\gamma}\cdot\boldsymbol{\nabla}\psi + m\bar{\psi}\psi\right\} \quad \text{free photon and fermion} \\ &+ \frac{e^2}{2}\int \mathrm{d}^3 x \mathrm{d}^3 y \frac{\rho(\boldsymbol{x})\rho(\boldsymbol{y})}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} - e\int \mathrm{d}^3 x \boldsymbol{j}\cdot\boldsymbol{A} \quad \text{interactions} \end{aligned}$$

where $\rho = \psi^\dagger\psi = \bar{\psi}\gamma^0\psi$, $\boldsymbol{j} = \bar{\psi}\boldsymbol{\gamma}\psi$ for 2 types of interactions.

5.4 Lorentz-invariant propagator

Consider $e^- e^-$ scattering at $O(e^2)$



$$k = p_3 - p_1 = p_2 - p_4$$

We expect this to involve

- spinors for external fermions
- $-ie\gamma^\mu$
- Photon propagator $D_{\mu\nu}(x-y)$

What we have found in Coulomb gauge is actually

- vertices $ie\gamma^i$, transverse propagator $D^{\mathrm{tr}}_{\mu\nu}(x-y)$
- vertices $\pm ie\gamma^0$, instantaneous Coumlomb interaction $\frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\delta(x^0-y^0)$

Effectively combine these propagators terms into $D^{\mathrm{Coul}}_{\mu\nu}(x-y)$, where the $D^{\mathrm{Coul}}_{00}(x-y) = \frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\delta(x^0-y^0)$. This component in momentum space is simply

$$\int \frac{\mathrm{d}^3 k}{(2\pi)^3}\frac{e^{i\boldsymbol{k}\cdot\boldsymbol{r}}}{|\boldsymbol{k}|^2} = \frac{1}{4\pi|\boldsymbol{r}|}$$

Therefore Coumlomb propagator in momentum space:

$$D_{\mu\nu}^{\mathrm{Coul}}(k) = \begin{cases} \frac{i}{k^2+i\epsilon}\left(\delta_{ij} - \frac{k_i k_j}{|\boldsymbol{k}|^2}\right) & \mu = i, \nu = j \\ \frac{i}{|\boldsymbol{k}|^2} & \mu = \nu = 0 \\ 0 & \text{otherwise} \end{cases}$$

Consider contraction to scattering amplitude from vertex at x:

$$\sim e\bar{u}(p_3)\gamma^\mu u(p_1) e^{i(p_3-p_1)x}$$

current conservation $\partial_\mu j^\mu = 0$ written in momentum space

$$\underbrace{(p_3 - p_1)_\mu}_{k_\mu} \bar{u}(p_3)\gamma^\mu u(p_1) = 0$$

so in the complete diagram $D_{\mu\nu}^{\mathrm{Coul}}$ occurs in a form

$$\begin{aligned} & a^\mu D_{\mu\nu}^{\mathrm{Coul}}(k) b^\nu \\ =& a^0 \frac{i}{|\boldsymbol{k}|^2} b^0 + a^i \left[\frac{i}{k^2 + i\epsilon}\left(\delta_{ij} - \frac{k_i k_j}{|\boldsymbol{k}|^2}\right)\right] b^j \end{aligned}$$

where $k^\mu a^\mu = 0, k_\mu a^\mu = 0$

$$\begin{aligned} =& i\left[\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{k^2} \underbrace{- \frac{k_0^2 a_0 b_0}{k^2|\boldsymbol{k}|^2} + \frac{a_0 b_0}{|\boldsymbol{k}|^2}}_{= \frac{-k_0^2 a_0 b_0 + a_0 b_0 (k_0^2 - |\boldsymbol{k}|^2)}{k^2 |\boldsymbol{k}|}{}^2}\right] \\ =& \frac{i}{k^2}(\boldsymbol{a}\cdot\boldsymbol{b} - a_0 b_0) = -\frac{i}{k} a_\mu b^\mu \end{aligned}$$

Conclusion in this diagram (and in fact, in general), we may replace the $D_{\mu\nu}^{\mathrm{Coul}}(k)$ by the manifestly Lorentz covariant propagator

$$D_{\mu\nu}(k) = -\frac{i g_{\mu\nu}}{k + i\epsilon} \tag{5.4.1}$$

This can be generalised to

$$D_{\mu\nu}(k) = -\frac{i}{k^2 + i\epsilon}\left(g_{\mu\nu} - (1-\alpha)\frac{k_\mu k_\nu}{k^2}\right) \tag{5.4.2}$$

as, by current consevation, additional term doesn't contribute.

Feynman gauge $\alpha = 1$; Landau gauge $\alpha = 0$.

Remark one can also try to quantise photons in a manifestly covariant way, imposing Lorentz gauge $\partial_\mu A^\mu = 0$

$$[A_\mu(\boldsymbol{x}), \Pi_\nu(\boldsymbol{y})] = i g_{\mu\nu} \delta^{(3)}(\boldsymbol{x} - \boldsymbol{y})$$

This is trouble since $\Pi^0 = \frac{\partial \mathcal{L}}{\partial \dot{A}_0} = 0$. This cannot hold!

We thus change the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\alpha}(\partial_\mu A^\mu)^2$$

with "gauge fixing term". The equation of motion from this is

$$\partial^2 A^\mu - \left(1 - \frac{1}{\alpha}\right)\partial^\mu(\partial_\lambda A^\lambda) = 0$$

e.g. $\alpha = 1$ is the Feynman gauge.

With this Lagrangian we can the 0th component of conjugate momentum

$$\Pi^0 = -\frac{1}{\alpha}\partial_\mu A^\mu$$

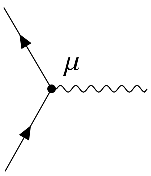
but this seems as bad as before!

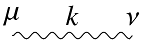
We cannot impose Coulomb gauge condition $\partial_\mu A^\mu = 0$ as an operator identity. Instead demand a weaker condition $\langle \text{out}|\partial_\mu A^\mu|\text{in}\rangle = 0$ for all physical states.

This in turn tells us which states are actually physical. The 4 polarisation states consist of physical, timelike(scalar) and longitudinal states. The negative-norm states cancel each ther out (Gopta-Bleuler formalism).

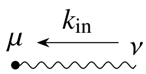
Feynman rules for QED diagrams constucted from electron (positron) and photon ; rules for fermions are valid as before.

In addition

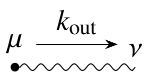
- vertex 

$= -ie\gamma^\mu$;
- photon propagator 

$= -\frac{ig_{\mu\nu}}{k^2+i\epsilon}$
- external photons



$= \epsilon_\mu$

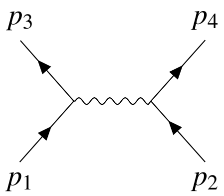


$= \epsilon_\nu^*$

ϵ_μ polarisation vector of in/out photon and ϵ_μ^* for out photon required for complex (circular) polarisation.

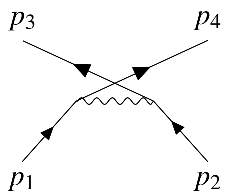
5.5 QED process at tree level

Example $e^-e^- \to e^-e^-$



$$\sim \frac{1}{(p_1 - p_3)^2} = \frac{1}{t}$$

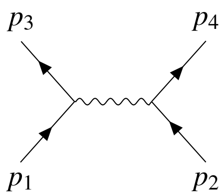
t-channel



$$\sim \frac{1}{(p_1 - p_4)^2} = \frac{1}{u}$$

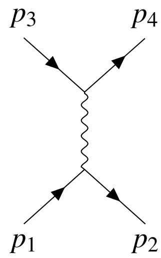
u-channel

Example $e^-e^+ \to e^-e^+$



$$\sim \frac{1}{(p_1 - p_3)^2} = \frac{1}{t}$$

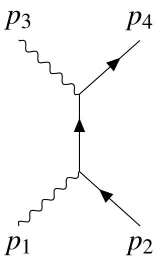
t-channel



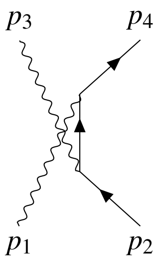
$$\sim \frac{1}{(p_1 + p_2)^2} = \frac{1}{s}$$

s-channel

Compton scattering $\gamma e^- \to \gamma e^-$

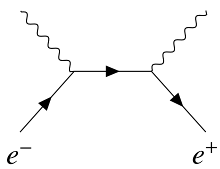


s-channel



u-channel

Example $e^+e^- \to \gamma\gamma$



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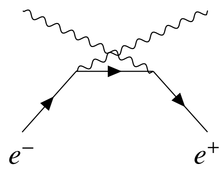
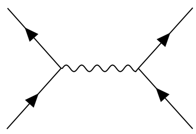
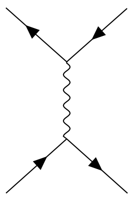


u-channel

These are important for lifetime of positronium.

All these amplitudes are $O(e^2)$, $\alpha = \frac{e^2}{4\pi} = \frac{1}{137.036}$ the fine structure constant.
Muons $\mu^\pm$, like electrons, just ca. 200 times heavier.

$e^-\mu^- \to e^-\mu^- \Rightarrow$  Just one diagram

$e^+e^- \to \mu^+\mu^- \Rightarrow$  Just one diagram

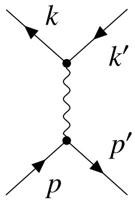
$\mu^\pm$ decay into $e^\pm$ and neutrinos in weak interactions.

For tree level diagrams the photon propagator does not need to have the $i\epsilon$ in the denominator, since we will never be able to see a singularity/pole.

5.5.1 Some hints and tricks for cross section calculations

See application in exercises!

Example $e^+e^- \to \mu^+\mu^-$



$$
\begin{aligned}
iM &= \bar{v}_e^{s'}(-ie\gamma^\mu)u_e^s(p)\,\frac{-ig_{\mu\nu}}{s}\bigg|_{s=q^2}\bar{u}_\mu^r(k)(-ie\gamma^\nu)v_\mu^{r'}(k') \\
&= \frac{ie^2}{s}\left(\bar{v}_e(p')\gamma^\mu u_e(p)\right)\left(\bar{u}_\mu(k)\gamma_\mu v_\mu(k')\right)
\end{aligned}
$$

See section **??**, $|M|^2$ is needed for cross section. M^* involves things like

$$
\begin{aligned}
(\bar{v}\gamma^\mu u)^* &= (\bar{v}\gamma^\mu u)^\dagger = u^\dagger\gamma^{\mu\dagger}\gamma_0^\dagger v \\
&= u^\dagger\gamma_0\gamma^\mu\gamma_0\gamma_0 v = \bar{u}\gamma^\mu v
\end{aligned}
$$

So

$$
|M|^2 = \frac{e^4}{s^2}\left[\bar{v}(p')\gamma^\mu u(p)\bar{u}(p)\gamma^\nu v(p')\right]_{e^\pm}\cdot\left[\bar{u}(k)\gamma_\mu v(p)\bar{v}(k')\gamma_\nu u(k)\right]_{\mu^\pm}
$$

Unpolarized scattering= $\frac{1}{4}\sum_{r,s,r',s'}|M|^2$.

Now $\bar{v}\gamma^\mu u$, $\bar{u}\gamma^\nu v$ etc. are scalars in Dirac/spinor space:

$$
\begin{aligned}
&\sum_{s,s'} \bar{v}_{s'} p' \gamma^\mu u_s(p) \bar{u}_s(p) \gamma^\nu v_{s'}(p') \\
\text{(taking trace of scalar)} &= \sum_{s,s'} \mathrm{Tr}\left(\bar{v}_{s'} p' \gamma^\mu u_s(p) \bar{u}_s(p) \gamma^\nu v_{s'}(p')\right) \\
&= \sum_{s,s'} \mathrm{Tr}\left(v_{s'}(p') \bar{v}_{s'}(p') \gamma^\mu u_s(p) \bar{u}_s(p) \gamma^\nu\right)
\end{aligned}
$$

using spin sums

$$
= \mathrm{Tr}\left((\not{p}' - m)\gamma^\mu(\not{p} + m)\gamma^\nu\right)
$$

Trace technology

- remember $\mathrm{Tr}\gamma_\mu = 0$
- $\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4g_{\mu\nu}$
- $\mathrm{Tr}(\text{odd number of } \gamma) = 0$
- $\mathrm{Tr}(\gamma_\mu\gamma_\nu\gamma_\alpha\gamma_\beta) = 4\left(g_{\mu\nu}g_{\alpha\beta} + g_{\mu\beta}g_{\nu\alpha} - g_{\mu\alpha}g_{\nu\beta}\right)$
- more rules involving γ_5 (weak interactions!)

So

$$
\mathrm{Tr}\left((\not{p}' - m)\gamma^\mu(\not{p} + m)\gamma^\nu\right) = 4\left(p'^\mu p^\nu + p'^\nu p^\mu - g^{\mu\nu}(p \cdot p' + m^2)\right)
$$

Mandelstam variables with 4 equal masses, center-of-mass system (CMS):

$$
p = (E, \boldsymbol{p}), \quad p' = (E, -\boldsymbol{p}), \quad k = (E, \boldsymbol{k}), \quad \theta = \measuredangle(\boldsymbol{p}, \boldsymbol{k})
$$

$$
s \overset{\text{CMS}}{=} (p + p')^2 = 4E^2 \tag{5.5.1}
$$

$$
t = (p - k)^2 = -(\boldsymbol{p} - \boldsymbol{k})^2 = -2|\boldsymbol{p}|^2(1 - \cos\theta) \tag{5.5.2}
$$

$$
u = (p' - k)^2 = -2|\boldsymbol{p}|^2(1 + \cos\theta) \tag{5.5.3}
$$

$$
|\boldsymbol{p}|^2 = E^2 - m^2 = \frac{s}{4} - m^2 \tag{5.5.4}
$$

Only 2 Mandelstam variables are independent.

$$
\begin{aligned}
s + t + u &= (p_1 + p_2)^2 + (p_1 - p_3)^2 + (p_1 - p_4)^2 \\
&= p_1^2 + p_2^2 + p_3^2 + p_4^2 + 2p_1\underbrace{(p_1 + p_2 - p_3 - p_4)}_{=0} \\
&= \sum_i m_i^2 = \text{const}
\end{aligned}
$$

photon polarisation sums Analogy to fermion spin sums before, Feynman rules for external photons involve $\epsilon^(*)_\mu$; e.g. Compton amplitude of the form

$$M \sim \epsilon^*_\mu(p_3)\epsilon_\nu(p_1)T^{\mu\nu}$$

Thus

$$\sum_{\text{spin, pol.}} |M|^2 = \sum_{\text{spin, pol.}} \epsilon^*_\mu(p_3)\epsilon_\alpha(p_3)\epsilon^*_\beta(p_1)\epsilon_\nu(p_1)T^{\mu\nu}T^{\alpha\beta^*}$$

How can we simplify $\sum_{pol}\epsilon^*_\mu(k)\epsilon_\nu(k)$? Again we have only 2 physical polarisation states, but want to do it in a covariant form.

Assume a simpler process (than Compton) with a single external photon, $\epsilon^*_\mu(k)M^\mu$. Choose

$$k^\mu = (k, 0, 0, k), \quad \epsilon^\mu_{(1)} = (0, 1, 0, 0), \quad \epsilon^\mu_{(2)} = (0, 0, 1, 0)$$

so $\sum_{\text{pol}} |\epsilon^*_\mu(k)M^\mu|^2 = |M_1|^2 + |M_2|^2$

Remember that photon coupled source j^μ, current sonvervation $\partial_\mu j^\mu = 0$. We will see (next term) this holds in general as Ward identity

$$k_\mu M^\mu = 0 \tag{5.5.5}$$

In exercises, show $p_{3\mu}T^{\mu\nu} = 0 = p_{1\nu}T^{\mu\nu}$ for Compton

Here $kM^0 - kM^3 = 0 \Rightarrow M^0 = M^3$ and we can rewrite

$$\sum_{pol}\epsilon^*_\mu\epsilon_\nu M^\mu M^{*\nu} = |M_1|^2 + |M_2|^2 + \underbrace{|M_3|^2 - |M_0|^2}_{=0} = -g_{\mu\nu}M^\mu M^{*\nu}$$

so effectively

$$\sum_{pol}\epsilon^*_\mu(k)\epsilon_\nu(k) = -g_{\mu\nu} \tag{5.5.6}$$

side remark

- KG propagator $\frac{i}{p^2-M^2+i\epsilon}$
- Dirac propagator $\frac{i(\not{p}+m)}{p^2-m^2+i\epsilon} = \frac{i\sum_s u_s(p)\bar{u}_s(p)}{p^2-m^2+i\epsilon}$
- Photon propagator $\frac{-ig_{\mu\nu}}{p^2+i\epsilon} = \frac{i\sum_{pol}\epsilon^*_\mu(p)\epsilon_\nu(p)}{p^2+i\epsilon}$