Formula for QFT I

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1 Classical field theory

1.1 Field theory in continuum

Euler-Lagrange-equation

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \tag{1.1}$$

momentum density

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}(\boldsymbol{x})} \tag{1.2}$$

Hamiltonian density

$$\mathcal{H}(\phi(\boldsymbol{x}), \pi(\boldsymbol{x})) = \pi(\boldsymbol{x})\dot{\phi}(\boldsymbol{x}) - \mathcal{L}(\phi, \partial_\mu \phi) \tag{1.3}$$

1.2 Noether Theorem

If a Lagrangian field theory has an infinitisimal symmetry, then there is an associated current j^μ, which is conserved.

$$\partial_\mu j^\mu = 0 \tag{1.4}$$

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta\phi - X^m u \tag{1.5}$$

Energy-momentum tensor (stress-energy tensor)

Asymmetric version

$$\Theta^\mu_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu_\nu \mathcal{L} \tag{1.6}$$

General version

$$T^{\mu\nu} = \Theta^{\mu\nu} + \partial_\lambda f^{\mu\nu\lambda} \tag{1.7}$$

with $f^{\lambda\mu\nu} = -f^{\mu\lambda\nu}$ or $\partial_\mu \partial \nu f^{\lambda\mu\nu} = 0$

2 Klein-Gordon theory

(Real) Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 \tag{2.1}$$

Quantization

$$\begin{aligned}[\phi(\boldsymbol{x}), \phi(\boldsymbol{x}')] = [\pi(\boldsymbol{x}), \pi(\boldsymbol{x}')] = 0 \\ [\phi(\boldsymbol{x}), \pi(\boldsymbol{x}')] = i\delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}')\end{aligned} \tag{2.2}$$

Decomposition into Fourier modes

$$\phi(\boldsymbol{x}) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(a_{\boldsymbol{p}}e^{i\boldsymbol{p}\cdot\boldsymbol{x}} + a^\dagger_{\boldsymbol{p}}e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\right) \tag{2.3}$$

$$\pi(\boldsymbol{x}) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3}(-i)\sqrt{\frac{E_p}{2}}\left(a_{\boldsymbol{p}}e^{i\boldsymbol{p}\cdot\boldsymbol{x}} - a^\dagger_{\boldsymbol{p}}e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\right) \tag{2.4}$$

thus the commutation relations for ladder operators:

$$\left[a_{\boldsymbol{p}}, a_{\boldsymbol{p}'}\right] = \left[a^\dagger_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{p}'}\right] = 0 \tag{2.5}$$

$$\left[a_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{p}'}\right] = (2\pi)^3\delta^{(3)}(\boldsymbol{p} - \boldsymbol{p}') \tag{2.6}$$

Hamiltonian in terms of ladder operator

$$H = \int \frac{\mathrm{d}^3 p}{(2\pi)^3}E_p\left(a_{\boldsymbol{p}}a^\dagger_{\boldsymbol{p}} + \frac{1}{2}\left[a_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{p}}\right]\right) \tag{2.7}$$

Normlisation it's also lorentz-invariante

$$\langle p|q\rangle = 2E_p(2\pi)^3\delta^{(3)}(\boldsymbol{p} - \boldsymbol{q}) \tag{2.8}$$

3 Heisenberg-picture fields

Heisenberg-picture

$$|\psi_H\rangle = e^{iHt}|\psi_s(t)\rangle \tag{3.1}$$

$$O_H(t) = e^{iHt}O_S\, e^{-iHt} \tag{3.2}$$

Field operator

$$\phi(x) = \phi(\boldsymbol{x}, t) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(a_p e^{ipx} + a^\dagger_p e^{-ipx}\right) \tag{3.3}$$

4 Commutations and propogators

Commutations

$$[\phi(x), \phi(y)] = D(x-y) - D(y-x) \begin{cases} = 0 & \text{if } (x-y) \text{ is space-like} \\ \neq 0 & \text{otherwise} \end{cases} \tag{4.1}$$

$$D(x-y) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip(x-y)} \tag{4.2}$$

Propogator

$$\langle 0|\phi(x)\phi(y)|0\rangle = D(x-y) \tag{4.3}$$

Feynman propagator

$$\begin{aligned} D_F(x-y) &= \langle 0|T\phi(x)\phi(y)|0\rangle \\ &= \Theta(x^0 - y^0)D(x-y) + \Theta(y^0 - x^0)D(y-x) \end{aligned} \tag{4.4}$$

$$D_F(x-y) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip(x-y)} \tag{4.5}$$

5 Quantization of the Dirac field

5.1 Dirac equation

$$\left(i\gamma^\mu \partial_\mu - m\right)\phi(x) = 0 \tag{5.1}$$

Standard representation (Dirac's)

$$\gamma_0 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix} \quad \boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix} \tag{5.2}$$

Lorentz transformation

$$\Lambda = \exp\left(\frac{1}{2}\omega_{\mu\nu} M^{\mu\nu}\right) \tag{5.3}$$

with ω set of parameters and M the generator of Lie algebra.

Spinor representation

$$S^{\rho\sigma} = \frac{1}{4}\left[\gamma^\rho, \gamma^\sigma\right] = \frac{1}{2i}\sigma^{\rho\sigma} \tag{5.4}$$

$$\tag{5.5}$$

Spinor transformation

$$S(\Lambda) = \exp\left(\frac{1}{2}\omega_{\mu\nu} S^{\mu\nu}\right) \tag{5.6}$$

$$\psi'_a(x) = S_{ab}(\Lambda)\psi_b(\Lambda^{-1}x) \tag{5.7}$$

adjoint spinor

$$\bar{\psi} = \psi^\dagger \gamma^0 \tag{5.8}$$

Fifth gamma matrix

$$\gamma^5 := i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{5.9}$$

$$\left\{\gamma^\mu, \gamma^5\right\} = 0 \tag{5.10}$$

$$(\gamma^5)^2 = \mathbb{1}_4 \tag{5.11}$$

Plane wavesolutions

$$\psi(x) = \begin{cases} u(p)e^{-ipx} & \text{positive frequency} \\ v(p)e^{ipx} & \text{negative frequency} \end{cases} \tag{5.12}$$

$$u_s(p) = \sqrt{E_p + m}\begin{pmatrix} \chi_s \\ \frac{\boldsymbol{u}\cdot\boldsymbol{p}}{E_p+m}\chi_s \end{pmatrix} e^{-ipx} v_s(p) = \sqrt{E_p + m}\begin{pmatrix} \frac{\boldsymbol{u}\cdot\boldsymbol{p}}{E_p+m}\tilde{\chi}_s \\ \tilde{\chi}_s \end{pmatrix} e^{ipx} \tag{5.13}$$

with

$$\chi_{\frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad x_{-\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$s = \pm\frac{1}{2} \quad \tilde{\chi}_s = \chi_{-s}$$

Orthogonality of spinor

$$\bar{u}_s(p)u_{s'}(p) = -\bar{v}_s(p)v_{s'}(p) = 2m\delta_{ss'} \tag{5.14}$$

$$\bar{u}_s(p)v_{s'}(p) = 0 \tag{5.15}$$

Spin sums

$$\sum_s u_s(p)\bar{u}_s(p) = \not{p} + m \tag{5.16}$$

$$\sum_s v_s(p)\bar{v}_s(p) = \not{p} - m \tag{5.17}$$

5.2 Dirac Lagrangian and quantization

$$\mathcal{L} = \bar{\psi}(x)(i\not{\partial} - m)\psi(x) \tag{5.18}$$

Quantization

$$\left\{\psi_a(\boldsymbol{x}), \psi_b^\dagger(\boldsymbol{x}')\right\} = \delta_{ab}\delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}') \tag{5.19}$$

$$\{\psi_a(\boldsymbol{x}), \psi_b(\boldsymbol{x}')\} = \left\{\psi_a^\dagger(\boldsymbol{x}), \psi_b^\dagger(\boldsymbol{x}')\right\} = 0 \tag{5.20}$$

Field operators

$$\psi(\boldsymbol{x}) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3 \sqrt{2E_p}} \sum_s (a^s_{\boldsymbol{p}} u_s(\boldsymbol{p}) e^{i\boldsymbol{p}\cdot\boldsymbol{x}} + b^{s\,\dagger}_{\boldsymbol{p}} v_s(\boldsymbol{p}) e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}) \tag{5.21}$$

thus the anticommutations of ladder operators:

$$\begin{aligned} \left\{a^s_{\boldsymbol{p}}, a^{s'\,\dagger}_{\boldsymbol{p'}}\right\} = \left\{b^s_{\boldsymbol{p}}, b^{s'\,\dagger}_{\boldsymbol{p'}}\right\} &= (2\pi)^3 \delta_{ss'} \delta^{(3)}(\boldsymbol{p} - \boldsymbol{p}') \\ \{a, a\} = \left\{a^\dagger, a^\dagger\right\} &= \cdots = 0 \end{aligned}$$

Hamiltonian in terms of Fourier modes (with normal ordering)

$$H = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \sum_s E_p (a^{s\,\dagger}_{\boldsymbol{p}} a^s_{\boldsymbol{p}} - b^{s\,\dagger}_{\boldsymbol{p}} b^s_{\boldsymbol{p}}) \tag{5.22}$$

5.3 Particles and antiparticles

$$Q = e \int \mathrm{d}^3 x \psi^\dagger(x)\psi(x) \tag{5.23}$$

$$: Q : = e \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \sum_s (a^{s\,\dagger}_{\boldsymbol{p}} a^s_{\boldsymbol{p}} - b^{s\,\dagger}_{\boldsymbol{p}} b^s_{\boldsymbol{p}}) \tag{5.24}$$

5.4 Dirac propagator and anticommutators

$$\begin{aligned} S_{ab}(x-y) &= \{\psi_a(x), \bar{\psi}_b(y)\} \\ &= (i\partial\!\!\!/ + m)\left[D(x-y) - D(y-x)\right] \end{aligned} \tag{5.25}$$

Time ordering of Dirac fields

$$T(\phi_a(x)\bar{\psi}_b(y)) = \Theta(x^0 - y^0)\psi_a(x)\bar{\psi}_b(y) - \Theta(y^0 - x^0)\bar{\psi}_b(y)\psi_a(x) \tag{5.26}$$

Feynman propogator for the Dirac field

$$S_F(x-y) = \langle 0|T\psi(x)\bar{\psi}(y)|0\rangle = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i(p\!\!\!/ + m)}{p^2 - m^2 + i\epsilon} e^{-ip\cdot(x-y)} \tag{5.27}$$

5.5 Discrete symmetries of the Dirac Field

	orientation perserving	orientation not perserving
(ortho)chronous	$\mathcal{L}^\uparrow_+$	$\mathcal{L}^\uparrow_- = \mathcal{P}\mathcal{L}^\uparrow_+$
non-orthochronous	$\mathcal{L}^\downarrow_- = \mathcal{T}\mathcal{L}^\uparrow_+$	$\mathcal{L}^\downarrow_+ = \mathcal{P}\mathcal{T}\mathcal{L}^\uparrow_+$