

Quantum Field Theory

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1 Classical field theory

1.1 Field theory in continuum

Euler-Lagrange-equation

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \quad (1.1.1)$$

momentum density

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}(x)} \quad (1.1.2)$$

Hamiltonian density

$$\mathcal{H}(\phi(x), \pi(x)) = \pi(x) \dot{\phi}(x) - \mathcal{L}(\phi, \partial_\mu \phi) \quad (1.1.3)$$

1.2 Noether Theorem

If a Lagrangian field theory has an infinitesimal symmetry, then there is an associated current j^μ , which is conserved.

$$\partial_\mu j^\mu = 0 \quad (1.2.1)$$

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta \phi - X^\mu \mathcal{L} \quad (1.2.2)$$

Energy-momentum tensor (stress-energy tensor)

Asymmetric version

$$\Theta^\mu_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu_\nu \mathcal{L} \quad (1.2.3)$$

General version

$$T^{\mu\nu} = \Theta^{\mu\nu} + \partial_\lambda f^{\mu\nu\lambda} \quad (1.2.4)$$

with $f^{\lambda\mu\nu} = -f^{\mu\lambda\nu}$ or $\partial_\mu \partial_\nu f^{\lambda\mu\nu} = 0$

2 Klein-Gordon theory

(Real) Lagrangian density

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2} \phi^2 \quad (2.0.1)$$

Quantization

$$\begin{aligned} [\phi(\mathbf{x}), \phi(\mathbf{x}')] &= [\pi(\mathbf{x}), \pi(\mathbf{x}')] = 0 \\ [\phi(\mathbf{x}), \pi(\mathbf{x}')] &= i\delta^{(3)}(\mathbf{x} - \mathbf{x}') \end{aligned} \quad (2.0.2)$$

Decomposition into Fourier modes

$$\phi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} + a_{\mathbf{p}}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}} \right) \quad (2.0.3)$$

$$\pi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{E_p}{2}} \left(a_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} - a_{\mathbf{p}}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}} \right) \quad (2.0.4)$$

thus the commutation relations for ladder operators:

$$[a_{\mathbf{p}}, a_{\mathbf{p}'}] = [a_{\mathbf{p}}^\dagger, a_{\mathbf{p}'}^\dagger] = 0 \quad (2.0.5)$$

$$[a_{\mathbf{p}}, a_{\mathbf{p}'}^\dagger] = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{p}') \quad (2.0.6)$$

Hamiltonian in terms of ladder operator

$$H = \int \frac{d^3p}{(2\pi)^3} E_p \left(a_{\mathbf{p}} a_{\mathbf{p}}^\dagger + \frac{1}{2} [a_{\mathbf{p}}, a_{\mathbf{p}}^\dagger] \right) \quad (2.0.7)$$

Normalisation it's also lorentz-invariante

$$\langle p|q \rangle = 2E_p (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) \quad (2.0.8)$$

2.1 Heisenberg-picture fields

Heisenberg-picture

$$|\psi_H\rangle = e^{iHt} |\psi_S(t)\rangle \quad (2.1.1)$$

$$O_H(t) = e^{iHt} O_S e^{-iHt} \quad (2.1.2)$$

Field operator

$$\phi(x) = \phi(\mathbf{x}, t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_{\mathbf{p}} e^{ipx} + a_{\mathbf{p}}^\dagger e^{-ipx} \right) \quad (2.1.3)$$

2.2 Commutations and propagators

Commutations

$$[\phi(x), \phi(y)] = D(x-y) - D(y-x) \begin{cases} = 0 & \text{if } (x-y) \text{ is space-like} \\ \neq 0 & \text{otherwise} \end{cases} \quad (2.2.1)$$

$$D(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip(x-y)} \quad (2.2.2)$$

Propogator

$$\langle 0 | \phi(x) \phi(y) | 0 \rangle = D(x-y) \quad (2.2.3)$$

Feynman propagator

$$\begin{aligned} D_F(x-y) &= \langle 0 | T \phi(x) \phi(y) | 0 \rangle \\ &= \Theta(x^0 - y^0) D(x-y) + \Theta(y^0 - x^0) D(y-x) \end{aligned} \quad (2.2.4)$$

$$D_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip(x-y)} \quad (2.2.5)$$

3 Quantization of the Dirac field

3.1 Dirac equation

$$(i\gamma^\mu \partial_\mu - m)\phi(x) = 0 \quad (3.1.1)$$

Standard representation (Dirac's)

$$\gamma_0 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix} \quad \boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix} \quad (3.1.2)$$

Lorentz transformation

$$\Lambda = \exp\left(\frac{1}{2}\omega_{\mu\nu}M^{\mu\nu}\right) \quad (3.1.3)$$

with ω set of parameters and M the generator of Lie algebra.

Spinor representation

$$S^{\rho\sigma} = \frac{1}{4} [\gamma^\rho, \gamma^\sigma] = \frac{1}{2i} \sigma^{\rho\sigma} \quad (3.1.4)$$

$$(3.1.5)$$

Spinor transformation

$$S(\Lambda) = \exp\left(\frac{1}{2}\omega_{\mu\nu}S^{\mu\nu}\right) \quad (3.1.6)$$

$$\psi'_a(x) = S_{ab}(\Lambda)\psi_b(\Lambda^{-1}x) \quad (3.1.7)$$

adjoint spinor

$$\bar{\psi} = \psi^\dagger \gamma^0 \quad (3.1.8)$$

Fifth gamma matrix

$$\gamma^5 := i\gamma^0\gamma^1\gamma^2\gamma^3 \quad (3.1.9)$$

$$\{\gamma^\mu, \gamma^5\} = 0 \quad (3.1.10)$$

$$(\gamma^5)^2 = \mathbb{1}_4 \quad (3.1.11)$$

Plane wavesolutions

$$\psi(x) = \begin{cases} u(p)e^{-ipx} & \text{positive frequency} \\ v(p)e^{ipx} & \text{negative frequency} \end{cases} \quad (3.1.12)$$

$$u_s(p) = \sqrt{E_p + m} \begin{pmatrix} \chi_s \\ \frac{\mathbf{u} \cdot \mathbf{p}}{E_p + m} \chi_s \end{pmatrix} e^{-ipx} \quad v_s(p) = \sqrt{E_p + m} \begin{pmatrix} \frac{\mathbf{u} \cdot \mathbf{p}}{E_p + m} \tilde{\chi}_s \\ \tilde{\chi}_s \end{pmatrix} e^{ipx} \quad (3.1.13)$$

with

$$\chi_{\frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \chi_{-\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$s = \pm \frac{1}{2} \quad \tilde{\chi}_s = \chi_{-s}$$

Orthogonality of spinor

$$\bar{u}_s(p)u_{s'}(p) = -\bar{v}_s(p)v_{s'}(p) = 2m\delta_{ss'} \quad (3.1.14)$$

$$\bar{u}_s(p)v_{s'}(p) = 0 \quad (3.1.15)$$

Spin sums

$$\sum_s u_s(p)\bar{u}_s(p) = \not{p} + m \quad (3.1.16)$$

$$\sum_s v_s(p)\bar{v}_s(p) = \not{p} - m \quad (3.1.17)$$

3.2 Dirac Lagrangian and quantization

$$\mathcal{L} = \bar{\psi}(x)(i\not{\partial} - m)\psi(x) \quad (3.2.1)$$

Quantization

$$\{\psi_a(\mathbf{x}), \psi_b^\dagger(\mathbf{x}')\} = \delta_{ab}\delta^{(3)}(\mathbf{x} - \mathbf{x}') \quad (3.2.2)$$

$$\{\psi_a(\mathbf{x}), \psi_b(\mathbf{x}')\} = \{\psi_a^\dagger(\mathbf{x}), \psi_b^\dagger(\mathbf{x}')\} = 0 \quad (3.2.3)$$

Field operators

$$\psi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3 \sqrt{2E_p}} \sum_s (a_{\mathbf{p}}^s u_s(\mathbf{p}) e^{i\mathbf{p} \cdot \mathbf{x}} + b_{\mathbf{p}}^{s\dagger} v_s(\mathbf{p}) e^{-i\mathbf{p} \cdot \mathbf{x}}) \quad (3.2.4)$$

thus the anticommutations of ladder operators:

$$\{a_{\mathbf{p}}^s, a_{\mathbf{p}'}^{s'\dagger}\} = \{b_{\mathbf{p}}^s, b_{\mathbf{p}'}^{s'\dagger}\} = (2\pi)^3 \delta_{ss'} \delta^{(3)}(\mathbf{p} - \mathbf{p}')$$

$$\{a, a\} = \{a^\dagger, a^\dagger\} = \dots = 0$$

Hamiltonian in terms of Fourier modes (with normal ordering)

$$H = \int \frac{d^3 p}{(2\pi)^3} \sum_s E_p (a_p^{s\dagger} a_p^s - b_p^{s\dagger} b_p^s) \quad (3.2.5)$$

3.3 Particles and antiparticles

$$Q = e \int d^3 x \psi^\dagger(x) \psi(x) \quad (3.3.1)$$

$$: Q : = e \int \frac{d^3 p}{(2\pi)^3} \sum_s (a_p^{s\dagger} a_p^s - b_p^{s\dagger} b_p^s) \quad (3.3.2)$$

3.4 Dirac propagator and anticommutators

$$\begin{aligned} S_{ab}(x-y) &= \{\psi_a(x), \bar{\psi}_b(y)\} \\ &= (i\not{\partial} + m) [D(x-y) - D(y-x)] \end{aligned} \quad (3.4.1)$$

Time ordering of Dirac fields

$$T(\phi_a(x) \bar{\psi}_b(y)) = \Theta(x^0 - y^0) \psi_a(x) \bar{\psi}_b(y) - \Theta(y^0 - x^0) \bar{\psi}_b(y) \psi_a(x) \quad (3.4.2)$$

Feynman propagator for the Dirac field

$$S_F(x-y) = \langle 0 | T \psi(x) \bar{\psi}(y) | 0 \rangle = \int \frac{d^4 p}{(2\pi)^4} \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon} e^{-ip \cdot (x-y)} \quad (3.4.3)$$

3.5 Discrete symmetries of the Dirac Field

	orientation preserving	orientation not perserving
(ortho)chronous	\mathcal{L}_+^\uparrow	$\mathcal{L}_-^\uparrow = \mathcal{P} \mathcal{L}_+^\uparrow$
non-orthochronous	$\mathcal{L}_-^\downarrow = \mathcal{T} \mathcal{L}_+^\uparrow$	$\mathcal{L}_+^\downarrow = \mathcal{PT} \mathcal{L}_+^\uparrow$

4 Interacting QFT

4.1 Introduction and examples

Theories discussed so far are Klein-Gordon theory (spin 0)

$$\mathcal{L}_{KG} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2} \phi^2$$

and Dirac theory (spin $\frac{1}{2}$)

$$\mathcal{L}_D = \bar{\psi}(i\partial\!\!\!/ - m)\psi$$

There is also $\mathcal{L}_{EM} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ for a massless vector field. Its quantisation gives photon

One thing they have in common is quadratic in the fields. As result:

- linear field equations
- exact quantisation
- multi-particle states without scattering or interaction
- linear fourier decompositions , no momentum changes

To have an interacting theory with scattering, need higher powers in the field in the Lagrangians. A few examples are following

scalar ϕ^4 theory

$$\mathcal{L} = \mathcal{L}_{KG} + \frac{\lambda}{4!} \phi^4$$

need positive sign $\lambda > 0$ for a stable theory, otherwise classical energy can be arbitrarily negative.

Equation of motions

$$(\partial^2 + m^2)\phi = -\frac{\lambda}{3!} \phi^3$$

is nonlinear, cannot be solved by Fourier decomposition.

Yukawa-theory

$$\mathcal{L} = \mathcal{L}_{KG} + \mathcal{L}_D - g\bar{\psi}\psi\phi$$

It is originally developed as a theory for nuclear forces with ψ nucleon, ϕ pion. In the Standard Model it is similar to interactions in Higgs mechanism.

Quantum Electrodynamics (QED)

$$\mathcal{L} = \mathcal{L}_{EM} + \mathcal{L}_D - eA_\mu \bar{\psi} \gamma^\mu \psi$$

describes electrons, their antiparticles positrons and photons.

Yang-Mills theory generalises \mathcal{L}_{EM} with terms like A^4 or $A^2 \partial A$

Scalar QED describes pions and photons

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_{EM} + D_\mu \phi D^\mu \phi^* - m^2 |\phi|^2 \\ &= \mathcal{L}_{EM} + \partial_\mu \phi \partial^\mu \phi^* - m^2 \phi \phi^* + ieA_\mu (\phi \partial^\mu \phi^* - \phi^* \partial^\mu \phi) + e^2 A_\mu A^\mu \phi \phi^* \end{aligned}$$

Remarks

1. Interaction terms in $H_{\text{int}} = \int d^3x \mathcal{H}_{\text{int}} = - \int d^3x \mathcal{L}_{\text{int}}$ always involves products of fields at the same point \mathbf{x} . It ensures causality, no "instant at a distance".
2. There are no derivative interactions. These may complicate quantisation as

$$\pi(\mathbf{x}) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi(\mathbf{x}))}$$

3. Why are we taking the examples above? There must be zillions of theories (Lagrangians)? We have the criterion of **renormalizability**. Note the mass dimensions of fields;

$$[S] = 1 \text{ so } [\mathcal{L}] = [M]^4 \Rightarrow [\phi] = [M], [\psi] = [M]^{\frac{3}{2}}, [A_\mu] = [M]$$

So in all the interaction terms indicated above, the coupling constant λ , e , g are all **dimensionless**! Can add $-\frac{\mu}{3!}\phi^3$ to the ϕ^4 theory. This leads to $[\mu] = [M]$ and all these generate renormalisable interactions.

All higher interaction terms require coupling constants of **negative** mass dimension, e.g. $G\bar{\psi}\psi\bar{\psi}\psi$ and then $[G] = [M]^{-2}$. These are nonrenormalisable and create trouble when performing higher-order calculation in perturbation theory. (with energy cutoff; corrections $G\Lambda^2$, $\Lambda \rightarrow \infty$)

4. we haven't quantised the photon yet. The reason is that its is a vector field, i.e. 4 degrees of freedom, but photon has just 2 physical polarisaion states. It is linked to gauge symmetry and complicates quantisation somewhat.

4.2 The interaction picture

Consider the ϕ^4 theory,

$$\mathcal{L}_{\text{int}} = -\frac{\lambda}{4!} \phi(x)^4 \quad (4.2.1)$$

Hamiltonian $H = H_0 + H_{\text{int}}$ with

$$H_0 = \int d^3x \left\{ \frac{1}{2} \pi^2(x) + \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} m^2 \phi^2 \right\} \quad (4.2.2)$$

$$H_{\text{int}} = - \int d^3x \mathcal{L}_{\text{int}} = \frac{\lambda}{4!} \int d^3x \phi^4 \quad (4.2.3)$$

Interaction picture means that operators evolve in time using H_0 (only), in particular

$$\phi_I(t, \mathbf{x}) = e^{iH_0 t} \phi(\mathbf{x}) e^{-iH_0 t} \quad (4.2.4)$$

Time-dependence of the free field, obeys classical equation of motion $(\partial^2 + m^2)\phi_I(t, \mathbf{x}) = 0$. Solution in terms of fourier modes as before:

$$\phi_I = \int \frac{d^3 p}{(2\pi)^3 \sqrt{2E_p}} (a_{\mathbf{p}}^I e^{-ipx} + a_{\mathbf{p}}^{I\dagger} e^{+ipx}) \quad (4.2.5)$$

as in the free theory with standard commutation relations $[a_{\mathbf{p}}^I, a_{\mathbf{p}'}^{I\dagger}] = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{p}')$. The state satisfying $a_{\mathbf{p}}^I |0\rangle = 0$ is the vacuum of the free, noninteracting theory.

Relation between interaction and Schrödinger picture states:

$$|\phi_I(t)\rangle = e^{iH_0 t} |\psi_S(t)\rangle \quad (4.2.6)$$

Schrödinger equation becomes:

$$\begin{aligned} i \frac{\partial}{\partial t} |\psi_S\rangle &= (H_0 + H_{\text{int}}) |\psi_S\rangle \\ \text{LHS} &= i \frac{\partial}{\partial t} (e^{-iH_0 t} |\phi_I\rangle) = H_0 e^{-iH_0 t} |\phi_I\rangle + e^{-iH_0 t} i \frac{\partial}{\partial t} |\phi_I\rangle \\ \text{RHS} &= (H_0 + H_{\text{int}}) e^{-iH_0 t} |\phi_I\rangle \\ \Rightarrow i \frac{\partial}{\partial t} |\phi_I\rangle &= e^{iH_0 t} H_{\text{int}} e^{-iH_0 t} |\phi_I\rangle = H_I(t) |\phi_I\rangle \end{aligned} \quad (4.2.7)$$

with H_I interaction Hamiltonian in the interaction picture. Clearly

$$H_I = \frac{\lambda}{4!} \int d^3 x \phi_I^4(x)$$

What is the solution of 4.2.7 for the time evolution of $|\phi_I(t)\rangle$? Define time-evolution operator in the interaction picture.

$$|\phi_I(t)\rangle = U(t, t_0) |\phi_I(t_0)\rangle \quad (4.2.8)$$

$$\text{where } U(t, t_0) = e^{iH_0(t-t_0)} e^{-iH(t-t_0)} \quad (4.2.9)$$

With 4.2.7 and 4.2.8:

$$i \frac{\partial}{\partial t} U(t, t_0) = H_I(t) U(t, t_0) \quad (4.2.10)$$

To solve with boundary conditions: $U(t_0, t_0) = \mathbb{1}$. The formal solution:

$$U(t, t_0) = 1 - i \int_{t_0}^t dt' H_I(t') U(t', t_0)$$

Substitute back in and we get:

$$U(t, t_0) = 1 - i \int_{t_0}^t dt' H_I(t') + (-i)^2 \int_{t_0}^t dt' \int_{t_0}^{t'} dt'' H_I(t') H_I(t'') + \dots \quad (4.2.11)$$

Ranges of integration: H_I in the product is automatically time-ordered.

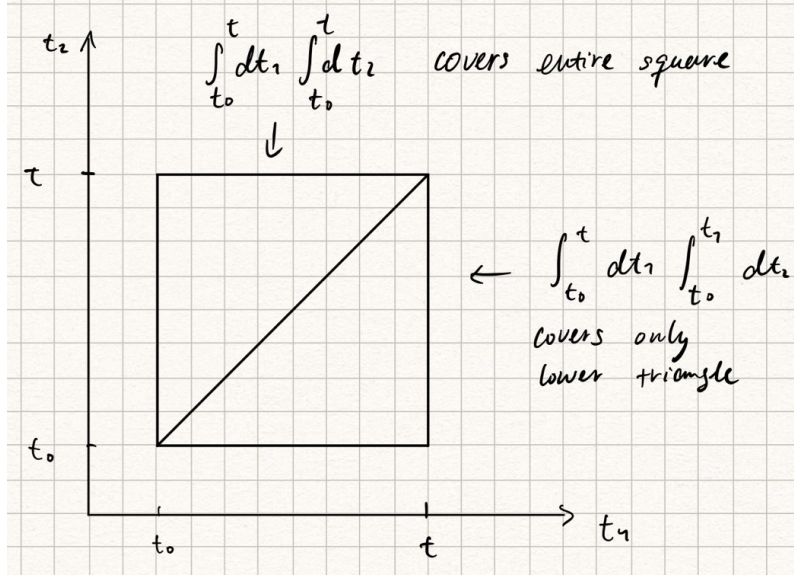


Figure 4.1: Time ordering

Upper triangle has the wrong time order. We are going to "repair" it by hand.

$$\begin{aligned}
 U(t, t_0) &= 1 - i \int_{t_0}^t dt' H_I(t') + \frac{(-i)^2}{2} \int_{t_0}^t dt' \int_{t_0}^{t'} dt'' T(H_I(t') H_I(t'')) + \dots \\
 &= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{t_0}^t dt_1 \dots \int_{t_0}^t dt_n T(H_I(t_1) \dots H_I(t_n)) \\
 &= T \exp \left\{ -i \int_{t_0}^t dt' H_I(t') \right\}
 \end{aligned} \tag{4.2.12}$$

It is interesting for scattering to transition into asymptotic state for $t \rightarrow \infty$

$$\begin{aligned}
 S &= \lim_{t \rightarrow \infty} U(t, -t) = T \exp \left\{ -i \int_{-\infty}^{\infty} dt H_I(t) \right\} \\
 &\stackrel{\phi^4}{=} T \exp \left\{ -i \int d^4x \frac{\lambda}{4!} \phi_I^4(x) \right\}
 \end{aligned} \tag{4.2.13}$$

Both U and S are formally unitary

Composition law for time evolution operator:

$$U(t_2, t_0) = U(t_2, t_1)U(t_1, t_0) = U(t_2, t_1)U(t_0, t_1)^\dagger \tag{4.2.14}$$

4.2.1 Scattering amplitudes and the S-matrix

Take $|i\rangle$ the initial (multi-particle) state and $|f\rangle$ the final (multi-particle) state. Time evolution of $|i\rangle$ then is

$$\lim(t \rightarrow \infty) U(t, -\infty) |i\rangle = S |i\rangle$$

Probability that $|i\rangle$ evolves into $|f\rangle$ is proportional to the squared "**S-matrix element**"

$$|\langle f, t \rightarrow \infty | i, t \rightarrow -\infty \rangle|^2 = |\langle f | S | i \rangle|^2 = |S_{fi}|^2 \tag{4.2.15}$$

The nontrivial part of the S-matrix is the T-matrix:

$$S_{fi} := \delta_{fi} + iT_{fi} \quad (4.2.16)$$

Use momentum conservation (from translation invariance) to define matrix element

$$S_{fi} = \delta_{fi} + i(2\pi)^4 \delta^{(4)}(p_f - p_i) M_{fi} \quad (4.2.17)$$

M_{fi} measures "genuine scattering" from $|i\rangle$ to $|f\rangle$.

How are we going to calculate correlation functions in the interacting theory:

$$\langle \Omega | T \phi(x) \phi(y) | \Omega \rangle \quad (4.2.18)$$

or more generally $\langle \Omega | T \phi(x_1) \phi(x_2) \dots | \Omega \rangle$, where $|\Omega\rangle$ is the vacuum/ground state of the interacting theory and $\phi(x)$ the Heisenberg operators.

Ignore $|\Omega\rangle \neq |0\rangle$ for the moment other than saying: we want to study the time evolution from the vacuum at $t \rightarrow -\infty$ to $t \rightarrow +\infty$. So rewriting in terms $\phi_I(x)$, assuming $x^0 > y^0$ for now:

$$\langle 0 | U(\infty, x^0) \phi_I(x^0) U(x^0, y^0) \phi_I(y^0) U(y^0, -\infty) | 0 \rangle = \langle 0 | T(\phi_I(x) \phi_I(y) S) | 0 \rangle \quad (4.2.19)$$

still holds if $x^0 < y^0$ because of T .

Now $|\Omega\rangle \neq |0\rangle$: this can be taken care of by dividing out the time evolution of the (free) vacuum $\langle 0 | S | 0 \rangle$, so

$$\begin{aligned} \langle \Omega | T(\phi(x) \phi(y)) | \Omega \rangle &= \frac{\langle 0 | T(\phi_I(x) \phi_I(y) S) | 0 \rangle}{\langle 0 | S | 0 \rangle} \\ &\stackrel{\phi^4}{=} \frac{\langle 0 | T \phi_I(x) \phi_I(y) \exp\left\{-i \int d^4 x' \frac{\lambda}{4!} \phi^4(x')\right\} | 0 \rangle}{\langle 0 | T \exp\left\{-i \int d^4 x' \frac{\lambda}{4!} \phi^4(x')\right\} | 0 \rangle} \end{aligned} \quad (4.2.20)$$

Proof can be found in Peskin. It will also be illustrated parctically later ("vacuum bubbles").

Perturbation theory is viable when λ (or some other coupling) is "small" and then expands $U(t, t_0)$ or S in powers of λ .

4.3 Wick's theorem

From now on drop the subscript for interaction picture fields $\phi_I(x) \rightarrow \phi(x)$.

Want to calculate stuff like $\langle 0 | T \phi(x_1) \dots \phi(x_n) S | 0 \rangle$ in perturbation theory; so e.g. at order λ^n . So

$$\frac{1}{n!} \left(-i \frac{\lambda}{4!}\right)^n \int d^4 y_1 \dots d^4 y_n \langle 0 | T \phi(x_1) \dots \phi(x_n) \phi^4(y_1) \dots \phi^4(y_n) | 0 \rangle \quad (4.3.1)$$

is tough!

We know $\langle 0 | T \phi(x_1) \phi(x_2) | 0 \rangle$ is the Feynman propagator!

Recall **normal ordering** with $\phi(x) = \phi^+(x) + \phi^-(x)$

$$: \phi^+ \phi^- :=: \phi^- \phi^+ :=: \phi^- \phi^+ \quad (4.3.2)$$

Wick's therem expresses time-ordered products in terms of normal-ordered ones. Then it is easy to take vacuum expectation values, as $\langle 0 | : \phi(x_1) \dots \phi(x_n) : | 0 \rangle = 0$

Take two fields and $x^0 > y^0$:

$$\begin{aligned} T\phi(x)\phi(y) &= \phi(x)\phi(y) = (\phi^+(x) + \phi^-(x))(\phi^+(y) + \phi^-(y)) \\ &= \phi^+(x)\phi^+(y) + \phi^-(x)\phi^-(y) + \phi^-(x)\phi^+(y) + \phi^+(x)\phi^-(y) + [\phi^+(x), \phi^-(y)] \\ &=: \phi(x)\phi(y) : + [\phi^+(x), \phi^-(y)] \end{aligned}$$

Particularly for $y^0 > x^0$:

$$T\phi(x)\phi(y) =: \phi(x)\phi(y) : + [\phi^+(y), \phi^-(x)]$$

Thus altogether:

$$T\phi(x)\phi(y) =: \phi(x)\phi(y) : + D_F(x - y) \quad (4.3.3)$$

as $\Theta(x^0 - y^0)[\phi^+(x), \phi^-(y)] + \Theta(y^0 - x^0)[\phi^+(y), \phi^-(x)] = D_F(x - y)$.

Worth noting that $D_F(x - y)$ is still a c-number, not operator (yet). Thus it can be pulled out of any matrix element or expectation value.

We now define "contraction":

$$\overline{\phi(x_1)\phi(x_2)} = D_F(x_1 - x_2) \quad (4.3.4)$$

Thus we can remove the fields from the product leaving only the propagators:

$$T\phi(x)\phi(y) =: \phi(x)\phi(y) : + \overline{\phi(x)\phi(y)} \quad (4.3.5)$$

General form of **Wick's theorem** for arbitrary number of fields

$$T\phi(x_1) \dots \phi(x_n) =: \phi(x_1) \dots \phi(x_n) : + : (\text{sum over all possible contractions}) : \quad (4.3.6)$$

Example with four fields:

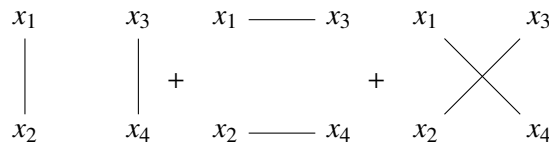
$$T(\phi_1\phi_2\phi_3\phi_4) =: \phi_1\phi_2\phi_3\phi_4 :$$

$$\begin{aligned} &+ \overline{\phi_1\phi_2} : \phi_3\phi_4 : + \overline{\phi_1\phi_3} : \phi_2\phi_4 : + \overline{\phi_1\phi_4} : \phi_2\phi_3 : + \overline{\phi_2\phi_3} : \phi_1\phi_4 : + \overline{\phi_2\phi_4} : \phi_1\phi_3 : + \overline{\phi_3\phi_4} : \phi_1\phi_2 : \\ &+ \overline{\phi_1\phi_2}\overline{\phi_3\phi_4} + \overline{\phi_1\phi_3}\overline{\phi_2\phi_4} + \overline{\phi_1\phi_4}\overline{\phi_2\phi_3} \end{aligned}$$

Thus

$$\langle 0 | T(\phi_1\phi_2\phi_3\phi_4) | 0 \rangle = D_F(x_1 - x_2)D_F(x_3 - x_4) + D_F(x_1 - x_3)D_F(x_2 - x_4) + D_F(x_1 - x_4)D_F(x_2 - x_3)$$

which can be visually represented as



Proof of the general theorem by *induction* in the number of fields (see exercise). The idea is to suppose it is true for $\phi_2 \dots \phi_m$, $x_1^0 > x_{k>1}^0$. Then

$$\begin{aligned} T\phi_1\phi_2 \dots \phi_m &= (\phi_1^+ + \phi_1^-)T\phi_2 \dots \phi_m \\ &= (\phi_1^+ + \phi_1^-)[: \phi_2 \dots \phi_m : + : \text{contractions} :] \end{aligned}$$

ϕ_1^- can stay as it is part of $(: \phi_1\phi_2 \dots \phi_m :)$. But ϕ_1^+ needs to be comuted past all ϕ_1^- operators, giving rise to additional contractions $\overline{\phi_1\phi_2}$.

Consequences

- $n = 2k + 1, k \in \mathbb{N}$

$$\langle 0|T\phi_1 \dots \phi_m|0\rangle = 0$$

- $n = 2k, k \in \mathbb{N}$

$$\langle 0|T\phi_1 \dots \phi_m|0\rangle = \sum_{\text{pairing of fields}} D_F(x_{i_1} - x_{i_2}) \dots D_F(x_{i_{m-1}} - x_{i_m})$$

4.3.1 Wick's theorem and the S-Matrix

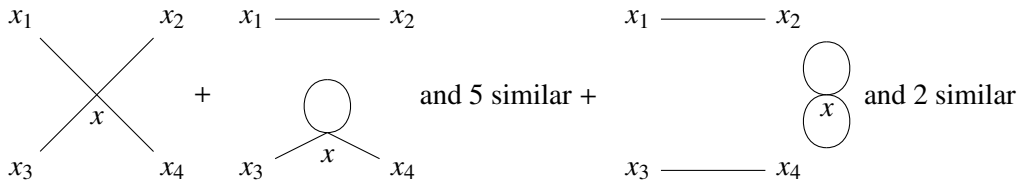
Apply Wick's theorem to correlation functions $\langle 0|T\{\phi_1 \dots \phi_m\}S|0\rangle$ n-th term in the perturbative expansion of S with $\phi(x_1) := \phi_1$.

$$\frac{1}{n!} \left(\frac{-i\lambda}{4!} \right)^n \int d^4y_1 \dots d^4y_n \langle 0|T\{\phi_1 \dots \phi_m \phi^4(y_1) \dots \phi^4(y_n)\}|0\rangle$$

Example with $m = 4, n = 1$

$$\begin{aligned} & -\frac{i\lambda}{4!} \int d^4x \langle 0|T\phi_1\phi_2\phi_3\phi_4\phi^4(x)|0\rangle \\ &= -\frac{i\lambda}{4!} \int d^4x \langle 0|\phi_1\phi_2\phi_3\phi_4\phi(x)\phi(x)\phi(x)\phi(x)|0\rangle + 23 \text{ permutations} \\ & -\frac{i\lambda}{4!} \int d^4x \langle 0|\phi_1\phi_2\phi_3\phi_4\phi(x)\phi(x)\phi(x)\phi(x)|0\rangle + 11 \text{ permutations} + 5 \text{ similar} \\ & -\frac{i\lambda}{4!} \int d^4x \langle 0|\phi_1\phi_2\phi_3\phi_4\phi(x)\phi(x)\phi(x)\phi(x)|0\rangle + 2 \text{ permutations} + 2 \text{ similar} \\ &= -i\lambda \int d^4x D_F(x_1 - x)D_F(x_2 - x)D_F(x_3 - x)D_F(x_4 - x) \\ & -\frac{i\lambda}{2} D_F(x_1 - x_2) \int d^4x D_F(x_3 - x)D_F(x_4 - x)D_F(x - x) + 5 \text{ similar} \\ & -\frac{i\lambda}{8} D_F(x_1 - x_2)D_F(x_3 - x_4) \int d^4x D_F(x - x) + 2 \text{ similar} \end{aligned}$$

Permutation means permutation of $\phi(x)$ and similar means exchanging $\phi_i, i \in 1, 2, 3, 4$ without changing the diagram. Represented in Feynman diagrams:



In fact $D_F(x - x) = D_F(0)$ diverges!

Example with $m = 0, n = 1$ vacuum diagram

$$\begin{aligned}
 & -\frac{i\lambda}{4!} \int d^4x \langle 0|T\phi^4(x)|0\rangle \\
 &= -\frac{i\lambda}{8} [D_F(0)]^2 \int d^4x \\
 &= \text{diagram: two circles stacked vertically, with the label } x \text{ below the bottom circle}
 \end{aligned}$$

Example: 2nd order S-matrix term

$$\frac{1}{2!} \left(\frac{-i\lambda}{4!} \right)^2 \int d^4x d^4y \langle 0|T\phi_1\phi_2\phi_3\phi_4\phi^4(x)\phi^4(y)|0\rangle$$

It has many contractions and some of the fully connected ones are of the type there are


$(4 \times 3)[\text{choose } \phi(x)] \times (4 \times 3)[\text{choose } \phi(y)] \times 2[\text{x-y-cont.}] \times 2(\text{x-y-symm.}) + 2 \text{ similar, exchanging external points}$

$$= \frac{(-i\lambda)^2}{2} \int d^4x d^4y D_F(x_1 - x) D_F(x_2 - x) D_F(x_3 - y) D_F(x_4 - y) [D_F(x - y)]^2 + 2 \text{ similar}$$

$$= \text{diagram 1} + \text{diagram 2} + \text{diagram 3}$$

Diagram 1: A circle with two vertices labeled x and y . Four external lines connect the vertices to points x_1, x_2 (top) and x_3, x_4 (bottom).
Diagram 2: A circle with two vertices labeled x and y . Four external lines connect the vertices to points x_1, x_2 (top) and x_3, x_4 (bottom).
Diagram 3: A circle with two vertices labeled x and y . Four external lines connect the vertices to points x_1, x_4 (top) and x_3, x_2 (bottom).

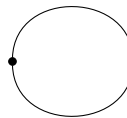
Symmetry factors A lot of the contractions eliminate the factors $\frac{1}{n!} \left(\frac{1}{4!} \right)^4$ in the denominators; the $\frac{1}{4!}$

was chosen as to yield  $\sim -i\lambda$

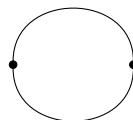
See examples above. Sometimes, factors are not completely cancelled and thus procedure gets "reversed". Divide diagrams by *symmetry factor* $\hat{=}$ the "missing factors".

Where does it come from?

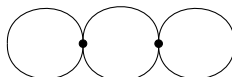
- factor 2 from the line that starts and ends at the same point.



- two (or more) lines linking the same 2 points.

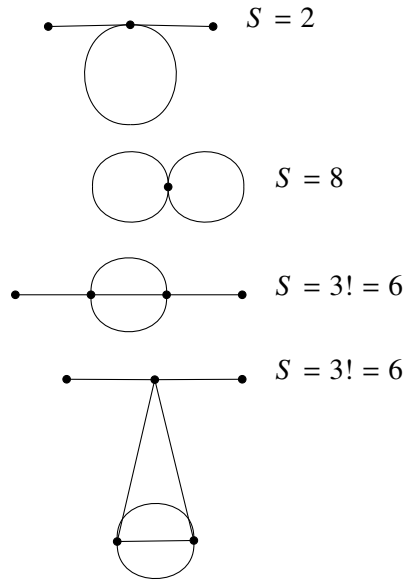


- 2 vertices can be equivalent.



When in doubt, can always go back to Wick's theorem and count the contractions explicitly.

Examples:



Summary of Feynman rules

$$\langle 0|T\phi_1 \dots \phi_m \exp\left(-\frac{i\lambda}{4!} \int d^4x \phi^4(x)\right)|0\rangle$$

= sum of all diagrams with m external points;

usually organised by number of internal points (i.e. power of λ).


Each diagram built out of

- propagators
- vertices (n)
- external points (m)

Feynman rules in position space

Analytic expression obtained by combining

- for each propagator $\overset{x}{\bullet} \text{---} \overset{y}{\bullet} = D_F(x - y)$

- for each vertex  $= -i\lambda \int d^4x$

- for each external point $\overset{x}{\bullet} \text{---} = 1$

- divide diagram by its symmetry factor S

Since the propagator $D_F(x - y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip(x-y)}$. It is actually simpler to express these in momentum space instead.

The way to do it is to assign a momentum p to each propagator. (direction arbitrary)



- assign e^{ipy} to y-vertex (arrow out)
- assign e^{-ipx} to x-vertex (arrow in)
- $\frac{i}{p^2 - m^2 + i\epsilon}$ to the line and the integration $\int \frac{d^4 p}{(2\pi)^4}$

At vertex x:

$$\begin{aligned}
 &= -i\lambda \int d^4 x e^{-i(p_1 + p_2 + p_3)x + ip_4 x} \\
 &= -i\lambda (2\pi)^4 \delta^{(4)}(p_1 + p_2 + p_3 - p_4)
 \end{aligned}$$

This imposes momentum conservation at vertex. $\delta^{(4)}$ -functions make some of the momentum integrals trivial, always with $(2\pi)^4$ cancelled appropriately.

Momentum space Feynman rules

- propagator $\xrightarrow{p} = \frac{i}{p^2 - m^2 + i\epsilon}$
- vertex (position integrated out) $= -i\lambda$
- external points $\begin{cases} e^{-ipx} & \text{incoming} \\ e^{+ipx} & \text{outgoing} \end{cases}$
- impose momentum conservation at each vertex
- integrate over each undetermined momentum $\int \frac{d^4 p}{(2\pi)^4}$
- divide by symmetry factor

e.g.:

$$= (-i\lambda) \int \frac{d^4 p}{(2\pi)^4} \frac{d^4 1}{(2\pi)^4} \left(\frac{i}{p^2 - m^2 + i\epsilon} \right)^2 \frac{i}{q^2 - m^2 + i\epsilon} e^{-ip(x-y)}$$

Vacuum diagrams

Disconnected pieces in Feynman diagrams are pretty bad. Not only $D_F(0) = \int \frac{d^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon}$ is divergent (that will be taken care of later), it also contains an integral $\int d^4 x \text{const.}$ thus divergent once more!

Typical diagram contributing to 2-point function. one piece connected to x and y , plus disconnected pieces.

Call disconnected pieces $V_i \in \left\{ \text{diagram 1}, \text{diagram 2}, \dots \right\}$. Points are connected internally, but not to external points.

V_i can occur n_i -times, then

$$[\text{diagram}] = [\text{connected pieces}] \times \prod_i \frac{1}{n_i!} (V_i)^{n_i}$$

The factorial is the symmetry factor of n_i disconnected copies of V_i .

Then

$$\begin{aligned} \langle 0|T\phi_1 \dots \phi_n S|0\rangle &= \sum_{\text{connected}} \sum_{\text{all}\{n_i\}} [\text{connected}] \times \prod_i \frac{1}{n_i!} (V_i)^{n_i} \\ &= \left(\sum_{\text{connected}} [\text{connected}] \right) \times \sum_{\text{all}\{n_i\}} \left(\prod_i \frac{1}{n_i!} (V_i)^{n_i} \right) \\ &= \prod_i \left(\sum_{n_i} \frac{1}{n_i!} (V_i)^{n_i} \right) \\ &= \exp \left(\sum_i V_i \right) \end{aligned}$$

Thus

$$\text{sum of ALL diagrams} = (\text{sum of all CONNECTED diagrams}) \quad (4.3.7)$$

$$\times \exp(\text{sum of all DISCONNECTED diagrams}) \quad (4.3.8)$$

Obvious from the above:

$$\langle 0|S|0\rangle = \langle 0|T\{\exp\left(-\frac{i\lambda}{4!} \int d^4 x \phi^4(x)\right)\}|0\rangle = \exp(\text{sum of all vacuum bubbles})$$

Conclusion from the (unproven) formula for n-point correlation functions in the true, interacting vacuum:

$$\langle \Omega|T\phi_1 \dots \phi_m|\Omega\rangle = \frac{\langle 0|T\phi_1 \dots \phi_m S|0\rangle}{\langle 0|S|0\rangle} \quad (4.3.9)$$

$$= \sum (\text{connected diagrams with m external points}) \quad (4.3.10)$$

Here: "connected" means connected to any external point. External points do not have to be linked to each other.

4.4 S-matrix elements and Feynman diagrams

What is the correlation function in interacting vacuum $\langle \Omega | T \phi_1 \dots \phi_m | \Omega \rangle$ good for? For scattering, shouldn't we rather look at $\langle p_1 \dots p_m | S | p_A p_B \rangle$ with the perturbative expansion of S as before?

Decomposition

$$S_{fi} = \delta_{fi} + iT_{fi} = \delta_{fi} + i(2\pi)^4 \delta^{(4)}(p_f - p_i) M_{fi} \quad (4.4.1)$$

M_{fi} is the invariant matrix element, used to calculate cross section etc..

Zeroth term in the expansion of S

$$\begin{aligned} \langle p_1 p_2 | p_A p_B \rangle &= \sqrt{2E_1 2E_2 2E_A 2E_B} \langle 0 | a_1 a_2 a_A^\dagger a_B^\dagger | 0 \rangle \\ &= 2E_A 2E_B (2\pi)^6 \left\{ \delta^{(3)}(\mathbf{p}_A - \mathbf{p}_1) \delta^{(3)}(\mathbf{p}_B - \mathbf{p}_2) + \delta^{(3)}(\mathbf{p}_A - \mathbf{p}_2) \delta^{(3)}(\mathbf{p}_B - \mathbf{p}_1) \right\} \end{aligned}$$

This actually is "no scattering", part of the $\mathbb{1}$ in the S-matrix.

First term is

$$\begin{aligned} \langle p_1 p_2 | T \left(-\frac{i\lambda}{4!} \int d^4x \phi^4(x) \right) | p_A p_B \rangle \\ \stackrel{\text{wick}}{=} \langle p_1 p_2 | : \left(-\frac{i\lambda}{4!} \int d^4x \phi^4(x) + \text{contractions} \right) : | p_A p_B \rangle \end{aligned}$$

However now the expectation value of a normal-ordered expression doesn't vanish!

$$\begin{aligned} \phi^+(x) | \mathbf{p} \rangle &= \int \frac{d^3k}{(2\pi)^3 \sqrt{2E_k}} a_{\mathbf{k}} e^{-ikx} \sqrt{2E_p} a_{\mathbf{p}}^\dagger | 0 \rangle \\ &= \int \frac{d^3k}{(2\pi)^3 \sqrt{2E_k}} e^{-ikx} \sqrt{2E_p} \delta^{(3)}(\mathbf{k} - \mathbf{p}) | 0 \rangle \\ &= e^{-ipx} | 0 \rangle \end{aligned}$$

So in general, need two field operators to annihilate the in-state and m fields operators to create the out-states.

New type of Feynman diagram to deal with external states. Define contractions of field operators with external states according to

$$\begin{aligned} \overline{\phi(x)} | \mathbf{p} \rangle &= e^{-ipx} | 0 \rangle \\ \langle \mathbf{p} | \phi(x) &= e^{+ipx} | 0 \rangle \end{aligned}$$

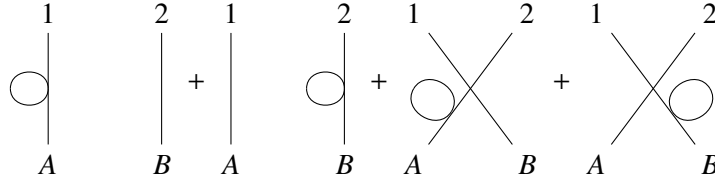
How does this work for $p_A p_B \rightarrow p_1 p_2$ in ϕ^4 at $O(\lambda)$? The above contains 3 types of terms: $:\phi\phi\phi\phi:$, $\overline{\phi\phi} : \phi\phi :$ and $\overline{\phi\phi\phi\phi}$.

1. $:\phi\phi\phi\phi:$ allows for contractions with all external states. There is $4!$ possibilities

$$\begin{array}{c} 1 \quad 2 \\ \diagdown \quad \diagup \\ A \quad B \end{array} = 4! \frac{-i\lambda}{4!} \int d^4x e^{-i(p_A + p_B - p_1 - p_2)x} = -i\lambda \underbrace{(2\pi)^4 \delta^{(4)}(p_A + p_B - p_1 - p_2)}_{\text{Prefactor in definition of } i\mathcal{M}}$$

$i\mathcal{M}$ receives a contribution $-i\lambda!$

2. $\overline{\phi}\phi\phi\phi$ leaves 2 operators to connect to external particles. Momentum conservation at each vertex. Still trivial!



Only fully connected Feynman diagrams contribute to $iT/i\mathcal{M}$!

- 3.

$$-\frac{i\lambda}{4!} \int d^4x \langle p_1 p_2 | \overline{\phi}\phi\phi\phi | p_A p_B \rangle$$

$$= \text{[Diagram: two circles connected by a dot]} \times \left(\begin{array}{c} 1 \\ | \\ A \end{array} \begin{array}{c} 2 \\ | \\ B \end{array} + \begin{array}{c} 1 \quad 2 \\ \diagdown \quad \diagup \\ A \quad B \end{array} \right)$$

4.4.1 Feynman rules (with external lines)

position space calculate iT by summing overall fully connected diagrams with

- propagator $x \xrightarrow{\quad} y = D_F(x - y)$
- vertex $\begin{array}{c} \diagup \quad \diagdown \\ \bullet \\ \diagdown \quad \diagup \end{array} x = -i\lambda \int d^4x$
- external lines "in" $x \xleftarrow{p} = e^{-ip \cdot x}$, $x \xrightarrow{p} = e^{ip \cdot x}$
- divide diagram by its symmetry factor $\frac{1}{S}$

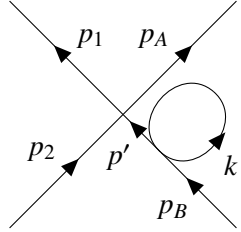
momentum space We have seen it before. Now (with external lines) all positions are integrated over. Result is a function of external momenta only. Integrating out all momentum-conserving δ -distribution yields overall momentum conservation: $(2\pi)^4 \delta^{(4)}(P_f - P_i)$

Momentum space Feynman rules for calculating iM :

- internal propagator $x \xrightarrow{\quad} y = \frac{i}{p^2 - M^2 + i\epsilon}$
- vertex $\begin{array}{c} \diagup \quad \diagdown \\ \bullet \\ \diagdown \quad \diagup \end{array} x = -i\lambda \int d^4x$
- external lines ("in" or "out") $x \xleftarrow{p} = 1$
- impose 4-momentum conservation at each vertex

- integrate over all undetermined momenta $\int \frac{d^4 p}{(2\pi)^4}$
- divide diagram by its symmetry factor $\frac{1}{S}$

There is still trouble in there. Consider the next-to-leading contribution to the scattering amplitude



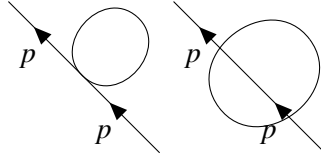
$$= \frac{1}{2} \int \frac{d^4 p'}{(2\pi)^4} \frac{i}{p'^2 - m^2} \int \frac{d^4 k}{(2\pi)^4} \frac{i}{k^2 - m^2} \times$$

$$(-i\lambda)(2\pi)^4 \delta^{(4)}(p_A + p' - p_1 - p_2) \cdot (-i\lambda)(2\pi)^4 \delta^{(4)}(p_B - p')$$

This contains the internal propagator $\frac{i}{p_B^2 - m^2 + i\epsilon}$, but all the external particle are on their mass-shell, i.e.

$$P_A^2 = P_B^2 = P_1^2 = P_2^2 = m^2 \Rightarrow \frac{i}{P_B^2 + m^2} = \frac{i}{0}$$

In Addition to having fully connected diagrams, also need to confine ourselves to amputated diagrams: disregard all these diagrams with loops attached to external legs.



These diagrams represent the transition from the free to the interacting asymptotic states.

Lehmann-Symanzik-Zimmermann (LSZ) reduction formula

Proof on relation between correlation functions and S-matrix elements will be provided later.

$$\prod_{i=1}^n \int d^4 x_i e^{ip_i \cdot x_i} \prod_{j=1}^m \int d^4 y_j e^{-ik_j \cdot y_j} \langle \Omega | T \phi(x_1) \dots \phi(x_n) \phi(y_1) \dots \phi(y_m) | \Omega \rangle$$

$$\stackrel{\text{LSZ}}{=} (\text{disconnected stuff}) + \underbrace{\prod_{i=1}^n \frac{\sqrt{z} i}{p_i^2 - m^2 + i\epsilon} \prod_{j=1}^m \frac{\sqrt{z} i}{k_j^2 - m^2 + i\epsilon}}_{\text{remove poles from external legs}} \langle p_1 \dots p_n | S | k_1 \dots k_m \rangle \quad (4.4.2)$$

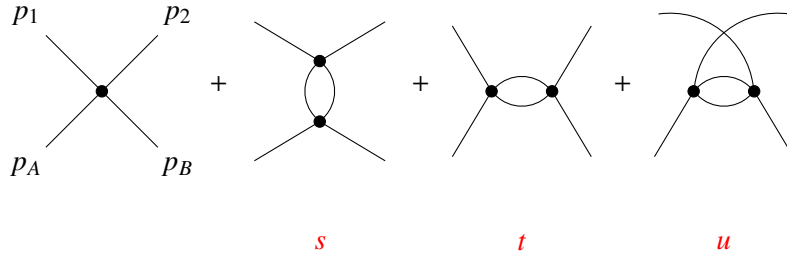
z is the wave-function renormalization factor.

Then amend feynman rules above

consider only fully connected, amputated diagrams

$$\begin{aligned}
 \langle p_1 p_2 | iT | p_A p_B \rangle = & \text{diagram 1} + \text{diagram 2} + \text{diagram 3} + \dots \\
 & + \left(\text{diagram 4} + \text{diagram 5} + \dots \right) \text{ yields } |0\rangle \rightarrow |\Omega\rangle \\
 & + \left(\text{diagram 6} + \dots \right) \text{ yields } |p\rangle_{\text{free}} \rightarrow |p\rangle_{\text{int}} \\
 & + \left(\begin{array}{ccc} 1 & 2 & 1 \\ | & | & | \\ A & B & A \end{array} + \text{diagram 7} + \begin{array}{ccc} 2 & & \\ | & & \\ B & & \end{array} + \dots \right) \text{ yields 1 in S-matrix}
 \end{aligned}$$

All allowed scattering diagrams $2 \rightarrow 2$ in ϕ^4 up to $O(\lambda^2)$:



Define the Lorentz-invariant quantities, *Mandelstam variables*:

$$s = (p_A + p_B)^2, \quad t = (p_A - p_1)^2, \quad u = (p_A - p_2)^2 \quad (4.4.3)$$

$$= \frac{1}{2} (-i\lambda)^2 \int \frac{d^4 k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} \frac{i}{(p_A + p_B + k)^2 - m^2 + i\epsilon} \stackrel{(p_A + p_B)^2 = s}{=} \frac{1}{2} (-i\lambda)^2 i J(s)$$

Then the complete invariant amplitude is

$$M = -\lambda - \frac{\lambda^2}{2} (J(s) + J(t) + J(u)) \quad (4.4.4)$$

4.5 Scattering cross section

(Itzykson & Zuber, Chapter 5.1)

The aim is to relate (differential) cross section to reduced/invariant matrix element M_{fi} . First we describe the initial states not as momentum eigenstates $|p_A p_B\rangle$, but as wave packets.

$$|i\rangle = \int \frac{d^3 k_A}{(2\pi)^3 2k_A^0} \frac{d^3 k_B}{(2\pi)^3 2k_B^0} f(k_A) g(k_B) |k_A k_B\rangle$$

with $f(k_A)$, $g(k_B)$ strongly peaked at $k_A \approx p_A$, $k_B \approx p_B$.

We can write the transition amplitude to the final state $|f\rangle \propto |p_1 p_2\rangle$ (note: normalisation not the same)

$$\begin{aligned} A_{fi} &= \int \frac{d^3 k_A}{(2\pi)^3 2k_A^0} \frac{d^3 k_B}{(2\pi)^3 2k_B^0} f(k_A) g(k_B) \langle f | iT | k_A k_B \rangle \\ &= \int \frac{d^3 k_A}{(2\pi)^3 2k_A^0} \frac{d^3 k_B}{(2\pi)^3 2k_B^0} f(k_A) g(k_B) (2\pi)^4 \delta^{(4)}(\underbrace{p_f}_{=p_1+p_2} - k_A - k_B) iM(f, k_A, k_B) \end{aligned}$$

Thus the transition probability:

$$\begin{aligned} \omega_{fi} &= (2\pi)^8 \int \frac{d^3 k_A}{(2\pi)^3 2k_A^0} \frac{d^3 k_B}{(2\pi)^3 2k_B^0} \frac{d^3 q_A}{(2\pi)^3 2q_A^0} \frac{d^3 q_B}{(2\pi)^3 2q_B^0} f(k_A) g(k_B) f(q_A)^* g(q_B)^* \times \\ &\quad \underbrace{\delta^{(4)}(p_f - k_A - k_B) \delta^{(4)}(p_f - q_A - q_B)}_{=\delta^{(4)}(q_A + q_B - k_A - k_B) \delta^{(4)}(p_f - p_A - p_B)} \underbrace{M(f, k_A, k_B) M^*(f, q_A, q_B)}_{\approx |M(f, p_A, p_B)|^2} \\ &= \int d^4 x \int \frac{d^3 k_A}{(2\pi)^3 2k_A^0} \frac{d^3 q_A}{(2\pi)^3 2q_A^0} e^{i(k_A - q_A) \cdot x} f(k_A) f^*(q_A) \times \\ &\quad \underbrace{\int \frac{d^3 k_B}{(2\pi)^3 2k_B^0} \frac{d^3 q_B}{(2\pi)^3 2q_B^0} e^{i(k_B - q_B) \cdot x} g(k_B) g^*(q_B) (2\pi)^4 \delta^{(4)}(p_f - p_A - p_B)}_{:=|\tilde{g}(x)|^2} \cdot |M(f, p_A, p_B)|^2 \\ &= \int d^4 x \underbrace{|f(x)|^2}_{:=|\tilde{f}(x)|^2} \underbrace{|\tilde{g}(x)|^2}_{:=|\tilde{g}(x)|^2} (2\pi)^4 \delta^{(4)}(p_f - p_A - p_B) \cdot |M(f, p_A, p_B)|^2 \end{aligned}$$

note that $M(f, p_A, p_B)$ and $M(p_1, p_2, p_A, p_B)$ have different normalisation.

We now consider transition probability per unit volume per unit time:

$$\frac{d\omega_{fi}}{dV dt} = (\text{incident flux}) \cdot (\text{target density}) \cdot d\sigma$$

with $d\sigma$ the infinitesimal cross section for scattering into final state $\langle f |$.

Product $(\text{incident flux}) \cdot (\text{target density})$ denotes overlap of wave function. Necessary condition!

Covariant renormalization of states $\langle p | q \rangle \sim 2p^0 \delta^3(\mathbf{p} - \mathbf{q})$ means the number of particles per unit volume is $2p_A^0 |\tilde{f}(x)|^2$ and $2p_B^0 |\tilde{g}(x)|^2$, respectively.

Assume $\bullet \xrightarrow{p_A} \bullet \mathbf{p}_B = 0$ in target rest frame. Then $p_B^0 = 2m_B$.

Incident flux = $|\mathbf{v}_A| \cdot 2p_A^0 |\tilde{f}(x)|^2 = 2|\mathbf{p}_A| |\tilde{f}(x)|^2$ since $|\mathbf{v}_A| = |\mathbf{p}_A|/p_A^0$. Then

$$d\sigma = (2\pi)^4 \delta^{(4)}(p_f - p_A - p_B) \frac{1}{4m_B |\mathbf{p}_A|} |M(f, p_A, p_B)|^2$$

$$(\text{for } A + B \rightarrow 1 + 2) = \int_{\Delta} \frac{d^3 p_1}{(2\pi)^3 2p_1^0} \frac{d^3 p_2}{(2\pi)^3 2p_2^0} (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_A - p_B) \frac{1}{4m_B |\mathbf{p}_A|} |M(p_1, p_2, p_A, p_B)|^2$$

with Δ energy-momentum resolution of 4-momentum of final state $|f\rangle$.

Covariant form of

$$m_B \cdot |\mathbf{p}_A| = m_B \sqrt{(p_A^0)^2 - m_A^2} = \sqrt{(p_A \cdot p_B)^2 - m_A^2 m_B^2} =: F \quad (4.5.1)$$

This is scattering into arbitrary final state subject to 4-momentum conservation: $p_A + p_B = p_1 + p_2$.

Consider now differential cross section for scattering into a particular infinitesimal solid angle $d\Omega$, hence specific momentum dp_1, dp_2 variations:

$$\begin{aligned} d\sigma &= \frac{1}{4F} \prod_f \frac{d^3 p_f}{(2\pi)^3 2p_f^0} (2\pi)^4 \delta^{(4)}(p_A + p_B - \sum_f p_f) |M|^2 \\ &\stackrel{f=1,2}{=} \frac{1}{4F} \frac{d^3 p_1}{(2\pi)^3 2p_1^0} \frac{d^3 p_2}{(2\pi)^3 2p_2^0} (2\pi)^4 \delta^{(4)}(p_i - p_f) |M|^2 \\ &= \frac{1}{64\pi^2 F} \frac{d^3 p_1}{E_1} \frac{d^3 p_2}{E_2} \delta^{(4)}(p_1 + p_2 - p_i) |M|^2 \\ &\quad \left[\begin{aligned} &\int \frac{d^3 p_1}{E_1} \frac{d^3 p_2}{E_2} \delta^{(4)}(p_1 + p_2 - p_i) \\ &\stackrel{\text{CMS}}{=} \int d|\mathbf{p}_1| d\Omega_1 \frac{|\mathbf{p}_1|^2}{E_1 E_2} \delta(E_1 + E_2 - E_i) \\ &= \int d(E_1 + E_2) \frac{d|\mathbf{p}_1|}{d(E_1 + E_2)} d\Omega_1 \frac{|\mathbf{p}_1|^2}{E_1 E_2} \delta(E_1 + E_2 - E_i) \\ &= \frac{|\mathbf{p}_1|^2}{E_1 E_2} \left(\frac{|\mathbf{p}_1|}{E_1} + \frac{|\mathbf{p}_1|}{E_2} \right)^{-1} d\Omega_1 \\ &\stackrel{E_1+E_2}{=} \frac{|\mathbf{p}_1| d\Omega_1}{E_i} = \frac{|\mathbf{p}_1| d\Omega_1}{E_i} \end{aligned} \right] \\ \frac{d\sigma}{d\Omega} &= \frac{1}{64\pi^2} \frac{|\mathbf{p}_1|}{F \cdot E_i} |M|^2 \end{aligned} \quad (4.5.2)$$

Rewrite all kinematical factors in terms of $s = (p_A + p_B)^2 = (p_1 + p_2)^2$. Define the function

$$\lambda(x, y, z) := x^2 + y^2 + z^2 - 2(xy + xz + yz) \quad (4.5.3)$$

then

$$\begin{aligned} F &= \sqrt{(p_A \cdot p_B)^2 - m_A^2 m_B^2} = \frac{1}{2} \lambda^{\frac{1}{2}}(s, m_A^2, m_B^2) = \sqrt{s} |\mathbf{p}_i| \\ \left[\begin{aligned} \lambda(s, m_A^2, m_B^2) &= s^2 - 2s(m_A^2 + m_B^2) - (m_A^2 - m_B^2)^2 = (s - (m_A + m_B)^2)(s - (m_A - m_B)^2) \\ &= (2p_A \cdot p_B - 2m_A \cdot m_B) \cdot (2p_A \cdot p_B + 2m_A \cdot m_B) = 4 \left[(p_A p_B)^2 - m_A^2 m_B^2 \right] \\ &\quad \left[\begin{aligned} p_A &= (c\sqrt{s}, \mathbf{p}_i), c \in [0, 1] \rightarrow m_A^2 = c^2 s - |\mathbf{p}_i|^2 \\ p_B &= ((1-c)\sqrt{s}, -\mathbf{p}_i) \rightarrow m_B^2 = (1-c)^2 s - |\mathbf{p}_i|^2 \end{aligned} \right] \\ &= 4 \left[\left(\frac{s}{4} + |\mathbf{p}_i|^2 \right)^2 - \left(\frac{s}{4} - |\mathbf{p}_i|^2 \right)^2 \right] = 4s |\mathbf{p}_i|^2 \end{aligned} \right] \\ |\mathbf{p}_f| &= \sqrt{E_{1,2}^2 - m_{1,2}^2} = \frac{1}{2\sqrt{s}} \lambda^{\frac{1}{2}}(s, m_1^2, m_2^2) \\ E_i &= \sqrt{s} \end{aligned}$$

$$\frac{d\sigma}{d\Omega_{CMS}} = \frac{1}{64\pi^2 s} \frac{|\mathbf{p}_f|}{|\mathbf{p}_i|} |M|^2 = \frac{1}{64\pi^2 s} \sqrt{\frac{\lambda(s, m_1^2, m_2^2)}{\lambda(s, m_A^2, m_A^2)}} |M|^2 \quad (4.5.4)$$

Decay rate instead of cross section means no "incident flux" to divide by, only "target density"

$$d\Gamma = \frac{1}{2m_A} \prod_f \frac{d^3 p_f}{(2\pi)^3 2p_f^0} (2\pi)^4 \delta^{(4)}(p_A - \sum_f p_f) |M|^2 \quad (4.5.5)$$

Particles with spin (unpolarized): sum over outgoing or average over initial spins

$$|M|^2 \rightarrow \frac{1}{(2s_A + 1)(2s_B + 1)} \sum_{s_i, s_f} |M_{fi}|^2 \quad (4.5.6)$$

Symmetry factor $|M|^2 \rightarrow \frac{1}{s} |M|^2$ with $s = \prod_i k_i!$ if there are k_i identical particles of species i in the final states.


If 1 & 2 are identical, then factor $\frac{1}{s} = \frac{1}{2}$ on the right hand side.

4.6 Feynman rules for fermions

Consider the simplest interacting theory with fermions, Yukawa-theory. We will treat QED later.

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{M^2}{2} \phi^2 + \bar{\psi}(i\not{\partial} - m)\psi - g\bar{\psi}\psi\phi \quad (4.6.1)$$

Feynman rules will involve:

- scalar $\begin{array}{c} x \\ \bullet \end{array} \text{-----} \begin{array}{c} y \\ \bullet \end{array} = D_F(x-y) = \int \frac{d^4}{(2\pi)^4} \frac{i}{p^2 - M^2 + i\epsilon} e^{-ip(x-y)}$
- fermions $\begin{array}{c} x, \alpha \\ \bullet \end{array} \text{---} \begin{array}{c} y, \beta \\ \bullet \end{array} = S_F(x-y)_{\alpha\beta} = \int \frac{d^4}{(2\pi)^4} \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon} e^{-ip(x-y)}$
- vertices  $= -ig \int d^4x$

What previous steps need reconsideration due to the anticommutating fermion operators? Interaction Hamiltonian $\sim \bar{\psi}\psi\phi$ and in general compose of even number of fermion fields (spin conservation and fermion number conservation). Thus there is no problem with time-ordered exponential in definition of S-matrix. (Time ordering always takes two or even number of fields.)

Remember the relation

$$T(\psi_\alpha(x)\bar{\psi}_\beta(x)) = -\bar{\psi}_\beta(x)\psi_\alpha(x) \leftarrow y^0 > x^0 \quad (4.6.2)$$

Similarly in normal product:

$$:\psi^+\psi^- = -\psi^-\psi^+ : \quad (4.6.3)$$

Then Wick's theorem is formally the same as before

$$T(\psi_\alpha(x)\bar{\psi}_\beta(x)) =: \psi_\alpha(x)\bar{\psi}_\beta(x) : + \psi_\alpha(x)\bar{\psi}_\beta(x)$$

note by definition $\overline{\psi\psi} = \overline{\bar{\psi}\bar{\psi}} = 0$

Thus contractions inside normal-ordered products would be

$$: \psi_1 \psi_2 \bar{\psi}_3 \bar{\psi}_4 : = - \overline{\psi_1 \bar{\psi}_3} : \psi_2 \bar{\psi}_4 : = - S_F(x_1 - x_3) : \psi_2 \bar{\psi}_4 :$$

because of the additional operator exchange.

We will want to consider fermion-(anti-)fermion scattering. Leading contribution at $O(g^2)$:

$$\frac{1}{2!}(-ig)^2 \int d^4x d^4y \langle p', k' | T \bar{\phi}(x) \phi(x) \phi(x) \bar{\phi}(y) \phi(y) | p, k \rangle$$

Contractions with initial-/final-state fermions?

$$\begin{aligned} \phi^+(x) |p, s\rangle &= \int \frac{d^3k}{(2\pi)^3 \sqrt{2E_k}} \sum_r a_k^r u_r(k) e^{-ik \cdot x} \sqrt{(2E_p)} a_p^{s\dagger} |0\rangle \\ &= e^{-ip \cdot x} u_s(p) |0\rangle \end{aligned}$$

So define

$$\begin{aligned} \overline{\psi(x) |p, s\rangle} &= e^{-ip \cdot x} u_s(p) \\ \langle p, s | \bar{\psi}(x) &= e^{ip \cdot x} \bar{u}_s(p) \end{aligned} \quad (4.6.4)$$

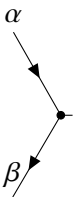
note, though, for antifermion states $|p', s'\rangle$:

$$\begin{aligned} \overline{\bar{\psi}(x) |p, s\rangle} &= e^{-ip' \cdot x} \bar{v}_{s'}(p') \\ \langle p', s' | \psi(x) &= e^{ip' \cdot x} v_{s'}(p') \end{aligned} \quad (4.6.5)$$

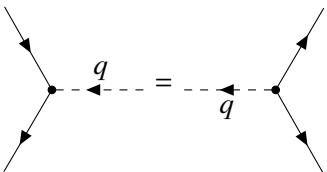
In short $\overline{\psi} | \rangle$ contractes with a fermion, $\langle | \bar{\psi}$ with an antifermion; vice verse for $\bar{\psi}$.

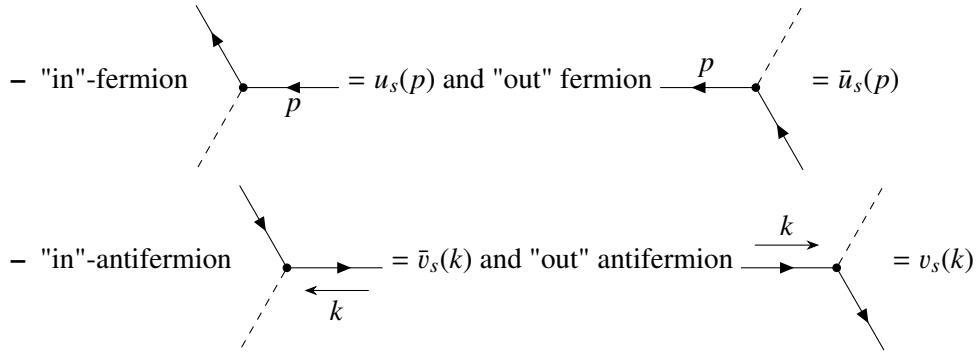
Momentum space feynman rule for iM

- internal propagators $\xrightarrow{q} = \frac{i}{q^2 - M^2 + i\epsilon}$; $\xrightarrow{q} = \frac{i}{q^2 - M^2 + i\epsilon} = \frac{i(p+m)_{\alpha\beta}}{p^2 - m^2 + i\epsilon}$

- vertex  $= -ig \int d^4x = ig \delta_{\beta\alpha}$

- external lines:

- "in" and "out" scalar  $= 1$



- impose energy-momentum conservation at each vertex
- integrate over undetermined (loop) momenta
- include an overall sign for the diagram

note

- Arrows on the fermion lines by convention denote fermion (or charge) flow. They must flow consistently through the diagram. (\equiv fermion number conservation) (Only potential confusion: external antifermion lines)

- No symmetry factors (except vacuum bubbles $\frac{1}{s} = \frac{1}{2}$). $\bar{\psi}\psi\phi$ allows for unambiguous contractions.
- Dirac indices are summed over at each vertex

$$\mathcal{L}_{\text{int}} \approx \bar{\psi}_\alpha(x) \psi_\alpha(x) \phi(x)$$

$(\not{p} + m)$ terms in propagator are matrix-multiplied contracted with external spinors, e.g.

$$\sim \bar{u}_\alpha(p_3) \frac{i(\not{p} + m)_{\alpha\beta}}{p_2^2 - m^2 + i\epsilon} \frac{i(\not{p} + m)_{\beta\gamma}}{p_1^2 - m^2 + i\epsilon} u_\gamma(p_0)$$

- closed fermion loop

$$\sim \bar{\psi}_\alpha(x) \psi_\alpha(x) \bar{\psi}_\beta(y) \psi_\beta(y) = -\bar{\psi}_\alpha(x) \bar{\psi}_\beta(y) \psi_\alpha(x) \psi_\beta(y)$$

$$= -S_F(y-x)_{\beta\alpha} S_F(x-y)_{\alpha\beta} = -\text{Tr}(S_F(y-x) S_F(x-y))$$

It always (also with more propagators/couplings) involves an overall (-1) and a trace $\text{Tr}(\dots)$.