

# **Quantum Field Theory**

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## 4 Interacting QFT

### 4.1 Introduction and examples

Theories discussed so far are Klein-Gordon theory with spin 0

$$\mathcal{L}_{KG} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2} \phi^2$$

and Dirac theory spin  $\frac{1}{2}$

$$\mathcal{L}_D = \bar{\psi}(i\not{\partial} - m)\psi$$

There is also  $\mathcal{L}_{EM} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$  with  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$  for a massless vector field. Its quantisation gives photon.

One thing they have in common is quadratic in the fields. As a result:

- linear field equations
- exact quantisation
- multi-particle states without scattering or interaction
- linear Fourier decomposition, no momentum changes

To have an interacting theory with scattering, need higher powers in the field in the Lagrangians. A few examples are following

#### Scalar $\phi^4$ theory

$$\mathcal{L} = \mathcal{L}_{KG} + \frac{\lambda}{4!} \phi^4$$

need positive sign  $\lambda > 0$  for a stable theory, otherwise classical energy can be arbitrarily negative.

Equation of motions

$$(\partial^2 + m^2)\phi = -\frac{\lambda}{3!} \phi^3$$

is nonlinear, cannot be solved by Fourier decomposition.

#### Yukawa-theory

$$\mathcal{L} = \mathcal{L}_{KG} + \mathcal{L}_D - g\bar{\psi}\psi\phi$$

is originally developed as a theory for nuclear forces with  $\psi$  nucleon,  $\phi$  pion. In the Standard Model it is similar to interactions in Higgs mechanism.

**Quantum Electrodynamics (QED)**

$$\mathcal{L} = \mathcal{L}_{EM} + \mathcal{L}_D - eA_\mu \bar{\psi} \gamma^\mu \psi$$

describes electrons, their antiparticles positrons and photons.

**Yang-Mills theory** generalises  $\mathcal{L}_{EM}$  with terms like  $A^4$  or  $A^2 \partial A$

**Scalar QED** describes pions and photons

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_{EM} + D_\mu \phi D^\mu \phi^* - m^2 |\phi|^2 \\ &= \mathcal{L}_{EM} + \partial_\mu \phi \partial^\mu \phi^* - m^2 \phi \phi^* + ieA_\mu (\phi \partial^\mu \phi^* - \phi^* \partial^\mu \phi) + e^2 A_\mu A^\mu \phi \phi^* \end{aligned}$$

**Remarks**

1. Interaction terms in  $H_{\text{int}} = \int d^3x \mathcal{H}_{\text{int}} = - \int d^3x \mathcal{L}_{\text{int}}$  always involves products of fields at the same point  $\mathbf{x}$ . It ensures causality, no "instant at a distance".
2. There are no derivative interactions. These may complicate quantisation as

$$\pi(\mathbf{x}) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi(\mathbf{x}))}$$

3. Why are we taking the examples above? There must be zillions of theories (Lagrangians)? We have the criterion of **renormalizability**. Note the mass dimensions of fields;

$$[S] = 1 \text{ so } [\mathcal{L}] = [M]^4 \Rightarrow [\phi] = [M], [\psi] = [M]^{\frac{3}{2}}, [A_\mu] = [M]$$

So in all the interaction terms indicated above, the coupling constant  $\lambda$ ,  $e$ ,  $g$  are all **dimensionless**! Can add  $-\frac{\mu}{3!} \phi^3$  to the  $\phi^4$  theory. This leads to  $[\mu] = [M]$  and all these generate renormalisable interactions.

All higher interaction terms require coupling constants of **negative** mass dimension, e.g.  $G \bar{\psi} \psi \bar{\psi} \psi$  and then  $[G] = [M]^{-2}$ . These are nonrenormalisable and create trouble when performing higher-order calculation in perturbation theory. (with energy cutoff; corrections  $G\Lambda^2$ ,  $\Lambda \rightarrow \infty$ )

4. We haven't quantised the photon yet. The reason is that its is a vector field, i.e. 4 degrees of freedom, but photon has just 2 physical polarisaion states. It is linked to gauge symmetry and complicates quantisation somewhat.

**4.2 The interaction picture**

Consider the  $\phi^4$  theory,

$$\mathcal{L}_{\text{int}} = -\frac{\lambda}{4!} \phi(x)^4 \quad (4.2.1)$$

Hamiltonian  $H = H_0 + H_{\text{int}}$  with

$$H_0 = \int d^3x \left\{ \frac{1}{2} \pi^2(x) + \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} m^2 \phi^2 \right\} \quad (4.2.2)$$

$$H_{\text{int}} = - \int d^3x \mathcal{L}_{\text{int}} = \frac{\lambda}{4!} \int d^3x \phi^4 \quad (4.2.3)$$

Interaction picture means that *operators* evolve in time using  $H_0$  (only), in particular

$$\phi_I(t, \mathbf{x}) = e^{iH_0 t} \phi(\mathbf{x}) e^{-iH_0 t} \quad (4.2.4)$$

Time-dependence of the free field obeys classical equation of motion  $(\partial^2 + m^2)\phi_I(t, \mathbf{x}) = 0$ . Solution in terms of fourier modes as before:

$$\phi_I(t, \mathbf{x}) = \int \frac{d^3 p}{(2\pi)^3 \sqrt{2E_p}} (a_{\mathbf{p}}^I e^{-ipx} + a_{\mathbf{p}}^{I\dagger} e^{ipx}) \quad (4.2.5)$$

as in the free theory with standard commutation relations  $[a_{\mathbf{p}}^I, a_{\mathbf{p}'}^{I\dagger}] = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{p}')$ . The state satisfying  $a_{\mathbf{p}}^I |0\rangle = 0$  is the vacuum of the free, noninteracting theory.

Relation between interaction and Schrödinger picture states:

$$|\phi_I(t)\rangle = e^{iH_0 t} |\psi_S(t)\rangle \quad (4.2.6)$$

Schrödinger equation becomes:

$$\begin{aligned} i \frac{\partial}{\partial t} |\psi_S\rangle &= (H_0 + H_{\text{int}}) |\psi_S\rangle \\ \text{LHS} &= i \frac{\partial}{\partial t} (e^{-iH_0 t} |\phi_I\rangle) = H_0 e^{-iH_0 t} |\phi_I\rangle + e^{-iH_0 t} i \frac{\partial}{\partial t} |\phi_I\rangle \\ \text{RHS} &= (H_0 + H_{\text{int}}) e^{-iH_0 t} |\phi_I\rangle \\ \Rightarrow i \frac{\partial}{\partial t} |\phi_I\rangle &= e^{iH_0 t} H_{\text{int}} e^{-iH_0 t} |\phi_I\rangle = H_I(t) |\phi_I\rangle \end{aligned} \quad (4.2.7)$$

with  $H_I$  interaction Hamiltonian in the interaction picture. Clearly

$$H_I = \frac{\lambda}{4!} \int d^3 x \phi_I^4(x)$$

What is the solution of 4.2.7 for the time evolution of  $|\phi_I(t)\rangle$ ? Define time-evolution operator in the interaction picture.

$$|\phi_I(t)\rangle = U(t, t_0) |\phi_I(t_0)\rangle \quad (4.2.8)$$

$$\text{with } U(t, t_0) = e^{iH_0(t-t_0)} e^{-iH(t-t_0)} \quad (4.2.9)$$

With 4.2.7 and 4.2.8:

$$i \frac{\partial}{\partial t} U(t, t_0) = H_I(t) U(t, t_0) \quad (4.2.10)$$

To solve with boundary conditions  $U(t_0, t_0) = 1$ . The formal solution is then:

$$U(t, t_0) = 1 - i \int_{t_0}^t dt' H_I(t') U(t', t_0)$$

Substitute back in and we get:

$$U(t, t_0) = 1 - i \int_{t_0}^t dt' H_I(t') + (-i)^2 \int_{t_0}^t dt' \int_{t_0}^{t'} dt'' H_I(t') H_I(t'') + \dots$$

$H_I$  inside the integral is automatically time-ordered. Ranges of integration is not.

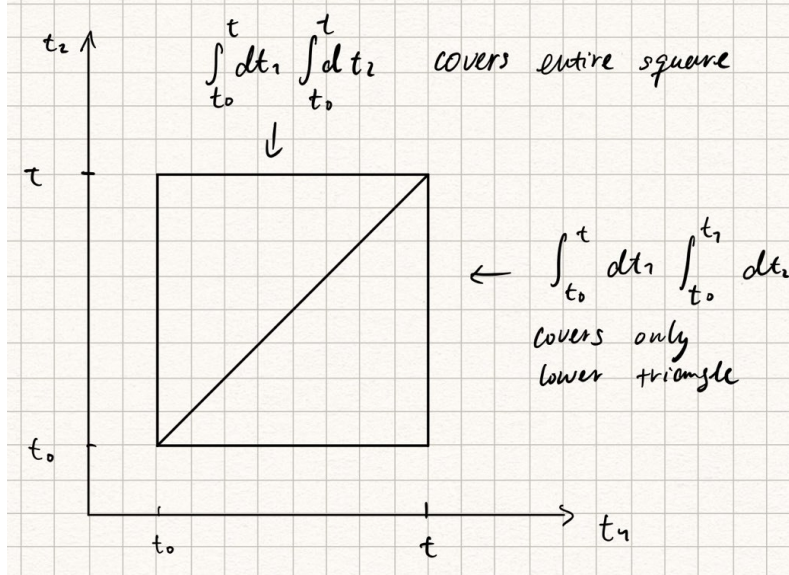


Figure 4.1: Time ordering

Upper triangle has the wrong time order. We are going to "repair" it by hand.

$$\begin{aligned}
 U(t, t_0) &= 1 - i \int_{t_0}^t dt' H_I(t') + \frac{(-i)^2}{2} \int_{t_0}^t dt' \int_{t_0}^{t'} dt'' T(H_I(t') H_I(t'')) + \dots \\
 &= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{t_0}^t dt_1 \dots \int_{t_0}^{t_{n-1}} dt_n T(H_I(t_1) \dots H_I(t_n)) \\
 &= T \exp \left\{ -i \int_{t_0}^t dt' H_I(t') \right\}
 \end{aligned} \tag{4.2.11}$$

It is interesting for scattering to transition into asymptotic state for  $t \rightarrow \infty$

$$\begin{aligned}
 S &= \lim_{t \rightarrow \infty} U(t, -t) = T \exp \left\{ -i \int_{-\infty}^{\infty} dt H_I(t) \right\} \\
 &\stackrel{\phi^4}{=} T \exp \left\{ -i \int d^4x \frac{\lambda}{4!} \phi_I^4(x) \right\}
 \end{aligned} \tag{4.2.12}$$

Both  $U$  and  $S$  are formally unitary

Composition law for time evolution operator

$$U(t_2, t_0) = U(t_2, t_1) U(t_1, t_0) = U(t_2, t_1) U(t_0, t_1)^\dagger \tag{4.2.13}$$

#### 4.2.1 Scattering amplitudes and the S-matrix

Take  $|i\rangle$  the initial (multi-particle) state and  $|f\rangle$  the final (multi-particle) state. Time evolution of  $|i\rangle$  then is

$$\lim_{t \rightarrow \infty} U(t, -t) |i\rangle = S |i\rangle$$

Probability that  $|i\rangle$  evolves into  $|f\rangle$  is proportional to the squared "**S-matrix element**"

$$|\langle f, t \rightarrow \infty | i, t \rightarrow -\infty \rangle|^2 = |\langle f | S | i \rangle|^2 = |S_{fi}|^2 \tag{4.2.14}$$

The nontrivial part of the S-matrix is the T-matrix:

$$S_{fi} := \delta_{fi} + iT_{fi} \quad (4.2.15)$$

Use momentum conservation (from translation invariance) to define matrix element

$$S_{fi} = \delta_{fi} + i(2\pi)^4 \delta^{(4)}(p_f - p_i) M_{fi} \quad (4.2.16)$$

$M_{fi}$  measures "genuine scattering" from  $|i\rangle$  to  $|f\rangle$ .

How are we going to calculate correlation functions in the interacting theory:

$$\langle \Omega | T \phi(x) \phi(y) | \Omega \rangle$$

or more generally  $\langle \Omega | T \phi(x_1) \phi(x_2) \dots | \Omega \rangle$ ? Here  $|\Omega\rangle$  is the vacuum/ground state of the interacting theory  $\mathcal{H}$  (in contrast to  $|0\rangle$  is the ground state of  $\mathcal{H}_0$ ) and  $\phi(x)$  the Heisenberg operators.

Ignore  $|\Omega\rangle \neq |0\rangle$  for the moment saying that we want to study the time evolution from the vacuum at  $t \rightarrow -\infty$  to  $t \rightarrow +\infty$ . So rewriting in terms  $\phi_I(x)$ , assuming  $x^0 > y^0$  for now:

$$\langle 0 | U(\infty, x^0) \phi_I(x^0) U(x^0, y^0) \phi_I(y^0) U(y^0, -\infty) | 0 \rangle = \langle 0 | T(\phi_I(x) \phi_I(y) S) | 0 \rangle \quad (4.2.17)$$

still holds if  $x^0 < y^0$  because of  $T$ .

Now  $|\Omega\rangle \neq |0\rangle$ : this can be taken care of by dividing out the time evolution of the (free) vacuum  $\langle 0 | S | 0 \rangle$ , so

$$\begin{aligned} \langle \Omega | T(\phi(x) \phi(y)) | \Omega \rangle &= \frac{\langle 0 | T(\phi_I(x) \phi_I(y) S) | 0 \rangle}{\langle 0 | S | 0 \rangle} \\ &\stackrel{\phi^4}{=} \frac{\langle 0 | T \phi_I(x) \phi_I(y) \exp\left\{-i \int d^4 x' \frac{\lambda}{4!} \phi^4(x')\right\} | 0 \rangle}{\langle 0 | T \exp\left\{-i \int d^4 x' \frac{\lambda}{4!} \phi^4(x')\right\} | 0 \rangle} \end{aligned} \quad (4.2.18)$$

Proof can be found in Peskin. It will also be illustrated parctically later ("vacuum bubbles").

Perturbation theory is viable when  $\lambda$  (or some other coupling) is "small" and then expands  $U(t, t_0)$  or  $S$  in powers of  $\lambda$ .

### 4.3 Wick's theorem

From now on drop the subscript for interaction picture fields  $\phi_I(x) \rightarrow \phi(x)$  for convenience.

Want to calculate stuff like  $\langle 0 | T \phi(x_1) \dots \phi(x_n) S | 0 \rangle$  in perturbation theory, e.g. at order  $\lambda^n$ .

$$\frac{1}{n!} \left(-i \frac{\lambda}{4!}\right)^n \int d^4 y_1 \dots d^4 y_n \langle 0 | T \phi(x_1) \dots \phi(x_n) \phi^4(y_1) \dots \phi^4(y_n) | 0 \rangle \quad (4.3.1)$$

We know  $\langle 0 | T \phi(x_1) \phi(x_2) | 0 \rangle$  is the Feynman propagator!

Recall **normal ordering** with  $\phi(x) = \phi^+(x) + \phi^-(x)$

$$: \phi^+ \phi^- := \phi^- \phi^+ := \phi^- \phi^+ \quad (4.3.2)$$

where

$$\phi^+ = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} a_{\mathbf{p}} e^{-ip \cdot x} \quad (4.3.3)$$

$$\phi^- = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} a_{\mathbf{p}}^\dagger e^{+ip \cdot x} \quad (4.3.4)$$

Wick's theorem expresses time-ordered products in terms of normal-ordered ones. Then it is easy to take vacuum expectation values, as  $\langle 0 | : \phi(x_1) \dots \phi(x_n) : | 0 \rangle = 0$

Take two fields and  $x^0 > y^0$ :

$$\begin{aligned} T\phi(x)\phi(y) &= \phi(x)\phi(y) = (\phi^+(x) + \phi^-(x))(\phi^+(y) + \phi^-(y)) \\ &= \phi^+(x)\phi^+(y) + \phi^-(x)\phi^-(y) + \phi^-(x)\phi^+(y) + \phi^+(x)\phi^-(y) + [\phi^+(x), \phi^-(y)] \\ &=: \phi(x)\phi(y) : + [\phi^+(x), \phi^-(y)] \end{aligned}$$

Particularly for  $y^0 > x^0$ :

$$T\phi(x)\phi(y) =: \phi(x)\phi(y) : + [\phi^+(y), \phi^-(x)]$$

Thus altogether:

$$T\phi(x)\phi(y) =: \phi(x)\phi(y) : + D_F(x - y) \quad (4.3.5)$$

as  $\Theta(x^0 - y^0)[\phi^+(x), \phi^-(y)] + \Theta(y^0 - x^0)[\phi^+(y), \phi^-(x)] = D_F(x - y)$ .

Worth noting that  $D_F(x - y)$  is still a c-number, not operator (yet). Thus it can be pulled out of any matrix element or expectation value.

We now define "contraction":

$$\overline{\phi(x_1)\phi(x_2)} = D_F(x_1 - x_2) \quad (4.3.6)$$

Thus we can remove the fields from the product leaving only the propagators:

$$T\phi(x)\phi(y) =: \phi(x)\phi(y) : + \overline{\phi(x)\phi(y)} \quad (4.3.7)$$

General form of **Wick's theorem** for arbitrary number of fields

$$T\phi(x_1) \dots \phi(x_n) =: \phi(x_1) \dots \phi(x_n) : + : (\text{sum over all possible contractions}) : \quad (4.3.8)$$

Example with four fields:

$$T(\phi_1\phi_2\phi_3\phi_4) =: \phi_1\phi_2\phi_3\phi_4 :$$

$$\begin{aligned} &+ \overline{\phi_1\phi_2} : \phi_3\phi_4 : + \overline{\phi_1\phi_3} : \phi_2\phi_4 : + \overline{\phi_1\phi_4} : \phi_2\phi_3 : + \overline{\phi_2\phi_3} : \phi_1\phi_4 : + \overline{\phi_2\phi_4} : \phi_1\phi_3 : + \overline{\phi_3\phi_4} : \phi_1\phi_2 : \\ &+ \overline{\phi_1\phi_2}\overline{\phi_3\phi_4} + \overline{\phi_1\phi_3}\overline{\phi_2\phi_4} + \overline{\phi_1\phi_4}\overline{\phi_2\phi_3} \end{aligned}$$

Thus

$$\langle 0 | T(\phi_1\phi_2\phi_3\phi_4) | 0 \rangle = D_F(x_1 - x_2)D_F(x_3 - x_4) + D_F(x_1 - x_3)D_F(x_2 - x_4) + D_F(x_1 - x_4)D_F(x_2 - x_3)$$

which can be visually represented as

$$\begin{array}{ccc} \begin{array}{c} x_1 \\ | \\ x_2 \end{array} & \begin{array}{c} x_3 \\ | \\ x_4 \end{array} & + \\ \begin{array}{c} x_1 \text{ --- } x_3 \\ x_2 \text{ --- } x_4 \end{array} & + & \begin{array}{c} x_1 \text{ --- } x_4 \\ x_2 \text{ --- } x_3 \end{array} \end{array}$$

Proof of the general theorem by *induction* in the number of fields (see exercise). The idea is to suppose it is true for  $\phi_2 \dots \phi_m$ ,  $x_1^0 > x_{k>1}^0$ . Then

$$\begin{aligned} T\phi_1\phi_2 \dots \phi_m &= (\phi_1^+ + \phi_1^-)T\phi_2 \dots \phi_m \\ &= (\phi_1^+ + \phi_1^-)[ : \phi_2 \dots \phi_m : + : \text{contractions} : ] \end{aligned}$$

$\phi_1^-$  can stay as it is part of  $( : \phi_1\phi_2 \dots \phi_m : )$ . But  $\phi_1^+$  needs to be commuted past all  $\phi_1^-$  operators, giving rise to additional contractions  $\overline{\phi_1\phi_2}$ .



### Consequences

- $n = 2k + 1, k \in \mathbb{N}$

$$\langle 0|T\phi_1 \dots \phi_m|0\rangle = 0$$

- $n = 2k, k \in \mathbb{N}$

$$\langle 0|T\phi_1 \dots \phi_m|0\rangle = \sum_{\text{pairing of fields}} D_F(x_{i_1} - x_{i_2}) \dots D_F(x_{i_{m-1}} - x_{i_m})$$

#### 4.3.1 Wick's theorem and the S-Matrix

Apply Wick's theorem to correlation functions  $\langle 0|T\{\phi_1 \dots \phi_m\}S|0\rangle$  n-th term in the perturbative expansion of  $S$  with  $\phi(x_1) := \phi_1$ .

$$\frac{1}{n!} \left( \frac{-i\lambda}{4!} \right)^n \int d^4y_1 \dots d^4y_n \langle 0|T\{\phi_1 \dots \phi_m \phi^4(y_1) \dots \phi^4(y_n)\}|0\rangle$$

**Example with  $m = 4, n = 1$**

$$\begin{aligned} & -\frac{i\lambda}{4!} \int d^4x \langle 0|T\phi_1\phi_2\phi_3\phi_4\phi^4(x)|0\rangle \\ &= -\frac{i\lambda}{4!} \int d^4x \langle 0|\phi_1\phi_2\phi_3\phi_4\phi(x)\phi(x)\phi(x)\phi(x)|0\rangle + 23 \text{ permutations} \\ & \quad -\frac{i\lambda}{4!} \int d^4x \langle 0|\phi_1\phi_2\phi_3\phi_4\phi(x)\phi(x)\phi(x)\phi(x)|0\rangle + 11 \text{ permutations} + 5 \text{ similar} \\ & \quad -\frac{i\lambda}{4!} \int d^4x \langle 0|\phi_1\phi_2\phi_3\phi_4\phi(x)\phi(x)\phi(x)\phi(x)|0\rangle + 2 \text{ permutations} + 2 \text{ similar} \\ &= -i\lambda \int d^4x D_F(x_1 - x)D_F(x_2 - x)D_F(x_3 - x)D_F(x_4 - x) \\ & \quad -\frac{i\lambda}{2} D_F(x_1 - x_2) \int d^4x D_F(x_3 - x)D_F(x_4 - x)D_F(x - x) + 5 \text{ similar} \\ & \quad -\frac{i\lambda}{8} D_F(x_1 - x_2)D_F(x_3 - x_4) \int d^4x D_F(x - x) + 2 \text{ similar} \end{aligned}$$

Permutation means permutation of  $\phi(x)$  and similar means exchanging external states  $\phi_i, i \in \{1, 2, 3, 4\}$  without changing the shape of diagram. Note that the prefactors of the integrals are called *symmetry factor*. The number of permutations of first diagram is equivalent to arrangement of 4 elements

$$\frac{4!}{4!} = 1$$

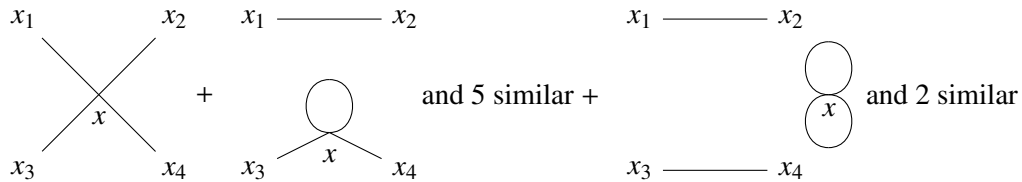
The Permutations of second diagram are to link two internal points to external ones and one external point can link to either of internal one

$$\binom{4}{2} \cdot 2 \cdot \frac{1}{4!} = \frac{1}{2}$$

There are only three ways to permute a vacuum bubble

$$\frac{3}{4!} = \frac{1}{8}$$

To be represented in Feynman diagrams:



In fact  $D_F(x - x) = D_F(0)$  diverges!

**Example with  $m = 0, n = 1$**  vacuum diagram

$$\begin{aligned}
 & -\frac{i\lambda}{4!} \int d^4x \langle 0|T\phi^4(x)|0\rangle \\
 &= -\frac{i\lambda}{8} [D_F(0)]^2 \int d^4x \\
 &= \text{tadpole diagram}
 \end{aligned}$$

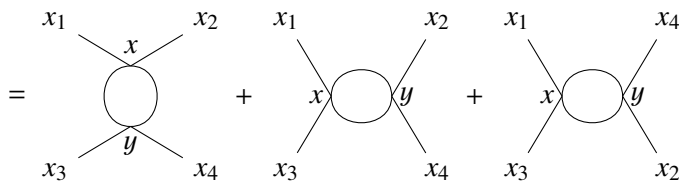
**Example: 2nd order S-matrix term**

$$\frac{1}{2!} \left( \frac{-i\lambda}{4!} \right)^2 \int d^4x d^4y \langle 0|T\phi_1\phi_2\phi_3\phi_4\phi^4(x)\phi^4(y)|0\rangle$$

It has many contractions and some of the fully connected ones are of the type there are

$(4 \times 3)[\text{choose } \phi(x)] \times (4 \times 3)[\text{choose } \phi(y)] \times 2[\text{x-y-cont.}] \times 2(\text{x-y-symm.}) + 2$  similar, exchanging external points

$$= \frac{(-i\lambda)^2}{2} \int d^4x d^4y D_F(x_1 - x) D_F(x_2 - x) D_F(x_3 - y) D_F(x_4 - y) [D_F(x - y)]^2 + 2 \text{ similar}$$



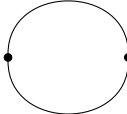
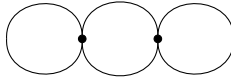
**Symmetry factors** A lot of the contractions eliminate the factors  $\frac{1}{n!} \left( \frac{1}{4!} \right)^n$  in the denominators; the  $\frac{1}{4!}$

was chosen to yield  $\text{cross} \sim -i\lambda$

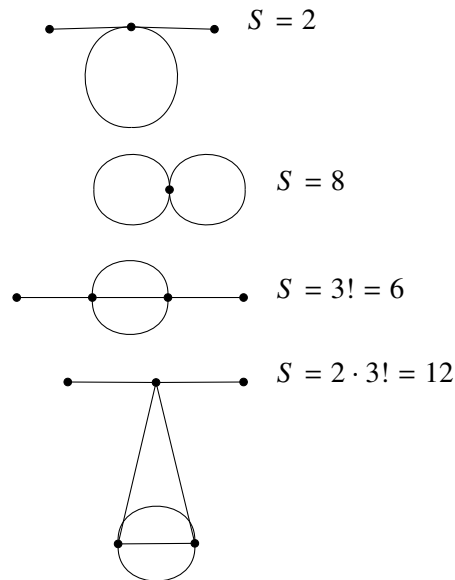
See examples above. Sometimes, factors are not completely cancelled and thus procedure gets "reversed". Divide diagrams by *symmetry factor*  $\hat{=}$  the "missing factors".

Where does it come from?

- Factor 2 from the line that starts and ends at the same point

- Factor  $j!$  for  $j$  lines linking the same 2 points 
- Factor  $k!$  for  $k$  equivalent vertices 

When in doubt, can always go back to Wick's theorem and count the contractions explicitly.  
Examples:



### Summary of Feynman rules

$$\langle 0|T\phi_1 \dots \phi_m \exp\left(-\frac{i\lambda}{4!} \int d^4x \phi^4(x)\right)|0\rangle$$

= sum of all diagrams with m external points;


usually organised by number of internal points(i.e. power of  $\lambda$ ).

Each diagram built cut of

- propagators
- vertices (n)
- external points (m)

**Feynman rules in position space** Analytic expression obtained by combining

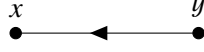
- for each propagator  $\overset{x}{\bullet} \text{---} \overset{y}{\bullet} = D_F(x-y)$

- for each vertex   $= -i\lambda \int d^4x$

- for each external point  $\bullet \xrightarrow{x} = 1$
- divide diagram by its symmetry factor  $S$

Since the propagator  $D_F(x-y) = \int \frac{d^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip(x-y)}$ . It is actually simpler to express these in momentum space instead.

The way to do it is to assign a momentum  $p$  to each propagator. (direction arbitrary)



- assign  $e^{ipy}$  to y-vertex (arrow out)
- assign  $e^{-ipx}$  to x-vertex (arrow in)
- $\frac{i}{p^2 - m^2 + i\epsilon}$  to the line and the integration  $\int \frac{d^4 p}{(2\pi)^4}$

At vertex  $x$ :

$$\begin{aligned}
 & \begin{array}{c} p_1 \swarrow \quad \nearrow p_2 \\ \bullet \\ p_3 \swarrow \quad \nearrow p_4 \end{array} = -i\lambda \int d^4 x e^{-i(p_1 + p_2 + p_3)x + ip_4 x} \\
 & = -i\lambda (2\pi)^4 \delta^{(4)}(p_1 + p_2 + p_3 - p_4)
 \end{aligned}$$

This imposes momentum conservation at vertex.  $\delta^{(4)}$ -functions make some of the momentum integrals trivial, always with  $(2\pi)^4$  cancelled appropriately.

### Momentum space Feynman rules

- propagator  $\xrightarrow{p} = \frac{i}{p^2 - m^2 + i\epsilon}$
- vertex (position integrated out)  $\begin{array}{c} \diagup \quad \diagdown \\ \bullet \\ \diagdown \quad \diagup \end{array} \begin{array}{c} x \\ x \end{array} = -i\lambda$
- external points  $\begin{cases} e^{-ipx} & \text{incoming} \\ e^{+ipx} & \text{outgoing} \end{cases}$
- impose momentum conservation at each vertex
- integrate over each undetermined momentum  $\int \frac{d^4 p}{(2\pi)^4}$
- To be divided by symmetry factor

e.g.:

$$\begin{array}{c} x \xleftarrow{p} \bullet \xleftarrow{p} y \\ \quad \quad \quad \bigcirc \quad \quad \quad q \end{array} = (-i\lambda) \int \frac{d^4 p}{(2\pi)^4} \frac{d^4 q}{(2\pi)^4} \left( \frac{i}{p^2 - m^2 + i\epsilon} \right)^2 \frac{i}{q^2 - m^2 + i\epsilon} e^{-ip(x-y)}$$

### Vacuum diagrams

Disconnected pieces in Feynman diagrams are pretty bad. Not only  $D_F(0) = \int \frac{d^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon}$  is divergent (that will be taken care of later), it also contains an integral  $\int d^4 x \text{const.}$  thus divergent once more!

Typical diagram contributing to 2-point function. one piece connected to  $x$  and  $y$ , plus disconnected pieces.

Call disconnected pieces  $V_i \in \left\{ \text{diagram 1}, \text{diagram 2}, \dots \right\}$ . Points are connected internally, but not to external points.

$V_i$  can occur  $n_i$ -times, then

$$[\text{diagram}] = [\text{connected pieces}] \times \prod_i \frac{1}{n_i!} (V_i)^{n_i}$$

The factorial is the symmetry factor of  $n_i$  disconnected copies of  $V_i$ .

Then

$$\begin{aligned} \langle 0|T\phi_1 \dots \phi_n S|0\rangle &= \sum_{\text{connected}} \sum_{\text{all}\{n_i\}} [\text{connected}] \times \prod_i \frac{1}{n_i!} (V_i)^{n_i} \\ &= \left( \sum [\text{connected}] \right) \times \sum_{\text{all}\{n_i\}} \left( \prod_i \frac{1}{n_i!} (V_i)^{n_i} \right) \\ &= \left( \sum [\text{connected}] \right) \times \prod_i \left( \sum_{n_i} \frac{1}{n_i!} (V_i)^{n_i} \right) \\ &= \left( \sum [\text{connected}] \right) \times \exp \left( \sum_i V_i \right) \end{aligned}$$

Thus

$$\text{sum of ALL diagrams} = (\text{sum of all CONNECTED diagrams}) \quad (4.3.9)$$

$$\times \exp(\text{sum of all DISCONNECTED diagrams}) \quad (4.3.10)$$

Obvious from the above:

$$\langle 0|S|0\rangle = \langle 0|T\{\exp\left(-\frac{i\lambda}{4!} \int d^4 x \phi^4(x)\right)\}|0\rangle = \exp(\text{sum of all vacuum bubbles})$$

**Conclusion** from the (unproven) formula for n-point correlation functions in the true, interacting vacuum:

$$\langle \Omega|T\phi_1 \dots \phi_m|\Omega\rangle = \frac{\langle 0|T\phi_1 \dots \phi_m S|0\rangle}{\langle 0|S|0\rangle} \quad (4.3.11)$$

$$= \sum (\text{connected diagrams with m external points}) \quad (4.3.12)$$

Here: "connected" means connected to any external point. External points do not have to be linked to each other.

## 4.4 S-matrix elements and Feynman diagrams

What is the correlation function in interacting vacuum  $\langle \Omega | T \phi_1 \dots \phi_m | \Omega \rangle$  good for? For scattering, shouldn't we rather look at  $\langle p_1 \dots p_m | S | p_A p_B \rangle$  with the perturbative expansion of  $S$  as before?

Decompose the S-matrix

$$S_{fi} = \delta_{fi} + iT_{fi} = \delta_{fi} + i(2\pi)^4 \delta^{(4)}(p_f - p_i) M_{fi} \quad (4.4.1)$$

$M_{fi}$  is the invariant matrix element, used to calculate cross section etc..

Zeroth term in the expansion of  $S$

$$\begin{aligned} \langle p_1 p_2 | p_A p_B \rangle &= \sqrt{2E_1 2E_2 2E_A 2E_B} \langle 0 | a_1 a_2 a_A^\dagger a_B^\dagger | 0 \rangle \\ &= 2E_A 2E_B (2\pi)^6 \left\{ \delta^{(3)}(\mathbf{p}_A - \mathbf{p}_1) \delta^{(3)}(\mathbf{p}_B - \mathbf{p}_2) + \delta^{(3)}(\mathbf{p}_A - \mathbf{p}_2) \delta^{(3)}(\mathbf{p}_B - \mathbf{p}_1) \right\} \end{aligned}$$

This actually is "no scattering", part of the  $\mathbb{1}$  in the S-matrix.

First term is

$$\begin{aligned} \langle p_1 p_2 | T \left( -\frac{i\lambda}{4!} \int d^4x \phi^4(x) \right) | p_A p_B \rangle \\ \stackrel{\text{wick}}{=} \langle p_1 p_2 | : \left( -\frac{i\lambda}{4!} \int d^4x \phi^4(x) + \text{contractions} \right) : | p_A p_B \rangle \end{aligned}$$

However now the expectation value of a normal-ordered expression doesn't vanish!

$$\begin{aligned} \phi^+(x) | \mathbf{p} \rangle &= \int \frac{d^3k}{(2\pi)^3 \sqrt{2E_k}} a_{\mathbf{k}} e^{-ikx} \sqrt{2E_p} a_{\mathbf{p}}^\dagger | 0 \rangle \\ &= \int \frac{d^3k}{(2\pi)^3 \sqrt{2E_k}} e^{-ikx} \sqrt{2E_p} \delta^{(3)}(\mathbf{k} - \mathbf{p}) | 0 \rangle \\ &= e^{-ipx} | 0 \rangle \end{aligned}$$

So in general, need two field operators to annihilate the in-state and  $m$  fields operators to create the out-states.

New type of Feynman diagram to deal with external states. Define contractions of field operators with external states according to

$$\begin{aligned} \overline{\phi(x)} | \mathbf{p} \rangle &= e^{-ipx} | 0 \rangle \\ \langle \mathbf{p} | \phi(x) &= e^{+ipx} | 0 \rangle \end{aligned}$$

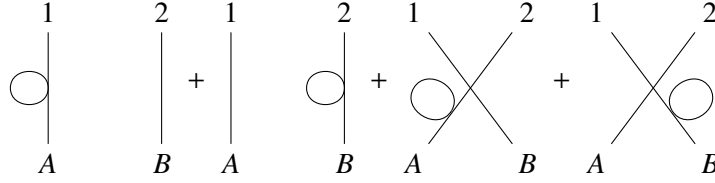
How does this work for  $p_A p_B \rightarrow p_1 p_2$  in  $\phi^4$  at  $O(\lambda)$ ? The above contains 3 types of terms:  $\phi\phi\phi\phi$ ,  $\overline{\phi\phi\phi\phi}$  and  $\overline{\phi\phi}\phi\phi$ .

1.  $\phi\phi\phi\phi$  allows full contractions with all external states. There is 4! possibilities

$$\begin{array}{c} 1 \quad 2 \\ \diagdown \quad \diagup \\ \quad \times \\ \diagup \quad \diagdown \\ A \quad B \end{array} = 4! \frac{-i\lambda}{4!} \int d^4x e^{-i(p_A + p_B - p_1 - p_2)x} = -i\lambda \underbrace{(2\pi)^4 \delta^{(4)}(p_A + p_B - p_1 - p_2)}_{\text{Prefactor in definition of } i\mathcal{M}}$$

$i\mathcal{M}$  receives a contribution  $-i\lambda!$

2.  $\overline{\phi}\phi\phi\phi$  leaves 2 operators to connect to external particles. Momentum conservation at each vertex. Still trivial!



Only fully connected Feynman diagrams contribute to  $iT/iM$ !

3.

$$-\frac{i\lambda}{4!} \int d^4x \langle p_1 p_2 | \overline{\phi}\phi\phi\phi | p_A p_B \rangle$$

$$= \text{diagram with two loops} \times \left( \begin{array}{c} 1 \\ | \\ A \end{array} + \begin{array}{c} 2 \\ | \\ B \end{array} + \begin{array}{c} 1 \quad 2 \\ \diagdown \quad \diagup \\ A \quad B \end{array} \right)$$

#### 4.4.1 Feynman rules (with external lines)

**Position space** calculate  $iT$  by summing over all fully connected diagrams with

- propagator  $x \text{---} y = D_F(x-y)$
- vertex  $\begin{array}{c} \diagup \quad \diagdown \\ \bullet \\ \diagdown \quad \diagup \end{array} x = -i\lambda \int d^4x$
- external lines "in"  $x \xleftarrow{p} = e^{-ip \cdot x}$ ;  $x \xrightarrow{p} = e^{ip \cdot x}$
- divide diagram by its symmetry factor  $S$

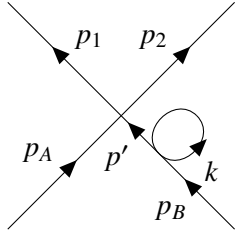
**Momentum space** We have seen it before. Now (with external lines) all positions are integrated over. Result is a function of external momenta only. Integrating out all momentum-conserving  $\delta$ -distribution yields overall momentum conservation:  $(2\pi)^4 \delta^{(4)}(P_f - P_i)$

Momentum space Feynman rules for calculating  $iM$ :

- internal propagator  $x \text{---} y = \frac{i}{p^2 - M^2 + i\epsilon}$
- vertex  $\begin{array}{c} \diagup \quad \diagdown \\ \bullet \\ \diagdown \quad \diagup \end{array} x = -i\lambda$
- external lines ("in" or "out")  $x \xleftarrow{p} = 1$
- impose 4-momentum conservation at each vertex

- integrate over all undetermined momenta  $\int \frac{d^4 p}{(2\pi)^4}$
- divide diagram by its symmetry factor  $S$

There is still trouble in there. Consider the next-to-leading contribution to the scattering amplitude

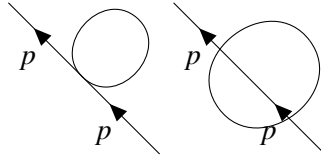


$$\begin{aligned}
 &= \frac{1}{2} \int \frac{d^4 p'}{(2\pi)^4} \frac{i}{p'^2 - m^2} \int \frac{d^4 k}{(2\pi)^4} \frac{i}{k^2 - m^2} \\
 &\quad \times (-i\lambda)(2\pi)^4 \delta^{(4)}(p_A + p' - p_1 - p_2) \cdot (-i\lambda)(2\pi)^4 \delta^{(4)}(p_B - p')
 \end{aligned}$$

This contains the internal propagator  $\frac{i}{p_B^2 - m^2 + i\epsilon}$ , but all the external particle are on their mass-shell, i.e.

$$p_A^2 = p_B^2 = p_1^2 = p_2^2 = m^2 \Rightarrow \frac{i}{p_B^2 - m^2} = \frac{i}{0}$$

In Addition to having fully connected diagrams, also need to confine ourselves to amputated diagrams: disregard all these diagrams with loops attached to external legs.



These diagrams represent the transition from the free to the interacting asymptotic states.

### Lehmann-Symanzik-Zimmermann (LSZ) reduction formula

Proof on relation between correlation functions and S-matrix elements will be provided later.

$$\begin{aligned}
 &\prod_{i=1}^n \int d^4 x_i e^{ip_i \cdot x_i} \prod_{j=1}^m \int d^4 y_j e^{-ik_j \cdot y_j} \langle \Omega | T \phi(x_1) \dots \phi(x_n) \phi(y_1) \dots \phi(y_m) | \Omega \rangle \\
 &\stackrel{\text{LSZ}}{=} (\text{disconnected stuff}) + \underbrace{\prod_{i=1}^n \frac{\sqrt{z} i}{p_i^2 - m^2 + i\epsilon} \prod_{j=1}^m \frac{\sqrt{z} i}{k_j^2 - m^2 + i\epsilon}}_{\text{remove poles from external legs}} \langle p_1 \dots p_n | S | k_1 \dots k_m \rangle
 \end{aligned} \tag{4.4.2}$$

$z$  is the wave-function renormalization factor.

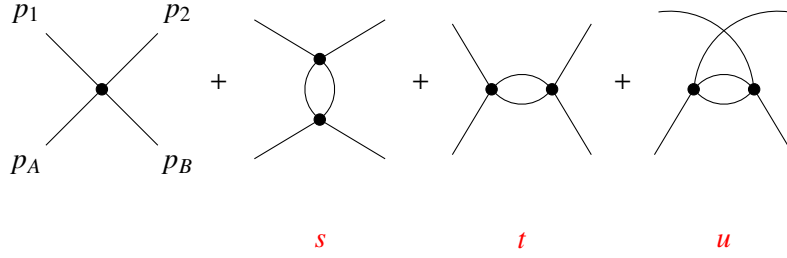
Then amend feynman rules above

consider only fully connected, amputated diagrams



$$\begin{aligned}
 \langle p_1 p_2 | iT | p_A p_B \rangle = & \text{[Cross diagram]} + \text{[Bubble diagram]} + \text{[Triangle diagram]} + \dots \\
 & + \left( \text{[Square diagram]} + \text{[Two-bubble diagram]} + \dots \right) \quad \text{yields } |0\rangle \rightarrow |\Omega\rangle \\
 & + \left( \text{[Cross with bubble diagram]} + \dots \right) \quad \text{yields } |p\rangle_{\text{free}} \rightarrow |p\rangle_{\text{int}} \\
 & + \left( \begin{array}{cc} 1 & 2 \\ | & | \\ A & B \end{array} + \text{[Bubble on line]} + \begin{array}{cc} 1 & 2 \\ | & | \\ A & B \end{array} + \dots \right) \quad \text{yields } \mathbb{1} \text{ in S-matrix}
 \end{aligned}$$

All allowed scattering diagrams  $2 \rightarrow 2$  in  $\phi^4$  up to  $O(\lambda^2)$ :



Define the Lorentz-invariant quantities, *Mandelstam variables*:

$$s = (p_A + p_B)^2, \quad t = (p_A - p_1)^2, \quad u = (p_A - p_2)^2 \quad (4.4.3)$$

$$= \frac{1}{2}(-i\lambda)^2 \int \frac{d^4 k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} \frac{i}{(p_A + p_B + k)^2 - m^2 + i\epsilon} =: \frac{1}{2}(-i\lambda)^2 iJ(s)$$

Then the complete invariant amplitude is

$$M = -\lambda - \frac{\lambda^2}{2} (J(s) + J(t) + J(u)) \quad (4.4.4)$$

## 4.5 Scattering cross section

This section is based on Itzykson & Zuber, Chapter 5.1.

The aim is to relate (differential) cross section to reduced/invariant matrix element  $M_{fi}$ . First we describe the initial states not as momentum eigenstates  $|p_A p_B\rangle$ , but as wave packets.

$$|i\rangle = \int \frac{d^3 k_A}{(2\pi)^3 2k_A^0} \frac{d^3 k_B}{(2\pi)^3 2k_B^0} f(k_A) g(k_B) |k_A k_B\rangle$$

with  $f(k_A)$ ,  $g(k_B)$  strongly peaked at  $k_A \approx p_A$ ,  $k_B \approx p_B$ .

We can write the transition amplitude to the final state  $|f\rangle \propto |p_1 p_2\rangle$  (note: normalisation not the same)

$$\begin{aligned} A_{fi} &= \int \frac{d^3 k_A}{(2\pi)^3 2k_A^0} \frac{d^3 k_B}{(2\pi)^3 2k_B^0} f(k_A) g(k_B) \langle f | iT | k_A k_B \rangle \\ &= \int \frac{d^3 k_A}{(2\pi)^3 2k_A^0} \frac{d^3 k_B}{(2\pi)^3 2k_B^0} f(k_A) g(k_B) (2\pi)^4 \delta^{(4)}(\underbrace{p_f}_{=p_1+p_2} - k_A - k_B) iM(f, k_A, k_B) \end{aligned}$$

Thus the transition probability:

$$\begin{aligned} \omega_{fi} &= (2\pi)^8 \int \frac{d^3 k_A}{(2\pi)^3 2k_A^0} \frac{d^3 k_B}{(2\pi)^3 2k_B^0} \frac{d^3 q_A}{(2\pi)^3 2q_A^0} \frac{d^3 q_B}{(2\pi)^3 2q_B^0} f(k_A) g(k_B) f(q_A)^* g(q_B)^* \\ &\times \underbrace{\delta^{(4)}(p_f - k_A - k_B) \delta^{(4)}(p_f - q_A - q_B)}_{=\delta^{(4)}(q_A + q_B - k_A - k_B) \delta^{(4)}(p_f - p_A - p_B)} \underbrace{M(f, k_A, k_B) M^*(f, q_A, q_B)}_{\approx |M(f, p_A, p_B)|^2} \end{aligned}$$

Using the fourier representation of delta function  $\delta^{(4)}(q_A + q_B - k_A - k_B) = (2\pi)^{-4} \int d^4 x e^{i(k_A + k_B - q_A - q_B) \cdot x}$

$$\begin{aligned} &= \int d^4 x \underbrace{\int \frac{d^3 k_A}{(2\pi)^3 2k_A^0} \frac{d^3 q_A}{(2\pi)^3 2q_A^0} e^{i(k_A - q_A) \cdot x} f(k_A) f^*(q_A)}_{:=|\tilde{f}(x)|^2} \\ &\times \underbrace{\int \frac{d^3 k_B}{(2\pi)^3 2k_B^0} \frac{d^3 q_B}{(2\pi)^3 2q_B^0} e^{i(k_B - q_B) \cdot x} g(k_B) g^*(q_B)}_{:=|\tilde{g}(x)|^2} (2\pi)^4 \delta^{(4)}(p_f - p_A - p_B) \cdot |M(f, p_A, p_B)|^2 \end{aligned}$$

Using Fourier transformation  $\tilde{g}(x) := \int \frac{d^3 q}{(2\pi)^3 2q^0} e^{iq \cdot x} g(q)$

$$= \int d^4 x |\tilde{f}(x)|^2 |\tilde{g}(x)|^2 (2\pi)^4 \delta^{(4)}(p_f - p_A - p_B) \cdot |M(f, p_A, p_B)|^2$$

note that  $M(f, p_A, p_B)$  and  $M(p_1, p_2, p_A, p_B)$  have different normalisation.

We now consider transition probability per unit volume per unit time:

$$\frac{d\omega_{fi}}{dV dt} = (\text{incident flux}) \cdot (\text{target density}) \cdot d\sigma$$

with  $d\sigma$  the infinitesimal cross section for scattering into final state  $\langle f |$ .

Product  $(\text{incident flux}) \cdot (\text{target density})$  denotes overlap of wave function. Necessary condition!

Covariant renormalization of states  $\langle \mathbf{p} | \mathbf{q} \rangle \sim 2p^0 \delta^3(\mathbf{p} - \mathbf{q})$  means the number of particles per unit volume is  $2p_A^0 |\tilde{f}(x)|^2$  and  $2p_B^0 |\tilde{g}(x)|^2$ , respectively.

Assume

$$\bullet \xrightarrow{p_A} \bullet \mathbf{p}_B = 0$$

in target rest frame. Then  $2p_B^0 = 2m_B$  and **target density** =  $2m_B |\tilde{g}(x)|^2$

Incident flux =  $|\mathbf{v}_A| \cdot 2p_A^0 |\tilde{f}(x)|^2 = 2|\mathbf{p}_A| |\tilde{f}(x)|^2$  since  $|\mathbf{v}_A| = |\mathbf{p}_A|/p_A^0$ . Then

$$d\sigma = (2\pi)^4 \delta^{(4)}(p_f - p_A - p_B) \frac{1}{4m_B |\mathbf{p}_A|} |M(f, p_A, p_B)|^2$$

for  $A + B \rightarrow 1 + 2$  processes

$$= \int_{\Delta} \frac{d^3 p_1}{(2\pi)^3 2p_1^0} \frac{d^3 p_2}{(2\pi)^3 2p_2^0} (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_A - p_B) \frac{1}{4m_B |\mathbf{p}_A|} |M(p_1, p_2, p_A, p_B)|^2$$

with  $\Delta$  energy-momentum resolution of 4-momentum of final state  $|f\rangle$ .

Covariant form of

$$m_B \cdot |\mathbf{p}_A| = m_B \sqrt{(p_A^0)^2 - m_A^2} = \sqrt{(p_A \cdot p_B)^2 - m_A^2 m_B^2} =: F \quad (4.5.1)$$

This is scattering into arbitrary final state subject to 4-momentum conservation:  $p_A + p_B = p_1 + p_2$ .

Consider now differential cross section for scattering into a particular infinitesimal solid angle  $d\Omega$ , hence specific momentum  $d p_1, d p_2$  variations:

$$\begin{aligned} d\sigma &= \frac{1}{4F} \prod_f \frac{d^3 p_f}{(2\pi)^3 2p_f^0} (2\pi)^4 \delta^{(4)}(p_A + p_B - \sum_f p_f) |M|^2 \\ &\stackrel{f=1,2}{=} \frac{1}{4F} \frac{d^3 p_1}{(2\pi)^3 2p_1^0} \frac{d^3 p_2}{(2\pi)^3 2p_2^0} (2\pi)^4 \delta^{(4)}(p_i - p_f) |M|^2 \\ &= \frac{1}{64\pi^2 F} \frac{d^3 p_1}{E_1} \frac{d^3 p_2}{E_2} \delta^{(4)}(p_1 + p_2 - p_i) |M|^2 \\ &\quad \boxed{\begin{aligned} &\int \frac{d^3 p_1}{E_1} \frac{d^3 p_2}{E_2} \delta^{(4)}(p_1 + p_2 - p_i) \\ &\stackrel{\text{CMS}}{=} \int d|\mathbf{p}_1| d\Omega_1 \frac{|\mathbf{p}_1|^2}{E_1 E_2} \delta(E_1 + E_2 - E_i) \\ &= \int d(E_1 + E_2) \frac{d|\mathbf{p}_1|}{d(E_1 + E_2)} d\Omega_1 \frac{|\mathbf{p}_1|^2}{E_1 E_2} \delta(E_1 + E_2 - E_i) \\ &= \frac{|\mathbf{p}_1|^2}{E_1 E_2} \left( \frac{|\mathbf{p}_1|}{E_1} + \frac{|\mathbf{p}_1|}{E_2} \right)^{-1} d\Omega_1 \\ &= \frac{|\mathbf{p}_1| d\Omega_1}{E_1 + E_2} = \frac{|\mathbf{p}_1| d\Omega_1}{E_i} \end{aligned}} \\ &\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \frac{|\mathbf{p}_1|}{F \cdot E_i} |M|^2 \quad (4.5.2) \end{aligned}$$

Rewrite all kinematical factors in terms of  $s = (p_A + p_B)^2 = (p_1 + p_2)^2$ . Define the function

$$\lambda(x, y, z) := x^2 + y^2 + z^2 - 2(xy + xz + yz) \quad (4.5.3)$$

then

$$F = \sqrt{(p_A \cdot p_B)^2 - m_A^2 m_B^2} = \frac{1}{2} \lambda^{\frac{1}{2}}(s, m_A^2, m_B^2) = \sqrt{s} |\mathbf{p}_i|$$

$$\begin{aligned} \lambda(s, m_A^2, m_B^2) &= s^2 - 2s(m_A^2 + m_B^2) - (m_A^2 - m_B^2)^2 = (s - (m_A + m_B)^2)(s - (m_A - m_B)^2) \\ &= (2p_A \cdot p_B - 2m_A \cdot m_B) \cdot (2p_A \cdot p_B + 2m_A \cdot m_B) = 4 \left[ (p_A p_B)^2 - m_A^2 m_B^2 \right] \\ &\quad \begin{aligned} p_A &= (c \sqrt{s}, \mathbf{p}_i), c \in [0, 1] \rightarrow m_A^2 = c^2 s - |\mathbf{p}_i|^2 \\ p_B &= ((1-c) \sqrt{s}, -\mathbf{p}_i) \rightarrow m_B^2 = (1-c)^2 s - |\mathbf{p}_i|^2 \end{aligned} \\ &= 4 \left[ \left( (c(1-c)s + p_i^2)^2 + (c^2 s - p_i^2)((1-c)^2 s - p_i^2) \right) \right] = 4s |\mathbf{p}_i|^2 \end{aligned}$$

$$|\mathbf{p}_f| = \sqrt{E_{1,2}^2 - m_{1,2}^2} = \frac{1}{2\sqrt{s}} \lambda^{\frac{1}{2}}(s, m_1^2, m_2^2)$$

$$E_i = \sqrt{s}$$

$$\frac{d\sigma}{d\Omega_{CMS}} = \frac{1}{64\pi^2 s} \frac{|\mathbf{p}_f|}{|\mathbf{p}_i|} |M|^2 = \frac{1}{64\pi^2 s} \sqrt{\frac{\lambda(s, m_1^2, m_2^2)}{\lambda(s, m_A^2, m_B^2)}} |M|^2 \quad (4.5.4)$$

Decay rate instead of cross section means no "incident flux" to divide by, only "target density"

$$d\Gamma = \frac{1}{2m_A} \prod_f \frac{d^3 p_f}{(2\pi)^3 2p_f^0} (2\pi)^4 \delta^{(4)}(p_A - \sum_f p_f) |M|^2 \quad (4.5.5)$$

Particles with spin (unpolarized): sum over outgoing or average over initial spins

$$|M|^2 \rightarrow \frac{1}{(2s_A + 1)(2s_B + 1)} \sum_{s_i, s_f} |M_{fi}|^2 \quad (4.5.6)$$

Symmetry factor  $|M|^2 \rightarrow \frac{1}{s} |M|^2$  with  $s = \prod_i k_i!$  if there are  $k_i$  identical particles of species  $i$  in the final states.

If 1 and 2 are identical, then facotr  $\frac{1}{s} = \frac{1}{2}$  on the right hand side.


## 4.6 Feynman rules for fermions

Consider the simplest interacting theory with fermions, Yukawa-theory. We will treat QED later.

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{M^2}{2} \phi^2 + \bar{\psi}(i\not{\partial} - m)\psi - g\bar{\psi}\psi\phi \quad (4.6.1)$$

Feynman rules will involve:

- scalar  $\begin{array}{c} x \\ \bullet \text{-----} \bullet \end{array}^y = D_F(x-y) = \int \frac{d^4}{(2\pi)^4} \frac{i}{p^2 - M^2 + i\epsilon} e^{-ip(x-y)}$
- fermions  $\begin{array}{c} x, \alpha \\ \bullet \longleftarrow \bullet \end{array}^{y, \beta} = S_F(x-y)_{\alpha\beta} = \int \frac{d^4}{(2\pi)^4} \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon} e^{-ip(x-y)}$

• vertices   $= -ig \int d^4x$

What previous steps need reconsideration due to the anticommutating fermion operators? Interaction Hamiltonina  $\sim \bar{\psi}\psi\phi$  and in general compose of even number of fermion fields (spin conservation and fermion number conservation). Thus there is no problem with time-ordered exponential in definition of S-matrix. (Time ordering always takes two or even number of fields.)

Remember the relation

$$T(\psi_\alpha(x)\bar{\psi}_\beta(x)) = -\bar{\psi}_\beta(x)\psi_\alpha(x) \text{ when } y^0 > x^0 \quad (4.6.2)$$

Similarly in normal product:

$$:\psi^+\psi^- = -\psi^-\psi^+ : \quad (4.6.3)$$

Then Wick's theorem is formally the same as before

$$T(\psi_\alpha(x)\bar{\psi}_\beta(x)) = :\psi_\alpha(x)\bar{\psi}_\beta(x) : + \overline{\psi_\alpha(x)\bar{\psi}_\beta(x)}$$

note by definition  $\overline{\psi\psi} = \bar{\psi}\bar{\psi} = 0$

Thus contractions inside normal-ordered products would be

$$:\psi_1\psi_2\bar{\psi}_3\bar{\psi}_4 := -\overline{\psi_1\bar{\psi}_3} : \psi_2\bar{\psi}_4 := -S_F(x_1 - x_3) : \psi_2\bar{\psi}_4 :$$

because of the additional operator exchange.

We will want to consider fermion-(anti-)fermion scattering. Leading contribution at  $O(g^2)$ :

$$\frac{1}{2!}(-ig)^2 \int d^4x d^4y \langle p', k' | T \bar{\phi}(x) \phi(x) \phi(x) \bar{\phi}(y) \phi(y) | p, k \rangle$$

Contractions with initial-/final-state fermions?

$$\begin{aligned} \phi^+(x) |p, s\rangle &= \int \frac{d^3k}{(2\pi)^3 \sqrt{2E_k}} \sum_r a_k^r u_r(k) e^{-ik \cdot x} \sqrt{2E_p} a_p^{s\dagger} |0\rangle \\ &= e^{-ip \cdot x} u_s(p) |0\rangle \end{aligned}$$

So define

$$\begin{aligned} \overline{\psi(x) |p, s} &= e^{-ip \cdot x} u_s(p) \\ \langle p, s | \overline{\psi(x)} &= e^{ip \cdot x} \bar{u}_s(p) \end{aligned} \quad (4.6.4)$$

note, though, for antifermion states  $|p', s'\rangle$ :

$$\begin{aligned} \overline{\bar{\psi}(x) |p, s} &= e^{-ip' \cdot x} \bar{v}_{s'}(p') \\ \langle p', s' | \overline{\bar{\psi}(x)} &= e^{ip' \cdot x} v_{s'}(p') \end{aligned} \quad (4.6.5)$$

In short  $\overline{\psi} | \rangle$  contractes with a fermion,  $\langle | \bar{\psi}$  with an antifermion; vice verse for  $\bar{\psi}$ .

**Momentum space Feynman rule for  $iM$** 

- internal propagators

$$\text{---}\overrightarrow{q}\text{---} = \frac{i}{q^2 - M^2 + i\epsilon}$$

$$\overset{\beta}{\text{---}}\overrightarrow{q}\overset{\alpha}{\text{---}} = \frac{i(\not{p} + m)_{\alpha\beta}}{p^2 - m^2 + i\epsilon}$$

- vertex

$$\begin{array}{c} \alpha \\ \swarrow \\ \bullet \\ \nwarrow \\ \beta \end{array} \text{---} = -ig \int d^4x = ig\delta_{\beta\alpha}$$

- external lines

$$\text{---} \text{---} \overrightarrow{q} \text{---} \text{---} = \text{---} \overleftarrow{q} \text{---} \text{---} = 1$$

– "in" and "out" scalar

$$\text{---} \overrightarrow{p} \text{---} = u_s(p) \text{ and "out" fermion } \overleftarrow{p} \text{---} = \bar{u}_s(p)$$

– "in"-fermion

$$\text{---} \overleftarrow{k} \text{---} = \bar{v}_s(k) \text{ and "out" antifermion } \overrightarrow{k} \text{---} = v_s(k)$$

– "in"-antifermion

- impose energy-momentum conservation at each vertex
- integrate over undetermined (loop) momenta
- include an overall sign for the diagram

**Note**

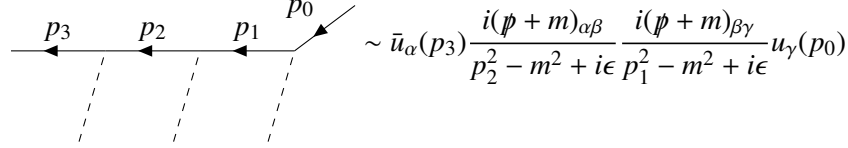
- Arrows on the fermion lines by convention denote fermion (or charge) flow. They must flow consistently through the diagram. ( $\equiv$  fermion number conservation) (Only potential confusion: external antifermion lines)

- No symmetry factors (except vacuum bubbles  $\text{---} \bigcirc \text{---} \frac{1}{s} = \frac{1}{2}$ ).  $\bar{\psi}\psi\phi$  allows for unambiguous contractions.

- Dirac indices are summed over at each vertex

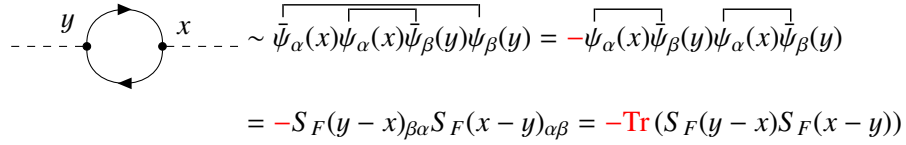
$$\mathcal{L}_{\text{int}} \approx \bar{\psi}_\alpha(x) \psi_\alpha(x) \phi(x)$$

$(\not{p} + m)$  terms in propagator are matrix-multiplied contracted with external spinors, e.g.



$$\sim \bar{u}_\alpha(p_3) \frac{i(\not{p}_2 + m)_{\alpha\beta}}{p_2^2 - m^2 + i\epsilon} \frac{i(\not{p}_1 + m)_{\beta\gamma}}{p_1^2 - m^2 + i\epsilon} u_\gamma(p_0)$$

- closed fermion loop



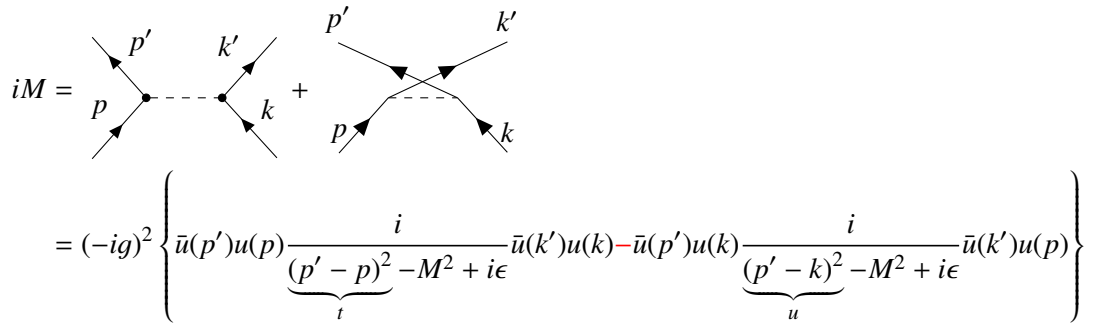
$$\sim \bar{\psi}_\alpha(x) \psi_\alpha(x) \bar{\psi}_\beta(y) \psi_\beta(y) = - \bar{\psi}_\alpha(x) \bar{\psi}_\beta(y) \bar{\psi}_\alpha(x) \bar{\psi}_\beta(y)$$

$$= -S_F(y-x)_{\beta\alpha} S_F(x-y)_{\alpha\beta} = -\text{Tr}(S_F(y-x)S_F(x-y))$$

It always (also with more propagators/couplings) involves an overall  $(-1)$  and a trace  $\text{Tr}(\dots)$ .

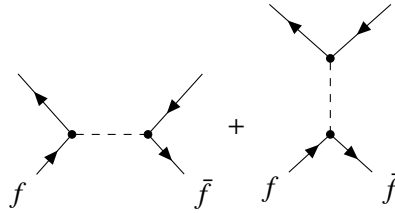
### Examples

- fermion-fermion scattering to lowest order  $\mathcal{O}(g^2)$



$$iM = (-ig)^2 \left\{ \bar{u}(p')u(p) \frac{i}{(p' - p)^2 - M^2 + i\epsilon} \bar{u}(k')u(k) - \bar{u}(p')u(k) \frac{i}{(p' - k)^2 - M^2 + i\epsilon} \bar{u}(k')u(p) \right\}$$

- fermion-antifermion scattering



These are tree diagrams. Thus there is no undetermined momenta to integrate.