

Quantum Field Theory

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3 Quantization of the Dirac field

3.1 Dirac equation

$$(i\gamma^\mu \partial_\mu - m)\phi(x) = 0 \quad (3.1.1)$$

Standard representation (Dirac's)

$$\gamma_0 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix} \quad \boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix} \quad (3.1.2)$$

Lorentz transformation

$$\Lambda = \exp\left(\frac{1}{2}\omega_{\mu\nu}M^{\mu\nu}\right) \quad (3.1.3)$$

with ω set of parameters and M the generator of Lie algebra.

Spinor representation

$$S^{\rho\sigma} = \frac{1}{4} [\gamma^\rho, \gamma^\sigma] = \frac{1}{2i} \sigma^{\rho\sigma} \quad (3.1.4)$$

$$(3.1.5)$$

Spinor transformation

$$S(\Lambda) = \exp\left(\frac{1}{2}\omega_{\mu\nu}S^{\mu\nu}\right) \quad (3.1.6)$$

$$\psi'_a(x) = S_{ab}(\Lambda)\psi_b(\Lambda^{-1}x) \quad (3.1.7)$$

adjoint spinor

$$\bar{\psi} = \psi^\dagger \gamma^0 \quad (3.1.8)$$

Fifth gamma matrix

$$\gamma^5 := i\gamma^0\gamma^1\gamma^2\gamma^3 \quad (3.1.9)$$

$$\{\gamma^\mu, \gamma^5\} = 0 \quad (3.1.10)$$

$$(\gamma^5)^2 = \mathbb{1}_4 \quad (3.1.11)$$

Plane wavesolutions

$$\psi(x) = \begin{cases} u(p)e^{-ipx} & \text{positive frequency} \\ v(p)e^{ipx} & \text{negative frequency} \end{cases} \quad (3.1.12)$$

$$u_s(p) = \sqrt{E_p + m} \begin{pmatrix} \chi_s \\ \frac{\mathbf{u} \cdot \mathbf{p}}{E_p + m} \chi_s \end{pmatrix} e^{-ipx} \quad v_s(p) = \sqrt{E_p + m} \begin{pmatrix} \frac{\mathbf{u} \cdot \mathbf{p}}{E_p + m} \tilde{\chi}_s \\ \tilde{\chi}_s \end{pmatrix} e^{ipx} \quad (3.1.13)$$

with

$$\chi_{\frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \chi_{-\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$s = \pm \frac{1}{2} \quad \tilde{\chi}_s = \chi_{-s}$$

Orthogonality of spinor

$$\bar{u}_s(p)u_{s'}(p) = -\bar{v}_s(p)v_{s'}(p) = 2m\delta_{ss'} \quad (3.1.14)$$

$$\bar{u}_s(p)v_{s'}(p) = 0 \quad (3.1.15)$$

Spin sums

$$\sum_s u_s(p)\bar{u}_s(p) = \not{p} + m \quad (3.1.16)$$

$$\sum_s v_s(p)\bar{v}_s(p) = \not{p} - m \quad (3.1.17)$$

3.2 Dirac Lagrangian and quantization

$$\mathcal{L} = \bar{\psi}(x)(i\not{\partial} - m)\psi(x) \quad (3.2.1)$$

Quantization

$$\{\psi_a(\mathbf{x}), \psi_b^\dagger(\mathbf{x}')\} = \delta_{ab}\delta^{(3)}(\mathbf{x} - \mathbf{x}') \quad (3.2.2)$$

$$\{\psi_a(\mathbf{x}), \psi_b(\mathbf{x}')\} = \{\psi_a^\dagger(\mathbf{x}), \psi_b^\dagger(\mathbf{x}')\} = 0 \quad (3.2.3)$$

Field operators

$$\psi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3 \sqrt{2E_p}} \sum_s (a_{\mathbf{p}}^s u_s(\mathbf{p}) e^{i\mathbf{p} \cdot \mathbf{x}} + b_{\mathbf{p}}^{s\dagger} v_s(\mathbf{p}) e^{-i\mathbf{p} \cdot \mathbf{x}}) \quad (3.2.4)$$

thus the anticommutations of ladder operators:

$$\{a_{\mathbf{p}}^s, a_{\mathbf{p}'}^{s'\dagger}\} = \{b_{\mathbf{p}}^s, b_{\mathbf{p}'}^{s'\dagger}\} = (2\pi)^3 \delta_{ss'} \delta^{(3)}(\mathbf{p} - \mathbf{p}')$$

$$\{a, a\} = \{a^\dagger, a^\dagger\} = \dots = 0$$

Hamiltonian in terms of Fourier modes (with normal ordering)

$$H = \int \frac{d^3 p}{(2\pi)^3} \sum_s E_p (a_p^{s\dagger} a_p^s - b_p^{s\dagger} b_p^s) \quad (3.2.5)$$

3.3 Particles and antiparticles

$$Q = e \int d^3 x \psi^\dagger(x) \psi(x) \quad (3.3.1)$$

$$: Q : = e \int \frac{d^3 p}{(2\pi)^3} \sum_s (a_p^{s\dagger} a_p^s - b_p^{s\dagger} b_p^s) \quad (3.3.2)$$

3.4 Dirac propagator and anticommutators

$$\begin{aligned} S_{ab}(x-y) &= \{\psi_a(x), \bar{\psi}_b(y)\} \\ &= (i\not{\partial} + m) [D(x-y) - D(y-x)] \end{aligned} \quad (3.4.1)$$

Time ordering of Dirac fields

$$T(\phi_a(x) \bar{\psi}_b(y)) = \Theta(x^0 - y^0) \psi_a(x) \bar{\psi}_b(y) - \Theta(y^0 - x^0) \bar{\psi}_b(y) \psi_a(x) \quad (3.4.2)$$

Feynman propagator for the Dirac field

$$S_F(x-y) = \langle 0 | T \psi(x) \bar{\psi}(y) | 0 \rangle = \int \frac{d^4 p}{(2\pi)^4} \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon} e^{-ip \cdot (x-y)} \quad (3.4.3)$$

3.5 Discrete symmetries of the Dirac Field

	orientation preserving	orientation not perserving
(ortho)chronous	\mathcal{L}_+^\uparrow	$\mathcal{L}_-^\uparrow = \mathcal{P} \mathcal{L}_+^\uparrow$
non-orthochronous	$\mathcal{L}_-^\downarrow = \mathcal{T} \mathcal{L}_+^\uparrow$	$\mathcal{L}_+^\downarrow = \mathcal{PT} \mathcal{L}_+^\uparrow$