A.3

a) Plug in the solution:

$$0 \overset{!}{=} (\partial_t^2 - \partial_x^2 + m^2)\phi_\pm(x,t) = (-\omega^2 + p^2 + m^2) f_\pm(p)$$

$$\Rightarrow \quad \omega^2 = p^2 + m^2$$

b) $$\phi_\pm(0,t) = \phi_\pm(L,t)$$

$$\Rightarrow \quad e^{\pm i\omega t} f_\pm(p) = e^{\pm i(\omega t - pL)} f_\pm(p)$$

$$\Rightarrow \quad e^{\pm ipL} = 1 \quad \Leftrightarrow \quad p = p_n = \frac{2\pi n}{L}, \; n \in \mathbb{Z}$$

c) $$\phi(x,t) = \sum_{n=0}^{\infty} \left(e^{i\omega_n t - p_n x} f_+(p_n) + e^{-i\omega_n t + p_n x} f_-(p_n)\right)$$

$$\phi(x,t) \overset{!}{=} \phi^*(x,t)$$

$$\Rightarrow \quad f_\pm^* = f_\mp$$

d) $$\pi(x,t) = \partial_t \phi(x,t) = i\sum_{n=-\infty}^{\infty} \omega_n \left(e^{i\omega_n t - ip_n x} f_+(p_n) - e^{-i\omega_n t + ip_n x} f_-(p_n)\right)$$

e) Fourier series:

$$\omega_m \phi \pm i\pi = \omega_m \sum_n \left(e^{i\omega_n t - p_n x} f_+ + e^{-i\omega_n t + p_n x} f_-\right)$$

$$\mp \sum_n \omega_n \left(e^{i\omega_n t - ip_n x} f_+ - e^{-i\omega_n t + ip_n x} f_-\right)$$

$$= \sum_n e^{i\omega_n t - p_n x} f_+(\omega_n \mp 1) + e^{-i\omega_n t + p_n x} f_-(\omega_m \pm 1)$$

?

$$\text{Integral}^{+}_{m} = \int_0^L dx\,(\omega_m \phi(x,t) - i\pi(x,t))\,\underbrace{e^{-i\omega_n t + ip_m x}}_{e^{-ik^{\mu}_{m} x_\mu}} \quad , \quad \mu = 0,1$$

$$= \sum_{n=-\infty}^{\infty} \int_0^L dx \Big[e^{it(\omega_n - \omega_m) - ix(p_n - p_m)} (\omega_m f_+(p_n) + \omega_n f_+(p_n)) + e^{-it(\omega_m + \omega_n) + ix(p_n + p_m)} (\omega_m f_-(p_n) - \omega_n f_-(p_n)) \Big]$$

$$(\omega_{-n} = \omega_n = \sqrt{p_n^2 + n})$$

$$= L\Big[(\omega_m f_+(p_m) + \omega_m f_+(p_m)) + e^{-2it\omega_m} \underbrace{(\omega_m f_-(p_n) - \omega_{-m} f_-(p_m))}_{=0} \Big]$$

$$= 2L\,\omega_m f_+(p_m)$$

$$\Rightarrow I^{+}_{m} = 2L\,\omega_m f_+(p_m) \quad \Rightarrow \quad f_+(p_m) = \frac{I^{+}_{m}}{2L\omega_m}$$

$$(f_+(p_m))^* = \frac{(I^{+}_{m})^*}{2L\omega_m} = f_-(p_m)$$

f) $[f_\pm(p_m), f_+(p_m)]$

$$\phi = \phi(x,t) \qquad \phi = \phi(y,t)$$

$$= \frac{1}{4L^2 \omega_n \omega_m} \int_0^L dx\,dy\, e^{i(\mp k_m x - k_n y)} [\omega_m \phi \mp i\pi^{x}, \omega_n \phi - i\pi^{y}]$$

$$= \frac{1}{4L^2 \omega_n \omega_m} \int_0^L dx\,dy\, e^{i(\mp k_m x - k_n y)} \Big\{ \omega_m \omega_n \overbrace{[\phi(x,t), \phi(y,t)]}^{=0} - i\omega_m \overbrace{[\phi(x,t), \pi(y,t)]}^{i\delta(x-y)} \mp i\omega_n \overbrace{[\pi(x,t), \phi(y,t)]}^{-i\delta(x-y)} \mp \underbrace{[\pi(x,t), \pi(y,t)]}_{=0} \Big\}$$

$$= \frac{\omega_m \mp \omega_n}{4L^2 \omega_n \omega_m} \int_0^L dx\, e^{i(\mp k_m - k_n)x} = \begin{cases} 0 & , \quad f_+ \\ \dfrac{1}{2L\omega_n} \delta_{mn} & , \quad f_- \end{cases}$$

$$\phi^{+} = \phi \;, \quad \Rightarrow \quad f^{+}_{\pm} = f_{\mp}$$

g) $f_-(p_m) = \frac{1}{\sqrt{2L\omega_m}} a_m$, $f_+(p_m) = \frac{1}{\sqrt{2L\omega_m}} a_m^+$

$$P = -:\int_0^L dx\, \pi(x,t) \frac{\partial\phi(x,t)}{\partial x}:$$

$$= -\frac{1}{2L}\sum_{n,m}\sqrt{\frac{\omega_n}{\omega_m}}\, p_m \int_0^L :[e^{ik_n x} a_n^+ - e^{-ik_n x} a_n][e^{ik_m x} a_m^+ - e^{-ik_m x} a_m]: dx$$

$$= -\frac{1}{2}\sum_n p_n :[e^{2i\omega_n t} a_n^+ a_{-n}^+ - a_n^+ a_n - a_n a_n^+ - e^{-2i\omega_n t} a_n a_{-n}]:$$

4^{th} term $\propto \sum_n p_n e^{-2i\omega_n t} a_n a_{-n} = \frac{1}{2}[\sum_n p_n e^{-2i\omega_n t} + \sum_n \underset{-p_n}{\underset{\downarrow}{p_n}} e^{-2i\omega_n t} \underbrace{a_{-n} a_n}_{-n \leftrightarrow n}]$

$= 0$

1^{st} term $= 0$ for same reason

2^{nd} term $\propto [a_i, a_j^+] = a_i a_j^+ - a_j^+ a_i \Rightarrow a_i a_j^+ = [a_i, a_j^+] + a_j^+ a_i$

$\Rightarrow P = -\frac{1}{2}\sum_n -p_n 2 a_n^+ a_n = \sum_n p_n a_n^+ a_n$,

p_n and $a_n^{(+)}$ do not depend on time

$\Rightarrow \frac{d}{dt} P = 0$

h) $[P, \phi(x,t)] = [-\int_0^L dy\, \pi(y,t) \frac{\partial}{\partial y}\phi(y,t), \phi(x,t)]$

$= -\int_0^L dy\, [\pi(y,t)\frac{\partial}{\partial y}\phi(y,t), \phi(x,t)]$

$= -\int_0^L dy\, [\pi(y,t), \phi(x,t)] \frac{\partial}{\partial y}\phi(y,t) + 0$

$[\frac{\partial}{\partial y}\phi(y,t), \phi(x,t)] \propto [\phi(y,t), \phi(x,t)]$

$= -\int_0^L dy\, (-i)\delta(x-y) \frac{\partial}{\partial y}\phi(x,t)$

$= i\frac{\partial}{\partial x}\phi(x,t)$

$[P, \pi(x,t)] = [P, \partial_t \phi(x,t)] \underset{\partial_t P = 0}{=} \partial_t [P, \phi(x,t)] = \partial_t\, i\partial_x \phi(x,t)$

$= i\partial_x \pi(x,t)$

i) $[H, P] = 0$ Need to show $H = \sum_n \omega_n \underbrace{a_n^+ a_n}_{\hat{n}_n}$, $P = \sum_n p_n \underbrace{a_n^+ a_n}$

$$
\begin{aligned}
[\hat{n}_k, \hat{n}_l] &= [a_k^+ a_k, a_l^+ a_l] \\
&= a_k^+ [a_k, a_l^+] a_l + a_k^+ a_l^+ \underbrace{[a_k, a_l]}_{=0} + [\underbrace{a_k^+, a_l^+}_{=0}] a_l a_k \\
&\quad + a_l^+ [a_k^+, a_l] a_k \\
&= a_k^+ a_k - a_k^+ a_k = 0
\end{aligned}
$$

$$\Rightarrow \quad [H, P] = 0$$