4.9 Chenhuan Wang 16/20

a) $\mathcal{L} = \sum_{i=1}^{3} \frac{1}{2}(\partial_\mu \phi_i)(\partial^\mu \phi_i) - \frac{1}{2} m^2 \left(\sum_{i=1}^{3} \phi_i^2\right) - \frac{\lambda}{8}\left(\sum_{i=1}^{3} \phi_i^2\right)^2$

$\mathcal{H}_{int} = \frac{\lambda}{8}\left(\sum \phi_i^2\right)^2$

$\phi_1(x) = \int \frac{d^3k}{(2\pi)^3 \sqrt{2\omega_k}} \left(e^{-ikx} a_{1,k} + e^{+ik\cdot x} a_{1,k}^\dagger\right)$

$\phi_2(x) = \int \frac{d^3k}{(2\pi)^3 \sqrt{2\omega_k}} \left(e^{-ikx} a_{2,k} + e^{+ik\cdot x} a_{2,k}^\dagger\right)$

$\phi_3(x) = \int \frac{d^3k}{(2\pi)^3 \sqrt{2\omega_k}} \left(e^{-ikx} a_{3,k} + e^{+ik\cdot x} a_{3,k}^\dagger\right)$

$i \neq j \rightarrow \langle 0 | T \phi_i(x) \phi_j(y) | 0 \rangle$

$\propto \langle P_{\phi_i} | P_{\phi_j} \rangle = 0$, orthogonality condition.

$i = j \rightarrow \langle 0 | T \phi_i(x) \phi_i(y) | 0 \rangle = D_F(x-y)$

$\Rightarrow \langle 0 | T \phi_i(x) \phi_i(y) | 0 \rangle = \delta_{ij} D_F(x-y)$

Feynman rules in position space:

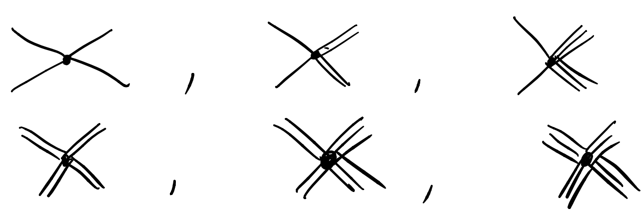
- Propagator

 $= \phi_1(x)\phi_1(y) = D^1_F(x-y)$

 $= \phi_2(x)\phi_2(y) = D^2_F(x-y)$

 $= \phi_3(x)\phi_3(y) = D^3_F(x-y)$

- Vertices



$= -3i\lambda \int d^4x$

3 comes from $\frac{4!}{8} = 3$

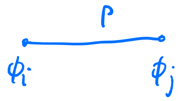
General case?

- external lines

$x \leftarrow p$ $\quad e^{-ip\cdot x}$ $\qquad$ same for ϕ_2 and ϕ_3

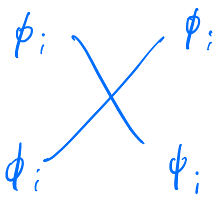
$x \rightarrow p$ $\quad e^{+ip\cdot x}$

- Symmetry factor $\qquad$ +7



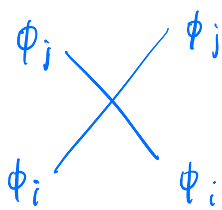
$$= \frac{i\delta_{ij}}{p^2-m^2+i\epsilon}$$

vertex rules: $\langle \phi_i \phi_i | T \exp(-i\int d^4x\, \mathcal{H}_{int}(x)) | \phi_i\ \phi_i \rangle$



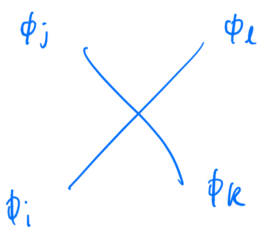
$$\langle \phi_i \phi_i | T [-i\frac{\lambda}{4!}\int d^4x\, \phi_i^4(x)] | \phi_i \phi_i \rangle$$
$$= 4!\ \frac{-3i\lambda}{4!}\int d^4x \langle \phi_i\ \phi_i | \phi_i\ \phi_i\ \phi_i\ \phi_i\ | \phi_i\ \phi_i \rangle$$

$$= -3i\lambda \int d^4x\ \exp(-ix(p_1+p_2-p_3-p_4)]$$
$$= -3i\lambda\ (2\pi)^4\ \delta^{(4)}(p_1+p_2-p_3-p_4)$$



$$= 4\left(\frac{-i\lambda}{4}\right)\int d^4x\ \langle \phi_i\ \phi_j | \phi_i(x)\ \phi_i(x)\ \phi_j(x)\ \phi_j(x) | \phi_i\ \phi_j \rangle$$

$$= -i\lambda\ (2\pi)^4\ \delta^{(4)}(p_1+p_2-p_3-p_4)$$



$$= -i\lambda(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})\ (2\pi)^4\delta^{(4)}(p_1+p_2-p_3-p_4)$$

Another Way

$$\frac{\delta i\mathcal{L}_{int}}{\delta\phi_i} = -\frac{i\lambda}{2}\phi_i\sum_{n=1}^{3}\phi_n^2,\qquad \frac{\delta^2(i\mathcal{L}_{int})}{\delta\phi_j\ \delta\phi_i} = -\frac{i\lambda}{2}\delta_{ij}\sum_{n=1}^{3}\phi_n^3 - i\lambda\phi_i\phi_j$$

$$\frac{\delta^4(i\mathcal{L}_{int})}{\delta\phi_l\delta\phi_n\delta\phi_j\delta\phi_i} = -i\lambda(\delta_{ij}\delta_{nl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})$$

b) $\mathcal{L}_0$, $\mathcal{L}_1 = -g\phi\psi\chi$

$$S_{fi} = \delta_{fi} + i(2\pi)^4 \delta^{(4)}(P_f - P_i)\, M_{fi}$$
$$= \langle f \,|\, T\exp(-i\int d^4x\, \mathcal{H}_{int}(x)) \,|\, i \rangle$$
$$= \langle P_3, P_4 \,|\, T\exp(-i\int d^4x\, \mathcal{H}_{int}(x)) \,|\, P_1, P_2 \rangle$$
$$= \langle P_3, P_4 \,|\, T\, [1 - i\int d^4x\, (-g\phi(x)\psi(x)\chi(x))$$
$$- \frac{1}{2}\int d^4x \int d^4y\, g^2\, \phi(x)\psi(x)\chi(x)\, \phi(y)\psi(y)\chi(y) + \mathcal{O}(g^3)] \,|\, P_1, P_2 \rangle$$

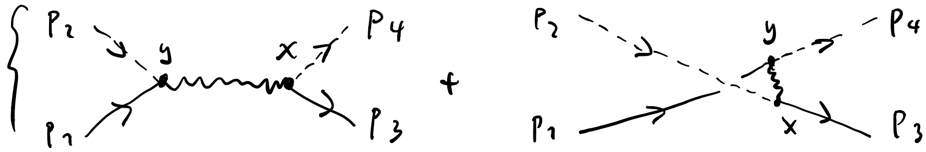
$S^{(0)} = 0$ $\quad S^{(0)} = \langle \phi(P_3)\psi(P_4) | \phi(P_1)\psi(P_2) \rangle = 2E_\phi\, \delta^{(3)}(\vec{P}_1 - \vec{P}_3)\, 2E_\phi \times \delta^{(3)}(\vec{P}_2 - \vec{P}_4)\, (2\pi)^6$

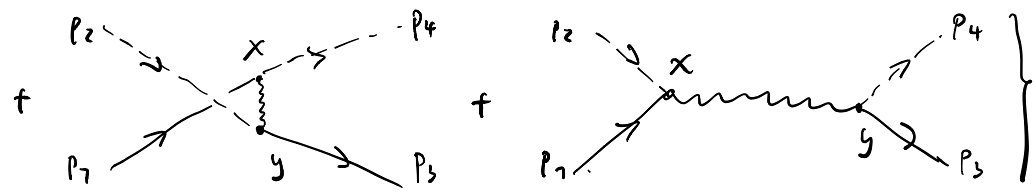
$$S^{(1)} = ig\int d^4x \; \langle \underset{\phi}{\underset{\uparrow}{P_3}}, \underset{\psi}{\underset{\uparrow}{P_4}} \,|\, T\,(\phi(x)\psi(x)\chi(x)) \,|\, \underset{\phi}{\underset{\uparrow}{P_1}}, \underset{\psi}{\underset{\uparrow}{P_2}} \rangle = 0$$

$$S^{(2)} = -\frac{1}{2}g^2 \int d^4x\, d^4y\, \langle P_3, P_4 \,|\, T(\phi(x)\psi(x)\chi(x)\, \phi(y)\psi(y)\chi(y)) \,|\, P_1, P_2 \rangle$$

$$\overset{\text{Wick}}{=} -\frac{1}{2}g^2 \int d^4x\, d^4y \,\{ \langle P_3, P_4 |\, \phi(x)\psi(x)\chi(x)\phi(y)\psi(y)\chi(y) \,| P_1, P_2 \rangle$$
$$+ \langle P_3, P_4 |\, \phi(x)\psi(x)\chi(x)\phi(y)\psi(y)\chi(y) \,| P_1, P_2 \rangle$$
$$+ \langle P_3, P_4 |\, \phi(x)\psi(x)\chi(x)\phi(y)\psi(y)\chi(y) \,| P_1, P_2 \rangle$$
$$+ \langle P_3, P_4 |\, \phi(x)\psi(x)\chi(x)\phi(y)\psi(y)\chi(y) \,| P_1, P_2 \rangle \}$$

$$= -\frac{1}{2}g^2 \int d^4x\, d^4y \;\times$$





The scattering amplitude associated with diagram on sheet :

$$= -g^2 \int d^4x\, d^4y \quad e^{-ip_2 y}\, e^{-ip_1 y}\, D_F^x(y-x)\, e^{ip_3 x}\, e^{ip_4 x}$$

$$= -g^2 \int d^4x\, d^4y\, d^4p\; e^{-i(p_1+p_2)y + i(p_3+p_4)x}$$

$$\frac{1}{(2\pi)^4}\, \frac{i}{p^2-m^2+i\epsilon} \cdot e^{-ip(x-y)}$$

$$= -g^2 \int d^4y\, d^4p\;\; \delta^{(4)}(-p+p_3+p_4)\, e^{-i(p_1+p_2)y}\, \frac{i}{p^2-m^2+i\epsilon}\;\; e^{+ipy}$$

$$= -g^2 (2\pi)^4 \int d^4p\; \frac{i}{p^2+m^2+i\epsilon}\; \delta^{(4)}(-p+p_3+p_4)\, \delta^{(4)}(-p+p_1+p_2)$$

$$= -g^2 (2\pi)^4\; \delta^{(4)}(p_1+p_2-p_3-p_4)\; \frac{i}{(p_1+p_2)^2-m^2+i\epsilon}$$

$$\stackrel{!}{=} i(2\pi)^4\, \delta^{(4)}(p_f-p_i)\; \mathcal{M}$$

$$\Rightarrow \quad \mathcal{M} = \frac{-g^2}{(p_1+p_2)^2-M^2} = \frac{-g^2}{s-M^2}$$

+9

the second type of diagrams $= \dfrac{-g^2}{u-M^2}$