H.8 Chenhuan Wang

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi\, \partial^\mu \phi - m^2 \phi^2) - j(x)\phi \quad , \quad \phi = \phi(x)$$

$$= \mathcal{L}_{Dirac} + \mathcal{L}_{int}$$

$$H_I = -\int d^3x\, j(x)\, \phi_I(x)$$

a) $$A = \langle 0 | T \exp\left(-i \int dt\, H_I(t)\right) |0\rangle$$

$$= \langle 0 | T \left[1 - i\int dt H_I(t) + \frac{(-i)^2}{2!}\left(\int dt\, H_I(t)\right)^2 + \mathcal{O}(j^3) \right] |0\rangle$$

$$= \langle 0 | T | 0 \rangle + i \langle 0 | T \int d^4x\, j(x)\, \phi_I(x) |0\rangle$$

$$- \frac{1}{2} \langle 0 | T \left[\int d^4x\, j(x)\, \phi_I(x) \right]^2 |0\rangle + \mathcal{O}(j^3)$$

$$= 1 + i \underbrace{\langle 0 | \int d^4x\, j(x)\, \phi_I(x) |0\rangle}_{=0} \underbrace{- \frac{1}{2} \langle 0 | T \int d^4x\, j(x)\, \phi_I(x) \int d^4y\, j(y)\, \phi_I(y) |0\rangle}_{} + \mathcal{O}(j^3)$$

$$= -\frac{1}{2} \int d^4x \int d^4y\, \langle 0 | T\, \phi_I(x)\, \phi_I(y)\, j(x) j(y) |0\rangle$$

extra factor 2 comes from interchanging vertices ←

$$= - \int d^4x \int d^4y\, \langle 0 | \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\varepsilon} e^{-ip(x-y)}\, j(x) j(y) |0\rangle$$

$$= - \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\varepsilon} \underbrace{\tilde{j}(p)\, \tilde{j}(-p)}_{= |\tilde{j}(p)|^2} = (13)$$

Or:

$$\phi_I(x) = \phi^+(x) + \phi^-(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} a_p e^{-ipx} + \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} a_p^{+} e^{+ipx}$$

$$= -\frac{1}{2} \int d^4x \int d^4y\, \langle 0 | T [\phi_I^+(x), \phi_I^-(y)]\, j(x) j(y) |0\rangle$$

$$= - \int d^4x \int d^4y\, \langle 0 | \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip(x-y)}\, j(x) j(y) |0\rangle$$

$$= - \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} |\tilde{j}(p)|^2 = (14)$$

$$\Rightarrow \quad P(0 \rightarrow 0) = 1 - \lambda + \mathcal{O}(j^3)$$

b) $A = \langle 0 | T \exp(-i\int dt\, H_I(t)) |0\rangle$

$$= \langle 0 | T \sum_{n=0}^{\infty} \frac{1}{n!} \left[-i\int dt\, H_I(t)\right]^n |0\rangle$$

$$= \langle 0 | T \sum_{n=0}^{\infty} \frac{1}{(2n)!} \left[+i\int d^4x\, j(x)\phi_I(x)\right]^{2n} |0\rangle,$$ since $\langle 0 | a_p^{(+)} |0\rangle = 0$,

terms with odd numbers of field will disappear.

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \int d^4x_1 \dots d^4x_{2n} \langle 0 | T(\phi_1 \dots \phi_{2n}\, j_1 \dots j_{2n}) |0\rangle$$

Wick's theorem: $T\,\phi_I(x_1)\dots\phi_I(x_m) = :\phi_I(x_1)\dots\phi_I(x_m): + :(\text{all possible contractions}):$

$\Rightarrow \langle 0 | T\,\phi_I(x_1)\dots\phi_I(x_m) |0\rangle = \langle 0 | (\text{fully contracted fields}) |0\rangle$

(i.e. no normal ordered fields)

Here the contracted fields give the same result since they are all inside the integral. Only need to compute the number of the all possible full contractions. The number of full contractions.

$$N_{2n} = \prod_{k=2,4,6\dots}^{2n} \binom{k}{2} \Big/ \underset{\substack{\nearrow \\ \text{permutation}}}{n!} \cdot \underset{\substack{\uparrow \\ \text{since all the fields} \\ \text{are the same}}}{2^n} = \frac{(2n)!}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{\cancel{(2n)!}} \frac{\cancel{(2n)!}}{n!} \lambda^n = \exp(-\lambda)$$