Quantum Field Theory

May 10, 2019

1 Classical field theory

1.1 Field theory in continuum

Euler-Lagrange-equation

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \tag{1.1.1}$$

momentum density

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}(\boldsymbol{x})} \tag{1.1.2}$$

Hamiltonian density

$$\mathcal{H}(\phi(\boldsymbol{x}), \pi(\boldsymbol{x})) = \pi(\boldsymbol{x})\dot{\phi}(\boldsymbol{x}) - \mathcal{L}(\phi, \partial_\mu \phi) \tag{1.1.3}$$

1.2 Noether Theorem

If a Lagrangian field theory has an infinitisimal symmetry, then there is an associated current j^μ, which is conserved.

$$\partial_\mu j^\mu = 0 \tag{1.2.1}$$

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta\phi - X^m u \tag{1.2.2}$$

Energy-momentum tensor (stress-energy tensor)
Asymmetric version

$$\Theta^\mu_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu_\nu \mathcal{L} \tag{1.2.3}$$

General version

$$T^{\mu\nu} = \Theta^{\mu\nu} + \partial_\lambda f^{\mu\nu\lambda} \tag{1.2.4}$$

with $f^{\lambda\mu\nu} = -f^{\mu\lambda\nu}$ or $\partial_\mu \partial \nu f^{\lambda\mu\nu} = 0$

2 Klein-Gordon theory

(Real) Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 \tag{2.0.1}$$

Quantization

$$\begin{aligned}[\phi(\boldsymbol{x}), \phi(\boldsymbol{x}')] = [\pi(\boldsymbol{x}), \pi(\boldsymbol{x}')] = 0 \\ [\phi(\boldsymbol{x}), \pi(\boldsymbol{x}')] = i\delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}')\end{aligned} \tag{2.0.2}$$

Decomposition into Fourier modes

$$\phi(\boldsymbol{x}) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(a_{\boldsymbol{p}}e^{i\boldsymbol{p}\cdot\boldsymbol{x}} + a_{\boldsymbol{p}}^\dagger e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\right) \tag{2.0.3}$$

$$\pi(\boldsymbol{x}) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3}(-i)\sqrt{\frac{E_p}{2}}\left(a_{\boldsymbol{p}}e^{i\boldsymbol{p}\cdot\boldsymbol{x}} - a_{\boldsymbol{p}}^\dagger e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\right) \tag{2.0.4}$$

thus the commutation relations for ladder operators:

$$\left[a_{\boldsymbol{p}}, a_{\boldsymbol{p}'}\right] = \left[a_{\boldsymbol{p}}^\dagger, a_{\boldsymbol{p}'}^\dagger\right] = 0 \tag{2.0.5}$$

$$\left[a_{\boldsymbol{p}}, a_{\boldsymbol{p}'}^\dagger\right] = (2\pi)^3\delta^{(3)}(\boldsymbol{p} - \boldsymbol{p}') \tag{2.0.6}$$

Hamiltonian in terms of ladder operator

$$H = \int \frac{\mathrm{d}^3 p}{(2\pi)^3}E_p\left(a_{\boldsymbol{p}}a_{\boldsymbol{p}}^\dagger + \frac{1}{2}\left[a_{\boldsymbol{p}}, a_{\boldsymbol{p}}^\dagger\right]\right) \tag{2.0.7}$$

Normlisation it's also lorentz-invariante

$$\langle p|q\rangle = 2E_p(2\pi)^3\delta^{(3)}(\boldsymbol{p} - \boldsymbol{q}) \tag{2.0.8}$$

2.1 Heisenberg-picture fields

Heisenberg-picture

$$|\psi_H\rangle = e^{iHt}|\psi_s(t)\rangle \tag{2.1.1}$$

$$O_H(t) = e^{iHt}O_S e^{-iHt} \tag{2.1.2}$$

Field operator

$$\phi(x) = \phi(\boldsymbol{x}, t) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(a_p e^{ipx} + a_p^\dagger e^{-ipx}\right) \tag{2.1.3}$$

2.2 Commutations and propogators

Commutations

$$[\phi(x), \phi(y)] = D(x-y) - D(y-x) \begin{cases} = 0 & \text{if } (x-y) \text{ is space-like} \\ \neq 0 & \text{otherwise} \end{cases} \tag{2.2.1}$$

$$D(x-y) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip(x-y)} \tag{2.2.2}$$

Propogator

$$\langle 0|\phi(x)\phi(y)|0\rangle = D(x-y) \tag{2.2.3}$$

Feynman propagator

$$\begin{aligned} D_F(x-y) &= \langle 0|T\phi(x)\phi(y)|0\rangle \\ &= \Theta(x^0 - y^0)D(x-y) + \Theta(y^0 - x^0)D(y-x) \end{aligned} \tag{2.2.4}$$

$$D_F(x-y) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip(x-y)} \tag{2.2.5}$$

3 Quantization of the Dirac field

3.1 Dirac equation

$$\left(i\gamma^{\mu}\partial_{\mu} - m\right)\phi(x) = 0 \tag{3.1.1}$$

Standard representation (Dirac's)

$$\gamma_0 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix} \quad \boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix} \tag{3.1.2}$$

Lorentz transformation

$$\Lambda = \exp\left(\frac{1}{2}\omega_{\mu\nu}M^{\mu\nu}\right) \tag{3.1.3}$$

with ω set of parameters and M the generator of Lie algebra.

Spinor representation

$$S^{\rho\sigma} = \frac{1}{4}\left[\gamma^{\rho}, \gamma^{\sigma}\right] = \frac{1}{2i}\sigma^{\rho\sigma} \tag{3.1.4}$$

$$\tag{3.1.5}$$

Spinor transformation

$$S(\Lambda) = \exp\left(\frac{1}{2}\omega_{\mu\nu}S^{\mu\nu}\right) \tag{3.1.6}$$

$$\psi'_a(x) = S_{ab}(\Lambda)\psi_b(\Lambda^{-1}x) \tag{3.1.7}$$

adjoint spinor

$$\bar{\psi} = \psi^{\dagger}\gamma^0 \tag{3.1.8}$$

Fifth gamma matrix

$$\gamma^5 := i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{3.1.9}$$

$$\left\{\gamma^{\mu}, \gamma^5\right\} = 0 \tag{3.1.10}$$

$$(\gamma^5)^2 = \mathbb{1}_4 \tag{3.1.11}$$

Plane wavesolutions

$$\psi(x) = \begin{cases} u(p)e^{-ipx} & \text{positive frequency} \\ v(p)e^{ipx} & \text{negative frequency} \end{cases} \tag{3.1.12}$$

$$u_s(p) = \sqrt{E_p + m}\begin{pmatrix} \chi_s \\ \frac{\boldsymbol{u}\cdot\boldsymbol{p}}{E_p+m}\chi_s \end{pmatrix} e^{-ipx} v_s(p) = \sqrt{E_p + m}\begin{pmatrix} \frac{\boldsymbol{u}\cdot\boldsymbol{p}}{E_p+m}\tilde{\chi}_s \\ \tilde{\chi}_s \end{pmatrix} e^{ipx} \tag{3.1.13}$$

with

$$\chi_{\frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad x_{-\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$s = \pm\frac{1}{2} \quad \tilde{\chi}_s = \chi_{-s}$$

Orthogonality of spinor

$$\bar{u}_s(p)u_{s'}(p) = -\bar{v}_s(p)v_{s'}(p) = 2m\delta_{ss'} \tag{3.1.14}$$
$$\bar{u}_s(p)v_{s'}(p) = 0 \tag{3.1.15}$$

Spin sums

$$\sum_s u_s(p)\bar{u}_s(p) = \not{p} + m \tag{3.1.16}$$
$$\sum_s v_s(p)\bar{v}_s(p) = \not{p} - m \tag{3.1.17}$$

3.2 Dirac Lagrangian and quantization

$$\mathcal{L} = \bar{\psi}(x)(i\not{\partial} - m)\psi(x) \tag{3.2.1}$$

Quantization

$$\left\{\psi_a(\boldsymbol{x}), \psi_b^\dagger(\boldsymbol{x}')\right\} = \delta_{ab}\delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}') \tag{3.2.2}$$
$$\{\psi_a(\boldsymbol{x}), \psi_b(\boldsymbol{x}')\} = \left\{\psi_a^\dagger(\boldsymbol{x}), \psi_b^\dagger(\boldsymbol{x}')\right\} = 0 \tag{3.2.3}$$

Field operators

$$\psi(\boldsymbol{x}) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3\sqrt{2E_p}} \sum_s (a_{\boldsymbol{p}}^s u_s(\boldsymbol{p})e^{i\boldsymbol{p}\cdot\boldsymbol{x}} + b_{\boldsymbol{p}}^{s\,\dagger} v_s(\boldsymbol{p})e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}) \tag{3.2.4}$$

thus the anticommutations of ladder operators:

$$\left\{a_{\boldsymbol{p}}^s, a_{\boldsymbol{p}'}^{s'\,\dagger}\right\} = \left\{b_{\boldsymbol{p}}^s, b_{\boldsymbol{p}'}^{s'\,\dagger}\right\} = (2\pi)^3\delta_{ss'}\delta^{(3)}(\boldsymbol{p} - \boldsymbol{p}')$$
$$\{a, a\} = \left\{a^\dagger, a^\dagger\right\} = \cdots = 0$$

Hamiltonian in terms of Fourier modes (with normal ordering)

$$H = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \sum_s E_p (a_{\boldsymbol{p}}^{s\dagger} a_{\boldsymbol{p}}^s - b_{\boldsymbol{p}}^{s\dagger} b_{\boldsymbol{p}}^s) \tag{3.2.5}$$

3.3 Particles and antiparticles

$$Q = e \int \mathrm{d}^3 x \psi^\dagger(x)\psi(x) \tag{3.3.1}$$

$$: Q : = e \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \sum_s (a_{\boldsymbol{p}}^{s\dagger} a_{\boldsymbol{p}}^s - b_{\boldsymbol{p}}^{s\dagger} b_{\boldsymbol{p}}^s) \tag{3.3.2}$$

3.4 Dirac propagator and anticommutators

$$\begin{aligned} S_{ab}(x-y) &= \{\psi_a(x), \bar{\psi}_b(y)\} \\ &= (i\partial\!\!\!/ + m)\left[D(x-y) - D(y-x)\right] \end{aligned} \tag{3.4.1}$$

Time ordering of Dirac fields

$$T(\phi_a(x)\bar{\psi}_b(y)) = \Theta(x^0 - y^0)\psi_a(x)\bar{\psi}_b(y) - \Theta(y^0 - x^0)\bar{\psi}_b(y)\psi_a(x) \tag{3.4.2}$$

Feynman propogator for the Dirac field

$$S_F(x-y) = \langle 0|T\psi(x)\bar{\psi}(y)|0\rangle = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i(p\!\!\!/ + m)}{p^2 - m^2 + i\epsilon} e^{-ip\cdot(x-y)} \tag{3.4.3}$$

3.5 Discrete symmetries of the Dirac Field

	orientation perserving	orientation not perserving
(ortho)chronous	$\mathcal{L}_+^\uparrow$	$\mathcal{L}_-^\uparrow = \mathcal{P}\mathcal{L}_+^\uparrow$
non-orthochronous	$\mathcal{L}_-^\downarrow = \mathcal{T}\mathcal{L}_+^\uparrow$	$\mathcal{L}_+^\downarrow = \mathcal{P}\mathcal{T}\mathcal{L}_+^\uparrow$

4 Interacting QFT

4.1 Introduction and examples

Theories discussed so far are Klein-Gordon theory (spin 0)

$$\mathcal{L}_{KG} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2$$

and Dirac theory (spin $\frac{1}{2}$)

$$\mathcal{L}_D = \bar{\psi}(i\partial\!\!\!/ - m)\psi$$

There is also $\mathcal{L}_{EM} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ for a massless vector filed. Its quantiasation gives photon

One thing they have in common is quadratic in the fields. As result:

- linear field equations
- exact quantisation
- multi-particle states without scattering or interaction
- linear fourier decompositions , no mementum changes

To have an interacting theory with scattering, need higher powers in the field in the Lagrangians. A few examples are following

scalar ϕ^4 theory

$$\mathcal{L} = \mathcal{L}_{KG} + \frac{\lambda}{4!}\phi^4 \tag{4.1.1}$$

need positive sign $\lambda > 0$ for a stable theory, otherwise classical energy can be arbitarily negative.

Equation of motions

$$(\partial^2 + m^2)\phi = -\frac{\lambda}{3!}\phi^3 \tag{4.1.2}$$

is nonlinear, cannot be solved by Fourier decomposition.

Yukawa-theory

$$\mathcal{L} = \mathcal{L}_{KG} + \mathcal{L}_D - g\bar{\psi}\psi\phi \tag{4.1.3}$$

It is originally developed as a theory for nuclear forces with ψ nucleon, ϕ pion. In the Standard Model it is similar to interactions in Higgs mechanism.

Quantum Electrodynamics (QED)

$$\mathcal{L} = \mathcal{L}_{EM} + \mathcal{L}_D - eA_\mu \bar{\psi}\gamma^\mu \psi \tag{4.1.4}$$

descreibes electrons, their antiparticles positrons and photons.

Yang-Mills theory generalises $\mathcal{L}_{EM}$ with terms like A^4 or $A^2\partial A$

Scalar QED descreibes pions and photons

$$\begin{aligned}\mathcal{L} &= \mathcal{L}_{EM} + D_\mu\phi D^\mu\phi^* - m^2|\phi|^2 \\ &= \mathcal{L}_{EM} + \partial_\mu\phi\partial^\mu\phi^* - m^2\phi\phi^* + ieA_\mu(\phi\partial^\mu\phi^* - \phi^*\partial^\mu\phi) + e^2A_\mu A^\mu\phi\phi^*\end{aligned} \tag{4.1.5}$$

Remarks

1. Interaction terms in $H_{\text{int}} = \int \mathrm{d}^3\mathcal{H}_{\text{int}} = -\int \mathrm{d}^3x\mathcal{L}_{\text{int}}$ always involves products of fields at the same point $\boldsymbol{x}$. It ensures causality, no "instant at a distance".
2. There are no derivative interactions. These may complicate quantisation as
$$\pi(\boldsymbol{x}) = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi(\boldsymbol{x}))}$$
3. Why the examples above? There must be zillions of theories (Lagrangians)?
We have the criterion of **renormalizability**. Note the mass dimensions of fields;
$$[S] = 1 \text{ so } [\mathcal{L}] = [M]^4 \Rightarrow [\phi] = [M],\ [\psi] = [M]^{\frac{3}{2}},\ [A_\mu] = [M]$$
So in all the interaction terms indicated above, the coupling constant λ, e, g are all **dimensionless**! Can add $-\frac{\mu}{3!}\phi^3$ to the ϕ^4 theory. This leads to $[\mu] = [M]$ and all these generate renormalisable interactions.
All higher interaction terms require coupling constants of **negative** mass dimension. e.g. $G\bar{\psi}\psi\bar{\psi}\psi$ and then $[G] = [M]^{-2}$. These are nonrenormalisable and create trouble when performing higher-order calculation in perturbation theory. (with energy cutoff; corrections $G\Lambda^2$, $\Lambda \to \infty$)
4. we haven't quantised the photon yet. The reason is that its is a vector field, i.e. 4 degrees of freedom, but photon has just 2 physical polarisaion states. It is linked to gauge symmetry and complicates quantisation somewhat.

4.2 The interaction picture

Consider the ϕ^4 theory,

$$\mathcal{L}_{int} = -\frac{\lambda}{4!}\phi(x)^4 \tag{4.2.1}$$

Hamiltonian $H = H_0 + H_{int}$ with

$$H_0 = \int \mathrm{d}^3x \left\{\frac{1}{2}\pi^2(x) + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\right\} \tag{4.2.2}$$

$$H_{int} = -\int \mathrm{d}^3x\mathcal{L}_{int} = \frac{\lambda}{4!}\int \mathrm{d}^3x\phi^4 \tag{4.2.3}$$

Interaction picture means that operators evolve in time using H_0 (only), in particular

$$\phi_I(t,\boldsymbol{x}) = e^{iH_0t}\phi(\boldsymbol{x})e^{-iH_0t} \tag{4.2.4}$$

Time-dependence of the free field, obeys classical equation of motion $\left(\partial^2 + m^2\right)\phi_I(t,\boldsymbol{x}) = 0$. Solution in terms if fourier modes as before:

$$\phi_I = \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_p}}(a^I_{\boldsymbol{p}}e^{-ipx} + a^{I\,\dagger}_{\boldsymbol{p}}e^{+ipx}) \tag{4.2.5}$$

as in the free theory with standard commutation relations $[a^I_{\boldsymbol{p}}, a^{I\,\dagger}_{\boldsymbol{p}}] = (2\pi)^3\delta^{(3)}(\boldsymbol{p}-\boldsymbol{p}')$. The state satisfing $a^I_p\,|0\rangle = 0$ is the vacuum of the free, noninteratcting theory.

Relation between interaction and Schrödinger picure states:

$$|\phi_I(t)\rangle = e^{iH_0t}|\psi_S(t)\rangle \tag{4.2.6}$$

Schrödinger equation becomes:

$$\begin{aligned}
i\frac{\partial}{\partial t}\,|\psi_S\rangle &= (H_0 + H_{\text{int}})\,|\psi_S\rangle \\
\text{LHS} &= i\frac{\partial}{\partial t}(e^{-iH_0t}\,|\phi_I\rangle) = H_0e^{-iH_0t}\,|\phi_I\rangle + e^{-iH_0t}i\frac{\partial}{\partial t}\,|\phi_I\rangle \\
\text{RHS} &= (H_0 + H_{\text{int}})\,e^{-iH_0t}\,|\phi_I\rangle \\
\Rightarrow i\frac{\partial}{\partial t}\,|\phi_I\rangle &= e^{iH_0t}H_{int}e^{-iH_0t} = H_I(t)\,|\phi_I\rangle
\end{aligned} \tag{4.2.7}$$

with H_I interaction Hamiltonian in the interaction picture. Clearly

$$H_I = \frac{\lambda}{4!}\int \mathrm{d}^3x\phi_I^4(x)$$

What is the solution of 4.2.7 for the time evolution of $|\phi_I(t)\rangle$? Define time-evolution operator in the interaction picture.

$$|\phi_I(t)\rangle = U(t,t_0)\,|\phi_I(t_0)\rangle \tag{4.2.8}$$

$$\text{where } U(t,t_0) = e^{iH_0(t-t_0)}e^{-iH(t-t_0)} \tag{4.2.9}$$

With 4.2.7 and 4.2.8:

$$i\frac{\partial}{\partial t}U(t,t_0) = H_I(t)U(t,t_0) \tag{4.2.10}$$

To solve with boundary conditions: $U(t_0,t_0) = \mathbb{1}$. The formal solution:

$$U(t,t_0) = 1 - i\int_{t_0}^{t}\mathrm{d}t'\,H_I(t')U(t',t_0)$$

Substitute back in and we get:

$$U(t,t_0) = 1 - i\int_{t_0}^{t}\mathrm{d}t'\,H_I(t') + (-i)^2\int_{t_0}^{t}\mathrm{d}t'\int_{t_0}^{t'}\mathrm{d}t''\,H_I(t')H_I(t'') + \ldots \tag{4.2.11}$$

Ranges of integration: H_I in the product is automatically time-ordered.

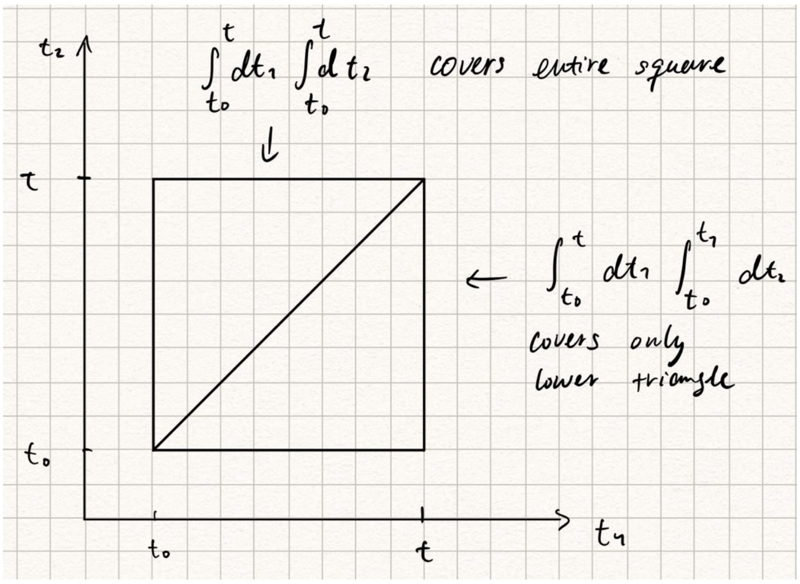

Figure 4.1: Time ordering

Upper triangle has the wrong time order. We are going to "repair" it by hand.

$$\begin{aligned}U(t,t_0) &= 1 - i\int_{t_0}^{t} \mathrm{d}t' H_I(t') + \frac{(-i)^2}{2}\int_{t_0}^{t}\mathrm{d}t'\int_{t_0}^{t'}\mathrm{d}t'' T(H_I(t')H_I(t'')) + \dots \\ &= \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int_{t_0}^{t}\mathrm{d}t_1\cdots\int_{t_0}^{t}\mathrm{d}t_n T(H_I(t_1)\dots H_I(t_n)) \\ &= T\exp\left\{-i\int_{t_0}^{t}\mathrm{d}t' H_I(t')\right\}\end{aligned} \tag{4.2.12}$$

It is interesing for scattering to transition into asymptotic state for $t \to \infty$

$$\begin{aligned}S &= \lim_{t\to\infty} U(t,-t) = T\exp\left\{-i\int_{-\infty}^{\infty}\mathrm{d}t H_I(t)\right\} \\ &\overset{\phi^4}{=} T\exp\left\{-i\int \mathrm{d}^4x \frac{\lambda}{4!}\phi_I^4(x)\right\}\end{aligned} \tag{4.2.13}$$

Both U and S are formally unitary

Composition law for time evolution operator:

$$U(t_2,t_0) = U(t_2,t_1)U(t,t_0) = U(t_2,t_1)U(t_0,t_1)^\dagger \tag{4.2.14}$$

4.2.1 Scattering amplitudes and the S-matrix

Take $|i\rangle$ the initial (multi-particle) state and $|f\rangle$ the final (multi-particle) state. Time evolution of $|i\rangle$ then is

$$\lim(t\to\infty)U(t,-\infty)\,|i\rangle = S\,|i\rangle$$

Probability that $|i\rangle$ evolves into $|f\rangle$ is proporional to the squared "**S-matrix element**"

$$|\langle f,\ t\to\infty|i, t\to-\infty\rangle|^2 = |\langle f|S\,|i\rangle|^2 = \left|S_{fi}\right|^2 \tag{4.2.15}$$

The nontrivial part of the S-matrix is the T-matrix:

$$S_{fi} := \delta_{fi} + iT_{fi} \tag{4.2.16}$$

Use momentum conservation (from translation invariance) to define matrix element

$$S_{fi} = \delta_{fi} + i(2\pi)^4\delta^{(4)}(p_f - p_i)M_{fi} \tag{4.2.17}$$

M_{fi} measures "genuine scattering" from $|i\rangle$ to $|f\rangle$.
How are we going to calculate correlation functions in the interacting theory:

$$\langle\Omega|\, T\phi(x)\phi(y)|\Omega\rangle \tag{4.2.18}$$

or more generally $\langle\Omega|T\phi(x_1)\phi(x_2)\ldots|\Omega\rangle$, where $|\Omega\rangle$ is the vaccum/ground state of the interacting theory and $\phi(x)$ the Heisenberg operators.
Ignore $|\Omega\rangle \neq |0\rangle$ for the moment other than saying: we want to study the time evolution from the vacuum at $t \to -\infty$ to $t \to +\infty$. So rewriting in terms $\phi_I(x)$, assuming $x^0 > y^0$ for now:

$$\langle 0|U(\infty, x^0)\phi_I(x^0)U(x^0, y^0)\phi_I(y^0)U(y^0, -\infty)|0\rangle = \langle 0|T(\phi_I(x)\phi_I(y)S)|0\rangle \tag{4.2.19}$$

still holds if $x^0 < y^0$ because of T.
Now $|\Omega\rangle \neq |0\rangle$: this can be taken care of by dividing out the time evolution of the (free) vacuum $\langle 0|S|0\rangle$, so

$$\begin{aligned}\langle\Omega|T(\phi(x)\phi(y))|\Omega\rangle&\\ &= \frac{\langle 0|T(\phi_I(x)\phi_I(y)S)|0\rangle}{\langle 0|S|0\rangle}\\ &\overset{\phi^4}{=} \frac{\langle 0|T\phi_I(x)\phi_I(y)\exp\left\{-i\int \mathrm{d}^4x' \frac{\lambda}{4!}\phi^4(x')\right\}|0\rangle}{\langle 0|T\exp\left\{-i\int \mathrm{d}^4x' \frac{\lambda}{4!}\phi^4(x')\right\}|0\rangle}\end{aligned} \tag{4.2.20}$$

Proof can be found in Peskin. It will also be illustrated parctically later ("vacuum bubbles").
Perturbation theory is viable when λ (or some other coupling) is "small" and then expands $U(t, t_0)$ or S in powers of λ.

4.3 Wick's theorem

From now on drop the subscript for interaction pictire fields $\phi_I(x) \to \phi(x)$.
Want to calculate stuff like $\langle 0|T\phi(x_1)\ldots\phi(x_n)S|0\rangle$ in pert. theory; so e.g. at order λ^n. So

$$\frac{1}{n!}\left(-i\frac{\lambda}{4!}\right)^n \int \mathrm{d}^4y_1\ldots\mathrm{d}^4y_n\, \langle 0|T\phi(x_1)\ldots\phi(x_n)\phi^4(y_1)\ldots\phi^4(y_n)|0\rangle \tag{4.3.1}$$

is tough!
We know $\langle 0|T\phi(x_1)\phi(x_2)|0\rangle$ is the Feynman propagator!
Recall **normal ordering** with $\phi(x) = \phi^+(x) + \phi^-(x)$

$$:\phi^+\phi^- :=: \phi^-\phi^+ := \phi^-\phi^+ \tag{4.3.2}$$

Wick's therem expresses time-ordered products in terms of normal-ordered ones. Then it is easy to take vacuum expectation values, as $\langle 0| : \phi(x_1)\ldots\phi(x_n) : |0\rangle = 0$

Take two fields and $x^0 > y^0$:

$$\begin{aligned}
T\phi(x)\phi(y) = \phi(x)\phi(y) &= \left(\phi^+(x) + \phi^-(x)\right)\left(\phi^+(y) + \phi^-(y)\right) \\
&= \phi^+(x)\phi^+(y) + \phi^-(x)\phi^-(y) + \phi^-(x)\phi^+(y) + \phi^-(x)\phi^+(y) + [\phi^+(x), \phi^-(y)] \\
&=: \phi(x)\phi(y) : + [\phi^+(x), \phi^-(y)]
\end{aligned}$$

Particularly for $y^0 > x^0$:

$$T\phi(x)\phi(y) =: \phi(x)\phi(y) : + [\phi^+(y), \phi^-(x)]$$

Thus altogether:

$$T\phi(x)\phi(y) =: \phi(x)\phi(y) : + D_f(x - y) \tag{4.3.3}$$

as $\Theta(x^0 - y^0)[\phi^+(x), \phi^-(y)] + \Theta(y^0 - x^0)[\phi^+(y), \phi^-(x)] = D_F(x - y)$.

Worth noting that $D_F(x - y)$ is still a c-number, not operator (yet). Thus it can be pulled out of any matrix element or expectation value.

We now define "contraction":

$$\wick{\c{\phi}(x_1)\c{\phi}(x_2)} = D_F(x_1 - x_2) \tag{4.3.4}$$

Thus we can remove the fields from the product leaving only the propagators:

$$T\phi(x)\phi(y) =: \phi(x)\phi(y) : + \wick{\c{\phi}(x)\c{\phi}(y)} \tag{4.3.5}$$

General form of **Wick's theorem** for arbitary number of fields

$$T\phi(x_1)\dots\phi(x_n) =: \phi(x_1)\dots\phi(x_n) : + : (\text{sum over all possible contractions}) : \tag{4.3.6}$$

Example with four fields:

$$\begin{aligned}
T(\phi_1\phi_2\phi_3\phi_4) =&: \phi_1\phi_2\phi_3\phi_4 : \\
&+ \wick{\c{\phi_1}\c{\phi_2}} : \phi_3\phi_4 : + \wick{\c{\phi_1}\c{\phi_3}} : \phi_2\phi_4 : + \wick{\c{\phi_1}\c{\phi_4}} : \phi_2\phi_3 : + \wick{\c{\phi_2}\c{\phi_3}} : \phi_1\phi_4 : + \wick{\c{\phi_2}\c{\phi_4}} : \phi_1\phi_3 : + \wick{\c{\phi_3}\c{\phi_4}} : \phi_1\phi_2 : \\
&+ \wick{\c{\phi_1}\c{\phi_2}\c{\phi_3}\c{\phi_4}} + \wick{\c{\phi_1}\c{\phi_3}\c{\phi_2}\c{\phi_4}} + \wick{\c{\phi_1}\c{\phi_4}\c{\phi_2}\c{\phi_3}}
\end{aligned}$$

Thus

$$\langle 0| T\left(\phi_1\phi_2\phi_3\phi_4\right) |0\rangle = D_F(x_1 - x_2)D_F(x_3 - x_4) + D_F(x_1 - x_3)D_F(x_2 - x_4) + D_F(x_1 - x_4)D_F(x_2 - x_3)$$

which can be visually represented as

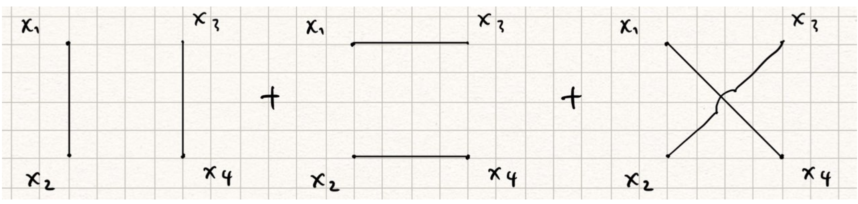

Figure 4.2: Feynman diagrams