

Quantum Field Theory

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5 Quantum Electrodynamics (QED)

5.1 Classical Electrodynamics and Maxwell's equations

We have the gauge potential $A^\mu = (A^0, \mathbf{A}) = (\phi, \mathbf{A})$ & $A_\mu = (A^0, -\mathbf{A}) = (\phi, -\mathbf{A})$ and the field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

Then

- electric field $E_i = F_{0i} = \partial_0 A_i - \partial_i A_0 \rightarrow \mathbf{E} = -\dot{\mathbf{A}} - \nabla\phi$
- magnetic field $B^i = -\frac{1}{2}\epsilon^{ijk}F_{jk} \rightarrow \mathbf{B} = \nabla \times \mathbf{A}$

Lagrangian density $\mathcal{L}_{EM} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{2}(\mathbf{E} \cdot \mathbf{E} - \mathbf{B} \cdot \mathbf{B})$. The field equation $\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu A_\nu)} \right) - \frac{\partial \mathcal{L}}{\partial A_\nu} = 0$ leads to

$$\partial_\mu F^{\mu\nu} = 0 \quad (5.1.1)$$

it is half of Maxwell's equations (in vacuum).

The other half are Bianchi identities following from the definition of $F_{\mu\nu}$:

$$\begin{aligned} \partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} &= 0 \Leftrightarrow \epsilon^{\sigma\lambda\mu\nu} \partial_\lambda F_{\mu\nu} = 0 \\ \text{or } \partial_\lambda \tilde{F}^{\sigma\lambda} &= 0, \quad \tilde{F}^{\sigma\lambda} = \frac{1}{2}\epsilon^{\sigma\lambda\mu\nu} F_{\mu\nu} \end{aligned}$$

In terms of \mathbf{E} and \mathbf{B} :

$$\begin{aligned} \nabla \cdot \mathbf{E} &= 0, \quad \dot{\mathbf{E}} = \nabla \times \mathbf{B} && \text{dynamical equations} \\ \nabla \cdot \mathbf{B} &= 0, \quad \dot{\mathbf{B}} = -\nabla \times \mathbf{E} && \text{Bianchi identities} \end{aligned}$$

Remarks

- Lagrangian density does not depend on \dot{A}_0 , since A_0 is not really dynamical.

$$\nabla \cdot \mathbf{E} = 0 \rightarrow \nabla^2 A_0 + \nabla \cdot \dot{\mathbf{A}} = 0$$

Solve this Poisson equation for $A_0(\mathbf{x}, t) = \frac{1}{4\pi} \int d^3y \frac{\nabla \cdot \dot{\mathbf{A}}(\mathbf{y}, t)}{|\mathbf{y} - \mathbf{x}|}$. Thus A_0 is given in terms of the other components of A .

- gauge invariance: field strength tensor invariant under the transformation $A_\mu \mapsto A_\mu - \partial_\mu X$ due to commuting derivatives. This leads to gauge invariance of Maxwell equations. Choose X to satisfy $\partial_\mu \partial^\mu X = \partial^2 X = \partial_\mu A^\mu$ allows us to demand the condition (Lorenz condition)

$$\partial_\mu A^\mu = 0 \quad (5.1.2)$$

such that A_μ belongs to the "Lorenz gauge" and reduces the degrees of freedom from 4 to 3.

- Further freedom is eliminated by adding any X with $\partial^2 X = 0$, e.g. $\frac{\partial}{\partial t} X = A_0$. Then we get the Coulomb or radiation gauge

$$A_0 = 0, \quad \nabla \cdot \mathbf{A} = 0 \quad (5.1.3)$$

Note: vice versa imposing $\nabla \cdot \mathbf{A} = 0$ first, yields $A_0 = 0$ (using Lorenz condition?).
In Coulomb gauge:

$$\begin{aligned} \mathbf{E} &= -\dot{\mathbf{A}}, \quad \mathbf{B} = \nabla \times \mathbf{A}, \quad \nabla \times \mathbf{A} = 0 \\ -\ddot{\mathbf{A}} &= \dot{\mathbf{E}} \stackrel{\text{Maxwell}}{=} \nabla \times \mathbf{B} = \nabla \times (\nabla \times \mathbf{A}) = \nabla(\underbrace{\nabla \cdot \mathbf{A}}_{=0}) - \nabla^2 \mathbf{A} \\ \Rightarrow \partial^2 \mathbf{A} &= 0 \end{aligned}$$

This wave equation is massless KG equation for each spatial component.

Then the solutions are obvious: $\mathbf{A} = \boldsymbol{\epsilon} e^{-ik \cdot x}$ with $k^2 = 0$ and $\boldsymbol{\epsilon} \cdot \mathbf{k} = 0$. The polarization vector $\boldsymbol{\epsilon}$ is transverse to \mathbf{k} .

Can write the lagrangian in Coulmb gauge

$$\mathcal{L}_{\text{EM}} = \frac{1}{2} \dot{\mathbf{A}} \cdot \dot{\mathbf{A}} - \frac{1}{2} \mathbf{B} \cdot \mathbf{B}$$

Then the conjugate momentum to \mathbf{A} is $\boldsymbol{\Pi} = \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{A}}} = \dot{\mathbf{A}} = -\mathbf{E}$. It has only 3 components, there is no conjugate momentum to A_0 !. Because of Coulomb gauge $\boldsymbol{\Pi}$ is subject to the constraint $\nabla \cdot \boldsymbol{\Pi} = 0$

Hamiltonian

$$H_{\text{EM}} = \int d^3x \left(\frac{1}{2} \boldsymbol{\Pi} \cdot \boldsymbol{\Pi} + \frac{1}{2} \mathbf{B} \cdot \mathbf{B} \right)$$

5.2 Quantizing the Maxwell field

We would like to impose canonical commutation relations, à la

$$\begin{aligned} [A_i(\mathbf{x}), A_j(\mathbf{y})] &= [\Pi_i(\mathbf{x}), \Pi_j(\mathbf{y})] = 0 \\ [A_i(\mathbf{x}), \Pi_j(\mathbf{y})] &= i\delta_{ij}\delta^{(3)}(\mathbf{x} - \mathbf{y}) \end{aligned}$$

However this cannot be true. Take either derivative of the last equation and it needs to vanish due to $\nabla \cdot \mathbf{A} = \nabla \cdot \boldsymbol{\Pi} = 0$. But

$$[\partial^i A_i(\mathbf{x}), \Pi_k(\mathbf{y})] = i\delta_{ij}\partial^i \delta^{(3)}(\mathbf{x} - \mathbf{y})$$

here the derivative is takev with respect to \mathbf{x} , i.e. $\partial^i = \frac{\partial}{\partial x_i}$.

Replace δ_{ij} by Δ_{ij}

$$\begin{aligned} [\partial^i A_i(\mathbf{x}), \Pi_j(\mathbf{y})] &= i\Delta_{ij}\partial^i \frac{1}{(2\pi)^3} \int d^3k e^{ik \cdot (\mathbf{x} - \mathbf{y})} \\ &= -\frac{1}{(2\pi)^3} \int d^3k (k^i \Delta_{ij}) e^{ik \cdot (\mathbf{x} - \mathbf{y})} \stackrel{!}{=} 0 \end{aligned}$$

it works for $\Delta_{ij} = \delta_{ij} - \frac{k_i k_j}{k^2}$ in momentum space or $\Delta_{ij} = \delta_{ij} - \nabla^{-2} \partial_i \partial_j$ in position space.

$$[A_i(\mathbf{x}), \Pi_j(\mathbf{y})] = i(\delta_{ij} - \nabla^{-2} \partial_i \partial_j) \delta^{(3)}(\mathbf{x} - \mathbf{y}) \quad (5.2.1)$$

As before we have the mode expansion

$$\begin{aligned} \mathbf{A}(\mathbf{x}) &= \int \frac{d^3 k}{(2\pi)^3 \sqrt{2|\mathbf{k}|}} (\mathbf{a}_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}} + \mathbf{a}_{\mathbf{k}}^\dagger e^{-i\mathbf{k} \cdot \mathbf{x}}) \\ \Pi(\mathbf{x}) &= \int \frac{d^3 k}{(2\pi)^3} (-i) \sqrt{\frac{|\mathbf{k}|}{2}} (\mathbf{a}_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}} - \mathbf{a}_{\mathbf{k}}^\dagger e^{-i\mathbf{k} \cdot \mathbf{x}}) \end{aligned}$$

with $\mathbf{k} \cdot \mathbf{a}_{\mathbf{k}} = \mathbf{k} \cdot \mathbf{a}_{\mathbf{k}}^\dagger = 0$.

Introduce 2 orthogonal polarization vectors $\boldsymbol{\epsilon}^{(1)}(\mathbf{k})$ and $\boldsymbol{\epsilon}^{(2)}(\mathbf{k})$ for each \mathbf{k} .

$$\begin{aligned} \mathbf{a}_{\mathbf{k}} &= a_{\mathbf{k}}^{(1)} \boldsymbol{\epsilon}^{(1)} + a_{\mathbf{k}}^{(2)} \boldsymbol{\epsilon}^{(2)} = \sum_{\lambda=1}^2 a_{\mathbf{k}}^{(\lambda)} \boldsymbol{\epsilon}^{(\lambda)}(\mathbf{k}) \\ \text{with } \mathbf{k} \cdot \boldsymbol{\epsilon}^{(1)}(\mathbf{k}) &= \mathbf{k} \cdot \boldsymbol{\epsilon}^{(2)}(\mathbf{k}) = 0, \quad \boldsymbol{\epsilon}^{(\lambda)} \cdot \boldsymbol{\epsilon}^{(\lambda')} = \delta_{\lambda\lambda'} \end{aligned}$$

Creation and annihilation operator have the standard commutation relations

$$[a_{\mathbf{k}}^{(\lambda)}, a_{\mathbf{k}'}^{(\lambda')\dagger}] = (2\pi)^3 \delta_{\lambda\lambda'} \delta^{(3)}(\mathbf{k} - \mathbf{k}') \quad (5.2.2)$$

and all other commutators vanish. Geometrically, still possible to write including the unphysical longitudinal components:

$$\begin{aligned} [\mathbf{a}_{\mathbf{k}}, \mathbf{a}_{\mathbf{l}}] &= [\mathbf{a}_{\mathbf{k}}^\dagger, \mathbf{a}_{\mathbf{l}}^\dagger] = 0 \\ [a_{\mathbf{k}}^i, a_{\mathbf{l}}^{j\dagger}] &= (2\pi)^3 \left(\delta^{ij} - \frac{k^i k^j}{k^2} \right) \delta^{(3)}(\mathbf{k} - \mathbf{l}) \end{aligned}$$

$a_{\mathbf{k}}^{(\lambda)}$ and $a_{\mathbf{k}}^{(\lambda)\dagger}$ create and destroy photons of momentum \mathbf{k} , energy $|\mathbf{k}|$ and (electric) polarization along $\boldsymbol{\epsilon}^{(\lambda)}(\mathbf{k})$.

Next steps are analogout to KG theory.

Hamiltonian

$$H = \frac{1}{2} \int d^3 x (\mathbf{E}^2 + \mathbf{B}^2) = \frac{1}{2} \int d^3 x (\dot{\mathbf{A}}^2 + (\nabla \times \mathbf{A}) \cdot (\nabla \times \mathbf{A}))$$

using identity $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A})$

$$= \frac{1}{2} \int d^3 x (\dot{\mathbf{A}}^2 + \mathbf{A} \cdot \nabla \times (\nabla \times \mathbf{A}))$$

using the identity $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$

$$= \frac{1}{2} \int d^3 x (\dot{\mathbf{A}}^2 - \mathbf{A} \cdot \nabla^2 \mathbf{A} + \mathbf{A} \cdot \nabla(\nabla \cdot \mathbf{A}))$$

using coulomb gauge condition

$$= \frac{1}{2} \int d^3 x (\dot{\mathbf{A}}^2 - \mathbf{A} \cdot \nabla^2 \mathbf{A})$$

the first term vanishes and use normal ordering

$$= \int \frac{d^3 k}{(2\pi)^3} |\mathbf{k}| \mathbf{a}_{\mathbf{k}}^\dagger \cdot \mathbf{a}_{\mathbf{k}} = \sum_{\lambda=1}^2 \int \frac{d^3 k}{(2\pi)^3} |\mathbf{k}| a_{\mathbf{k}}^{(\lambda\dagger)} a_{\mathbf{k}}^{(\lambda)}$$

Heisenberg field

$$\mathbf{A}(\mathbf{x}, t) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2|\mathbf{k}|}} \left(\mathbf{a}_{\mathbf{k}} e^{-ik \cdot x} + \mathbf{a}_{\mathbf{k}}^\dagger e^{ik \cdot x} \right)$$

Photon propagator

$$\langle 0 | T A_i(x) A_j(y) | 0 \rangle =: D_{ij}^{\text{tr}}(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 + i\epsilon} \left(\delta_{ij} - \frac{k_i k_j}{|\mathbf{k}|^2} \right) e^{-ik \cdot (x-y)} \quad (5.2.3)$$

tr stands for transverse: photon polarization perpendicular to its momentum. This is **NOT** the final version of the photon propagator!

5.3 Inclusion of matter - QED

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\not{D} - m)\psi \quad (5.3.1)$$

where $D_\mu = \partial_\mu + ieA_\mu$ is the (gauge) covariant derivative

$$= \mathcal{L}_{\text{EM}} + \mathcal{L}_D - e \underbrace{\bar{\psi} \gamma^\mu \psi A_\mu}_{j^\mu} \quad (5.3.2)$$

Field equations would be

$$\partial_\mu F^{\mu\nu} = e j^\nu \quad (i\not{D} - m)\psi = 0$$

where $e j^\nu$ is the electromagnetic 4-current.

Gauge invariance under the transformation

$$\begin{cases} \psi(x) \mapsto \psi'(x) = e^{ie\chi(x)} \psi \\ A_\mu(x) \mapsto A'_\mu(x) = A_\mu(x) - \partial_\mu \chi(x) \end{cases}$$

To check the consistence: covariant derivative transforms like $D_\mu \mapsto D'_\mu \psi'(x) = e^{ie\chi(x)} D_\mu \psi(x)$. Since the adjoint spinor transforms like $\bar{\psi}(x) \mapsto \bar{\psi}'(x) = \bar{\psi}(x) e^{-ie\chi(x)}$, the Lagrangian and field equations are gauge invariant.

Again we choose Coulomb gauge $\nabla \cdot \mathbf{A} = 0$, then equation for A^0 :

$$\begin{aligned} \partial_i F^{i0} &= e j^0 \\ \Rightarrow -\nabla^2 A^0 &= e j^0 = e \bar{\psi} \gamma^0 \psi \\ &= e \bar{\psi} \gamma^0 \psi = e \psi^\dagger \psi \\ &= e \rho(x) \\ A^0(\mathbf{x}, t) &= e \int d^3y \frac{\rho(\mathbf{y}, t)}{4\pi |\mathbf{x} - \mathbf{y}|} \end{aligned} \quad (5.3.3)$$

We want to derive the interaction Hamiltonian. Note

$$\int d^3x \frac{1}{2} \mathbf{E}^2 = \int d^3x \frac{1}{2} (\dot{\mathbf{A}} + \nabla A^0)^2$$

cross terms vanish after integration by parts due to $\nabla \cdot \dot{\mathbf{A}} = 0$

$$\begin{aligned} &= \int d^3x \frac{1}{2} \left(\dot{\mathbf{A}}^2 + (\nabla A^0)^2 \right) \\ &= \int d^3x \frac{1}{2} \left(\dot{\mathbf{A}}^0 - A^0 \nabla^2 A^0 \right) \end{aligned}$$

$$-e j^0 = -e \rho$$

$$= \int d^3x \frac{1}{2} \dot{\mathbf{A}}^2 + \underbrace{\frac{e^2}{2} \int d^3x d^3y \frac{\rho(\mathbf{x})\rho(\mathbf{y})}{4\pi|\mathbf{x}-\mathbf{y}|}}_{=\frac{e^2}{2} j^0 A_0}$$

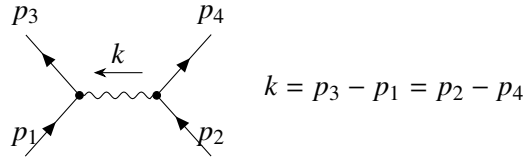
Combined Hamiltonian

$$\begin{aligned} H = \int d^3x \left\{ \frac{1}{2} \boldsymbol{\Pi} \cdot \boldsymbol{\Pi} + \frac{1}{2} \mathbf{B} \cdot \mathbf{B} + i\bar{\psi} \boldsymbol{\gamma} \cdot \nabla \psi + m\bar{\psi}\psi \right\} & \text{ free photon and fermion} \\ + \frac{e^2}{2} \int d^3x d^3y \frac{\rho(\mathbf{x})\rho(\mathbf{y})}{4\pi|\mathbf{x}-\mathbf{y}|} - e \int d^3x \mathbf{j} \cdot \mathbf{A} & \text{ interactions} \end{aligned}$$

where $\rho = \psi^\dagger \psi = \bar{\psi} \gamma^0 \psi$, $\mathbf{j} = \bar{\psi} \boldsymbol{\gamma} \psi$ for 2 types of interactions.

5.4 Lorentz-invariant propagator

Consider $e^- e^-$ scattering at $O(e^2)$



We expect this to involve

- spinors for external fermions
- $-ie\gamma^\mu$
- Photon propagator $D_{\mu\nu}(x-y)$

What we have found in Coulomb gauge is actually

- vertices $ie\gamma^i$, transverse propagator $D_{\mu\nu}^{\text{tr}}(x-y)$
- vertices $\pm ie\gamma^0$, instantaneous Coulomb interaction $\frac{1}{4\pi|\mathbf{x}-\mathbf{y}|} \delta(x^0 - y^0)$

Effectively combine these propagators terms into $D_{\mu\nu}^{\text{Coul}}(x-y)$, where the $D_{00}^{\text{Coul}}(x-y) = \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|} \delta(x^0 - y^0)$. This component in momentum space is simply

$$\int \frac{d^3k}{(2\pi)^3} \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{|\mathbf{k}|^2} = \frac{1}{4\pi|\mathbf{r}|}$$

Therefore Coulomb propagator in momentum space:

$$D_{\mu\nu}^{\text{Coul}}(k) = \begin{cases} \frac{i}{k^2 + i\epsilon} \left(\delta_{ij} - \frac{k_i k_j}{|\mathbf{k}|^2} \right) & \mu = i, \nu = j \\ \frac{i}{|\mathbf{k}|^2} & \mu = \nu = 0 \\ 0 & \text{otherwise} \end{cases}$$

Consider contraction to scattering amplitude from vertex at x :

$$\sim e \bar{u}(p_3) \gamma^\mu u(p_1) e^{i(p_3 - p_1)x}$$

current conservation $\partial_\mu j^\mu = 0$ written in momentum space

$$\underbrace{(p_3 - p_1)_\mu}_{k_\mu} \bar{u}(p_3) \gamma^\mu u(p_1) = 0$$

so in the complete diagram $D_{\mu\nu}^{\text{Coul}}$ occurs in a form

$$\begin{aligned} & a^\mu D_{\mu\nu}^{\text{Coul}}(k) b^\nu \\ &= a^0 \frac{i}{|\mathbf{k}|^2} b^0 + a^i \left[\frac{i}{k^2 + i\epsilon} \left(\delta_{ij} - \frac{k_i k_j}{|\mathbf{k}|^2} \right) \right] b^j \end{aligned}$$

where $k^\mu a^\mu = 0, k_\mu a^\mu = 0$

$$\begin{aligned} &= i \left[\frac{\mathbf{a} \cdot \mathbf{b}}{k^2} - \frac{k_0^2 a_0 b_0}{k^2 |\mathbf{k}|^2} + \frac{a_0 b_0}{|\mathbf{k}|^2} \right] \\ &= \frac{-k_0^2 a_0 b_0 + a_0 b_0 (k_0^2 - |\mathbf{k}|^2)}{k^2 |\mathbf{k}|^2} \\ &= \frac{i}{k^2} (\mathbf{a} \cdot \mathbf{b} - a_0 b_0) = -\frac{i}{k} a_\mu b^\mu \end{aligned}$$

Conclusion in this diagram (and in fact, in general), we may replace the $D_{\mu\nu}^{\text{Coul}}(k)$ by the manifestly Lorentz covariant propagator

$$D_{\mu\nu}(k) = -\frac{i g_{\mu\nu}}{k^2 + i\epsilon} \quad (5.4.1)$$

This can be generalised to

$$D_{\mu\nu}(k) = -\frac{i}{k^2 + i\epsilon} \left(g_{\mu\nu} - (1 - \alpha) \frac{k_\mu k_\nu}{k^2} \right) \quad (5.4.2)$$

as, by current conservation, additional term doesn't contribute.

Feynman gauge $\alpha = 1$; Landau gauge $\alpha = 0$.

Remark one can also try to quantise photons in a manifestly covariant way, imposing Lorentz gauge $\partial_\mu A^\mu = 0$

$$[A_\mu(\mathbf{x}), \Pi_\nu(\mathbf{y})] = i g_{\mu\nu} \delta^{(3)}(\mathbf{x} - \mathbf{y})$$

This is trouble since $\Pi^0 = \frac{\partial \mathcal{L}}{\partial A_0} = 0$. This cannot hold!

We thus change the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\alpha}(\partial_\mu A^\mu)^2$$

with "gauge fixing term". The equation of motion from this is

$$\partial^2 A^\mu - \left(1 - \frac{1}{\alpha}\right)\partial^\mu(\partial_\lambda A^\lambda) = 0$$

e.g. $\alpha = 1$ is the Feynman gauge.

With this Lagrangian we can the 0th component of conjugate momentum

$$\Pi^0 = -\frac{1}{\alpha}\partial_\mu A^\mu$$

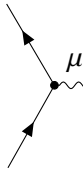
but this seems as bad as before!

We cannot impose Coulomb gauge condition $\partial_\mu A^\mu = 0$ as an operator identity. Instead demand a weaker condition $\langle \text{out} | \partial_\mu A^\mu | \text{in} \rangle = 0$ for all physical states.

This in turn tells us which states are actually physical. The 4 polarisation states consist of physical, timelike(scalar) and longitudinal states. The negative-norm states cancel each other out (Gupta-Bleuler formalism).

Feynman rules for QED diagrams constructed from electron (positron) \longrightarrow and photon \sim ; rules for fermions are valid as before.

In addition

- vertex  $= -ie\gamma^\mu$;

- photon propagator $\mu \xrightarrow{k} \nu = -\frac{ig_{\mu\nu}}{k^2 + i\epsilon}$

- external photons

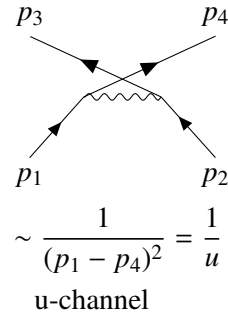
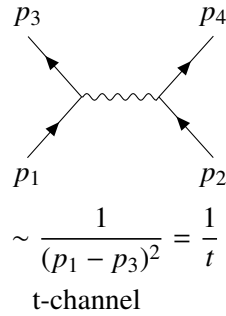
$$\mu \xleftarrow{k_{\text{in}}} \nu = \epsilon_\mu$$

$$\mu \xrightarrow{k_{\text{out}}} \nu = \epsilon_\nu^*$$

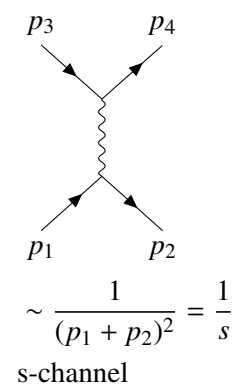
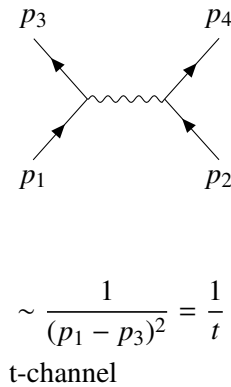
ϵ_μ polarisation vector of in/out photon and ϵ_μ^* for out photon required for complex (circular) polarisation.

5.5 QED process at tree level

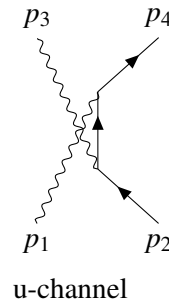
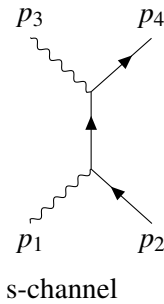
Example $e^-e^- \rightarrow e^-e^-$



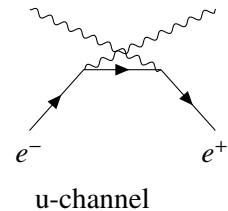
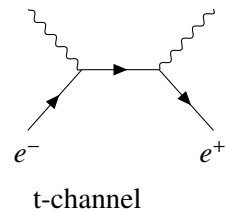
Example $e^-e^+ \rightarrow e^-e^+$



Compton scattering $\gamma e^- \rightarrow \gamma e^-$

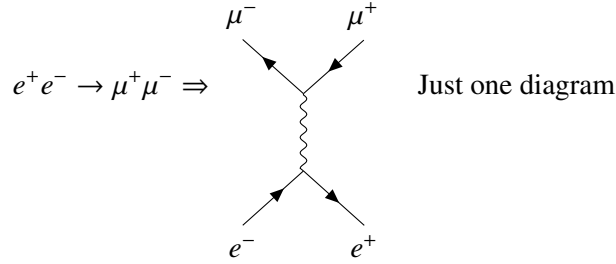
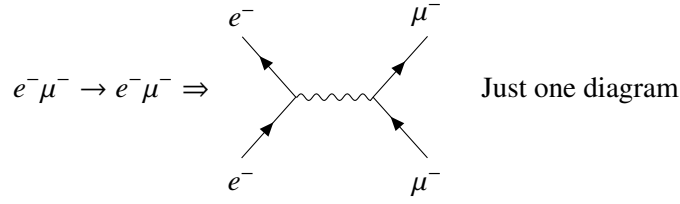


Example $e^+e^- \rightarrow \gamma\gamma$



These are important for lifetime of positronium.

All these amplitudes are $O(e^2)$, $\alpha = \frac{e^2}{4\pi} = \frac{1}{137.036}$ the fine structure constant.
Muons μ^\pm , like electrons, just ca. 200 times heavier.



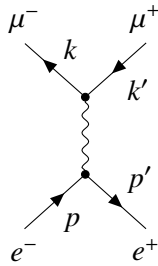
μ^\pm decay into e^\pm and neutrinos in weak interactions.

For tree level diagrams the photon propagator does not need to have the $i\epsilon$ in the denominator, since we will never be able to see a singularity/pole.

5.5.1 Some hints and tricks for cross section calculations

See application in exercises!

Example $e^+ e^- \rightarrow \mu^+ \mu^-$



$$\begin{aligned} iM &= \bar{v}_e^{s'}(-ie\gamma^\mu)u_e^s(p) \frac{-ig_{\mu\nu}}{s} \Big|_{s=q^2} \bar{u}_\mu^r(k)(-ie\gamma^\nu)v_\mu^{r'}(k') \\ &= \frac{ie^2}{s} (\bar{v}_e(p')\gamma^\mu u_e(p)) (\bar{u}_\mu(k)\gamma_\mu v_\mu(k')) \end{aligned}$$

See section 4.5, $|M|^2$ is needed for cross section. M^* involves things like

$$\begin{aligned} (\bar{v}\gamma^\mu u)^* &= (\bar{v}\gamma^\mu u)^\dagger = u^\dagger \gamma^{\mu\dagger} \gamma_0^\dagger v \\ &= u^\dagger \gamma_0 \gamma^\mu \gamma_0 v = \bar{u}\gamma^\mu v \end{aligned}$$

So

$$|M|^2 = \frac{e^4}{s^2} [\bar{v}(p')\gamma^\mu u(p)\bar{u}(p)\gamma^\nu v(p')]_{e^\pm} \cdot [\bar{u}(k)\gamma_\mu v(p)\bar{v}(k')\gamma_\nu u(k)]_{\mu^\pm}$$

Unpolarized scattering = $\frac{1}{4} \sum_{r,s,r',s'} |M|^2$.

Now $\bar{v}\gamma^\mu u$, $\bar{u}\gamma^\nu v$ etc. are scalars in Dirac/spinor space:

$$\begin{aligned}
 & \sum_{s,s'} \bar{v}_{s'} p' \gamma^\mu u_s(p) \bar{u}_s(p) \gamma^\nu v_{s'}(p') \\
 \text{(taking trace of scalar)} &= \sum_{s,s'} \text{Tr}(\bar{v}_{s'} p' \gamma^\mu u_s(p) \bar{u}_s(p) \gamma^\nu v_{s'}(p')) \\
 &= \sum_{s,s'} \text{Tr}(v_{s'}(p') \bar{v}_{s'}(p') \gamma^\mu u_s(p) \bar{u}_s(p) \gamma^\nu)
 \end{aligned}$$

using spin sums

$$= \text{Tr}((\not{p}' - m) \gamma^\mu (\not{p} + m) \gamma^\nu)$$

Trace technology

- remember $\text{Tr} \gamma_\mu = 0$
- $\text{Tr}(\gamma^\mu \gamma^\nu) = 4g_{\mu\nu}$
- $\text{Tr}(\text{odd number of } \gamma) = 0$
- $\text{Tr}(\gamma_\mu \gamma_\nu \gamma_\alpha \gamma_\beta) = 4(g_{\mu\nu} g_{\alpha\beta} + g_{\mu\beta} g_{\nu\alpha} - g_{\mu\alpha} g_{\nu\beta})$
- more rules involving γ_5 (weak interactions!)

So

$$\text{Tr}((\not{p}' - m) \gamma^\mu (\not{p} + m) \gamma^\nu) = 4(p'^\mu p^\nu + p'^\nu p^\mu - g^{\mu\nu}(p \cdot p' + m^2))$$

Mandelstam variables with 4 equal masses, center-of-mass system (CMS):

$$p = (E, \mathbf{p}), \quad p' = (E, -\mathbf{p}), \quad k = (E, \mathbf{k}), \quad \theta = \angle(\mathbf{p}, \mathbf{k})$$

$$s \stackrel{\text{CMS}}{=} (p + p')^2 = 4E^2 \tag{5.5.1}$$

$$t = (p - k)^2 = -(\mathbf{p} - \mathbf{k})^2 = -2|\mathbf{p}|^2(1 - \cos \theta) \tag{5.5.2}$$

$$u = (p' - k)^2 = -2|\mathbf{p}|^2(1 + \cos \theta) \tag{5.5.3}$$

$$|\mathbf{p}|^2 = E^2 - m^2 = \frac{s}{4} - m^2 \tag{5.5.4}$$

Only 2 Mandelstam variables are independent.

$$\begin{aligned}
 s + t + u &= (p_1 + p_2)^2 + (p_1 - p_3)^2 + (p_1 - p_4)^2 \\
 &= p_1^2 + p_2^2 + p_3^2 + p_4^2 + 2p_1 \underbrace{(p_1 + p_2 - p_3 - p_4)}_{=0} \\
 &= \sum_i m_i^2 = \text{const}
 \end{aligned}$$

photon polarisation sums Analogy to fermion spin sums before, Feynman rules for external photons involve $\epsilon^{(*)}_\mu$; e.g. Compton amplitude of the form

$$M \sim \epsilon_\mu^*(p_3) \epsilon_\nu(p_1) T^{\mu\nu}$$

Thus

$$\sum_{\text{spin, pol.}} |M|^2 = \sum_{\text{spin, pol.}} \epsilon_\mu^*(p_3) \epsilon_\alpha(p_3) \epsilon_\beta^*(p_1) \epsilon_\nu(p_1) T^{\mu\nu} T^{\alpha\beta*}$$

How can we simplify $\sum_{\text{pol}} \epsilon_\mu^*(k) \epsilon_\nu(k)$? Again we have only 2 physical polarisation states, but want to do it in a covariant form.

Assume a simpler process (than Compton) with a single external photon, $\epsilon_\mu^*(k) M^\mu$. Choose

$$k^\mu = (k, 0, 0, k), \quad \epsilon_{(1)}^\mu = (0, 1, 0, 0), \quad \epsilon_{(2)}^\mu = (0, 0, 1, 0)$$

$$\text{so } \sum_{\text{pol}} |\epsilon_\mu^*(k) M^\mu|^2 = |M_1|^2 + |M_2|^2$$

Remember that photon coupled source j^μ , current conservation $\partial_\mu j^\mu = 0$. We will see (next term) this holds in general as Ward identity

$$k_\mu M^\mu = 0 \tag{5.5.5}$$

In exercises, show $p_{3\mu} T^{\mu\nu} = 0 = p_{1\nu} T^{\mu\nu}$ for Compton

Here $kM^0 - kM^3 = 0 \Rightarrow M^0 = M^3$ and we can rewrite

$$\sum_{\text{pol}} \epsilon_\mu^* \epsilon_\nu M^\mu M^{*\nu} = |M_1|^2 + |M_2|^2 + \underbrace{|M_3|^2 - |M_0|^2}_{=0} = -g_{\mu\nu} M^\mu M^{*\nu}$$

so effectively

$$\sum_{\text{pol}} \epsilon_\mu^*(k) \epsilon_\nu(k) = -g_{\mu\nu} \tag{5.5.6}$$

side remark

- KG propagator $\frac{i}{p^2 - M^2 + i\epsilon}$
- Dirac propagator $\frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon} = \frac{i \sum_s u_s(p) \bar{u}_s(p)}{p^2 - m^2 + i\epsilon}$
- Photon propagator $\frac{-ig_{\mu\nu}}{p^2 + i\epsilon} = \frac{i \sum_{\text{pol}} \epsilon_\mu^*(p) \epsilon_\nu(p)}{p^2 + i\epsilon}$