1. a) $$\mathcal{L} = \bar{\psi}(i\not{\partial} - M)\psi + g\bar{\psi}\gamma^\mu\psi\bar{\psi}\gamma_\mu\psi$$

i) EOM:

$\psi$: $$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} - \frac{\partial \mathcal{L}}{\partial \psi} = 0$$

$$\partial_\mu(\bar{\psi} i\gamma^\mu) + \bar{\psi}M$$
$$-g\bar{\psi}\gamma^\mu\bar{\psi}\gamma_\mu\psi - g\bar{\psi}\gamma^\mu\psi\bar{\psi}\gamma_\mu = 0$$

$\bar{\psi}$: $$(i\not{\partial} - M)\psi$$
$$+ g\gamma^\mu\psi\bar{\psi}\gamma_\mu\psi + g\bar{\psi}\gamma^\mu\psi\gamma_\mu\psi = 0$$

ii) $$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\partial^\nu\psi + (\partial^\nu\bar{\psi})\frac{\partial \mathcal{L}}{\partial(\partial_\mu\bar{\psi})} - g^{\mu\nu}\mathcal{L}$$

$$= \partial_\mu(\bar{\psi} i\gamma^\mu)\partial^\nu\psi - g^{\mu\nu}\mathcal{L}$$

iii) $\psi \to e^{-i\alpha}\psi$, $\bar{\psi} \to e^{+i\alpha}\psi$

$$\Rightarrow \mathcal{L} \to \mathcal{L}$$

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu\psi)}\Delta\psi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu\bar{\psi})}\Delta\bar{\psi}$$

$$= \bar{\psi}\cdot i\gamma^\mu\cdot(-i\alpha\psi) = \alpha\bar{\psi}\gamma^\mu\psi$$

$$\partial_\mu j^\mu = \alpha\left(\partial_\mu\bar{\psi}\cdot\gamma^\mu\psi + \bar{\psi}\gamma^\mu\partial_\mu\psi\right)$$
$$\propto -i\left[\left(-\underline{\bar{\psi}M} + g\bar{\psi}\gamma^\mu\bar{\psi}\gamma_\mu\psi + \underline{g\bar{\psi}\gamma^\mu\psi\bar{\psi}\gamma_\mu}\right)\psi\right.$$
$$\left. + \bar{\psi}\left(\underline{M\psi} - \underline{g\gamma^\mu\psi\bar{\psi}\gamma_\mu\psi} - g\bar{\psi}\gamma^\mu\psi\gamma_\mu\psi\right)\right]$$

$$\propto \bar{\psi}\gamma^\mu\bar{\psi}\gamma_\mu\psi\psi - \bar{\psi}\bar{\psi}\gamma^\mu\psi\gamma_\mu\psi$$
$$= \bar{\psi}(\gamma^\mu\bar{\psi}\gamma_\mu\psi - \bar{\psi}\gamma^\mu\psi\gamma_\mu)\psi$$
$$= 0$$

ab b bc c

a ab b bc

1. b) $$\psi(t,\vec{x}) \xrightarrow{P} \gamma^0 \psi(t,-\vec{x})$$
$$\psi(t,\vec{x}) \xrightarrow{T} -\gamma^1\gamma^3 \psi(-t,\vec{x})$$

i) $$(\bar{\psi}\, i\gamma^5 \psi)^\dagger = -\psi^\dagger\, i(\gamma^5)^\dagger(\gamma^0)^\dagger \psi$$
$$= -\psi^\dagger\, i\gamma^5\gamma^0\psi$$
$$= \psi^\dagger\, i\gamma^0\gamma^5\psi$$
$$= \bar{\psi}\, i\gamma^5\psi$$

ii) $$\psi^\dagger(t,\vec{x}) \xrightarrow{P} \psi^\dagger(t,-\vec{x})\gamma^0 \,;\quad \bar{\psi} \xrightarrow{P} \psi^\dagger(t,-\vec{x})\gamma^0\gamma^0 = \psi^\dagger(t,-\vec{x})$$

$$\bar{\psi}(t,\vec{x})\, i\gamma^5 \psi(t,\vec{x})$$
$$\xrightarrow{P} \psi^\dagger(t,-\vec{x})\, i\gamma^5\gamma^0\psi(t,\vec{x})$$
$$= -\bar{\psi}(t,-\vec{x})\, i\gamma^5\psi(t,\vec{x})$$

iii) $T$ is antiunitary and antilinear
$$\langle x, y\rangle = \langle Tx, Ty\rangle^*$$
$$(\text{or}\quad Tc = c^* T \,,\quad c\in\mathbb{C})$$
$$\psi^\dagger \xrightarrow{T} -\psi^\dagger(\gamma^3)^\dagger(\gamma^1)^\dagger$$
$$\bar{\psi}(t,\vec{x}) \xrightarrow{T} -\psi^\dagger(-t,\vec{x})(\gamma^3)^\dagger(\gamma^1)^\dagger(\gamma^0)^\dagger$$
$$= -\psi^\dagger(-t,\vec{x})\gamma^0\gamma^3\underbrace{\gamma^0\gamma^0}_{\mathbb{1}}\gamma^1\underbrace{\gamma^0\gamma^0}_{\mathbb{1}}$$
$$= -\bar{\psi}\gamma^3\gamma^1 \qquad -$$
$$\bar{\psi}(t,\vec{x})\, i\gamma^5\psi(t,\vec{x}) \longrightarrow T\bar{\psi}T^{-1}\, \overbrace{T(i\gamma^5)T^{-1}}\, T\psi T^{-1}$$
$$= -\bar{\psi}(-t,\vec{x})\gamma^3\gamma^1\, i\gamma^5\gamma^1\gamma^3\psi(-t,\vec{x})$$
$$= -\bar{\psi}\gamma^3\, i\gamma^5\gamma^3\psi$$
$$= -\bar{\psi}\, i\gamma^5\psi$$

iv) $$\psi(t,\vec{x}) \xrightarrow{C} -i\gamma^2\psi^*(t,\vec{x}) \,,\quad \eta_c = 1$$

$$\psi^\dagger \xrightarrow{C} (\psi^*)^\dagger (-i\gamma^2)^\dagger = \psi^T \; i\gamma^0\gamma^2\gamma^0$$
$$\bar{\psi} \xrightarrow{C} \psi^T \; i\gamma^2\gamma^0$$

$$\bar{\psi}\, i\gamma^5 \psi \xrightarrow{C} \psi^T \; i\gamma^2\gamma^0 \quad i\gamma^5 \quad -i\gamma^2 \psi^*$$
$$= -i\psi^T \gamma^2\gamma^0\gamma^5\gamma^2\psi^*$$

$$\left(\{\gamma^\mu, \gamma^\nu\} = 2\,\mathbb{1} g^{\mu\nu}\right)$$

$$= -\psi^T (\gamma^2)^2 \gamma^0 i\gamma^5 \psi^*$$
$$= \psi^T \gamma^0 \, i\gamma^5 \psi^*$$
$$= \bar{\psi} \; i\gamma^5 \psi$$

$$\bar{\psi}\, i\gamma^5 \psi \xrightarrow{CPT} \bar{\psi}\, i\gamma^5 \psi$$

Chirality conserved under C.

Same for particle and antiparticle.

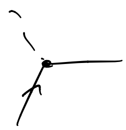
2. $\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - M)\psi + \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{m^2}{2}\phi^2 - g\bar{\psi}\psi\phi$

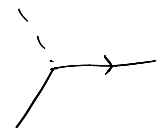
a) - propagator $\sim \dfrac{(p\!\!\!/ + m)_{\mu\nu}}{p^2 - m^2 + i\varepsilon}$

$\sim \dfrac{1}{p^2 - m^2 + i\varepsilon}$

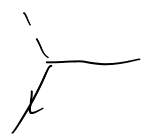
- vertex $\sim -ig$

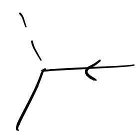
- Fermion :





$\overrightarrow{t}$ $u$ $\bar{u}$

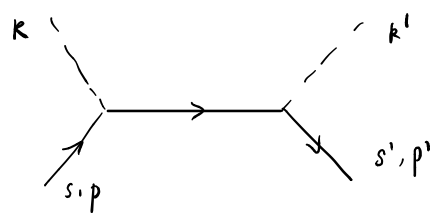


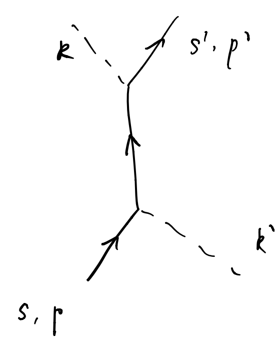


$\bar{v}$ $v$

- Integrate undetermined momenta
- momentum conservation
- $1/S$

b) $\psi(s, p) + \phi(k) \longrightarrow \psi(s', p') + \phi(k')$




s- t-

channel

c) $\overline{|i\mathcal{M}|^2} = \frac{1}{2}\sum_{s,s'} |\mathcal{M}|^2$

$$i\mathcal{M} = \bar{u}^{s'}(p')(-ig)\frac{[(\not{p}+\not{k})+m]_{\alpha\beta}}{s-m^2+i\varepsilon}u^s(p)(-ig)$$
$$+\bar{u}^{s'}(p')(-ig)\frac{[(\not{p}-\not{k}')+m]_{\alpha\beta}}{t-m^2+i\varepsilon}u^s(p)(-ig)$$

$$= -g^2\,\bar{u}^{s'}(p')\Bigg(\underbrace{\frac{[(\not{p}+\not{k})+m]_{\alpha\beta}}{s-m^2+i\varepsilon}}_{\mathcal{M}_a}+\underbrace{\frac{[(\not{p}-\not{k}')+m]_{\alpha\beta}}{t-m^2+i\varepsilon}}_{\mathcal{M}_b}\Bigg)u^s(p)$$

$$\Rightarrow \overline{|i\mathcal{M}|^2} = \underbrace{|\mathcal{M}_a|^2}_{f(k,k)} + \underbrace{|\mathcal{M}_b|^2}_{f(-k',-k')} + 2\,\underbrace{\mathrm{Re}(\mathcal{M}_a\mathcal{M}_b)}_{f(k,-k')}$$

d)
$$\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}$$
$$\mathrm{tr}[\gamma^\mu\gamma^\nu] = \mathrm{tr}[g^{\mu\nu}\mathbb{1}] = 4g^{\mu\nu}$$
$$\mathrm{tr}[\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta] = \mathrm{tr}[(-\gamma^\nu\gamma^\mu+2g^{\mu\nu}\mathbb{1})\gamma^\alpha\gamma^\beta]$$
$$= \mathrm{tr}[-\gamma^\nu\gamma^\mu\gamma^\alpha\gamma^\beta] + 2g^{\mu\nu}\cdot\mathrm{tr}[\gamma^\alpha\gamma^\beta]$$
$$= \mathrm{tr}[-\gamma^\nu(-\gamma^\alpha\gamma^\mu+2g^{\mu\alpha}\mathbb{1})\gamma^\beta] + 8g^{\mu\nu}g^{\alpha\beta}$$
$$= \mathrm{tr}[\gamma^\nu\gamma^\alpha\gamma^\mu\gamma^\beta] - 2g^{\mu\alpha}\mathrm{tr}[\gamma^\nu\gamma^\beta] + 8g^{\mu\nu}g^{\alpha\beta}$$
$$-8g^{\mu\alpha}g^{\nu\beta}$$
$$= \mathrm{tr}[\gamma^\nu\gamma^\alpha(-\gamma^\beta\gamma^\mu+2g^{\mu\beta}\mathbb{1})] - 8(g^{\mu\alpha}g^{\nu\beta}+8g^{\mu\nu}g^{\alpha\beta})$$
$$= 2\mathrm{tr}[\gamma^\nu\gamma^\alpha\gamma^\beta\gamma^\mu] + 8(g^{\mu\beta}g^{\nu\alpha}-g^{\mu\alpha}g^{\nu\beta}+g^{\mu\nu}g^{\alpha\beta})$$

$\Rightarrow$

e)
$$[\bar{u}(s',p')\gamma^\mu u(s,p)]^* = (\bar{u}(s',p')\gamma^\mu u(s,p))^\dagger$$
$$= u^\dagger(s,p)\gamma^0\gamma^\mu\underline{\gamma^0\gamma^0}u(s',p')$$
$$= \bar{u}(s,p)\gamma^\mu u(s',p')$$

f) $f(a,b) = \frac{1}{2}\sum_{s,s'} g^4 \, \bar{u}^{s'}(p') \frac{(\not p + \not a + m)}{(p+a)^2 - m^2 + i\varepsilon} u^s(p) \, \bar{u}^s(p) \frac{(\not p + \not b + m)}{(p+b)^2 - m^2 + i\varepsilon} u^{s'}(p')$

$m \to M$

$$= \frac{g^4}{2} \frac{1}{[(p+a)^2 - m^2][(p+b)^2 - m^2]}$$

$$\times \sum_{s',s} \mathrm{tr}[\bar{u}^{s'}(p') (\not p + \not a + m) u^s(p) \bar{u}^s(p) (\not p + \not b + m) u^{s'}(p')]$$

$$= \frac{g^4}{2} \frac{1}{\cdots} \mathrm{tr}[(\not p + \not a + m)(\not p + m)(\not p + \not b + m)(\not p' + m)]$$

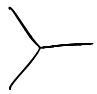
$$\simeq \frac{g^4}{2} \frac{1}{(p+a)^2 (p+b)^2} \mathrm{tr}[(\not p + \not a) \not p (\not p + \not b) \not p']$$

$$= \frac{g^4}{2} \frac{1}{\cdots} \mathrm{tr}[\underbrace{\not p \not p}_{=m^2 \sim 0} \not p \not p' + \not a \underbrace{\not p \not p}_{=m^2 \sim 0} \not p' + \underbrace{\not p \not p}_{=m^2 \sim 0} \not b \not p' + \not a \not p \not b \not p']$$

$$= \frac{g^4}{2} \frac{1}{4 \cdot (p \cdot a)(p \cdot b)} \cdot 4 [(a \cdot p)(b \cdot p') - (a \cdot b)(p \cdot p') + (a \cdot p')(p \cdot b)]$$

3. $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{m^2}{2}\phi^2 - \frac{g}{3!}\phi^3$

a)

- propagator $\sim \frac{1}{p^2 - m^2 + i\varepsilon}$
- vertex  $\sim -ig$
- external  $\sim 1$
- Integrate over undetermined $p$
- momentum conservation at vertices
- $1/S$

b)

$$\int \frac{d^dk \ldots}{k^2 \ldots}$$

$$D = dL - 2I$$

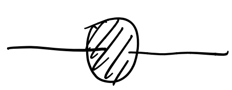
$$\begin{cases} L = I - V + 1 & \longrightarrow \quad L = \frac{3}{2}V - \frac{1}{2}E - V + 1 \\ V = \frac{1}{3}E + \frac{2}{3}I \quad \Leftrightarrow & I = \frac{3}{2}\left(V - \frac{1}{3}E\right) = \frac{3}{2}V - \frac{1}{2}E \end{cases}$$

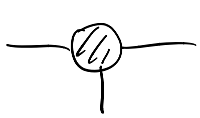
$$\rightarrow D = d\left(\frac{1}{2}V - \frac{1}{2}E + 1\right) - 2\left(\frac{3}{2}V - \frac{1}{2}E\right)$$

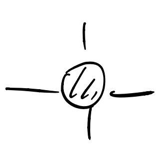
$$= \left(\frac{d}{2} - 3\right)V + \left(-\frac{1}{2}d + 1\right)E + d$$

$$\Rightarrow D \overset{d=6}{=} 6 - 2E$$

c)

 $D = 2$

 $D = 0$

 $D = -2$

$\rightarrow$ renormalizable

d) $g = \mu^x \tilde{g}$ , $[\mu] = 1$

$[\mathcal{L}] = d$ , $[\partial_\mu \phi] = d/2$ $[\phi] = d/2 - 1$

$[g\phi^3] = [g] + 3 \cdot (\frac{d}{2} - 1) \overset{!}{=} d$

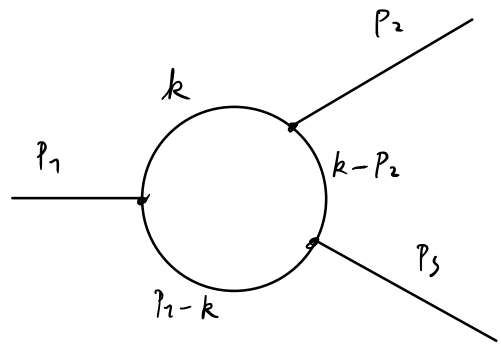
$\Rightarrow [g] = d - \frac{3}{2} d + 3 = 3 - \frac{1}{2} d$

$x = 3 - \frac{1}{2} d$

$[g] = 1$ , $d = 4$ super renormalisable

$= 0$ , $d = 6$ renormalisable

e)




$\cancel{k} - P_2 + P_1 - \cancel{k} = P_3$ $\longrightarrow$ momentum conservation

$$i\mathcal{M} = (-ig)^3 \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 - m^2 + i\varepsilon} \frac{1}{(P_1 - k)^2 - m^2 + i\varepsilon} \frac{1}{(k - P_2)^2 - m^2 + i\varepsilon}$$

$$= ig^3 \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 - m^2 + i\varepsilon)^3}$$

( $P_1 = P_2 = P_3 = 0$ , since they don't contribute to divergence.

$$= ig^3 \frac{(-1)^3 i}{(4\pi)^{d/2}} \frac{\Gamma(3 - d/2)}{\Gamma(3)} (m^2)^{d/2 - 3}$$

$$= ig^3 \frac{(-1)^3 i}{(4\pi)^{3-\varepsilon}} \frac{\Gamma(3 - 3 + \varepsilon)}{\Gamma(3)} m^{-2\varepsilon}$$

$$= g^3 \frac{\Gamma(\varepsilon)}{\Gamma(3)} = \frac{g^3}{2}\left(\frac{1}{\varepsilon} - \gamma_E + \mathcal{O}(\varepsilon)\right)$$