1. Classical field theory

1.1. From point mechanics to the continuum

- Lagrangian: $L(q_i, \dot{q}_i)$;
- canonical momentum: $p_i = \frac{\partial L}{\partial \dot{q}_i}$

$\rightarrow$ Hamiltonian: $H(q_i, p_i) = \sum_i p_i \dot{q}_i - L(q_i, \dot{q}_i)\big|_{\dot{q}_i = \ldots}$

eq. of motion: $\dot{q}_i = \frac{\partial H}{\partial p_i}$, $\dot{p}_i = -\frac{\partial H}{\partial q_i}$

- Poisson brackets: $\{A, B\} = \sum_i \left(\frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q_i} \right)$

$\rightsquigarrow \{q_i, q_j\} = \{p_i, p_j\} = 0$

$\{q_i, p_j\} = \delta_{ij}$

$\rightsquigarrow \dot{q}_i = \{q_i, H\}$, $\dot{p}_i = \{p_i, H\}$

Quantisation:

$$E \longrightarrow i\frac{\partial}{\partial t}$$

$$p_i \longrightarrow -i\frac{\partial}{\partial q_i}$$

$$\{\cdot, \cdot\} \longrightarrow \frac{1}{i}[\cdot, \cdot]$$

equal-time commutators:

$$[q_i(t), q_j(t)] = 0 = [p_i(t), p_j(t)]$$

$$[q_i(t), p_j(t)] = i\delta_{ij}$$

e.o.m. $\dot{q}_i = \frac{1}{i}[q_i, H] = \frac{p_i}{m}$,

$$\dot{p}_i = \frac{1}{i}[p_i, H] = -\frac{\partial V}{\partial q_i}$$

- now, instead of discrete point masses, consider a <u>continuum</u>

(classical mechanics: fluid dynamics with a density field $\rho(\underset{\sim}{x}, t)$

$$\boxed{\vec{x} =: \underset{\sim}{x}}$$

so label $i \longrightarrow$ space coordinate $\underset{\sim}{x} \in \mathbb{R}^d$ $(d=3)$

$\sum_i \longrightarrow \int d^d x$

rename $q_i(t) \longrightarrow \phi(\underset{\sim}{x}, t) = \phi(x)$

- generalise $L = \sum_i L_i(q_i, \dot{q}_i)$

$$\longrightarrow L = \int d^d x \underset{\substack{\uparrow \\ \text{Lagrangian density}}}{\mathcal{L}} (\phi(x), \partial_\mu \phi(x))$$

- action $S = \int dt\, L = \int d^D x\, \mathcal{L}$, $D = d+1 = 4$, space-time

  S is dim. less in N.U. $\left([L] = [T] = [M]^{-1} \right)$

  $\rightarrow [\mathcal{L}] = [M]^4$

- dynamical principle of classical field theory:

  S is stationary w.r.t. variation of the fields, keeping initial and final values fixed:

$$\delta S = \int d^4 x \left\{ \frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta(\partial_\mu \phi) \right\}$$

$$= \int d^4 x \left\{ \frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right) - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta\phi \right\}$$

2. term : total divergence $\xrightarrow{\text{Gauß}}$ surface integral $\xrightarrow{x \to \infty} 0$

$$\rightarrow \delta S \overset{!}{=} 0 \quad \text{iff} \quad \boxed{\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0}$$

Euler-Lagrange-equations

Example: scalar field with interactions

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - V(\phi)$$

$$\overset{ELE}{\rightsquigarrow} \quad \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = \partial_\mu \partial^\mu \phi$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = -V'(\phi)$$

$$\Rightarrow \quad \partial_\mu \partial^\mu \phi + V'(\phi) = (\partial_t^2 - \underset{\sim}{\nabla}^2) + V'(\phi) = 0$$

- free Klein-Gordon field theory: $V(\phi) = \frac{1}{2} m^2 \phi^2$

$$\longrightarrow \text{KGE}: \quad (\partial^2 + m^2)\phi(x) = 0$$

No magic: Lagrangian is designed to do this

(First get KGE $\xrightarrow[\text{reproduce}]{}$ KG - Field)

Conjugate momentum and Hamiltonian

- conjugate mom. in point mechanics is $p_i = \frac{\partial L}{\partial \dot{q}_i}$

with $L = \int d^3x \, \mathcal{L}$, it is more sensible to look at <u>momentum densities</u> conjugate to a field $\phi(\underset{\sim}{x})$:

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}(\underset{\sim}{x})}$$

- Hamiltonian density:

$$H = \int d^3x \, \{ \pi(\underset{\sim}{x}) \dot{\phi}(\underset{\sim}{x}) - \mathcal{L} \}$$

$$=: \int d^3x \, \mathcal{H}(\phi(\underset{\sim}{x}), \pi(\underset{\sim}{x}))$$

with $\dot{\phi}$ eliminated in favour of π

Example: $\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\underset{\sim}{\nabla}\phi \cdot \underset{\sim}{\nabla}\phi - V(\phi)$

$$\rightsquigarrow \quad \pi(\underset{\sim}{x}) = \dot{\phi}(\underset{\sim}{x})$$

$$H = \int d^3x \, \mathcal{H} = \int d^3x \, \{ \tfrac{1}{2}\pi^2 + \tfrac{1}{2}\underset{\sim}{\nabla}\phi \cdot \underset{\sim}{\nabla}\phi + V(\phi) \}$$

(free KG : $V(\phi) = \frac{1}{2} m^2 \phi^2$)

- retrieve e.o.m. from Poisson brackets

1.2 Noether theorem

If a Lagrangian field theory has a infinitesimal symmetry, then there is an associated current j^μ, which is conserved :

$$\partial_\mu j^\mu = 0$$

- infinitesimal variation of field ϕ

(may be multi-component : $\underline{\phi} = (\phi_1, \dots, \phi_n)$)

$$\phi(x) \longrightarrow \phi'(x) = \phi(x) + \alpha \Delta \phi(x)$$

"symmetry" : does not affect the field equation , i.e. leaves the action invariant

$\rightsquigarrow$ $\mathcal{L}$ invariant up to a total 4-divergence

$$\mathcal{L}(x) \longrightarrow \mathcal{L}(x) + \alpha \partial_\mu X^\mu(x) \qquad (*)$$

(often simply $X^\mu = 0$: symmetry of $\mathcal{L}$, not just action)

- calculate change in $\mathcal{L}$ explicitely : use the explicite dependence of field

$$\mathcal{L}(x) \longrightarrow \mathcal{L}(x) + \alpha \frac{\partial \mathcal{L}}{\partial \phi} \Delta\phi + \alpha \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu \Delta\phi$$

$$= \mathcal{L}(x) + \alpha\, \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta\phi \right) + \alpha \underbrace{\left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right)}_{=0} \Delta\phi \qquad (**)$$

$$(*) \overset{!}{=} (**)$$

$$\rightarrow \quad \partial_\mu j^\mu = 0, \quad \boxed{j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta\phi - X^\mu}$$

- Noether's theorem implies the conservation of the charge:

$$Q(t) = \int d^3x \, j^0(x)$$

$$\dot{Q}(t) = \frac{d}{dt} Q(t) = 0$$

$$0 = \int d^3x \, \partial_\mu j^\mu = \int d^3x \, \partial_0 j^0 + \underbrace{\int d^3x \, \underset{\sim}{\nabla} \cdot \underset{\sim}{j}}_{= \oint dA \cdot \underset{\sim}{j} = 0}$$

$$= \frac{d}{dt} \underbrace{\int d^3x \, j^0}_{Q}$$

if current / field fall off at infinity

Examples: ① Complex KG-field

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi$$

$$\left(\phi = \frac{1}{\sqrt{2}} (\phi_1 + i\phi_2), \quad \phi^* = \frac{1}{\sqrt{2}} (\phi_1 - i\phi_2) \right.$$
$$\left. \text{w/ real fields } \phi_1, \phi_2 \right)$$

$$\hookrightarrow \mathcal{L} = \frac{1}{2} (\partial_\mu \phi_1 \partial^\mu \phi_1 + \partial_\mu \phi_2 \partial^\mu \phi_2) - \frac{m^2}{2} (\phi_1^2 + \phi_2^2)$$

- symmetry under phase rotation $\phi \mapsto e^{i\alpha} \phi$

  so infinitesimally $\alpha \Delta \phi = i\alpha \phi$, $\alpha \Delta \phi^* = -i\alpha \phi^*$

  this leaves $\mathcal{L}$ invariant $\longrightarrow$ no X^μ

  then $j^\mu = i[(\partial^\mu \phi^*) \phi - \phi^* (\partial^\mu \phi)]$

  same as the probability current density of KG

- can verify $\partial_\mu j^\mu$ directly by using the K.G. eq. interporetation as electromagnetic current, Q = electric charge

② Infinitesimal translations (for arbitary $\mathcal{L}$ that's not explicitely space-time dependent)

$$x^\mu \longrightarrow x^\mu - a^\mu \quad (a^\mu \text{ infinitesimal})$$

so $\phi(x) \longmapsto \phi(x+a) \simeq \phi(x) + a^\nu \partial_\nu \phi(x)$

(transformation law for scalar field)

similarly, $\mathcal{L}(x) \longrightarrow \mathcal{L}(x) + a^\nu \partial_\nu \mathcal{L}$

$= \mathcal{L}(x) + a^\nu \partial_\mu (\delta^\mu_\nu \mathcal{L})$

note: we have a <u>vector</u> a^μ as infinitesimal parameter

$\rightarrow$ expect rank-2 tensor as conserved quantity:

$$\text{"}j^\mu\text{"} \rightarrow \boxed{\Theta^\mu_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu_\nu \mathcal{L}} \leftarrow \text{"}X\text{"}$$

satisfies $\partial_\mu \Theta^\mu_\nu = 0$ (if ϕ fulfills EL-eq)

This is "<u>energy-momentum-tensor</u>".

- conserved charge associated w/ time translation:

$$\int \Theta^{00} d^3x = \int \left(\frac{\partial \mathcal{L}}{\partial \dot\phi} \dot\phi - \mathcal{L} \right) d^3x = \int \mathcal{H} d^3x = E$$

- conserved (vector) charge associated w/ space translation:

$$\int \Theta^{0i} d^3x = \int \left(\frac{\partial \mathcal{L}}{\partial \dot\phi} \partial^i \phi - 0 \right) = -\int \pi \partial^i \phi \, d^3x = P^i$$

together $P^\mu = \int \Theta^{0\mu} d^3x$ 4-momentum of field

- for the KG Lagrangian:

$$\Theta^{\mu\nu} = \partial^\mu \phi \partial^\nu \phi - g^{\mu\nu} \mathcal{L}$$

$\mapsto$ symmetric (under $\mu \leftrightarrow \nu$)

in general, this is not true for $\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial^\nu \phi$

- On the other hand, $\Theta^{\mu\nu}$ is not unique:

take any rank-3 tensor $f^{\lambda\mu\nu}$ with $f^{\lambda\mu\nu} = -f^{\mu\lambda\nu}$,

s.t. $\partial_\mu \partial_\lambda f^{\lambda\mu\nu} = 0$

so def. $T^{\mu\nu} = \Theta^{\mu\nu} + \partial_\lambda f^{\lambda\mu\nu}$, then obviously

$$\partial_\mu T^{\mu\nu} = 0$$

also 4-momenta associated with $\Theta^{\mu\nu}$ and $T^{\mu\nu}$ the same:

$$\int \partial_\lambda f^{\lambda 0 \nu} d^3x = \int \partial_i f^{i0\nu} d^3x = \oint dA_i f^{i0\nu} = 0$$

if fields vanish at ∞

$\hookrightarrow$ choose $f^{\lambda\mu\nu}$ to make $T^{\mu\nu}$ symmetric

(Why care? ($T^{\mu\nu}$'s symmetry)

General Relativity: $\underbrace{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}}_{\text{symmetric}} R = 8\pi G T_{\mu\nu}$)