2. Klein-Gordon theory and its quantization

- $\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2}\phi$ $\rightsquigarrow$ K-G. eq. $(\partial^2 + m^2)\phi = 0$

  It has wave-like : $\phi = e^{\mp ip\cdot x} = e^{\mp i p_\mu x^\mu}$

  $\xrightarrow{\text{Plug into eq.}} -p^2 + m^2 = 0 \quad \rightsquigarrow \quad E^2 = |\underline{p}|^2 + m^2 = E_p^2$

- General solution constructed from $E_p = \sqrt{|\underline{p}|^2 + m^2}$

  Plane waves: $\phi(x) = \int d^3p \, (f(p) e^{-ipx} + g(p) e^{+ipx})$

- Hamiltonian $\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\underline{\nabla}\phi)\cdot(\underline{\nabla}\phi) + \frac{1}{2}m^2\phi^2$

- quantization : treat $\phi(\underline{x})$, $\pi(\underline{x})$ as operators at each space point, impose canonical commutation relations

$$[\phi(\underline{x}), \phi(\underline{x}')] = 0 = [\pi(\underline{x}), \pi(\underline{x}')]$$

$$[\phi(\underline{x}), \pi(\underline{x}')] = i\delta^{(3)}(\underline{x} - \underline{x}')$$

- these are time-independent, schrödinger-picture operators / commutation relations

  (will see later that these translate into equal-time commutation relations for time-dependent Heisenberg. operators.)

- decompose into Fourier modes ?

$$\phi(\underline{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_{\underline{p}} e^{i\underline{p}\cdot\underline{x}} + a_{\underline{p}}^\dagger e^{-i\underline{p}\cdot\underline{x}} \right)$$

$$\pi(\underline{x}) = \int \frac{d^3p}{(2\pi)^3} (-i)\sqrt{\frac{E_p}{2}} \left(a_{\underline{p}} e^{i\underline{p}\cdot\underline{x}} - a_{\underline{p}}^\dagger e^{-i\underline{p}\cdot\underline{x}} \right)$$

- note anology to harm. oszi.:

$$\underset{\sim\phi}{x = \frac{1}{\sqrt{2m\omega}}(a + a^\dagger)}, \qquad \underset{\sim\pi}{p = -i\sqrt{\frac{m\omega}{2}}(a - a^\dagger)}$$

- here: each momentum / each Fourier mode has its own set of creation / annihilation operators $a_{\underline{p}}^{(+)}$
- as a consequence: $[a_{\underline{p}}, a_{\underline{p}'}] = 0 = [a_{\underline{p}}^+, a_{\underline{p}'}^+]$

$$[a_{\underline{p}}, a_{\underline{p}'}^+] = (2\pi)^3 \delta^{(3)}(\underline{p} - \underline{p}')$$

check consistency: rewrite

$$\phi(\underline{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (a_{\underline{p}} + a_{-\underline{p}}^+) e^{i\underline{p}\cdot\underline{x}}$$

$$\pi(\underline{x}) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{E_p}{2}} (a_{\underline{p}} - a_{-\underline{p}}^+) e^{i\underline{p}\cdot\underline{x}}$$

$\Rightarrow [\phi(\underline{x}), \pi(\underline{x}')]$

$$= \int \frac{d^3p\, d^3p'}{(2\pi)^6} \left(-\frac{i}{2}\right) \sqrt{\frac{E_{p'}}{E_p}} \Big([a_{-\underline{p}}^+, a_{\underline{p}'}] - [a_{\underline{p}}, a_{-\underline{p}'}^+] \Big) e^{i(\underline{p}\cdot\underline{x} + \underline{p}'\cdot\underline{x}')}$$

$$= \int \frac{d^3p\, d^3p'}{(2\pi)^3} (+i) \sqrt{\frac{E_{p'}}{E_p}} \delta^{(3)}(\underline{p} + \underline{p}')\, e^{i(\underline{p}\cdot\underline{x} + \underline{p}'\cdot\underline{x}')}$$

$$= \int \frac{d^3p}{(2\pi)^3}\, i\, e^{ip(\underline{x} - \underline{x}')} = i\,\delta^{(3)}(\underline{x} - \underline{x}')$$

- Hamiltonian in terms of ladder operators?

$$H = \int d^3x\, \mathcal{H} = \int d^3x \frac{1}{2} (\pi^2(\underline{x}) + \underline{\nabla}\phi(\underline{x}) \cdot \underline{\nabla}\phi(\underline{x}) + m^2\phi^2(\underline{x}))$$

$$= \int d^3x \int \frac{d^3p\, d^3p'}{(2\pi)^3} e^{i(\underline{p}+\underline{p}')\cdot\underline{x}} \Big\{ -\frac{\sqrt{E_p E_{p'}}}{4} (a_{\underline{p}} - a_{-\underline{p}}^+)(a_{\underline{p}'} - a_{-\underline{p}'}^+)$$

$$+ \frac{-\underline{p}\cdot\underline{p}' + m^2}{4\sqrt{E_p E_{p'}}} (a_{\underline{p}} + a_{-\underline{p}}^+)(a_{\underline{p}'} + a_{-\underline{p}'}^+) \Big\}$$

$$\left[\begin{array}{l} \int d^3x\, e^{i(\underline{p}+\underline{p}')\cdot\underline{x}} = (2\pi)^3 \delta^{(3)}(\underline{p} + \underline{p}') \\ \int d^3p \quad \Rightarrow \quad \underline{p}' \to -\underline{p} \\ E_{p'} \to E_p \ , \quad \underline{p}^2 + m^2 = E_p^2 \\ a_p a_p ,\ a_p^+ a_p \ \text{terms cancel} \end{array}\right]$$

$$= \int \frac{d^3p}{(2\pi)^3} E_p \left(\frac{1}{2} a_{\underline{p}} a_{\underline{p}}^+ + \frac{1}{2} a_{\underline{p}}^+ a_{\underline{p}} \right) = \int \frac{d^3p}{(2\pi)^3} E_p \Big(a_{\underline{p}} a_{\underline{p}}^+ + \underbrace{\tfrac{1}{2} [a_{\underline{p}}, a_{\underline{p}}^+]}_{\sim \delta^{(3)}(0)} \Big)$$

$\rightsquigarrow$ zero-point energy of infinitely many oscillators

$\rightsquigarrow$ ignore !

$$\rightsquigarrow \boxed{H = \int \frac{d^3p}{(2\pi)^3} E_p \, a_{\underline{p}}^+ a_{\underline{p}}}$$

also : H " in normal - order form " :

$$\left.\begin{aligned} : a_{\underline{p}} a_{\underline{p}'}^+ : &= a_{\underline{p}'}^+ a_{\underline{p}} \\ : a_{\underline{p}}^+ a_{\underline{p}'} : &= a_{\underline{p}}^+ a_{\underline{p}'} \end{aligned}\right\}$$ and generalised to more operators : " all creators to the left "

(Note : $\langle 0 | : a_{\underline{p}}^{(+)} a_{\underline{p}'}^{(+)} \dots : | 0 \rangle = 0$)

- Obviously $[H, a_{\underline{p}}^+] = E_p a_{\underline{p}}^+$, $[H, a_{\underline{p}}] = - E_p a_{\underline{p}}$

- Vacuum / ground state defined by $\forall \underline{p} \;\; a_{\underline{p}} | 0 \rangle = 0$
- so : $a_{\underline{p}}^+$ increases the energy of a state by E_p

  $a_{\underline{p}}$ lowers —— ———— — by E_p

  $a_{\underline{p}}^+ | 0 \rangle$ carries the energy E_p
- Similarly , recall momentum op. $\underline{P} = - \int d^3x \, \pi(\underline{x}) \underline{\nabla} \phi(\underline{x})$

$$= \int \frac{d^3p}{(2\pi)^3} \underline{p} \, a_{\underline{p}}^+ a_{\underline{p}}$$

note $[H, \underline{P}] = 0$

so $a_{\underline{p}}^+$ also creates momentum $\underline{p}$; if $| 0 \rangle$ has energy and momentum 0, $a_{\underline{p}}^+ | 0 \rangle$ has energy E_p and momentum $\underline{p}$,

$E_p = \sqrt{\underline{p}^2 + m^2}$

$\hookrightarrow$ interpret $a_{\underline{p}}^{+}|0\rangle$ as 1-particle state

$a_{\underline{p}}^{+} a_{\underline{p}'}^{+} \ldots |0\rangle$ as many-particle state

<u>Normalisation</u>: vacuum $\langle 0|0\rangle = 1$

- Lorentz-inv. momentum eigenstate:

$$|\underline{p}\rangle = \sqrt{2E_p}\, a_{\underline{p}}^{+}\, |0\rangle$$

$\hookrightarrow$ $\langle \underline{p}|\underline{q}\rangle = 2E_p (2\pi)^3 \delta^{(3)}(\underline{p}-\underline{q})$ Lorentz-invariant!

- Lorentz-invariance of the three-momentum δ-functions:

(i) Phase space integral:

$$I = \int \underbrace{d^4p}_{} \, \delta(p^2-m^2)\, \Theta(p^0)\, g(p^0, \underline{p})$$

L-inv.: determinant of ortho chron. L-trafo $= 1$

$$= \int dE\, d^3p\, \delta(E^2-\underline{p}^2-m^2)\, \Theta(E)\, g(p^0, \underline{p})$$

$$= \int \frac{d^3p}{2E_p}\, g(p^0, \underline{p}) \quad , \quad E_p = \sqrt{\underline{p}^2+m^2}$$

$$\Rightarrow \underbrace{\frac{d^3p}{2E_p}}_{} = \delta(p^2-m^2)\, \Theta(p^0)\, d^4p$$

L.-inv. 3-mom. volume element ← b.o. the L.-inv. of right side!

(ii) now $\underbrace{1}_{\text{L.-inv.}} = \int d^3p\, \delta^{(3)}(\underline{p}) = \int \underbrace{\frac{d^3p}{2E_p}}_{\text{L.-inv.}} \underbrace{2E_p\, \delta^{(3)}(\underline{p})}_{\text{L.-inv.}}$

- then from $\phi(\underline{x}) = \int \frac{d^3p}{(2\pi)^3 \sqrt{2E_p}} \left(a_{\underline{p}} e^{i\underline{p}\cdot\underline{x}} + a_{\underline{p}}^{+} e^{-i\underline{p}\cdot\underline{x}} \right)$

$\hookrightarrow$ $\phi(\underline{x})|0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}}\, e^{-i\underline{p}\cdot\underline{x}}\, |\underline{p}\rangle$

$\hookrightarrow$ superposition of 1-particle state w/ momenta $\underline{p}$, interpreted as particle of position $\underline{x}$

- on the other hand, $\phi(\underline{x})$ can also annihilate particles,

  i.e. $\phi(\underline{x})|\underline{p}\rangle$ contains a piece $C\,|0\rangle$, where

  $C = \langle 0|\,\phi(\underline{x})\,|\underline{p}\rangle$. As $\phi(\underline{x})$ is hermitian, $\phi(\underline{x}) = \phi^{+}(\underline{x})$

  $$C^{*} = \langle \underline{p}|\,\phi(\underline{x})|0\rangle \overset{\text{see}}{\underset{\text{above}}{=}} \int \frac{d^3p'}{(2\pi)^3}\,\frac{e^{-i\underline{p}'\cdot\underline{x}}}{2E_{p'}}\,\underbrace{\langle \underline{p}\,|\,\underline{p}'\rangle}_{2E_p(2\pi)^3\delta^{(3)}(\underline{p}-\underline{p}')} = e^{-i\underline{p}\cdot\underline{x}}$$

  or $C = e^{i\underline{p}\cdot\underline{x}}$

2.2 Heisenberg-picture fields

- reminder: Schrödinger picture: operators time-indep.

  states time-dep.

  Heisenberg picture: operators time-dep.

  states time-indep.

  $$|\Psi_s(t)\rangle = |\Psi_s(t=0)\rangle\, e^{-iEt}$$

  so $|\Psi_H\rangle = e^{iHt}\,|\Psi_s(t)\rangle$

  $$O_H(t) = e^{iHt}\,O_s\,e^{-iHt} \iff i\frac{d}{dt}O_H(t) = [O_H, H]$$

- here: Heisenberg-picture field operators!

  $\phi(x) = \phi(\underline{x},t) = e^{iHt}\,\phi(\underline{x})\,e^{-iHt}$ and similarly for $\pi(x)$

- recall $[H, a_{\underline{p}}] = -E_p a_{\underline{p}} \rightarrow H a_{\underline{p}} = a_p(H - E_p)$

  $$\rightarrow H^n a_{\underline{p}} = a_{\underline{p}}(H - E_{\underline{p}})^n$$

  $$\Rightarrow e^{iHt} a_{\underline{p}} = a_{\underline{p}}\, e^{i(Ht - E_p t)}$$

  $$\Rightarrow e^{iHt} a_{\underline{p}}\, e^{-iHt} = a_{\underline{p}}\, e^{-iE_p t}$$

  Similarly $\rightarrow e^{iHt} a_{\underline{p}}\, e^{-iHt} = a_{\underline{p}}^{+}\, e^{+iE_p t}$

  We had $\phi(\underline{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(a_{\underline{p}}\, e^{+i\underline{p}\cdot\underline{x}} + a_{\underline{p}}^{+}\, e^{-i\underline{p}\cdot\underline{x}}\right)$

  so $$\phi(x) = \phi(\underline{x},t) = \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{\sqrt{2E_p}}\left(a_p e^{-ipx} + a_p^{+} e^{+ipx}\right)$$

similarly for $\pi(x)$

- Heisenberg picture operators fulfills classical equations of motion:

$$\dot{\phi}(x) = \pi(x) \,, \qquad \dot{\pi}(x) = (\underline{\nabla}^2 - m^2)\phi(x)$$

(check explicitely !)

$$\rightsquigarrow \ddot{\phi}(x) = (\underline{\nabla}^2 - m^2)\phi(x)$$

$$\rightsquigarrow (\partial^2 + m^2)\phi(x) = 0$$

given our expansion of $\phi(x)$ in Fourier modes $e^{\pm ipx}$, $\phi(x)$ clearly obeys the KG-eq., as $e^{\pm ipx}$ does

- Furthermore: from $\underset{\uparrow\ \text{general momentum operator}}{\hat{\underline{P}}}\, a_{\underline{p}} = a_{\underline{p}}(\hat{\underline{P}} - \underline{p})$

$$\left.\begin{aligned} \text{find} \quad e^{-i\hat{\underline{P}}\cdot\underline{x}} a_{\underline{p}} e^{+i\hat{\underline{P}}\cdot\underline{x}} &= a_{\underline{p}} e^{+i\underline{p}\cdot\underline{x}} \\ e^{-i\hat{\underline{P}}\cdot\underline{x}} a^\dagger_{\underline{p}} e^{+i\hat{\underline{P}}\cdot\underline{x}} &= a^\dagger_{\underline{p}} e^{-i\underline{p}\cdot\underline{x}} \end{aligned}\right\} \text{ Space translation!}$$

$$\phi(x) = e^{iHt - i\hat{\underline{P}}\cdot\underline{x}}\, \phi(0)\, e^{-iHt + i\hat{\underline{P}}\cdot\underline{x}} = e^{iPx}\phi(0)e^{-iPx}$$

$\uparrow$ $P^\mu = (H, \hat{\underline{P}})$ operators

2.3 Commutators and propagators

- With time-dep. Heisenberg ops, sufficient to consider $\phi(x) = \phi(\underline{x}, t)$, as $\pi(x) = \dot{\phi}(x)$

- Want to consider

$[\phi(x), \phi(y)]$ (Not equal time)

$\langle 0 | \phi(x)\phi(y) | 0 \rangle$ "propagator": particle created at y, propagating to x

- split up $\phi(x)$ according to $\phi(x) = \phi^+(x) + \phi^-(x)$

with $\quad \phi^+(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} a_{\underline{p}} e^{-ipx}$, $\phi^-(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} a^+_{\underline{p}} e^{+ipx}$

pos. frequency part $\qquad\qquad$ neg. frequency part

- now $\quad [\phi^+(x), \phi^+(y)] = \int \frac{d^3p\, d^3p'}{(2\pi)^6} \frac{1}{\sqrt{2E_p E_{p'}}} [a_{\underline{p}}, a^+_{\underline{p}'}] e^{-i(px - p'y)}$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip(x-y)} \equiv D(x-y)$$

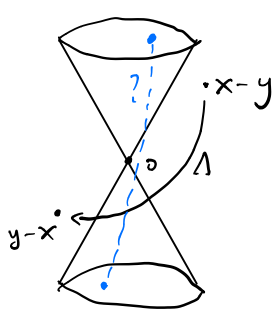
- $D(x-y)$: Lorentz-inv. function of $x-y$, non-vanishing for time- and space-like arguments

  exercise: if $x^0 = y^0$, then decay $e^{-m|\underline{x}-\underline{y}|}$ for large $|\underline{x}-\underline{y}|$

- now: $\quad [\phi(x), \phi(y)] = [\phi^+(x), \phi^-(y)] + [\phi^-(x), \phi^+(y)]$

$$= D(x-y) - D(y-x)$$

light-cone:



- if $x-y$ space-like, there exists a Lorentz-trafo $x-y \mapsto y-x$, so $D(x-y) = D(y-x)$

  $\Rightarrow [\phi(x), \phi(y)] = 0$

- if $x-y$ time-like, not

- interpretation: (i) $\phi(x)$ and $\phi(y)$ are "simultaneously" measurable if $x-y$ is space-like

  (ii) causality is preserved: a measurement of ϕ at x cannot affect ϕ at y if $x-y$ is space-like.

- now $\quad \langle 0| \phi(x)\phi(y) |0\rangle = \langle 0| \phi^+(x)\, \phi^-(y) |0\rangle$

  (since $\phi^+(y)|0\rangle = 0$, $\langle 0| \phi^-(x) = 0$)

$$= \langle 0| \underbrace{[\phi^+(x), \phi^-(x)]}_{\in \mathbb{C}} |0\rangle = D(x-y)$$

so the commutator equals (one half of) a propagator.

- it turns out that a slightly different kind of "propagator" is most useful in QFT: the "Feynman propagator"

$$D_F(x-y) = \langle 0 | T\, \phi(x)\phi(y) | 0 \rangle$$

where T denotes time-ordering:

$$D_F(x-y) = \Theta(x^0-y^0)\langle 0|\phi(x)\phi(y)|0\rangle + \Theta(y^0-x^0)\langle 0|\phi(y)\phi(x)|0\rangle$$
$$= \Theta(x^0-y^0)D(x-y) + \Theta(y^0-x^0)D(y-x)$$

- We'll prove (next week) a useful integral representation for the Feynman propagator:

$$D_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\underbrace{\frac{i}{p^2-m^2+i\varepsilon}}_{\text{Feynman propagator in momentum space!}} e^{-ip(x-y)}$$