3. Quantisation of the dirac field

3.1 Repetition: Dirac equation

- aim: relativistic wave equation of first order in time & space derivation

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0$$

$$(i\not{\partial} - m)\psi(x) = 0$$

$\hookrightarrow$ only works if γ^μ are matrices

ψ " vector "

Components $\psi_a(x)$ ought to fulfill the KG eq.

Clifford algebra, $\{\gamma^\mu, \gamma^\nu\} = 2\mathbb{1} g^{\mu\nu}$

Standard representation (dirac's)

$$\gamma_0 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix} \; ; \quad \underline{\gamma} = \begin{pmatrix} 0 & \underline{\sigma} \\ -\underline{\sigma} & 0 \end{pmatrix}$$

in Peskin: chiral rep. $\gamma_0 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}$,

Lorentz covariance

Lorentz trafo acting on space-time coordinates

$$\Lambda = \exp\left(\frac{1}{(i)\,2}\,\omega_{\mu\nu} M^{\mu\nu}\right)$$

$M^{\mu\nu}$: generator of Lie algebra

$\omega_{\mu\nu}$: set of parameters

here anti-hermitian generators

$$(*) \quad [M^{\mu\nu}, M^{\lambda\rho}] = g^{\nu\lambda} M^{\mu\rho} - g^{\mu\lambda} M^{\nu\rho} + g^{\mu\rho} M^{\nu\lambda} - g^{\nu\rho} M^{\mu\lambda}$$

explicit matrix repre. $(M^{\mu\nu})^\alpha{}_\beta = g^{\mu\alpha}\delta^\nu{}_\beta - g^{\nu\alpha}\delta^\mu{}_\beta$

e.g. $M^{01} = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$; $M^{12} = \begin{pmatrix} 0 & & & \\ & 0 & -1 & \\ & 1 & 0 & \\ & & & 0 \end{pmatrix}$

introduce spinor repre.

$$S^{\rho\sigma} = \frac{1}{4}[\gamma^\rho, \gamma^\sigma] = \frac{1}{2i}\sigma^{\rho\sigma}$$

these fulfill Lie algebra (*) with $M^{\mu\nu} \mapsto S^{\mu\nu}$

Furthermore $\quad [\gamma^\mu, S^{\rho\sigma}] = (M^{\rho\sigma})^\mu{}_\nu \gamma^\nu$

γ^μ transforms like a Lorentz 4-vector

Associated with Lorentz-trafo $\Lambda = \exp(\frac{1}{2}\omega_{\mu\nu}M^{\mu\nu})$

there is the spinor / Dirac rep.

$$S(\Lambda) = \exp(\tfrac{1}{2}\omega_{\mu\nu}S^{\mu\nu})$$

- spinor transformation law:

$$\psi'_a(x) = S_{ab}(\Lambda)\,\psi_b(\Lambda^{-1}x)$$

$$(\text{from } \psi'(x') = S(\Lambda)\psi(x))$$

used to prove Lorentz-covariance of Dirac eq.

Adjoint spinor ; bilinear

- Lorentz group is not unitary

  note $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$, $\gamma^{0\,2} = \mathbb{1}$

  $S(\Lambda)^\dagger = \gamma^0 S(\Lambda)^{-1}\gamma^0$

  so the <u>adjoint spinor</u> $\bar\psi = \psi^\dagger\gamma^0$

  transform as $\bar\psi \longmapsto \bar\psi S(\Lambda)^{-1}$

- Define the fifth gamma matrix:

  $$\gamma_5 := i\gamma^0\gamma^1\gamma^2\gamma^3 \qquad \{\gamma^\mu, \gamma^5\} = 0 \;, \; \gamma^{5\,2} = \mathbb{1}$$

  Stand. rep $\gamma_5 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$

  (in Peskin $\gamma_5 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}$)

- Parity $\Lambda_P = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$, $S(\Lambda_P) = \underset{\substack{\uparrow \\ \text{phase}}}{\eta}\gamma^0$

  $[\gamma_5, S^{\mu\nu}] = 0 \longrightarrow \gamma_5$ is invariant und $\mathcal{L}_P^+$ (proper orthochronous LT) but $[\gamma_5, \gamma^0] \propto [\gamma_5, P] \neq 0$

not invariant under parity

- 16 (4×4) indep. bilinear densities

$$S(x) = \bar{\psi}(x)\psi(x) \quad , \quad V^\mu(x) = \bar{\psi}(x)\gamma^\mu\psi(x)$$

scalar (density) — vector

$$t^{\mu\nu}(x) = \bar{\psi}(x) S^{\mu\nu}\psi(x) \qquad a^\mu = \bar{\psi}(x)\gamma^\mu\gamma^5\psi(x)$$

(anti-symm.) tensor — axial-vector

if symm. ⟶ clifford algebra
⟶ zero

- Plane wave solutions

$$(i\not{\partial} - m)\psi(x) = 0 \ ; \quad \text{ansatz} \begin{cases} \psi(x) = u(p)e^{-ip\cdot x} & \text{positive} \\ \psi(x) = v(p)e^{+ip\cdot x} & \text{negative} \end{cases} \text{frequency}$$

$$\hookrightarrow (\not{p} - m)u(p) = 0 \qquad (\not{p} + m)v(p) = 0$$

Construct solutions

(i) explicitely using the repre. of γ-matrices and 2-spinor $u(p) = \begin{pmatrix} \phi(p) \\ \chi(p) \end{pmatrix}$

(ii) boosting the rest-frame solutions:

result: $$u_s(p) = \sqrt{E_p + m}\begin{pmatrix} \chi_s \\ \frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\chi_s \end{pmatrix} e^{-ip\cdot x}$$

$$\chi_{\frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \quad \chi_{-\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad s = \pm\frac{1}{2}$$

$$v_s(p) = \sqrt{E_p + m}\begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\tilde{\chi}_s \\ \tilde{\chi}_s \end{pmatrix} e^{ip\cdot x} , \quad \tilde{\chi}_s = \chi_{-s}$$

Orthogonality $$\bar{u}_s(p)u_{s'}(p) = 2m\,\delta_{ss'} = -\bar{v}_s(p)v_{s'}(p)$$
$$\bar{u}_s(p)v_{s'}(p) = 0$$

QFT Normalisation ($\neq$ QM N.)

u_s / v_s : positive/negative frequency (energy)

Spin sums / projection operators:

From the Dirac spinor eq.s

$$(\not{p} - m)\, u(p) = 0$$

$$(\not{p} + m)\, v(p) = 0$$

$\Lambda_\pm(p) = \dfrac{m \pm \not{p}}{2m}$ are projection ops on pos./neg. frequency states

$$\not{p}\not{p} = p^2 = m^2$$

$\Lambda_+ u_s = u_s$, $\Lambda_- u_s = 0$, $\Lambda_+ v_s = 0$, $\Lambda_- v_s = v_s$

As u_s, v_s, $s = \pm\frac{1}{2}$ form basis of spinor space

$$\sum_s u_s(p)\, \bar{u}_s(p) = 2m\, \Lambda_+(p) = \not{p} + m$$

$$\sum_s v_s(p)\, \bar{v}_s(p) = -2m\, \Lambda_-(p) = \not{p} - m$$

3.2 Dirac Lagrangian and quantization

$$\mathcal{L} = \bar{\psi}(x)\,(i\not{\partial} - m)\,\psi(x)$$

we can treat ψ and $\bar{\psi}$ as two independent fields

the EL-eq.s are:

$\bar{\psi}$: $\quad \underbrace{\partial_\mu \left(\dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} \right)}_{=0} - \dfrac{\partial \mathcal{L}}{\partial \bar{\psi}} = 0 \quad \Rightarrow \quad (i\not{\partial} - m)\psi = 0$

ψ: $\quad 0 = \partial_\mu \left(\dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \right) - \dfrac{\partial \mathcal{L}}{\partial \psi} \rightsquigarrow \partial_\mu \bar{\psi}\, i\gamma^\mu + \bar{\psi} m$

$$\Rightarrow \quad \bar{\psi}(i\overleftarrow{\not{\partial}} + m) = 0$$

$$\gamma^{\mu\dagger} = \gamma^0 \gamma^\mu \gamma^0$$

Field conjugate to (Schrödinger-picture) field $\psi_a(\underline{x})$

$$\pi_a(\underline{x}) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi_a)} = i(\bar{\psi}\gamma^0)_a = i\psi^+_a(\underline{x})$$

No derivative on $\bar{\psi}_b$ $\rightsquigarrow$ no field $\times$ conjugate !

- Hamiltonian for the Dirac field:

$$H = \int d^3x \, (\pi_a(\underline{x})\dot{\psi}_a(\underline{x}) - \mathcal{L})$$

$$= \int d^3x \, (i\bar{\psi}\gamma^0\partial_0\psi - \mathcal{L})$$

$$= \int d^3x \, \psi^+(-i\underbrace{\gamma^0\underline{\gamma}}_{\alpha}\cdot\underline{\nabla} + m\underbrace{\gamma^0}_{\beta})\psi$$

$$= \int d^3x \, \psi^+(\frac{1}{i}\underline{\alpha}\cdot\underline{\nabla} + m\beta)\psi$$

How to quantize ?

expect ψ_a, π_a to create / annihilate particles

$\longrightarrow$ Pauli principle

no more than one particle in a given state

(of momentum, spin ---)

$\hookrightarrow$ Amplitude should change sign under the exchange of identical particles

Conclusion:

$$\{\psi_a(\underline{x}), \psi_b^+(\underline{x}')\} = \delta_{ab}\,\delta^{(3)}(\underline{x}-\underline{x}')$$

$$\{\psi_a(\underline{x}), \psi_b(\underline{x}')\} = \cdots = 0$$

(at equal time !)

expand $\psi, \psi^+ (\sim \bar{\psi})$ in Fourier modes

$$\psi(\underline{x}) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} \sum_s (a^s_{\underline{p}} u_s(\underline{p}) e^{i\underline{p}\cdot\underline{x}} + b^{s+}_{\underline{p}} v_s(\underline{p}) e^{-i\underline{p}\cdot\underline{x}})$$

$\rightsquigarrow$ two types of operator a, b associated with pos. / neg. frequency states

$\rightsquigarrow$ writing $b_{\underline{p}}^{s\dagger}$ (instead of $b_{\underline{p}}^{s}$) is a convention

we want $a^\dagger$, $b^\dagger$ to create pos. energy particles

Conjugate field: $\bar{\psi}(\underline{x}) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} \sum_s \left(a_{\underline{p}}^{s\dagger} \bar{u}_s(\underline{p}) e^{-i\underline{p}\cdot\underline{x}} + b_{\underline{p}}^{s} \bar{v}_s(\underline{p}) e^{i\underline{p}\cdot\underline{x}} \right)$

Anticommutation relation for Fourier modes:

$$\{ a_{\underline{p}}^{s}, a_{\underline{p}'}^{s'\dagger} \} = \{ b_{\underline{p}}^{s}, b_{\underline{p}'}^{s'\dagger} \} = (2\pi)^3 \delta_{ss'} \delta^{(3)}(\underline{p} - \underline{p}')$$

$$\{ a, a \} = \{ a^\dagger, a^\dagger \} = \underset{(b)}{\cdots} = 0$$

- Calculating Hamiltonian in terms of Fourier modes (using only spinor properties) yields:

$$H = \int \frac{d^3p}{(2\pi)^3} \sum_s E_p \left(a_{\underline{p}}^{s\dagger} a_{\underline{p}}^{s} - b_{\underline{p}}^{s} b_{\underline{p}}^{s\dagger} \right)$$