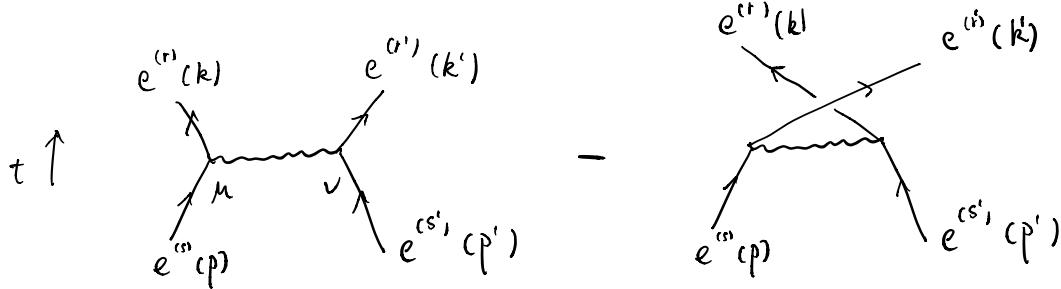


H.12

a)



$$iM = (-ie)^2 \bar{u}^{(r)}(k) \gamma^\mu u^{(s)}(p) \frac{-ig_{\mu\nu}}{(p-k)^2 + i\varepsilon} \bar{u}^{(r')}(k') \gamma^\nu u^{(s')}(p')$$

$$- (-ie)^2 \bar{u}^{(r')}(k') \gamma^\mu u^{(s)}(p) \frac{-ig_{\mu\nu}}{(p-k')^2 + i\varepsilon} \bar{u}^{(r)}(k) \gamma^\nu u^{(s')}(p')$$

comes about because the anti-commutation relation of fermions.

More precisely: $\psi_1 = \psi_{e(p)}(x)$, $\psi_2 = \psi_{e(k)}(x)$

$\psi_3 = \psi_{e(p')}(x)$ / $\psi_4 = \psi_{e(k')}(x)$

$$H_{int} \propto \int d^4x \bar{\psi} \gamma_\mu A^\mu \psi$$

→ zeroth order: $M \rightarrow 1$

first order: since A contains ladder operator (S),
in total we get $(2+3+2) = 7$ ladder operators
⇒ first order contribution vanishes

second order:

$$\sim \langle k k' | T(\bar{\psi} \gamma_\mu A^\mu \psi \bar{\psi} \gamma_\nu A^\nu \psi) | p p' \rangle$$

nontrivial part $\rightarrow = \langle k k' | \underbrace{\bar{\psi} \gamma_\mu A^\mu \psi}_{: :} \underbrace{\bar{\psi} \gamma_\nu A^\nu \psi}_{: :} | p p' \rangle \quad \times$

$$+ \langle k k' | \underbrace{\bar{\psi} \gamma_\mu A^\mu \psi}_{: :} \underbrace{\bar{\psi} \gamma_\nu A^\nu \psi}_{: :} | p p' \rangle$$

$$+ \langle k | \overbrace{\bar{\psi} \gamma_\mu A^\mu \psi}^{\text{1st pair}} | \overbrace{\bar{\psi} \gamma_\nu A^\nu \psi}^{\text{2nd pair}} | p' \rangle$$

$$+ \langle k | \overbrace{\bar{\psi} \gamma_\mu A^\mu \psi}^{\text{1st pair}} | \overbrace{\bar{\psi} \gamma_\nu A^\nu \psi}^{\text{2nd pair}} | p' \rangle$$

A must contract with A , since there is no external photon.

The k, k' can also contract with second pair of $\bar{\psi}, \psi$, but then it's the same expression and in the end it only contribute a factor (2) in front.

$$\text{1. term} = \langle 0 | \overbrace{\bar{a}_k^{(r)} \bar{a}_{k'}^{(s)}}^{\text{1st pair}} : \bar{\psi} \gamma_\mu A^\mu \psi : \bar{\psi} \gamma_\nu A^\nu \psi | a_p^{(s)} a_{p'}^{(r)} | 0 \rangle$$

$$= (-1)^3 \langle 0 | \overbrace{\bar{a}_k^{(r)} \bar{\psi} \gamma_\mu}^{\text{1st}} \overbrace{a_{k'}^{(s)} \psi}^{\text{2nd}} \overbrace{A^\mu A^\nu}^{\text{3rd}} \bar{\psi} \overbrace{a_{p'}^{(s)}}^{\text{4th}} \gamma_\nu \bar{\psi} \overbrace{a_p^{(r)}}^{\text{5th}} | 0 \rangle$$

→ negative!

Analogous.

2. Term → positive

3. Term → positive

4. term → negative

⇒ in the end we get the minus sign for the second diagram. The factor 2 will cancel with the $\frac{1}{2}$ of expansion of e^x .

$$b) \frac{1}{4} \sum_{\text{spins}} |\mathcal{M}|^2 = \frac{1}{4} \sum_{\text{spins}} \left(|\mathcal{M}_{\alpha\beta}|^2 + |\mathcal{M}_{\beta\alpha}|^2 + 2 \operatorname{Re}(\mathcal{M}_{\alpha\beta} \mathcal{M}_{\beta\alpha}^*) \right)$$

$$\frac{1}{4} \sum_{\text{spins}} |\mathcal{M}_{\alpha\beta}|^2$$

$$= \frac{1}{4} \sum_{\text{spins}} |(-ie)^2 \bar{u}^{(r)}(k) \gamma^\mu u^{(s)}(p) \frac{-ig_{\mu\nu}}{t} \bar{u}^{(r)}(k') \gamma^\nu u^{(s)}(p')|^2$$

$$= \frac{e^4}{4} \sum_{\text{spins}} \bar{u}^{(r)}(k) \gamma^\mu u^{(s)}(p) \frac{1}{t} \bar{u}^{(r)}(k') \gamma_\mu u^{(s)}(p') \cdot \bar{u}^{(s)}(p') \gamma_\nu u^{(r)}(k') \frac{1}{t} \bar{u}^{(s)}(p) \gamma^\nu u^{(r)}(k)$$

$$\begin{aligned}
&= \frac{e^4}{4t^2} \sum_{r,r'} \sum_{s,s'} \bar{u}^{(r)}(k) \gamma^\mu u^{(s)}(p) \bar{u}^{(r')}(k') \gamma_\mu u^{(s')}(p') \bar{u}^{(s)}(p') \gamma_r u^{(r)}(k) \bar{u}^{(s)}(p) \gamma^\nu u^{(s)}(k) \\
&= \frac{e^4}{4t^2} \sum_{r,r'} \sum_{s,s'} \sum (\bar{u}^{(r)}(k))_\alpha (\gamma^\mu)_{\alpha\beta} (u^{(s)}(p))_\beta (\bar{u}^{(r')}(k'))_\mu (\gamma_\mu)_{\mu\nu} (u^{(s')}(p'))_\nu \\
&\quad \times (\bar{u}^{(s)}(p'))_i (\gamma_r)_{ij} (u^{(r)}(k'))_j (\bar{u}^{(s)}(p))_m (\gamma^\nu)_{mn} (u^{(s)}(k))_n \\
&= \frac{e^4}{4t^2} \sum_{r,r'} \sum_{s,s'} \sum (\bar{u}^{(r)}(k))_\alpha (u^{(r)}(k))_n (\gamma^\mu)_{\alpha\beta} (\bar{u}^{(r')}(k'))_\mu (u^{(r')}(k'))_j (\gamma_\mu)_{\mu\nu} \\
&\quad \times (\bar{u}^{(s)}(p'))_i (u^{(s)}(p'))_r (\gamma_r)_{ij} (\bar{u}^{(s)}(p))_m (u^{(s)}(p))_\beta (\gamma^\nu)_{mn} \\
&= \frac{e^4}{4t^2} \sum (k+m)_{\alpha\beta} (\gamma^\mu)_{\alpha\beta} (\not{p}+m)_{ij} (\gamma_\mu)_{\mu\nu} (\not{p}+m)_{ri} (\gamma_r)_{ij} (\not{p}+m)_{\beta m} (\gamma^\nu)_{mn} \\
&= \frac{e^4}{4t^2} \sum (\gamma^\mu)_{\alpha\beta} (\not{p}+m)_{\beta m} (\gamma^\nu)_{mn} (\not{k}+m)_{\alpha\beta} \\
&\quad \times (\gamma_\mu)_{\mu\nu} (\not{p}+m)_{ri} (\gamma_r)_{ij} (\not{k}+m)_{ji} \\
&= \frac{e^4}{4t^2} \text{tr} [\gamma^\mu (\not{p}+m) \gamma^\nu (\not{k}+m)] + \text{tr} [\gamma_\mu (\not{p}+m) \gamma_\nu (\not{k}+m)]
\end{aligned}$$

Is it always necessary to calculating the components of matrices
 Is there a trick to get the traces directly?
Tr (scalar) = scalar
↑ especially in exams!
(the M^* calculation as well.)

$$\begin{aligned}
&\not{q}^2 = (\not{p} - \not{k})^2 \equiv t \\
&= \frac{e^4}{4t^4} \text{tr} [\gamma^\mu \not{p} \gamma^\nu \not{k} + m^2 \gamma^\mu \gamma^\nu] \text{tr} [\gamma_\mu \not{p}^\dagger \gamma_\nu \not{k}' + m^2 \gamma_\mu \gamma_\nu] \\
&\quad \downarrow \\
&\quad = 4 p_\alpha k_\beta (g^{\alpha\beta} g^{\nu\beta} - g^{\mu\nu} g^{\alpha\beta} + g^{\mu\beta} g^{\nu\alpha}) \\
&\quad = 4 (p^\mu k^\nu - g^{\mu\nu} (p \cdot k) + p^\nu k^\mu)
\end{aligned}$$

$$\begin{aligned}
&= \frac{e^4}{4t^4} [4(p^\mu k^\nu - g^{\mu\nu} (p \cdot k) + p^\nu k^\mu) + 4m^2 g^{\mu\nu}] \\
&\quad \times [4(p'_{\mu} k'_\nu + p'_{\nu} k'_\mu - g_{\mu\nu} (p' \cdot k')) + 4m^2 g_{\mu\nu}]
\end{aligned}$$

$$\begin{aligned}
&= \frac{4e^4}{t^4} \left[\underbrace{(p \cdot p')(k \cdot k')}_{(p \cdot k)(p' \cdot k')} + \underbrace{(p \cdot k')(p' \cdot k)}_{(p \cdot k)(p' \cdot k')} - \underbrace{(p \cdot k)(p' \cdot k')}_{(p \cdot k)(p' \cdot k')} + m^2 (p \cdot k) \right. \\
&\quad - \underbrace{(p \cdot k)(p' \cdot k')}_{(p \cdot k)(p' \cdot k')} - \underbrace{(p \cdot k)(p' \cdot k')}_{(p \cdot k)(p' \cdot k')} + \underbrace{4(p \cdot k)(p' \cdot k')}_{(p \cdot k)(p' \cdot k')} - 4m^2 (p \cdot k) \\
&\quad \left. + \underbrace{(p' \cdot k)(p \cdot k')}_{(p \cdot k)(p' \cdot k')} + \underbrace{(p \cdot p')(k \cdot k')}_{(p \cdot k)(p' \cdot k')} - \underbrace{(p \cdot k)(p' \cdot k')}_{(p \cdot k)(p' \cdot k')} + m^2 (p \cdot k) \right]
\end{aligned}$$

$$+ m^2(p' \cdot k') + m^2(p' \cdot k') - 4m^2(p' \cdot k') + 4m^4]$$

$$= \frac{4e^4}{q^4} \left[4m^4 - 2m^2((p \cdot k) + (p' \cdot k')) + 2(p \cdot k)(p' \cdot k) + 2(p \cdot p')(k \cdot k') \right]$$

$$\approx \frac{8e^4}{q^4} [(p \cdot k)(p' \cdot k) + (p \cdot p')(k \cdot k')]$$

$$\frac{1}{4} \sum_{\text{spin}} |M_b|^2$$

$$= \frac{e^4}{4} \sum_{\text{spin}} \left| \bar{u}^{(r)}(k) \gamma^\mu u^{(s)}(p) \frac{1}{q} \bar{u}^{(r)}(k) \gamma_\mu u^{(s)}(p') \right|^2$$

$$= \frac{e^4}{4q^4} \sum_{r,r'} \sum_{s,s'} \bar{u}^{(r)}(k') \gamma^\mu u^{(s)}(p) \bar{u}^{(r)}(k) \gamma_\mu u^{(s)}(p') \cdot \bar{u}^{(s')}(p') \gamma_\nu u^{(r)}(k) \bar{u}^{(s)}(p) \gamma^\nu u^{(r')}(k')$$

$$= \frac{e^4}{4q^4} \text{tr} [(k' + m) \gamma^\mu (p + m) \gamma^\nu] \cdot \text{tr} [(k + m) \gamma_\mu (p' + m) \gamma_\nu]$$

$$= \frac{4e^4}{q^4} [k'^\mu p^\nu + k'^\nu p^\mu - g^{\mu\nu}(k' \cdot p) + m^2 g^{\mu\nu}] [k_\mu p'_\nu + k_\nu p'_\mu - g_{\mu\nu}(k \cdot p') + m^2 g_{\mu\nu}]$$

↑
essentially the same expression as i) with $k \leftrightarrow k'$

$$= \frac{8e^4}{q^4} [(p \cdot k)(p' \cdot k') + (p \cdot p')(k \cdot k')]$$

$$\frac{1}{4} \sum_{\text{spin}} 2 \text{Re} (M_a M_b^*)$$

$$= \frac{1}{2} \sum_{r,r'} \sum_{s,s'} \frac{e^4}{q^2 q^2} \bar{u}^{(r)}(k) \gamma_\mu u^{(s)}(p) \bar{u}^{(r)}(k') \gamma^\mu u^{(s')}(\bar{p}) \bar{u}^{(s')}(\bar{p}') \gamma^\nu u^{(r)}(k) \bar{u}^{(s)}(p) \gamma_\nu u^{(r')}(k')$$

$$= \frac{e^4}{2q^2 q^2} \sum_{r,r'} \sum_{s,s'} \sum (\bar{u}^{(r)}(k))_a (\gamma_\mu)_{\alpha \beta} (u^{(s)}(p))_\beta (\bar{u}^{(r')}(k'))_\mu (\gamma^\mu)_{\nu \nu} (u^{(s')}(p'))_\nu \\ \times (\bar{u}^{(s')}(p'))_i (\gamma^\nu)_{ij} (u^{(r)}(k))_j (\bar{u}^{(s)}(p))_m (\gamma_\nu)_{mn} (u^{(r')}(k'))_n$$

$$= \frac{e^4}{2q^2 q^2} \sum (K+m)_j \alpha (\gamma_\mu)_{\alpha \beta} (p+m)_{\beta m} (K'+m)_{n \mu} (\gamma^\mu)_{\mu \nu} (p'+m)_{\nu i} (\gamma^\nu)_{ij} (\gamma_\nu)_{mn}$$

$$= \frac{e^4}{2q^2 q^2} \text{tr} [(K+m) \gamma_\mu (p+m) \gamma_\nu (K'+m) \gamma^\mu (p'+m) \gamma^\nu]$$

$$= \frac{e^4}{2q^2 q^2} \text{tr} [K \gamma_\mu p \gamma_\nu K' \gamma^\mu p' \gamma^\nu + (K \gamma_\mu p \gamma_\nu \gamma^\mu \gamma^\nu + K \gamma_\mu \gamma_\nu K' \gamma^\mu \gamma^\nu \\ + K \gamma_\mu \gamma_\nu \gamma^\mu p' \gamma^\nu + \gamma_\mu p \gamma_\nu K' \gamma^\mu \gamma^\nu + \gamma_\mu p \gamma_\nu \gamma^\mu p' \gamma^\nu]$$

$$\begin{aligned}
& + \gamma_\mu \gamma_\nu K' \gamma^\mu p^\nu] \cdot m^2 + m^4 \gamma_\mu \gamma_\nu \gamma^\mu \gamma^\nu] \\
\propto & \frac{e^4}{2q^2 q^2} \text{tr} [K \gamma_\mu \cancel{p} \gamma_\nu \cancel{K'} \gamma^\mu \cancel{p'} \gamma^\nu] \\
& \quad | \\
& = \text{tr} [\cancel{K'} \gamma^\mu \cancel{p}^\mu \gamma^\nu \cancel{p'} \gamma_\nu \cancel{K} \gamma_\mu] \\
& \quad \quad \quad -2 \cancel{p} \\
& = -2 \text{tr} [\cancel{K'} \gamma^\mu \cancel{p}^\mu \cancel{p'} \cancel{K} \gamma_\mu] \\
& = 4 \text{tr} [\cancel{K'} \cancel{K} \cancel{p} \cancel{p'}] \\
& = \frac{e^4}{2q^2 q^2} \cdot 4 \cdot 4 \left[(k' \cdot k)(p \cdot p') - \underbrace{(k \cdot p)(k \cdot p') + (k' \cdot p')(k \cdot p)}_{=0, \text{ using } p \cdot k = p \cdot k'} \right] \\
& = \frac{8e^4}{q^2 q^2} (k \cdot k')(p \cdot p') \\
& \quad \quad \quad \text{16}
\end{aligned}$$

c) $S = (p + p')^2 = 2(p \cdot p') = 2(E^2 + p^2) = 4E^2$ elastic scattering

 $p = (E, 0, 0, p), \quad p' = (E, 0, 0, -p)$
 $k = (E, p \sin \theta, 0, p \cos \theta), \quad k' = (E, -p \sin \theta, 0, -p \cos \theta)$
 $\quad \quad \quad k' = (E, p)$
 $\Rightarrow q = p - k = (0, -p \sin \theta, 0, p(1 - \cos \theta))$
 $q^2 = p^2 \sin^2 \theta - p^2 (1 - \cos \theta)^2 = -p^2 (2 - 2 \cos \theta)$
 $= -4p^2 \sin^2 \frac{\theta}{2}$
 $\tilde{q} = p - k' = (0, p \sin \theta, 0, p(1 + \cos \theta))$
 $\tilde{q}^2 = -p^2 \sin^2 \theta - p^2 (1 + \cos \theta)^2$
 $= -p^2 (2 + 2 \cos \theta) = -4p^2 \cos^2 \frac{\theta}{2}$

$(k \cdot k') = E^2 + p^2 \sin^2 \theta + p^2 \cos^2 \theta = E^2 + p^2 = (p \cdot p') = 2E^2$

$$(p \cdot k') = (p' \cdot k) = E^2 + p^2 \cos \theta = \bar{E}^2 (1 + \cos \theta)$$

$$(p \cdot k) = E^2 - p^2 \cos \theta , \quad (p' \cdot k') = E^2 - p^2 \cos \theta \approx E^2 (1 - \cos \theta)$$

$$\Rightarrow i) = \frac{8e^4}{16p^4 \sin^4 \frac{\theta}{2}} [(2\bar{E}^2)^2 + \bar{E}^4 (1 + \cos \theta)^2]$$

$$= \frac{e^4}{2\bar{E}^4 \sin^4 \frac{\theta}{2}} \cdot \bar{E}^4 (4 + 1 + 2\cos \theta + \cos^2 \theta)$$

$$= \frac{e^4}{2 \sin^4 \frac{\theta}{2}} (5 + 2\cos \theta + \cos^2 \theta)$$

$$ii) = \frac{8e^4}{16p^4 \cos^4 \frac{\theta}{2}} [(2\bar{E}^2)^2 + \bar{E}^4 (1 - \cos \theta)^2]$$

$$= \frac{e^4}{2 \cos^4 \frac{\theta}{2}} (5 - 2\cos \theta + \cos^2 \theta)$$

$$iii) = \frac{16e^4}{16p^4 \sin^2 \frac{\theta}{2} \cos^2 \frac{\theta}{2}} \cdot (2\bar{E}^2)^2 = \frac{4e^4}{\sin^2 \frac{\theta}{2} \cos^2 \frac{\theta}{2}}$$

$$\Rightarrow |\bar{M}|^2 = e^4 \left[\frac{1}{2 \sin^4 \frac{\theta}{2}} (5 + 2\cos \theta + \cos^2 \theta) + \frac{1}{2 \cos^4 \frac{\theta}{2}} (5 - 2\cos \theta + \cos^2 \theta) \right.$$

$$+ \frac{4}{\sin^2 \frac{\theta}{2} \cos^2 \frac{\theta}{2}} \left(\frac{1}{2} \cos^4 \frac{\theta}{2} (5 + 2\cos \theta + \cos^2 \theta) + \frac{1}{2} \sin^4 \frac{\theta}{2} (5 - 2\cos \theta + \cos^2 \theta) \right.$$

$$+ 4 \cos^2 \frac{\theta}{2} \sin^2 \frac{\theta}{2} \left. \right)$$

$$= \frac{e^4}{\sin^4 \frac{\theta}{2} \cos^4 \frac{\theta}{2}} \left(\frac{1}{2} \cdot \frac{1}{4} (1 + \cos \theta)^2 (5 + 2\cos \theta + \cos^2 \theta) \right.$$

$$+ \frac{1}{2} \cdot \frac{1}{4} (1 - \cos \theta)^2 (5 - 2\cos \theta + \cos^2 \theta) \left. + (1 - \cos \theta)(1 + \cos \theta) \right)$$

$$= \frac{e^4}{8 \sin^4 \frac{\theta}{2} \cos^4 \frac{\theta}{2}} [(1 + 2\cos \theta + \cos^2 \theta)(5 + 2\cos \theta + \cos^2 \theta)]$$

$$+ (1 - 2\cos^2\theta + \cos^4\theta)(5 - 2\cos^2\theta + \cos^4\theta) + 8(1 - \cos^2\theta)]$$

$$= \frac{e^4}{8 \sin^4 \frac{\theta}{2} \cos^4 \frac{\theta}{2}} \left[\begin{array}{l} 5 + 2\cancel{\cos^2\theta} + \cos^4\theta + 10\cancel{\cos^2\theta} + 4\cancel{\cos^4\theta} + 2\cancel{\cos^3\theta} + 5\cos^2\theta + 2\cancel{\cos^3\theta} + \cancel{\cos^4\theta} \\ + - \cancel{+} - \cancel{+} + - \cancel{+} + - \cancel{+} + \\ + 8 - 8\cos^2\theta \end{array} \right]$$

$$= \frac{e^4}{8 \sin^4 \frac{\theta}{2} \cos^4 \frac{\theta}{2}} (10 + \cancel{2\cos^2\theta} + \cancel{8\cos^2\theta} + \cancel{10\cos^2\theta} + 2\cos^4\theta + 8 - \cancel{8\cos^2\theta})$$

$$= \frac{e^4}{8 \sin^4 \frac{\theta}{2} \cos^4 \frac{\theta}{2}} (18 + 12\cos^2\theta + 2\cos^4\theta)$$

$$= \frac{e^4}{4 \sin^4 \frac{\theta}{2} \cos^4 \frac{\theta}{2}} (3 + \cos^2\theta)^2$$

$$= \frac{4 e^4}{\sin^4 \theta} (3 + \cos^2\theta)^2$$

$$\Rightarrow \frac{dF}{d\theta} = \frac{1}{64\pi^2 S} \overline{|M|^2}$$

$$= \frac{1}{64\pi^2 S} \cdot \frac{4e^4}{\sin^4 \theta} (3 + \cos^2\theta)^2$$

$$= \frac{\alpha^2}{S \cdot \sin^4 \theta} (3 + \cos^2\theta)^2$$

a) illa

$$\sim \int d^4x d^4y \langle k, k' | \bar{\psi}(ie\gamma^\mu) A_\mu \bar{\psi}(x) \bar{\psi}(y) (-ie\gamma^\nu) A_\nu(x) \bar{\psi}(x) | p, p' \rangle$$

\downarrow

a vertex
 k', p'

a vertex connecting
 k, p

iMb

$$\sim \int d^4x d^4y \langle k, k' | \bar{\psi}(x) (-ie\gamma^\mu) A_\mu(x) \bar{\psi}(x) \bar{\psi}(y) (-ie\gamma^\nu) A_\nu(y) \bar{\psi}(y) | p, p' \rangle$$

c) To define:

$$p = E \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad p' = E \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

$$k = E \begin{pmatrix} 1 \\ \sin\theta \cos\varphi \\ \sin\theta \sin\varphi \\ \cos\theta \end{pmatrix}, \quad k' = \begin{pmatrix} 1 \\ -\sin\theta \cos\varphi \\ -\sin\theta \sin\varphi \\ -\cos\theta \end{pmatrix}$$

$$= (E, \vec{k}) \quad = (E, -\vec{k})$$

$$q^2 = -4E^2 \sin^2 \frac{\theta}{2} \quad \tilde{q}^2 = -4E^2 \cos^2 \frac{\theta}{2}$$