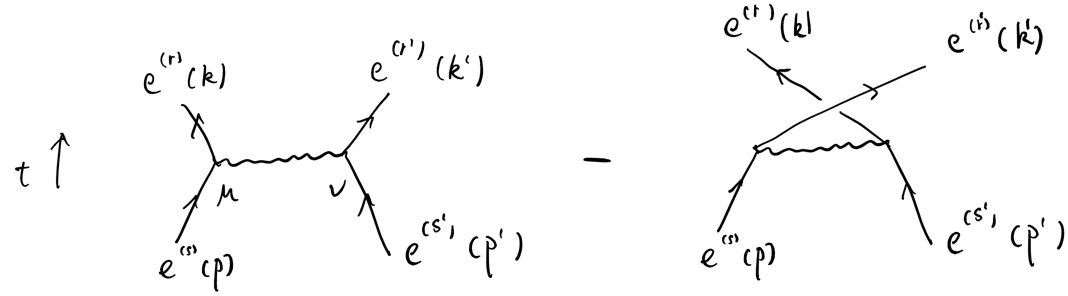
H.12

a)



$$i\mathcal{M} = (-ie)^2 \bar{u}^{(r)}(k)\, \gamma^\mu\, u^{(s)}(p)\, \underset{\uparrow t}{\frac{-ig_{\mu\nu}}{(p-k)^2+i\varepsilon}}\, \bar{u}^{(r')}(k')\, \gamma^\nu\, u^{(s')}(p')$$

$$\underset{\nearrow}{-}\, (-ie)^2 \bar{u}^{(r')}(k')\, \gamma^\mu\, u^{(s)}(p)\, \underset{\uparrow u}{\frac{-ig_{\mu\nu}}{(p-k')^2+i\varepsilon}}\, \bar{u}^{(r)}(k)\, \gamma^\nu\, u^{(s')}(p')$$

comes about because the anti-commutation relation of fermions.

More precisely: $\quad \psi_1 = \psi_{e(p)}(x) \quad , \quad \psi_2 = \psi_{e(k)}(x)$

$\psi_3 = \psi_{e(p')}(x) \quad / \quad \psi_4 = \psi_{e(k')}(x)$

$$H_{int} \propto \int d^4x\ \bar{\psi}\, \gamma_\mu A^\mu \psi$$

$\rightarrow$ zeroth order : $\quad \mathcal{M} \rightarrow \mathbb{1}$

first order : $\quad$ since A contains ladder operator (s),

in total we get $(2+3+2) = 7$ ladder operators

$\Rightarrow$ first order contribution vanishes

second order :

$$\sim \langle k\ k' | T(\bar{\psi}\, \gamma_\mu A^\mu \psi\ \bar{\psi}\, \gamma_\nu A^\nu \psi) | p\ p' \rangle$$

nontrivial part $\rightarrow$ $= \langle k\ k' | :\bar{\psi}\, \gamma_\mu A^\mu \psi::\bar{\psi}\, \gamma_\nu A^\nu \psi: | p\ p' \rangle$ $\quad$ X

$+ \langle k\ k' | :\bar{\psi}\, \gamma_\mu A^\mu \psi::\bar{\psi}\, \gamma_\nu A^\nu \psi: | p\ p' \rangle$

$$+\langle k\;k'|:\bar\psi\gamma_\mu A^\mu\psi::\bar\psi\gamma_\nu A^\nu\psi:|p\;\;p'\rangle$$

$$+\langle k\;k'|:\bar\psi\gamma_\mu A^\mu\psi::\bar\psi\gamma_\nu A^\nu\psi:|p\;\;p'\rangle$$

A must contract with A, since there is no external photon.
The k, k' can also contract with second pair of $\bar\psi, \psi$, but then it's the same expression and in the end it only contribute a factor (2) in front.

$$1.\text{ term} = \langle 0|\, a_k^{(r)}\;a_{k'}^{(r')}\;:\bar\psi\gamma_\mu A^\mu\psi::\bar\psi\gamma_\nu A^\mu\psi:\,a_p^{(s)}\;a_{p'}^{(s')}|0\rangle$$

$$=(-1)^3\langle 0|\;a_k^{(r)}\;\bar\psi\;\gamma_\mu\;a_{k'}^{(r')}\;\psi\;\;A^\mu A^\nu\;\;\bar\psi\;a_{p'}^{(s')}\;\gamma_\nu\;\;\psi\;a_p^{(s)}\;|0\rangle$$

$\to$ negative!

Analogous. 2. Term $\to$ positive

3. Term $\to$ positive

4. term $\to$ negative

$\Rightarrow$ in the end we get the minus sign for the second diagram. The factor 2 will cancel with the $\frac{1}{2}$ of expansion of e^x.

b) $$\frac{1}{4}\sum_{\text{spin}}|\mathcal{M}|^2=\frac{1}{4}\sum_{\text{spins}}\left(|\mathcal{M}_a|^2+|\mathcal{M}_b|^2+2\,\mathrm{Re}(\mathcal{M}_a\mathcal{M}_b^*)\right)$$

$$\frac{1}{4}\sum_{\text{spin}}|\mathcal{M}_a|^2$$

$$=\frac{1}{4}\sum_{\text{spin}}\left|(-ie)^2\,\bar u^{(r)}(k)\gamma^\mu u^{(s)}(p)\frac{-ig_{\mu\nu}}{t}\,\bar u^{(r')}(k')\gamma^\nu u^{(s')}(p')\right|^2$$

$$=\frac{e^4}{4}\sum_{\text{spin}}\bar u^{(r)}(k)\gamma^\mu u^{(s)}(p)\frac{1}{t}\bar u^{(r')}(k')\gamma_\mu u^{(s')}(p')\cdot\bar u^{(s')}(p')\gamma_\nu u^{(r')}(k')\frac{1}{t}\bar u^{(s)}(p)\gamma^\nu u^{(r)}(k)$$

$$= \frac{e^4}{4t^2} \sum_{r,r'} \sum_{s,s'} \bar{u}^{(r)}(k)\gamma^\mu u^{(s)}(p)\, \bar{u}^{(r')}(k')\gamma_\mu u^{(s')}(p')\bar{u}^{(s')}(p')\gamma_\nu u^{(r')}(k')\, \bar{u}^{(s)}(p)\gamma^\nu u^{(r)}(k)$$

$$= \frac{e^4}{4t^2} \sum_{r,r'} \sum_{s,s'} \sum \left(\bar{u}^{(r)}(k)\right)_\alpha (\gamma^\mu)_{\alpha\beta} \left(u^{(s)}(p)\right)_\beta \left(\bar{u}^{(r')}(k')\right)_\mu (\gamma_\mu)_{\mu\nu} \left(u^{(s')}(p')\right)_\nu$$

$$\times \left(\bar{u}^{(s')}(p')\right)_i (\gamma_\nu)_{ij} \left(u^{(r')}(k')\right)_j \left(\bar{u}^{(s)}(p)\right)_m (\gamma^\nu)_{mn} \left(u^{(r)}(k)\right)_n$$

$$= \frac{e^4}{4t^2} \sum_{r,r'} \sum_{s,s'} \sum \left(\bar{u}^{(r)}(k)\right)_\alpha \left(u^{(r)}(k)\right)_n (\gamma^\mu)_{\alpha\beta} \left(\bar{u}^{(r')}(k')\right)_\mu \left(u^{(r')}(k')\right)_j (\gamma_\mu)_{\mu\nu}$$

$$\times \left(\bar{u}^{(s')}(p')\right)_i \left(u^{(s')}(p')\right)_\nu (\gamma_\nu)_{ij} \left(\bar{u}^{(s)}(p)\right)_m \left(u^{(s)}(p)\right)_\beta (\gamma^\nu)_{mn}$$

$$= \frac{e^4}{4t^2} \sum (\not{k}+m)_{n\alpha} (\gamma^\mu)_{\alpha\beta} (\not{k}'+m)_{j\mu} (\gamma_\mu)_{\mu\nu} (\not{p}'+m)_{\nu i} (\gamma_\nu)_{ij} (\not{p}+m)_{\beta m} (\gamma^\nu)_{mn}$$

$$= \frac{e^4}{4t^2} \sum (\gamma^\mu)_{\alpha\beta} (\not{p}+m)_{\beta m} (\gamma^\nu)_{mn} (\not{k}+m)_{n\alpha}$$

$$\times (\gamma_\mu)_{\mu\nu} (\not{p}'+m)_{\nu i} (\gamma_\nu)_{ij} (\not{k}'+m)_{j\mu}$$

$$= \frac{e^4}{4t^2} \mathrm{tr}\left[\gamma^\mu (\not{p}+m)\gamma^\nu (\not{k}+m)\right] \mathrm{tr}\left[\gamma_\mu (\not{p}'+m)\gamma_\nu (\not{k}'+m)\right]$$

(Is it always necessary to calculating the components of matrices

Is there a trick to get the trace(s) directly ?

especidly in exams !

(the $\mathcal{M}^*$ calculation as well.)

$\boxed{\mathrm{Tr}\,(\text{scalar}) = \text{scalar}}$

$$q^2 = (p-k)^2 \equiv t$$

$$= \frac{e^4}{4q^4} \mathrm{tr}\left[\gamma^\mu \not{p} \gamma^\nu \not{k} + m^2 \gamma^\mu \gamma^\nu\right] \mathrm{tr}\left[\gamma_\mu \not{p}' \gamma_\nu \not{k}' + m^2 \gamma_\mu \gamma_\nu\right]$$

$$\downarrow$$

$$= 4 p_\alpha k_\beta \left(g^{\mu\alpha} g^{\nu\beta} - g^{\mu\nu} g^{\alpha\beta} + g^{\mu\beta} g^{\nu\alpha}\right)$$

$$= 4\left(p^\mu k^\nu - g^{\mu\nu}(p\cdot k) + p^\nu k^\mu\right)$$

$$= \frac{e^4}{4q^4}\left[4\left(p^\mu k^\nu - g^{\mu\nu}(p\cdot k) + p^\nu k^\mu\right) + 4m^2 g^{\mu\nu}\right]$$

$$\times \left[4\left(p'_\mu k'_\nu + p'_\nu k'_\mu - g_{\mu\nu}(p'\cdot k')\right) + 4m^2 g_{\mu\nu}\right]$$

$$= \frac{4e^4}{q^4}\Big[(p\cdot p')(k\cdot k') + (p\cdot k')(p'\cdot k) - (p\cdot k)(p'\cdot k') + m^2 (p\cdot k)$$

$$- (p\cdot k)(p'\cdot k') - (p\cdot k)(p'\cdot k') + 4(p\cdot k)(p'\cdot k') - 4m^2 (p\cdot k)$$

$$+ (p'\cdot k)(p\cdot k') + (p\cdot p')(k\cdot k') - (p\cdot k)(p'\cdot k') + m^2 (p\cdot k)$$

$$+ m^2(p'\cdot k') + m^2(p'\cdot k') - 4m^2(p'\cdot k') + 4m^4]$$

$$= \frac{4e^4}{q^4}\left[4m^4 - 2m^2\left((p\cdot k) + (p'\cdot k')\right) + 2(p\cdot k')(p'\cdot k) + 2(p\cdot p')(k\cdot k') \right]$$

$$\simeq \frac{8e^4}{q^4}\left[(p\cdot k')(p'\cdot k) + (p\cdot p')(k\cdot k') \right]$$

$$\frac{1}{4}\sum_{spin} |\mathcal{M}_b|^2$$

$$= \frac{e^4}{4}\sum_{spin}\left| \bar{u}^{(r')}(k')\gamma^\mu u^{(s)}(p)\frac{1}{\tilde{q}}\bar{u}^{(r)}(k)\gamma_\mu u^{(s')}(p')\right|^2$$

$$= \frac{e^4}{4\tilde{q}^4}\sum_{r,r'}\sum_{s,s'} \bar{u}^{(r')}(k')\gamma^\mu u^{(s)}(p)\,\bar{u}^{(r)}(k)\gamma_\mu u^{(s')}(p')\cdot \bar{u}^{(s')}(p')\gamma_\nu u^{(r)}(k)\,\bar{u}^{(s)}(p)\gamma^\nu u^{(r')}(k')$$

$$= \frac{e^4}{4\tilde{q}^4}\, tr\left[(\not{k}'+m)\gamma^\mu(\not{p}+m)\gamma^\nu \right]\cdot tr\left[(\not{k}+m)\gamma_\mu(\not{p}'+m)\gamma_\nu \right]$$

$$= \frac{4e^4}{\tilde{q}^4}\left[k'^\mu p^\nu + k'^\nu p^\mu - g^{\mu\nu}(k'\cdot p) + m^2 g^{\mu\nu}\right]\left[k_\mu p'_\nu + k_\nu p'_\mu - g_{\mu\nu}(k\cdot p') + m^2 g_{\mu\nu}\right]$$

↑

essentially the same expression as i) with $k \longleftrightarrow k'$

$$= \frac{8e^4}{\tilde{q}^4}\left[(p\cdot k)(p'\cdot k') + (p\cdot p')(k\cdot k') \right]$$

$$\frac{1}{4}\sum_{spin} 2\,Re(\mathcal{M}_a \cdot \mathcal{M}_b^*)$$

$$= \frac{1}{2}\sum_{r,r'}\sum_{s,s'} \frac{e^4}{q^2\tilde{q}^2}\, \bar{u}^{(r)}(k)\gamma_\mu u^{(s)}(p)\,\bar{u}^{(r')}(k')\gamma^\mu u^{(s')}(p')\,\bar{u}^{(s')}(p')\gamma^\nu u^{(r)}(k)\,\bar{u}^{(s)}(p)\gamma_\nu u^{(r')}(k')$$

$$= \frac{e^4}{2q^2\tilde{q}^2}\sum_{r,r'}\sum_{s,s'}\sum (\bar{u}^{(r)}(k))_\alpha(\gamma_\mu)_{\alpha\beta}(u^{(s)}(p))_\beta\,(\bar{u}^{(r')}(k'))_\mu(\gamma^\mu)_{\mu\nu}(u^{(s')}(p'))_\nu$$
$$\times (\bar{u}^{(s')}(p'))_i(\gamma^\nu)_{ij}(u^{(r)}(k))_j\,(\bar{u}^{(s)}(p))_m(\gamma_\nu)_{mn}(u^{(r')}(k'))_n$$

$$= \frac{e^4}{2q^2\tilde{q}^2}\sum (\not{k}+m)_{j\alpha}(\gamma_\mu)_{\alpha\beta}(\not{p}+m)_{\beta m}(\not{k}'+m)_{n\mu}(\gamma^\mu)_{\mu\nu}(\not{p}'+m)_{\nu i}(\gamma^\nu)_{ij}(\gamma_\nu)_{mn}$$

$$= \frac{e^4}{2q^2\tilde{q}^2}\, tr\left[(\not{k}+m)\gamma_\mu(\not{p}+m)\gamma_\nu(\not{k}'+m)\gamma^\mu(\not{p}'+m)\gamma^\nu \right]$$

$$= \frac{e^4}{2q^2\tilde{q}^2}\, tr\left[\not{k}\gamma_\mu\not{p}\gamma_\nu\not{k}'\gamma^\mu\not{p}'\gamma^\nu + (\not{k}\gamma_\mu\not{p}\gamma_\nu\gamma^\mu\gamma^\nu + \not{k}\gamma_\mu\gamma_\nu\not{k}'\gamma^\mu\gamma^\nu \right.$$
$$+ \not{k}\gamma_\mu\gamma_\nu\gamma^\mu\not{p}'\gamma^\nu + \gamma_\mu\not{p}\gamma_\nu\not{k}'\gamma^\mu\gamma^\nu + \gamma_\mu\not{p}\gamma_\nu\gamma^\mu\not{p}'\gamma^\nu$$

$$+ \gamma_\mu \gamma_\nu \not{k}' \gamma^\mu \not{p}' \gamma^\nu \Big) \cdot m^2 + m^4 \gamma_\mu \gamma_\nu \gamma^\mu \gamma^\nu \Big]$$

$$\simeq \frac{e^4}{2 q^2 \tilde{q}^2} \, tr \left[\not{k} \gamma_\mu \not{p} \gamma_\nu \not{k}' \gamma^\mu \not{p}' \gamma^\nu \right]$$

$$= tr \left[\not{k}' \gamma^\mu \not{p}' \underbrace{\gamma^\nu \not{p} \gamma_\nu}_{-2\not{p}} \not{k} \gamma_\mu \right]$$

$$= -2 \, tr \left[\not{k}' \gamma^\mu \not{p}' \not{p} \not{k} \gamma_\mu \right]$$

$$= 4 \, tr \left[\not{k}' \not{k} \not{p} \not{p}' \right]$$

$$= \frac{e^4}{2 \tilde{q}^2 q^2} \cdot 4 \cdot 4 \left[(k' \cdot k)(p \cdot p') \underbrace{- (k' \cdot p)(k \cdot p') + (k' \cdot p')(k \cdot p)}_{= 0 \text{, using } p' \cdot k = p \cdot k'} \right]$$

$$= \frac{8 e^4}{\tilde{q}^2 q^2} (k \cdot k')(p \cdot p')$$

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c) $S = (p + p')^2 \simeq 2(p \cdot p') = 2(E^2 + p^2) = 4E^2$ elastic scattering

$p = (E, 0, 0, p)$, $p' = (E, 0, 0, -p)$

$k = (E, p\sin\theta, 0, p\cos\theta)$, $k' = (E, -p\sin\theta, 0, -p\cos\theta)$ ←

$k' = (E, \vec{p})$

$$\Rightarrow \quad q = p - k = (0, -p\sin\theta, 0, p(1-\cos\theta))$$

$$q^2 = -p^2 \sin^2\theta - p^2 (1-\cos\theta)^2 = -p^2 (2 - 2\cos\theta)$$

$$= -4p^2 \sin^2 \frac{\theta}{2}$$

$$\tilde{q} = p - k' = (0, p\sin\theta, 0, p(1+\cos\theta))$$

$$\tilde{q}^2 = -p^2 \sin^2\theta - p^2 (1+\cos\theta)^2$$

$$= -p^2 (2 + 2\cos\theta) = -4 p^2 \cos^2 \frac{\theta}{2}$$

$$(k \cdot k') = E^2 + p^2 \sin^2\theta + p^2 \cos^2\theta = E^2 + p^2 = (p \cdot p') \simeq 2E^2$$

$$(p\cdot k') = (p'\cdot k) = E^2 + p^2\cos\theta \simeq E^2(1+\cos\theta)$$

$$(p\cdot k) = E^2 - p^2\cos\theta \quad , \qquad (p'\cdot k') = E^2 - p^2\cos\theta \simeq E^2(1-\cos\theta)$$

$$\Rightarrow \quad i) = \frac{8e^4}{16p^4\sin^4\frac{\theta}{2}}\left[(2E^2)^2 + E^4(1+\cos\theta)^2\right]$$

$$= \frac{e^4}{2E^4\sin^4\frac{\theta}{2}}\cdot E^4\left(4+1+2\cos\theta+\cos^2\theta\right)$$

$$= \frac{e^4}{2\sin^4\frac{\theta}{2}}\left(5+2\cos\theta+\cos^2\theta\right)$$

$$ii) = \frac{8e^4}{16p^4\cos^4\frac{\theta}{2}}\left[(2E^2)^2 + E^4(1-\cos\theta)^2\right]$$

$$= \frac{e^4}{2\cos^4\frac{\theta}{2}}\left(5-2\cos\theta+\cos^2\theta\right)$$

$$iii) = \frac{16e^4}{16p^4\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}}\cdot(2E^2)^2 = \frac{4e^4}{\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}}$$

$$\Rightarrow \quad |\overline{\mathcal{M}}|^2 = e^4\left[\frac{1}{2\sin^4\frac{\theta}{2}}(5+2\cos\theta+\cos^2\theta) + \frac{1}{2\cos^4\frac{\theta}{2}}(5-2\cos\theta+\cos^2\theta) + \frac{4}{\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}}\right]$$

$$= \frac{e^4}{\sin^4\frac{\theta}{2}\cos^4\frac{\theta}{2}}\left(\frac{1}{2}\cos^4\frac{\theta}{2}(5+2\cos\theta+\cos^2\theta) + \frac{1}{2}\sin^4\frac{\theta}{2}(5-2\cos\theta+\cos^2\theta) + 4\cos^2\frac{\theta}{2}\sin^2\frac{\theta}{2}\right)$$

$$= \frac{e^4}{\sin^4\frac{\theta}{2}\cos^4\frac{\theta}{2}}\left(\frac{1}{2}\cdot\frac{1}{4}(1+\cos\theta)^2(5+2\cos\theta+\cos^2\theta) + \frac{1}{2}\cdot\frac{1}{4}(1-\cos\theta)^2\cdot(5-2\cos\theta+\cos^2\theta) + (1-\cos\theta)(1+\cos\theta)\right)$$

$$= \frac{e^4}{8\sin^4\frac{\theta}{2}\cos^4\frac{\theta}{2}}\left[(1+2\cos\theta+\cos^2\theta)(5+2\cos\theta+\cos^2\theta)\right.$$

$$+ (1 - 2\cos\theta + \cos^2\theta)(5 - 2\cos\theta + \cos^2\theta) + 8(1 - \cos^2\theta))\Big]$$

$$= \frac{e^4}{8 \sin^4\frac{\theta}{2} \cos^4\frac{\theta}{2}} \Big[$$

$$5 + \cancel{2\cos\theta} + \cos^2\theta + \cancel{10\cos\theta} + 4\cos^2\theta + \cancel{2\cos^3\theta} + 5\cos^2\theta + \cancel{2\cos^3\theta} + \cos^4\theta$$

$$+ \quad - \quad + \quad - \quad + \quad - \quad + \quad - \quad +$$

$$+ \; 8 - 8\cos^2\theta \Big]$$

$$= \frac{e^4}{8 \sin^4\frac{\theta}{2} \cos^4\frac{\theta}{2}} \left(10 + \underline{2\cos^2\theta} + \cancel{8\cos^2\theta} + \underline{10\cos^2\theta} + 2\cos^4\theta + 8 - \cancel{8\cos^2\theta}\right)$$

$$= \frac{e^4}{8 \sin^4\frac{\theta}{2} \cos^4\frac{\theta}{2}} \left(18 + 12\cos^2\theta + 2\cos^4\theta\right)$$

$$= \frac{e^4}{4 \sin^4\frac{\theta}{2} \cos^4\frac{\theta}{2}} \left(3 + \cos^2\theta\right)^2$$

$$= \frac{4e^4}{\sin^4\theta} \left(3 + \cos^2\theta\right)^2$$

$$\Rightarrow \quad \frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s} \overline{|M|^2}$$

$$= \frac{1}{64\pi^2 s} \frac{4e^4}{\sin^4\theta} \left(3 + \cos^2\theta\right)^2$$

$$= \frac{\alpha^2}{s \cdot \sin^4\theta} \left(3 + \cos^2\theta\right)^2$$

a)

$$iM_a \sim \int d^4x\, d^4y \,\langle k, k' |\, \bar{\psi}\, (ie\gamma^\mu) A_\mu\, \psi(x)\, \bar{\psi}(y)\, (-ie\gamma^\nu) A_\nu(x)\, \psi(x)\, | p, p' \rangle$$

a vertex k', p'

a vertex connecting k, p

$$iM_b \sim \int d^4x\, d^4y \,\langle k, k' |\, \bar{\psi}(x)\, (-ie\gamma^\mu) A_\mu(x)\, \psi(x)\, \bar{\psi}(y)\, (-ie\gamma^\nu) A_\nu(x)\, \psi(y)\, | p, p' \rangle$$

c) To define:

$$p = E\begin{pmatrix}1\\0\\0\\1\end{pmatrix}, \qquad p' = E\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}$$

$$k = E\begin{pmatrix}1\\ \sin\theta\cos\varphi\\ \sin\theta\sin\varphi\\ \cos\theta\end{pmatrix} = (E, \vec{k}), \qquad k' = \begin{pmatrix}1\\ -\sin\theta\cos\varphi\\ -\sin\theta\sin\varphi\\ -\cos\theta\end{pmatrix} = (E, -\vec{k})$$

$$q^2 = -4E^2 \sin^2\frac{\theta}{2} \qquad \tilde{q}^2 = -4E^2 \cos^2\frac{\theta}{2}$$