H.13

a) in d-dimension: $\quad \mathrm{tr}[\mathbb{1}_d] = d$

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}_d$$

$$\gamma^\mu\gamma_\mu = \gamma^\mu\gamma^\nu g_{\nu\mu} = \tfrac{1}{2} g_{\nu\mu}(\gamma^\mu\gamma^\nu + \underbrace{\gamma^\mu\gamma^\nu}_{\mu\leftrightarrow\nu})$$

$$= \tfrac{1}{2} g_{\nu\mu}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu)$$

$$= \tfrac{1}{2} g_{\nu\mu}\cdot 2g^{\mu\nu}\mathbb{1}_d$$

$$= d\,\mathbb{1}_d$$

$$\gamma^\mu\gamma^\nu\gamma_\mu = -\gamma^\mu\gamma_\mu\gamma^\nu + 2\gamma^\mu g^\nu{}_\mu \mathbb{1}_d$$

$$= -d\,\mathbb{1}_d\gamma^\nu + 2\gamma^\nu\mathbb{1}_d$$

$$= \gamma^\nu(-d+2)\mathbb{1}_d \quad = -2\gamma^\nu + (4-d)\gamma^\nu$$

$$\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu = -\gamma^\mu\gamma^\nu\gamma_\mu\gamma^\rho + 2\gamma^\mu\gamma^\nu g^\rho{}_\mu\mathbb{1}_d$$

$$= -\gamma^\nu(2-d)\mathbb{1}_d\gamma^\rho + 2\gamma^\rho\gamma^\nu\mathbb{1}_d$$

$$= (d-2)\gamma^\nu\gamma^\rho - 2\gamma^\nu\gamma^\rho + 4g^{\rho\nu}\mathbb{1}_d$$

$$= 4g^{\rho\nu}\mathbb{1}_d + (d-4)\gamma^\nu\gamma^\rho$$

$$\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\omega\gamma_\mu = -\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu\gamma^\omega + 2\gamma^\mu\gamma^\nu\gamma^\rho g^\omega{}_\mu$$

$$= -(4g^{\rho\nu}\mathbb{1}_d + (d-4)\gamma^\nu\gamma^\rho)\gamma^\omega + 2\gamma^\omega\gamma^\nu\gamma^\rho$$

$$= (4-d)\gamma^\nu\gamma^\rho\gamma^\omega - \cancel{4g^{\rho\nu}\gamma^\omega} - 2\gamma^\omega\gamma^\rho\gamma^\nu + \cancel{4\gamma^\omega g^{\nu\rho}}$$

$$= -2\gamma^\omega\gamma^\rho\gamma^\nu + (4-d)\gamma^\nu\gamma^\rho\gamma^\omega$$

b)
$$I_1 = \gamma^\nu(\not{p}'(1-y) - \not{p}x - \not{k} + m)\gamma^\mu(\not{p}(1-x) - \not{p}'y - \not{k} + m)\gamma_\nu$$

$$= \gamma^\nu(\not{p}'(1-y) - \not{p}x - \not{k})\gamma^\mu(\not{p}(1-x) - \not{p}'y - \not{k})\gamma_\nu$$

$$+ m\gamma^\nu\gamma^\mu(\not{p}(1-x) - \not{p}'y - \not{k} + m)\gamma_\nu$$

$$+ m\gamma^\nu(\not{p}'(1-y) - \not{p}x - \not{k} + m)\gamma^\mu\gamma_\nu$$

$$+ m^2\gamma^\nu\gamma^\mu\gamma_\nu$$

$$= \gamma^\nu (p'(1-y) - px - \tilde{k})_\alpha \gamma^\alpha \gamma^\mu (p(1-x) - p'y - \tilde{k})_\beta \gamma^\beta \gamma_\nu$$
$$+ m \gamma^\nu \gamma^\mu (p(1-x) - p'y - \tilde{k} + m)_\alpha \gamma^\alpha \gamma_\nu$$
$$+ m \gamma^\nu (p'(1-y) - px - \tilde{k} + m)_\alpha \gamma^\alpha \gamma^\mu \gamma_\nu$$
$$+ m^2 \gamma^\nu \gamma^\mu \gamma_\nu$$
$$= (p'(1-y) - px - \tilde{k})_\alpha (p(1-x) - p'y - \tilde{k})_\beta \, (-2\gamma^\beta\gamma^\mu\gamma^\alpha + (4-d)\gamma^\alpha\gamma^\mu\gamma^\beta)$$
$$+ m (p(1-x) - p'y - \tilde{k} + m)_\alpha \, (4 g^{\mu\alpha} - (4-d)\gamma^\mu\gamma^\alpha)$$
$$+ m (p'(1-y) - px - \tilde{k} + m)_\alpha \, (4 g^{\alpha\mu} - (4-d)\gamma^\alpha\gamma^\mu)$$
$$+ m^2 \, (-2\gamma^\mu + (4-d)\gamma^\mu)$$

Contributions with lineer $\tilde{k}$ vanish because of symmetry consideration

$$= \{ [p'(1-y) - px]_\alpha [p(1-x) - p'y]_\beta + \tilde{k}_\alpha \tilde{k}_\beta \} (-2\gamma^\beta\gamma^\mu\gamma^\alpha + (4-d)\gamma^\alpha\gamma^\mu\gamma^\beta)$$
$$+ m (p(1-x) - p'y + m)_\alpha \, (4 g^{\mu\alpha} - (4-d)\gamma^\mu\gamma^\alpha)$$
$$+ m (p'(1-y) - px + m)_\alpha \, (4 g^{\alpha\mu} - (4-d)\gamma^\alpha\gamma^\mu)$$
$$+ m^2 (-2\gamma^\mu + (4-d)\gamma^\mu)$$
$$= \{ [p'(1-y) - px]_\alpha [p(1-x) - p'y]_\beta + \tilde{k}_\alpha \tilde{k}_\beta \} (-2\gamma^\beta\gamma^\mu\gamma^\alpha + (4-d)\gamma^\alpha\gamma^\mu\gamma^\beta)$$
$$+ 4m (p - 2px + p' - 2p'y + 2m)^\mu - m(4-d)(p(1-x) - p'y + m)_\alpha \gamma^\mu\gamma^\alpha$$
$$- m(4-d)(p'(1-y) - px + m)_\alpha \gamma^\alpha\gamma^\mu + m^2(-2\gamma^\mu + (4-d)\gamma^\mu)$$

b) define $\quad \not{w} = \not{p}'(1-y) - \not{p}x - \not{\tilde{k}} = w_\alpha \gamma^\alpha$

$$\not{w}' = \not{p}(1-x) - \not{p}'y - \not{\tilde{k}} = w'_\alpha \gamma^\alpha$$

$$\Rightarrow I_1 = \gamma^\nu (\not{w} + m) \gamma^\mu (\not{w}' + m) \gamma_\nu$$
$$= w_\alpha w'_\beta \, \gamma^\nu \gamma^\alpha \gamma^\mu \gamma^\beta \gamma_\nu + m w_\alpha \gamma^\nu \gamma^\alpha \gamma^\mu \gamma_\nu + m w'_\alpha \gamma^\nu \gamma^\mu \gamma^\alpha \gamma_\nu + m^2 \gamma^\nu \gamma^\mu \gamma_\nu$$
$$\overset{(a)}{=} w_\alpha w'_\beta \, [-2\gamma^\beta\gamma^\mu\gamma^\alpha + (4-d)\gamma^\alpha\gamma^\mu\gamma^\beta]$$
$$+ m w_\alpha [4 g^{\alpha\mu} - (4-d)\gamma^\alpha\gamma^\mu] + m w'_\alpha [4 g^{\mu\alpha} - (4-d)\gamma^\mu\gamma^\alpha]$$
$$+ m^2 [-2\gamma^\mu + (4-d)\gamma^\mu]$$
$$= \underline{-2 \not{w}' \gamma^\mu \not{w}} + (4-d) \not{w} \gamma^\mu \not{w}' + \underline{4 m w^\mu} - (4-d) m \not{w} \gamma^\mu$$

$$+\underline{4mw'^{\mu}} - (4-d)m\gamma^{\mu}\not{p}' - \underline{2m^2\gamma^{\mu}} + (4-d)m^2\gamma^{\mu}$$

$$= -2\not{p}'\gamma^{\mu}\not{p} + 4m(w^{\mu}+w'^{\mu}) - 2m^2\gamma^{\mu} \longrightarrow I_1^{(0)}$$

$$+(4-d)(\not{p}\gamma^{\mu}\not{p}' - m\not{p}\gamma^{\mu} - m\gamma^{\mu}\not{p}' + m^2\gamma^{\mu}) \longrightarrow I_2^{(0)}$$

$$I_1^{(0)} = -2[\not{p}(1-x)-\not{p}'y-\not{\tilde{k}}]\gamma^{\mu}[\not{p}'(1-y)-\not{p}x-\not{\tilde{k}}]$$

$$+4m[p'(1-y)-px-\tilde{k}+p(1-x)-p'y-\tilde{k}]^{\mu}$$

$$-2m^2\gamma^{\mu}\}$$

$I_2^{(0)}$ enclosed by $\bar{u}^{(s')}(p')$ and $u^{(s)}(p)$, to use Dirac equation

$$\left[\begin{array}{l}(\not{p}-m)u=0\\ \bar{u}(\not{p}+m)=0\end{array}\right]$$

$$\vdots$$

$$= -2\not{\tilde{k}}\gamma^{\mu}\not{\tilde{k}} - \gamma^{\mu}[2m^2(2x+2y-x^2-y^2)-4pp'(1-x-y+xy)]$$

$$+p^{\mu}[y-xy-x^2]4m + p'^{\mu}[x-xy-y^2]4m$$

$$I_1^{(1)} = (4-d)\{\gamma^{\mu}[m^2(1-x)(1-y)+m^2y(1-y)+m^2x(1-x)-2xypp'-xym^2-m^2]$$

$$+p'^{\mu}(-2m(1-y)y+2mxy+2my)$$

$$+p^{\mu}(-2m(1-x)x+2mxy+2xm) + \not{\tilde{k}}\gamma^{\mu}\not{\tilde{k}}\}$$

c) power counting: $\not{\tilde{k}}\gamma^{\mu}\not{\tilde{k}} \sim \tilde{k}^2$

$$I_2^{3} \propto \tilde{k}^6$$

$$d^d\tilde{k} \sim \tilde{k}^d$$

for $d \to 4$: $\sim \dfrac{\not{\tilde{k}}\gamma^{\mu}\not{\tilde{k}}}{(I_2)^3}d^d\tilde{k}$, convergent terms are irrelevant

$$\Gamma^{\mu}_{div} = -2i\frac{\tilde{e}^2\mu^{4-d}}{(2\pi)^d}\int_0^1 dx\int_0^{1-x}dy\underbrace{\int d^d\tilde{k}\frac{\tilde{k}_\alpha\tilde{k}_\beta\gamma^\alpha\gamma^\mu\gamma^\beta}{(\tilde{k}^2-\Delta+i\varepsilon)^3}}(-2+4-d)$$

$$= \frac{(-1)^2 i}{(4\pi)^{d/2}}\frac{g_{\alpha\beta}}{2}\frac{\Gamma(3-\frac{d}{2}-1)}{\Gamma(3)}\Delta^{1+\frac{d}{2}-3}$$

$$= \cancel{2}\, \tilde{e}^2 \mu^{4-d} (2-d) \frac{1}{(4\pi)^{d/2}} \frac{g_{\alpha\beta}}{\cancel{2}} \frac{\Gamma(2-\frac{d}{2})}{\Gamma(3)} \gamma^\alpha \gamma^\mu \gamma^\beta \int_0^1 dx \int_0^{1-x} dy\, \Delta^{1+\frac{d}{2}-3}$$

$$\left[g_{\alpha\beta} \gamma^\alpha \gamma^\mu \gamma^\beta = \gamma^\alpha \gamma^\mu \gamma_\alpha = -2\gamma^\mu + (4-d)\gamma^\mu \right]$$

$$= \frac{\tilde{e}^2 \mu^{4-d}}{(4\pi)^{d/2}} \frac{\Gamma(2-d/2)}{\underbrace{\Gamma(3)}_{=2}} (2-d)^2 \gamma^\mu \int_0^1 dx \int_0^{1-x} dy\, \Delta^{d/2-2}$$

$d = 4 - \varepsilon$

$$= \frac{\tilde{e}^2 \overbrace{\mu^\varepsilon}^{\to 1}}{\underbrace{(4\pi)^{2-\varepsilon/2}}_{\to (4\pi)^2}} \frac{\Gamma(\varepsilon/2)}{2} \underbrace{(\varepsilon-2)^2}_{\to 4} \gamma^\mu \underbrace{\int_0^1 dx \int_0^{1-x} dy\, \underbrace{\Delta^{-\varepsilon/2}}_{\to 1}}_{\to \int_0^1 dx\,(1-x) = x - \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}}$$

$\varepsilon \to 0$

$$= \frac{\tilde{e}^2}{(4\pi)^2} \cancel{2} \frac{2}{\varepsilon} \gamma^\mu \cdot \frac{1}{\cancel{2}}$$

$$= \frac{\tilde{e}^2}{8\pi^2} \frac{1}{\varepsilon} \gamma^\mu$$

d) $\Gamma^\mu(p', p) = \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}(p'-p)_\nu}{2m} F_2(q^2)$

$$\Gamma^\mu_{F_2} \underset{d=4}{=} -2i \frac{\tilde{e}^2}{(2\pi)^4} \int_0^1 dx \int_0^{1-x} dy \int d^4\tilde{k}\, \frac{4mp^\mu(y-xy-x^2) + 4mp'^\mu(x-xy-y^2)}{(\tilde{k}^2 - \Delta + i\varepsilon)^3}$$

so no contributions from $I_1^{(1)}$

momentum conservation : $0 = q^2 = (p-p')^2 = 2m^2 - 2pp'$

$\Rightarrow pp' = m^2$

$$\Rightarrow \Delta = m^2(x+y) - m^2 x(1-x) - m^2 y(1-y) + 2m^2 xy$$

$$= m^2(\cancel{x+y} - \cancel{x} + x^2 - \cancel{y} + y^2 + 2xy)$$

$$\text{The integral} = \int \frac{d^4\tilde{\ell}}{(2\pi)^4} \frac{1}{(\tilde{\ell}^2 - \Delta + i\varepsilon)^3} = \frac{(-1)^3 i \; \Gamma(1)}{(4\pi)^2 \; \Gamma(3)} \frac{1}{\underset{= m^2(x+y)^2}{\Delta}}$$

$$= -i \frac{\Delta}{2(4\pi)^2}$$

$$\Gamma^{\mu}_{F_2}(0) = -\cancel{i} \frac{\tilde{e}^2}{\cancel{2}\,\cancel{i}(4\pi)^2} \int_0^1 dx \int_0^{1-x} dy \; \frac{4mp^{\mu}(y - xy - x^2) + 4mp'^{\mu}(x - xy - y^2)}{m^2(x+y)^2}$$

$$= -\frac{\tilde{e}^2}{(4\pi)^2} \frac{4}{m} \int_0^1 dx \int_0^{1-x} dy \left[\frac{p^{\mu}(y - xy - x^2)}{(x+y)^2} + \frac{p'^{\mu}(x - xy - y^2)}{(x+y)^2} \right]$$

$$= -\frac{e^2}{16\pi^2} \left[\frac{p^{\mu}}{m} + \frac{p'^{\mu}}{m} \right]$$

$$= -\frac{e^2}{16\pi^2} \left[2\gamma^{\mu} - i\sigma^{\mu\nu} \frac{q_\nu}{m} \right]$$

$$F_2(0) = 2 \frac{e^2}{16\pi^2} = \frac{\alpha}{2\pi}$$