Superstring theory
Homework 10

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1 Free fermion

The action is

$$S = \frac{1}{2} g \int \mathrm{d}^2 x \, \Psi^\dagger \gamma^0 \gamma^\mu \partial_\mu \Psi \tag{1.1}$$

(a) In terms of two-component spinor $\Psi = (\psi, \bar{\psi})^{T}$[1], the action becomes

$$\begin{aligned} S &= \frac{g}{2} \int \mathrm{d}^2 x \, \Psi^\dagger \begin{pmatrix} \partial_0 + i\partial_1 & 0 \\ 0 & \partial_0 - i\partial_1 \end{pmatrix} \Psi \\ &= g \int \mathrm{d}^2 x \begin{pmatrix} \bar{\psi} & \psi \end{pmatrix} \begin{pmatrix} \partial_{\bar{z}} & 0 \\ 0 & \partial_z \end{pmatrix} \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix} \\ &= g \int \mathrm{d}^2 x \left(\psi \partial_{\bar{z}} \psi + \bar{\psi} \partial_z \bar{\psi} \right) \end{aligned}$$

where we used the fact that Ψ describes Majorana fermions, thus its components must be real and

$$\begin{aligned} z &= x^0 + ix^1 \\ \bar{z} &= x^0 - ix^1 \\ \partial_z &= \frac{1}{2} (\partial_0 - \partial_1) \\ \partial_{\bar{z}} &= \frac{1}{2} (\partial_0 + \partial_1) \end{aligned}$$

The equations of motions via Euler-Lagrange-Equations are

$$\begin{aligned} 2\partial_{\bar{z}} \psi &= \partial_0 \psi + i\partial_1 \psi = 0 \\ 2\partial_z \bar{\psi} &= \partial_0 \bar{\psi} - i\partial_1 \bar{\psi} = 0 \end{aligned}$$

[1] $\bar{\psi}$ is to be understood as complex conjugate of ψ instead of conventional notation

Thus they must be (anti-)holomorphic.

(b) The Lagrangian density can also be written as

$$\mathcal{L} = \int g\Psi \begin{pmatrix} \partial_{\bar{z}} & 0 \\ 0 & \partial_z \end{pmatrix} \Psi$$

The operator is

$$A_{ij} = 2g \begin{pmatrix} \partial_{\bar{z}} & 0 \\ 0 & \partial_z \end{pmatrix} \tag{1.2}$$

In analogy with bosonic case, we have the differential equation[2]

$$A^{ij} G_{jk} = \delta^i_k \delta(x-y) \tag{1.3}$$

Use a representation of delta function

$$\delta(x-y) = \frac{1}{\pi}\partial_{\bar{z}}\frac{1}{z-w} = \frac{1}{\pi}\partial_z\frac{1}{\bar{z}-\bar{w}}$$

We identify the Green's function as

$$G_{ij}(z,\bar{z},w,\bar{w}) = \frac{1}{2\pi g}\begin{pmatrix} \frac{1}{z-w} & 0 \\ 0 & \frac{1}{\bar{z}-\bar{w}} \end{pmatrix} \tag{1.4}$$

2 The Schwarzian Derivative

(a) The infinitesimal change of energy-momentum tensor is

$$\delta_\epsilon T(z) = -\epsilon(z)\partial_z T(z) - 2\partial_z\epsilon(z)T(z) - \frac{c}{12}\partial_z^3\epsilon(z) \tag{2.1}$$

We want to show it is equivalent to its OPE

$$\begin{aligned}
\delta_\epsilon T(z) &= -[Q_\epsilon, T(w)] \\
&= \oint_{C_0} \frac{\mathrm{d}z}{2\pi i}\epsilon(z)[T(z), T(w)] \\
&= -\oint_{C_w} \frac{\mathrm{d}z}{2\pi i}\epsilon(z)T(z)T(w)
\end{aligned}$$

Plug in the OPE

$$\begin{aligned}
&= -\oint_{C_w} \frac{\mathrm{d}z}{2\pi i}\epsilon(z)\left[\frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)} + \text{reg.}\right] \\
&= -\frac{c}{2\cdot 3!}\partial_z^3\epsilon(z) - 2\partial_z\epsilon(z)T(w) - \epsilon(z)\partial_w T(w) \\
&= -\epsilon(z)\partial_z T(z) - 2\partial_z\epsilon(z)T(z) - \frac{c}{12}\partial_z^3\epsilon(z)
\end{aligned}$$

[2] Note that $z = x_0 + ix_1$ and so on.

(b) The Schwarzian is defined as

$$\{w, z\} = \frac{w'''}{w'} - \frac{3}{2}\left(\frac{w''}{w'}\right)^2 \tag{2.2}$$

Then for $w = \frac{az+b}{cz+d}$ with $ad - bc \neq 0$

$$\begin{aligned}
w' &= \frac{a(cz+d) - c(az+d)}{(cz+d)^2} = \frac{ad - cd}{(cz+d)^2} \\
w'' &= -2c\frac{ad - cd}{(cz+d)^3} \\
w''' &= 6c^2\frac{ad - cd}{(cz+d)^4}
\end{aligned}$$

The Schwarzian is

$$\{w, z\} = \frac{6c^2}{(cz+d)^2} - \frac{3}{2}\left(\frac{-2c}{cz+d}\right)^2 = 0$$

The energy-momentum tensor is quasi-primary field or secondary field. Its conformal dimension (weight?) is $h = 2$.

(c) The map from cylinder to complex plane is

$$z = e^{2\pi w/l}, \bar{z} = e^{2\pi \bar{w}/l}$$

Then the Schwarzian is

$$\{w, z\} = \frac{2}{z^2} - \frac{3}{2}\left(-\frac{1}{z}\right)^2 = \frac{1}{2}\frac{1}{z^2}$$

Put together

$$\begin{aligned}
T_{\text{cycl.}}(w) &= \left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^{-2}\left[T(z) - \frac{c}{12}\{w, z\}\right] \\
&= \left(\frac{2\pi}{l}\right)^2 z^2\left[T_{\text{plane}}(z) - \frac{c}{12}\frac{1}{2}\frac{1}{z^2}\right] \\
&= \left(\frac{2\pi}{l}\right)^2\left[z^2 T_{\text{plane}}(z) - \frac{c}{24}\right]
\end{aligned}$$