Superstring theory
Homework 5

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1 A first look at the canonical quantization of the bosonic string

(a) From previous sheet(s),

$$X_L^\mu(\sigma^+) = \frac{1}{2}(x^\mu - c^\mu) + \frac{\pi\alpha'}{l}p^\mu\sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{1}{n}\tilde{\alpha}_n^\mu e^{-\frac{2\pi}{l}in\sigma^+} \tag{1.1}$$

$$X_R^\mu(\sigma^-) = \frac{1}{2}(x^\mu - c^\mu) + \frac{\pi\alpha'}{l}p^\mu\sigma^- + i\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{1}{n}\alpha_n^\mu e^{-\frac{2\pi}{l}in\sigma^-} \tag{1.2}$$

Now we want to express $X^\mu_{L,R}$ in terms of $(z,\bar{z}) = (e^{2\pi i\sigma_-/l}, e^{2\pi i\sigma_+/l})$. It should be stress that $\bar{z}$ is *not* the complex conjugation of z. Thus

$$X_L^\mu(\bar{z}) = \frac{1}{2}(x^\mu - c^\mu) + \frac{\alpha'}{2i}p^\mu\ln\bar{z} + i\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{1}{n}\tilde{\alpha}_n^\mu\bar{z}^{-n} \tag{1.3}$$

$$X_R^\mu(z) = \frac{1}{2}(x^\mu - c^\mu) + \frac{\alpha'}{2i}p^\mu\ln z + i\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{1}{n}\alpha_n^\mu z^{-n} \tag{1.4}$$

(b) Propagator is defined as

$$\langle X^\mu(\sigma,\tau)X^\nu(\sigma',\tau)\rangle = T[X^\mu(\sigma,\tau)X^\nu(\sigma',\tau')] - :X^\mu(\sigma,\tau)X^\nu(\sigma',\tau'): \tag{1.5}$$

T denotes time ordering and $::$ normal ordering

$$:\alpha_{-m}\alpha_m: = \begin{cases}\alpha_{-m}\alpha_m, & m>0\\ \alpha_{-m}\alpha_m - [\alpha_{-m},\alpha_m], & m<0\end{cases}$$

$$:p^\nu x^\mu: = x^\mu p^\nu = p^\nu x^\mu - [p^\nu, x^\mu]$$

We can write out the first term

$$T[X_L^\mu(\bar{z})X_L^\nu(\bar{w})] = \begin{cases} X_L^\mu(\bar{z})X_L^\nu(\bar{w}), & \tau > \tau' \\ X_L^\nu(\bar{w})X_L^\mu(\bar{z}), & \tau < \tau' \end{cases}$$

The normal orderer product is just the "original" product along with some appropriate commutators.

$$\begin{aligned} :X_L^\mu(\bar{z})X_L^\nu(\bar{w}): &= X_L^\mu(\bar{z})X_L^\nu(\bar{w}) - \frac{\alpha'}{4i}[p^\mu, x^\nu]\ln\bar{z} + \frac{\alpha'}{2}\sum_{n>0,m<0}\frac{1}{nm}[\tilde{\alpha}_n^\mu, \tilde{\alpha}_m^\nu]\bar{z}^{-n}\bar{w}^{-m} \\ &= X_L^\mu(\bar{z})X_L^\nu(\bar{w}) + \frac{\alpha'}{4i}i\eta^{\nu\mu}\ln\bar{z} + \frac{\alpha'}{2}\sum_{n>0,m<0}\frac{1}{nm}n\eta^{\mu\nu}\delta_{m+n}\bar{z}^{-n}\bar{w}^{-m} \\ &= X_L^\mu(\bar{z})X_L^\nu(\bar{w}) + \frac{\alpha'}{4}\eta^{\mu\nu}\ln\bar{z} - \frac{\alpha'}{2}\sum_{n>0}\frac{1}{n}\eta^{\mu\nu}\bar{z}^{-n}\bar{w}^{n} \\ &= X_L^\mu(\bar{z})X_L^\nu(\bar{w}) + \frac{\alpha'}{4}\eta^{\mu\nu}\ln\bar{z} + \frac{\alpha'}{2}\eta^{\mu\nu}\ln\Big(1-\frac{\bar{w}}{\bar{z}}\Big) \end{aligned}$$

Thus

$$\begin{aligned} \langle X_L^\mu(\bar{z})X_L^\nu(\bar{w})\rangle &= -\frac{\alpha'}{4}\eta^{\mu\nu}\ln\bar{z} - \frac{\alpha'}{2}\eta^{\mu\nu}\ln\Big(1-\frac{\bar{w}}{\bar{z}}\Big) \\ &= \eta^{\mu\nu}\left[\frac{\alpha'}{4}\ln(\bar{z}) - \frac{\alpha'}{2}\ln(\bar{z}-\bar{w})\right] \end{aligned} \tag{1.6}$$

With "RR" combination, it is the same with $\bar{z}\to z$ and $\bar{w}\to w$

$$\langle X_R^\mu(z)X_R^\nu(w)\rangle = \eta^{\mu\nu}\left[\frac{\alpha'}{4}\ln(z) - \frac{\alpha'}{2}\ln(z-w)\right] \tag{1.7}$$

The "mixed" contributions are simpler since commutators involving ladder operators vanish

$$\langle X_L^\mu(\bar{z})X_R^\nu(w)\rangle = -\frac{\alpha'}{4}\eta^{\mu\nu}\ln\bar{z} \tag{1.8}$$

$$\langle X_R^\mu(z)X_L^\nu(\bar{w})\rangle = -\frac{\alpha'}{4}\eta^{\mu\nu}\ln z \tag{1.9}$$

All together we have

$$\langle X^\mu(z,\bar{z})X^\nu(w,\bar{w})\rangle = -\eta^{\mu\nu}\frac{\alpha'}{2}\ln(z-w)(\bar{z}-\bar{w}) \tag{1.10}$$

2 The quantum Virasoro algebra

Virasoro algebra is in quantum theory defined as

$$L_m = \frac{1}{2}\sum_{n\in\mathbb{Z}} :\alpha_{m-n}\alpha_n:, \quad \bar{L}_m = \frac{1}{2}\sum_{n\in\mathbb{Z}} :\bar{\alpha}_{n-m}\bar{\alpha}_n: \tag{2.1}$$

(a) In L_m there are only two terms of interest, since all other vanish due to commutation relation

$$\begin{aligned}
[L_m, \alpha_n^\mu] &= \frac{1}{2}\alpha_{m-(-n),\nu}\left[\alpha_{-n}^\nu, \alpha_n^\mu\right] + \frac{1}{2}\alpha_{m+n,\nu}\left[\alpha_{m-(m+n)}^\nu, \alpha_n^\mu\right] \\
&= \frac{1}{2}\left(\alpha_{m-(-n),\nu}(-n)\eta^{\nu\mu} + \alpha_{m+n,\nu}(-n)\eta^{\nu\mu}\right) \\
&= -n\alpha_{m+n}^\mu
\end{aligned}$$

One should note that the equal sign on the first row is not always valid. Here we have the commutation simply as a number instead of operator, so we can place the commutator where ever we want. In principle, the ordering should depend on the values of m and n.

(b) Use the definition of normal ordering, we see

$$L_m = \frac{1}{2}\sum_{n\geq m/2}\alpha_{m-n}\alpha_n + \frac{1}{2}\sum_{n<m/2}\alpha_n\alpha_{m-n}$$

As a matter of fact, the "threshold" to separate these two summations are arbitrary, since the mode indices are not opposite if $m \neq 0$

$$= \frac{1}{2}\sum_{n\geq 0}\alpha_{m-n}\alpha_n + \frac{1}{2}\sum_{n<0}\alpha_n\alpha_{m-n} \tag{2.2}$$

(c) From previous results, we can write the commutator of Virasoro generators as

$$
\begin{aligned}
[L_m, L_n] &= \frac{1}{2}\left(\sum_{p<0}[L_m, \alpha_p \cdot \alpha_{n-p}] + \sum_{p\geq 0}[L_m, \alpha_{n-p}\cdot\alpha_p]\right)\\
2[L_m, L_n] &= \sum_{p<0}\left[L_m, \alpha_p^\mu\right]\alpha_{n-p,\mu} + \sum_{p<0}\alpha_{p,\mu}\left[L_m, \alpha_{n-p}^\mu\right]\\
&+ \sum_{p\geq 0}\left[L_m, \alpha_{n-p}^\mu\right]\alpha_{p,\mu} + \sum_{p\geq 0}\alpha_{n-p,\mu}\left[L_m, \alpha_p^\mu\right]\\
2[L_m, L_n] &= \sum_{p<0} -p\alpha_{m+p}^\mu \alpha_{n-p,\mu} + \sum_{p<0}\alpha_{p,\mu}(-n+p)\alpha_{m+n-p}^\mu\\
&+ \sum_{p\geq 0}(-n+p)\alpha_{m+n-p}^\mu\alpha_{p,\mu} + \sum_{p\geq 0}\alpha_{n-p,\mu}(-p)\alpha_{m+p}^\mu
\end{aligned}
$$

Now we realized that I should have expanded L_m. Instead, rename indices $n \leftrightarrow m$

$$
\begin{aligned}
2[L_n, L_m] &= \sum_{p<0} -p\alpha_{n+p}^\mu \alpha_{m-p,\mu} + \sum_{p<0}\alpha_{p,\mu}(-m+p)\alpha_{m+n-p}^\mu\\
&+ \sum_{p\geq 0}(-m+p)\alpha_{m+n-p}^\mu\alpha_{p,\mu} + \sum_{p\geq 0}\alpha_{m-p,\mu}(-p)\alpha_{n+p}^\mu\\
[L_m, L_n] &= \frac{1}{2}\sum_{p<0}[(m-p)\alpha_p\cdot\alpha_{m+n-p} + p\alpha_{n+p}\cdot\alpha_{m-p}]\\
&+ \frac{1}{2}\sum_{p\geq 0}[(m-p)\alpha_{m+n-p}\cdot\alpha_p + p\alpha_{m-p}\cdot\alpha_{n+p}]
\end{aligned}
$$

This is the same as the expression given on the sheet, up to "position" of $p = 0$. It does not matter, since first sum is essentially the same as the second sum.

(d) Change the summation variable as follows: $p \to q$ in the first term, $p \to q - n$ in the second term, $p \to q$ in the third term and $p \to q - n$

in the last term.

$$[L_m, L_n] = \frac{1}{2}\left[\sum_{q\le 0}(m-q)\alpha_q \cdot \alpha_{m+n-q} + \sum_{q\le n}(q-n)\alpha_q \cdot \alpha_{m+n-q}\right]$$
$$+\frac{1}{2}\left[\sum_{q>0}(m-q)\alpha_{m+n-q} \cdot \alpha_q + \sum_{q>n}(q-n)\alpha_{m+n-q} \cdot \alpha_q\right]$$

First two terms are very similar, thus we "split" second sum and combine one part into the first sum. The same operation is done to the second two terms.

$$[L_m, L_n] = \frac{1}{2}\left[\sum_{q\le 0}(m-n)\alpha_q \cdot \alpha_{m+n-q} + \sum_{0<q\le n}(q-n)\alpha_q \cdot \alpha_{m+n-q}\right]$$
$$+\frac{1}{2}\left[\sum_{q>n}(m-n)\alpha_{m+n-q} \cdot \alpha_q + \sum_{0<q\le n}(m-q)\alpha_{m+n-q} \cdot \alpha_q\right] \tag{2.3}$$

Is it normal ordered with $n > 0$? No, for instance if $m \to -\infty$, then the first term with $q = 0$ is not normal ordered.

(e) If $m + n \neq 0$ and $n > 0$, with equation (2.2)

$$L_{m+n} = \frac{1}{2}\sum_{q<0}\alpha_q \cdot \alpha_{m+n-q} + \frac{1}{2}\sum_{q\ge 0}\alpha_{m+n-q} \cdot \alpha_q$$

This is the same as $[L_m, L_n]$ up to a numerical factor. One can see it from combining second and last term of equation (2.3) together (with $m+n \neq 0$ we are allowed to exchange α's in the second term) and then put it into the third sum. Thus

$$[L_m, L_n] = (m-n)L_{m+n}, \quad m+n \neq 0$$

(f) If $m + n = 0$ and $n > 0$,

$$[L_m, L_n] = \frac{1}{2}\left[\sum_{q\le 0}(m-n)\alpha_q \cdot \alpha_{-q} + \sum_{0<q\le n}(q-n)\alpha_q \cdot \alpha_{-q}\right]$$
$$+\frac{1}{2}\left[\sum_{q>n}(m-n)\alpha_{-q} \cdot \alpha_q + \sum_{0<q\le n}(m-q)\alpha_{-q} \cdot \alpha_q\right]$$

The second term is not normal ordered,

$$\begin{aligned}
\sum_{0<q\leq n}(q-n)\alpha_q\cdot\alpha_{-q} &= \sum_{0<q\leq n}(q-n)(\alpha_{-q}\cdot\alpha_q+\left[\alpha_q^\mu,\alpha_{-q,\mu}\right]) \\
&= \sum_{0<q\leq n}(q-n)\alpha_{-q}\cdot\alpha_q+\sum_{0<q\leq n}(q-n)q\eta_\mu^\mu \\
&= \sum_{0<q\leq n}(q-n)\alpha_{-q}\cdot\alpha_q+\sum_{0<q\leq n}(q^2-nq)d \\
&= \sum_{0<q\leq n}(q-n)\alpha_{-q}\cdot\alpha_q+\left[\frac{1}{6}\left(2n^3+3n^2+n\right)-n\frac{n}{2}(n+1)\right]d \\
&= \sum_{0<q\leq n}(q-n)\alpha_{-q}\cdot\alpha_q+\frac{1}{6}n(1-n^2)d
\end{aligned}$$

with d the dimension of the target space.

Thus with $m+n=0$,

$$\begin{aligned}
[L_m,L_n] &= \frac{1}{2}\left[\sum_{q\leq 0}(m-n)\alpha_q\cdot\alpha_{-q}+\sum_{0<q\leq n}(q-n)\alpha_{-q}\cdot\alpha_q\right] \\
&\quad+\frac{1}{2}\left[\sum_{q>n}(m-n)\alpha_{-q}\cdot\alpha_q+\sum_{0<q\leq n}(m-q)\alpha_{-q}\cdot\alpha_q\right]+\frac{d}{12}n(1-n^2)
\end{aligned}$$

combine second, third and fourth term together

$$\begin{aligned}
&= \frac{1}{2}\left[\sum_{q\leq 0}(m-n)\alpha_q\cdot\alpha_{-q}+\sum_{q>0}(m-n)\alpha_{-q}\cdot\alpha_q\right]+\frac{d}{12}n(1-n^2) \\
&= (m-n)L_0+\frac{d}{12}n(1-n^2)
\end{aligned}$$

$$[L_m,L_n] = 2mL_0+\frac{d}{12}m(m^2-1)$$