Superstring theory
Homework 5

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1 A first look at the canonical quantization of the bosonic string

(a) From previous sheet(s),

$$X_L^\mu(\sigma^+) = \frac{1}{2}(x^\mu - c^\mu) + \frac{\pi\alpha'}{l}p^\mu\sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{1}{n}\tilde{\alpha}_n^\mu e^{-\frac{2\pi}{l}in\sigma^+} \tag{1.1}$$

$$X_R^\mu(\sigma^-) = \frac{1}{2}(x^\mu - c^\mu) + \frac{\pi\alpha'}{l}p^\mu\sigma^- + i\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{1}{n}\alpha_n^\mu e^{-\frac{2\pi}{l}in\sigma^-} \tag{1.2}$$

Now we want to express $X^\mu_{L,R}$ in terms of $(z,\bar{z}) = (e^{2\pi i\sigma_-/l}, e^{2\pi i\sigma_+/l})$. It should be stress that $\bar{z}$ is *not* the complex conjugation of z. Thus

$$X_L^\mu(\bar{z}) = \frac{1}{2}(x^\mu - c^\mu) + \frac{\alpha'}{2i}p^\mu \ln\bar{z} + i\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{1}{n}\tilde{\alpha}_n^\mu \bar{z}^{-n} \tag{1.3}$$

$$X_R^\mu(z) = \frac{1}{2}(x^\mu - c^\mu) + \frac{\alpha'}{2i}p^\mu \ln z + i\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{1}{n}\alpha_n^\mu z^{-n} \tag{1.4}$$

(b) Propagator is defined as

$$\langle X^\mu(\sigma,\tau)X^\nu(\sigma',\tau)\rangle = T[X^\mu(\sigma,\tau)X^\nu(\sigma',\tau')] - :X^\mu(\sigma,\tau)X^\nu(\sigma',\tau'): \tag{1.5}$$

T denotes time ordering and $::$ normal ordering

$$:\alpha_{-m}\alpha_m: = \begin{cases}\alpha_{-m}\alpha_m, & m>0\\ \alpha_{-m}\alpha_m - [\alpha_{-m},\alpha_m], & m<0\end{cases}$$
$$:p^\nu x^\mu: = x^\mu p^\nu = p^\nu x^\mu - [p^\nu, x^\mu]$$

We can write out the first term

$$T[X_L^\mu(\bar{z})X_L^\nu(\bar{w})] = \begin{cases} X_L^\mu(\bar{z})X_L^\nu(\bar{w}), & \tau > \tau' \\ X_L^\nu(\bar{w})X_L^\mu(\bar{z}), & \tau < \tau' \end{cases}$$

But for this computation, we will assume that operator products are properly time ordered. The other situation will give the same results.

The normal orderer product is just the "original" product along with some appropriate commutators.

$$\begin{aligned}
:X_L^\mu(\bar{z})X_L^\nu(\bar{w}): &= X_L^\mu(\bar{z})X_L^\nu(\bar{w}) - \frac{\alpha'}{4i}[p^\mu, x^\nu]\ln\bar{z} + \frac{\alpha'}{2}\sum_{n>0,m<0}\frac{1}{nm}[\tilde{\alpha}_n^\mu, \tilde{\alpha}_m^\nu]\bar{z}^{-n}\bar{w}^{-m} \\
&= X_L^\mu(\bar{z})X_L^\nu(\bar{w}) + \frac{\alpha'}{4i}i\eta^{\nu\mu}\ln\bar{z} + \frac{\alpha'}{2}\sum_{n>0,m<0}\frac{1}{nm}n\eta^{\mu\nu}\delta_{m+n}\bar{z}^{-n}\bar{w}^{-m} \\
&= X_L^\mu(\bar{z})X_L^\nu(\bar{w}) + \frac{\alpha'}{4}\eta^{\mu\nu}\ln\bar{z} - \frac{\alpha'}{2}\sum_{n>0}\frac{1}{n}\eta^{\mu\nu}\bar{z}^{-n}\bar{w}^{n} \\
&= X_L^\mu(\bar{z})X_L^\nu(\bar{w}) + \frac{\alpha'}{4}\eta^{\mu\nu}\ln\bar{z} + \frac{\alpha'}{2}\eta^{\mu\nu}\ln\Big(1 - \frac{\bar{w}}{\bar{z}}\Big)
\end{aligned}$$

Thus

$$\begin{aligned}
\langle X_L^\mu(\bar{z})X_L^\nu(\bar{w})\rangle &= -\frac{\alpha'}{4}\eta^{\mu\nu}\ln\bar{z} - \frac{\alpha'}{2}\eta^{\mu\nu}\ln\Big(1 - \frac{\bar{w}}{\bar{z}}\Big) \\
&= \eta^{\mu\nu}\left[\frac{\alpha'}{4}\ln(\bar{z}) - \frac{\alpha'}{2}\ln(\bar{z} - \bar{w})\right] \qquad (1.6)
\end{aligned}$$

With "RR" combination, it is the same with $\bar{z} \to z$ and $\bar{w} \to w$

$$\langle X_R^\mu(z)X_R^\nu(w)\rangle = \eta^{\mu\nu}\left[\frac{\alpha'}{4}\ln(z) - \frac{\alpha'}{2}\ln(z - w)\right] \qquad (1.7)$$

The "mixed" contributions are simpler since commutators involving ladder operators vanish

$$\langle X_L^\mu(\bar{z})X_R^\nu(w)\rangle = -\frac{\alpha'}{4}\eta^{\mu\nu}\ln\bar{z} \qquad (1.8)$$

$$\langle X_R^\mu(z)X_L^\nu(\bar{w})\rangle = -\frac{\alpha'}{4}\eta^{\mu\nu}\ln z \qquad (1.9)$$

All together we have

$$\langle X^\mu(z,\bar{z})X^\nu(w,\bar{w})\rangle = -\eta^{\mu\nu}\frac{\alpha'}{2}\ln(z - w)(\bar{z} - \bar{w}) \qquad (1.10)$$

2 The quantum Virasoro algebra

Virasoro algebra is in quantum theory defined as

$$L_m = \frac{1}{2}\sum_{n\in\mathbb{Z}} :\alpha_{m-n}\cdot\alpha_n:, \quad \bar{L}_m = \frac{1}{2}\sum_{n\in\mathbb{Z}} :\bar{\alpha}_{m-n}\cdot\bar{\alpha}_n: \tag{2.1}$$

(a) In L_m there are only two terms of interest, since all other vanish due to commutation relation

$$\begin{aligned}[L_m, \alpha_n^\mu] &= \frac{1}{2}\alpha_{m-(-n),\nu}\left[\alpha_{-n}^\nu, \alpha_n^\mu\right] + \frac{1}{2}\alpha_{m+n,\nu}\left[\alpha_{m-(m+n)}^\nu, \alpha_n^\mu\right] \\ &= \frac{1}{2}\left(\alpha_{m-(-n),\nu}(-n)\eta^{\nu\mu} + \alpha_{m+n,\nu}(-n)\eta^{\nu\mu}\right) \\ &= -n\alpha_{m+n}^\mu\end{aligned}$$

One should note that the equal sign on the first row is not always valid. Here we have the commutation simply as a number instead of operator, so we can place the commutator where ever we want. In principle, the ordering should depend on the values of m and n.

(b) Use the definition of normal ordering, we see

$$L_m = \frac{1}{2}\sum_{n\geq m/2}\alpha_{m-n}\alpha_n + \frac{1}{2}\sum_{n<m/2}\alpha_n\alpha_{m-n}$$

As a matter of fact, the "threshold" to separate these two summations are arbitrary, since the mode indices are not opposite if $m \neq 0$

$$= \frac{1}{2}\sum_{n\geq 0}\alpha_{m-n}\alpha_n + \frac{1}{2}\sum_{n<0}\alpha_n\alpha_{m-n} \tag{2.2}$$

(c) From previous results, we can write the commutator of Virasoro generators as

$$
\begin{aligned}
[L_m, L_n] &= \frac{1}{2}\left(\sum_{p<0}[L_m, \alpha_p \cdot \alpha_{n-p}] + \sum_{p\geq 0}[L_m, \alpha_{n-p} \cdot \alpha_p]\right) \\
2[L_m, L_n] &= \sum_{p<0}\left[L_m, \alpha_p^\mu\right]\alpha_{n-p,\mu} + \sum_{p<0}\alpha_{p,\mu}\left[L_m, \alpha_{n-p}^\mu\right] \\
&\quad + \sum_{p\geq 0}\left[L_m, \alpha_{n-p}^\mu\right]\alpha_{p,\mu} + \sum_{p\geq 0}\alpha_{n-p,\mu}\left[L_m, \alpha_p^\mu\right] \\
2[L_m, L_n] &= \sum_{p<0} -p\alpha_{m+p}^\mu \alpha_{n-p,\mu} + \sum_{p<0}\alpha_{p,\mu}(-n+p)\alpha_{m+n-p}^\mu \\
&\quad + \sum_{p\geq 0}(-n+p)\alpha_{m+n-p}^\mu \alpha_{p,\mu} + \sum_{p\geq 0}\alpha_{n-p,\mu}(-p)\alpha_{m+p}^\mu
\end{aligned}
$$

Now we realized that we should have expanded L_m. Instead, rename indices $n \leftrightarrow m$

$$
\begin{aligned}
2[L_n, L_m] &= \sum_{p<0} -p\alpha_{n+p}^\mu \alpha_{m-p,\mu} + \sum_{p<0}\alpha_{p,\mu}(-m+p)\alpha_{m+n-p}^\mu \\
&\quad + \sum_{p\geq 0}(-m+p)\alpha_{m+n-p}^\mu \alpha_{p,\mu} + \sum_{p\geq 0}\alpha_{m-p,\mu}(-p)\alpha_{n+p}^\mu \\
[L_m, L_n] &= \frac{1}{2}\sum_{p<0}\left[(m-p)\alpha_p \cdot \alpha_{m+n-p} + p\alpha_{n+p} \cdot \alpha_{m-p}\right] \\
&\quad + \frac{1}{2}\sum_{p\geq 0}\left[(m-p)\alpha_{m+n-p} \cdot \alpha_p + p\alpha_{m-p} \cdot \alpha_{n+p}\right]
\end{aligned}
$$

This is the same as the expression given on the sheet, up to "position" of $p = 0$. It does not matter, since first sum is essentially the same as the second sum.

(d) Change the summation variable as follows: $p \to q$ in the first term, $p \to q - n$ in the second term, $p \to q$ in the third term and $p \to q - n$

in the last term.

$$[L_m, L_n] = \frac{1}{2}\left[\sum_{q\leq 0}(m-q)\alpha_q \cdot \alpha_{m+n-q} + \sum_{q\leq n}(q-n)\alpha_q \cdot \alpha_{m+n-q}\right]$$
$$+\frac{1}{2}\left[\sum_{q>0}(m-q)\alpha_{m+n-q} \cdot \alpha_q + \sum_{q>n}(q-n)\alpha_{m+n-q} \cdot \alpha_q\right]$$

First two terms are very similar, thus we "split" second sum and combine one part into the first sum. The same operation is done to the second two terms.

$$[L_m, L_n] = \frac{1}{2}\left[\sum_{q\leq 0}(m-n)\alpha_q \cdot \alpha_{m+n-q} + \sum_{0<q\leq n}(q-n)\alpha_q \cdot \alpha_{m+n-q}\right]$$
$$+\frac{1}{2}\left[\sum_{q>n}(m-n)\alpha_{m+n-q} \cdot \alpha_q + \sum_{0<q\leq n}(m-q)\alpha_{m+n-q} \cdot \alpha_q\right] \tag{2.3}$$

Is it normal ordered with $n > 0$? No, for instance if $m \to -\infty$, then the first term with $q = 0$ is not normal ordered.

(e) If $m + n \neq 0$ and $n > 0$, with equation (2.2)

$$L_{m+n} = \frac{1}{2}\sum_{q<0}\alpha_q \cdot \alpha_{m+n-q} + \frac{1}{2}\sum_{q\geq 0}\alpha_{m+n-q} \cdot \alpha_q$$

This is the same as $[L_m, L_n]$ up to a numerical factor. One can see it from combining second and last term of equation (2.3) together (with $m+n \neq 0$ we are allowed to exchange α's in the second term) and then put it into the third sum. Thus

$$[L_m, L_n] = (m-n)L_{m+n}, \quad m+n \neq 0$$

(f) If $m + n = 0$ and $n > 0$,

$$[L_m, L_n] = \frac{1}{2}\left[\sum_{q\leq 0}(m-n)\alpha_q \cdot \alpha_{-q} + \sum_{0<q\leq n}(q-n)\alpha_q \cdot \alpha_{-q}\right]$$
$$+\frac{1}{2}\left[\sum_{q>n}(m-n)\alpha_{-q} \cdot \alpha_q + \sum_{0<q\leq n}(m-q)\alpha_{-q} \cdot \alpha_q\right]$$

The second term is not normal ordered,

$$
\begin{aligned}
\sum_{0<q\leq n}(q-n)\alpha_q\cdot\alpha_{-q} &= \sum_{0<q\leq n}(q-n)(\alpha_{-q}\cdot\alpha_q+\left[\alpha_q^\mu,\alpha_{-q,\mu}\right]) \\
&= \sum_{0<q\leq n}(q-n)\alpha_{-q}\cdot\alpha_q+\sum_{0<q\leq n}(q-n)q\eta_\mu^\mu \\
&= \sum_{0<q\leq n}(q-n)\alpha_{-q}\cdot\alpha_q+\sum_{0<q\leq n}(q^2-nq)d \\
&= \sum_{0<q\leq n}(q-n)\alpha_{-q}\cdot\alpha_q+\left[\frac{1}{6}\left(2n^3+3n^2+n\right)-n\frac{n}{2}(n+1)\right]d \\
&= \sum_{0<q\leq n}(q-n)\alpha_{-q}\cdot\alpha_q+\frac{1}{6}n(1-n^2)d
\end{aligned}
$$

with d the dimension of the target space.

Thus with $m+n=0$,

$$
\begin{aligned}
[L_m,L_n] &= \frac{1}{2}\left[\sum_{q\leq 0}(m-n)\alpha_q\cdot\alpha_{-q}+\sum_{0<q\leq n}(q-n)\alpha_{-q}\cdot\alpha_q\right] \\
&\quad+\frac{1}{2}\left[\sum_{q>n}(m-n)\alpha_{-q}\cdot\alpha_q+\sum_{0<q\leq n}(m-q)\alpha_{-q}\cdot\alpha_q\right]+\frac{d}{12}n(1-n^2)
\end{aligned}
$$

combine second, third and fourth term together

$$
\begin{aligned}
&= \frac{1}{2}\left[\sum_{q\leq 0}(m-n)\alpha_q\cdot\alpha_{-q}+\sum_{q>0}(m-n)\alpha_{-q}\cdot\alpha_q\right]+\frac{d}{12}n(1-n^2) \\
&= (m-n)L_0+\frac{d}{12}n(1-n^2)
\end{aligned}
$$

$$
[L_m,L_n] = 2mL_0+\frac{d}{12}m(m^2-1)
$$