Superstring theory
Homework 12

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1 The Ghost System

The ghost system has the action

$$S = \frac{g}{2} \int \mathrm{d}^2 x \, b_{\mu\nu} \partial^\mu c^\nu \tag{1.1}$$

and the propagator is

$$\langle b(z) c(w) \rangle = \frac{1}{\pi g} \frac{1}{z - w} \tag{1.2}$$

Because they are anti-commuting fields $\langle b(z)c(w)\rangle = \langle c(z)b(w)\rangle$.

(a) Other similar correlators are then determined as

$$\langle b(z) \partial c(w) \rangle = - \langle \partial c(z) b(w) \rangle = \frac{1}{\pi g} \frac{1}{(z-w)^2} \tag{1.3}$$

$$\langle \partial b(z) c(w) \rangle = - \langle c(z) \partial b(w) \rangle = -\frac{1}{\pi g} \frac{1}{(z-w)^2} \tag{1.4}$$

$$\langle \partial b(z) \partial c(w) \rangle = \langle \partial c(z) \partial b(w) \rangle = -\frac{2}{\pi g} \frac{1}{(z-w)^3} \tag{1.5}$$

(b) The energy-momentum tensor of the system is given by

$$T(z) = \pi g \, {:}(2 \partial c b + c \partial b){:} \tag{1.6}$$

Then we can calculate the following OPEs

$$
\begin{aligned}
T(z)b(w) &= \pi g \, {:}(2\partial c(z)b(z) + c(z)\partial b(z)){:} \, b(w) \\
&= \pi g\big(2 \, {:}\partial c(z)b(z)b(w){:} + {:}c(z)\partial b(z)b(w){:} \,\big) + \text{reg.} \\
&= \frac{2b(z)}{(z-w)^2} - \frac{\partial b(z)}{z-w} + \text{reg.} \\
&= \frac{2b(w)}{(z-w)^2} + \frac{\partial b(w)}{z-w} + \text{reg.}
\end{aligned}
$$

Thus b-field has conformal dimension $(h, \bar{h}) = (2, 0)$.

c-field has conformal dimension $(h, \bar{h}) = (1, 0)$, since

$$
\begin{aligned}
T(z)c(w) &= \pi g \, {:}(2\partial c(z)b(z) + c(z)\partial b(z)){:} \, c(w) \\
&= \pi g\big(2 \, {:}\partial c(z)b(z)c(w){:} + {:}c(z)\partial b(z)c(w){:} \,\big) + \text{reg.} \\
&= \frac{2\partial c(z)}{z-w} + \frac{-c(z)}{(z-w)^2} + \text{reg.} \\
&= \frac{-c(w)}{(z-w)^2} + \frac{\partial c(w)}{z-w} + \text{reg.}
\end{aligned}
$$

(c) OPE of energy-momentum tensor with itself

$$
\begin{aligned}
&T(z)T(w) \\
&= \pi^2 g^2 \, {:}(2\partial c(z)b(z) + c(z)\partial b(z)){:}{:}(2\partial c(w)b(w) + c(w)\partial b(w)){:} \\
&= \pi^2 g^2 \big(4 \, {:}\partial c(z)b(z)\partial c(w)b(w){:} + 4 \, {:}\partial c(z)b(z)\partial c(w)b(w){:} + 4 \, {:}\partial c(z)b(z)\partial c(w)b(w){:} \\
&\quad + 2 \, {:}\partial c(z)b(z)c(w)\partial b(w){:} + 2 \, {:}\partial c(z)b(z)c(w)\partial b(w){:} + 2 \, {:}\partial c(z)b(z)c(w)\partial b(w){:} \\
&\quad + 2 \, {:}c(z)\partial b(z)\partial c(w)b(w){:} + 2 \, {:}c(z)\partial b(z)\partial c(w)b(w){:} + 2 \, {:}c(z)\partial b(z)\partial c(w)b(w){:} \\
&\quad + {:}c(z)\partial b(z)c(w)\partial b(w){:} + {:}c(z)\partial b(z)c(w)\partial b(w){:} + {:}c(z)\partial b(z)c(w)\partial b(w){:} \\
&\quad + \text{reg.}\big) \\
&= -\frac{4}{(z-w)^4} + \frac{4\pi g}{(z-w)^2}\left(- {:}b(z)\partial c(w){:} + {:}\partial c(z)b(w){:}\right) \\
&\quad - \frac{4}{(z-w)^4} + \frac{2\pi g}{z-w} \, {:}\partial c(z)\partial b(w){:} - \frac{4\pi g}{(z-w)^3} \, {:}b(z)c(w){:} \\
&\quad - \frac{4}{(z-w)^4} - \frac{4\pi g}{(z-w)^3} \, {:}c(z)b(w){:} + \frac{2\pi g}{z-w} \, {:}\partial b(z)\partial c(w){:} \\
&\quad - \frac{1}{(z-w)^4} + \frac{\pi g}{(z-w)^2}\left(- {:}c(z)\partial b(w){:} + {:}\partial b(z)c(w){:}\right) + \text{reg.}
\end{aligned}
$$

now to taylor expand the normal orderder product around $z = w$

$$
\begin{aligned}
&= \frac{-26/2}{(z-w)^4} + \frac{2\pi g}{z-w}\left[:\partial c(w)\partial b(w): + :\partial b(w)\partial c(w):\right] \\
&\quad + \frac{\pi g}{(z-w)^2}\Big[-4 :b(w)\partial c(w): -4(z-w) :\partial b(w)\partial c(w): +4 :\partial c(w)b(w): \\
&\quad + 4(z-w) :\partial^2 c(w)b(w): - :c(w)\partial b(w): -(z-w) :\partial c(w)\partial b(w): \\
&\quad + :\partial b(w)c(w): +(z-w) :\partial^2 b(w)c(w): \Big] \\
&\quad - \frac{4\pi g}{(z-w)^3}\Big[\cancel{:b(w)c(w):} + (z-w) :\partial b(w)c(w): +\frac{1}{2}(z-w)^2 :\partial^2 b(w)c(w): \\
&\quad + \cancel{:c(w)b(w):} + (z-w) :\partial c(w)b(w): +\frac{1}{2}(z-w)^2 :\partial^2 c(w)b(w): \Big] + \text{reg.}
\end{aligned}
$$

Note ghost fields are anti-commuting, thus $:bc:= - :cb:$ *despite* the normal ordering

$$
\begin{aligned}
&= \frac{-26/2}{(z-w)^4} + \frac{\pi g}{z-w}\big(3 :\partial c\partial b: + :c\partial^2 b: +2 :\partial^2 cb:\big) + \frac{2\pi g}{(z-w)^2}\left(2 :\partial cb: + :c\partial b:\right) + \text{reg.} \\
&= \frac{-26/2}{(z-w)^4} + \frac{\partial T(w)}{z-w} + \frac{2T(w)}{(z-w)^2} + \text{reg.}
\end{aligned}
$$

Thus the central charge of the ghost system is $c = -26$ and it has conformal weight $(h, \hat{h}) = (2, 0)$.

2 Vertex operator

Vertex operator is defined as

$$V_\alpha(z, \bar{z}) =:e^{i\alpha\phi(z,\bar{z})}: \tag{2.1}$$

One can show that normal ordered exponentials of free boson fields can be written as

$$:e^{a\phi_1}::e^{b\phi_2}:=:e^{a\phi_1+b\phi_2}: e^{ab\langle\phi_1\phi_2\rangle} \tag{2.2}$$

Up to second order in fields, we have

$$
\begin{aligned}
\text{LHS} =&:\left(1 + a\phi_1 + \frac{1}{2!}(a\phi_1)^2\right)::\left(1 + b\phi_2 + \frac{1}{2!}(b\phi_2)^2\right): +\mathcal{O}\big(\phi^3\big) \\
&= 1 + a\phi_1 + b\phi_2 + ab :\phi_1::\phi_2: +\frac{1}{2!} :(a\phi_1)^2: +\frac{1}{2!} :(b\phi_2)^2: +\mathcal{O}\big(\phi^3\big) \\
&= 1 + a\phi_1 + b\phi_2 + ab :\phi_1\phi_2: +\frac{1}{2!} :(a\phi_1)^2: +\frac{1}{2!} :(b\phi_2)^2: +ab\left\langle\phi_1\phi_2\right\rangle + \mathcal{O}\big(\phi^3\big)
\end{aligned}
$$

This is exactly the RHS up to second order in ϕ. In general, at $\mathcal{O}(\phi^n)$

$$
\begin{aligned}
\text{LHS} &= \sum_{k=0}^{n} \frac{1}{(n-k)!k!} :(a\phi_1)^{n-k}::(b\phi_2)^k: \\
&= \sum_{k=0}^{n} \frac{1}{(n-k)!k!} \sum_{m=0}^{\min(k,n-k)} m! \binom{n-k}{m} \binom{k}{m} (ab\,\langle\phi_1\phi_2\rangle)^m :(a\phi_1)^{n-k-m}(b\phi_2)^{k-m}:
\end{aligned}
$$

The combinatoric factors come from different ways to contract fields. The min is to avoid undefined binomial coeffcients. (Two extreme cases correspond to only ϕ_1 or ϕ_2 are left uncontracted.)

$$
= \sum_{k=0}^{n} \sum_{m=0}^{\min(k,n-k)} \frac{1}{m!(n-k-m)!(k-m)!} (ab\,\langle\phi_1\phi_2\rangle)^m :(a\phi_1)^{n-k-m}(b\phi_2)^{k-m}:
$$

which is the RHS at the same order.

Thus OPE of the vertex operators is

$$
\begin{aligned}
V_\alpha(z,\bar{z})V_\beta(w,\bar{w}) &= :e^{i\alpha\phi(z,\bar{z})+i\beta\phi(w,\bar{w})}:\, e^{-\alpha\beta\langle\phi(z,\bar{z})\phi(w,\bar{w})\rangle} \\
&= :e^{i\alpha\phi(z,\bar{z})+i\beta\phi(w,\bar{w})}:\, \exp\left[-\alpha\beta\frac{\alpha'}{2}\log|(z-w)/R|^2\right] \\
&= :e^{i\alpha\phi(z,\bar{z})+i\beta\phi(w,\bar{w})}:\, \left|\frac{z-w}{R}\right|^{-\alpha'\alpha\beta}
\end{aligned}
$$

The vacuum expectation value of the OPE vanishes unless $\alpha = -\beta$, since normal ordered product has zero vacuum expectation value. Use $\alpha' = \frac{1}{2\pi g}$ and set the cutoff R to unity Thus

$$
\langle V_\alpha(z,\bar{z})V_\beta(w,\bar{w})\rangle = \begin{cases} |z-w|^{-\alpha^2/2\pi g} & \alpha = -\beta \\ 0 & \text{else} \end{cases} \tag{2.3}
$$