Superstring theory
Homework 8

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1 The conformal group in d dimensions

Conformal transformation is defined as the transformation

$$\eta'_{\mu\nu}(x'^{\mu}) = \eta_{\rho\sigma}\frac{\partial x^{\rho}}{\partial x'^{\mu}}\frac{\partial x^{\sigma}}{\partial x'^{\nu}} = \Lambda(x^{\mu})\eta_{\mu\nu}(x^{\mu}) \tag{1.1}$$

In this assignment we will work in Euclidean d-dimensional spacetime.

1. We require that the transformation

$$x^{\mu} \to x^{\mu} + \epsilon^{\mu}(x^{\mu}) \tag{1.2}$$

with $\epsilon^{\mu} \ll 1$ is conformal. Then we have

$$\begin{aligned}\eta'_{\mu\nu}(x'^{\mu}) &= \eta_{\rho\sigma}\left(\delta^{\rho}_{\mu} + \partial_{\mu}\epsilon^{\rho}\right)\left(\delta^{\sigma}_{\nu} + \partial_{\nu}\epsilon^{\sigma}\right) \\ &= \eta_{\mu\nu} + \partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu} + \mathcal{O}\left(\epsilon^2\right) \\ &\stackrel{!}{=} \Lambda(x^{\mu})\eta_{\mu\nu}\end{aligned}$$

Thus we find

$$\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu} = f\eta_{\mu\nu} \tag{1.3}$$

We have a condition for f if we contract the above equation with $\eta^{\mu\nu}$

$$f = \frac{2}{d}(\partial_{\mu}\epsilon^{\mu}) \tag{1.4}$$

2. Take partial derivative ∂_{ρ} of (1.3)

$$\partial_{\rho}\partial_{\mu}\epsilon_{\nu} + \partial_{\rho}\partial_{\nu}\epsilon_{\mu} = \eta_{\mu\nu}\partial_{\rho}f \tag{1.5}$$

Rename (permute) the indices $(\mu,\nu,\rho) \to (\rho,\mu,\nu)$ and $(\mu,\nu,\rho) \to (\nu,\rho,\mu)$ and add these two equations together with "negative" (1.3)

$$\begin{aligned}
&\cancel{\partial_\nu\partial_\rho\epsilon_\mu} + \partial_\nu\partial_\mu\epsilon_\rho + \partial_\mu\partial_\nu\epsilon_\rho + \cancel{\partial_\mu\partial_\rho\epsilon_\nu} - \cancel{\partial_\rho\partial_\mu\epsilon_\nu} - \cancel{\partial_\rho\partial_\nu\epsilon_\mu} \\
&= (\eta_{\rho\mu}\partial_\nu + \eta_{\nu\rho}\partial_\mu - \eta_{\mu\nu}\partial_\rho)f \\
2\partial_\nu\partial_\mu\epsilon_\rho &= (\eta_{\rho\mu}\partial_\nu + \eta_{\nu\rho}\partial_\mu - \eta_{\mu\nu}\partial_\rho)f \qquad (1.6)
\end{aligned}$$

3. Contract (1.6) with $\eta^{\mu\nu}$ and take ∂^ν

$$2\partial^2\partial_\nu\epsilon_\rho = d\partial_\rho\partial_\nu f$$

Note that RHS is symmetric in ρ and ν, thus LHS must be also.
Take ∂^2 of (1.3)

$$\partial^2\partial_\mu\epsilon_\nu + \partial^2\partial_\nu\epsilon_\mu = \partial^2 f\eta_{\mu\nu}$$

Combine these two equations and use the fact that first equation is symmetric in indices

$$\partial^2 f\eta_{\rho\nu} = (2-d)\partial_\rho\partial_\nu f \qquad (1.7)$$

Contract it with $\eta^{\rho\nu}$

$$(d-1)\partial^2 f = 0 \qquad (1.8)$$

4. For $d > 2$, we have

$$\partial_\mu\partial_\nu f = 0 \Leftrightarrow f(x^\mu) = A + B_\mu x^\mu$$

Thus (1.3) becomes

$$\begin{aligned}
2\partial_\mu\epsilon_\nu &= (A + B_\rho x^\rho)\eta_{\mu\nu} \\
\epsilon_\nu &= a_\nu + \underbrace{\frac{1}{2}A\eta_{\mu\nu}}_{b_{\mu\nu}} x^\mu + \underbrace{\frac{1}{2}B_\rho\eta_{\mu\nu}}_{c_{\rho\mu\nu}} x^\mu x^\rho \qquad (1.9)
\end{aligned}$$

where a_ν is just an integration constant and $c_{\rho\mu\nu}$ is symmetric in $\mu\nu$ due to the symmetry of the metric.

5. Next we treat the allowed three terms separately.

   (a) The requirement for ϵ_μ in its original form, (1.3), involves a derivative. So naturally a constant term a_μ without constraint exist.

(b) Insert the linear term in (1.3)

$$b_{\nu\rho}\partial_\mu x^\rho + b_{\mu\rho}\partial_\nu x^\rho = \frac{2}{d}b^\rho_\rho \eta_{\mu\nu}$$
$$b_{\nu\mu} + b_{\mu\nu} \overset{(1.4)}{=} \frac{2}{d}b^\rho_\rho \eta_{\mu\nu}$$

(c) Insert the quadratic into (1.6)

$$\begin{aligned} 4c_{\rho\mu\nu} &= (\eta_{\rho\mu}\partial_\nu + \eta_{\nu\rho}\partial_\mu - \eta_{\mu\nu}\partial_\rho)\frac{4}{d}c^\sigma_{\ \sigma\xi}x^\xi \\ c_{\rho\mu\nu} &= (\eta_{\rho\mu}\delta^\xi_\nu + \eta_{\nu\rho}\delta^\xi_\mu - \eta_{\mu\nu}\delta^\xi_\rho)b_\xi \\ c_{\rho\mu\nu} &= \eta_{\rho\mu}b_\nu + \eta_{\nu\rho}b_\mu - \eta_{\mu\nu}b_\rho \end{aligned}$$

with $b_\mu := \frac{1}{d}c^\nu_{\ \nu\mu}$.

6. Transformations and generators of CFT are Check the commutation

Transformations	Generators
$x'^\mu = x^\mu + a^\mu$	$P_\mu = -i\partial_\mu$
$x'^\mu = M^\mu_\nu x^\nu$	$L_{\mu\nu} = -(x_\mu P_\nu - x_\nu P_\mu)$
$x'^\mu = \alpha x^\mu$	$D = x^\mu P_\mu$
$x'^\mu = \frac{x^\mu - b^\mu x^2}{1-2b\cdot x + b^2x^2}$	$K_\mu = 2x_\mu x^\nu P_\nu - x^2 P_\mu$

relations

$$\begin{aligned} [D, P_\mu] &= [x^\nu, P_\mu]P_\nu \\ &= i\delta^\nu_\mu P_\nu \\ &= iP_\mu \end{aligned} \tag{1.10}$$

Similarly $[D, x_\mu] = x_\nu[P^\nu, x_\mu] = -ix_\mu$.

$$\begin{aligned} [D, K_\mu] &= \left[D, 2x_\mu x^\rho P_\rho - x^2 P_\mu\right] \\ &= 2\left([D, x_\mu]x^\rho P_\rho + x_\mu[D, x^\rho]P_\rho + x_\mu x^\rho[D, P_\rho]\right) \\ &\quad - \Big([D, x^\nu]x_\nu P_\mu + x^\nu[D, x_\nu]P_\mu + x^2[D, P_\mu]\Big) \\ &= -2ix_\mu x^\rho P_\rho + ix^2 P_\mu \\ &= -iK_\mu \end{aligned} \tag{1.11}$$

$$\begin{aligned} [K_\mu, P_\nu] &= 2[x_\mu x^\rho, P_\nu]P_\rho - [x^\rho x_\rho, P_\nu]P_\mu \\ &= 2i(\eta_{\mu\nu}x^\rho + x_\mu\delta^\rho_\nu)P_\rho - i(x_\rho\delta^\rho_\nu + x^\rho\eta_{\rho\nu})P_\mu \\ &= 2i(x\cdot P)\eta_{\mu\nu} + 2ix_\mu P_\nu - 2ix_\nu P_\mu \\ &= 2iD\eta_{\mu\nu} - 2iL_{\mu\nu} \end{aligned} \tag{1.12}$$

First we compute

$$\begin{aligned}
[K_\mu, x_\nu] &= 2x_\mu x^\rho [P_\rho, x_\nu] - x^2 [P_\mu, x_\nu] \\
&= -2i x_\mu x_\nu + i x^2 \eta_{\mu\nu}
\end{aligned} \tag{1.13}$$

then

$$\begin{aligned}
[K_\rho, L_{\mu\nu}] &= -[K_\rho, x_\mu P_\nu - x_\nu P_\mu] \\
&= -[K_\rho, x_\mu] P_\nu - x_\mu [K_\rho, P_\nu] + [K_\rho, x_\nu] P_\mu + x_\nu [K_\rho, P_\mu] \\
&= -(-2i x_\rho x_\mu + i x^2 \eta_{\rho\mu}) P_\nu - x_\mu 2i (D \eta_{\rho\nu} - L_{\rho\nu}) \\
&\quad + (-2i x_\rho x_\nu + i x^2 \eta_{\rho\nu}) P_\mu + x_\nu 2i (D \eta_{\rho\mu} - L_{\rho\mu}) \\
&= -2i x_\mu D \eta_{\rho\nu} + 2i x_\nu D \eta_{\rho\mu} + 2i(-x_\rho L_{\mu\nu} + x_\mu L_{\rho\nu} - x_\nu L_{\rho\mu}) \\
&\quad + i x^2 (-\eta_{\rho\mu} P_\nu + \eta_{\rho\nu} P_\mu) \\
&= i(\eta_{\rho\mu} K_\nu - \eta_{\rho\nu} K_\mu) + \cancel{2i(-x_\rho L_{\mu\nu} + x_\mu L_{\rho\nu} - x_\nu L_{\rho\mu})}
\end{aligned} \tag{1.14}$$

Now we want to show that the second part indeed vanishes

$$\begin{aligned}
&- x_\rho L_{\mu\nu} + x_\mu L_{\rho\nu} - x_\nu L_{\rho\mu} \\
&= x_\rho x_\mu P_\nu - x_\rho x_\nu P_\mu - x_\mu x_\rho P_\nu + x_\mu x_\nu P_\rho + x_\nu x_\rho P_\mu - x_\nu x_\mu P_\rho \\
&= 0
\end{aligned}$$

There are two further commutation relations from the sheet

$$[P_\rho, L_{\mu\nu}] = i(\eta_{\rho\mu} P_\nu - \eta_{\rho\nu} P_\mu) \tag{1.15}$$

$$[L_{\mu\nu}, L_{\rho\sigma}] = i(\eta_{\nu\rho} L_{\mu\sigma} + \eta_{\mu\sigma} L_{\nu\rho} - \eta_{\mu\rho} L_{\nu\sigma} - \eta_{\nu\sigma} L_{\mu\rho}) \tag{1.16}$$

7. Now with new definitions

$$\begin{aligned}
J_{\mu\nu} &= L_{\mu\nu} \\
J_{-1,\mu} &= \frac{1}{2}(P_\mu - K_\mu) \\
J_{-1,0} &= D \\
J_{0,\mu} &= \frac{1}{2}(P_\mu + K_\mu)
\end{aligned} \tag{1.17}$$

Now Greek letters are $1, \ldots, d$ and Latin letters are $-1, 0, \ldots, d$. We assume other components of J_{ab}: $J_{0,0} = J_{0,-1} = J_{-1,-1} = 0$. We want to show

$$[J_{ab}, J_{cd}] = i(\eta_{bc} J_{ad} + \eta_{ad} J_{bc} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac}) \tag{1.18}$$

holds.

- if $(a,b,c,d)=(\mu,\nu,\rho,\sigma)$ then with (1.16) we find the equation to hold.
- if $(a,b,c,d)=(-1,0,-1,0)$ then

$$\begin{aligned}\text{LHS} &= [x^\mu P_\mu, x^\nu P_\nu] \\ &= x^\nu[x^\mu, P_\nu]P_\mu + x^\mu[P_\mu, x^\nu]P_\nu \\ &= iD - iD \\ &= 0 \\ \text{RHS} &= -i(\eta_{-1,-1}J_{00} - \eta_{00}J_{-1,-1}) = 0 \\ &= \text{LHS}\end{aligned}$$

- if $(a,b,c,d)=(-1,0,-1,\mu)$

$$\begin{aligned}\text{LHS} &= \frac{1}{2}[D, P_\mu - K_\mu] \\ &= \frac{i}{2}(P_\mu + K_\mu) \\ &= iJ_{0,\mu} \\ \text{RHS} &= -i\eta_{-1,-1}J_{0,\mu} \\ &= \text{LHS}\end{aligned}$$

- if $(a,b,c,d)=(-1,0,0,\mu)$

$$\begin{aligned}\text{LHS} &= \frac{1}{2}[D, P_\mu + K_\mu] \\ &= iJ_{-1,\mu} \\ \text{RHS} &= i\eta_{bc}J_{ad} \\ &= iJ_{-1,\mu} \\ &= \text{LHS}\end{aligned}$$

- if $(a,b,c,d)=(-1,0,\mu,\nu)$

$$\begin{aligned}\text{LHS} &= [D, L_{\mu\nu}] \\ &= -[x^\rho P_\rho, x_\mu P_\nu] + [x^\rho P_\rho, x_\nu P_\mu] \\ &= -x_\mu[x^\rho, P_\nu]P_\rho - x^\rho[P_\rho, x_\mu]P_\nu + x_\nu[x^\rho, P_\mu]P_\rho + x^\rho[P_\rho, x_\nu]P_\mu \\ &= -ix_\mu P_\nu + ix_\mu P_\nu + ix_\nu P_\mu - x_\nu P_\mu \\ &= 0 = \text{RHS}\end{aligned}$$

- if $(a,b,c,d) = (-1,\mu,-1,\nu)$. First to calculate

$$\begin{aligned}
\text{LHS} &= \frac{1}{4}[P_\mu - K_\mu, P_\nu - K_\nu] \\
&= \frac{1}{4}(-[P_\mu, K_\nu] - [K_\mu, P_\nu]) \\
&= \frac{i}{2}(\eta_{\nu\mu}D - L_{\nu\mu}) - \frac{i}{2}(\eta_{\mu\nu}D - L_{\mu\nu}) \\
&= iL_{\mu\nu} \\
\text{RHS} &= -i\eta_{-1,-1}J_{\mu\nu} \\
&= \text{LHS}
\end{aligned}$$

  where we have used $[K_\mu, K_\nu] = 0$.
- if $(a,b,c,d) = (-1,\mu,0,\nu)$

$$\begin{aligned}
\text{LHS} &= \frac{1}{4}[P_\mu - K_\mu, P_\nu + K_\nu] \\
&= \frac{1}{4}[P_\mu, K_\nu] - \frac{1}{4}[K_\mu, P_\nu] \\
&= -\frac{i}{2}\eta_{\nu\mu}D + \frac{i}{2}L_{\nu\mu} - \frac{i}{2}\eta_{\mu\nu}D + \frac{i}{2}L_{\mu\nu} \\
&= -i\eta_{\mu\nu}D \\
&= \text{RHS}
\end{aligned}$$

- if $(a,b,c,d) = (-1,\mu,\nu,\rho)$
- if $(a,b,c,d) = (0,\mu,-1,\nu)$
- if $(a,b,c,d) = (0,\mu,0,\nu)$
- if $(a,b,c,d) = (0,\mu,\nu,\rho)$

2 The conformal group in 2d

For a infinitesimal coordinate transformation $z^\mu \to z^\mu + \epsilon^\mu(z)$, conformal transformation satisfies

$$\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu = \frac{2}{d}\partial\cdot\epsilon\eta_{\mu\nu} \tag{2.1}$$

In $2d$, we have

$$\begin{aligned}
\partial_1\epsilon_1 &= \partial_2\epsilon_2 \\
\partial_1\epsilon_2 &= -\partial_2\epsilon_1
\end{aligned}$$

which are precisely Cauchy-Riemann equations for $z = x^1 + ix^2$ and $\epsilon(z) = \epsilon^1(z) + i\epsilon^2(z)$. Thus the infinitesimal conformal transformation in $2d$ resembles a holomorphic coordinate transformation.

(a) Consider a spinless, (scaling) dimensionless field $\phi(z)$. It transform under the conformal transformation as

$$\phi(z) \to \phi'(z) = \phi - \epsilon\partial_z\phi(z)$$

Since we have holomorphic function $\epsilon(z)$ (or at least analytic on some open set), there exist a Laurent series

$$\epsilon(z) = \sum_{z\in\mathbb{Z}} \epsilon_n z^{n+1}$$

The exponent $n+1$ is chosen only for convenience. Note since here we use $\epsilon = \epsilon_1 + i\epsilon_2$ and $z = x_1 + ix_2$, the notation is well-defined. Thus

$$\begin{aligned} \delta\phi &= -\epsilon\partial_z\phi(x) \\ &= -\sum_{z\in\mathbb{Z}} \epsilon_n \underbrace{z^{n+1}\partial_z}_{=-l_n} \phi(x) \end{aligned}$$

Analogously for $\bar{z} \to \bar{z} + \bar{\epsilon}(\bar{z})$ Thus

$$l_n = -z^{n+1}\partial_z, \qquad \bar{l}_n = -\bar{z}^{n+1}\partial_{\bar{z}} \tag{2.2}$$

One can show

$$\begin{aligned} [l_n, l_m] &= z^{n+1}\partial_z z^{m+1}\partial_z - z^{m+1}\partial_z z^{n+1}\partial_z \\ &= z^{n+1}(m+1)z^m\partial_z - z^{m+1}(n+1)z^n\partial_z \\ &= (m-n)l_{m+n} \end{aligned}$$

and the same for $\bar{l}$ and $\left[l_n, \bar{l}_n\right] = 0$. Thus we say the generators form Witt algebra.

(b) Riemann sphere is simply the complex plane and infinity: $\mathcal{C} \cup \infty$. It is clear that with $n \leq -1$, the generator l_n is singular at $z = 0$. To investigate singular behaviour at $z = \infty$, we substitute $w = 1/z$, then $l_n = -w^{-n+1}\partial_w$. Thus for $n \geq 1$, it is singular at $z = \infty$. Thus the only well-behaved generators are l_{-1}, l_0 and l_1 (same for $\bar{l}_m$).

(c) To expand the parameters around $a = 1, b = 0, c = 0, d = 0$. Then we have a matrix

$$\begin{pmatrix} 1 + \delta a & \delta b \\ \delta c & 1 + \delta d \end{pmatrix} \tag{2.3}$$

Since the determinant has to be 1 in leading order, we have $\delta d = -\delta a$. Then

$$\begin{aligned} \delta z &= \frac{(1 + \delta a)z + \delta b}{\delta c z + 1 - \delta a} - z \\ &= \frac{\delta b + 2\delta a z - \delta c z^2}{\delta c z + (1 - \delta a)} \\ &= \delta b + 2\delta a z - \delta c z^2 + \mathcal{O}\big(\delta^2\big) \end{aligned}$$

Once one expands $\phi(z + \delta z)$, then there are generators from global conformal groups. Thus global conformal group is $2d$ indeed corresponds to $SL(2, \mathbb{C})$. Since there is an additional symmetry $a, b, c, d \to -a, -b, -c, -d$. It is isomorphic to $SL(2, \mathbb{C})/\mathbb{Z}_2$.