Superstring theory
Homework 2

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1 Worldsheet light-cone coordinates

Worldsheet light-cone coordinates are defined as

$$\sigma^{\pm} = \tau \pm \sigma \tag{1.1}$$

Equivalently,

$$\tau = \frac{1}{2}\left(\sigma^+ + \sigma^-\right), \quad \sigma = \frac{1}{2}\left(\sigma^+ - \sigma^-\right) \tag{1.2}$$

(a)

$$\partial_+ = \frac{\partial}{\partial \sigma^+} = \frac{\partial \tau}{\partial \sigma^+}\frac{\partial}{\partial \tau} + \frac{\partial \sigma}{\partial \sigma^+}\frac{\partial}{\partial \sigma} = \frac{1}{2}\left(\partial_\tau + \partial_\sigma\right) \tag{1.3}$$

$$\partial_- = \frac{1}{2}\left(\partial_\tau - \partial_\sigma\right) \tag{1.4}$$

(b) Written in matrix form we have the $\eta^{\alpha\beta}$ in light-cone coordinates

$$\eta^{\alpha\beta} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^T \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}$$

Its inverse is

$$\eta_{\alpha\beta} = -\frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{1.5}$$

(c) Energy momentum tensor is defined as

$$
\begin{aligned}
T_{\alpha\beta} &= \frac{4\pi}{\sqrt{-h}} \frac{\delta S_p}{\delta h^{\alpha\beta}} \\
&= \frac{4\pi}{\sqrt{-h}} \left(-\frac{T}{2}\right) \int \mathrm{d}^2\sigma \, \Gamma_{\mu\nu} \frac{\delta}{\delta h_{\alpha\beta}} \sqrt{-h} h^{\mu\nu} \\
&= \frac{4\pi}{\sqrt{-h}} \left(-\frac{T}{2}\right) \Gamma_{\mu\nu} \left(-\frac{1}{2\sqrt{-h}} h_{\alpha\beta} h h^{\mu\nu} + \sqrt{-h} \delta^\mu_\alpha \delta^\nu_\beta\right) \\
&= -2\pi T \left(-\frac{1}{2} \Gamma_{\mu\nu} h_{\alpha\beta} h^{\mu\nu} + \Gamma_{\alpha\beta}\right) \qquad (1.6)
\end{aligned}
$$

Explicitely

$$
\begin{aligned}
T_{00} = T_{11} &= -\pi T(\dot{X}^2 + X'^2) \\
T_{01} = T_{10} &= -2\pi T(\dot{X} \cdot X')
\end{aligned}
$$

The energy momentum tensor corresponds to the change of the metric. We know from previous homework that in two dimension, there is no gravity. Thus this energy momemtum tensor must vanish. Alternatively, inserting equation of motion reveals that the energy-momentum tensor must vanish.

(d) In one way, since the energy-momentum tensor itself has to vanish. It must be traceless.

Or explicitely, one has

$$
\begin{aligned}
T_{\alpha\beta}\eta^{\beta\alpha} = T^\alpha_\alpha &\propto -\frac{1}{2}\Gamma_{\mu\nu} h_{\alpha\beta} h^{\mu\nu} \eta^{\beta\alpha} + \Gamma_{\alpha\beta}\eta^{\beta\alpha} \\
&= -\frac{1}{2}\Gamma_{\mu\nu} h^{\mu\nu} h_{\alpha\beta} \eta^{\beta\alpha} + \Gamma_{\alpha\beta}\eta^{\beta\alpha}
\end{aligned}
$$

Because of Wyel invariance, we have $h^{\mu\nu}\eta_\alpha = \eta^{\mu\nu}\eta_{\mu\nu} = 2$ and thus $T^\alpha_\alpha = 0$.

(e) In light-cone coordinates, energy momentum tensor has to vanish since coordinate transformation keeps a zero "matrix" intact.

$$
T'_{\alpha\beta} = \frac{\partial\sigma^\mu}{\partial\sigma'^\alpha} \frac{\partial\sigma^\nu}{\partial\sigma'^\beta} T_{\mu\nu} = 0 \qquad (1.7)
$$

with $\sigma' = (\sigma^+, \sigma^-)$.

Explicitely

$$
\begin{aligned}
T'_{00} &= \frac{\partial\sigma^\mu}{\partial\sigma^+}\frac{\partial\sigma^\nu}{\partial\sigma^+}T_{\mu\nu} \\
&= \frac{1}{2}T_{00} + \frac{1}{2}T_{01} \\
&= -\frac{\pi T}{2}\left(\dot{X} + X'\right)^2 \\
&= -2\pi T(\partial_+ X)^2 \\
T'_{11} &= \frac{1}{2}T_{00} - \frac{1}{2}T_{01} = -2\pi T(\partial_- X)^2 \\
T_{01} &= T_{10} = 0
\end{aligned}
$$

Here we denote $T'_{00} = T_{++}$, $T'_{11} = T_{--}$ and so on.

2 Classical string equations of motion and boundary conditions

(a) The Polyakov in conformal gauge ($h_{\alpha\beta} = \Omega(\sigma,\tau)\eta_{\alpha\beta}$)is

$$
\begin{aligned}
S_p &= -\frac{T}{2}\int \mathrm{d}^2\sigma\,\sqrt{-h}h^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu\eta_{\mu\nu} \\
&= -\frac{T}{2}\int \mathrm{d}^2\sigma\,\eta^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu\eta_{\mu\nu} \\
&= -\frac{T}{2}\int \mathrm{d}^2\sigma\,\partial_\alpha X^\mu\partial^\alpha X_\mu \\
&= \frac{T}{2}\int \mathrm{d}^2\sigma\left(\dot{X}^2 - X'^2\right)
\end{aligned}
\tag{2.1}
$$

(b) Take the variation of δS_p with respect to X^μ

$$
\begin{aligned}
\delta S_p &= \frac{T}{2}\int \mathrm{d}^2\sigma\left(2\dot{X}^\mu\delta\dot{X}_\mu - 2X'^\mu\delta X'_\mu\right) \\
&= T\int \mathrm{d}^2\sigma\left(\dot{X}^\mu\delta\dot{X}_\mu - X'^\mu\delta X'_\mu\right) \\
&\overset{\text{i.b.p.}}{=} T\int \mathrm{d}^2\sigma\left(\partial_\tau^2 - \partial_\sigma^2\right)X^\mu\delta X_\mu - T\int_0^l \mathrm{d}\sigma\,\dot{X}^\mu\delta X_\mu\Big|_{\tau=\tau_i}^{\tau_f} \\
&\quad - T\int_{\tau_i}^{\tau_f}\mathrm{d}\tau\,X'^\mu\delta X_\mu\Big|_{\sigma=0}^{l}
\end{aligned}
$$

where the second term vanished due to $\delta X^\mu(\tau_i) = \delta X^\mu(\tau_f)$. Thus assume that first and third term separately go to zero, we have

$$\left(\partial_\tau^2 - \partial_\sigma^2\right) X^\mu = 0 \tag{2.2}$$

$$\int_{\tau_i}^{\tau_f} \mathrm{d}\tau\, X'^\mu \delta X_\mu \bigg|_{\sigma=0}^{l} = 0 \tag{2.3}$$

(c) There are various ways to make the second term vanish

- Periodicity $X^\mu(\sigma,\tau) = X^\mu(\sigma + l, \tau)$ so that the integral evaluated at two points are equal. It means that we have closed string now.
- Neumann boundary condition $X'_\mu(\sigma,\tau) = 0$ for $\sigma = 0, l$ so that the integrand vanishes. It means that the direction or slope of string's end points are always fixed.
- Dirichlet boundary condition $X^\mu(\sigma = 0) = \text{const.}$ and $X^\mu(\sigma = l) = \text{const.}$ so that the integrand vanishes. It means the end points of string are fixed. It breaks Lorentz invariance.

3 Global Poincare invariance and Poincare Algebra

The energy-momentum current

$$P_\mu^\alpha = -T\sqrt{-h}h^{\alpha\beta}\partial_\beta X_\mu \tag{3.1}$$

The angular momentum current

$$J_{\mu\nu}^\alpha = -T\sqrt{-h}h^{\alpha\beta}\left(X_\mu \partial_\beta X_\nu - X_\nu \partial_\beta X_\mu\right) \tag{3.2}$$

(a) Obviously

$$\begin{aligned} J_{\mu\nu}^\alpha &= X_\mu \left(-T\sqrt{-h}h^{\alpha\beta}\partial_\beta X_\nu\right) - X_\nu \left(-T\sqrt{-h}h^{\alpha\beta}\partial_\beta X_\mu\right) \\ &= X_\mu P_\nu^\alpha - X_\nu P_\mu^\alpha \end{aligned} \tag{3.3}$$

(b) Integrals of these two currents over a space-like section are constant in time, i.e. conversed quantities. To see it

$$
\begin{aligned}
\partial_0 P_\mu &= \int_0^l \mathrm{d}\sigma\, \partial_0 P_\mu^0 \\
&= T \int_0^l \mathrm{d}\sigma\, \partial_0^2 X_\mu \\
&\overset{\text{e.o.m.}}{=} T \int_0^l \mathrm{d}\sigma\, \partial_1^2 X_\mu \\
&= T\, \partial_\sigma X_\mu \big|_{\sigma=0}^{\sigma=l}
\end{aligned}
$$

where the equation of motion is used. It is zero for closed string due to periodicity. For open string, momentum current is conserved with Neumann boundary condition. Physical intepretation is that with Neumann condition no momentum can flow in or out of string.

$$
\begin{aligned}
\partial_0 J_{\mu\nu} &= \partial_0 \int_0^l \mathrm{d}\sigma\, J_{\mu\nu}^0 \\
&= T \int_0^l \mathrm{d}\sigma\, \left(X_\mu \partial_0^2 X_\nu - X_\nu \partial_0^2 X_\mu \right) \\
&\overset{\text{e.o.m.}}{=} T \int_0^l \mathrm{d}\sigma\, \left(X_\mu \partial_1^2 X_\nu - X_\nu \partial_1^2 X_\mu \right) \\
\frac{1}{T} \partial_0 J_{\mu\nu} &\overset{\text{i.b.p.}}{=} X_\mu \partial_1 X_\nu \big|_{\sigma=0}^{l} - \int_0^l \mathrm{d}\sigma\, \partial_1 X_\mu \partial_1 X_\nu - \int_0^l \mathrm{d}\sigma\, X_\nu \partial_1^2 X_\mu \\
&\overset{\text{i.b.p.}}{=} X_\mu \partial_1 X_\nu \big|_{\sigma=0}^{l} - X_\nu \partial_1 X_\mu \big|_{\sigma=0}^{l}
\end{aligned}
$$

Again it vanishes for closed string with periodicity, with Neumann condition for open string also. Here the angular momentum cannot flow in or out of the string.

Also it is obvious that for open string with Dirichlet condition, momentum and angular momentum currents are not necessarily conserved. It has something to do with its explicite volation of Lorentz invariance.

(c) We work again in conformal gauge.

We have relation of Poisson brakcet

$$
\begin{aligned}
\{X^\mu(\sigma,\tau), X^\nu(\sigma',\tau)\} = \left\{\dot{X}^\mu(\sigma,\tau), \dot{X}^\nu(\sigma',\tau)\right\} &= 0 \\
\left\{X^\mu(\sigma,\tau), \dot{X}^\nu(\sigma',\tau)\right\} &= \frac{1}{T} \eta^{\mu\nu} \delta(\sigma - \sigma')
\end{aligned}
$$

Thus

$$\begin{aligned}
\{X^\mu(\sigma,\tau), P^\nu(\sigma',\tau)\} &= \int \mathrm{d}\sigma' \left\{X^\mu(\sigma,\tau), P^{0,\nu}(\sigma',\tau)\right\} \\
&= -T \int \mathrm{d}\sigma' \left\{X^\mu(\sigma,\tau), \partial^0 X^\nu(\sigma',\tau)\right\} \\
&= -\eta^{\mu\nu}
\end{aligned}$$

The Poincare Algebra

$$\begin{aligned}
\{P^\mu, P^\nu\} &= \int \mathrm{d}\sigma \int \mathrm{d}\sigma' \left\{P^{0,\mu}, P^{0,\mu}\right\} \\
&\propto \int \mathrm{d}\sigma \int \mathrm{d}\sigma' \left\{\partial^0 X^\mu, \partial^0 X^\nu\right\} \\
\{P^\mu, P^\nu\} &= 0 \qquad (3.4)
\end{aligned}$$

$$\begin{aligned}
\{P^\mu, J^{\rho\sigma}\} &= \{P^\mu, X^\rho P^\sigma - X^\sigma P^\rho\} \\
&= \{P^\mu, X^\rho P^\rho\} - \{P^\mu, X^\sigma P^\rho\} \\
&= \{P^\mu, X^\rho\}P^\rho + X^\rho \underbrace{\{P^\mu, P^\rho\}}_{=0} - \{P^\mu, X^\sigma\}P^\rho - X^\sigma \underbrace{\{P^\mu, P^\rho\}}_{=0} \\
&= -\eta^{\mu\rho}P^\sigma + \eta^{\mu\sigma}P^\rho \qquad (3.5)
\end{aligned}$$

If one substitue some certain P^μ with X^μ in the second last step, one has

$$\{X^\mu, J^{\rho\sigma}\} = X^\rho\{X^\mu, P^\rho\} - X^\sigma\{X^\mu, P^\rho\} = -X^\rho\eta^{\mu\rho} + X^\sigma\eta^{\mu\rho}$$

$$\begin{aligned}
\{J^{\mu\nu}, J^{\rho\sigma}\} &= \{X^\mu P^\nu - X^\nu P^\mu, J^{\rho\sigma}\} \\
&= \{X^\mu, J^{\rho\sigma}\}P^\nu - X^\mu\{P^\nu, J^{\rho\sigma}\} - \{X^\nu, J^{\rho\sigma}\}P^\mu + X^\nu\{P^\mu, J^{\rho\sigma}\} \\
&= \left(-X^\rho\eta^{\mu\sigma} + X^\sigma\eta^{\mu\rho}\right)P^\nu - X^\mu\left(P^\rho\eta^{\nu\sigma} - P^\sigma\eta^{\nu\rho}\right) \\
&\quad + \left(-X^\rho\eta^{\nu\sigma} + X^\sigma\eta^{\nu\rho}\right)P^\mu + X^\nu\left(P^\rho\eta^{\mu\sigma} - P^\sigma\eta^{\mu\rho}\right) \\
&= \eta^{\mu\sigma}J^{\rho\nu} - \eta^{\mu\rho}J^{\sigma\nu} - \eta^{\nu\sigma}J^{\rho\mu} + \eta^{\nu\rho}J^{\mu\sigma} \qquad (3.6)
\end{aligned}$$