

Superstring theory

Homework 3

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1 Oscillator expansions for the classical string

(a) We make the ansatz

$$X^\mu(\sigma, \tau) = f(\sigma)g(\tau) \quad (1.1)$$

Then the equation of motion becomes

$$g(\tau) \frac{\partial^2 f(\sigma)}{\partial \sigma^2} - f(\sigma) \frac{\partial^2 g(\tau)}{\partial \tau^2} = 0$$

Clearly if we have

$$\frac{\partial^2 f(\sigma)}{\partial \sigma^2} = k f(\sigma), \quad \frac{\partial^2 g(\tau)}{\partial \tau^2} = k g(\tau) \quad (1.2)$$

the equation of motion is satisfied. Thus we write the solutions

$$\begin{aligned} f(\sigma) &= A e^{-i\sqrt{-k}\sigma} + B e^{i\sqrt{-k}\sigma}, \\ g(\tau) &= A' e^{-i\sqrt{-k}\tau} + B' e^{i\sqrt{-k}\tau}, \text{ for } k \neq 0 \\ f(\sigma) &= C\sigma + D, \quad g(\tau) = C'\tau + D', \text{ for } k = 0 \end{aligned}$$

To determine the constant k , we need to use periodic boundary condition ($m \in \mathbb{Z}$)

$$\begin{aligned} l\sqrt{-k} &= 2\pi m \\ k &= -\frac{4\pi^2 m^2}{l^2} \end{aligned}$$

Polynomial solution of $f(\sigma)$ cannot satisfy unless $C = 0$. Multiplying out $f(\sigma)$ and $g(\tau)$ and redefining the constants (so that they also carries Lorentz indices)

$$X^\mu(\sigma, \tau) = A^\mu + B^\mu \tau + \sum_{m \in \mathbb{Z} \setminus \{0\}} e^{i\sqrt{-k}\sigma} \left(C_m^\mu e^{-i\sqrt{-k}\tau} + D_m^\mu e^{i\sqrt{-k}\tau} \right) \quad (1.3)$$

Note that terms with $e^{-i\sqrt{-k}\sigma}$ gets absorbed in redefinition of constants.

(b) Equation of motion can be rewritten as

$$\partial_+ \partial_- X^\mu = 0$$

Thus it makes sense to write the solution into left- and right-moving parts

$$X^\mu(\sigma, \tau) = X_L^\mu(\sigma^+) + X_R^\mu(\sigma^-) \quad (1.4)$$

These two parts are explicit given on the sheet

$$X_L^\mu(\sigma^+) = \frac{1}{2}(x^\mu - c^\mu) + \frac{\pi\alpha'}{l} p^\mu \sigma^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{1}{n} \tilde{\alpha}_n^\mu e^{-\frac{2\pi}{l} i n \sigma^+} \quad (1.5)$$

$$X_R^\mu(\sigma^-) = \frac{1}{2}(x^\mu - c^\mu) + \frac{\pi\alpha'}{l} p^\mu \sigma^- + i\sqrt{\frac{\alpha'}{2}} \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{1}{n} \alpha_n^\mu e^{-\frac{2\pi}{l} i n \sigma^-} \quad (1.6)$$

We will set c^μ to zero in the following discussion.

(c) We suppose that the solution from part (b) should be used here.

We require that solutions X^μ must be real. Thus x^μ and p^μ (α' too) must be real. Last term is a bit more involved

$$\begin{aligned} \left(i\sqrt{\frac{\alpha'}{2}} \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{1}{n} \tilde{\alpha}_n^\mu e^{-\frac{2\pi}{l} i n \sigma^+} \right)^* &= -i\sqrt{\frac{\alpha'}{2}} \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{1}{n} (\tilde{\alpha}_n^\mu)^* e^{\frac{2\pi}{l} i n \sigma^+} \\ &= i\sqrt{\frac{\alpha'}{2}} \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{1}{-n} (\tilde{\alpha}_{-(-n)}^\mu)^* e^{-\frac{2\pi}{l} i (-n) \sigma^+} \\ &= i\sqrt{\frac{\alpha'}{2}} \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{1}{n} (\tilde{\alpha}_{-n}^\mu)^* e^{-\frac{2\pi}{l} i n \sigma^+} \\ &\stackrel{!}{=} i\sqrt{\frac{\alpha'}{2}} \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{1}{n} \tilde{\alpha}_n^\mu e^{-\frac{2\pi}{l} i n \sigma^+} \end{aligned}$$

Thus the condition for $\tilde{\alpha}$ is

$$\tilde{\alpha}_n^\mu = (\tilde{\alpha}_{-n}^\mu)^*$$

Analogically,

$$\alpha_n^\mu = (\alpha_{-n}^\mu)^*$$

(d) There are a couple of interesting quantities to look at

- total spacetime momentum

$$\begin{aligned} P^\mu &= \frac{1}{2\pi\alpha'} \int_0^l d\sigma \dot{X}^\mu \\ &= \frac{1}{2\pi\alpha'} \int_0^l d\sigma (\partial_+ X_L^\mu + \partial_- X_R^\mu) \\ &= \frac{1}{2\pi\alpha'} \int_0^l d\sigma \frac{2\pi}{l} \sqrt{\frac{\alpha'}{2}} \sum_{n \in \mathbb{Z}} \left(\tilde{\alpha}_n^\mu e^{-\frac{2\pi}{l}in(\tau+\sigma)} + \alpha_n^\mu e^{-\frac{2\pi}{l}in(\tau-\sigma)} \right) \end{aligned}$$

Here the sum includes $n = 0$ mode, because we introduce

$$\alpha_0^\mu = \tilde{\alpha}_0^\mu = \sqrt{\frac{\alpha'}{2}} p^\mu$$

Obviously, integrals with $n \neq 0$ vanish because of periodicity of complex exponential function.

$$P^\mu = \frac{1}{2\pi\alpha'} \frac{2\pi}{l} \sqrt{\frac{\alpha'}{2}} l \cdot 2 \sqrt{\frac{\alpha'}{2}} p^\mu = p^\mu \quad (1.7)$$

It shows that the p^μ is the total momentum.

- center of mass position

$$\begin{aligned} q^\mu &= \frac{1}{l} \int_0^l d\sigma X^\mu \\ &= \frac{1}{l} \int_0^l d\sigma \left(x^\mu + 2 \frac{\pi\alpha'}{l} p^\mu \tau \right) \\ &= x^\mu + 2 \frac{\pi\alpha'}{l} p^\mu \tau \end{aligned} \quad (1.8)$$

x^μ can be interpreted as center of mass position at $\tau = 0$.

- total angular momentum

$$\begin{aligned}
J^{\mu\nu} &= \frac{1}{2\pi\alpha'} \int_0^l d\sigma \left(X^\mu \dot{X}^\nu - X^\nu \dot{X}^\mu \right) \\
&= \frac{1}{2\pi\alpha'} \int_0^l d\sigma \left(x^\mu + 2\frac{\pi\alpha'}{l} p^\mu \tau \right) \dot{X}^\nu \\
&\quad + \frac{i}{2l} \int_0^l d\sigma \sum_{n \neq 0} \left(\frac{1}{n} \tilde{\alpha}_n^\mu e^{-\frac{2\pi}{l}in(\tau+\sigma)} + \frac{1}{n} \alpha_n^\mu e^{-\frac{2\pi}{l}in(\tau-\sigma)} \right) \\
&\quad \times \sum_{n \in \mathbb{Z}} \left(\tilde{\alpha}_n^\mu e^{-\frac{2\pi}{l}in(\tau+\sigma)} + \alpha_n^\mu e^{-\frac{2\pi}{l}in(\tau-\sigma)} \right) - (\mu \leftrightarrow \nu)
\end{aligned}$$

Here term with $p^\mu \dot{X}^\nu$ is symmetric in indices and it get cancelled by the $(\mu \leftrightarrow \nu)$ term. The same as before, only terms with constant in σ can be non-zero after integration. Thus we must have n of opposite signs for two sums.

$$J^{\mu\nu} = x^\mu p^\nu + i \sum_{n>0} \frac{1}{n} \left(\tilde{\alpha}_n^\mu \tilde{\alpha}_{-n}^\nu - \alpha_n^\mu \alpha_{-n}^\nu \right) - (\mu \leftrightarrow \nu)$$

Note that $\frac{1}{2}$ factor in the last term disappeared because we can have $-n$ from first summation and n from second summation in the $(\mu \leftrightarrow \nu)$ term. The sign from $1/n$ cancels with the overall sign in front. Thus two times the same contributions.

- (e) Physically open strings can reflect left- and right-moving oscillations. Thus only one mode is necessary.
- (f) The given mode expansion solves the bulk equation

$$(\partial_\tau^2 - \partial_\sigma^2) X^\mu(\tau, \sigma) = i\sqrt{2\alpha'} \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-i\frac{\pi}{l}n\tau} \cos\left(\frac{n\pi\sigma}{l}\right) \left(-\frac{\pi}{l}n + \frac{n\pi}{l}\right) = 0$$

And to check the boundary conditions

$$X'^\mu = i\sqrt{2\alpha'} \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-i\frac{\pi}{l}n\tau} \sin\left(\frac{n\pi\sigma}{l}\right) \cdot \left(-\frac{n\pi}{l}\right)$$

It indeed vanishes at $\sigma = 0$ and $\sigma = l$.

- (g) This mode expansion also solves the bulk equation

$$(\partial_\tau^2 - \partial_\sigma^2) X^\mu(\tau, \sigma) = \sqrt{2\alpha'} \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-i\frac{\pi}{l}n\tau} \sin\left(\frac{n\pi\sigma}{l}\right) \left(-\frac{\pi}{l}n + \frac{n\pi}{l}\right) = 0$$

Because of $\sin(n\pi\sigma/l)$, third term vanishes at $\sigma = 0$ and $\sigma = l$. And $X^\mu(\sigma = 0) = x_0^\mu$, $X^\mu(\sigma = l) = x_l^\mu$. So the Dirichlet conditions are satisfied.

2 Poisson brackets for the classical closed string

From previous section, we have

$$X^\mu(\sigma, \tau) = (x^\mu - c^\mu) + 2\frac{\pi\alpha'}{l}p^\mu\tau + i\sqrt{\frac{\alpha'}{2}} \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{1}{n} \left(\tilde{\alpha}_n^\mu e^{-\frac{2\pi}{l}in(\tau+\sigma)} + \alpha_n^\mu e^{-\frac{2\pi}{l}in(\tau-\sigma)} \right)$$

$$\dot{X}^\mu(\sigma, \tau) = \frac{2\pi}{l} \sqrt{\frac{\alpha'}{2}} \sum_{n \in \mathbb{Z}} \left(\tilde{\alpha}_n^\mu e^{-\frac{2\pi}{l}in(\tau+\sigma)} + \alpha_n^\mu e^{-\frac{2\pi}{l}in(\tau-\sigma)} \right)$$

We need to "invert" the solutions, in order to express $\tilde{\alpha}_n^\mu$ and α_n^μ in terms of X^μ and \dot{X}^μ . We want to multiply these two with an exponential function and integrate it out, so that only some certain modes can survive ($m \neq 0$)

$$\begin{aligned} \int_0^l X^\mu(\sigma, 0) e^{\frac{2\pi}{l}im\sigma} d\sigma &= i\sqrt{\frac{\alpha'}{2}} \left(\frac{1}{m} \tilde{\alpha}_m^\mu - \frac{1}{m} \alpha_{-m}^\mu \right) \\ \int_0^l \dot{X}^\mu(\sigma, 0) e^{\frac{2\pi}{l}im\sigma} d\sigma &= \frac{2\pi}{l} \sqrt{\frac{\alpha'}{2}} (\tilde{\alpha}_m^\mu + \alpha_{-m}^\mu) \end{aligned} \quad (2.1)$$

Thus we get

$$\tilde{\alpha}_m^\mu = \frac{1}{\sqrt{2\alpha'}} \int_0^l d\sigma e^{\frac{2\pi}{l}im\sigma} \left(\frac{m}{i} X^\mu(\sigma, 0) + \frac{l}{2\pi} \dot{X}^\mu(\sigma, 0) \right) \quad (2.2)$$

$$\alpha_m^\mu = \frac{1}{\sqrt{2\alpha'}} \int_0^l d\sigma e^{-\frac{2\pi}{l}im\sigma} \left(\frac{m}{i} X^\mu(\sigma, 0) + \frac{l}{2\pi} \dot{X}^\mu(\sigma, 0) \right) \quad (2.3)$$

where for α_m^μ , we used $m \rightarrow -m$.

To get the desired Poisson brackets

$$\begin{aligned} \{\tilde{\alpha}_n^\mu, \tilde{\alpha}_m^\nu\} &= \frac{1}{2\alpha'} \int_0^l d\sigma e^{\frac{2\pi}{l}in\sigma} \int_0^l d\sigma' e^{\frac{2\pi}{l}im\sigma'} \\ &\quad \times \left\{ \frac{n}{i} X^\mu(\sigma, 0) + \frac{l}{2\pi} \dot{X}^\mu(\sigma, 0), \frac{m}{i} X^\nu(\sigma', 0) + \frac{l}{2\pi} \dot{X}^\nu(\sigma', 0) \right\} \\ &= \frac{1}{2\alpha'} \int_0^l d\sigma e^{\frac{2\pi}{l}in\sigma} \int_0^l d\sigma' e^{\frac{2\pi}{l}im\sigma'} \\ &\quad \times \left(\frac{nl}{2\pi i} \{X^\mu(\sigma), \dot{X}^\nu(\sigma')\} + \frac{ml}{2\pi i} \{\dot{X}^\mu(\sigma), X^\nu(\sigma')\} \right) \\ &= \frac{1}{2\alpha'} \int_0^l d\sigma e^{\frac{2\pi}{l}in\sigma} \int_0^l d\sigma' e^{\frac{2\pi}{l}im\sigma'} \frac{l}{2\pi T i} (n\eta^{\mu\nu} - m\eta^{\nu\mu}) \delta(\sigma - \sigma') \\ &= \frac{1}{2\alpha'} \int_0^l d\sigma e^{\frac{2\pi}{l}in\sigma} e^{\frac{2\pi}{l}im\sigma} \frac{l}{2\pi T i} (n\eta^{\mu\nu} - m\eta^{\nu\mu}) \\ &= -in\delta_{m+n,0}\eta^{\mu\nu} \end{aligned} \quad (2.4)$$

The calculation is essentially the same for α_m^μ , the signs in the exponential cancel with each other.

$$\{a_n^\mu, a_m^\nu\} = -in\delta_{n+m,0}\eta^{\mu\nu} \quad (2.5)$$

Similarly

$$\begin{aligned} \{\alpha_n^\mu, \tilde{\alpha}_m^\nu\} &= \frac{1}{2\alpha'} \int_0^l d\sigma e^{\frac{2\pi}{l}in\sigma} e^{-\frac{2\pi}{l}im\sigma} \frac{l}{2\pi T i} (n\eta^{\mu\nu} - m\eta^{\nu\mu}) \\ &= 0 \end{aligned} \quad (2.6)$$

For $m = 0$ in equation (2.1),

$$\begin{aligned} \int_0^l d\sigma X^\mu(\sigma, 0) &= l(x^\mu - c^\mu) \\ \int_0^l d\sigma \dot{X}^\mu(\sigma, 0) &= 2\pi\alpha' p^\mu \end{aligned}$$

Set $c^\mu = 0$ (or Poisson brackets with c^μ vanish)

$$\{x^\mu, x^\nu\} = \frac{1}{l^2} \int_0^l d\sigma \int_0^l d\sigma' \{X^\mu(\sigma), X^\nu(\sigma')\} = 0 \quad (2.7)$$

$$\{p^\mu, p^\nu\} = \frac{1}{4\pi^2\alpha'^2} \int_0^l d\sigma \int_0^l d\sigma' \{\dot{X}^\mu(\sigma), \dot{X}^\nu(\sigma')\} = 0 \quad (2.8)$$

$$\begin{aligned} \{x^\mu, p^\nu\} &= \frac{1}{2l\pi\alpha'} \int_0^l d\sigma \int_0^l d\sigma' \{X^\mu(\sigma), \dot{X}^\nu(\sigma')\} \\ &= \frac{1}{2l\pi\alpha'} \int_0^l d\sigma \int_0^l d\sigma' \frac{1}{T} \eta^{\mu\nu} \delta(\sigma - \sigma') \\ &= \eta^{\mu\nu} \end{aligned} \quad (2.9)$$

3 A spaghetti stick as solution to the string equations of motion

We have

$$\begin{aligned} X^0 &= A\tau, \\ X^1 &= A \cos \tau \cos \sigma, \\ X^2 &= A \sin \tau \cos \sigma, \\ X^i &= 0 \text{ for } i = 3, \dots, d-1 \end{aligned} \quad (3.1)$$

(a) To verify it is indeed a solution,

$$\begin{aligned}(\partial_\tau^2 - \partial_\sigma^2)X^1 &= (-1 + 1)X^1 = 0 \\(\partial_\tau^2 - \partial_\sigma^2)X^2 &= (-1 + 1)X^2 = 0\end{aligned}$$

And it is trivial for all other dimensions.

(b) X^0 has Neumann boundary conditions, as the σ -derivative vanishes identically. X^1 and X^2 have Neumann conditions again, since after differentiation they both proportional to $\sin \sigma$ which vanishes for $\sigma = 0$ and $\sigma = l$ if $l \in \pi\mathbb{N}$.

(c) The energy

$$\begin{aligned}M = P^0 &= \frac{1}{2\pi\alpha'} \int_0^l d\sigma \partial_\tau X^0 \\&= \frac{Al}{2\pi\alpha'}\end{aligned}$$

(d) Angular momentum

$$\begin{aligned}J = |J^{12}| &= \frac{1}{2\pi\alpha'} \int_0^l d\sigma \left(X^1 \dot{X}^2 - X^2 \dot{X}^1 \right) \\&= \frac{1}{2\pi\alpha'} A^2 \int_0^l d\sigma \left(\cos^2 \tau \cos^2 \sigma + \sin^2 \tau \cos^2 \sigma \right) \\&= \frac{A^2}{4\pi\alpha'} [\sin \sigma \cos \sigma + \sigma]_0^l \\&= \frac{A^2}{4\pi\alpha'} l\end{aligned}$$

In the last step, the Neumann boundary condition is used.

(e)

$$\frac{J}{M^2} = \frac{A^2 l}{4\pi\alpha'} \frac{(2\pi\alpha')^2}{A^2 l^2} = \frac{\pi\alpha'}{l}$$