

# Superstring theory

## Homework 1

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### 1 Symmetries of the Polyakov action

The Polyakov action is given by

$$S_p = -\frac{T}{2} \int d^2\sigma \sqrt{-h} (h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}) \quad (1.1)$$

(a) Polyakov action is Poincaré-invariant.

Poincaré transformation is given by

$$X^\mu \mapsto \Lambda^\mu_\nu X^\nu + b^\mu$$

The derivative in the action transforms like

$$\partial_\alpha X^\mu \mapsto \Lambda^\mu_\nu (\partial_\alpha X^\nu)$$

since the shift  $b^\mu$  doesn't depend on space-time coordinates.

The intrinsic metric  $h_{\alpha\beta} = h_{\alpha\beta}(\sigma, \tau)$  doesn't transform. Thus the only "changable" terms in the action are

$$\begin{aligned} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} &\mapsto \Lambda^\mu_\rho (\partial_\alpha X^\rho) \Lambda^\nu_\sigma (\partial_\beta X^\sigma) \eta_{\mu\nu} \\ &= \Lambda^\mu_\rho \Lambda^\nu_\sigma \eta_{\mu\nu} (\partial_\alpha X^\rho) (\partial_\beta X^\sigma) \\ &= \eta_{\rho\sigma} (\partial_\alpha X^\rho) (\partial_\beta X^\sigma) \end{aligned} \quad (1.2)$$

where the defining property of Lorentz transformation is used:  $\eta = \Lambda^T \eta \Lambda$ . Then the action is indeed invariant.

(b) The infinitesimal Poincaré transformations are

$$X^\mu \mapsto X^\mu + \epsilon a^\mu_\nu X^\nu, \quad X^\mu \mapsto X^\mu + \epsilon^\mu$$

As hinted on the sheet,  $h_{\alpha\beta} = \eta_{\alpha\beta}$  can be used and thus  $\sqrt{-h} = 1$ .

- With respect to the first transformation

$$\begin{aligned}
(\epsilon a^\mu_\nu) j^\alpha_\mu{}^\nu &= \frac{\partial L}{\partial(\partial_\alpha X^a)} \delta X^a \\
&= -\frac{T}{2} \eta^{\rho\sigma} \left( \frac{\partial}{\partial(\partial_\alpha X^a)} \partial_\rho X^\beta \partial_\sigma X^\gamma \right) \eta_{\beta\gamma} \delta X^a \\
&= -T \partial^\alpha X_a \delta X^a \\
&= -T \partial^\alpha X_a \epsilon a^a{}_\beta X^\beta
\end{aligned}$$

Since tensor  $a_{\mu\nu}$  is totally anti-symmetric, there are two ways to "cancel" the  $a$ 's on both sides. In the end, we have

$$\begin{aligned}
j^\alpha_\mu{}^\nu &= -T \partial^\alpha X_a X^\beta (-\delta_\mu^a \delta_\beta^\nu + \eta_{\mu\beta} \eta^{a\nu}) \\
&= -T (-X^\nu \partial^\alpha X_\mu + X_\mu \partial^\alpha X^\nu)
\end{aligned} \tag{1.3}$$

The sign arises by exchanging the order of indices of  $a$ . One can properly do this by multiplying  $a^{-1}$  to both sides and keep  $g^{ik} g_{kj} = \delta_j^i$ .

- With respect to the second transformation

$$\begin{aligned}
\epsilon^\mu j_\mu^\alpha &= -T \partial^\alpha X_a \delta X^a \\
&= -T \partial^\alpha X_a \epsilon^a \\
j_\mu^\alpha &= -T \partial^\alpha X_\mu
\end{aligned} \tag{1.4}$$

- (c) The action is invariant under reparametrization transformation  $\sigma^\alpha \mapsto \sigma'^\alpha(\sigma^\beta)$ .  $X^\mu$  is a world-sheet scalar (thus  $\partial_\mu = \nabla_\mu$  for  $X^\nu$ ),  $h_{\alpha\beta}$  is a tensor.

$$\begin{aligned}
X^\mu(\sigma^\alpha) &\mapsto X^\mu(\sigma'^\alpha) \\
\Gamma_{\alpha\beta} &\mapsto \Gamma_{\mu\nu} \frac{\partial \sigma^\mu}{\partial \sigma'^\alpha} \frac{\partial \sigma^\nu}{\partial \sigma'^\beta} \\
h_{\alpha\beta} &\mapsto h_{\mu\nu} \frac{\partial \sigma^\mu}{\partial \sigma'^\alpha} \frac{\partial \sigma^\nu}{\partial \sigma'^\beta} \\
d^2 \sigma \sqrt{-h} &\mapsto d^2 \sigma' \frac{\partial \sigma'^\alpha}{\partial \sigma^\beta} \frac{\partial \sigma'^\beta}{\partial \sigma^\alpha} \sqrt{-h} \det \left( \frac{\partial \sigma}{\partial \sigma'} \right) = d^2 \sigma' \sqrt{-h}
\end{aligned}$$

The last expression is the usual invariant measure for arbitrary continuous coordinate transformation (diffeomorphism).

Plug all these transformation in and we have

$$S_p \mapsto S'_p = -\frac{T}{2} \int d^2 \sigma' \sqrt{-h} h^{\alpha\beta} \partial'_\alpha X^\nu \partial'_\beta X^\mu \eta_{\mu\nu} = S_p \tag{1.5}$$

- (d) The action is invariant under Weyl transformation  $h_{\alpha\beta} \mapsto e^{\phi(\sigma^\alpha)} h_{\alpha\beta}$  (I think there is no summing over  $\alpha$  here). This is quite simple. Transformation of  $\sqrt{-h}$  introduce  $e^\phi$  and since it lies inside square root and determinant:  $\sqrt{-h} \mapsto e^\phi \sqrt{-h}$ . This factor gets cancelled by the transformation of  $h^{-1}$ . Thus the action is again invariant.
- (e) Because of the reparametrization symmetry, we can transform our metric into certain form. Since there are three independent entries of the metric and the reparametrization contains two equations, we are left with

$$h_{\alpha\beta} = \Omega^2(\sigma, \tau) \eta_{\alpha\beta} \quad (1.6)$$

Then with the Weyl transformation, we can make the  $h_{\alpha\beta}$  locally flat.

This cannot be done to all points on the world sheet at the same time (i.e. globally). Since these transformations are diffeomorphism, one can visualize them as stretching a curved two-dimensional plane. It is always possible to make the plane so that at one point, it resembles flat space. But if now one looks at somewhere else, it is not flat anymore, since the plane in the beginning can have different structure at different points.

## 2 Two-dimensional gravity

To show that in two dimension the energy momentum tensor vanishes identically, one needs to show  $T_{\mu\nu} \propto R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0$ .

In two dimension, Riemann tensor  $R_{\mu\nu\alpha\beta}$  can only have one non-zero component because of its symmetries. Its skew symmetry makes the Riemann tensor to be zero, if its indices are not combination of two 0s and two 1s. Reason is simple: otherwise one could use the skew symmetry, so that  $R_{\mu\nu\alpha\beta} = -R_{\mu\nu\beta\alpha} = 0$ . As a matter of fact, the only non-vanishing components are  $R_{0101} = -R_{1001} = -R_{0110} = R_{1010}$ , i.e. first and second pair indices must contain 0 and 1. Then the Ricci tensor is

$$\begin{aligned} R_{\mu\nu} &= g^{\alpha\beta} R_{\alpha\mu\beta\nu} \\ &= g^{00} R_{0\mu 0\nu} + g^{11} R_{1\mu 1\nu} + g^{01} R_{0\mu 1\nu} + g^{10} R_{1\mu 0\nu} \\ &= \begin{pmatrix} g^{11} & -g^{01} \\ -g^{01} & g^{00} \end{pmatrix} R_{0101} \\ &= g g_{\mu\nu} R_{0101} \end{aligned}$$

Here we express the metric with its inverse metric

$$g_{\mu\nu} = \frac{1}{\det g^{\mu\nu}} \begin{pmatrix} g^{11} & -g^{01} \\ -g^{01} & g^{00} \end{pmatrix}$$

The Ricci scalar is then

$$R = g^{\mu\nu} R_{\mu\nu} = 2gR_{0101}$$

Thus

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = gg_{\mu\nu}R_{0101} - gg_{\mu\nu}R_{0101} = 0 \quad (2.1)$$

Then the energy momentm tensor vanishes identically.