Superstring theory
Homework 9

Aristotelis Koutsikos, Chenhuan Wang and Mohamed Ghoneim

February 10, 2021

1 Operator Product Expansions

(a) We have two generic operators

$$A = \oint_{C_0} \frac{\mathrm{d}z}{2\pi i} A(z), \quad B = \oint_{C_0} \frac{\mathrm{d}z}{2\pi i} B(z) \tag{1.1}$$

with $C_0 = \{z \in \mathbb{C} | |z| < r\}$ and r is an arbitrary real number.

We want to expand the radial ordering term, thus need contour with unambiguous radii

$$\begin{aligned}
&\oint_{C_\omega} \frac{\mathrm{d}z}{2\pi i} \mathcal{R}(A(z)B(w)) \\
&= \left(\oint_{C_w^+} \frac{\mathrm{d}z}{2\pi i} - \oint_{C_w^-} \frac{\mathrm{d}z}{2\pi i} \right) \mathcal{R}(A(z)B(w)) \\
&= \oint_{C_w^+} \frac{\mathrm{d}z}{2\pi i} A(z)B(w) - \oint_{C_w^-} \frac{\mathrm{d}z}{2\pi i} B(w)A(z) \\
&= \oint_{C_w} \frac{\mathrm{d}z}{2\pi i} [A(z), B(w)]_{|z|=|w|}
\end{aligned} \tag{1.2}$$

where the contours are defined as $C_w^{\pm} = \{z \in \mathbb{C} | |z| = |w| \pm \delta, \delta \in \mathbb{R}\}$ and the commutator is understood as equal radius commutator. One can use the definition of the operators to "absorb" the integral on LHS and add another integral, then

$$[A, B] = \oint_{C_0} \frac{\mathrm{d}w}{2\pi i} \oint_{C_w} \frac{\mathrm{d}z}{2\pi i} \mathcal{R}(A(z)B(w)) \tag{1.3}$$

(b) The variation of a primary under infinitesimal conformal transformation is describe as

$$\begin{aligned}
\delta_\epsilon \phi(\omega) &= -[Q_\epsilon, \phi(w)] \\
&= -\oint_{C_0} \frac{\mathrm{d}z}{2\pi i}[\epsilon(z)T(z), \phi(w)] \\
&= -\oint_{C_w} \frac{\mathrm{d}z}{2\pi i}\epsilon(z)\mathcal{R}(T(z)\phi(w))
\end{aligned}$$

if we put the desired expression of $\mathcal{R}(T(z)\phi(w))$ in

$$\begin{aligned}
&= -\oint_{C_w} \frac{\mathrm{d}z}{2\pi i}\epsilon(z)\left[\frac{h\phi(w)}{(z-w)^2} + \frac{\partial_w \phi(w)}{z-w} + \text{reg.}\right] \\
&= -\partial_w(\epsilon(z)h\phi(w)) - \partial_w \phi(w) \\
&= -(h\partial_w \epsilon(w) + \partial_w)\phi(w)
\end{aligned}$$

which is precisely the alternative definition of the infinitesimal transformation

$$\delta_\epsilon \phi(w) = -(h\partial_w \epsilon + \epsilon \partial_w)\phi(w) \tag{1.4}$$

(c) Consider the Laurent series

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2}L_n, \quad L_n = \oint_{C_0} \frac{\mathrm{d}z}{2\pi i} z^{n+1}T(z) \tag{1.5}$$

We assume the statement is correct (with implicit radial ordering)

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)} + \text{reg.} \tag{1.6}$$

Then

$$\begin{aligned}
&[L_m, L_n] \\
&= \oint_{C_0} \frac{\mathrm{d}z}{2\pi i} z^{n+1} \oint_{C_0} \frac{\mathrm{d}w}{2\pi i} w^{m+1}[T(z), T(w)] \\
&\overset{(1.2)}{=} \oint_{C_w} \frac{\mathrm{d}z}{2\pi i} z^{n+1} \oint_{C_0} \frac{\mathrm{d}w}{2\pi i} w^{m+1} T(z)T(w) \\
&\overset{(1.6)}{=} \oint_{C_0} \frac{\mathrm{d}w}{2\pi i} w^{m+1} \oint_{C_w} \frac{\mathrm{d}z}{2\pi i} z^{n+1}\left[\frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)} + \text{reg.}\right] \\
&= \oint_{C_0} \frac{\mathrm{d}w}{2\pi i} w^{m+1}\left[\frac{c}{2}\frac{1}{3!}(n+1)n(n-1)w^{n-2} + 2(n+1)w^n T(w) + w^{n+1}\partial_w T(w)\right] \\
&\overset{\text{i.b.p.}}{=} \oint_{C_0} \frac{\mathrm{d}w}{2\pi i}\left[\frac{c}{12}n(n^2-1)w^{n+m-1} + 2(n+1)w^{m+n+1}T(w) - (m+n+2)w^{m+n+1}T(w)\right] \\
&= \frac{c}{12}n(n^2-1)\delta_{n+m} + (n-m)L_{m+n}
\end{aligned}$$

In principle, we could and should prove it in the other direction.

2 The propagator of the free boson

(a) The propagator is given by

$$K(\boldsymbol{x}, \boldsymbol{y}) = \langle \phi(\boldsymbol{x}) \phi(\boldsymbol{y}) \rangle \tag{2.1}$$

It obeys

$$g(-\partial_x^2 + m^2) K(\boldsymbol{x}, \boldsymbol{y}) = \delta(\boldsymbol{x} - \boldsymbol{y}) \tag{2.2}$$

The propagator is only a function of $r = |\boldsymbol{x} - \boldsymbol{y}|$, since one can check if $K(\boldsymbol{x}, \boldsymbol{y})$ satisfies (2.2), then $K(\boldsymbol{x} + \boldsymbol{a}, \boldsymbol{y} + \boldsymbol{a})$ also. Thus these two are equivalent. And more importantly, the action has translational invariance.

Because of this property, we can ignore y in the following calculation. To integrate (2.2) over a disk of radius R with $m = 0$

$$-g \int_{C_R} \partial_x^2 K(r) = 1$$

in spherical coordinates,

$$-2\pi g \int_0^R \mathrm{d}r \, \partial_r (r \partial_r K(r)) = 1$$

It is satisfied with $K(r) = -\frac{1}{2\pi g} \log r + \text{const}$. To see this,

$$\begin{aligned}
&\int_0^R \mathrm{d}r \, \partial_r (r \partial_r \log r) \\
&= \lim_{a \to 0} \int_0^R \mathrm{d}r \, \partial_r (r \partial_r \log(r + a)) \\
&= \lim_{a \to 0} \int_0^R \mathrm{d}r \, \frac{a}{(r + a)^2} \\
&= \lim_{a \to 0} a \cdot \left. \frac{-1}{r + a} \right|_{r=0}^{R} \\
&= \lim_{a \to 0} a \left(-\frac{1}{R + a} + \frac{1}{a} \right) \\
&= 1
\end{aligned}$$

(b) The computation in the last part leads to the OPE

$$\partial_z\phi(z,\bar{z})\partial_w\phi(w,\bar{w}) \sim -\frac{1}{4\pi g}\frac{1}{(z-w)^2} \tag{2.3}$$

We want to compute the OPE of $T(z) = -2\pi g : \partial\phi\partial\phi :$

$$T(z)V_\alpha(w,\bar{w}) = -2\pi g \sum_{n=0}^{\infty} \frac{(2\pi i\alpha)^n}{n!} : \partial_z\phi(z)\partial_z\phi(z) :: \phi^n(w) :$$

Obviouly OPE works similar like in QFT, we need to "contract" fields

$$\begin{aligned}
&= -2\pi g \sum_{n=0}^{\infty} \frac{(2\pi i\alpha)^n}{n!} \big[n\partial_z(\partial\phi(z)\phi(w))\phi^{n-1}(w) \\
&\quad + n(n-1)\phi^{n-2}(w)\left\langle \partial_z\phi(z)\partial_z\phi(z)\right\rangle^2 \phi(w)\big] \\
&= -2\pi g \sum_{n=1}^{\infty} \frac{(2\pi i\alpha)^{n-1}}{(n-1)!}\phi^{n-1}\frac{-2\partial_z\phi(z)}{4\pi g(z-w)}2\pi i\alpha \\
&\quad + -2\pi g \sum_{n=2}^{\infty} \frac{(2\pi i\alpha)^{n-2}}{(n-2)!}\phi^{n-2}(w)\left(\frac{-1}{4\pi g(z-w)}\right)^2 (2\pi i\alpha)^2 \\
&= 2\pi i\alpha\frac{\partial_z\phi(z)V(w)}{z-w} + \frac{\pi\alpha^2}{2g}\frac{V(w)}{(z-w)^2} \\
&= \frac{\partial_w V(w)}{z-w} + \frac{\pi\alpha^2}{2g}\frac{V(w)}{(z-w)^2}
\end{aligned}$$

Thus $h = \frac{\pi\alpha^2}{2g}$ and from OPE with $\bar{T}(\bar{z})$ we find $\bar{h} = \frac{\pi\alpha^2}{2g}$.