Superstring theory
Homework 7

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1 Lorentz invariance in light-cone quantization: cont

(i) Now to expand the commutator

$$\left[J^{i-}, J^{j-}\right] = \left[l^{i-}, l^{j-}\right] + \left[E^{i-}, l^{j-}\right] + \left[l^{i-}, E^{j-}\right] + \left[E^{i-}, E^{j-}\right]$$

- First commutator vanishes

$$\begin{aligned}\left[l^{i-}, l^{j-}\right] &= \left[x^i p^-, x^j p^-\right] - \left[x^i p^-, x^- p^j\right] - \left[x^- p^i, x^j p^-\right] + \left[x^- p^i, x^- p^j\right] \\ &= 0 - x^-\left[x^i, p^j\right]p^- - x^-\left[p^i, x^j\right]p^- + 0 \\ &= 0\end{aligned}$$

- Second commutator is

$$\begin{aligned}\left[E^{i-}, l^{j-}\right] &= \frac{1}{p^+}\left(\left[E^i, x^j p^-\right] - \left[E^i, x^- p^j\right]\right) \\ &= \frac{1}{p^+}\left(\left[E^i, x^j\right]p^- + x^j\left[E^i, p^-\right] - (- \leftrightarrow j)\right) \\ &= \frac{1}{(p^+)^2}\left(iE^{ji}p^- p^+ + p^+ x^j \cancel{\left[E^i, p^-\right]} - iE^{-i}p^j p^+ + p^+ x^- \cancel{\left[E^i, p^j\right]}\right)\end{aligned}$$

  The commutators in the last step vanish because $p^\mu = \alpha_0^\mu$ and $E^i = p^+ E^{i-}$ contains only zero mode p^+ and it clearly commute with p^- and p^j.

- Third term is the same computation as the second one

$$\left[l^{i-}, E^{j-}\right] = -\frac{1}{(p^+)^2}\left(iE^{ij}p^- p^+ - iE^{-j}p^i p^+\right)$$

- Last term

$$\left[E^{i-}, E^{j-}\right] = \frac{1}{(p^+)^2}\left[E^i, E^j\right]$$

Together

$$\left[J^{i-}, J^{j-}\right] = -\frac{1}{(p^+)^2}\left(2ip^+p^-E^{ij} - iE^ip^j + iE^jp^i - \left[E^i, E^j\right]\right) = -\frac{1}{(p^+)^2}C^{ij} \tag{1.1}$$

where anti-symmetric property of E^{ij} has been used.

(j) We expect this commutator to be in the form of

$$\left[J^{i-}, J^{j-}\right] = -\frac{1}{(p^+)^2}\sum_{m=1}^{\infty}\Delta_m\left(\alpha^i_{-m}\alpha^j_m - \alpha^j_{-m}\alpha^i_m\right) \tag{1.2}$$

Thus C^{ij} is basically the sum. In other word with matrix element of C^{ij}, the coefficients Δ_m can be determined.

Explicitly,

$$\begin{aligned}
&\langle 0|\alpha^k_m\left[J^{i-}, J^{j-}\right]\alpha^l_{-m}|0\rangle \\
=& -\frac{1}{(p^+)^2}\sum_{n=1}^{\infty}\Delta_n\langle 0|\alpha^k_m\left(\alpha^i_{-n}\alpha^j_n - \alpha^j_{-n}\alpha^i_n\right)\alpha^l_{-m}|0\rangle \\
=& -\frac{1}{(p^+)^2}\sum_{n=1}^{\infty}\Delta_n\left(\langle 0|\alpha^k_m\alpha^i_{-n}\alpha^j_n\alpha^l_{-m}|0\rangle - \langle 0|\alpha^k_m\alpha^j_{-n}\alpha^i_n\alpha^l_m|0\rangle\right) \\
=& -\frac{1}{(p^+)^2}\Delta_m m^2\left(\delta^{ki}\delta^{jl} - \delta^{kj}\delta^{il}\right)
\end{aligned} \tag{1.3}$$

(k) We first notice that C^{ij} is anti-symmetric in indices ij. Thus first write

$$C^{ij} = 2p^+p^-\sum_{r=1}^{\infty}\frac{1}{r}\alpha^i_{-r}\alpha^j_r - (p^+)^2E^{i-}E^{j-} - iE^ip^j - (i \leftrightarrow j)$$

Now we are going to compute the matrix element of first three terms. The strategy is to use the commutation relations between modes so that we only have Kronecker deltas afterwards. To use is $\alpha^\mu_m|0\rangle = 0$ for $m > 0$ and its hermitian conjugate. Also $m > 0$.

- Firstly,

$$
\begin{aligned}
&2p^{+}\sum_{r>0}\frac{1}{r}\left\langle 0\right|\alpha^{k}_{m}\alpha^{-}_{0}\alpha^{i}_{-r}\alpha^{j}_{r}\alpha^{l}_{-m}\left|0\right\rangle\\
&=2p^{+}\sum_{r>0}\frac{1}{r}\left\langle 0\right|\left[\alpha^{k}_{m},\alpha^{-}_{0}\right]\alpha^{i}_{-r}\left[\alpha^{j}_{r},\alpha^{l}_{-m}\right]\left|0\right\rangle\\
&=2m\left\langle 0\right|\alpha^{k}_{m}\alpha^{i}_{-m}\left|0\right\rangle\delta^{jl}\\
&=2m\left\langle 0\right|\left[\alpha^{k}_{m},\alpha^{i}_{-m}\right]\left|0\right\rangle\delta^{jl}\\
&=2m^{2}\left\langle 0\right|\delta^{ki}\delta^{jl}\left|0\right\rangle
\end{aligned}
$$

- Second chunk of terms

$$
\begin{aligned}
&\left\langle 0\right|\alpha^{k}_{m}E^{i-}E^{j-}\alpha^{l}_{-m}\left|0\right\rangle\\
&=-\sum_{r,s>0}^{\infty}\frac{1}{rs}\left\langle 0\right|\alpha^{k}_{m}(\alpha^{i}_{-r}\alpha^{-}_{r}-\alpha^{-}_{-r}\alpha^{i}_{r})(\alpha^{j}_{-s}\alpha^{-}_{s}-\alpha^{-}_{-s}\alpha^{j}_{s})\alpha^{l}_{-m}\left|0\right\rangle\\
&=-\sum_{r,s>0}^{\infty}\frac{1}{rs}\left\langle 0\right|\alpha^{k}_{m}(\alpha^{i}_{-r}\alpha^{-}_{r}\alpha^{j}_{-s}\alpha^{-}_{s}-\alpha^{i}_{-r}\alpha^{-}_{r}\alpha^{-}_{-s}\alpha^{j}_{s}\\
&\quad-\alpha^{-}_{-r}\alpha^{i}_{r}\alpha^{j}_{-s}\alpha^{-}_{s}+\alpha^{-}_{-r}\alpha^{i}_{r}\alpha^{-}_{-s}\alpha^{j}_{s})\alpha^{l}_{-m}\left|0\right\rangle
\end{aligned}
$$

The first term

$$
\begin{aligned}
&-\sum_{r,s>0}^{\infty}\frac{1}{rs}\left\langle 0\right|\alpha^{k}_{m}\alpha^{i}_{-r}\alpha^{-}_{r}\alpha^{j}_{-s}\alpha^{-}_{s}\alpha^{l}_{-m}\left|0\right\rangle\\
&=-\sum_{r,s>0}^{\infty}\frac{1}{rs}\left\langle 0\right|\left[\alpha^{k}_{m},\alpha^{i}_{-r}\right]\alpha^{-}_{r}\alpha^{j}_{-s}\left[\alpha^{-}_{s},\alpha^{l}_{-m}\right]\left|0\right\rangle\\
&=-\frac{1}{p^{+}}\sum_{r,s>0}^{\infty}\frac{1}{rs}m\delta_{m-r}\delta^{ki}\left\langle 0\right|\alpha^{-}_{r}\alpha^{j}_{-s}m\alpha^{l}_{s-m}\left|0\right\rangle\\
&=-\frac{m}{p^{+}}\sum_{r>0}^{\infty}\frac{1}{r}\delta^{ki}\left\langle 0\right|\alpha^{-}_{m}\alpha^{j}_{-s}\alpha^{l}_{s-m}\left|0\right\rangle
\end{aligned}
$$

Second term

$$
\begin{aligned}
&\sum_{r,s>0}^{\infty} \frac{1}{rs} \langle 0| \alpha_m^k \alpha_{-r}^i \alpha_r^- \alpha_{-s}^- \alpha_s^j \alpha_{-m}^l |0\rangle \\
&= \sum_{r,s>0}^{\infty} \frac{1}{rs} \langle 0| \left[\alpha_m^k, \alpha_{-r}^i\right] \alpha_r^- \alpha_{-s}^- \left[\alpha_s^j, \alpha_{-m}^l\right] |0\rangle \\
&= \sum_{r,s>0}^{\infty} \frac{1}{rs} \langle 0| m\delta_{m-r}\delta^{ki} \alpha_r^- \alpha_{-s}^- s\delta_{s-m}\delta^{jl} |0\rangle \\
&= \delta^{ik}\delta^{jl} \langle 0| \alpha_m^- \alpha_{-m}^- |0\rangle
\end{aligned}
$$

Third term

$$
\begin{aligned}
&-\sum_{r,s>0}^{\infty} \frac{1}{rs} \langle 0| \alpha_m^k \alpha_{-r}^- \alpha_r^i \alpha_{-s}^j \alpha_s^- \alpha_{-m}^l |0\rangle \\
&= -\sum_{r,s>0}^{\infty} \frac{1}{rs} \langle 0| \left[\alpha_m^k, \alpha_{-r}^-\right] \alpha_r^i \alpha_{-s}^j \left[\alpha_s^-, \alpha_{-m}^l\right] |0\rangle \\
&= -\frac{m^2}{(p^+)^2} \sum_{r,s>0}^{\infty} \frac{1}{rs} \langle 0| \alpha_{m-r}^k \alpha_r^i \alpha_{-s}^j \alpha_{s-m}^l |0\rangle
\end{aligned}
$$

Last term is similar as the first term

$$
\begin{aligned}
&-\sum_{r,s>0}^{\infty} \frac{1}{rs} \langle 0| \alpha_m^k \alpha_{-r}^- \alpha_r^i \alpha_{-s}^- \alpha_s^j \alpha_{-m}^l |0\rangle \\
&= -\frac{m}{p^+} \sum_{r>0}^{\infty} \frac{1}{r} \delta^{jl} \langle 0| \alpha_{m-s}^k \alpha_s^i \alpha_{-m}^- |0\rangle
\end{aligned}
$$

All the sum here actually terminate at $r, s = m$, because then we would have annihilation operators acting on the vacuum.

- Lastly,

$$
\begin{aligned}
&-p^+\sum_{r>0}^{\infty}\frac{1}{r}\langle 0|\alpha^k_m(\alpha^i_{-r}\alpha^-_r-\alpha^-_{-r}\alpha^i_r)\alpha^j_0\alpha^l_{-m}|0\rangle\\
&=-p^+\sum_{r>0}^{\infty}\frac{1}{r}\left(\langle 0|\left[\alpha^k_m,\alpha^i_{-r}\right]\alpha^-_r\alpha^j_0\alpha^l_{-m}|0\rangle-\langle 0|\alpha^k_m\alpha^-_{-r}\alpha^j_0\left[\alpha^i_r,\alpha^l_{-m}\right]|0\rangle\right)\\
&=-p^+\sum_{r>0}^{\infty}\frac{1}{r}\left(m\delta_{m-r}\delta^{ki}\langle 0|\alpha^-_r\alpha^l_{-m}\alpha^j_0|0\rangle-r\delta_{r-m}\delta^{il}\langle 0|\alpha^k_m\alpha^-_{-r}\alpha^j_0|0\rangle\right)\\
&=-p^+\sum_{r>0}^{\infty}\frac{1}{r}\left(m\delta_{m-r}\delta^{ki}\langle 0|\left[\alpha^-_r,\alpha^l_{-m}\right]\alpha^j_0|0\rangle-r\delta_{r-m}\delta^{il}\langle 0|\left[\alpha^k_m,\alpha^-_{-r}\right]\alpha^j_0|0\rangle\right)\\
&=-p^+\sum_{r>0}^{\infty}\frac{1}{r}\left(\frac{m^2}{p^+}\delta_{m-r}\delta^{ki}\langle 0|\alpha^l_{r-m}\alpha^j_0|0\rangle-\frac{mr}{p^+}\delta_{r-m}\delta^{il}\langle 0|\alpha^k_{m-r}\alpha^j_0|0\rangle\right)\\
&=-m\langle 0|\left(\delta^{ki}p^lp^j-\delta^{il}p^kp^j\right)|0\rangle
\end{aligned}
$$

Put everything together

$$
\begin{aligned}
\langle 0|\alpha^k_m C^{ij}\alpha^l_{-m}|0\rangle=&\langle 0|\left(2m^2\delta^{ik}\delta^{jl}+mp^jp^k\delta^{il}-mp^jp^l\delta^{ik}\right)|0\rangle\\
&+p^+m\delta^{ik}\sum_{s=1}^{m}\frac{1}{s}\langle 0|\alpha^-_m\alpha^j_{-s}\alpha^l_{s-m}|0\rangle-(p^+)^2\delta^{ik}\delta^{jl}\langle 0|\alpha^-_m\alpha^-_{-m}|0\rangle\\
&+m^2\sum_{r,s=1}^{m}\frac{1}{rs}\langle 0|\alpha^k_{m-s}\alpha^j_s\alpha^i_{-r}\alpha^l_{r-m}|0\rangle\\
&+p^+m\delta^{jl}\sum_{s=1}^{m}\frac{1}{s}\langle 0|\alpha^k_{m-s}\alpha^i_s\alpha^-_{-m}|0\rangle-(i\leftrightarrow j)
\end{aligned}
\tag{1.4}
$$

(l) Now to compute these matrix elements

(i) Since $m>0$, we can leave out the normal ordering constant in the α^-_n (remember the commutation relations from last sheet)

$$
\begin{aligned}
&(p^+)^2\langle 0|\alpha^-_m\alpha^-_{-m}|0\rangle\\
&=\langle 0|\left[p^+\alpha^-_m,p^+\alpha^-_{-m}\right]|0\rangle\\
&=\frac{d-2}{12}m(m^2-1)+2am
\end{aligned}
$$

where we have used $\left\langle\alpha^-_0\right\rangle=\langle p^-\rangle=0$, since it is the vacuum.

(ii)

$$
\begin{aligned}
&p^+\sum_{s=1}^{m}\frac{1}{s}\left\langle 0\middle|\alpha^-_m\alpha^j_{-s}\alpha^l_{s-m}\middle|0\right\rangle\\
=&p^+\sum_{s=1}^{m}\frac{1}{s}\left\langle 0\middle|\left[\alpha^-_m,\alpha^j_{-s}\right]\alpha^l_{s-m}\middle|0\right\rangle\\
=&\sum_{s=1}^{m}\left\langle 0\middle|\alpha^j_{m-s}\alpha^l_{s-m}\middle|0\right\rangle\\
=&\sum_{s=1}^{m-1}(m-s)\delta^{jl}+p^jp^l\\
=&\delta^{jl}m(m-1)/2+p^jp^l
\end{aligned}
$$

(iii) Identical computation as last one,

$$
p^+\sum_{s=1}^{m}\frac{1}{s}\left\langle 0\middle|\alpha^k_{m-s}\alpha^i_s\alpha^-_{-m}\middle|0\right\rangle=\delta^{ik}m(m-1)/2+p^ip^k \tag{1.5}
$$

(iv) Lastly

$$
\begin{aligned}
&\sum_{r,s=1}^{m}\frac{1}{rs}\left\langle 0\middle|\alpha^k_{m-s}\alpha^j_s\alpha^i_{-r}\alpha^l_{r-m}\middle|0\right\rangle-(i\leftrightarrow j)\\
=&\sum_{r,s=1}^{m}\frac{1}{rs}\left\langle 0\middle|\alpha^k_{m-s}\alpha^i_{-r}\alpha^j_s\alpha^l_{r-m}\middle|0\right\rangle-(i\leftrightarrow j)
\end{aligned}
$$

because the commutator term is $\propto\delta^{ij}$ and gets canceled by the $i\leftrightarrow j$ term

$$
=\sum_{r,s=1}^{m-1}\frac{1}{rs}\left\langle 0\middle|\alpha^k_{m-s}\alpha^i_{-r}\left[\alpha^j_s,\alpha^l_{r-m}\right]\middle|0\right\rangle-(i\leftrightarrow j)
$$

note that here the summation is only up to $(m-1)$, since if $r = m$ we would have a momentum operator acting on the vacuum. Thus the term vanishes.

$$
\begin{aligned}
&= \sum_{r,s=1}^{m-1} \frac{1}{rs} s\delta_{s+r-m}\delta^{jl} \langle 0|\alpha^k_{m-s}\alpha^i_{-r}|0\rangle - (i \leftrightarrow j) \\
&= \sum_{r=1}^{m-1} \frac{1}{r}\delta^{jl} \langle 0|\alpha^k_{r}\alpha^i_{-r}|0\rangle - (i \leftrightarrow j) \\
&= \sum_{r=1}^{m-1} \frac{1}{r}\delta^{jl} \langle 0|\alpha^k_{r}\alpha^i_{-r}|0\rangle - (i \leftrightarrow j) \\
&= \sum_{r=1}^{m-1} \delta^{jl}\delta^{ki} - (i \leftrightarrow j) \\
&=(m-1)(\delta^{jl}\delta^{ki} - \delta^{il}\delta^{kj})
\end{aligned}
$$

(j) We put everything together

$$
\begin{aligned}
\langle 0|\alpha^k_m C^{ij}\alpha^l_{-m}|0\rangle = \langle 0| &\left(2m^2\delta^{ik}\delta^{jl} + mp^jp^k\delta^{il} - mp^jp^l\delta^{ik}\right)|0\rangle \\
&+ m\delta^{ik}(p^jp^l + \delta^{jl}m(m-1)/2) - \delta^{ik}\delta^{jl}\left[\frac{d-2}{12}m(m^2-1) + 2am\right] \\
&- m^2(m-1)(\delta^{il}\delta^{jk}) \\
&+ m\delta^{jl}(p^ip^k + \delta^{ik}m(m-1)/2) - (i \leftrightarrow j)
\end{aligned}
\tag{1.6}
$$

Compare it with equation (1.3) (note that terms quadratic in momenta vanish once considering the $(i \leftrightarrow j)$ contributions)

$$
\begin{aligned}
\Delta_m m^2 &= 2m^2 + m^2(m-1) - \frac{d-2}{12}m(m^2-1) + 2am + m^2(m-1) \\
\Delta_m &= m\left(1 + \frac{d-2}{12} + 1\right) + \frac{1}{m}\left(\frac{d-2}{12} + 2a\right) \\
&= m\frac{26-d}{12} + \frac{1}{m}\left(\frac{d-26}{12} + 2(1-a)\right)
\end{aligned}
\tag{1.7}
$$

2 The ghost system

Ghost action

$$
S_g[h,b,c] = -\frac{i}{2\pi}\int \mathrm{d}^2\sigma\,\sqrt{-h}h^{\alpha\beta}b_{\beta\gamma}\nabla_\alpha c^\gamma \tag{2.1}
$$

Throughout this assignment, $h = \hat{h}$

(a) First consider the variation of the action S_g

$$\delta S_g = -\frac{i}{2\pi}\int \mathrm{d}^2\sigma \left(\delta\sqrt{-h}h^{\alpha\beta}b_{\beta\gamma}\nabla_\alpha c^\gamma + \sqrt{-h}\delta h^{\alpha\beta}b_{\beta\gamma}\nabla_\alpha c^\gamma + \sqrt{-h}h^{\alpha\beta}b_{\beta\gamma}\delta(\nabla_\alpha c^\gamma)\right)$$
$$= -\frac{i}{4\pi}\int \mathrm{d}^2\sigma\,\sqrt{-h}\left[\left(h^{\alpha\beta}h^{\mu\nu}\delta h_{\mu\nu} + \delta h^{\alpha\beta}\right)b_{\beta\gamma}\nabla_\alpha c^\gamma + \delta h^{\alpha\beta}b_{\alpha\gamma}\nabla_\beta c^\gamma\right]$$
$$-\frac{i}{2\pi}\int \mathrm{d}^2\sigma\,\sqrt{-h}h^{\alpha\beta}b_{\beta\gamma}\delta\Gamma^\gamma_{\mu\alpha}c^\mu$$

In the last step, we have used that variation in metric is symmetric to "split" the variation in metric.

Variation of Christoffel symbol is (here $g_{\mu\nu}$ is also the metric)

$$\delta\Gamma^\rho_{\mu\nu} = \frac{1}{2}\delta g^{\rho\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}) + \frac{1}{2}g^{\rho\sigma}(\partial_\mu\delta g_{\nu\sigma} + \partial_\nu\delta g_{\mu\sigma} - \partial_\sigma\delta g_{\mu\nu})$$
$$= \frac{1}{2}\delta g^{\rho\sigma}(\Gamma^\alpha_{\sigma\mu}g_{\nu\alpha} + \Gamma^\alpha_{\nu\mu}g_{\alpha\sigma} + \Gamma^\alpha_{\nu\mu}g_{\alpha\sigma} + \Gamma^\alpha_{\nu\sigma}g_{\mu\alpha} - \Gamma^\alpha_{\sigma\mu}g_{\alpha\nu} - \Gamma^\alpha_{\sigma\nu}g_{\mu\alpha})$$
$$+ \frac{1}{2}g^{\rho\sigma}(\partial_\mu\delta g_{\nu\sigma} + \partial_\nu\delta g_{\mu\sigma} - \partial_\sigma\delta g_{\mu\nu})$$
$$= \frac{1}{2}\delta g^{\rho\sigma}(\partial_\mu\delta g_{\nu\sigma} - \Gamma^\alpha_{\sigma\mu}\delta g_{\nu\alpha} - \Gamma^\alpha_{\nu\mu}\delta g_{\alpha\sigma} + \partial_\nu\delta g_{\mu\sigma} - \Gamma^\alpha_{\nu\mu}\delta g_{\alpha\sigma} + \Gamma^\alpha_{\nu\sigma}\delta g_{\mu\alpha}$$
$$- \partial_\sigma\delta g_{\mu\nu} + \Gamma^\alpha_{\sigma\mu}\delta g_{\alpha\nu} + \Gamma^\alpha_{\sigma\nu}\delta g_{\mu\alpha})$$
$$= \frac{1}{2}g^{\rho\sigma}(\nabla_\mu\delta g_{\nu\sigma} + \nabla_\nu\delta g_{\mu\sigma} - \nabla_\sigma\delta g_{\mu\nu})$$

where we have used the metric is covariantly constant and variation of Kronecker delta is zero

$$\nabla_\nu g_{\nu\lambda} = 0 \tag{2.2}$$
$$g_{\rho\lambda}\delta g^{\mu\rho} + g^{\mu\rho}\delta g_{\rho\lambda} = \delta(g^{\mu\rho}g_{\rho\lambda}) = 0 \tag{2.3}$$

Thus

$$-\frac{i}{2\pi}\int \mathrm{d}^2\sigma\,\sqrt{-h}h^{\alpha\beta}b_{\beta\gamma}\delta\Gamma^\gamma_{\mu\alpha}c^\mu$$
$$= -\frac{i}{4\pi}\int \mathrm{d}^2\sigma\,\sqrt{-h}b^{\alpha\rho}c^\mu(\nabla_\mu\delta h_{\alpha\rho} + \nabla_\alpha\delta h_{\mu\rho} - \nabla_\rho\delta h_{\mu\alpha})$$

The last two term cancel with each other if we relabel $\rho \leftrightarrow \alpha$ in one of these terms and use the fact that $b_{\alpha\beta}$ is symmetric. Now to integrate by parts

$$= \frac{i}{4\pi}\int \mathrm{d}^2\sigma\,\sqrt{-h}\nabla_\mu(b^{\alpha\rho}c^\mu)\delta h_{\alpha\rho}$$
$$= \frac{i}{4\pi}\int \mathrm{d}^2\sigma\,\sqrt{-h}\nabla_\mu(b_{\alpha\rho}c^\mu)\delta h^{\alpha\rho}$$

where equations (2.2) and (2.3) are used. Variation of the action is now

$$\delta S_g = -\frac{i}{4\pi}\int d^2\sigma\,\sqrt{-h}\left[\left(h^{\alpha\beta}h^{\mu\nu}\delta h_{\mu\nu} + \delta h^{\alpha\beta}\right) b_{\beta\gamma}\nabla_\alpha c^\gamma + \delta h^{\alpha\beta} b_{\alpha\gamma}\nabla_\beta c^\gamma - \frac{1}{2}\nabla_\mu (b_{\alpha\rho}c^\mu)\delta h^{\alpha\rho}\right]$$
$$\overset{2.3}{=} -\frac{i}{4\pi}\int d^2\sigma\,\sqrt{-h}\left[\left(-h^{\alpha\beta}h_{\mu\nu}\delta h^{\mu\nu} + \delta h^{\alpha\beta}\right) b_{\beta\gamma}\nabla_\alpha c^\gamma + \delta h^{\alpha\beta} b_{\alpha\gamma}\nabla_\beta c^\gamma - \frac{1}{2} b_{\alpha\rho}\nabla_\mu c^\mu \delta h^{\alpha\rho}\right]$$

In the last step, we have used $\delta(b_{\alpha\rho}h^{\alpha\rho}) = 0$ due to tracelessness and we don't consider $\delta b_{\mu\nu}$ here.

Thus the energy-momentum tensor is

$$\begin{aligned}
T_{\alpha\beta} &= \frac{4\pi}{\sqrt{-h}}\frac{\delta S_g}{\delta h^{\alpha\beta}} \\
&= -i\left(-h^{\mu\nu}h_{\alpha\beta}b_{\nu\gamma}\nabla_\mu c^\gamma + b_{\beta\gamma}\nabla_\alpha c^\gamma + b_{\alpha\gamma}\nabla_\beta c^\gamma - c^\mu\nabla_\mu b_{\alpha\beta}\right) \\
&= -i\left(+b_{\beta\gamma}\nabla_\alpha c^\gamma + b_{\alpha\gamma}\nabla_\beta c^\gamma - c^\mu\nabla_\mu b_{\alpha\beta} - h_{\alpha\beta}b_{\nu\gamma}\nabla^\nu c^\gamma\right) \qquad (2.4)
\end{aligned}$$

(b) Equations of motions are with respect to $b_{\alpha\beta}$

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial b_{\alpha\beta}} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu b_{\alpha\beta})} &= 0 \\
\nabla^\alpha c^\beta &= 0 \qquad (2.5)
\end{aligned}$$

and with respect to c_α

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial c_\alpha} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu c^\alpha)} &= 0 \\
\partial_\mu (h^{\mu\beta}b_{\beta\alpha}) &= 0 \qquad (2.6)
\end{aligned}$$

(c) Now we switch to flat metric: $h_{\alpha\beta} = \eta_{\alpha\beta}$. In worldsheet light-cone coordinates, the metric is

$$\eta^{+-} = \eta^{-+} = -2$$

Thus only diagonal entries of $b_{\alpha\beta}$ is non-vanishing, since

$$\begin{aligned}
\eta^{\alpha\beta}b_{\alpha\beta} &= \eta^{-+}b_{-+} + \eta^{+-}b_{+-} = 0 \\
b_{-+} &= -b_{+-}
\end{aligned}$$

and by definition $b_{\alpha\beta}$ is traceless and symmetric.

The equations of motion are

$$\partial_- b_{++} = \partial_+ b_{--} = 0 \tag{2.7}$$
$$\partial_- c^+ = \partial_- c^+ = 0 \tag{2.8}$$

We have the energy momentum tensor

$$\begin{aligned}
T^g_{--} &= -i\left(b_{--}\partial_- c^- + b_{--}\partial_- c^- - c^-\partial_- b_{--} - c^+\partial_+ b_{--}\right) \\
&\overset{(2.7)}{=} -i(2b_{--}\partial_- c^- - c^-\partial_- b_{--}) && (2.9) \\
T^g_{++} &= -i(2b_{++}\partial_+ c^+ - c^+\partial_+ b_{++}) && (2.10) \\
T^g_{-+} &= -i(b_{--}\partial_+ c^- + b_{++}\partial_- c^+ - \eta_{-+}(b_{--}\partial^- c^- + b_{++}\partial^+ c^+)) \\
&= 0 && (2.11) \\
T^g_{+-} &= T^g_{-+} = 0 && (2.12)
\end{aligned}$$

(d) Like always, first to invert $c^\pm$ and $b_{\pm\pm}$

$$c^\pm_n(\sigma,\tau) = \frac{2\pi}{l^2}\int_0^l \mathrm{d}\sigma^\pm\, c^\pm e^{\frac{2\pi}{l}in\sigma_\pm} \tag{2.13}$$
$$b^\pm_n(\sigma,\tau) = \frac{l}{4\pi^2}\int_0^l \mathrm{d}\sigma^\pm\, b_{\pm\pm} e^{\frac{2\pi}{l}in\sigma_\pm} \tag{2.14}$$

Thus

$$\begin{aligned}
\{b_m, c_n\} &= \{b^-_m, c^-_n\} \\
&= \frac{1}{2\pi l}\int \mathrm{d}\sigma^\pm\, \mathrm{d}\sigma'^\pm\, e^{\frac{2\pi}{l}im\sigma^\pm} e^{\frac{2\pi}{l}in\sigma'^\pm}\{b_{--}(\sigma,\tau), c^-(\sigma',\tau)\} \\
&= \frac{1}{2\pi l}\int \mathrm{d}\sigma^\pm\, \mathrm{d}\sigma'^\pm\, e^{\frac{2\pi}{l}im\sigma^\pm} e^{\frac{2\pi}{l}in\sigma'^\pm} 2\pi\delta(\sigma-\sigma') \\
&= l\int_0^l \mathrm{d}\sigma^\pm\, e^{\frac{2\pi}{l}i(m+n)\sigma^\pm} \\
\{b_m, c_n\} &= \delta_{m+n} && (2.15)
\end{aligned}$$

Since all other anti-commutation relations vanish, we have

$$\{b_m, b_n\} = 0 \tag{2.16}$$
$$\{c_m, c_n\} = 0 \tag{2.17}$$

(e) The Virasoro generators are

$$\begin{aligned}
L_m^g &= \frac{-l}{4\pi^2}\int_0^l \mathrm{d}\sigma\, T_{--}^g e^{-\frac{2\pi}{l}im\sigma} \\
&= \frac{il}{4\pi^2}\int_0^l \mathrm{d}\sigma\, e^{-\frac{2\pi}{l}im\sigma}(2b_{--}\partial_- c^- - c^-\partial_- b_{--}) \\
&= \frac{l}{4\pi^2}\int_0^l \mathrm{d}\sigma\, e^{-\frac{2\pi}{l}im\sigma}\left(\frac{2\pi}{l}\right)^2 \sum_{n,k\in\mathbb{Z}}\left[2kb_nc_ke^{-\frac{2\pi}{l}i(n+k)\sigma_-} + nb_nc_ke^{-\frac{2\pi}{l}i(n+k)\sigma_-}\right] \\
&= \frac{1}{l}\int_0^l \mathrm{d}\sigma\, e^{-\frac{2\pi}{l}im\sigma}\sum_{n,k\in\mathbb{Z}} b_nc_ke^{-\frac{2\pi}{l}i(n+k)\sigma_-}(2k+n) \\
&= \sum_{n,k\in\mathbb{Z}} b_nc_k\delta_{-m+n+k}(2k+n) \\
&= \sum_{n\in\mathbb{Z}}(m-n)b_{m+n}c_{-n}
\end{aligned}$$

Analogously

$$\bar{L}_m^g = \sum_{n\in\mathbb{Z}}(m-n)\bar{b}_{m+n}\bar{c}_{-n} \tag{2.18}$$

(f) For convenience, calculate the commutator of ghost fields first.

$$\begin{aligned}
\{b_m, c_n\} &= \delta_{m+n} \\
\Rightarrow [b_m, c_n] &= -2c_nb_m + \delta_{m+n} \\
\Rightarrow [c_n, b_m] &= -2b_mc_n + \delta_{m+n} \\
\{b_m, b_n\} &= 0 \\
\Rightarrow [b_m, b_n] &= 2b_mb_n \\
[c_m, c_n] &= 2c_mc_n
\end{aligned}$$

Thus

$$\begin{aligned}
[L_m, b_k] &= \sum_n (m-n)\left([b_{m+n}, b_k]c_{-n} + b_{m+n}[c_{-n}, b_k]\right) \\
&= \sum_n (m-n)b_{m+n}\delta_{k-n} \\
&= (m-k)b_{m+k} \qquad (2.19) \\
[L_m, c_k] &= \sum_n (m-n)\left([b_{m+n}, c_k]c_{-n} + b_{m+n}[c_{-n}, c_k]\right) \\
&= -\sum_n (m-n)c_{-n}\delta_{m+n+k} \\
&= -(2m+k)c_{m+k} \qquad (2.20)
\end{aligned}$$

Now in quantum theory, the Virasoro generator get normal ordered, meaning $: b_m c_{-n} := -c_{-n} b_m$ and $: b_{-m} c_n := b_{-m} c_n$ for $m, n > 0$. To keep discussion general, we for now consider the case $m+n$ can be zero

$$\begin{aligned}
&[L_m, L_n] \\
&= -\sum_{k>0}(n-k)[L_m, c_{-k}b_{n+k}] + \sum_{k\le 0}(n-k)[L_m, b_{n+k}c_{-k}] \\
&= \sum_{k>0}(n-k)(2m-k)c_{m-k}b_{n+k} - \sum_{k>0}(n-k)(m-n-k)c_{-k}b_{m+n+k} \\
&\quad + \sum_{k\le 0}(n-k)(m-n-k)b_{m+n+k}c_{-k} - \sum_{k\le 0}(n-k)(2m-k)b_{n+k}c_{m-k}
\end{aligned}$$

shift indices of first and last terms $k \to k+m$

$$\begin{aligned}
&= \sum_{k>-m}(n-m-k)(m-k)c_{-k}b_{n+m+k} - \sum_{k>0}(n-k)(m-n-k)c_{-k}b_{m+n+k} \\
&\quad + \sum_{k\le 0}(n-k)(m-n-k)b_{m+n+k}c_{-k} - \sum_{k\le -m}(n-m-k)(m-k)b_{n+m+k}c_{-k}
\end{aligned}$$

combine first term into second term and third into last term

$$\begin{aligned}
&= \sum_{-m<k\le 0}(n-m-k)(m-k)c_{-k}b_{n+m+k} - \sum_{k>0}(m-n)(m+n-k)c_{-k}b_{m+n+k} \\
&\quad + \sum_{-m<k\le 0}(n-k)(m-n-k)b_{m+n+k}c_{-k} + \sum_{k\le -m}(m-n)(m+n-k)b_{n+m+k}c_{-k}
\end{aligned} \tag{2.21}$$

When $m+n \neq 0$, the field anti-commute. Thus

$$[L_m, L_n] = \sum_{k\in\mathbb{Z}}(m-n)(m+n-k)b_{m+n+k}c_{-k} = (m-n)L^g_{m+n} \tag{2.22}$$

(g) If $m+n=0$, there are extra contributions in (2.21) from first term since it is the only term not normal ordered in this case

$$\begin{aligned}
[L_m, L_n] &= (m-n)L^g_0 + \sum_{-m<k\le 0}(n-m-k)(m-k) \\
&= 2mL^g_0 + \sum_{0<k\le m}(k^2 - mk - 2m^2) \\
&= 2mL^g_0 - \frac{1}{12}m(26m^2-2)
\end{aligned}$$

(h) Now to consider the full algebra

$$L_m = L^X_m + L^g_m + a\delta_m$$

We have

$$[L_m, L_n] = (m-n)L_{m+n} + \delta_{m+n}\left(\frac{d}{12}m(m^2-1) - \frac{m}{12}(26m^2-2) - 2am\right)$$

where the last term $-2am$ is due to the shift in L_0. Thus we see only if $d = 26$ and $a = -1$, there is no anomaly.