Superstring theory
Homework 6

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1 Lorentz symmetry of the quantum string

We have shown already the total angular momentum in terms of modes. In quantum theory, they get normal ordered

$$J^{\mu\nu} = l^{\mu\nu} + E^{\mu\nu} + \bar{E}^{\mu\nu} \tag{1.1}$$

with

$$\begin{aligned} l^{\mu\nu} &= x^\mu p^\nu - x^\nu p^\mu \\ E^{\mu\nu} &= -i \sum_{n=1}^{\infty} \frac{1}{n} \left(\alpha^\mu_{-n} \alpha^\nu_n - \alpha^\nu_{-n} \alpha^\mu_n \right) \end{aligned}$$

For closed strings, there is contribution $\bar{E}^{\mu\nu}$. Since it is entirely made of $\bar{\alpha}$'s, its commutator with Virasoro generator is zero. So in this calculation, open string and closed string are the same.

The quantum Virasoro operators are

$$L_m = \frac{1}{2} \sum_{n\in\mathbb{Z}} : \alpha_{m-n} \cdot \alpha_n := \frac{1}{2} \sum_{n<0} \alpha_n \cdot \alpha_{m-n} + \frac{1}{2} \sum_{n\geq 0} \alpha_{m-n} \cdot \alpha_n \tag{1.2}$$

We also have a useful relation

$$[L_m, \alpha^\mu_n] = -n \alpha^\mu_{m+n} \tag{1.3}$$

First

$$
\begin{aligned}
[L_m, E^{\mu\nu}] &= \sum_{n>0} \frac{-i}{n} \left[L_m, -\alpha^\nu_{-n}\alpha^\mu_n + \alpha^\mu_{-n}\alpha^\nu_n\right] \\
&= \sum_{n>0} \frac{-i}{n} \left(-\left[L_m, \alpha^\nu_{-n}\right]\alpha^\mu_n - \alpha^\nu_{-n}[L_m, \alpha^\mu_n] + [L_m, \alpha^\mu_{-n}]\alpha^\nu_n + \alpha^\mu_{-n}[L_m, \alpha^\nu_n]\right) \\
&= \sum_{n>0} \frac{-i}{n} \left(-n\alpha^\nu_{m-n}\alpha^\mu_n + n\alpha^\nu_{-n}\alpha^\mu_{m+n} + n\alpha^\mu_{m-n}\alpha^\nu_n - n\alpha^\mu_{-n}\alpha^\nu_{m+n}\right)
\end{aligned}
$$

It is clear that if $m = 0$, the commutator vanishes. What if $m \neq 0$? Indices of first and third terms get renamed $n \to -n$

$$
= -i \sum_{n>0} \left(\alpha^\nu_{-n}\alpha^\mu_{m+n} - \alpha^\mu_{-n}\alpha^\nu_{m+n}\right) + i \sum_{n<0} \left(\alpha^\mu_{m+n}\alpha^\nu_{-n} - \alpha^\nu_{m+n}\alpha^\mu_{-n}\right)
$$

Combining these two sums (commutator vanishes)

$$
= -i \sum_{n\neq 0} \left(\alpha^\mu_{m+n}\alpha^\nu_{-n} - \alpha^\nu_{m+n}\alpha^\mu_{-n}\right)
$$

Then for the orbital angular momentum part: if $m \neq 0$ then there are two terms contributing

$$
\begin{aligned}
[L_m, l^{\mu\nu}] &= \frac{1}{2} \left(\alpha^\sigma_m[\alpha_{0,\sigma}, l^{\mu\nu}] + \alpha^\sigma_m[\alpha_{0,\sigma}, l^{\mu\nu}]\right) \\
&= \alpha_{m,\sigma}[p^\sigma, x^\mu p^\nu - x^\nu p^\mu] \\
&= \alpha_{m,\sigma}(-i\eta^{\sigma\mu}p^\nu + i\eta^{\sigma\nu}p^\mu) \\
&= -i(\alpha^\mu_m\alpha^\nu_0 - \alpha^\nu_m\alpha^\mu_0)
\end{aligned}
$$

If $m = 0$

$$
\begin{aligned}
[L_0, l^{\mu\nu}] &= \frac{1}{2}[\alpha^\sigma_0\alpha_{0,\sigma}, x^\mu p^\nu - x^\nu p^\mu] \\
&= \alpha_{0,\sigma}\left([\alpha^\sigma_0, x^\mu]p^\nu - [\alpha^\sigma_0, x^\nu]p^\mu\right) \\
&= -i\alpha_{0,\sigma}\left(\eta^{\sigma\mu}p^\nu - \eta^{\sigma\nu}p^\mu\right) \\
&= 0
\end{aligned}
$$

Put everything together

$$
\begin{aligned}
[L_m, J^{\mu\nu}] &= \begin{cases} 0, & m = 0 \\ -i\sum_n \left(\alpha^\mu_{m+n}\alpha^\nu_{-n} - \alpha^\nu_{m+n}\alpha^\mu_{-n}\right), & m \neq 0 \end{cases} \\
&= 0
\end{aligned}
$$

In $m \neq 0$ these two terms cancel each other, since one can rename the index of second term $n \to -n - m$ and $m \neq 0$.

In quantum theory, constraints can be expressed through the restriction that Virasoro generator L_m ($m > 0$) annihilates physical states. The fact that Virasoro generator commutes with total angular momentum operator means that with these constraints physical states have well defined angular momenta. Thus representations of Lorentz group can be thought as particles, i.e. physical states.

2 Lorentz invariance in light-cone gauge

The space-time light-cone coordinates will be used

$$X^{\pm} = \frac{1}{\sqrt{2}}\left(X^0 \pm X^{d-1}\right) \tag{2.1}$$

with $i = 1, \ldots, d-2$. Light-cone gauge is

$$X^+ = x^+ + p^+\tau \tag{2.2}$$

(a) The constraints are

$$(\dot{X} \pm X')^2 = 0 \tag{2.3}$$

Written in space-time light-cone coordinates ($i = 1, \ldots, d-2$ and metric signature is $(+, -, -, \ldots)$)

$$\begin{aligned}
[(\dot{X} \pm X')^i]^2 &= 2(\dot{X} \pm X')^-(\dot{X} \pm X')^+ \\
&= 2p^+(\dot{X} \pm X')^-
\end{aligned}$$

Thus we have

$$(\dot{X} \pm X')^- = \frac{1}{2p^+}\sum_{i=1}^{d-2}[(\dot{X} \pm X')^i]^2 \tag{2.4}$$

(b) In order to express the previous equation in terms of modes, we first calculate ($\mu \neq +$)

$$\begin{aligned}
(\dot{X} \pm X')^\mu &= p^\mu + \sum_{n\neq 0} \alpha_n^\mu e^{-in\tau}\cos n\sigma \mp i\sum_{n\neq 0}\alpha_n^\mu e^{-in\tau}\sin n\sigma \\
&= p^\mu + \sum_{n\neq 0}\alpha_n^\mu e^{-in(\tau\pm\sigma)} \\
&= \sum_n \alpha_n^\mu e^{-in(\tau\pm\sigma)}
\end{aligned}$$

Thus equation (2.4) becomes

$$
\begin{aligned}
\sum_n \alpha_n^- e^{-in(\tau\pm\sigma)} &= \frac{1}{2p^+}\sum_{i=1}^{d-2}\sum_{n,m}\alpha_n^i\alpha_m^i e^{-i(n+m)(\tau\pm\sigma)} \qquad \Big| \int_0^{2\pi} \mathrm{d}\sigma^\pm\, e^{ik\sigma^\pm} \\
\sum_n \alpha_n^- 2\pi\delta_{n,k} &= \frac{1}{2p^+}\sum_{i=1}^{d-2}\sum_{n,m}\alpha_n^i\alpha_m^i 2\pi\delta_{n+m,k} \\
\alpha_k^- &= \frac{1}{2p^+}\sum_{i=1}^{d-2}\sum_{m}\alpha_{k-m}^i\alpha_m^i
\end{aligned}
$$

We introduce a normal-ordering constant constant a

$$
\alpha_n^- = \frac{1}{p^+}\left(\frac{1}{2}\sum_{i=1}^{d-2}\sum_{m=-\infty}^{\infty} :\alpha_{n-m}^i\alpha_m^i: -a\delta_n\right) \tag{2.5}
$$

(c) Mass shell condition

$$
\begin{aligned}
M^2 &= 2p^+p^- - p^ip_i \\
&= \sum_{i=1}^{d-2}\sum_{m}\alpha_{-m}^i\alpha_m^i - \alpha_0^i\alpha_0^i \\
&= \sum_{i=1}^{d-2}\sum_{m\neq 0}\alpha_{-m}^i\alpha_m^i \\
&= 2\sum_{i=1}^{d=2}\sum_{m=1}^{\infty} :\alpha_{-m}^i\alpha_m^i: -2a \\
&= 2(N-a)
\end{aligned}
$$

(d) We have found

$$
[J^{\mu\nu}, J^{\rho\sigma}] = i(\eta^{\mu\rho}J^{\nu\rho} + \eta^{\nu\sigma}J^{\mu\rho} - \eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\sigma}J^{\nu\rho}) \tag{2.6}
$$

Thus

$$
\begin{aligned}
\left[J^{i-}, J^{j-}\right] &= i(\eta^{ij}J^{--} + \eta^{--}J^{ij} - \eta^{-j}J^{i-} - \eta^{i-}J^{-j}) \\
&= i\delta^{ij}J^{--} \\
&= 0
\end{aligned}
$$

because by definition $J^{\mu\nu}$ is totally anti-symmetric tensor.

(e) First

$$\begin{aligned}[A, BC] &= ABC - BCA \\ &= [A, B]C + B[A, C]\end{aligned}$$

Thus

$$\begin{aligned}[AB, CD] &= [AB, C]D + C[AB, D] \\ &= -[C, AB]D - C[D, AB] \\ &= -\left([C, A]B + A[C, B]\right) D - C\left([D, A]B + A[D, B]\right) \\ &= [A, C]BD + A[B, C]D + C[A, D]B + CA[B, D] \\ &= A[B, C]D + C[A, D]B + [A, C](DB - [D, B]) + (AC - [A, C])[B, D] \\ &= A[B, C]D + C[A, D]B + [A, C]DB + AC[B, D] \\ &= A[B, C]D + AC[B, D] + [A, C]DB + C[A, D]B\end{aligned}$$

(f) Choice of coordinate doesn't change the previously proven commutation relations. Light-cone gauge also doesn't affect it, since it is just a condition that α_n^+ vanishes for all n.

Thus

$$\begin{aligned}\left[x^-, p^+\right] &= i\eta^{-+} = -i \\ \frac{1}{p^+}\left[x^-, p^+\right]\frac{1}{p^+} &= -i(p^+)^{-2} \\ \left[x^-, 1/p^+\right] &= i(p^+)^{-2}\end{aligned} \tag{2.7}$$

The second relation is

$$\begin{aligned}\left[\alpha_m^i, \alpha_n^-\right] &= \frac{1}{2p^+}\sum_{j=1}^{d-2}\sum_k \left[\alpha_m^i, \alpha_{n-k}^j\alpha_k^j\right] \\ &= \frac{1}{2p^+}\sum_{i=1}^{d-2}\sum_k \left(\left[\alpha_m^i, \alpha_{n-k}^j\right]\alpha_k^j + \alpha_{n-k}^j\left[\alpha_m^i, \alpha_k^j\right]\right) \\ &= \frac{1}{2p^+}\sum_{i=1}^{d-2}\sum_k \delta^{ij}\left(m\delta_{m+n-k}\alpha_k^j + \alpha_{n-k}^j m\delta_{m+k}\right) \\ &= \frac{m}{p^+}\alpha_{m+n}^j\end{aligned} \tag{2.8}$$

Note that since $[p^+, \alpha_n^\nu] = 0$, the quotient notation is well-defined.

And the third

$$\begin{aligned}\left[\alpha_m^-, x^-\right] &= \frac{1}{2}\sum_{i=1}^{d-2}\sum_n \left[1/p^+, x^-\right]\alpha_{m-n}^i\alpha_n^i \\ &= -\frac{i}{p^+}\alpha_m^- \end{aligned} \tag{2.9}$$

(g) In assignment 5.2, we have

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{d}{12}m(m^2-1)\delta_{m+n,0} \tag{2.10}$$

It is almost identical as the commutator here, since L_m is very similar as $p^+\alpha_m^-$ except the normal-ordering constant term. Thus

$$\left[p^+\alpha_m^-, p^+\alpha_m^-\right] = (m-n)p^+\alpha_{m+n}^- + \left[\frac{d-2}{12}m(m^2-1) + 2am\right]\delta_{m+n} \tag{2.11}$$

We have $d-2$ here instead of d because in L_m commutator, there is a normal Minkowski product thus summation over all space-time coordinates, whereas here we only sum over the transverse direction.

(h) Define

$$E^j = p^+E^{j-}$$

Then

$$\begin{aligned}\left[x^i, E^j\right] &= p^+\left[x^i, E^{j-}\right] \\ &= -ip^+\sum_{n>0}\frac{1}{n}\left(\left[x^i, \alpha_{-n}^j\alpha_n^-\right] - \left[x^i, \alpha_{-n}^-\alpha_n^j\right]\right) \\ &= \sum_{n>0}\sum_{k=1}^{d-2}\sum_m \frac{-i}{2n}\left(\alpha_{-n}^j\left[x^i, \alpha_{n-m}^k\alpha_m^k\right] - \left[x^i, \alpha_{-n-m}^k\alpha_m^k\right]\alpha_n^j\right)\end{aligned}$$

$\alpha^j_{\pm n}$ is brought out of the commutator since $n \neq 0$. In order for the commutator not to vanish, $m = 0$ or $\pm n - m = 0$ and these two cannot be satisfied simultaneously.

$$
\begin{aligned}
&= \sum_{n>0} \sum_{k=1}^{d-2} \sum_{m} \frac{-i}{2n} \left[\alpha^j_{-n} i\eta^{ik} (\delta_{n-m}\alpha^k_m + \alpha^k_{n-m}\delta_m) - i\eta^{ik}(\delta_{n+m}\alpha^k_m + \delta_m \alpha^k_{-n-m})\alpha^j_n \right] \\
&= \sum_{n>0} \sum_{k=1}^{d-2} \frac{-i}{2n} \left[\alpha^j_{-n} i\delta^{ik} (\alpha^k_n + \alpha^k_n) - i\delta^{ik}(\alpha^k_{-n} + \alpha^k_{-n})\alpha^j_n \right] \\
&= i \sum_{n>0} \frac{-i}{n} \left(\alpha^j_{-n}\alpha^i_n - \alpha^i_{-n}\alpha^j_n \right) \\
&= -iE^{ij}
\end{aligned}
$$