Superstring theory
Homework 4

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1 The classical Virasoro algebra for the closed bosonic string

(a) First we can show

$$\begin{aligned} \partial_- T_{++} &\propto \partial_- \partial_+ X^\mu \partial_+ X_\mu = 0 \\ \partial_+ T_{--} &\propto \partial_+ \partial_- X^\mu \partial_- X_\mu = 0 \end{aligned}$$

with equations of motion. Then we have

$$T_{++} = T_{++}(\sigma_+), \quad T_{--} = T_{--}(\sigma_-) \tag{1.1}$$

Equivalently, one could also use conservation of energy-momentum in flat spacetime (after using the reparametrization and Weyl invariances) $\partial^\alpha T_{\alpha\beta} = 0$. Thus we have found

$$\begin{aligned} \partial^+ T_{++} + \partial^- T_{-+} &= 0 \\ \cancel{\eta^{+-}} \partial_- T_{++} + \cancel{\eta^{-+}} \partial_+ T_{-+} &= 0 \\ \partial_- T_{++} &= 0 \end{aligned}$$

since off-diagonal entries T_{+-}, T_{-+} vanish. It is analogous for T_{--}. These two methods are the same, since in both constraint or EOM is used. Remember that e.g. in GR, conservation of energy-momentum is a consequence of Einstein field equations.

Essentially, we only showed that the relations are valid for "on-shell" strings. What about off-shell (i.e. without constraints)? It does not matter, since we still have classical theory.

(b) First we write the action in the normal worldsheet light-cone coordinates

$$
\begin{aligned}
S_p &= 2T \int \mathrm{d}\tau\, \mathrm{d}\sigma\, \partial_+ X \cdot \partial_- X \\
&= T \int \mathrm{d}^2\sigma^\pm\, \partial_+ X \cdot \partial_- X
\end{aligned}
$$

Remember X^μ lives in target space, thus unaffected by the coordinate transformation.

Actually after we set $h^{\alpha\beta} = \eta^{\alpha\beta}$, there is still residual symmetry. To see this, we reparametrize the worldsheet again via

$$\sigma^+ \to \tilde{\sigma}^+ = \tilde{\sigma}^+(\sigma^+) \tag{1.2}$$

$$\sigma^- \to \tilde{\sigma}^- = \tilde{\sigma}^-(\sigma^-) \tag{1.3}$$

Basically Jacobian is diagonal.

$$
\begin{aligned}
\mathrm{d}^2\sigma^\pm &= \mathrm{d}^2\tilde{\sigma}^\pm \frac{\partial(\sigma^\pm)}{\partial(\tilde{\sigma}^\pm)} \\
\partial_+ X \cdot \partial_- X &= \tilde{\partial}_+ X \cdot \tilde{\partial}_- X \frac{\partial \tilde{\sigma}^+}{\partial \sigma^+} \frac{\partial \tilde{\sigma}^-}{\partial \sigma^-}
\end{aligned}
$$

These two extra factors are inverse to each other. Thus this coordinate transformation again leaves the action invariant.

(c) To write the aforementioned transformation in infinitesimal form

$$\sigma^\pm \to \sigma^\pm + \xi^\pm(\sigma^\pm)$$

The field transforms as

$$X^\mu \to X^\mu + \xi^\pm \partial_\pm X^\mu$$

The corresponding conserved currents are ($\alpha = \pm$ also)

$$
\begin{aligned}
\xi^b j_b^\alpha &= \frac{\partial \mathcal{L}}{\partial(\partial_\alpha X^a)} \delta X^a \\
&= \left(\partial_- X_\mu \delta_+^\alpha \delta_a^\mu + \partial_+ X_\nu \delta_-^\alpha \delta_a^\nu\right) \xi^c \partial_c X^a \\
j_b^\alpha &= \partial_- X \cdot \partial_b X \delta_+^\alpha + \partial_+ X \cdot \partial_b X \delta_-^\alpha
\end{aligned}
$$

(d) The corresponding charges are

$$
\begin{aligned}
j_+ &= \int \mathrm{d}\sigma^+ \, j_+^- \\
&= T \int \mathrm{d}\sigma^+ \, \partial_+ X_\mu \partial_+ X^\mu \\
&= T \int \mathrm{d}\sigma^+ \, T_{++} \\
j_- &= \int \mathrm{d}\sigma^- \, j_-^+ \\
&= T \int \mathrm{d}\sigma^- \, T_{--}
\end{aligned}
$$

Also $\int \mathrm{d}\sigma^+ \, j_+^+ = \int \mathrm{d}\sigma^- \, j_-^- = 0$ due to the equation of motion.

(e) From last sheet, we have the mode expansion for closed strings

$$
X^\mu(\sigma,\tau) = (x^\mu - c^\mu) + \frac{\pi\alpha'}{l} p^\mu (\sigma^+ + \sigma^-) + i\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{n} \left(\tilde{\alpha}_n^\mu e^{-\frac{2\pi}{l} in\sigma^+} + \alpha_n^\mu e^{-\frac{2\pi}{l} in\sigma^-} \right)
$$

Then

$$
\begin{aligned}
\partial_+ X^\mu &= \frac{\pi\alpha'}{l} p^\mu + \frac{\sqrt{2\alpha'}\pi}{l} \sum_{n\in\mathbb{Z}\setminus\{0\}} \tilde{\alpha}_n^\mu e^{-\frac{2\pi}{l} in\sigma^+} \\
&= \frac{\sqrt{2\alpha'}\pi}{l} \sum_{n\in\mathbb{Z}} \tilde{\alpha}_n^\mu e^{-\frac{2\pi}{l} in\sigma^+} \\
\partial_- X^\mu &= \frac{\sqrt{2\alpha'}\pi}{l} \sum_{n\in\mathbb{Z}} \alpha_n^\mu e^{-\frac{2\pi}{l} in\sigma^-}
\end{aligned}
$$

Thus

$$
\begin{aligned}
T_{--} &= -\frac{1}{\alpha'} \frac{2\alpha'\pi^2}{l^2} \sum_{n\in\mathbb{Z}} \alpha_n^\mu e^{-\frac{2\pi}{l} in\sigma^-} \cdot \sum_{m\in\mathbb{Z}} \alpha_{m,\mu} e^{-\frac{2\pi}{l} im\sigma^-} \\
&= -\frac{2\pi^2}{l^2} \sum_{n,m\in\mathbb{Z}} \alpha_n \cdot \alpha_m e^{-\frac{2\pi i}{l}(n+m)\sigma^-} \\
&= -\frac{2\pi^2}{l^2} \sum_{n,m\in\mathbb{Z}} \alpha_{n-m} \cdot \alpha_m e^{-\frac{2\pi i}{l} n\sigma^-}
\end{aligned}
$$

with $n \to n - m$ shifted at last step. Thus we have (analogously for T_{++})

$$T_{--} = -\left(\frac{2\pi}{l}\right)^2 \sum_{n\in\mathbb{Z}} L_n e^{-\frac{2\pi i n}{l}\sigma^-} \tag{1.4}$$

$$T_{++} = -\left(\frac{2\pi}{l}\right)^2 \sum_{n\in\mathbb{Z}} \tilde{L}_n e^{-\frac{2\pi i n}{l}\sigma^+} \tag{1.5}$$

with $L_n = \frac{1}{2}\sum_{m\in\mathbb{Z}} \alpha_{n-m} \cdot \alpha_m$ and the same for $\tilde{L}_n$.

(f) In last sheet, we found

$$\{\alpha_n^\mu, \alpha_m^\nu\} = \{\tilde{\alpha}_n^\mu, \tilde{\alpha}_m^\nu\} = -in\eta^{\mu\nu}\delta_{m+n,0} \tag{1.6}$$

$$\{\alpha_n^\mu, \tilde{\alpha}_m^\nu\} = 0 \tag{1.7}$$

$$\begin{aligned}
\{L_m, L_n\} &= \frac{1}{4}\sum_{a,b\in\mathbb{Z}} \left\{\alpha^\mu_{m-a}\alpha_{a,\mu}, \alpha^\nu_{n-b}\alpha_{b,\nu}\right\} \\
&= \frac{1}{4}\sum_{a,b\in\mathbb{Z}} \left(\left\{\alpha^\mu_{m-a}, \alpha^\nu_{n-b}\alpha_{b,\nu}\right\}\alpha_{a,\mu} + \alpha^\mu_{m-a}\left\{\alpha_{a,\mu}, \alpha^\nu_{n-b}\alpha_{b,\nu}\right\}\right) \\
&= \frac{1}{4}\sum_{a,b\in\mathbb{Z}} \Big(\left\{\alpha^\mu_{m-a}, \alpha^\nu_{n-b}\right\}\alpha_{b,\nu}\alpha_{a,\mu} + \alpha^\nu_{n-b}\left\{\alpha^\mu_{m-a}, \alpha_{b,\nu}\right\}\alpha_{a,\mu} \\
&\quad + \alpha^\mu_{m-a}\left\{\alpha_{a,\mu}, \alpha^\nu_{n-b}\right\}\alpha_{b,\nu} + \alpha^\mu_{m-a}\alpha^\nu_{n-b}\left\{\alpha_{a,\mu}, \alpha_{b,\nu}\right\}\Big) \qquad (1.8) \\
&= -\frac{i}{4}\sum_{a,b\in\mathbb{Z}} \Big[(m-a)\eta^{\mu\nu}\delta_{m-a+n-b,0}\alpha_{b,\nu}\alpha_{a,\mu} + (m-a)\delta^\mu_\nu\delta_{m-a+b,0}\alpha^\nu_{n-b}\alpha_{a,\mu} \\
&\quad + a\delta^\nu_\mu\delta_{a+n-b,0}\alpha^\mu_{m-a}\alpha_{b,\nu} + a\eta_{\mu\nu}\delta_{a+b,0}\alpha^\mu_{m-a}\alpha^\nu_{n-b}\Big] \\
&= -\frac{i}{2}\sum_{a\in\mathbb{Z}} \left[(m-a)\alpha_{m+n-a}\cdot\alpha_a + a\alpha_{m-a}\cdot\alpha_{a+n}\right]
\end{aligned}$$

we shift summation index of the second term $a \to a - n$.

$$\begin{aligned}
&= -\frac{i}{2}(m-n)\sum_{a\in\mathbb{Z}} \alpha_{m+n-a}\cdot\alpha_a \\
&= -i(m-n)L_{m+n} \qquad (1.9)
\end{aligned}$$

Analogously,

$$\left\{\tilde{L}_m, \tilde{L}_n\right\} = -i(m-n)\tilde{L}_{m+n} \tag{1.10}$$

since Poisson brackets are the same. If we have the second Virasoro generator with tilde, then the second terms in the Poisson brackets in equation (1.8) are all with tilde. They vanish due to equation (1.7).

$$\left\{L_m, \tilde{L}_n\right\} = 0 \tag{1.11}$$

(g) Fourier transformed Virasoro generators, i.e. T_{++} and T_{--}, are zero. Thus we conclude that $L_n = \tilde{L}_n = 0$ because of constraints.

(h) Mass-shell condition for the closed bosonic string can be expressed with zero modes

$$M^2 = -p^\mu p_\mu = -\frac{2}{\alpha'}\alpha_0 \cdot \alpha_0 \tag{1.12}$$

We can express this in terms of all other mode coefficients with the result we obtained in last part $L_0 = \tilde{L}_0 = 0$.

$$\begin{aligned}
M^2 &= -\frac{1}{\alpha'}(\alpha_0 \cdot \alpha_0 + \tilde{\alpha}_0 \cdot \tilde{\alpha}_0) \\
&= \frac{1}{\alpha'}\left(\sum_{n\in\mathbb{Z}\setminus\{0\}} \alpha_{-n}\cdot\alpha_n + \sum_{n\in\mathbb{Z}\setminus\{0\}} \tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n\right) \\
&= \frac{2}{\alpha'}\sum_{n\in\mathbb{N}}(\alpha_{-n}\cdot\alpha_n + \tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n)
\end{aligned}$$

(i) For closed string, due to periodicity definitions of start and end points are arbitrary. Thus it must be σ-translation invariant. $-T\int \dot{X}\cdot X' \,\mathrm{d}\sigma$ is able to generate σ-translation

$$\begin{aligned}
&-T\int \mathrm{d}\sigma \left\{\dot{X}(\sigma,\tau)\cdot X'(\sigma,\tau), X^\nu(\sigma',\tau)\right\} \\
&= -T\int \mathrm{d}\sigma \left\{\dot{X}^\mu(\sigma,\tau), X^\nu(\sigma',\tau)\right\} X'_\mu(\sigma,\tau) \\
&= -X'^\nu(\sigma')
\end{aligned}$$

Plug in the solution, we have $-T\int \dot{X}\cdot X' \,\mathrm{d}\sigma = 2\pi/l(L_0 - \tilde{L}_0)$. Thus we need to impose the constraint that $L_0 - \tilde{L}_0 = 0$.

The periodicity also implies that we should have two distinct modes in the general solution, i.e. left and right moving modes.