Superstring theory
Summary

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1 General

Metric

$$\eta_{\mu\nu} = \mathrm{diga}(-1, +1, , \ldots, +1) \tag{1.1}$$

Cauchy integral formula

$$\oint_{C_w} \frac{\mathrm{d}z}{2\pi i} \frac{f(z)}{(z-w)^n} = \frac{1}{(n-1)!} f^{n-1}(w) \tag{1.2}$$

2 Classical Bosonic String

2.1 The Relativistic Particle

Action A free Relativistic particle of mass m moving in a d-dimensional flat spacetime has the action

$$S = -m \int_{s_0}^{s_1} \mathrm{d}x = -m \int_{\tau_0}^{\tau^1} \left[-\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} \eta_{\mu\nu} \right]^{1/2} \tag{2.1}$$

It is simply the length of the world-line.

Constraints By computing the momentum conjugate to x^μ, one finds $p^2 = -m^2$, with out using equation of motion. Such constraints are called primary constraints. If the constraint is from equations of motion, they are secondary constraints. Another categorization of constraints would be to look at Poisson brackets $\{\phi_a, \phi_k\}$ for all k. If it vanishes, constraint ϕ_a is first class, if not, it is second class.

Generalized action The action (2.1) can be generalized to massless case with an auxiliary variable $e(\tau)$,

$$S = \frac{1}{2}\int_{\tau_0}^{\tau_1} e(e^{-2}\dot{x}^2 - m^2)\,\mathrm{d}\tau \tag{2.2}$$

Plug the equation of motion back it and (2.1) is recovered. Note that the mass shell constraint (condition) becomes secondary.

2.2 The Nambu-Goto Action

Action of a string in d-dimensional space-time is describe be the action of the world-sheet swept by the string

$$\begin{aligned} S_{\mathrm{NG}} &= -T\int_\Sigma \mathrm{d}^2\sigma\, A \\ &= -T\int_\Sigma \mathrm{d}^2\sigma \left[-\det_{\alpha\beta}\left(\frac{\partial X^\mu}{\partial\sigma^\alpha}\frac{\partial X^\nu}{\partial\sigma^\beta}\eta_{\mu\nu}\right)\right]^{1/2} \\ &= -T\int_\Sigma \mathrm{d}^2\sigma \left[\left(\dot{X}\cdot X'\right)^2 - \dot{X}^2 X'^2\right]^{1/2} \\ &= -T\int_\Sigma \mathrm{d}^2\sigma\,\sqrt{-\Gamma} \end{aligned} \tag{2.3}$$

with string tension $T = \frac{1}{2\pi\alpha'}$. α' is the Regge slope.

2.3 The Polyakov Action and Its Symmetries

A d-dimension massless world-sheet scalar X^μ coupled to two-dimensional gravity in flat background has the action

$$S_{\mathrm{P}} = -\frac{T}{2}\int_\Sigma \mathrm{d}^2\sigma\,\sqrt{-h}h^{\alpha\beta}\,\underbrace{\partial_\alpha X^\mu \partial_\beta X^\mu \eta_{\mu\nu}}_{=\Gamma_{\alpha\beta}} \tag{2.4}$$

with $h_{\mu\nu}(\sigma,\tau)$ the metric on the world-sheet.

Energy momentum tensor of the world-sheet theory is defined as how the action changes under variation with respect to the metric

$$T_{\alpha\beta} = \frac{4\pi}{\sqrt{-h}}\frac{\delta S_{\mathrm{P}}}{\delta h^{\alpha\beta}} \tag{2.5}$$

In this case, it is

$$T_{\alpha\beta} = -\frac{1}{\alpha'}\left(\Gamma_{\alpha\beta} - \frac{1}{2}h_{\alpha\beta}h^{\gamma\delta}\Gamma_{\gamma\delta}\right) \tag{2.6}$$

Equations of motions are

$$T_{\alpha\beta} = 0 \tag{2.7a}$$

$$\frac{1}{\sqrt{-h}}\partial_\alpha\left(\sqrt{-h}h^{\alpha\beta}\partial_\beta X^\mu\right) = 0 \tag{2.7b}$$

if the boundary or periodicity conditions are satisfied (later).

Symmetries of Polyakov action

- (global) Poincare invariance
- (local) reparametrization invariance
- (local) Weyl rescaling invariance

Note that X^μ is a Minkowski space vector and world-sheet scalar. The metric $h_{\alpha\beta}$ is a Minkowski scalar and a world-sheet tensor. Weyl invariance ensures that the energy-momentum tensor is traceless. The local invariances allow conformal gauge $h_{\alpha\beta} = \Omega^2(\sigma,\tau)\eta_{\alpha\beta}$. With Weyl invariance, the metric can be set to flat.

World-sheet light-cone coordinates are given

$$\sigma^\pm = \tau \pm \sigma \tag{2.8}$$

Now the metric is

$$\eta_{+-} = \eta_{-+} = -\frac{1}{2}, \eta_{++} = \eta_{--} = 0 \tag{2.9}$$

Action in conformal gauge

$$S_{\mathrm{P}} = \frac{T}{2}\int \mathrm{d}^2\sigma\,(\dot{X}^2 - X'^2) = 2T\int \mathrm{d}^2\sigma\,\partial_+ X\cdot\partial_- X \tag{2.10}$$

It leads to the equations of motion

$$(\partial_\sigma^2 - \partial_\tau^2)X^\mu = 4\partial_-\partial_+ X^\mu = 0 \tag{2.11}$$

Boundary/Periodicity conditions In order for the surface term from variation of action to vanish, we demand

- Periodicity
$$X^\mu(\tau, \sigma + l) = X^\mu(\tau, \sigma) \tag{2.12}$$
- Neumann boundary condition
$$\partial_\sigma X^\mu|_{\sigma=0,l} = 0 \tag{2.13}$$
- Dirichlet boundary condition
$$\delta X^\mu|_{\sigma=0,l} = 0 \tag{2.14}$$
It breaks space-time Poincare invariance.

Energy-momentum tensor need to vanish due to equations of motion. This is alternatively
$$\left(\dot{X}^2 + X'\right)^2 = 0 \tag{2.15}$$
Energy-momentum needs to also be conserved
$$\partial_- T_{++} = 0, \quad \partial_+ T_{--} = 0 \tag{2.16}$$
It implies an infinite number of conserved charges
$$L_f = 2T \int_0^l \mathrm{d}\sigma\, f(\sigma^+) T_{++}(\sigma^+) \tag{2.17}$$

Poisson brackets
$$\{X^\mu(\sigma,\tau), X^\nu(\sigma',\tau)\} = \left\{\dot{X}^\mu(\sigma,\tau), \dot{X}^\nu(\sigma',\tau)\right\} = 0 \tag{2.18a}$$
$$\left\{X^\mu(\sigma,\tau), \dot{X}^\nu(\sigma',\tau)\right\} = \frac{1}{T}\eta^{\mu\nu}\delta(\sigma - \sigma') \tag{2.18b}$$

Conserved currents
$$P^\alpha_\mu = -T\sqrt{h} h^{\alpha\beta}\partial_\beta X_\mu \tag{2.19a}$$
$$P_\mu = \int_0^l \mathrm{d}\sigma\, P^\tau_\mu \tag{2.19b}$$
$$J_{\mu\nu} = X_\mu P^\alpha_\nu - X_\nu P^\alpha_\mu \tag{2.19c}$$
$$J_{\mu\nu} = \int_0^l \mathrm{d}\sigma\, J^\tau_{\mu\nu} \tag{2.19d}$$

2.4 Oscillator Expansions

2.4.1 Closed strings

Most general solution

$$X_L^\mu(\tau+\sigma) = \frac{1}{2}(x^\mu + c^\mu) + \frac{\pi\alpha'}{l}p^\mu(\tau+\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\bar{\alpha}_n^\mu e^{-\frac{2\pi}{l}in(\tau+\sigma)} \tag{2.20}$$

$$X_R^\mu(\tau+\sigma) = \frac{1}{2}(x^\mu - c^\mu) + \frac{\pi\alpha'}{l}p^\mu(\tau-\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-\frac{2\pi}{l}in(\tau+\sigma)} \tag{2.21}$$

Reality conditions are

$$\alpha_{-n}^\mu = (\alpha_n^\mu)^*, \quad \bar{\alpha}_{-n}^\mu = (\bar{\alpha}_n^\mu)^* \tag{2.22}$$

Explicit computation gives that p^μ is total space-time momentum, x^μ is the "center of mass" at $\tau = 0$.

Poisson brackets of the mode coefficients are[1]

$$\{\alpha_m^\mu, \alpha_m^\nu\} = \{\bar{\alpha}_m^\mu, \bar{\alpha}_n^\nu\} = -im\delta_{m+n}\eta^{\mu\nu} \tag{2.23a}$$
$$\{\alpha_m^\mu, \bar{\alpha}_m^\nu\} = 0 \tag{2.23b}$$
$$\{x^\mu, p^\nu\} = \eta^{\mu\nu} \tag{2.23c}$$

Virasoro generators

$$L_n = -\frac{l}{4\pi}\int_0^l \mathrm{d}\sigma\, e^{-\frac{2\pi i}{l}n\sigma}T_{--} = \frac{1}{2}\sum_m \alpha_{n-m}\cdot\alpha_m \tag{2.24a}$$

$$\bar{L}_n = -\frac{l}{4\pi}\int_0^l \mathrm{d}\sigma\, e^{+\frac{2\pi i}{l}n\sigma}T_{++} = \frac{1}{2}\sum_m \bar{\alpha}_{n-m}\cdot\bar{\alpha}_m \tag{2.24b}$$

They have the reality condition $L_n = L_{-n}^*$ and $\bar{L}_n = \bar{L}_{-n}^*$. From equations of motion, we find

$$L_n = \bar{L}_n = 0 \tag{2.25}$$

[1] It is done by invert (3.1). The trick is to multiply it with an exponential function and integrate over σ.

$L_0 - \bar{L}_0$ generator rigid σ-translation, thus need to require

$$L_0 - \bar{L}_0 = 0 \tag{2.26}$$

The generators form the Virasoro algebra

$$\{L_m, L_n\} = -i(m-n)L_{m+n} \tag{2.27a}$$
$$\left\{\bar{L}_m, \bar{L}_n\right\} = -i(m-n)\bar{L}_{m+n} \tag{2.27b}$$
$$\left\{\bar{L}_m, L_n\right\} = 0 \tag{2.27c}$$

2.4.2 Open strings

Neumann boundary condition

$$X^\mu(\sigma,\tau) = x^\mu + \frac{2\pi\alpha'}{l}p^\mu\tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-i\frac{\pi}{l}n\tau}\cos\Big(\frac{n\pi\sigma}{l}\Big) \tag{2.28}$$

Equations (3.2) and (2.24) still hold, but without $\bar{\alpha}$'s.

Dirichlet boundary condition

$$X^\mu(\sigma,\tau) = x_0^\mu + \frac{1}{l}(x_1^\mu - x_0^\mu)\sigma + \sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-i\frac{\pi}{l}n\tau}\sin\Big(\frac{n\pi\sigma}{l}\Big) \tag{2.29}$$

Equations (3.2) and (2.24) still hold, but without $\bar{\alpha}$'s. Now the center of mass is at $q^\mu = \frac{1}{2}(x_0^\mu + x_1^\mu)$ and there is "potential energy" term due to stretching included in Hamiltonian.

3 The Quantized Bosonic String

3.1 Canonical Quantization

Ladder operators Replace Poisson brackets by commutator in (3.1) and (3.2)

$$[X^\mu(\sigma,\tau), X^\nu(\sigma',\tau)] = \Big[\dot{X}^\mu(\sigma,\tau), \dot{X}^\nu(\sigma',\tau)\Big] = 0 \tag{3.1a}$$
$$\Big[X^\mu(\sigma,\tau), \dot{X}^\nu(\sigma',\tau)\Big] = 2\pi i\alpha'\eta^{\mu\nu}\delta(\sigma-\sigma') \tag{3.1b}$$

$$[\alpha_m^\mu, \alpha_m^\nu] = [\bar{\alpha}_m^\mu, \bar{\alpha}_n^\nu] = m\delta_{m+n}\eta^{\mu\nu} \tag{3.2a}$$
$$[\alpha_m^\mu, \bar{\alpha}_m^\nu] = 0 \tag{3.2b}$$
$$[x^\mu, p^\nu] = i\eta^{\mu\nu} \tag{3.2c}$$

Hermicity conditions are

$$(\alpha_m^\mu)^\dagger = \alpha_{-m}^\mu, \quad (\bar{\alpha}_m^\mu)^\dagger = \bar{\alpha}_m^\mu \tag{3.3}$$

The vacuum is defined as

$$\alpha_m^\mu \left|0; p^\mu\right\rangle = 0 \quad \text{for } m > 0, \quad \hat{p}^\mu \left|0; p^\mu\right\rangle = p^\mu \left|0; p^\mu\right\rangle \tag{3.4}$$

Number operator is ($m > 0$)

$$N_m =: \alpha_m \cdot \alpha_{-m} := \alpha_{-m} \cdot \alpha_m \tag{3.5}$$

This set of operators and choice of vacuum lead to negative norm states, called otherwise ghosts. In 26 dimensions, ghosts decouple from physical states.

Propagators is defined as

$$\langle X^\mu(\sigma,\tau) X^\nu(\sigma',\tau')\rangle = T[X^\mu(\sigma,\tau) X^\nu(\sigma',\tau')] - :[X^\mu(\sigma,\tau) X^\nu(\sigma',\tau')]: \tag{3.6}$$

For closed strings, it is

$$\langle X^\mu(z,\bar{z}) X^\nu(w,\bar{w})\rangle = -\frac{\alpha'}{2}\eta^{\mu\nu} \ln((z-w)(\bar{z}-\bar{w})) \tag{3.7}$$

For open string, it is

$$\langle X^\mu(z,\bar{z}) X^\nu(w,\bar{w})\rangle_{\text{NN, DD}} = -\frac{\alpha'}{2}\eta^{\mu\nu} \left[\ln|z-w|^2 \pm \ln|z-\bar{w}|^2\right] \tag{3.8}$$

Virasoro generators First need to define properly ordered Virasoro generators

$$L_n = \frac{1}{2}\sum_{m=-\infty}^{+\infty} :\alpha_{n-m} \cdot \alpha_m: \tag{3.9}$$

where

$$L_0 = \frac{1}{2}\alpha_0^2 + \sum_{m=1}^{+\infty} \alpha_{-m} \cdot \alpha_m \tag{3.10}$$

has ordering ambiguity. Thus we need to include a normal ordering constant $L_0 \to L_0 + a$. Because of the normal ordering, the algebra is now

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n} \tag{3.11}$$

Term proportional to c is due to quantum effects. And $c = d$ the dimension, meaning that each free scalar field contributes one unit to the central charge.

Virasoro constraints Now due to normal ordering constant, (2.25) cannot be fulfilled, instead

$$L_n \left|\text{phys.}\right\rangle = 0, \quad n > 0 \tag{3.12a}$$

$$(L_0 + a) \left|\text{phys.}\right\rangle = 0 \tag{3.12b}$$

For closed strings, the same apply for $\bar{L}_n$. In addition, we have level matching condition

$$(L_0 - \bar{L}_0) \left|\text{phys.}\right\rangle = 0 \tag{3.13}$$

Level number operator For open string

$$N = \sum_{n=1}^{\infty} (\alpha^{\mu}_{-n}\alpha_{\mu,n} + \alpha^{i}_{-n}\alpha_{i,n}) + \sum_{r \in N_0 + \frac{1}{2}} \alpha^{a}_{-r}\alpha_{a,r} \tag{3.14}$$

with μ for NN directions, i for DD directions and a for the mixed directions. With (3.12), one has

$$\alpha' m^2 = N + \alpha'(T\Delta x)^2 + a \tag{3.15}$$

For closed string (with (3.10))

$$L_0 = N + \frac{\alpha'}{4} p^2 \tag{3.16}$$

thus

$$m^2 = m_L^2 + m_R^2 = \frac{2}{\alpha'}\left(N + \bar{N} + 2a\right) \tag{3.17}$$

3.2 Light-cone Quantization

Light-cone coordinates are defined as (X^+, X^-, X^i) and

$$X^{\pm} = \frac{1}{\sqrt{2}}(X^0 \pm X^1) \tag{3.18}$$

So that the metric in $\pm$ direction is $\eta_{+-} = \eta_{-+} = -1$.

To fix the gauge,

$$X^+ = \frac{2\pi\alpha'}{l} p^+ \tau \tag{3.19}$$

With (2.15),

$$\partial_{\pm} X^- = \frac{l}{2\pi\alpha' p^+} (\partial_{\pm} X^i)^2 \tag{3.20}$$

Commutation relations

$$\left[q^-, p^+\right] = -i \tag{3.21a}$$
$$\left[q^i, p^j\right] = i\delta^{ij} \tag{3.21b}$$
$$\left[\alpha_n^i, \alpha_m^j\right] = n\delta^{ij}\delta_{n+m} \tag{3.21c}$$
$$\left[\bar{\alpha}_n^i, \bar{\alpha}_m^j\right] = n\delta^{ij}\delta_{m+n} \tag{3.21d}$$

3.3 Spectrum of the Bosonic String

Massive states can be classified by representation of $SO(d-1)$. Massless states can be classified by $E(d-2) \supset SO(d-2)$.

Open string spectrum Lorentz invariance can be kept intact, when $d = 26$. It leads to critical dimension. More rigorously, $[M^{i-}, M^{j-}]$ behaves normally at $d = 26$.

Regge trajectories

$$j_{\text{mas}} = n = \alpha' m^2 + 1 \tag{3.22}$$

Ground state has negative mass square, i.e. a tachyon.

Closed string spectrum has the Regge trajectories

$$j_{\max} = \frac{1}{2}\alpha' m^2 + 2 \tag{3.23}$$

Orientation Still need to fix original diffeomorphism invariance, in particular world-sheet parity transformation. Define a unitary operator

$$\Omega X^\mu(\sigma, \tau)\Omega^{-1} = X^\mu(l - \sigma, \tau) \tag{3.24}$$

Interpretation Massless spin two particles are always present in closed string spectrum. It can be identified with graviton. Massless vector states in open string can be identified with gauge bosons.

3.4 Covariant Path Integral Quantization

Partition function is

$$Z = \int \mathcal{D}h\mathcal{D}X e^{iS_{\text{P}}[h,X]} \tag{3.25}$$

By fixing the gauge, $h_{\mu\nu}$ is not dynamical and a determinant is introduced. The determinant then can be rewritten using ghost fields

$$Z = \int \mathcal{D}X^\mu \mathcal{D}c\mathcal{D}b \quad e^{iS[X,\hat{h},b,c]} \tag{3.26a}$$

$$S = -\frac{1}{4\pi\alpha'} \int \mathrm{d}^2\sigma \sqrt{-\hat{h}}\hat{h}^{\alpha\beta} \left\{ \partial_\alpha X^\mu \partial_\beta X_\mu + 2i\alpha' b_{\beta\gamma} \hat{\nabla}_\alpha c^\gamma \right\} \tag{3.26b}$$

Virasoro operators for ghost sector are z

$$L_m = \sum_{n=-\infty}^{+\infty} (m-n) :b_{m+n}c_{-n}: \tag{3.27a}$$

$$\bar{L}_m = \sum_{n=-\infty}^{+\infty} (m-n) :\bar{b}_{m+n}\bar{c}_{-n}: \tag{3.27b}$$

$$\tag{3.27c}$$

They satisfy

$$[L_m, L_n] = (m-n)L_{m+n} + A(m)\delta_{m+n} \tag{3.28}$$

with $A(m) = \frac{1}{12}(-26m^3 + 2m)$. Combing the Virasoro generators, then only when $a = -1$ and $d = 26$, the central charge term vanishes.

4 Introduction to Conformal Field Theory

4.1 General

Map to complex plane by

$$z = e^{\frac{2\pi}{l}(\tau - i\sigma)}, \quad \bar{z} = e^{\frac{2\pi}{l}(\tau + i\sigma)} \tag{4.1}$$

Radial ordering

$$R(\phi_1(z)\phi_1(w)) = \begin{cases} \phi_1(z)\phi_2(w) & |z| > |w| \\ \phi_2(w)\phi_1(z) & |z| < |w| \end{cases} \tag{4.2}$$

In conformal field theory, radial ordering is always implied.

Equal radius commutator

$$[\phi_1(z), \phi_2(w)]_{|z|=|w|} = \lim_{\delta\to 0} \left\{ (\phi_1(z)\phi_2(w))_{|z|=|w|+\delta} - (\phi_2(w)\phi_1(z))_{|z|=|w|-\delta} \right\} \tag{4.3}$$

$$\oint_{C_0} \frac{\mathrm{d}z}{2\pi i}[A(z), B(w)] = \oint_{C_w} \frac{\mathrm{d}z}{2\pi i} R(A(z)B(w)) \tag{4.4}$$

Conformal fields are defined by

$$\phi(z,\bar z) \to \phi'(z',\bar z') = \left(\frac{\partial z'}{\partial z'}\right)^{-h} \left(\frac{\partial \bar z'}{\partial \bar z}\right)^{-\bar h} \phi(z,\bar z) \tag{4.5}$$

Infinitesimally

$$\delta_{\xi,\bar\xi}\phi(z,\bar z) = -(h\partial\xi + \bar h\bar\partial\bar\xi + \xi\partial + \bar\xi\bar\partial)\phi(z,\bar z) \tag{4.6}$$

with h and $\bar h$ conformal weights. Tensor with $\bar h = 0$ ($h = 0$) are called (anti-)homomorphic tensors.

Mode expansion is

$$\phi_{\text{plane}}(z) = \sum_{n\in Z} \phi_n z^{-n-h} \Leftrightarrow \phi_n = \oint_{C_0} \frac{\mathrm{d}z}{2\pi i}\phi(z) z^{n+h-1} \tag{4.7}$$

Energy-momentum tensors must be conserved

$$\partial_{\bar z} T_{zz} = 0, \quad \partial_z T_{\bar z\bar z} = 0 \tag{4.8}$$

In the following denote $T_{zz}(z) = T(z)$ and $T_{\bar z\bar z}(\bar z) = \bar T(\bar z)$. The infinitesimal conformal transformation $z \to z + \xi(z)$ has the conserved charge

$$T_\xi = \oint_{C_0} \frac{\mathrm{d}z}{2\pi i}\xi(z)T(z) \tag{4.9}$$

(4.6) is now

$$\delta_\xi\phi(w) = -[T_\xi, \phi(w)] = -\oint_{C_w} \frac{\mathrm{d}z}{2\pi i}\xi(z)T(z)\phi(w) \tag{4.10}$$

Operator product expansions It leads to the OPE

$$T(z)\phi(w) = \frac{h\phi(w)}{(z-w)^2} + \frac{\partial\phi(w)}{(z-w)} + \text{reg.} \tag{4.11}$$

In general

$$O_i(z,\bar{z})O_j(w,\bar{w}) = \sum_k C_{ij}^k((z-w))O_k(w,\bar{w}) \tag{4.12}$$

OPE of energy-momentum tensor

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} + \text{reg.} \tag{4.13}$$

This is actually equivalent to the Virasoro algebra with central charge c. Expand

$$T(z) = \sum_n z^{-n-2}L_n \Leftrightarrow L_n = \oint \frac{\mathrm{d}z}{2\pi i} z^{n+1}T(z) \tag{4.14}$$

Then (3.11) is recovered and singular terms in OPE of energy-momentum tensor are equivalent to the Virasoro algebra.

Conformal group Conformal transformation leaves the metric invariant up to a scale (switch to d-dimensional flat spacetime temporarily)

$$\eta'_{\mu\nu} = \Lambda(x^\mu)\eta_{\mu\nu}(x^\mu) \tag{4.15}$$

In 1 dimension, any form of transformation is conformal.

In $d \leq 3$ dimension the infinitesimal transformation is given as

$$x^\mu \to x^\mu + \epsilon^\mu(x^\mu) \tag{4.16}$$

with

$$\epsilon_\mu = a_\mu + b_{\mu\nu}x^\nu + c_{\mu\nu\rho}x^\nu x^\rho \tag{4.17}$$

where a_μ leads to translation, $b_{\mu\nu}$ to rotation and scaling, and $c_{\mu\nu\rho}$ to special conformal transformation.

In 2d constraints of conformal transformation resemble the Cauchy-Riemann equations, thus conformal transformation in 2d is equivalent to (anti-)homomorphic coordinate transformation. The generator of the transformations are

$$l_m = -z^{m+1}\partial_z, \quad \bar{l}_m = -\bar{z}^{m+1}\bar{\partial} \tag{4.18}$$

Note that they are valid in classical theory. It shows that only $l_{-1,0,+1}$ are defined globally. Their corresponding finite transformation belong to special linear group $SL(2,\mathbb{C})$

$$z \to z' = \frac{az+b}{cz+d} \tag{4.19}$$

with $ad-bc=1$. Actually global conformal group is isometric to $SL(2,\mathbb{C})$.

Energy-momentum tensor under finite conformal transformations $z \to w(z)$ is

$$T(z) \to T'(w) = \left(\frac{\mathrm{d}z}{\mathrm{d}w}\right)^2 T(z) + \frac{c}{12}\{z,w\} \tag{4.20}$$

Here $\{z,w\}$ refers to Schwarzian derivative.

Representation Regularity of energy-momentum tensor (4.14) at $z=0$ ($\tau=-\infty$) gives

$$L_n \left|0\right\rangle = 0, \quad \text{for} \quad n > -2 \tag{4.21}$$

Its Hermitian conjugate holds for $n<2$ and same for $\bar{L}_n$. Similarly,

$$\phi_n \left|0\right\rangle = 0, \text{ for } n > -h \tag{4.22a}$$

$$\left\langle 0\right| \phi_n = 0, \text{ for } n < h \tag{4.22b}$$

State-operator correspondence

$$\left|\phi_{\text{in}}\right\rangle = \left|\phi\right\rangle = \lim_{z\to 0} \phi(z)\left|0\right\rangle = \phi(0)\left|0\right\rangle = \phi_{-h}\left|0\right\rangle \tag{4.23}$$

where the limit corresponds to $\tau \to -\infty$. The out states are

$$\left\langle\phi_{\text{out}}\right| = \lim_{z\to\infty} \left\langle 0\right| \phi^\dagger(z) z^{2h} = \left\langle 0\right| (\phi^\dagger)_h \tag{4.24}$$

Alternatively, another choice of out states is BPZ-conjugate

$$\left\langle\phi\right| = \lim_{z\to\infty} \left\langle 0\right| \phi(z) z^{2h} \tag{4.25}$$

The choices lead to

$$L_0 \left|\phi\right\rangle = h\left|\phi\right\rangle \tag{4.26a}$$

$$L_n \left|\phi\right\rangle \left|\phi\right\rangle = 0 \forall n > 0 \tag{4.26b}$$

It is to be compared with (3.12). States satisfy these conditions are called highest weight states. L_{-n} with ($n \geq 0$) raise the eigenvalue of L_0

$$L_0 \left(L_{-n}\left|\phi\right\rangle\right) = (n+h)\left(L_{-n}\left|\phi\right\rangle\right) \tag{4.27}$$

Complete Hilbert space Descendent states are constructed by

$$|\phi^{k_1 \ldots k_m}\rangle = L_{-k_1} \ldots L_{-k_m} |\phi\rangle \tag{4.28}$$

with $k_1 \geq \ldots k_m > 0$. They form Verma module $V(c, h)$ of the highest weight state.

Descendant fields are contained in the operator product of the primary fields with the energy-momentum tensor

$$T(z)\phi(w) = \sum_{k=0}^{\infty} (z-w)^{-2+k} \phi^{(k)}(w) \Leftrightarrow \phi^{(k)}(w) = L_{-k}\phi(w) \tag{4.29}$$

They are secondary fields. Descendent fields can have their own descendants, which is contained in OPE of $T(z)\phi^{(k_1)}(w)$. All these fields constitute the conformal family $[\phi]$. Energy-momentum tensor is an example of secondary fields.

Unitarity Requiring that $\langle \phi_j | L_n L_{-n} | \phi_j \rangle \geq 0$ leads to $c > 0$ and $h_j \geq 0$.

Correlation functions Correlation function need to be invariant under $SL(2, \mathbb{C})$. This requirement leads to for quasi-primary fields $\phi(z)$

- two-point function

$$\langle \phi_i(z) \phi_j(w) \rangle = \begin{cases} \frac{G_{ij}}{(z-w)^{2h_i}} & h_i = h_j \\ 0 & \text{otherwise} \end{cases} \tag{4.30}$$

- three-point function

$$\langle \phi_i(z_1) \phi_j(z_2) \phi_k(z_3) \rangle = \frac{C_{ijk}}{z_{12}^{h_i + h_j - h_k} z_{13}^{h_i + h_k - h_j} z_{23}^{h_j + h_k - h_i}} \tag{4.31}$$

4.2 Application to Closed String

Two-point function for free boson $X(z, \bar{z})$

$$\langle X(z, \bar{z}) X(w, \bar{w}) \rangle = -\frac{\alpha'}{2} \log |(z-w)/R|^2 \tag{4.32}$$

They are not primary. The derivative fields are (anti-)holomorphic fields

$$\langle \partial X(z) \partial X(w) \rangle = -\frac{\alpha'}{2} \frac{1}{(z-w)^2} \tag{4.33a}$$

$$\left\langle \bar{\partial} X(\bar{z}) \bar{\partial} X(\bar{w}) \right\rangle = -\frac{\alpha'}{2} \frac{1}{(\bar{z}-\bar{w})^2} \tag{4.33b}$$

Energy momentum tensor

$$T(z) = -\frac{1}{\alpha'} :\partial X(z)\partial X(z): \tag{4.34}$$

Wick's theorem

$$\phi_i(z)\phi_j(w) = \overbrace{\phi_i(z)\phi_j}(w) + :\phi_i(z)\phi_j(w): \tag{4.35}$$

Vertex operator

$$V_\alpha(z,\bar{z}) = :e^{i\alpha\phi(z,\bar{z})}: \tag{4.36}$$

4.3 The Ghost System as a CFT

Propagator

$$\langle b(z)c(w)\rangle = \langle c(z)b(w)\rangle = \frac{1}{z-w} \tag{4.37}$$

Energy-momentum tensor

$$T^{b,c}(z) = -2 :b\partial c(z): - :\partial bc(z): \tag{4.38}$$

OPE

$$T^{b,c}(z)T^{b,c}(w) = \frac{-26/2}{(z-w)^4} + \frac{2T^{b,c}(w)}{(z-w)^2} + \frac{\partial T^{b,c}(w)}{z-w} + \text{reg.} \tag{4.39}$$

Thus $c = 26$ for ghost systems and once again conformal anomaly is absent in $d = 26$.