Superstring theory
Homework 3

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February 4, 2021

1 Oscillator expansions for the classical string

(a) We make the ansatz

$$X^\mu(\sigma,\tau) = f(\sigma)g(\tau) \tag{1.1}$$

Then the equation of motion becomes

$$g(\tau)\frac{\partial^2 f(\sigma)}{\partial\sigma^2} - f(\sigma)\frac{\partial^2 g(\tau)}{\partial\tau^2} = 0$$

Clearly if we have

$$\frac{\partial^2 f(\sigma)}{\partial\sigma^2} = kf(\sigma), \quad \frac{\partial^2 g(\tau)}{\partial\tau^2} = kg(\tau) \tag{1.2}$$

the equation of motion is satisfied. Thus we write the solutions

$$\begin{aligned}
f(\sigma) &= Ae^{-i\sqrt{-k}\sigma} + Be^{i\sqrt{-k}\sigma},\\
g(\tau) &= A'e^{-i\sqrt{-k}\tau} + B'e^{i\sqrt{-k}\tau}, \text{ for } k \neq 0\\
f(\sigma) &= C\sigma + D, \quad g(\tau) = C'\tau + D', \text{ for } k = 0
\end{aligned}$$

To determine the constant k, we need to use periodic boundary condition ($m \in \mathbb{Z}$)

$$\begin{aligned}
l\sqrt{-k} &= 2\pi m\\
k &= -\frac{4\pi^2 m^2}{l^2}
\end{aligned}$$

Polynomial solution of $f(\sigma)$ cannot satisfy unless $C = 0$. Multiplying out $f(\sigma)$ and $g(\tau)$ and redefining the constants (so that they also carries Lorentz indices)

$$X^\mu(\sigma,\tau) = A^\mu + B^\mu \tau + \sum_{m\in\mathbb{Z}\setminus\{0\}} e^{i\sqrt{-k}\sigma}\left(C^\mu_m e^{-i\sqrt{-k}\tau} + D^\mu_m e^{i\sqrt{-k}\tau}\right) \quad (1.3)$$

Note that terms with $e^{-i\sqrt{-k}\sigma}$ gets absorbed in redefinition of constants.

(b) Equation of motion can be rewritten as

$$\partial_+\partial_- X^\mu = 0$$

Thus it makes sense to write the solution into left- and right-moving parts

$$X^\mu(\sigma,\tau) = X^\mu_L(\sigma^+) + X^\mu_R(\sigma^-) \quad (1.4)$$

These two parts are explicit given on the sheet

$$X^\mu_L(\sigma^+) = \frac{1}{2}(x^\mu - c^\mu) + \frac{\pi\alpha'}{l}p^\mu\sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{1}{n}\tilde{\alpha}^\mu_n e^{-\frac{2\pi}{l}in\sigma^+} \quad (1.5)$$

$$X^\mu_R(\sigma^-) = \frac{1}{2}(x^\mu - c^\mu) + \frac{\pi\alpha'}{l}p^\mu\sigma^- + i\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{1}{n}\alpha^\mu_n e^{-\frac{2\pi}{l}in\sigma^-} \quad (1.6)$$

We will set c^μ to zero in the following discussion.

(c) We suppose that the solution from part (b) should be used here.

We require that solutions X^μ must be real. Thus x^μ and p^μ (α' too) must be real. Last term is a bit more involved

$$\begin{aligned}\left(i\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{1}{n}\tilde{\alpha}^\mu_n e^{-\frac{2\pi}{l}in\sigma^+}\right)^* &= -i\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{1}{n}(\tilde{\alpha}^\mu_n)^* e^{\frac{2\pi}{l}in\sigma^+}\\
&= i\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{1}{-n}(\tilde{\alpha}^\mu_{-(-n)})^* e^{-\frac{2\pi}{l}i(-n)\sigma^+}\\
&= i\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{1}{n}(\tilde{\alpha}^\mu_{-n})^* e^{-\frac{2\pi}{l}in\sigma^+}\\
&\overset{!}{=} i\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{1}{n}\tilde{\alpha}^\mu_n e^{-\frac{2\pi}{l}in\sigma^+}\end{aligned}$$

Thus the condition for $\tilde{\alpha}$ is

$$\tilde{\alpha}_n^\mu = (\tilde{\alpha}_{-n}^\mu)^*$$

Analogically,

$$\alpha_n^\mu = (\alpha_{-n}^\mu)^*$$

(d) There are a couple of interesting quantities to look at

- total spacetime momentum

$$\begin{aligned}
P^\mu &= \frac{1}{2\pi\alpha'}\int_0^l \mathrm{d}\sigma\, \dot{X}^\mu \\
&= \frac{1}{2\pi\alpha'}\int_0^l \mathrm{d}\sigma\, (\partial_+ X_L^\mu + \partial_- X_R^\mu) \\
&= \frac{1}{2\pi\alpha'}\int_0^l \mathrm{d}\sigma\, \frac{2\pi}{l}\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}}\left(\tilde{\alpha}_n^\mu e^{-\frac{2\pi}{l}in(\tau+\sigma)} + \alpha_n^\mu e^{-\frac{2\pi}{l}in(\tau-\sigma)}\right)
\end{aligned}$$

Here the sum includes $n = 0$ mode, because we introduce

$$\alpha_0^\mu = \tilde{\alpha}_0^\mu = \sqrt{\frac{\alpha'}{2}}p^\mu$$

Obviously, integrals with $n \neq 0$ vanish because of periodicity of complex exponential function.

$$P^\mu = \frac{1}{2\pi\alpha'}\frac{2\pi}{l}\sqrt{\frac{\alpha'}{2}}l \cdot 2\sqrt{\frac{\alpha'}{2}}p^\mu = p^\mu \tag{1.7}$$

It shows that the p^μ is the total momentum.

- center of mass position

$$\begin{aligned}
q^\mu &= \frac{1}{l}\int_0^l \mathrm{d}\sigma\, X^\mu \\
&= \frac{1}{l}\int_0^l \mathrm{d}\sigma\, \left(x^\mu + 2\frac{\pi\alpha'}{l}p^\mu\tau\right) \\
&= x^\mu + 2\frac{\pi\alpha'}{l}p^\mu\tau
\end{aligned} \tag{1.8}$$

x^μ can be interpreted as center of mass position at $\tau = 0$.

- total angular momentum

$$
\begin{aligned}
J^{\mu\nu} &= \frac{1}{2\pi\alpha'}\int_0^l \mathrm{d}\sigma \left(X^\mu \dot{X}^\nu - X^\nu \dot{X}^\mu\right) \\
&= \frac{1}{2\pi\alpha'}\int_0^l \mathrm{d}\sigma \left(x^\mu + \cancel{2\frac{\pi\alpha'}{l}p^\mu\tau}\right)\dot{X}^\nu \\
&\quad + \frac{i}{2l}\int_0^l \mathrm{d}\sigma \sum_{n\neq 0}\left(\frac{1}{n}\tilde{\alpha}_n^\mu e^{-\frac{2\pi}{l}in(\tau+\sigma)} + \frac{1}{n}\alpha_n^\mu e^{-\frac{2\pi}{l}in(\tau-\sigma)}\right) \\
&\quad \times \sum_{n\in\mathbb{Z}}\left(\tilde{\alpha}_n^\mu e^{-\frac{2\pi}{l}in(\tau+\sigma)} + \alpha_n^\mu e^{-\frac{2\pi}{l}in(\tau-\sigma)}\right) - (\mu \leftrightarrow \nu)
\end{aligned}
$$

Here term with $p^\mu \dot{X}^\nu$ is symmetric in indices and it get cancelled by the $(\mu \leftrightarrow \nu)$ term. The same as before, only terms with constant in σ can be non-zero after integration. Thus we must have n of opposite signs for two sums.

$$
J^{\mu\nu} = x^\mu p^\nu + i\sum_{n>0}\frac{1}{n}\left(\tilde{\alpha}_n^\mu \tilde{\alpha}_{-n}^\nu - \alpha_n^\mu \alpha_{-n}^\nu\right) - (\mu \leftrightarrow \nu)
$$

Note that $\frac{1}{2}$ factor in the last term disappeared because we can have $-n$ from first summation and n from second summation in the $(\mu \leftrightarrow \nu)$ term. The sign from $1/n$ cancels with the overall sign in front. Thus two times the same contributions.

(e) Physically open strings can reflect left- and right-moving oscillations. Thus only one mode is necessary.

(f) The given mode expansion solves the bulk equation

$$
(\partial_\tau^2 - \partial_\sigma^2)X^\mu(\tau,\sigma) = i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-i\frac{\pi}{l}n\tau}\cos\Big(\frac{n\pi\sigma}{l}\Big)\left(-\frac{\pi}{l}n + \frac{n\pi}{l}\right) = 0
$$

And to check the boundary conditions

$$
X'^\mu = i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-i\frac{\pi}{l}n\tau}\sin\Big(\frac{n\pi\sigma}{l}\Big)\cdot\left(-\frac{n\pi}{l}\right)
$$

It indeed vanishes at $\sigma = 0$ and $\sigma = l$.

(g) This mode expansion also solves the bulk equation

$$
(\partial_\tau^2 - \partial_\sigma^2)X^\mu(\tau,\sigma) = \sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-i\frac{\pi}{l}n\tau}\sin\Big(\frac{n\pi\sigma}{l}\Big)\left(-\frac{\pi}{l}n + \frac{n\pi}{l}\right) = 0
$$

Because of $\sin(n\pi\sigma/l)$, third term vanishes at $\sigma = 0$ and $\sigma = l$. And $X^\mu(\sigma = 0) = x_0^\mu$, $X^\mu(\sigma = l) = x_l^\mu$. So the Dirichlet conditions are satisfied.

2 Poisson brackets for the classical closed string

From previous section, we have

$$X^\mu(\sigma,\tau) = (x^\mu - c^\mu) + 2\frac{\pi\alpha'}{l}p^\mu\tau + i\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}\backslash\{0\}}\frac{1}{n}\left(\tilde{\alpha}^\mu_n e^{-\frac{2\pi}{l}in(\tau+\sigma)} + \alpha^\mu_n e^{-\frac{2\pi}{l}in(\tau-\sigma)}\right)$$

$$\dot{X}^\mu(\sigma,\tau) = \frac{2\pi}{l}\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}}\left(\tilde{\alpha}^\mu_n e^{-\frac{2\pi}{l}in(\tau+\sigma)} + \alpha^\mu_n e^{-\frac{2\pi}{l}in(\tau-\sigma)}\right)$$

We need to "invert" the solutions, in order to express $\tilde{\alpha}^\mu_n$ and α^μ_n in terms of X^μ and $\dot{X}^\mu$. We want to multiply these two with an exponential function and integrate it out, so that only some certain modes can survive ($m \neq 0$)

$$\begin{aligned}\int_0^l \mathrm{d}\sigma\, X^\mu(\sigma,0)e^{\frac{2\pi}{l}im\sigma} &= i\sqrt{\frac{\alpha'}{2}}\left(\frac{1}{m}\tilde{\alpha}^\mu_m - \frac{1}{m}\alpha^\mu_{-m}\right)\\ \int_0^l \mathrm{d}\sigma\, \dot{X}^\mu(\sigma,0)e^{\frac{2\pi}{l}im\sigma} &= \frac{2\pi}{l}\sqrt{\frac{\alpha'}{2}}\left(\tilde{\alpha}^\mu_m + \alpha^\mu_{-m}\right)\end{aligned} \tag{2.1}$$

Thus we get

$$\tilde{\alpha}^\mu_m = \frac{1}{\sqrt{2\alpha'}}\int_0^l \mathrm{d}\sigma\, e^{\frac{2\pi}{l}im\sigma}\left(\frac{m}{i}X^\mu(\sigma,0) + \frac{l}{2\pi}\dot{X}^\mu(\sigma,0)\right) \tag{2.2}$$

$$\alpha^\mu_m = \frac{1}{\sqrt{2\alpha'}}\int_0^l \mathrm{d}\sigma\, e^{-\frac{2\pi}{l}im\sigma}\left(\frac{m}{i}X^\mu(\sigma,0) + \frac{l}{2\pi}\dot{X}^\mu(\sigma,0)\right) \tag{2.3}$$

where for α^μ_m, we used $m \to -m$.

To get the desired Poisson brackets

$$\begin{aligned}\{\tilde{\alpha}^\mu_n, \tilde{\alpha}^\nu_m\} &= \frac{1}{2\alpha'}\int_0^l \mathrm{d}\sigma\, e^{\frac{2\pi}{l}in\sigma}\int_0^l \mathrm{d}\sigma'\, e^{\frac{2\pi}{l}im\sigma'}\\ &\quad\times\left\{\frac{n}{i}X^\mu(\sigma,0) + \frac{l}{2\pi}\dot{X}^\mu(\sigma,0), \frac{m}{i}X^\nu(\sigma',0) + \frac{l}{2\pi}\dot{X}^\nu(\sigma',0)\right\}\\ &= \frac{1}{2\alpha'}\int_0^l \mathrm{d}\sigma\, e^{\frac{2\pi}{l}in\sigma}\int_0^l \mathrm{d}\sigma'\, e^{\frac{2\pi}{l}im\sigma'}\\ &\quad\times\left(\frac{nl}{2\pi i}\Big\{X^\mu(\sigma), \dot{X}^\nu(\sigma')\Big\} + \frac{ml}{2\pi i}\Big\{\dot{X}^\mu(\sigma), X^\nu(\sigma')\Big\}\right)\\ &= \frac{1}{2\alpha'}\int_0^l \mathrm{d}\sigma\, e^{\frac{2\pi}{l}in\sigma}\int_0^l \mathrm{d}\sigma'\, e^{\frac{2\pi}{l}im\sigma'}\frac{l}{2\pi Ti}\left(n\eta^{\mu\nu} - m\eta^{\nu\mu}\right)\delta(\sigma-\sigma')\\ &= \frac{1}{2\alpha'}\int_0^l \mathrm{d}\sigma\, e^{\frac{2\pi}{l}in\sigma}e^{\frac{2\pi}{l}im\sigma}\frac{l}{2\pi Ti}\left(n\eta^{\mu\nu} - m\eta^{\nu\mu}\right)\\ &= -in\delta_{m+n,0}\eta^{\mu\nu}\end{aligned} \tag{2.4}$$

The calculation is essentially the same for α^μ_m, the signs in the exponential cancel with each other.

$$\{a^\mu_n, a^\nu_m\} = -in\delta_{n+m,0}\eta^{\mu\nu} \tag{2.5}$$

Similarly

$$\begin{aligned}\{\alpha^\mu_n, \tilde{\alpha}^\nu_m\} &= \frac{1}{2\alpha'}\int_0^l \mathrm{d}\sigma\, e^{\frac{2\pi}{l}in\sigma}e^{-\frac{2\pi}{l}im\sigma}\frac{l}{2\pi Ti}\left(n\eta^{\mu\nu} - m\eta^{\nu\mu}\right)\\ &= 0\end{aligned} \tag{2.6}$$

For $m = 0$ in equation (2.1),

$$\begin{aligned}\int_0^l \mathrm{d}\sigma\, X^\mu(\sigma, 0) &= l(x^\mu - c^\mu)\\ \int_0^l \mathrm{d}\sigma\, \dot{X}^\mu(\sigma, 0) &= 2\pi\alpha' p^\mu\end{aligned}$$

Set $c^\mu = 0$ (or Poisson brackets with c^μ vanish)

$$\{x^\mu, x^\nu\} = \frac{1}{l^2}\int_0^l \mathrm{d}\sigma\int_0^l \mathrm{d}\sigma'\,\{X^\mu(\sigma), X^\nu(\sigma')\} = 0 \tag{2.7}$$

$$\{p^\mu, p^\nu\} = \frac{1}{4\pi^2\alpha'^2}\int_0^l \mathrm{d}\sigma\int_0^l \mathrm{d}\sigma'\left\{\dot{X}^\mu(\sigma), \dot{X}^\nu(\sigma')\right\} = 0 \tag{2.8}$$

$$\begin{aligned}\{x^\mu, p^\nu\} &= \frac{1}{2l\pi\alpha'}\int_0^l \mathrm{d}\sigma\int_0^l \mathrm{d}\sigma'\left\{X^\mu(\sigma), \dot{X}^\nu(\sigma')\right\}\\ &= \frac{1}{2l\pi\alpha'}\int_0^l \mathrm{d}\sigma\int_0^l \mathrm{d}\sigma'\,\frac{1}{T}\eta^{\mu\nu}\delta(\sigma - \sigma')\\ &= \eta^{\mu\nu}\end{aligned} \tag{2.9}$$

3 A spaghetti stick as solution to the string equations of motion

We have

$$\begin{aligned}X^0 &= A\tau,\\ X^1 &= A\cos\tau\cos\sigma,\\ X^2 &= A\sin\tau\cos\sigma,\\ X^i &= 0 \text{ for } i = 3, \ldots, d-1\end{aligned} \tag{3.1}$$

(a) To verify it is indeed a solution,

$$(\partial_\tau^2 - \partial_\sigma^2)X^1 = (-1+1)X^1 = 0$$
$$(\partial_\tau^2 - \partial_\sigma^2)X^2 = (-1+1)X^2 = 0$$

And it is trivial for all other dimensions.

(b) X^0 has Neumann boundary conditions, as the σ-derivative vanishes identically. X^1 and X^2 have Neumann conditions again, since after differentiation they both proportional to $\sin\sigma$ which vanishes for $\sigma = 0$ and $\sigma = l$ if $l \in \pi\mathbb{N}$.

(c) The energy

$$\begin{aligned} M = P^0 &= \frac{1}{2\pi\alpha'}\int_0^l \mathrm{d}\sigma\,\partial_\tau X^0 \\ &= \frac{Al}{2\pi\alpha'} \end{aligned}$$

(d) Angular momentum

$$\begin{aligned} J = |J^{12}| &= \frac{1}{2\pi\alpha'}\int_0^l \mathrm{d}\sigma\left(X^1\dot{X}^2 - X^2\dot{X}^1\right) \\ &= \frac{1}{2\pi\alpha'}A^2\int_0^l \mathrm{d}\sigma\left(\cos^2\tau\cos^2\sigma + \sin^2\tau\cos^2\sigma\right) \\ &= \frac{A^2}{4\pi\alpha'}\left[\sin\sigma\cos\sigma + \sigma\right]_0^l \\ &= \frac{A^2}{4\pi\alpha'}l \end{aligned}$$

In the last step, the Neumann boundary condition is used.

(e)

$$\frac{J}{M^2} = \frac{A^2 l}{4\pi\alpha'}\frac{(2\pi\alpha')^2}{A^2 l^2} = \frac{\pi\alpha'}{l}$$