Superstring theory
Homework 11

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1 Two-point and three-point correlation functions

A primary field $\phi(z)$ transform under the infinitesimal conformal transformation $z \to z + \epsilon(z)$ as

$$\delta_\epsilon \phi(z) = -\left[\epsilon(z)\partial_z + h\partial_z \epsilon(z)\right]\phi(z) \tag{1.1}$$

Now consider (quasi-)primary fields ϕ_i with conformal dimension h_i

1. Conformal invariance of two-point function $G(z_1, z_2)$ implies

$$\begin{aligned}\delta_{\epsilon(z_1),\epsilon(z2)}G(z_1, z_2) &= \left\langle \delta_{\epsilon(z_1)}\phi_1\phi_2 \right\rangle + \left\langle \phi_1\delta_{\epsilon(z_2)}\phi_2 \right\rangle \\ &= -\left[\epsilon(z_1)\partial_{z_1} + h_1\partial_{z_1}\epsilon(z_1) + \epsilon(z_2)\partial_{z_2} + h_2\partial_{z_2}\epsilon(z_2)\right]G(z_1, z_2) \\ &= 0\end{aligned} \tag{1.2}$$

2. With $\epsilon(z_i) = \alpha$ in (1.2), we have

$$\begin{aligned}\alpha\left(\partial_{z_1} + \partial_{z_2}\right)G(z_1, z_2) &= 0 \\ \partial_{\bar{z}_{12}}G(z_1, z_2) &= 0\end{aligned}$$

with $z_{12} = z_1 - z_2$ and $\bar{z}_{12} = z_1 + z_2$. Thus $G(z_1, z_2)$ depends only on $z_{12} = z_1 - z_2$. It follows that $\partial_{z_1}G = -\partial_{z_2}G$.

3. Use $\epsilon(z_i) = \beta z_i$ in (1.2), we have

$$\begin{aligned}
\beta\left(z_1\partial_{z_1} + h_1 + z_2\partial_{z_2} + h_2\right)G(z_1, z_2) &= 0 \\
(z_1 - z_2)\frac{\partial_{z_1}G}{G} + (h_1 + h_2) &= 0 \\
\int \mathrm{d}z_1\,\frac{\partial_{z_1}G}{G} + (h_1 + h_2)\int \mathrm{d}z_1\,\frac{1}{z_1 - z_2} &= 0 \\
\ln|G| + (h_1 + h_2)\ln|z_1 - z_2| + C &= 0
\end{aligned}$$

Indeed

$$G(z_1, z_2) = \frac{C_{12}}{(z_1 - z_2)^{h_1+h_2}} \tag{1.3}$$

is a solution and C_{12} is related to the integration constant.

4. Use $\epsilon(z_i) = \gamma z_i^2$, we have

$$\gamma\left[z_1^2\partial_{z_1} + 2h_1z_1 + z_2^2\partial_{z_2} + 2h_2z_2\right]G(z_1, z_2) = 0$$

Plug (1.3) into it

$$\begin{aligned}
&\gamma C_{12}\left[z_1^2\partial_{z_1} + 2h_1z_1 + z_2^2\partial_{z_2} + 2h_2z_2\right]\frac{1}{(z_1 - z_2)^{h_1+h_2}} \\
&= \gamma C_{12}\left[-(z_1^2 - z_2^2)\frac{h_1 + h_2}{z_1 - z_2} + 2(h_1z_1 + h_2z_2)\right]\frac{1}{(z_1 - z_2)^{h_1+h_2}} \\
&= \gamma C_{12}\left[(h_1 - h_2)z_1 + (h_2 - h_1)z_2\right]\frac{1}{(z_1 - z_2)^{h_1+h_2}}
\end{aligned}$$

In order for it to vanish, we need to demand $h_1 = h_2$ otherwise $G(z_1, z_2)$ should be zero.

5. For three-point function, (1.2) becomes

$$\begin{aligned}
&\big[\epsilon(z_1)\partial_{z_1} + h_1\partial_{z_1}\epsilon(z_1) + \epsilon(z_2)\partial_{z_2} + h_2\partial_{z_2}\epsilon(z_2) \\
&\quad + \epsilon(z_3)\partial_{z_3} + h_3\partial_{z_3}\epsilon(z_3)\big]\langle\phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\rangle = 0
\end{aligned} \tag{1.4}$$

   - With $\epsilon(z_i) = \alpha$, we have

$$\begin{aligned}
\alpha\left(\partial_{z_1} + \partial_{z_2} + \partial_{z_3}\right)\langle\phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\rangle &= 0 \\
\left(\partial_{\bar{z}_{12}} + \partial_{\bar{z}_{13}} + \partial_{\bar{z}_{23}}\right)\langle\phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\rangle &= 0
\end{aligned} \tag{1.5}$$

   Obviously $\langle\phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\rangle = f(z_{12}, z_{13}, z_{23})$ is a solution. To show it must be the case, note that the variables z_{ij} actually depend on each other. We can write $\partial_{\bar{z}_{12}} = \partial_{\bar{z}_{23}} - \partial_{\bar{z}_{13}}$, thus $\partial_{\bar{z}_{23}}\langle\phi\phi\phi\rangle = 0$. The same can be done with other z_{ij}, then indeed the solution is unique.

- With $\epsilon(z_i) = \beta z_i$, we have

$$\beta \left[z_1 \partial_{z_1} + z_2 \partial_{z_2} + z_3 \partial_{z_3} + h_1 + h_2 + h_3 \right] f(z_{12}, z_{13}, z_{23}) = 0$$

It can be solved by

$$f(z_{12}, z_{13}, z_{23}) = \frac{C_{123}}{z_{12}^{\alpha} z_{13}^{\beta} z_{23}^{\gamma}}$$

with $\alpha + \beta + \gamma = h_1 + h_2 + h_3$, since

$$\begin{aligned}
&\beta \left[z_1 \partial_{z_1} + z_2 \partial_{z_2} + z_3 \partial_{z_3} + h_1 + h_2 + h_3 \right] \frac{C_{123}}{z_{12}^{\alpha} z_{13}^{\beta} z_{23}^{\gamma}} \\
&= \beta \Bigg[z_1 \left(-\frac{\alpha}{z_{12}} - \frac{\beta}{z_{13}} \right) + z_2 \left(\frac{\alpha}{z_{12}} - \frac{\gamma}{z_{23}} \right) \\
&\quad + z_3 \left(\frac{\beta}{z_{13}} + \frac{\gamma}{z_{23}} \right) + h_1 + h_2 + h_3 \Bigg] \frac{C_{123}}{z_{12}^{\alpha} z_{13}^{\beta} z_{23}^{\gamma}} \\
&= \beta \left(-\alpha - \beta - \gamma + h_1 + h_2 + h_3 \right) f(z_{12}, z_{13}, z_{23}) \\
&= 0
\end{aligned}$$

- With $\epsilon(z_i) = \gamma z_i^2$, we have

$$\begin{aligned}
&\gamma \left[z_1^2 \partial_{z_1} + z_2^2 \partial_{z_2} + z_3^2 \partial_{z_3} + 2h_1 z_1 + 2h_2 z_2 + 2h_3 z_3 \right] f(z_{12}, z_{13}, z_{23}) = 0 \\
&z_1^2 \left(-\frac{\alpha}{z_{12}} - \frac{\beta}{z_{13}} \right) + z_2^2 \left(\frac{\alpha}{z_{12}} - \frac{\gamma}{z_{23}} \right) + z_3^2 \left(\frac{\beta}{z_{13}} + \frac{\gamma}{z_{23}} \right) \\
&\quad + 2h_1 z_1 + 2h_2 z_2 + 2h_3 z_3 = 0 \\
&-\alpha(z_1 + z_2) - \beta(z_1 + z_3) - \gamma(z_2 + z_3) + 2h_1 z_1 + 2h_2 z_2 + 2h_3 z_3 = 0
\end{aligned}$$

This has to be satisfied with arbitrary z_i, thus

$$\begin{aligned}
\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} &= 2 \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} \\
\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} &= \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix}
\end{aligned}$$

Then

$$\langle \phi_1(z_1) \phi_2(z_2) \phi_3(z_3) \rangle = \frac{C_{123}}{z_{12}^{h_1+h_2-h_3} z_{13}^{h_1-h_2+h_3} z_{23}^{-h_1+h_2+h_3}} \tag{1.6}$$