Theoretical Astroparticle Physik
Homework 2

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1 Equation of state for matter and radiation

(a) Compute the number density of non-relativstic particles

Maxwell-Boltzmann distribution can be approximated as

$$f_{\mathrm{MB}}(\vec{p}) = \frac{1}{(2\pi)^3}\exp\left(-\frac{E(\vec{p})-\mu}{T}\right) \approx \frac{1}{(2\pi)^3}\exp\left(\frac{\mu-m}{T}-\frac{|\vec{p}|^2}{2mT}\right)$$

For now on $|\vec{p}| = p$. The number density is defines as

$$\begin{aligned}
n &= g_i \int \mathrm{d}^3p\, f(\vec{p}) \\
&= \frac{g_i}{(2\pi)^3}\exp\left(\frac{\mu-m}{T}\right)\int \mathrm{d}^3p \exp\left(-\frac{|\vec{p}|^2}{2mT}\right) \\
&= \frac{4\pi g_i}{(2\pi)^3}\exp\left(\frac{\mu-m}{T}\right)\int \mathrm{d}|\vec{p}|\,|\vec{p}|^2 \exp\left(-\frac{|\vec{p}|^2}{2mT}\right) \\
&= \frac{4\pi g_i}{(2\pi)^3}\exp\left(\frac{\mu-m}{T}\right)\frac{\sqrt{\pi}}{2}(2mT)^{3/2} \\
n &= g_i \exp\left(\frac{\mu-m}{T}\right)\left(\frac{mT}{2\pi}\right)^{3/2} \qquad (1.1)
\end{aligned}$$

(b) Determine the pressure and find the equation of state

The pressure is given as

$$
\begin{aligned}
P &= g_i \int \mathrm{d}^3p \, \frac{p^2}{3E} f(\vec{p}) \\
&= \frac{g_i}{(2\pi)^3} \frac{4\pi}{3} \exp\left(\frac{\mu - m}{T}\right) \int \mathrm{d}p \, p^2 \frac{p^2}{E} \exp\left(-\frac{p^2}{2mT}\right) \\
&\approx \frac{g_i}{6\pi^2 m} \exp\left(\frac{\mu - m}{T}\right) \int \mathrm{d}p \, p^4 \exp\left(-\frac{p^2}{2mT}\right) \\
&= \frac{g_i}{8\pi^2 m} \exp\left(\frac{\mu - m}{T}\right) (2mT)^{5/2} \sqrt{\pi} \\
&\overset{1.1}{=} nT
\end{aligned}
\tag{1.2}
$$

In the limit $T \ll m$, the equation of state is

$$
\omega = \frac{P}{\rho} = \frac{T}{m} \approx 0 \tag{1.3}
$$

(c) Determine the energy density in the limit $T \gg E \gg m$

First to show $E \, \mathrm{d}E = p \, \mathrm{d}p$

$$
\frac{\mathrm{d}E}{\mathrm{d}p} = \frac{\mathrm{d}}{\mathrm{d}p} \sqrt{m^2 + p^2} = \frac{E}{p}
$$

$$
\begin{aligned}
\rho &= g_i \int \mathrm{d}^3p \, E(\vec{p}) f(\vec{p}) \\
&= \frac{g_i}{(2\pi)^3} \int \frac{\mathrm{d}^3p \, E(\vec{p})}{\exp(E(\vec{p})/T) \mp 1} \\
&= \frac{g_i}{2\pi^2} \int_0^\infty \frac{\mathrm{d}p \, p^2 E}{\exp(E(\vec{p})/T) \mp 1} \\
&= \frac{g_i}{2\pi^2} \int_m^\infty \frac{\mathrm{d}E \, E^2 \sqrt{E^2 - m^2}}{\exp(E(\vec{p})/T) \mp 1} \\
&\approx \frac{g_i}{2\pi^2} \int_0^\infty \frac{\mathrm{d}E \, E^3}{\exp(E(\vec{p})/T) \mp 1} \\
&= \begin{cases} \frac{g_i}{2\pi^2} T^4 \zeta(4) \Gamma(4) & \text{bosons} \\ \frac{g_i}{2\pi^2} T^4 \frac{7\pi^4}{120} & \text{fermions} \end{cases} \\
&= (1; 7/8) \cdot g_i \frac{\pi^2 T^4}{30}
\end{aligned}
$$

Here $(1; 7/8)$ denotes that it is 1 for bosons and 7/8 for fermions. For boson case, the integral representation of riemannian zeta function is used. Mathematic did the calculation of fermions.

(d) Determine the pressure and further the equation of state

Following the same recipe as before

$$\begin{aligned} P &= \frac{g_i}{(2\pi)^3} \int \mathrm{d}^3 p \, \frac{p^2}{3E} \frac{1}{\exp(E/T) \mp 1} \\ &\approx \frac{g_i}{6\pi^2} \int_0^\infty \frac{\mathrm{d}E \, E^3}{\exp(E/T) \mp 1} \\ &= \rho/3 \end{aligned} \tag{1.4}$$

This is the equation of state for ultra-relativstic particles with $\omega = 1/3$.