Theoretical Astroparticle Physik
Homework 1

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1 Quickies

(a) Briefly describe in your own words what is meant by a spatially isotropic and homogeneous universe.

It means in this Universe every direction should look the same (isotropic) and every part of it looks the same (homogeneous).

(b) State the definition of the Hubble parameter $H(t)$. What does the Hubble constant H_0 describe?

The Hubble parameter is defined as

$$H(t) := \frac{\dot{a}(t)}{a(t)}, \tag{1.1}$$

where $a(t)$ is the scale factor in the FLRW-metric. H_0 refers to the current value of $H(t)$ and it describe the current expansion rate of the Universe.

(c) The Hubble constant is usually parametrized as $H_0 = h \cdot 100\,\mathrm{km\,Mpc^{-1}\,s^{-1}}$, where $h \approx 0.6 - 0.7$ depends on the exact measurement. Convert H_0 into natural units.

$$\begin{aligned}
H_0 &= h \cdot 100\,\mathrm{km\,Mpc^{-1}\,s^{-1}} \\
&= 6.5 \times 10^4\,\mathrm{m} \cdot (3.1 \times 10^{22}\,\mathrm{m})^{-1}\mathrm{s}^{-1} \\
&= 2.1 \times 10^{-18}\,\mathrm{s}^{-1} \\
&= 2.1 \times 10^{-18} \cdot 6.58 \times 10^{-16}\,\mathrm{eV} \\
&= 1.4 \times 10^{-33}\,\mathrm{eV}
\end{aligned}$$

2 Cutoff for high energy astro-physical neutrinos

(a) Determine the energy E_ν

The neutrinos have the following 4-momenta

$$
\begin{aligned}
p_1 &= (m_\nu, 0, 0, 0) \\
p_2 &= (\sqrt{p^2 + m_\nu^2}, 0, 0, p)
\end{aligned}
$$

To activate the scattering process, one needs $s = m_Z^2$. LHS can be written as

$$
\begin{aligned}
s &= (p_1 + p_2)^2 \\
&= (m_\nu + \sqrt{p^2 + m_\nu^2}, 0, 0, p)^2 \\
&= 2m_\nu^2 + 2m_\nu\sqrt{p^2 + m_\nu^2} \\
&\approx 2m_\nu p \overset{!}{=} m_Z^2
\end{aligned}
$$

where higher order terms in m_ν are ignored. Thus

$$E \approx p = m_Z^2/2m_\nu = 41.6\,\mathrm{TeV} \tag{2.1}$$

(b) Estimate the mean free path l

$$
\begin{aligned}
l &\approx (\sigma_{\nu\bar{\nu}} n_\nu)^{-1} \\
&= (1.5 \times 10^{-31}\,\mathrm{cm}^2 \cdot 55\,\mathrm{cm}^{-3})^{-1} \\
&= 1.2 \times 10^{29}\,\mathrm{cm} \\
&= 3.9 \times 10^{12}\,\mathrm{pc} \\
&= 3.9 \times 10^{6}\,\mathrm{Mpc}
\end{aligned}
$$

(c) Find expression for E_3 and what is its minimal and maximal values? Can the reaction occur again for the outgoing neutrinos with largest possible energy?

We can write out the momenta as

$$
\begin{aligned}
p_1 &= (E_\nu, 0, 0, \sqrt{E_\nu^2 - m_\nu^2}) \\
p_2 &= (m_\nu, 0, 0, 0) \\
p_3 &= (E_3, 0, \sin\theta p_3, \cos\theta p_3)
\end{aligned}
$$

Following the hint to find out p_4 (equivalent to 4-momentum conservation)

$$
\begin{aligned}
t = (p_2 - p_3)^2 &= (p_4 - p_2)^2 \\
(E_\nu - E_3, 0, -\sin\theta p_3, \sqrt{E_\nu^2 - m_\nu^2} - \cos\theta p_3)^2 &= (E_4 - m_\nu, \boldsymbol{p}_4)^2 \\
(E_\nu - E_3)^2 - \sin^2\theta p_3^2 - (E_\nu - \cos\theta p_3)^2 &= E_4^2 - 2m_\nu E_4 - E_4^2 + \mathcal{O}(m_\nu^2) \\
\Rightarrow E_4 &= \frac{E_\nu E_3}{m_\nu}(1-\cos\theta)
\end{aligned}
$$

From energy conservation

$$
\begin{aligned}
E_3 &= E_\nu - E_4 \\
&= E_\nu - \frac{E_\nu E_3}{m_\nu}(1-\cos\theta) \\
&= \frac{E_\nu}{1 + \frac{E_\nu}{m_\nu}(1-\cos\theta)}
\end{aligned}
\tag{2.2}
$$

As function of scattering angle, its max and min values are

$$
\max(E_3) = E_\nu \tag{2.3}
$$

$$
\min(E_3) \approx m_\nu/2 \tag{2.4}
$$

Since in the case of maximal E_3, the neutrino doesn't lose energy in scattering, the process can occur again (and again).

To our understanding, distinction between inelastic and elastic scattering is at whether the kinetic energy is conserved or not. If it is true, scattering of $\nu\bar{\nu}$ into $\nu\bar{\nu}$ is purely elastic, since the rest masses stay the same and kinetic energy is conserved. On the sheet, the sentence confuses us: *For the elastic channels it is clear that the high energy neutrino flux will be depleted.*

(d) Is there also a cutoff for neutrinos?

Yes, there is, if there is no high energy neutrino sources within the $l_{\nu\bar{\nu}}$.

P.S. Should we consider t-channel processes as well? If we need to consider it, then the cross section will be modified. (Maybe it is already included in $\sigma_{\nu\bar{\nu}}$.)

3 Friedmann-Lemaitre-Robertson-Walker metric

Sorry that we are using the $(-,+,+,+)$ convention.

$$ds^2 = -dt^2 + a(t)^2 \left[\frac{dr^2}{1-\kappa r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{3.1}$$

(a) Since the metric is diagonal, the entries of inverse metric is just reciprocal of the entries of the metric.

$$g_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & \frac{a^2}{1-\kappa r^2} & & \\ & & a^2r^2 & \\ & & & a^2r^2\sin^2\theta \end{pmatrix}$$

$$g^{\mu\nu} = \begin{pmatrix} -1 & & & \\ & \frac{1-\kappa r^2}{a^2} & & \\ & & \frac{1}{a^2r^2} & \\ & & & \frac{1}{a^2r^2\sin^2\theta} \end{pmatrix}$$

(b) Note that:

(i) one can write $g_{ii} = a^2 f_{ii}$, where

$$\begin{aligned} f_{11} &= \frac{1}{1-\kappa r^2} \\ f_{22} &= r^2 \\ f_{33} &= r^2\sin^2\theta \end{aligned}$$

(ii) $g^{\mu\mu} = (g_{\mu\mu})^{-1}$ and $f^{ii} = (f_{ii})^{-1}$.

(iii) $g_{33} = g_{22}\sin^2\theta$.

Christoffel symbos is defined as

$$\begin{aligned} \Gamma^\lambda_{\mu\nu} &= g^{\lambda\rho}\Gamma_{\rho\mu\nu} \\ &= \frac{1}{2}g^{\lambda\rho}(\partial_\nu g_{\rho\mu} + \partial_\mu g_{\rho\nu} - \partial_\rho g_{\mu\nu}) \end{aligned}$$

For $\lambda = 0$, $\mu = \nu = 1$,

$$\Gamma^0_{11} = \frac{1}{2}g^{0\rho}(\partial_1 g_{\rho 1} + \partial_1 g_{\rho 1} - \partial_\rho g_{11})$$

$g_{\mu\nu}$ is symmetric, only $g_{00}, g_{11}, g_{22}, g_{33}$ is nonvanishing,

$$\begin{aligned}
\Gamma^0_{11} &= \frac{1}{2} g^{00}(\partial_1 g_{01} + \partial_1 g_{01} - \partial_0 g_{11}) \\
&= \frac{1}{2}(-1)(-\partial_t(\frac{a^2}{1-\kappa r^2})) \\
&= \frac{a\dot{a}}{1-\kappa r^2}
\end{aligned}$$

For $\lambda = 1$, $\mu = \nu = 1$,

$$\begin{aligned}
\Gamma^1_{11} &= \frac{1}{2} g^{11}(\partial_1 g_{11} + \partial_1 g_{11} - \partial_1 g_{11}) \\
&= \frac{1}{2} g^{11} \partial_1 g_{11} \\
&= \frac{1}{2}(\frac{1-\kappa r^2}{a^2})(\partial_r \frac{a^2}{1-\kappa r^2}) \\
&= \frac{1}{2}(\frac{1-\kappa r^2}{a^2})(-2\kappa r)(\frac{a^2}{(1-\kappa r^2)^2}) \\
&= \frac{\kappa r}{1-\kappa r^2}
\end{aligned}$$

For $\lambda = 0$, $\mu = \nu = 2$,

$$\begin{aligned}
\Gamma^0_{22} &= \frac{1}{2} g^{00}(\partial_2 g_{02} + \partial_2 g_{02} - \partial_0 g_{22}) \\
&= \frac{1}{2} g^{00}(-\partial_0 g_{22}) \\
&= (-1)(\frac{1}{2})(-\partial_t a^2 r^2) \\
&= \frac{1}{2}(2a\dot{a}r^2 \sin^2\theta) = a\dot{a}r^2
\end{aligned}$$

For $\lambda = 0$, $\mu = \nu = 3$,

$$\begin{aligned}
\Gamma^0_{33} &= \frac{1}{2} g^{00}(\partial_3 g_{03} + \partial_3 g_{03} - \partial_0 g_{33}) \\
&= \frac{1}{2} g^{00}(-\partial_0 g_{33}) \\
&= \frac{1}{2} g^{00}(-\partial_t(g_{22}\sin^2\theta)) \\
&= a\dot{a}r^2 \sin^2\theta
\end{aligned}$$

For $\lambda = \nu = i$, $\mu = 0$,

$$\begin{aligned}
\Gamma^i_{0i} &= \frac{1}{2} g^{ii}(\partial_0 g_{ii} + \partial_i g_{i0} - \partial_i g_{0i}) \\
&= \frac{1}{2}(\frac{1}{a(t)^2 f_{ii}}(\partial_t(a(t)^2 f_{ii}))) \\
&= \frac{1}{2}\frac{2a\dot{a}}{a^2} \\
&= \frac{\dot{a}}{a}
\end{aligned}$$

For $\lambda = 1$, $\mu = \nu = 2$,

$$\begin{aligned}
\Gamma^1_{22} &= \frac{1}{2} g^{11}(\partial_2 g_{12} + \partial_2 g_{12} - \partial_1 g_{22}) \\
&= \frac{1}{2} g^{11}(-\partial_r g_{22}) \\
&= \frac{1}{2}(\frac{1-\kappa r^2}{a^2})(-\partial_r(a^2 r^2)) \\
&= \frac{1}{2}(1-\kappa r^2)(-2r) \\
&= -r(1-\kappa r^2)
\end{aligned}$$

For $\lambda = 1$, $\mu = \nu = 3$,

$$\begin{aligned}
\Gamma^1_{33} &= \frac{1}{2} g^{11}(\partial_3 g_{13} + \partial_3 g_{13} - \partial_1 g_{33}) \\
&= \frac{1}{2} g^{11}(-\partial_r g_{22}\sin^2\theta) \\
&= \frac{1}{2} g^{11}(-\partial_r g_{22})\sin^2\theta \\
&= \sin^2\theta \Gamma^1_{22} \\
&= -r(1-\kappa r^2)\sin^2\theta
\end{aligned}$$

For $\lambda = \nu = 2$, $\mu = 1$,

$$\begin{aligned}
\Gamma^2_{12} &= \frac{1}{2} g^{22}(\partial_1 g_{22} + \partial_2 g_{12} - \partial_2 g_{12}) \\
&= \frac{1}{2}\frac{1}{a^2 r^2}\partial_r(a^2 r^2) \\
&= \frac{1}{2}\frac{1}{a^2 r^2} 2a^2 r \\
&= \frac{1}{r}
\end{aligned}$$

For $\lambda = \nu = 3$, $\mu = 1$,

$$\begin{aligned}
\Gamma^3_{13} &= \frac{1}{2} g^{33} (\partial_1 g_{33} + \partial_3 g_{13} - \partial_3 g_{13}) \\
&= \frac{1}{2} \frac{1}{a^2 r^2 \sin^2 \theta} \partial_r (a^2 r^2 \sin^2 \theta) \\
&= \frac{1}{2} \frac{1}{a^2 r^2 \sin^2 \theta} 2a^2 r \sin^2 \theta \\
&= \frac{1}{r}
\end{aligned}$$

For $\lambda = \nu = 3$, $\mu = 2$,

$$\begin{aligned}
\Gamma^3_{23} &= \frac{1}{2} g^{33} (\partial_2 g_{33} + \partial_3 g_{32} - \partial_3 g_{23}) \\
&= \frac{1}{2} g^{33} (\partial_2 g_{33}) \\
&= \frac{1}{2} \frac{1}{a^2 r^2 \sin^2 \theta} (\partial_\theta (a^2 r^2 \sin^2 \theta)) \\
&= \frac{1}{2} \frac{1}{a^2 r^2 \sin^2 \theta} 2a^2 r^2 \sin \theta \cos \theta \\
&= \frac{\cos \theta}{\sin \theta}
\end{aligned}$$

For $\lambda = 2$, $\mu = \nu = 3$,

$$\begin{aligned}
\Gamma^2_{33} &= \frac{1}{2} g^{22} (\partial_3 g_{23} + \partial_3 g_{23} - \partial_2 g_{33}) \\
&= \frac{1}{2} g^{22} (-\partial_2 g_{33}) \\
&= \frac{1}{2} g^{33} \sin^2 \theta (-\partial_2 g_{33}) \\
&= -\sin^2 \theta \Gamma^3_{23} \\
&= -\sin \theta \cos \theta
\end{aligned}$$