Theoretical Astroparticle Physik
Homework 3

Chenhuan Wang and Koay Yong Sheng

May 17, 2020

1 Quickies

(a) Definition of critical density

$$\rho_c = \frac{3}{8\pi G} H_0^2 \tag{1.1}$$

When $\rho_M + \rho_{\text{rad}} + \rho_\Lambda = \rho_c$, it means that there is no curvature contribution, thus the Universe is flat. According to the definition of ρ_{curv}, it could also mean that $a = \pm\infty$.

(b) Particle horizon, or cosmological horizon, is the maximal distance from which light could traveled to observer since the beginning of the Universe[1]. It is relevant to discussion of the whole Universe. Event horizon is the boundary where events inside cannot affect or be perceived in any ways by outside, e.g. near a black hole. And it is normally in smaller scale than particle horizon.

2 Accelerated Expansion in a de-Sitter Universe

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \exp(2H_{dS}t)\,\mathrm{d}\vec{x}^2 \tag{2.1}$$

(a) Determine the Riemann curvature tensor

Set $H_{\text{dS}} = H$ throughout this assignment. Curvature tensor $R_{\mu\nu\rho\sigma}$ has mass dimension 2 because of two derivatives. To achieve this, only quantity we have is just H^2. We also know that $R_{\mu\nu\lambda\rho}$ is anti-symmetric in first and second pair of indices and symmetric under $(\mu,\nu) \leftrightarrow (\lambda,\rho)$. The only choice is then

$$R_{\mu\nu\lambda\rho} = H^2(g_{\mu\rho}g_{\nu\lambda} - g_{\mu\lambda}g_{\nu\rho}) \tag{2.2}$$

To show this explicitely, first calculate christoffel symbols

$$\Gamma^{\mu}_{\nu\lambda} = \frac{1}{2}g^{\mu\rho}\left(\partial_\nu g_{\rho\lambda} + \partial_\lambda g_{\nu\rho} - \partial_\rho g_{\nu\lambda}\right) \tag{2.3}$$

One important fact about the metric is that there is only time dependence and thus spatial derivatives always vanish.

$$\begin{aligned}
\Gamma^0_{00} &= \frac{1}{2}g^{0\rho}(\partial g_{\rho 0} + \partial_0 g_{0\rho} - \partial_\rho g_{00}) \\
&= 0 \\
\Gamma^0_{0i} &= \Gamma^0_{i0} = \frac{1}{2}\left(\partial_i g_{\rho 0} + \partial_0 g_{\rho i} - \partial_\rho g_{0i}\right) \\
&= \frac{1}{2}\partial_0 g_{0i} \\
&= 0 \\
\Gamma^i_{00} &= \frac{1}{2}g^{i\rho}(\partial_0 g_{\rho 0} + \partial_0 g_{\rho 0} - \partial_\rho g_{00}) \\
&= 0 \\
\Gamma^i_{0j} &= \Gamma^i_{j0} = \frac{1}{2}g^{i\rho}\left(\partial_0 g_{\rho j} + \partial_j g_{\rho 0} - \partial_\rho g_{0j}\right) \\
&= -\frac{1}{2}e^{-2Ht}\delta^{i\rho}\left(\partial_0 g_{\rho j} - \partial_\rho g_{0j}\right) \\
&= -\frac{1}{2}e^{-2Ht}\delta^i_j\partial_0 e^{2Ht} \\
&= -H\delta^i_j \\
\Gamma^0_{ij} &= \Gamma^0_{ji} = \frac{1}{2}g^{0\rho}\left(\partial_i g_{\rho j} + \partial_j g_{\rho i} - \partial_\rho g_{ij}\right) \\
&= -\frac{1}{2}g^{0\rho}\partial_\rho g_{ij} \\
&= \delta_{ij}He^{2Ht} \\
\Gamma^i_{jk} &= 0
\end{aligned}$$

Definition of curvature tensor

$$R_{\mu\nu\lambda\rho} = g_{\mu\sigma}\left(\partial_\lambda\Gamma^\sigma_{\nu\rho} - \partial_\rho\Gamma^\sigma_{\nu\lambda} + \Gamma^\sigma_{\kappa\lambda}\Gamma^\kappa_{\nu\rho} - \Gamma^\sigma_{\kappa\rho}\Gamma^\kappa_{\lambda\nu}\right) \tag{2.4}$$

Because of mentioned symmetries, we split the tensor into two parts

$$R_{\mu\nu\lambda\rho} = A_{\mu\nu\lambda\rho} - A_{\mu\nu\rho\lambda}$$

Now try to compute $A_{\mu\nu\lambda\rho}$

$$A_{\mu\nu\lambda\rho} = g_{\mu\sigma}\partial_\lambda \Gamma^\sigma_{\nu\rho} + g_{\mu\sigma}\Gamma^\sigma_{\kappa\lambda}\Gamma^\kappa_{\nu\rho} = B + C$$

in which

$$\begin{aligned}
B &= g_{\mu\sigma}\partial_\lambda \Gamma^\sigma \nu\rho \\
&= \left(\delta_{\mu 0}\delta^0_\sigma - e^{2Ht}\delta_{\mu i}\delta^i_0\right)\partial_\lambda \Gamma^\sigma_{\nu\rho} \\
&= \partial_\lambda \Gamma^0_{\nu\rho}\delta_{\mu 0} - e^{2Ht}\partial_\lambda \Gamma^i_{\nu\rho}\delta_{\mu i} \\
&= \delta_{\lambda 0}\left(\partial_0 \Gamma^0_{\nu\rho}\delta_{\mu 0} - e^{2Ht}\partial_0 \Gamma^i_{\nu\rho}\delta_{\mu i}\right) \\
&= \delta_{\mu 0}\delta_{\nu i}\delta^i_\rho \delta_{\lambda 0} 2H^2 e^{2Ht}
\end{aligned}$$

and

$$\begin{aligned}
C &= g_{\mu\sigma}\Gamma^\sigma \kappa\lambda \Gamma^\kappa_{\nu\rho} \\
&= \delta_{\mu 0}\Gamma^0_{i\lambda}\Gamma^i_{\nu\rho} - e^{2Ht}\delta_{\mu i}\Gamma^i_{\kappa\lambda}\Gamma^\kappa_{\nu\rho} \\
&= \delta_{\mu 0}\delta_{i\lambda}He^{2Ht}\left(\delta_{\nu 0}\Gamma^i_{0\rho} + \delta_{\rho 0}\Gamma^i_{\nu 0}\right) - e^{2Ht}\delta_{\mu i}\left(\delta_{\mu 0}\Gamma^i_{0\lambda} + \delta_{\lambda 0}\Gamma^i_{\kappa 0}\right)\Gamma^\kappa_{\nu\rho} \\
Ce^{-2Ht} &= -H^2\delta_{\mu 0}\delta_{i\lambda}\left(\delta_{\nu 0}\delta^i_\rho + \delta_{\rho 0}\delta^i_\nu\right) + H\delta_{\mu i}\left(\delta_{\kappa 0}\delta^i_\lambda H + \delta_{\lambda 0}\delta^i_\kappa\right)\Gamma^\kappa_{\nu\rho} \\
Ce^{-2Ht} &= -H^2\delta_{\mu 0}\delta_{i\lambda}\left(\delta_{\nu 0}\delta^i_\rho + \delta_{\rho 0}\delta^i_\nu\right) + H\delta_{\mu i}\left(\Gamma^0_{\nu\rho}\delta^i_\lambda + \delta_{\lambda 0}\Gamma^i_{\nu\rho}\right) \\
Ce^{-2Ht} &= -H^2\delta_{\mu 0}\delta_{i\lambda}\left(\delta_{\nu 0}\delta^i_\rho + \delta_{\rho 0}\delta^i_\nu\right) + H^2\delta_{\mu i}\left[\delta^i_\mu \delta_{\nu j}\delta^j_\rho e^{2Ht} - \delta_{\lambda 0}\left(\delta_{\nu 0}\delta_{\nu 0}\delta^i_\rho + \delta_{\rho 0}\delta^i_\nu\right)\right]
\end{aligned}$$

Thus

$$\begin{aligned}
A_{\mu\nu\rho\lambda}/(He^{2Ht}) =& 2\delta_{\mu 0}\delta_{\nu i}\delta^i_\rho \delta_{\lambda 0} + e^{2Ht}\delta_{\mu i}\delta_{\nu j}\delta^j_\rho \delta^i_\lambda - \delta_{\mu i}\delta_{\nu 0}\delta^i_\rho \delta_{0\lambda} \\
& - \cancel{\delta_{\mu i}\delta^i_\nu \delta_{i\lambda}\delta^i_\rho} - \cancel{\delta_{\mu 0}\delta_{\nu 0}\delta_{i\lambda}\delta^i_\rho} - \delta_{\mu 0}\delta^i_\nu \delta_{\rho 00}\delta_{i\lambda}
\end{aligned}$$

Some of these terms get canceled because of symmtry in ρ, λ. In the end, we have

$$R_{\mu\nu\lambda\rho} = H^2(g_{\mu\rho}g_{\nu\lambda} - g_{\mu\lambda}g_{\nu\rho}) \tag{2.5}$$

(b) Solve Einstein field equations

$$\begin{aligned}
R_{\mu\nu} &= R^{\lambda}_{\mu\lambda\nu} \\
&= g^{\alpha\lambda} R_{\alpha\mu\lambda\nu} \\
&= H^2 g^{\alpha\lambda}(g_{\alpha\nu}g_{\mu\lambda} - g_{\alpha\lambda}g_{\mu\nu}) \\
&= H^2(g_{\mu\nu} - 4g_{\mu\nu}) \\
&= -3H^2 g_{\mu\nu} \\
R &= g^{\mu\nu} R_{\mu\nu} \\
&= -12H^2 \\
G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \\
&= H^2 \left(-3g_{\mu\nu} + 6g_{\mu\nu}\right) \\
&= 3H^2 g_{\mu\nu} \\
&\overset{!}{=} 8\pi G \Lambda g_{\mu\nu}
\end{aligned}$$

Thus

$$\Lambda = \frac{3H^2}{8\pi G} \tag{2.6}$$

3 Age of the universe for some toy cosmologies

Assumming we are using the ΛCDM model, the Friedman equation is as follows:

$$\begin{aligned}
H^2 = \left(\frac{\dot{a}}{a}\right)^2 &= \frac{8\pi}{3} G \left(\rho_{mat} + \rho_{rad} + \rho_{\Lambda} + \rho_{curv}\right) \\
&= \frac{8\pi}{3} G \rho_c \left(\frac{\rho_{mat}}{\rho_c} + \frac{\rho_{rad}}{\rho_c} + \frac{\rho_{\Lambda}}{\rho_c} + \frac{\rho_{curv}}{\rho_c}\right) \\
&= \frac{8\pi}{3} G \rho_c \left[\Omega_{mat}\left(\frac{a_0}{a}\right)^3 + \Omega_{rad}\left(\frac{a_0}{a}\right)^4 + \Omega_{\Lambda} + \Omega_{curv}\left(\frac{a_0}{a}\right)^2\right]
\end{aligned} \tag{3.1}$$

where,

$$\frac{8\pi}{3} G \rho_{curv} = -\frac{\kappa}{a^2} \tag{3.2}$$

$$\rho_c = \frac{3}{8\pi G} H_0^2 \tag{3.3}$$

(a) We are considering a spatially open $\kappa = -1$universe where the only non-vanishing energy densities are those from matter and curvature.

$$\Omega_{mat} \neq 0,\ \Omega_{rad} = 0,\ \Omega_{curv} \neq 0,\ \Omega_{\Lambda} = 0 \tag{3.4}$$
$$\Omega_{mat} + \Omega_{curv} = 1 \tag{3.5}$$

Given these conditions,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G\rho_c \left[\Omega_{mat}\left(\frac{a_0}{a}\right)^3 + \Omega_{curv}\left(\frac{a_0}{a}\right)^2\right]$$
$$(\dot{a})^2 = \frac{8\pi}{3} G\rho_c \left[\Omega_{mat}\left(\frac{a_0^3}{a}\right) + \Omega_{curv} a_0^2\right] \tag{3.6}$$

let $a(t) = A(t)x$ with $A(t_0) = 1$, which leads to

$$a_0 = x,$$
$$a(t) = A(t)a_0 \tag{3.7}$$

Substituting eq (3.7) to eq (3.6)

$$(\dot{A}a_0)^2 = \frac{8\pi}{3} G\rho_c \left[\Omega_{mat}\left(\frac{a_0^2}{A}\right) + \Omega_{curv} a_0^2\right]$$
$$\dot{A}^2 = \frac{8\pi}{3} G\rho_c \left[\Omega_{mat}\left(\frac{1}{A}\right) + \Omega_{curv}\right]$$
$$\dot{A} = \sqrt{\frac{8\pi}{3} G\rho_c \left[\Omega_{mat}\left(\frac{1}{A}\right) + \Omega_{curv}\right]}$$
$$\frac{dA}{dt} = \sqrt{\frac{8\pi}{3} G\rho_c \left[\Omega_{mat}\left(\frac{1}{A}\right) + \Omega_{curv}\right]}$$
$$dt = \frac{dA}{\sqrt{\frac{8\pi}{3} G\rho_c \left[\Omega_{mat}\left(\frac{1}{A}\right) + \Omega_{curv}\right]}}$$
$$\int_0^{t_0} dt = \int_0^{A(t_0)=1} dA \frac{1}{\sqrt{\frac{8\pi}{3} G\rho_c \left[\Omega_{mat}\left(\frac{1}{A}\right) + \Omega_{curv}\right]}}$$

$$
\begin{aligned}
t_0 &= \int_0^1 dA \frac{1}{\sqrt{\frac{8\pi}{3} G\rho_c \left[\Omega_{mat}\left(\frac{1}{A}\right) + \Omega_{curv}\right]}} \\
&= \int_0^1 dA \frac{1}{\sqrt{\frac{8\pi}{3} G\rho_c \left[\Omega_{mat}\left(\frac{1}{A}\right) + 1 - \Omega_{mat}\right]}} \\
&= \int_0^1 dA \frac{1}{\sqrt{H_0^2 \left[(0.3)\left(\frac{1}{A}\right) + 0.7\right]}} \\
&= \int_0^1 dA \frac{1}{\sqrt{(2.269 \cdot 10^{-18} s^{-1})^2 \left[(0.3)\left(\frac{1}{A}\right) + 0.7\right]}} \\
&= 3.5645 \cdot 10^{17}\text{s} \\
&= 11.3 \text{ billion years}
\end{aligned}
$$

(b) Now considering a different universe, spatially flat and with matter and dark energy left,

$$
\Omega_{mat} \neq 0,\ \Omega_{rad} = 0,\ \Omega_{curv} = 0,\ \Omega_\Lambda \neq 0 \tag{3.8}
$$

$$
\Omega_{mat} + \Omega_\Lambda = 1 \tag{3.9}
$$

Starting with the continuity equation,

$$
\begin{aligned}
\dot{\rho_\Lambda} + 3\frac{\dot{a}}{a}(\rho_\Lambda + P_\Lambda) = 0 \\
\dot{\rho_\Lambda} + 3\frac{\dot{a}}{a}\rho_\Lambda(1+\omega) = 0
\end{aligned}
$$

$$
\begin{aligned}
\dot{\rho_\Lambda} &= -3\frac{\dot{a}}{a}\rho_\Lambda(1+\omega) \\
\frac{d\rho_\Lambda}{\rho_\Lambda} &= \frac{-3(1+\omega)}{a}\frac{da}{dt}dt \\
\ln\left(\frac{\rho_\Lambda}{\rho_{\Lambda,0}}\right) &= -3(1+\omega)\int_{a_0}^{a}\frac{da}{a} \\
\frac{\rho_\Lambda}{\rho_{\Lambda,0}} &= \frac{a}{a_0}e^{-3(1+\omega)} \\
\rho_\Lambda &= \rho_{\Lambda,0}\frac{a}{a_0}e^{-3(1+\omega)} \qquad (3.10)
\end{aligned}
$$

Here, we have found the ρ_Λ's dependence on the scale factor a.Plugging it back to the Friedmann equation.

$$\begin{aligned}
\left(\frac{\dot{a}}{a}\right)^2 &= \frac{8\pi}{3}G\rho \\
&= \frac{8\pi}{3}G(\rho_m + \rho_\Lambda) \\
&= \frac{8\pi}{3}G\rho_c\left(\Omega_m\left(\frac{a_0}{a}\right)^3 + \frac{\rho_\Lambda}{\rho_c}\right) \\
&= \frac{8\pi}{3}G\rho_c\left(\Omega_m\left(\frac{a_0}{a}\right)^3 + \frac{\rho_{\Lambda,0}}{\rho_c}\frac{a}{a_0}e^{-3(1+\omega)}\right) \\
\left(\frac{\dot{a}}{a}\right)^2 &= \frac{8\pi}{3}G\rho_c\left(\Omega_m\left(\frac{a_0}{a}\right)^3 + \Omega_\Lambda\frac{a_0}{a}e^{3(1+\omega)}\right) \qquad (3.11)
\end{aligned}$$

(c) Now, just like how we did it in the previous scenario,

$$\begin{aligned}
\left(\frac{\dot{a}}{a}\right)^2 &= H_0^2\left(\Omega_m\left(\frac{a_0}{a}\right)^3 + \Omega_\Lambda\frac{a_0}{a}e^{3(1+\omega)}\right) \\
\dot{a}^2 &= H_0^2\left(\Omega_m\left(\frac{a_0^3}{a}\right) + \Omega_\Lambda a_0 a e^{3(1+\omega)}\right) \\
(\dot{A}a_0)^2 &= H_0^2\left(\Omega_m\left(\frac{a_0^2}{A}\right) + \Omega_\Lambda a_0^2 A e^{3(1+\omega)}\right) \\
\dot{A}^2 &= H_0^2\left(\Omega_m\left(\frac{1}{A}\right) + \Omega_\Lambda A e^{3(1+\omega)}\right) \\
\frac{dA}{dt} &= \sqrt{H_0^2\left(\Omega_m\left(\frac{1}{A}\right) + \Omega_\Lambda A e^{3(1+\omega)}\right)} \\
t_0 &= \int_0^1 dA\frac{1}{\sqrt{H_0^2\left(\Omega_m\left(\frac{1}{A}\right) + \Omega_\Lambda A e^{3(1+\omega)}\right)}} \\
&= \int_0^1 dA\frac{1}{\sqrt{(2.269\cdot 10^{-18}s^{-1})^2\left(0.27\left(\frac{1}{A}\right) + 0.73Ae^{3(1+\omega)}\right)}}
\end{aligned}$$

For $\omega = -1.1$,

$$\begin{aligned}
t_0 &= \int_0^1 dA\frac{1}{\sqrt{(2.269\cdot 10^{-18}s^{-1})^2\left(0.27\left(\frac{1}{A}\right) + 0.73Ae^{3(1-1.1)}\right)}} \\
&= 4.31976\cdot 10^{17}s \\
&= 13.6979 \text{ billion years}
\end{aligned}$$

For $\omega = -0.9$,

$$t_0 = \int_0^1 dA \frac{1}{\sqrt{(2.269 \cdot 10^{-18} s^{-1})^2 \left(0.27 \left(\frac{1}{A}\right) + 0.73 A e^{3(1-0.9)}\right)}}$$
$$= 3.81096 \cdot 10^{17} s$$
$$= 12.0845 \text{ billion years}$$

References

[1] *Particle horizon.* URL: https://en.wikipedia.org/wiki/Particle_horizon.