Theoretical Astroparticle Physik
Homework 5

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June 6, 2020

1 Time of recombination from photon last scattering

a) Assume the expansion was only cause by non-relativistic matter with $\Omega_m = 0.27$. From previous homework, we know its energy density is proportional to its number density

$$\rho_M = m_M \cdot n_M \tag{1.1}$$

Per definition the ratios of Ω is the same as the ratio of energy density

$$m_M n_M = \frac{\Omega_M}{\Omega_B} m_p n_B^{tot}(T) \tag{1.2}$$

and we say the baryonic matter in this epoch is simply protons or hydrogen atoms, thus the mass $m_B = m_p$.

Again the baryonic matter density is directly related to photon density

$$n_B^{tot} = \eta_B \cdot n_\gamma(T) = \eta_B \frac{2\zeta(3)}{\pi^2} T^3 \tag{1.3}$$

Putting these three equations (1.1), (1.2), and (1.3) together

$$\rho_M = \frac{\Omega_M}{\Omega_B} m_p \eta_b \frac{2\zeta(3)}{\pi^2} T^3 \tag{1.4}$$

In matter-dominated epoch (the first equation is valid for all types of Universe with zero curvature), we have

$$\rho = \frac{3}{8\pi G} H^2 \tag{1.5}$$

$$H(t) = \frac{2}{3t} \tag{1.6}$$

Then the Hubble parameter as last scattering is

$$\begin{aligned}
H(T_r) &= \left[\frac{8\pi G}{3}\frac{\Omega_M}{\Omega_B}m_p\eta_b\frac{2\zeta(3)}{\pi^2}T^3\right]^{1/2} \\
&= \left[\frac{8\pi}{3}\frac{0.27}{0.046}6.2\cdot 10^{-10}\frac{2\cdot 1.20}{\pi^2}\frac{938.3\,\mathrm{MeV}\cdot(0.26\,\mathrm{eV})^3}{(1.22\times 10^{19}\,\mathrm{GeV})^2}\right]^{1/2} \\
&= 2.87\times 10^{-38}\,\mathrm{GeV} \\
&= 1.435\times 10^{-24}\,\mathrm{cm}^{-1} \\
&= 4.45\,\mathrm{Mpc}^{-1} \qquad (1.7)
\end{aligned}$$

The time can be computed with (1.6)

$$t = \frac{2}{3H} = 2.32\times 10^{37}\,\mathrm{GeV}^{-1} = 15.3\times 10^{12}\,\mathrm{s} = 4.85\times 10^5\,\mathrm{year} \quad (1.8)$$

b) Now we have matter and rediation contributions. The Friedmann equation becomes

$$H^2(t) = \frac{8\pi G}{3}\rho = \frac{8\pi G}{3}\left(\rho_{\mathrm{rad}} + \rho_M\right)$$

Energy density of radiation, considering all relativitstic species, follows

$$\rho_{\mathrm{rad}} = g_*\frac{\pi^2}{30}T^4$$

where g_* is the effective degrees of freedom. Without exact knowledge of the Universe, in particular the relativistic particles, g_* is hard to determine. Thus we don't really want to plug it into Friedmann equation.

However, we also know that the energy density of radiation $\rho_{\mathrm{rad}} \sim a^{-4}$. Then we can write

$$a = \frac{c}{T} \quad (1.9)$$

with c a numerical constant and its exact value will be irrelevant.

The Friedmann equation with both contributions reads

$$\begin{aligned}
\left(\frac{\dot{a}}{a}\right)^2 &= \frac{8\pi G}{3}\rho_c\left[\Omega_M\left(\frac{a_0}{a}\right)^3 + \Omega_{\mathrm{rad}}\left(\frac{a_0}{a}\right)^4\right] \\
\frac{\mathrm{d}a}{\mathrm{d}t} &= H_0\left(\Omega_M\frac{a_0^3}{a} + \Omega_{\mathrm{rad}}\frac{a_0^4}{a^2}\right)^{1/2} \\
\int_0^{t_r}\mathrm{d}t &= \frac{1}{H_0}\int_0^{a_r}\mathrm{d}a\left(\Omega_M\frac{a_0^3}{a} + \Omega_{\mathrm{rad}}\frac{a_0^4}{a^2}\right)^{-1/2}
\end{aligned}$$

Do a simple substitution

$$\mathrm{d}a = -\frac{a}{T}\,\mathrm{d}T$$

The integral on RHS gets flipped because of the negative sign

$$\begin{aligned}
\int_0^{t_r} \mathrm{d}t &= \frac{1}{H_0}\int_{T_r}^{\infty} \frac{\mathrm{d}T}{T}\left(\Omega_M \frac{a_0^3}{a^3} + \Omega_{\mathrm{rad}}\frac{a_0^4}{a^4}\right)^{-1/2} \\
t_r &= \frac{1}{H_0}\int_{T_r}^{\infty} \frac{\mathrm{d}T}{T}\left(\Omega_M \frac{T^3}{T_0^3} + \Omega_{\mathrm{rad}}\frac{T^4}{T_0^4}\right)^{-1/2}
\end{aligned}$$

With Python and its package, the integral is nuemerically evaluated

$$t_r = 2.1 \times 10^6\,\text{year} \tag{1.10}$$

which cannot be right.

c) The Universe transitioned from radiation dominated to matter dominated at 0.76 eV. Thus in order to get the correct time of last scattering, one need to include radiation.