Theoretical Astroparticle Physik
Homework 5

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1 Quickies

a) Briefly explain why the temperature of about 0.3 eV, at which the production of neutral hydrogen becomes thermodynamically favored, is less than one would naively expect from the binding energy of neutral hydrogen, which is 13.6 eV.

Like last homework, one can write out saha equation and solve the recombination temeperature at $n_p \sim n_H$. To give a more physical and intuitive explanation without detailed calculation, we consider the kinetic aspects of the interaction $e + p \leftrightarrow H + \gamma$.

We assume that proton densities are small. (Rightfully so, if the chemical potential is small relative to the temperature in the distribution functions, and the number density is exponentially suppressed. It will be shown in the next part.) The time for a electron with several protons to interact in the forward direction is $\tau_+ \propto 1/\Gamma \propto 1/n_B$, where we take all baryons are in form of proton (or at least fixed proportional of baryons). Since the photon is thermally distributed, when the temperature is lower than the ionization energy, there was still photons able to ionize hydrogen (backward interaction). The time it takes is then $\tau_- \propto e^{\Delta_H/T}$. We define the recombination time as the time at $\tau_+ \sim \tau_-$, thus for small n_B, we have $\Delta \gg T$.

b) The baryon asymmetry of our universe can be quantified with the time independent ratio $\Delta_B = \frac{n_B - n_{\bar{B}}}{s}$, where n_B ($n_{\bar{B}}$) is the number density of baryons (anti-baryons) and s is the entropy density of the universe. It turns out that $\Delta_B \approx 0.14\eta_B$, where $\eta_B \approx 6.1 \times 10^{-10}$ is the baryon to photon r ratio inferred from measurements of the CMB and nuclear abundances from BBN. What does the smallness of this number imply for the chemical potential of baryons μ_B at temperatures much larger than both µB and the baryon mass?

The number densities difference of baryons and anti-baryons (obtained from last exercise).

$$\frac{\Delta n}{g} = \alpha \mu T^2 \text{ with } \alpha \begin{cases} 1/6 \text{ for fermions} \\ 1/3 \text{ for bosons} \end{cases} \tag{1.1}$$

The entropy density (taken from the text), The entropy (taken from the text) is,

$$s = g_* \frac{2\pi^2}{45} T^3 \tag{1.2}$$

From these equations and the defintion of Δ_B, we see that

$$\Delta_B \propto \frac{\mu_B}{T} \tag{1.3}$$

as we know that α, g_B, g_* is of order $0.1 \sim 1$ Thus, the smallness of baryon asymmetry implies the relative smallness of the chemical potential when compared to the temperature (an assumption used for the previous question).

2 Time of recombination from photon last scattering

a) Assume the expansion was only cause by non-relativistic matter with $\Omega_m = 0.27$. From previous homework, we know its energy density is proportional to its number density

$$\rho_M = m_M \cdot n_M \tag{2.1}$$

Per definition the ratios of Ω is the same as the ratio of energy density

$$m_M n_M = \frac{\Omega_M}{\Omega_B} m_p n_B^{tot}(T) \tag{2.2}$$

and we say the baryonic matter in this epoch is simply protons or hydrogen atoms, thus the mass $m_B = m_p$.

Again the baryonic matter density is directly related to photon density (time dependence of energy density is cancelled out)

$$n_B^{tot} = \eta_B \cdot n_\gamma(T) = \eta_B \frac{2\zeta(3)}{\pi^2} T^3 \tag{2.3}$$

Putting these three equations (2.1), (2.2), and (2.3) together

$$\rho_M = \frac{\Omega_M}{\Omega_B} m_p \eta_b \frac{2\zeta(3)}{\pi^2} T^3 \tag{2.4}$$

In matter-dominated epoch (the first equation is valid for all types of Universe with zero curvature), we have

$$\rho = \frac{3}{8\pi G} H^2 \tag{2.5}$$

$$H(t) = \frac{2}{3t} \tag{2.6}$$

Then the Hubble parameter as last scattering is

$$\begin{aligned} H(T_r) &= \left[\frac{8\pi G}{3} \frac{\Omega_M}{\Omega_B} m_p \eta_b \frac{2\zeta(3)}{\pi^2} T^3\right]^{1/2} \\ &= \left[\frac{8\pi}{3} \frac{0.27}{0.046} 6.2 \cdot 10^{-10} \frac{2 \cdot 1.20}{\pi^2} \frac{938.3\,\mathrm{MeV} \cdot (0.26\,\mathrm{eV})^3}{(1.22 \times 10^{19}\,\mathrm{GeV})^2}\right]^{1/2} \\ &= 2.87 \times 10^{-38}\,\mathrm{GeV} \\ &= 1.435 \times 10^{-24}\,\mathrm{cm}^{-1} \\ &= 4.45\,\mathrm{Mpc}^{-1} \end{aligned} \tag{2.7}$$

The time can be computed with (2.6)

$$t = \frac{2}{3H} = 2.32 \times 10^{37}\,\mathrm{GeV}^{-1} = 15.3 \times 10^{12}\,\mathrm{s} = 4.85 \times 10^5\,\mathrm{year} \tag{2.8}$$

b) Now we have matter and rediation contributions. The Friedmann equation becomes

$$H^2(t) = \frac{8\pi G}{3} \rho = \frac{8\pi G}{3} (\rho_{\mathrm{rad}} + \rho_M)$$

Energy density of radiation, considering all relativitstic species, follows

$$\rho_{\mathrm{rad}} = g_* \frac{\pi^2}{30} T^4$$

where g_* is the effective degrees of freedom. Without exact knowledge of the Universe, in particular the relativistic particles, g_* is hard to determine. Thus we don't really want to plug it into Friedmann equation.

However, we also know that the energy density of radiation $\rho_{\text{rad}} \sim a^{-4}$. Then we can write

$$a = \frac{c}{T} \tag{2.9}$$

with c a numerical constant and its exact value will be irrelevant.

The Friedmann equation with both contributions reads

$$\begin{aligned}
\left(\frac{\dot{a}}{a}\right)^2 &= \frac{8\pi G}{3}\rho_c \left[\Omega_M \left(\frac{a_0}{a}\right)^3 + \Omega_{\text{rad}} \left(\frac{a_0}{a}\right)^4\right] \\
\frac{\mathrm{d}a}{\mathrm{d}t} &= H_0 \left(\Omega_M \frac{a_0^3}{a} + \Omega_{\text{rad}} \frac{a_0^4}{a^2}\right)^{1/2} \\
\int_0^{t_r} \mathrm{d}t &= \frac{1}{H_0} \int_0^{a_r} \mathrm{d}a \left(\Omega_M \frac{a_0^3}{a} + \Omega_{\text{rad}} \frac{a_0^4}{a^2}\right)^{-1/2}
\end{aligned}$$

Do a simple substitution

$$\mathrm{d}a = -\frac{a}{T}\,\mathrm{d}T$$

The integral on RHS gets flipped because of the negative sign

$$\begin{aligned}
\int_0^{t_r} \mathrm{d}t &= \frac{1}{H_0} \int_{T_r}^{\infty} \frac{\mathrm{d}T}{T} \left(\Omega_M \frac{a_0^3}{a^3} + \Omega_{\text{rad}} \frac{a_0^4}{a^4}\right)^{-1/2} \\
t_r &= \frac{1}{H_0} \int_{T_r}^{\infty} \frac{\mathrm{d}T}{T} \left(\Omega_M \frac{T^3}{T_0^3} + \Omega_{\text{rad}} \frac{T^4}{T_0^4}\right)^{-1/2}
\end{aligned}$$

With Python and its package, the integral is nuemerically evaluated

$$t_r = 2.1 \times 10^6 \,\text{year} \tag{2.10}$$

which cannot be right.

Maybe use $g_* = 3.36$? But why it does not work here?

c) The Universe transitioned from radiation dominated to matter dominated at 0.76 eV. Thus in order to get the correct time of last scattering, one needs to include radiation.