

Theoretical Astroparticle Physik

Notes and summary

Chenhuan Wang

July 16, 2020

1 Introduction to GR

Nah

2 Introduction to cosmology

Gravitational constant

$$G = M_{pl}^{-2}, \quad [G] = -2 \quad (2.1)$$

Current value of Hubble parameter

$$H_0 = (70.5 \pm 1.3) \text{ km s}^{-1} \text{ Mpc}^{-1} \quad (2.2)$$

$$1 \text{ year} \approx \pi \cdot 10^7 \text{ s} \quad (2.3)$$

CMB temerature

$$T_0 = (2.726 \pm 0.001) \text{ K} \quad (2.4)$$

3 Homogeneous isotropic Universe

FLRW metric

$$ds^2 = dt^2 - a^2(t) \gamma_{ij} dx^i dx^j \quad (3.1)$$

with

$$\kappa = +1 : \quad \gamma_{ij} dx^i dx^j = R^2 [d\chi^2 + \sin^2 \chi (d\theta^2 + \sin^2 \theta d\phi^2)] \quad (3.2)$$

$$\kappa = -1 : \quad \gamma_{ij} dx^i dx^j = R^2 [d\chi^2 + \sinh^2 \chi (d\theta^2 + \sin^2 \theta d\phi^2)] \quad (3.3)$$

$$\kappa = 0 : \quad \gamma_{ij} dx^i dx^j = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \quad (3.4)$$

This metric is in the comoving frame, meaning that space is the same everywhere at each moment of time and world lines of particles which are at rest with respect to this frame are geodesic. Observer in this frame moves along with expansion and perceives the Universe to be isotropic. Observer in this frame moves along with expansion and perceives the Universe to be isotropic.

Hubble parameter

$$H(t) = \frac{\dot{a}(t)}{a(t)} \quad (3.5)$$

Conformal coordinates

$$dt = a d\eta \Rightarrow \eta = \int \frac{dt}{a(t)} \quad (3.6)$$

$$ds^2 = a^2(\eta) \eta_{\mu\nu} dx^\mu dx^\nu \quad (3.7)$$

In theories of massless vector fields, the action in conformal coordinates (η, x^i) in general reduces to the flat space-time action. The physical momentum \mathbf{p} (same for the frequency) at time t are

$$\mathbf{p}(t) = \frac{\mathbf{k}}{a(t)} \quad (3.8)$$

Redshift

$$\lambda_0 = \lambda_i \frac{a_0}{a(t_i)} = \lambda_i [1 + z(t_i)] \quad (3.9)$$

With linear approximation, we get the Hubble law

$$z = H_0 r, \quad r \ll 1 \quad (3.10)$$

Slowing down of relative motion

$$\mathbf{p} = \frac{\mathbf{k}}{a(t)} \quad (3.11)$$

Gases of free particles in expanding Universe

- relativistic

$$T_{\text{eff}} \propto \frac{1}{a(t)} \quad (3.12)$$

since $\exp^{p/T}$ appears in the denominator of distribution $f(\mathbf{p})$.

- non-relativistic

$$T_{\text{eff}} \propto \frac{1}{a^2(t)} \quad (3.13)$$

since $\exp(-p^2/2mT)$ appears in the distribution.

4 Dynamics of cosmological expansion

Friedmann equation (00 component of Einstein field equations)

$$H(t)^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\rho - \frac{\kappa}{a^2} \quad (4.1)$$

Energy conservation

$$\nabla_\mu T^{\mu\nu} = 0 \quad (4.2)$$

$$\stackrel{\nu=0}{\Rightarrow} \dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0 \quad (4.3)$$

$$\frac{d\rho}{p + \rho} = -3 d(\ln a) \quad (4.4)$$

Equation of state of matter

$$p = p(\rho) = \omega\rho \quad (4.5)$$

with $\omega = 0$ for non-relativistic particles, $\omega = 1/3$ for relativistic particles, and $\omega = -1$ for vacuum.

Sample cosmological solutions with zero curvature ($\kappa = 0$)

- matter dominated Universe

(4.4) with $p = 0$

$$\rho \propto a(t)^{-3} \quad (4.6)$$

With (4.1)

$$a(t) \propto (t - t_s)^{2/3} \quad (4.7)$$

Normally we count time from cosmological singularity, so set $t_s = 0$. Then

$$H(t) = \frac{2}{3t} \quad (4.8)$$

- radiation dominated Universe

(4.4) with $p = \rho/3$

$$\rho \propto a^{-4} \quad (4.9)$$

$$a \propto t^{1/2} \quad (4.10)$$

Hubble parameter

$$H(t) = \frac{1}{2t} \quad (4.11)$$

$$H(T) = \frac{T^2}{M_{pl}^*} = \frac{1.66\sqrt{g^*}T^2}{M_{pl}} \quad (4.12)$$

where g_* is defined as

$$g_* = \sum_b g_b + \frac{7}{8} \sum_f g_f \quad (4.13)$$

- vacuum dominated (de Sitter space, $p = -\rho$)

$$a(t) \propto e^{H_{ds}t} \quad (4.14)$$

$$ds^2 = dt^2 - e^{2H_{ds}t} d\mathbf{x}^2 \quad (4.15)$$

Cosmological horizons

- Particle horizon is the size of the region (in infinite Universe!) causally connected by the time t

$$l_H(t) = a(t)\eta(t) = a(t) \int_0^t \frac{dt'}{a(t')} = \begin{cases} \frac{2}{H(t)} & \text{matter} \\ \frac{1}{H(t)} & \text{radiation} \end{cases} \quad (4.16)$$

- Event horizon is the size of the region from which signals emitted at that moment of time will *ever* reach the observer ($\mathbf{x} = 0$) in arbitrary distant *future*.

In de Sitter space, beginning of time shifted to $-\infty$. Thus particle horizon $l_H(t) = \infty$. Instead we define event horizon

$$l_{ds} = a(t) [\eta(t \rightarrow \infty) - \eta(t)] = a(t) \int_t^\infty \frac{dt'}{a(t')} = \frac{1}{H_{ds}} \quad (4.17)$$

5 Λ CDM model

There are four components of energy density in the Λ CDM Universe

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G(\rho_M + \rho_{rad} + \rho_\Lambda + \rho_{curv}) \quad (5.1)$$

by definition

$$\frac{8\pi}{3}G\rho_{curv} = -\frac{\kappa}{a^2} \quad (5.2)$$

Critical density (only in application to the present Universe)

$$\rho_c = \frac{3H_0^2}{8\pi G} \quad (5.3)$$

Introduce parameter ($\sum_i \Omega_i \equiv 1$ and only to the present Universe)

$$\Omega_M = \frac{\rho_{M,0}}{\rho_c} = 0.315 \quad (5.4)$$

$$\Omega_{rad} = \frac{\rho_{rad,0}}{\rho_c} \lesssim 10^{-4} \quad (5.5)$$

$$\Omega_\Lambda = \frac{\rho_{\Lambda,0}}{\rho_c} = 0.685 \quad (5.6)$$

$$\Omega_{curv} = \frac{\rho_{curv,0}}{\rho_c} < 10^{-2} \quad (5.7)$$

With these

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\rho_c \left[\Omega_M \left(\frac{a_0}{a}\right)^3 + \Omega_{rad} \left(\frac{a_0}{a}\right)^4 + \Omega_\Lambda + \Omega_{curv} \left(\frac{a_0}{a}\right)^2 \right] \quad (5.8)$$

Curvature has never dominated in the evolution of the Universe. Dark energy will eventually dominate, since all other contribution decrease with time.

Transition from deceleration to acceleration (when Dark energy started to dominated)

Use (5.8) to calculate the second derivative and find

$$z_{ac} = \left(\frac{2\Omega_\Lambda}{\Omega_M}\right)^{1/3} - 1 \approx 0.63 \quad (5.9)$$

Transition from radiation domination to matter domination

$$z_{eq} + 1 = \frac{a_0}{a_{eq}} - \frac{\Omega_M}{\Omega_{rad}} \sim 10^4 \Rightarrow T_{eq} = 1 \text{ eV} \quad (5.10)$$

Refined calculation (with neutrinos) gives $T_{eq} = 0.8 \text{ eV}$.

Present age of the Universe Only consider Ω_M and Ω_Λ

$$t_0 = \frac{2}{3\sqrt{\Omega_\Lambda}} \frac{1}{H_0} \text{arsinh} \sqrt{\frac{\Omega_\Lambda}{\Omega_M}} \quad (5.11)$$

6 Thermodynamics of expanding Universe

In reaction $A_1 + A_2 + \dots \leftrightarrow B_1 + B_2 + \dots$ with chemical equilibrium

$$\sum \mu_{A_i} = \sum \mu_{B_j} \quad (6.1)$$

To create photon or destroy photon, there is no threshold energy, thus $\mu_\gamma = 0$. Consider annihilation of electron and positron, then $\mu_{e^-} = -\mu_{e^+}$.

Distributions

$$f(\mathbf{p}) = \frac{1}{(2\pi)^3} \frac{1}{\exp[E(\mathbf{p} - \mu)/T] \mp 1} \quad (6.2)$$

(boson with minus and fermion with plus sign). Neglecting the ∓ 1 , it reduces to Maxwell-Boltzmann

$$f(\mathbf{p}) = \frac{1}{(2\pi)^3} e^{-(E(\mathbf{p}) - \mu)/T} \quad (6.3)$$

In $m \gg T, (m - \mu) \gg T$

$$f(\mathbf{p}) = \frac{1}{(2\pi)^3} e^{(\mu - m)/T} e^{-\mathbf{p}^2/(2mT)} \quad (6.4)$$

Number density n_i , energy density ρ_i , and pressure p_i are defined as ($p \, dp = E \, dp$)

$$n_i = g_i \int f(\mathbf{p}) \, d^3\mathbf{p} = 4\pi g_i \int f(E) \sqrt{E^2 - m_i^2} E \, dE \quad (6.5)$$

$$\rho_i = g_i \int f(\mathbf{p}) E(\mathbf{p}) \, d^3\mathbf{p} = 4\pi g_i \int f(E) \sqrt{E^2 - m_i^2} E^2 \, dE \quad (6.6)$$

$$p_i = \frac{4\pi g_i}{3} \int_0^\infty f(E) (E^2 - m_i^2)^{3/2} \, dE \quad (6.7)$$

Relativistic particles with zero chemical potential $\mu_i = 0$

$$\rho_i = \begin{cases} g_i \frac{\pi^2}{30} T^4 & \text{Bose} \\ \frac{7}{8} g_i \frac{\pi^2}{30} T^4 & \text{Fermi} \end{cases} \quad (6.8)$$

To combine all relativistic particles

$$\rho = g_* \frac{\pi^2}{30} T^4 \quad (6.9)$$

with

$$g_* = \sum_{\text{boson}, m \ll T} g_i + \frac{7}{8} \sum_{\text{fermion}, m \ll T} g_i \quad (6.10)$$

From the integrals, we also find $p = \rho/3$.

Number densities

$$n_i = \begin{cases} g_i \frac{\zeta(3)}{\pi^2} T^3 & \text{Boson} \\ \frac{3}{4} g_i \frac{\zeta(3)}{\pi^2} T^3 & \text{Fermi} \end{cases} \quad (6.11)$$

Non-relativistic particles

$$n_i = g_i \left(\frac{m_i T}{2\pi} \right)^{3/2} e^{\frac{\mu_i - m_i}{T}} \quad (6.12)$$

$$\rho_i m_i n_i + \frac{3}{2} n_i T_i \quad (6.13)$$

$$p_i T n_i \ll \rho_i \quad (6.14)$$

7 Entropy, Baryon-to-Photon ratio

$$s = \frac{p + \rho - \mu n}{T} \quad (7.1)$$

Relativistic ($\mu = 0$)

$$s_i = (1; \frac{7}{8}) g_i \frac{2\pi^2}{45} T^3 \quad (7.2)$$

Non-relativistic particles' entropy is always small in cosmic medium

$$s_i = \frac{5}{2} n_i + \frac{m_i - \mu_i}{T} n_i \quad (7.3)$$

Total entropy in comoving is conserved

$$\frac{d}{dt}(sa^3) = 0 \quad (7.4)$$

Baryon asymmetry is characterised by

$$\Delta_B = \frac{n_B - n_{\bar{B}}}{s} = 0.86 \times 10^{-10} \quad (7.5)$$

It is time-independent, if there is no B -violation processes and no large amount of entropy is released.

Baryon-to-photon ratio (applicable at $T \lesssim 1 \text{ MeV}$)

$$\eta_B = \frac{n_B}{n_\gamma} = (6.05 \pm 0.07) \cdot 10^{-10} \quad (7.6)$$

For a massless particle with small chemical potential

$$\frac{\Delta_n}{g} = \frac{n - \bar{n}}{g} = \left(\frac{1}{3}; \frac{1}{6}\right) \mu T^2 \quad (7.7)$$

Equilibrium

- Thermal equilibrium

Two parts of a system could exchange heat energy but they don't on net.

- Chemical equilibrium

Number densities of both sides of process don't change, i.e. back and forward reactions happen at same rate.

- Kinetical equilibrium

Particles are distributed according to (6.2).

8 Recombination

As the Universe cools down, it becomes thermodynamically for baryons and electrons to combine into atoms. Sometimes recombination is also understood as the last time photons scattered and propagate freely. It is not an equilibrium process!

Naively one would expect $T_{rec} \sim \Delta_H$, but in reality it is an order of magnitude lower! The main deciding factor is the *low abundance* of proton $n_p \approx n_B$. Consider process $p + e \leftrightarrow H + \gamma$. Forward reaction is characterised by $\tau_+ \sim n_B^{-1}$, backward by $\exp(\Delta_H/T)$. To define the recombination happened with $\tau_+ \sim \tau_-$, we find $T \ll \Delta_H$ with small n_B .

Equilibrium approach three non-rel. Boltzmann distributions n_e , n_p , n_H . There are but unknown chemical potentials, but also three further equations: baryon number conservation, chemical potential in equilibrium, and charge neutrality. In the end, we get Saha equation. Define recombination with $n_p(T_r^{eq}) = n_H(T_r^{eq})$, we have

$$T_r^{eq} \approx 0.38 \text{ eV} \quad (8.1)$$

(With numerical calculation $T_r = 0.33 \text{ eV}$.)

Photon last scattering is more important, since after this photons freely propagate and they are CMB photons. Last scattering means scattering off electrons. Thus electron number density is the key information.

Need to consider energy levels of hydrogen atoms. Only consider ionization processes, there is no chemical equilibrium for ground state hydrogen, but there is for excited state ($2s, 2p$). This is all because of photon number density capable of ionizing the respective atom. Higher excited levels are not considered, since their number densities are exponentially small.

Three other processes

- $e + p \leftrightarrow 1s + \gamma$

doesn't change electron number density, because the photon release in forward reaction will quickly ionize a neighboring $1s$ -atom.

- $2s \leftrightarrow 1s + 2\gamma$

Backward process is suppressed, because it involves three particles.

- $2p \leftrightarrow 1s + \gamma$

Backward process is also suppressed, because here the redshift because of cosmic expansion is important and photon wavelength gets stretched out.

As stated before, higher level hydrom atoms can be easily ionized, but process of excited hydrogen atoms are more likely to go to $1s$ states without emitting electron. In the end, there is increase in number of $1s$ atoms and decrease in free electron and protons.

After calculation, we find

$$T_r = 0.26 \text{ eV} \quad (8.2)$$

Horizon at recombination

$$l_{H,r}(t_0) = \frac{2}{H_0 \sqrt{\Omega_M}} \frac{f}{\sqrt{1+z_r}} \quad (8.3)$$

It gives us that there should be $\sim 3 \times 10^4$ regions causally disconnected at recombination, so no thermal equilibrium with each other. But from CMB and galaxy surveys, there is no difference. *Horizon Problem!*

9 Relic neutrinos

Neutrino freeze-out

$$\sigma_\nu \sim G_F^2 T^2 \quad (9.1)$$

$$\Rightarrow \tau_\nu \sim \frac{1}{G_F^2 T^5} \quad (9.2)$$

$$\tau_\nu(T) \sim H^{-1}(T) \quad (9.3)$$

$$\Rightarrow T_{\nu,f} \sim 2 \text{ MeV} \quad (9.4)$$

Effective neutrino temperature Neutrinos remain relativistic and cool down with expansion. There is energy injection into photon because of electron positron annihilation.

From entropy conservation (of electron photon component)

$$g_*^{eq}(T) a^3 T^3 = \text{const} \quad (9.5)$$

Then

$$\frac{T_{\gamma,0}}{T_{\nu,0}} = \left(\frac{g_*^{eq}(T_{\nu,f})}{g_*^{eq}(T_0)} \right)^{1/3} = \left(\frac{11}{4} \right)^{1/3} \approx 1.4 \quad (9.6)$$

This can be used to constraint neutrino masses.

10 Big bang nucleosynthesis

In chronological order

Neutron freeze-out $p + e \leftrightarrow n + \nu_e$

In naive calculation ($T \gtrsim \Delta_m, m_e$), we have (with similar approach as neutrino decoupling)

$$T_n \approx 1.4 \text{ MeV} \quad (10.1)$$

In reality it is $T_n \approx 0.75 \text{ MeV}$.

Deuterium production $p + n \Rightarrow D + \gamma$

Saha equation gives

$$T_D = 65 \text{ keV} \quad (10.2)$$

It is much lower than the binding energy, since the result of Saha equation has a suppression factor

$$\frac{X_D}{X_n} \approx \eta_B \left(\frac{2.5T}{m_p} \right)^{3/2} e^{\Delta_D/T} \quad (10.3)$$

$$\frac{n_n(T_{NS})}{n_p(T_{NS})} \approx 0.14 \quad (10.4)$$

$$X_{4\text{He}} \approx 25\% \quad (10.5)$$

Deuterium burning $D + D \rightarrow {}^3\text{He} + n$ $D + D \rightarrow T + p$

Nucleosynthesis begins a bit early than deuterium production, $T_{NS} = 75 \text{ eV}$. Because the cross sections of deuterium into heavier element are large, once deuterium gets produced it gets immediately turned into heavier nuclei. Until T_D , the abundance of deuterium is very small. Nevertheless, BBN depends on deuterium production, since heavier elements must go through deuterium first. *Deuterium bottleneck!*

Precise calculation involves cross sections, tunneling rates, and Boltzmann equation.

11 Dark matter

Dark matter should be

- non-relativistic
- very weakly, if at all, interacting with photon
- long-lived

DM is categorised into two (three) depends on the temperature at its freeze-out

- cold dark matter $T_f < M$
- $T_f > M$
 - warm dark matter $M \lesssim 1 \text{ eV}$
 - hot dark matter $m \gtrsim 1 \text{ eV}$

1 eV is the temperature of transition from radiation to matter domination!
CDM is preferred.

Boltzmann equation

$$\mathrm{d}n_X/\mathrm{d}t + 3Hn_X = -\langle\sigma^{ann}\cdot v\rangle(n_X^2 - n_X^{eq\,2}) \quad (11.1)$$

Partial wave analysis

$$\langle\sigma^{ann}\cdot v\rangle = \sigma_0 + 6\frac{T}{M_X}\sigma_1 \quad (11.2)$$

$$T_f \approx M_X \quad (11.3)$$

$$\Omega_X \propto \langle\sigma\cdot v\rangle^{-1} \quad (11.4)$$

$$\propto \ln M_x \quad (11.5)$$

WIMP Assume $M_x = 100\text{ GeV}$ and use $\Omega_X = 0.25$, then

$$\langle\sigma\cdot v\rangle \sim 1 \times 10^{-8} \text{ GeV}^{-2} \quad (11.6)$$

Somewhat similar to electroweak coupling! *WIMP miracle!*