Theoretical Astroparticle Physik
Homework 4

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1 Quickies

(a) Briefly argue why the chemical potential μ_γ of photons is zero.
We know that one can introduce chemical potential μ_i to conserved quantum numbers $Q^{(i)}$ to deal with all relevant reactions in thermalized system with various particle species. The chemical potential for particle of type A would then be

$$\mu_A = \sum_i \mu_i Q_A^{(i)} \tag{1.1}$$

where $Q_A^{(i)}$ are the quantum numbers carried by particle A.

And for a photon, all of its quantum numbers are 0, eg. charge neutral, no lepton/baryon numbers, color neutral, etc. As such one can easily deduce that $\mu_\gamma = 0$

Another way of looking at this is that we know photon is its own antiparticle. A consequence of the equation 1.1 is that $\mu_{\bar{A}} = -\mu_A$, which again leads to $\mu_\gamma = 0$

Another intuitive way of looking at this is going back to statistical mechanics, we know that the infinitesimal change of internal enegry U is given by,

$$\mathrm{d}U = T\,\mathrm{d}s - P\,\mathrm{d}V + \mu\,\mathrm{d}N \tag{1.2}$$

And given that $\mathrm{d}U$ should reach 0 at some point and $\mathrm{d}N_\gamma$ is never zero (because we know that photons can very easily created and disappear easily), μ_γ must be 0.

(b) At which approximate temperature did our universe transition from being dominated by radiation to a phase, where its energy density was dominated by non-relativistic matter? What is the corresponding redshift?
To get the transition, one need to consider when the energy density of non-relativistic matter and the energy density of radiations are equal:

$$\frac{8\pi}{3}G\rho_c\Omega_M\left(\frac{a_0}{a_{eq}}\right)^3 = \frac{8\pi}{3}G\rho_c\Omega_{rad}\left(\frac{a_0}{a_{eq}}\right)^4 \tag{1.3}$$

$$\Omega_M = \Omega_{rad}\left(\frac{a_0}{a_{eq}}\right)$$

$$\frac{\Omega_M}{\Omega_{rad}} = \frac{a_0}{a_{eq}} = z_{eq} + 1 \tag{1.4}$$

Given that $\Omega_{rad} \approx 5.0 \cdot 10^{-5}$ and $\Omega_M \approx 0.315$

$$z_{eq} + 1 \approx 6300 \tag{1.5}$$

$$T_{eq} = T_0(z_{eq} + 1) \approx 3 \cdot (6300)K \approx 1.89 \times 10^4\,\mathrm{K} \tag{1.6}$$

2 Number densities of particles and antiparticles

System that we are looking at:

- Massless particles, $m = 0$,
- chemical potential for particle μ, chemical potential for anti particle $-\mu$,
- thermal bath with $T \gg \mu$
- spin degrees of freedom, g
- $E^2 = |\vec{p}|^2$ (because of $m = 0$)

(a) First we look at the number density, from here on we use p as $|\vec{p}|$

$$\begin{aligned} n &= g\int d^3p f(\vec{p}) &(2.1)\\ &= g\int f(\vec{p})d\Omega p^2 dp \\ &= 4\pi g\int f(p)p^2 dp \\ &= 4\pi g\int \frac{1}{(2\pi)^3}\frac{1}{e^{\frac{E-\mu}{T}}\pm 1}p^2 dp \\ &= \frac{g}{(2\pi^2)}\int \frac{1}{e^{\frac{p-\mu}{T}}\pm 1}p^2 dp \\ n &= \frac{g}{(2\pi^2)}\int \left(\frac{1}{e^{\frac{p}{T}}\pm 1}+\frac{e^{\frac{p}{T}}}{(e^{\frac{p}{T}}\pm 1)^2}\frac{\mu}{T}\right)p^2 dp &(2.2)\end{aligned}$$

Now getting the equation from the question

$$\begin{aligned} \frac{\Delta n}{g} &= \frac{n-\bar{n}}{g} &(2.3)\\ &= \frac{1}{\pi^2}\int \left(\frac{e^{\frac{p}{T}}}{(e^{\frac{p}{T}}\pm 1)^2}\frac{\mu}{T}\right)p^2 dp \end{aligned}$$

(b) Now using $z=\frac{p}{T}$

$$\begin{aligned} \frac{\Delta n}{g} &= \frac{1}{\pi^2}\int \left(\frac{e^z}{(e^z\pm 1)^2}\frac{\mu}{T}\right)(zT)^2 d(zT) \\ &= \frac{\mu T^2}{\pi^2}\int \frac{z^2 e^z}{(e^z\pm 1)^2}dz &(2.4)\end{aligned}$$

Focus on the integral, one can do integrate by parts

$$\begin{aligned} &\int \mathrm{d}z\,\frac{z^2 e^z}{(e^z\pm 1)^2} \\ &= \int \mathrm{d}\left(-\frac{1}{e^z\pm 1}\right)z^2 \\ &= -\frac{z^2}{e^z\pm 1}\bigg|_0^\infty + 2\int \mathrm{d}z\,\frac{z}{e^z\pm 1} \end{aligned}$$

If one looks at the first term graphically, one would notice that the at the boundaries, the term is equals to zero. One can then use the integrals given in the question to evaluate the second term:

$$2\int \frac{\mathrm{d}z\, z}{e^z \pm 1} = \begin{cases} 2\cdot\frac{2-1}{2}\pi^2\frac{1}{6} & \text{fermions} \\ 2\cdot\frac{(2\pi)^2}{4}\frac{1}{6} & \text{bosons} \end{cases} = \begin{cases} \frac{\pi^2}{6} & \text{fermions} \\ \frac{\pi^2}{3} & \text{bosons} \end{cases}$$

Finally,

$$\frac{\Delta n}{g} = \begin{cases} \frac{\mu T^2}{6} & \text{fermions} \\ \frac{\mu T^2}{3} & \text{bosons} \end{cases} \tag{2.5}$$

3 Equilibrium Recombination Temperature of Helium

(a) Particles in equation (4) are in chemical equilibrium, thus

$$\mu_{\mathrm{He}^{++}} + 2\mu_{e^-} = \mu_{\mathrm{He}} \tag{3.1}$$

(b) Heuristically we assume that the particles are non-relativistic during recombination. Thus the number density follows the usual formular

$$n_e = g_e \left(\frac{m_e T}{2\pi}\right)^{3/2} e^{(\mu_e - m_e)/T} \tag{3.2}$$

$$n_{\mathrm{He}} = g_{\mathrm{He}} \left(\frac{m_{\mathrm{He}} T}{2\pi}\right)^{3/2} e^{(\mu_{\mathrm{He}} - m_{\mathrm{He}})/T} \tag{3.3}$$

$$n_{\mathrm{He}^{++}} = g_{\mathrm{He}^{++}} \left(\frac{m_{\mathrm{He}^{++}} T}{2\pi}\right)^{3/2} e^{(\mu_{\mathrm{He}^{++}} - m_{\mathrm{He}^{++}})/T} \tag{3.4}$$

Naively, one say because of Pauli principle, two neutrons (protons) in alpha particle (He^{++}) should have opposite spins. And thus the net spin should be zero. But one need to know the difference between S and S_z.

We know that two spin $\frac{1}{2}$ particles (neutron or proton in this case) can form a symmetric triplet and an antisymmetric singlet, $2\otimes 2 = 3_{\mathrm{S}} \oplus 1_{\mathrm{A}}$. It is true that with $S_{z,1} = -S_{z,2}$, the total spin in z-direction should be zero, but the spin can either be 1 or 0.

$$|1,0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)$$
$$|0,0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$$

(one could use CG-coefficients from opposite direction to get the same result.)

Since we are looking for ground state, we need to find out which state have lower energy. Both states have integer spin, thus the total wavefunction must be symmetric. Then the $|0,0\rangle$ must have antisymmetric spatial component of wave function $\psi_{\mathrm{A}}(\vec{r}_1,\vec{r}_2)$ to achieve the symmetry. Antisymmetric spatial component should always have lower energy. To see this, we note that in the limit $\vec{r}_1 \to \vec{r}_2$

$$\psi_{\mathrm{A}}(\vec{r}_1,\vec{r}_2) \to 0$$

Thus in this configuration two identical particle must be further away from each other resulting lowered energy. After determining the configuration of pp or nn system, we then conclude that the alpha particle should have spin 0 ($g_{\mathrm{He}^{++}} = 1$). Since the eletrons in Helium also follows Pauli principle, $g_{He} = 1$.

(c) Expression of baryon number conservation for n_{He} and $n_{\mathrm{He}^{++}}$ assuming that nuclei don't convert into each other. Since no interactions can change (A, Z), one quarter of the baryons only have two forms (ignore the He^{+})

$$n_{\mathrm{He}^{++}} + n_{\mathrm{He}} = 0.25 n_{\mathrm{B}} \tag{3.5}$$

In this era, there are three electrically charge particles and asuming electrically neutral Universe

$$n_p + 2n_{\mathrm{He}^{++}} = n_e \tag{3.6}$$

There is 2 before number density of He^{++} because it carries two electric charges. In the next part, we will ignore n_p contributions. (Why can we do this? We know that heliom recombination happened before hydrogen recombination, thus there should have been a lot protons floating around.)

(d) Expression of n_{He} in terms of $n_{\mathrm{He}^{++}}$.

First multiply (3.4) and (3.2)^2 (the square will be obvious in a minute)

$$\begin{aligned}
n_{\mathrm{He}^{++}} n_e^2 &= g_{\mathrm{He}^{++}} g_e^2 \left(\frac{m_{\mathrm{He}^{++}} T}{2\pi}\right)^{3/2} \left(\frac{m_e T}{2\pi}\right)^{3/2 \cdot 2} e^{(\mu_{\mathrm{He}^{++}} + 2\mu_e - m_{\mathrm{He}^{++}} - 2m_e)/T} \\
4n_{\mathrm{He}^{++}}^3 &\overset{3.6}{=} g_{\mathrm{He}^{++}} g_e^2 \left(\frac{m_{\mathrm{He}^{++}} T}{2\pi}\right)^{3/2} \left(\frac{m_e T}{2\pi}\right)^{3/2 \cdot 2} e^{[\mu_{\mathrm{He}} - (m_{\mathrm{He}} + \Delta\mathrm{He})]/T} \\
&= \frac{g_{\mathrm{He}^{++}} g_e^2}{g_{\mathrm{He}}} n_{\mathrm{He}} \left(\frac{m_{\mathrm{He}^{++}}}{m_{\mathrm{He}}}\right)^{3/2} \left(\frac{m_e T}{2\pi}\right)^3 e^{-\Delta\mathrm{He}/T} \\
&\approx 4 n_{\mathrm{He}} \left(\frac{m_e T}{2\pi}\right)^3 e^{-\Delta\mathrm{He}/T}
\end{aligned}$$

where we have used (3.6) without n_p (Is this justified?) and

$$\frac{m_{\mathrm{He}^{++}}}{m_{\mathrm{He}}} = \frac{m_{\mathrm{He}^{++}}}{m_{\mathrm{He}^{++}} + 2m_e - \Delta\mathrm{He}} = \frac{3.7\,\mathrm{GeV}}{3.7\,\mathrm{GeV} + 2 \cdot 0.5\,\mathrm{MeV} - 79\,\mathrm{eV}} \approx 1$$

Rearranging this, we get

$$n_{\mathrm{He}} = n_{\mathrm{He}^{++}}^3 \left(\frac{m_e T}{2\pi}\right)^{-3} e^{\Delta\mathrm{He}/T} \tag{3.7}$$

Using (3.5) and defintions of X_i

$$X_{\mathrm{He}^{++}} + X_{\mathrm{He}} = 1 \tag{3.8}$$

Plug in the result (3.7) and we have the Saha equation for Helium

$$X_{\mathrm{He}^{++}} + \tilde{n}_B^2 X_{\mathrm{He}^{++}}^3 \left(\frac{m_e T}{2\pi}\right)^{-3} e^{\Delta He/T} = 1 \tag{3.9}$$

$\tilde{n}_B$ can be expressed in terms of baryon number density and is then related to photon number density

$$\begin{aligned}
X_{\mathrm{He}^{++}} + (\tilde{\eta}_B n_\gamma)^2 X_{\mathrm{He}^{++}}^3 \left(\frac{m_e T}{2\pi}\right)^{-3} e^{\Delta He/T} &= 1 \\
X_{\mathrm{He}^{++}} + \tilde{\eta}_B^2 \left(g_\gamma \frac{\zeta(3)}{\pi^2} T^3\right)^2 X_{\mathrm{He}^{++}}^3 \left(\frac{m_e T}{2\pi}\right)^{-3} e^{\Delta He/T} &= 1
\end{aligned}$$

with $\tilde{\eta}_B = 0.25 \cdot \eta_B$ The second term in LHS is still X_{He}. We demand

$$X_{\mathrm{He}^{++}}(T_r^{\mathrm{eq}}) \sim X_{\mathrm{He}}(T_r^{\mathrm{eq}}) \sim 1 \tag{3.10}$$

and

$$X_{\rm He} = 4\tilde{\eta}_B^2 \left(\frac{\zeta(3)}{\pi^2}\right)^2 T^6 X_{\rm He^{++}}^3 \left(\frac{m_e T}{2\pi}\right)^{-3} e^{\Delta He/T}$$

By demanding this quantity to be $\mathcal{O}(1)$ at $T_r^{\rm eq}$ (why is this true? (I know this is just assumption of this exercise. But why can we make this assumption without having large error?))

$$\frac{\Delta \mathrm{He}}{T_r^{\rm eq}} \sim -\ln\left[\frac{1}{4}\eta_B^2 \frac{\zeta(3)^2}{\pi^4}\left(\frac{2\pi T_r^{\rm eq}}{m_e}\right)^3\right]$$

Use the hint to obtain approximate equation

$$\frac{\Delta \mathrm{He}}{T_r^{\rm eq}} \approx \ln\left[\frac{4\pi^4}{\eta_B^2\zeta^2(3)}\left(\frac{m_e}{2\pi\Delta\mathrm{He}}\right)^3\right] \approx 68.8 \gg 1 \tag{3.11}$$

The approximation is also valid! In the end, we have

$$T_r^{\rm eq} \approx 1.14\,\mathrm{eV} < 0.38\,\mathrm{eV} \tag{3.12}$$

Indeed, helium recombination did happen before hydrogen recombination and the formula for number density in the beginning is well justified.