Theoretical Astroparticle Physik
Homework 1

Chenhuan, Junaid, Yong Sheng

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1 Quickies

(a) Briefly describe in your own words what is meant by a spatially isotropic and homogeneous universe.

It means in this Universe every direction should look the same (isotropic) and every part of it looks the same (homogeneous).

(b) State the definition of the Hubble parameter $H(t)$. What does the Hubble constant H_0 describe?

The Hubble parameter is defined as

$$H(t) := \frac{\dot{a}(t)}{a(t)}, \tag{1.1}$$

where $a(t)$ is the scale factor in the FLRW-metric. H_0 refers to the current value of $H(t)$ and it describe the current expansion rate of the Universe.

(c) The Hubble constant is usually parametrized as $H_0 = h{\cdot}100\,\mathrm{km\,Mpc^{-1}\,s^{-1}}$, where $h \approx 0.6 - 0.7$ depends on the exact measurement. Convert H_0 into natural units.

$$\begin{aligned} H_0 &= h \cdot 100\,\mathrm{km\,Mpc^{-1}\,s^{-1}} \\ &= 6.5 \times 10^4\,\mathrm{m} \cdot (3.1 \times 10^{22}\,\mathrm{m})^{-1}\mathrm{s}^{-1} \\ &= 2.1 \times 10^{-18}\,\mathrm{s}^{-1} \\ &= 2.1 \times 10^{-18} \cdot 6.58 \times 10^{-16}\,\mathrm{eV} \\ &= 1.4 \times 10^{-33}\,\mathrm{eV} \end{aligned}$$

2 Cutoff for high energy astro-physical neutrinos

3 Friedmann-Lemaitre-Robertson-Walker metric

$$ds^2 = -dt^2 + a(t)^2[\frac{dr^2}{1-\kappa r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)] \tag{3.1}$$

(a)

$$g_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & \frac{a^2}{1-\kappa r^2} & & \\ & & a^2r^2 & \\ & & & a^2r^2\sin^2\theta \end{pmatrix}$$

$$g^{\mu\nu} = \begin{pmatrix} -1 & & & \\ & \frac{1-\kappa r^2}{a^2} & & \\ & & \frac{1}{a^2r^2} & \\ & & & \frac{1}{a^2r^2\sin^2\theta} \end{pmatrix}$$

Note that:

(i) one can write $g_{ii} = a^2 f_{ii}$, where

$$f_{11} = \frac{1}{1-\kappa r^2}$$
$$f_{22} = r^2$$
$$f_{33} = r^2\sin^2\theta$$

(ii) $g^{\mu\mu} = (g_{\mu\mu})^{-1}$ and $f^{ii} = (f_{ii})^{-1}$.

(iii) $g_{33} = g_{22}\sin^2\theta$.

(b)

$$\begin{aligned}\Gamma^{\lambda}_{\mu\nu} &= g^{\lambda\rho}\Gamma_{\rho\mu\nu} \\ &= \frac{1}{2}g^{\lambda\rho}(\partial_\nu g_{\rho\mu} + \partial_\mu g_{\rho\nu} - \partial_\rho g_{\mu\nu})\end{aligned}$$

For $\lambda = 0$, $\mu = \nu = 1$,

$$\Gamma^0_{11} = \frac{1}{2}g^{0\rho}(\partial_1 g_{\rho 1} + \partial_1 g_{\rho 1} - \partial_\rho g_{11})$$

$g_{\mu\nu}$ is symmetric, only $g_{00}, g_{11}, g_{22}, g_{33}$ is nonvanishing,

$$\begin{aligned}\Gamma^0_{11} &= \frac{1}{2}g^{00}(\partial_1 g_{01} + \partial_1 g_{01} - \partial_0 g_{11}) \\ &= \frac{1}{2}(-1)(-\partial_t(\frac{a^2}{1-\kappa r^2})) \\ &= \frac{a\dot{a}}{1-\kappa r^2}\end{aligned}$$

For $\lambda = 1$, $\mu = \nu = 1$,

$$\begin{aligned}\Gamma^1_{11} &= \frac{1}{2}g^{11}(\partial_1 g_{11} + \partial_1 g_{11} - \partial_1 g_{11}) \\ &= \frac{1}{2}g^{11}\partial_1 g_{11} \\ &= \frac{1}{2}(\frac{1-\kappa r^2}{a^2})(\partial_r \frac{a^2}{1-\kappa r^2}) \\ &= \frac{1}{2}(\frac{1-\kappa r^2}{a^2})(-2\kappa r)(\frac{a^2}{(1-\kappa r^2)^2}) \\ &= \frac{\kappa r}{1-\kappa r^2}\end{aligned}$$

For $\lambda = 0$, $\mu = \nu = 2$,

$$\begin{aligned}\Gamma^0_{22} &= \frac{1}{2}g^{00}(\partial_2 g_{02} + \partial_2 g_{02} - \partial_0 g_{22}) \\ &= \frac{1}{2}g^{00}(-\partial_0 g_{22}) \\ &= (-1)(\frac{1}{2})(-\partial_t a^2 r^2) \\ &= \frac{1}{2}(2a\dot{a}r^2 \sin^2\theta) = a\dot{a}r^2\end{aligned}$$

For $\lambda = 0$, $\mu = \nu = 3$,

$$\begin{aligned}\Gamma^0_{33} &= \frac{1}{2}g^{00}(\partial_3 g_{03} + \partial_3 g_{03} - \partial_0 g_{33}) \\ &= \frac{1}{2}g^{00}(-\partial_0 g_{33}) \\ &= \frac{1}{2}g^{00}(-\partial_t(g_{22}\sin^2\theta)) \\ &= a\dot{a}r^2\sin^2\theta\end{aligned}$$

For $\lambda = \nu = i$, $\mu = 0$,

$$\begin{aligned}
\Gamma^i_{0i} &= \frac{1}{2} g^{ii}(\partial_0 g_{ii} + \partial_i g_{i0} - \partial_i g_{0i}) \\
&= \frac{1}{2}(\frac{1}{a(t)^2 f_{ii}}(\partial_t(a(t)^2 f_{ii}))) \\
&= \frac{1}{2}\frac{2a\dot{a}}{a^2} \\
&= \frac{\dot{a}}{a}
\end{aligned}$$

For $\lambda = 1$, $\mu = \nu = 2$,

$$\begin{aligned}
\Gamma^1_{22} &= \frac{1}{2} g^{11}(\partial_2 g_{12} + \partial_2 g_{12} - \partial_1 g_{22}) \\
&= \frac{1}{2} g^{11}(-\partial_r g_{22}) \\
&= \frac{1}{2}(\frac{1-\kappa r^2}{a^2})(-\partial_r(a^2 r^2)) \\
&= \frac{1}{2}(1-\kappa r^2)(-2r) \\
&= -r(1-\kappa r^2)
\end{aligned}$$

For $\lambda = 1$, $\mu = \nu = 3$,

$$\begin{aligned}
\Gamma^1_{33} &= \frac{1}{2} g^{11}(\partial_3 g_{13} + \partial_3 g_{13} - \partial_1 g_{33}) \\
&= \frac{1}{2} g^{11}(-\partial_r g_{22}\sin^2\theta) \\
&= \frac{1}{2} g^{11}(-\partial_r g_{22})\sin^2\theta \\
&= \sin^2\theta \Gamma^1_{22} \\
&= -r(1-\kappa r^2)\sin^2\theta
\end{aligned}$$

For $\lambda = \nu = 2$, $\mu = 1$,

$$\begin{aligned}
\Gamma^2_{12} &= \frac{1}{2} g^{22}(\partial_1 g_{22} + \partial_2 g_{12} - \partial_2 g_{12}) \\
&= \frac{1}{2}\frac{1}{a^2 r^2}\partial_r(a^2 r^2) \\
&= \frac{1}{2}\frac{1}{a^2 r^2} 2a^2 r \\
&= \frac{1}{r}
\end{aligned}$$

For $\lambda = \nu = 3$, $\mu = 1$,

$$\begin{aligned}
\Gamma^3_{13} &= \frac{1}{2} g^{33} (\partial_1 g_{33} + \partial_3 g_{13} - \partial_3 g_{13}) \\
&= \frac{1}{2} \frac{1}{a^2 r^2 \sin^2 \theta} \partial_r (a^2 r^2 \sin^2 \theta) \\
&= \frac{1}{2} \frac{1}{a^2 r^2 \sin^2 \theta} 2a^2 r \sin^2 \theta \\
&= \frac{1}{r}
\end{aligned}$$

For $\lambda = \nu = 3$, $\mu = 2$,

$$\begin{aligned}
\Gamma^3_{23} &= \frac{1}{2} g^{33} (\partial_2 g_{33} + \partial_3 g_{32} - \partial_3 g_{23}) \\
&= \frac{1}{2} g^{33} (\partial_2 g_{33}) \\
&= \frac{1}{2} \frac{1}{a^2 r^2 \sin^2 \theta} (\partial_\theta (a^2 r^2 \sin^2 \theta)) \\
&= \frac{1}{2} \frac{1}{a^2 r^2 \sin^2 \theta} 2a^2 r^2 \sin \theta \cos \theta \\
&= \frac{\cos \theta}{\sin \theta}
\end{aligned}$$

For $\lambda = 2$, $\mu = \nu = 3$,

$$\begin{aligned}
\Gamma^2_{33} &= \frac{1}{2} g^{22} (\partial_3 g_{23} + \partial_3 g_{23} - \partial_2 g_{33}) \\
&= \frac{1}{2} g^{22} (-\partial_2 g_{33}) \\
&= \frac{1}{2} g^{33} \sin^2 \theta (-\partial_2 g_{33}) \\
&= -\sin^2 \theta \Gamma^3_{23} \\
&= -\sin \theta \cos \theta
\end{aligned}$$