# Theoretical Astroparticle Physik
# Homework 2

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## 1 Quickies

(a) What is the definition of the cosmological redshift?
Cosmological redshift is the redshift caused by the expansion of the universe itself. If a photon is emitted at time $t_i$ with a given wavelength, $\lambda_i$, then the observed wavelength of the photon at present time (with the cosmological redshift, z) is

$$\lambda_0 = \lambda_i \frac{a_0}{a(t_i)} = \lambda_i \left[1 + z(t_i)\right] \tag{1.1}$$

where $a_0$ is the scale factor at present time and $a(t_i)$ is the scale factor at time $t_i$.

(b) Explain the notion of co-moving coordinates.
Since in GR we learned that physics doesnt care what coordinate system we choose, we can choose one that is "natural"/easiest. In the comoving frame, the worldlines of particles at rest in this frame are geodesics, ie particles intially at rest will remain at rest. Particles that are initially moving with respect to this frame will eventually come to rest in it. The comoving coordinates assign constant spatial coordinate values to observers who perceive the universe as isotropic. The comoving time coordinate is just the proper time by an observer at rest in the comoving frame. It is also the elapsed time since the Big Bang according to a clock of a comoving observer and is a measure of cosmological time. Comoving distance is the distance between two points measured along a path defined at the present cosmological time. The comoving spatial coordinates tell where an event occurs while cosmological time tells when an event occurs[1][2].

Reference:

# 2 Equation of state for matter and radiation

(a) Compute the number density of non-relativstic particles

Maxwell-Boltzmann distribution can be approximated as

$$f_{\mathrm{MB}}(\vec{p}) = \frac{1}{(2\pi)^3}\exp\left(-\frac{E(\vec{p})-\mu}{T}\right) \approx \frac{1}{(2\pi)^3}\exp\left(\frac{\mu-m}{T}-\frac{|\vec{p}|^2}{2mT}\right)$$

For now on $|\vec{p}| = p$. The number density is defines as

$$\begin{aligned} n &= g_i \int \mathrm{d}^3p\, f(\vec{p}) \\ &= \frac{g_i}{(2\pi)^3}\exp\left(\frac{\mu-m}{T}\right)\int \mathrm{d}^3p \exp\left(-\frac{p^2}{2mT}\right) \\ &= \frac{4\pi g_i}{(2\pi)^3}\exp\left(\frac{\mu-m}{T}\right)\int \mathrm{d}p\, p^2 \exp\left(-\frac{p^2}{2mT}\right) \\ &= \frac{4\pi g_i}{(2\pi)^3}\exp\left(\frac{\mu-m}{T}\right)\frac{\sqrt{\pi}}{2}(2mT)^{3/2} \\ n &= g_i \exp\left(\frac{\mu-m}{T}\right)\left(\frac{mT}{2\pi}\right)^{3/2} \end{aligned} \tag{2.1}$$

(b) Determine the pressure and find the equation of state

The pressure is given as

$$\begin{aligned} P &= g_i \int \mathrm{d}^3p\, \frac{p^2}{3E} f(\vec{p}) \\ &= \frac{g_i}{(2\pi)^3}\frac{4\pi}{3}\exp\left(\frac{\mu-m}{T}\right)\int \mathrm{d}p\, p^2 \frac{p^2}{E}\exp\left(-\frac{p^2}{2mT}\right) \\ &\approx \frac{g_i}{6\pi^2 m}\exp\left(\frac{\mu-m}{T}\right)\int \mathrm{d}p\, p^4 \exp\left(-\frac{p^2}{2mT}\right) \\ &= \frac{g_i}{8\pi^2 m}\exp\left(\frac{\mu-m}{T}\right)(2mT)^{5/2}\sqrt{\pi} \\ &\overset{2.1}{=} nT \end{aligned} \tag{2.2}$$

In the limit $T \ll m$, the equation of state is

$$\omega = \frac{P}{\rho} = \frac{T}{m} \approx 0 \tag{2.3}$$

(c) Determine the energy density in the limit $T \gg E \gg m$

First to show $E\,\mathrm{d}E = p\,\mathrm{d}p$

$$\frac{\mathrm{d}E}{\mathrm{d}p} = \frac{\mathrm{d}}{\mathrm{d}p}\sqrt{m^2+p^2} = \frac{E}{p}$$

$$\begin{aligned}
\rho &= g_i \int \mathrm{d}^3p\, E(\vec{p}) f(\vec{p}) \\
&= \frac{g_i}{(2\pi)^3} \int \frac{\mathrm{d}^3p\, E(\vec{p})}{\exp(E(\vec{p})/T) \mp 1} \\
&= \frac{g_i}{2\pi^2} \int_0^\infty \frac{\mathrm{d}p\, p^2 E}{\exp(E(\vec{p})/T) \mp 1} \\
&= \frac{g_i}{2\pi^2} \int_m^\infty \frac{\mathrm{d}E\, E^2\sqrt{E^2-m^2}}{\exp(E(\vec{p})/T) \mp 1} \\
&\approx \frac{g_i}{2\pi^2} \int_0^\infty \frac{\mathrm{d}E\, E^3}{\exp(E(\vec{p})/T) \mp 1} \\
&= \begin{cases} \frac{g_i}{2\pi^2} T^4 \zeta(4)\Gamma(4) & \text{bosons} \\ \frac{g_i}{2\pi^2} T^4 \frac{7\pi^4}{120} & \text{fermions} \end{cases} \\
&= (1; 7/8) \cdot g_i \frac{\pi^2 T^4}{30}
\end{aligned}$$

Here $(1; 7/8)$ denotes that it is 1 for bosons and $7/8$ for fermions. For boson case, the integral representation of riemannian zeta function is used. Mathematic did the calculation of fermions.

(d) Determine the pressure and further the equation of state

Following the same recipe as before

$$\begin{aligned}
P &= \frac{g_i}{(2\pi)^3} \int \mathrm{d}^3p\, \frac{p^2}{3E} \frac{1}{\exp(E/T) \mp 1} \\
&\approx \frac{g_i}{6\pi^2} \int_0^\infty \frac{\mathrm{d}E\, E^3}{\exp(E/T) \mp 1} \\
&= \rho/3
\end{aligned} \tag{2.4}$$

This is the equation of state for ultra-relativstic particles with $\omega = 1/3$.

# 3 Equation of state for an exotic scalar field

(a) Using the identity,

$$\frac{\delta(\det A)}{\det A} = \mathrm{tr}\left(\frac{\delta A}{A}\right) \tag{3.1}$$

One can show that

$$\frac{\delta g}{g} = g^{\mu\nu}\delta g_{\mu\nu}$$
$$\delta g = g g^{\mu\nu}\delta g_{\mu\nu} \tag{3.2}$$

Then,

$$\begin{aligned}
\delta\sqrt{-g} &= -\frac{1}{2\sqrt{-g}}\delta g \\
&= -\frac{1}{2\sqrt{-g}} g g^{\mu\nu}\delta g_{\mu\nu} \\
&= -\frac{1}{2}\sqrt{-g} g^{\mu\nu}\delta g_{\mu\nu}
\end{aligned} \tag{3.3}$$

Now, our Enery-momentum Tensor:

$$\begin{aligned}
T_{\mu\nu} &= \frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L})}{\delta g^{\mu\nu}} \\
&= \frac{2}{\sqrt{-g}}\left[\frac{\delta(\sqrt{-g})}{\delta g^{\mu\nu}}\mathcal{L} + \sqrt{-g}\frac{\delta\mathcal{L}}{\delta g^{\mu\nu}}\right] \\
&= \frac{2}{\sqrt{-g}}\left(-\frac{1}{2}\sqrt{-g} g_{\mu\nu}\right)\mathcal{L} + 2\frac{\delta\mathcal{L}}{\delta g^{\mu\nu}} \\
&= -g_{\mu\nu}\mathcal{L} + 2\frac{\delta\mathcal{L}}{\delta g_{\mu\nu}}
\end{aligned} \tag{3.4}$$

Given that the lagrangian is

$$\mathcal{L} = -V_0\sqrt{1 - g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi} \tag{3.5}$$

Then,

$$\begin{aligned}
\frac{\delta\mathcal{L}}{\delta g^{\mu\nu}} &= -V_0\left(\frac{1}{2}\right)\left(\frac{-\delta^\alpha_\mu\delta^\beta_\nu\partial_\alpha\phi\partial_\beta\phi}{\sqrt{1 - g^{\lambda\rho}\partial_\lambda\phi\partial_\rho\phi}}\right) \\
&= \frac{V_0}{2}\frac{\partial_\mu\phi\partial_\nu\phi}{\sqrt{1 - g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi}}
\end{aligned} \tag{3.6}$$

Coming back to our Enery-momentum Tensor:

$$\begin{aligned}
T_{\mu\nu} &= -g_{\mu\nu}\mathcal{L} + \frac{V_0\partial_\mu\phi\partial_\nu\phi}{\sqrt{1 - g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi}} \\
&= V_0\frac{g_{\mu\nu}(1 - g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi) + \partial_\mu\phi\partial_\nu\phi}{\sqrt{1 - g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi}}
\end{aligned} \tag{3.7}$$

(b) Since we are considering a spatially homogenous scalar field $\phi(t)$,

$$\partial_i \phi = 0 \tag{3.8}$$

Either we are using Minkowski or FLRW metric, $g_{00} = 1$ regardless,

$$\begin{aligned} T_{\mu\nu} &= V_0 \frac{g_{\mu\nu}(1 - g^{00}\partial_0\phi\partial_0\phi) + \partial_\mu\phi\partial_\nu\phi}{\sqrt{1 - g^{00}\partial_0\phi\partial_0\phi}} \\ &= V_0 \frac{g_{\mu\nu}(1 - (\partial_t\phi)^2) + \partial_\mu\phi\partial_\nu\phi}{\sqrt{1 - (\partial_t\phi)^2}} \end{aligned} \tag{3.9}$$

Taking the 00-component,

$$\begin{aligned} T_{00} &= V_0 \frac{1 - (\partial_t\phi)^2 + (\partial_t\phi)^2}{\sqrt{1 - (\partial_t\phi)^2}} \\ \rho &= \frac{V_0}{\sqrt{1 - (\partial_t\phi)^2}} \end{aligned} \tag{3.10}$$

Taking the ij-component,

$$\begin{aligned} T_{ij} &= V_0 \frac{g_{ij}(1 - (\partial_t\phi)^2) + \partial_i\phi\partial_j\phi}{\sqrt{1 - (\partial_t\phi)^2}} \\ -P g_{ij} &= V_0 g_{ij} \sqrt{1 - (\partial_t\phi)^2} \\ P &= -V_0 \sqrt{1 - (\partial_t\phi)^2} \end{aligned} \tag{3.11}$$

Looking at the equation of state,

$$\omega = \frac{P}{\rho} \tag{3.12}$$

$$\begin{aligned} &= \frac{-V_0\sqrt{1 - (\partial_t\phi)^2}}{\frac{V_0}{\sqrt{1-(\partial_t\phi)^2}}} \\ &= (\partial_t\phi)^2 - 1 \end{aligned} \tag{3.13}$$

if $\partial_t\phi \ll 1$

$$\omega = -1 \tag{3.14}$$

$$P = -\rho \tag{3.15}$$

For positive energy density, the pressure exerted by the scalar field is negative. The equation of state for an exotic scalar field (when $\partial_t\phi \ll 1$) is similar to that of a vacuum in a flat space–time.

## References

[1] *comoving and proper distances*. URL: https://en.wikipedia.org/wiki/Comoving_and_proper_distances.

[2] E. W. Kolb and M. S. Turner. *The Early Universe*. 1981.