

# Theoretical Astroparticle Physik

## Homework 4

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### 1 Quickies

- (a) Briefly argue why the chemical potential  $\mu_\gamma$  of photons is zero.

We know that one can introduce chemical potential  $\mu_i$  to conserved quantum numbers  $Q^{(i)}$  to deal with all relevant reactions in thermalized system with various particle species. The chemical potential for particle of type A would then be

$$\mu_A = \sum_i \mu_i Q_A^{(i)} \quad (1.1)$$

where  $Q_A^{(i)}$  are the quantum numbers carried by particle A.

And for a photon, all of its quantum numbers are 0, eg. charge neutral, no lepton/baryon numbers, color neutral, etc. As such one can easily deduce that  $\mu_\gamma = 0$

Another way of looking at this is that we know photon is its own antiparticle. A consequence of the equation 1.1 is that  $\mu_{\bar{A}} = -\mu_A$ , which again leads to  $\mu_\gamma = 0$

- (b) At which approximate temperature did our universe transition from being dominated by radiation to a phase, where its energy density was dominated by non-relativistic matter? What is the corresponding redshift?

To get the transition, one need to consider when the energy density of non-relativistic matter and the energy density of radiations are equal:

$$\frac{8\pi}{3}G\rho_c\Omega_M\left(\frac{a_0}{a_{eq}}\right)^3 = \frac{8\pi}{3}G\rho_c\Omega_{rad}\left(\frac{a_0}{a_{eq}}\right)^4 \quad (1.2)$$

$$\Omega_M = \Omega_{rad}\left(\frac{a_0}{a_{eq}}\right)$$

$$\frac{\Omega_M}{\Omega_{rad}} = \frac{a_0}{a_{eq}} = z_{eq} + 1 \quad (1.3)$$

Given that  $\Omega_{rad} \approx 5.0 \cdot 10^{-5}$  and  $\Omega_M \approx 0.315$

$$z_{eq} + 1 \approx 6300 \quad (1.4)$$

$$T_{eq} = T_0(z_{eq} + 1) \approx 3 \cdot (6300)K \approx 18900K \approx 1.89 \times 10^4 K \quad (1.5)$$

## 2 Number densities of particles and antiparticles

System that we are looking at:

- Massless particles,  $m = 0$ ,
- chemical potential for particle  $\mu$ , chemical potential for anti particle  $-\mu$ ,
- thermal bath with  $T \gg \mu$
- spin degrees of freedom, g
- $E^2 = |\vec{p}|^2$  (because of  $m=0$ )

(a) First we look at the number density, from here on we use  $p$  as  $|\vec{p}|$

$$n = g \int d^3p f(\vec{p}) \quad (2.1)$$

$$= g \int f(\vec{p}) d\Omega p^2 dp$$

$$= 4\pi g \int f(p) p^2 dp$$

$$= 4\pi g \int \frac{1}{(2\pi)^3} \frac{1}{e^{\frac{E-\mu}{T}} \pm 1} p^2 dp$$

$$= \frac{g}{(2\pi^2)} \int \frac{1}{e^{\frac{p-\mu}{T}} \pm 1} p^2 dp$$

$$n = \frac{g}{(2\pi^2)} \int \left( \frac{1}{e^{\frac{p}{T}} \pm 1} + \frac{e^{\frac{p}{T}}}{(e^{\frac{p}{T}} \pm 1)^2} \frac{\mu}{T} \right) p^2 dp \quad (2.2)$$

Now getting the equation from the question

$$\begin{aligned} \frac{\Delta n}{g} &= \frac{n - \bar{n}}{g} \\ &= \frac{1}{(\pi^2)} \int \left( \frac{e^{\frac{p}{T}}}{(e^{\frac{p}{T}} \pm 1)^2} \frac{\mu}{T} \right) p^2 dp \end{aligned} \quad (2.3)$$

(b) Now using  $z = \frac{p}{T}$

$$\begin{aligned} \frac{\Delta n}{g} &= \frac{1}{(\pi^2)} \int \left( \frac{e^z}{(e^z \pm 1)^2} \frac{\mu}{T} \right) (zT)^2 d(zT) \\ &= \frac{\mu T^2}{(\pi^2)} \int \frac{z^2 e^z}{(e^z \pm 1)^2} dz \end{aligned} \quad (2.4)$$

### 3 Equilibrium Recombination Temperature of Helium

(a) Particles in equation (4) are in chemical equilibrium, thus

$$\mu_{\text{He}^{++}} + 2\mu_{e^-} = \mu_{\text{He}} \quad (3.1)$$

(b) Heuristically we assume that the particles are non-relativistic during

recombination. Thus the number density follows the usual formular

$$n_e = g_e \left( \frac{m_e T}{2\pi} \right)^{3/2} e^{(\mu_e - m_e)/T} \quad (3.2)$$

$$n_{\text{He}} = g_{\text{He}} \left( \frac{m_{\text{He}} T}{2\pi} \right)^{3/2} e^{(\mu_{\text{He}} - m_{\text{He}})/T} \quad (3.3)$$

$$n_{\text{He}^{++}} = g_{\text{He}^{++}} \left( \frac{m_{\text{He}^{++}} T}{2\pi} \right)^{3/2} e^{(\mu_{\text{He}^{++}} - m_{\text{He}^{++}})/T} \quad (3.4)$$

Naively, one say because of Pauli principle, two neutrons (protons) in alpha particle ( $\text{He}^{++}$ ) should have opposite spins. And thus the net spin should be zero. But one need to know the difference between  $S$  and  $S_z$ .

We know that two spin  $\frac{1}{2}$  particles (neutron or proton in this case) can form a symmetric triplet and an antisymmetric singlet,  $2 \otimes 2 = 3_S \oplus 1_A$ . It is true that with  $S_{z,1} = -S_{z,2}$ , the total spin in  $z$ -direction should be zero, but the spin can either be 1 or 0.

$$\begin{aligned} |1, 0\rangle &= \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \\ |0, 0\rangle &= \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \end{aligned}$$

(one could use CG-coefficients from opposite direction to get the same result.)

Since we are looking for ground state, we need to find out which state have lower energy. Both states have integer spin, thus the total wave-function must be symmetric. Then the  $|0, 0\rangle$  must have antisymmetric spatial component of wave function  $\psi_A(\vec{r}_1, \vec{r}_2)$  to achieve the symmetry. Antisymmetric spatial component should always have lower energy. To see this, we note that in the limit  $\vec{r}_1 \rightarrow \vec{r}_2$

$$\psi_A(\vec{r}_1, \vec{r}_2) \rightarrow 0$$

Thus in this configuration two identical particle must be further away from each other resulting lowered energy. After determining the configuration of  $pp$  or  $nn$  system, we then conclude that the alpha particle should have spin 0 ( $g_{\text{He}^{++}} = 1$ ). Since the eletrons in Helium also follows Pauli principle,  $g_{He} = 1$ .

- (c) Expression of baryon number conservation for  $n_{\text{He}}$  and  $n_{\text{He}^{++}}$  assuming that nuclei don't convert into each other. Since no interactions can

change  $(A, Z)$ , one quarter of the baryons only have two forms (ignore the  $\text{He}^+$ )

$$n_{\text{He}^{++}} + n_{\text{He}} = 0.25n_{\text{B}} \quad (3.5)$$

In this era, there are three electrically charge particles and assuming electrically neutral Universe

$$n_p + 2n_{\text{He}^{++}} = n_e \quad (3.6)$$

There is 2 before number density of  $\text{He}^{++}$  because it carries two electric charges. In the next part, we will ignore  $n_p$  contributions. (Why can we do this? We know that helium recombination happened before hydrogen recombination, thus there should have been a lot protons floating around.)

(d) Expression of  $n_{\text{He}}$  in terms of  $n_{\text{He}^{++}}$ .

First multiply (3.4) and (3.2)<sup>2</sup> (the square will be obvious in a minute)

$$\begin{aligned} n_{\text{He}^{++}} n_e^2 &= g_{\text{He}^{++}} g_e^2 \left( \frac{m_{\text{He}^{++}} T}{2\pi} \right)^{3/2} \left( \frac{m_e T}{2\pi} \right)^{3/2} e^{(\mu_{\text{He}^{++}} + 2\mu_e - m_{\text{He}^{++}} - 2m_e)/T} \\ 4n_{\text{He}^{++}}^3 &\stackrel{3.6}{=} g_{\text{He}^{++}} g_e^2 \left( \frac{m_{\text{He}^{++}} T}{2\pi} \right)^{3/2} \left( \frac{m_e T}{2\pi} \right)^{3/2} e^{[\mu_{\text{He}} - (m_{\text{He}} + \Delta\text{He})]/T} \\ &= \frac{g_{\text{He}^{++}} g_e^2}{g_{\text{He}}} n_{\text{He}} \left( \frac{m_{\text{He}^{++}}}{m_{\text{He}}} \right)^{3/2} \left( \frac{m_e T}{2\pi} \right)^3 e^{-\Delta\text{He}/T} \\ &\approx 4n_{\text{He}} \left( \frac{m_e T}{2\pi} \right)^3 e^{-\Delta\text{He}/T} \end{aligned}$$

where we have used (3.6) without  $n_p$  (Is this justified?) and

$$\frac{m_{\text{He}^{++}}}{m_{\text{He}}} = \frac{m_{\text{He}^{++}}}{m_{\text{He}^{++}} + 2m_e - \Delta\text{He}} = \frac{3.7 \text{ GeV}}{3.7 \text{ GeV} + 2 \cdot 0.5 \text{ MeV} - 79 \text{ eV}} \approx 1$$

Rearranging this, we get

$$n_{\text{He}} = n_{\text{He}^{++}}^3 \left( \frac{m_e T}{2\pi} \right)^{-3} e^{\Delta\text{He}/T} \quad (3.7)$$

Using (3.5) and definitions of  $X_i$

$$X_{\text{He}^{++}} + X_{\text{He}} = 1 \quad (3.8)$$

Plug in the result (3.7) and we have the Saha equation for Helium

$$X_{\text{He}^{++}} + \tilde{n}_B^2 X_{\text{He}^{++}}^3 \left( \frac{m_e T}{2\pi} \right)^{-3} e^{\Delta_{\text{He}}/T} = 1 \quad (3.9)$$

$\tilde{n}_B$  can be expressed in terms of baryon number density and is then related to photon number density

$$\begin{aligned} X_{\text{He}^{++}} + (\tilde{\eta}_B n_\gamma)^2 X_{\text{He}^{++}}^3 \left( \frac{m_e T}{2\pi} \right)^{-3} e^{\Delta_{\text{He}}/T} &= 1 \\ X_{\text{He}^{++}} + \tilde{\eta}_B^2 \left( g_\gamma \frac{\zeta(3)}{\pi^2} T^3 \right)^2 X_{\text{He}^{++}}^3 \left( \frac{m_e T}{2\pi} \right)^{-3} e^{\Delta_{\text{He}}/T} &= 1 \end{aligned}$$

with  $\tilde{\eta}_B = 0.25 \cdot \eta_B$ . The second term in LHS is still  $X_{\text{He}}$ . We demand

$$X_{\text{He}^{++}}(T_r^{\text{eq}}) \sim X_{\text{He}}(T_r^{\text{eq}}) \sim 1 \quad (3.10)$$

and

$$X_{\text{He}} = 4\tilde{\eta}_B^2 \left( \frac{\zeta(3)}{\pi^2} \right)^2 T^6 X_{\text{He}^{++}}^3 \left( \frac{m_e T}{2\pi} \right)^{-3} e^{\Delta_{\text{He}}/T}$$

By demanding this quantity to be  $\mathcal{O}(1)$  at  $T_r^{\text{eq}}$  (why is this true? (I know this is just assumption of this exercise. But why can we make this assumption without having large error?))

$$\frac{\Delta_{\text{He}}}{T_r^{\text{eq}}} \sim -\ln \left[ \frac{1}{4} \eta_B^2 \frac{\zeta(3)^2}{\pi^4} \left( \frac{2\pi T_r^{\text{eq}}}{m_e} \right)^3 \right]$$

Use the hint to obtain approximate equation

$$\frac{\Delta_{\text{He}}}{T_r^{\text{eq}}} \approx \ln \left[ \frac{4\pi^4}{\eta_B^2 \zeta^2(3)} \left( \frac{m_e}{2\pi \Delta_{\text{He}}} \right)^3 \right] \approx 68.8 \gg 1 \quad (3.11)$$

The approximation is also valid! In the end, we have

$$T_r^{\text{eq}} \approx 1.14 \text{ eV} < 0.38 \text{ eV} \quad (3.12)$$

Indeed, helium recombination did happen before hydrogen recombination and the formula for number density in the beginning is well justified.