Theoretical Astroparticle Physik
Homework 3

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1 Quickies

(a) Definition of critical density

$$\rho_c = \frac{3}{8\pi G} H_0^2 \tag{1.1}$$

When $\rho_M + \rho_{\mathrm{rad}} + \rho_\Lambda = \rho_c$, it means that there is no curvature contribution, thus the Universe is flat.

(b) Particle horizon, or cosmological horizon, is the maximal distance from which light could traveled to observer since the beginning of the Universe[1]. It is relevant to discussion of whole Universe. Event horizon is the boundary where events inside cannot affect or be perceived in any ways by outside, e.g. near black hole. And it is normally in smaller scale than particle horizon.

2 Accelerated Expansion in a de-Sitter Universe

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \exp(2H_{dS}t)\,\mathrm{d}\vec{x}^2 \tag{2.1}$$

(a) Determine the Riemann curvature tensor

Set $H_{\mathrm{dS}} = H$ throughout this assignment. Curvature tensor R has mass dimension 2 because of two derivatives. To achieve this, only quantity we have is just H^2. We also know that $R_{\mu\nu\lambda\rho}$ is anti-symmetric in first and second pair of indices and symmetric under $(\mu,\nu) \leftrightarrow (\lambda,\rho)$. The only choice is then

$$R_{\mu\nu\lambda\rho} = H^2(g_{\mu\rho}g_{\nu\lambda} - g_{\mu\lambda}g_{\nu\rho}) \tag{2.2}$$

To show this explicitely, first calculate christoffel symbols

$$\Gamma^{\mu}_{\nu\lambda} = \frac{1}{2}g^{\mu\rho}\left(\partial_\nu g_{\rho\lambda} + \partial_\lambda g_{\nu\rho} - \partial_\rho g_{\nu\lambda}\right) \tag{2.3}$$

One important fact about the metric is that there is only time dependence and thus spatial derivatives always vanish.

$$\begin{aligned}
\Gamma^0_{00} &= \frac{1}{2}g^{0\rho}(\partial g_{\rho 0} + \partial_0 g_{0\rho} - \partial_\rho g_{00}) \\
&= 0 \\
\Gamma^0_{0i} &= \Gamma^0_{i0} = \frac{1}{2}\left(\partial_i g_{\rho 0} + \partial_0 g_{\rho i} - \partial_\rho g_{0i}\right) \\
&= \frac{1}{2}\partial_0 g_{0i} \\
&= 0 \\
\Gamma^i_{00} &= \frac{1}{2}g^{i\rho}(\partial_0 g_{\rho 0} + \partial_0 g_{\rho 0} - \partial_\rho g_{00}) \\
&= 0 \\
\Gamma^i_{0j} &= \Gamma^i_{j0} = \frac{1}{2}g^{i\rho}\left(\partial_0 g_{\rho j} + \partial_j g_{\rho 0} - \partial_\rho g_{0j}\right) \\
&= -\frac{1}{2}e^{-2Ht}\delta^{i\rho}\left(\partial_0 g_{\rho j} - \partial_\rho g_{0j}\right) \\
&= -\frac{1}{2}e^{-2Ht}\delta^i_j\partial_0 e^{2Ht} \\
&= -H\delta^i_j \\
\Gamma^0_{ij} &= \Gamma^0_{ji} = \frac{1}{2}g^{0\rho}\left(\partial_i g_{\rho j} + \partial_j g_{\rho i} - \partial_\rho g_{ij}\right) \\
&= -\frac{1}{2}g^{0\rho}\partial_\rho g_{ij} \\
&= \delta_{ij}He^{2Ht} \\
\Gamma^i_{jk} &= 0
\end{aligned}$$

Definition of curvature tensor

$$R_{\mu\nu\lambda\rho} = g_{\mu\sigma}\left(\partial_\lambda\Gamma^\sigma_{\nu\rho} - \partial_\rho\Gamma^\sigma_{\nu\lambda} + \Gamma^\sigma_{\kappa\lambda}\Gamma^\kappa_{\nu\rho} - \Gamma^\sigma_{\kappa\rho}\Gamma^\kappa_{\lambda\nu}\right) \tag{2.4}$$

Because of mentioned symmetries, we split the tensor into two parts

$$R_{\mu\nu\lambda\rho} = A_{\mu\nu\lambda\rho} - A_{\mu\nu\rho\lambda}$$

Now try to compute $A_{\mu\nu\lambda\rho}$

$$A_{\mu\nu\lambda\rho} = g_{\mu\sigma}\partial_\lambda\Gamma^\sigma_{\nu\rho} + g_{\mu\sigma}\Gamma^\sigma_{\kappa\lambda}\Gamma^\kappa_{\nu\rho} = B + C$$

in which

$$\begin{aligned}
B &= g_{\mu\sigma}\partial_\lambda\Gamma^\sigma\nu\rho \\
&= \left(\delta_{\mu0}\delta^0_\sigma - e^{2Ht}\delta_{\mu i}\delta^i_0\right)\partial_\lambda\Gamma^\sigma_{\nu\rho} \\
&= \partial_\lambda\Gamma^0_{\nu\rho}\delta_{\mu0} - e^{2Ht}\partial_\lambda\Gamma^i_{\nu\rho}\delta_{\mu i} \\
&= \delta_{\lambda0}\left(\partial_0\Gamma^0_{\nu\rho}\delta_{\mu0} - e^{2Ht}\partial_0\Gamma^i_{\nu\rho}\delta_{\mu i}\right) \\
&= \delta_{\mu0}\delta_{\nu i}\delta^i_\rho\delta_{\lambda0}2H^2e^{2Ht}
\end{aligned}$$

and

$$\begin{aligned}
C &= g_{\mu\sigma}\Gamma^\sigma\kappa\lambda\Gamma^\kappa_{\nu\rho} \\
&= \delta_{\mu0}\Gamma^0_{i\lambda}\Gamma^i_{\nu\rho} - e^{2Ht}\delta_{\mu i}\Gamma^i_{\kappa\lambda}\Gamma^\kappa_{\nu\rho} \\
&= \delta_{\mu0}\delta_{i\lambda}He^{2Ht}\left(\delta_{\nu0}\Gamma^i_{0\rho} + \delta_{\rho0}\Gamma^i_{\nu0}\right) - e^{2Ht}\delta_{\mu i}\left(\delta_{\mu0}\Gamma^i_{0\lambda} + \delta_{\lambda0}\Gamma^i_{\kappa0}\right)\Gamma^\kappa_{\nu\rho} \\
Ce^{-2Ht} &= -H^2\delta_{\mu0}\delta_{i\lambda}\left(\delta_{\nu0}\delta^i_\rho + \delta_{\rho0}\delta^i_\nu\right) + H\delta_{\mu i}\left(\delta_{\kappa0}\delta^i_\lambda H + \delta_{\lambda0}\delta^i_\kappa\right)\Gamma^\kappa_{\nu\rho} \\
Ce^{-2Ht} &= -H^2\delta_{\mu0}\delta_{i\lambda}\left(\delta_{\nu0}\delta^i_\rho + \delta_{\rho0}\delta^i_\nu\right) + H\delta_{\mu i}\left(\Gamma^0_{\nu\rho}\delta^i_\lambda + \delta_{\lambda0}\Gamma^i_{\nu\rho}\right) \\
Ce^{-2Ht} &= -H^2\delta_{\mu0}\delta_{i\lambda}\left(\delta_{\nu0}\delta^i_\rho + \delta_{\rho0}\delta^i_\nu\right) + H^2\delta_{\mu i}\left[\delta^i_\mu\delta_{\nu j}\delta^j_\rho e^{2Ht} - \delta_{\lambda0}\left(\delta_{\nu0}\delta_{\nu0}\delta^i_\rho + \delta_{\rho0}\delta^i_\nu\right)\right]
\end{aligned}$$

Thus

$$\begin{aligned}
A_{\mu\nu\rho\lambda}/(He^{2Ht}) =& 2\delta_{\mu0}\delta_{\nu i}\delta^i_\rho\delta_{\lambda0} + e^{2Ht}\delta_{\mu i}\delta_{\nu j}\delta^j_\rho\delta^i_\lambda - \delta_{\mu i}\delta_{\nu0}\delta^i_\rho\delta_{0\lambda} \\
&- \cancel{\delta_{\mu i}\delta^i_\nu\delta_{i\lambda}\delta^i_\rho} - \cancel{\delta_{\mu0}\delta_{\nu0}\delta_{i\lambda}\delta^i_\rho} - \delta_{\mu0}\delta^i_\nu\delta_{\rho00}\delta_{i\lambda}
\end{aligned}$$

Some of these terms get canceled because of symmtry in ρ, λ. In the end, we have

$$R_{\mu\nu\lambda\rho} = H^2(g_{\mu\rho}g_{\nu\lambda} - g_{\mu\lambda}g_{\nu\rho}) \tag{2.5}$$

(b) Solve Einstein field equations

$$
\begin{aligned}
R_{\mu\nu} &= R^{\lambda}_{\mu\lambda\nu} \\
&= g^{\alpha\lambda} R_{\alpha\mu\lambda\nu} \\
&= H^2 g^{\alpha\lambda}(g_{\alpha\nu} g_{\mu\lambda} - g_{\alpha\lambda} g_{\mu\nu}) \\
&= H^2(\delta_{\mu\nu} - g_{\mu\nu}) \\
R &= g^{\mu\nu} R_{\mu\nu} \\
&= H^2(g^{\mu\nu}\delta_{\mu\nu} - 1) \\
&= -3H^2 e^{-2Ht} \\
G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \\
&= H^2(\delta_{\mu\nu} - g_{\mu\nu} + \frac{3}{2} g_{\mu\nu} e^{-2Ht}) \\
&\overset{!}{=} 8\pi G \Lambda g_{\mu\nu}
\end{aligned}
$$

For 00 component, we get

$$\frac{3}{2} H^2 e^{-2Ht} = 8\pi G\Lambda > 0 \tag{2.6}$$

For *ii* component, we get

$$H^2 \left(\frac{1}{2} e^{-2Ht} - 1 \right) = 8\pi G\Lambda < 0 \tag{2.7}$$

References

[1] *Particle horizon*. URL: `https://en.wikipedia.org/wiki/Particle_horizon`.