H3.1

a) $K_i = -M_{0i}\,, \qquad S_k = \frac{1}{2}\epsilon_{kij}M_{ij}$

$$\Rightarrow \omega^{\rho\sigma}M_{\rho\sigma} = \omega^{0\sigma}M_{0\sigma} + \omega^{i\sigma}M_{i\sigma}$$

$$= \omega^{0i}M_{0i} + \omega^{i0}M_{i0} + \omega^{ij}M_{ij}$$

$$S_k = \frac{1}{2}\epsilon_{kij}M_{ij}$$

$$2S_k = \epsilon_{kij}M_{ij}$$

$$2\epsilon_{kmn}S_k = \underbrace{\epsilon_{kij}\epsilon_{kmn}}_{=\delta_{im}\delta_{jn}-\delta_{in}\delta_{jm}}M_{ij}$$

$$= M_{mn} - M_{nm}$$

$$= 2M_{mn}$$

$$\Rightarrow M_{mn} = \epsilon_{kmn}S_k$$

$$(M_{\rho\sigma})^\mu{}_\nu = i(g_{\sigma\nu}\delta^\mu{}_\rho - g_{\rho\nu}\delta^\mu{}_\sigma)$$

$$= -i(g_{\rho\nu}\delta^\mu{}_\sigma - g_{\sigma\nu}\delta^\mu{}_\rho)$$

$$= -(M_{\sigma\rho})^\mu{}_\nu$$

$$\Rightarrow M_{ii} = 0 \quad \forall i$$

$$= -2\omega^{0i}K_i + \omega^{ij}\epsilon_{kij}S_k$$

$$-\frac{i}{2}\omega_{\rho\sigma}M^{\rho\sigma} = i\,\omega^{0i}K_i - \frac{i}{2}\omega^{ij}\underbrace{(\epsilon_{kij}S_k)}_{M_{ij}}$$

$$(\omega^{i0} - \omega^{0i})$$

$$= -\omega_{i0} + \omega_{0i}$$

$$= 2\zeta_i$$

$$\overset{!}{=} -i\,\omega^i S_i - i\,\zeta^i K_i$$

$$\rightarrow \Lambda = \exp(-i(\zeta_i K_i + \omega_k k_k))$$

$$\Rightarrow \zeta^i = -\omega^{0i} = \omega^{i0}$$

$$\omega^i = \frac{1}{2}\epsilon^{ijk}\omega_{jk}$$

b) $T^L_i = S_i^+ = \frac{1}{2}(S_i + iK_i)\,, \qquad T^R_i = S_i^- = \frac{1}{2}(S_i - iK_i)$

$$\Lambda = \exp(-i(\vec{\omega} - i\vec{\zeta})\vec{T}^L)\exp(-i(\vec{\omega} + i\vec{\zeta})\vec{T}^R)$$

$$= \exp\Big(-\frac{i}{2}(\omega^i S_i - i\cancel{\omega^i}K_i - i\zeta^i S_i + \zeta^i K_i)$$

$$-\frac{i}{2}(\omega^i S_i + i\cancel{\omega^i}K_i + i\zeta^i S_i + \zeta^i K_i)\Big)$$

$$= \exp(-i(w^i S_i + \varsigma^i K_i))$$

$$\Lambda = \exp[-i(\omega_k - i\varsigma_k)T_k^L - i(\omega_k + i\varsigma_k)T_k^R]$$

$$\left| \; [T_k^L, T_k^R] = 0 \right.$$

$$= \exp(-i(\omega_k - i\varsigma_k)T_k^L) \cdot \exp(-i(\omega_k + i\varsigma_k)T_k^R)$$

c) $\quad S_i = (T_i^L + T_i^R) \qquad K_i = -i(T_i^L - T_i^R)$

$$\begin{aligned}
M^{\mu\nu} &= M^{0\nu}\delta^{\mu 0} + M^{i\nu}\delta^\mu_i \\
&= M^{0i}(\delta^{\mu 0}\delta^\nu{}_i + \delta^\mu_i\delta^{\nu 0}) + M^{ij}\delta^\mu_i\delta^\nu_j \\
&= -K^i(\delta^{\mu 0}\delta^\nu{}_i + \delta^\mu_i\delta^{\nu 0}) + \epsilon^{kij} S_k \delta^\mu_i \delta^\nu_j \\
&= +i(T_i^L - T_i^R)(\delta^{\mu 0}\delta^\nu{}_i + \delta^\mu_i\delta^{\nu 0}) + \epsilon^{kij}(T_k^L + T_k^R)\delta^\mu_i\delta^\nu_j
\end{aligned}$$

With Ψ_L

$$\omega_{\mu\nu}M^{\mu\nu} = \omega_{\mu\nu}\left[i\tfrac{1}{2}\sigma^i(\delta^{\mu 0}\delta^\nu{}_i + \delta^\mu_i\delta^{\nu 0}) + \epsilon^{kij}\tfrac{1}{2}\sigma_k\delta_i^\mu\delta^\nu_j\right]$$

$$\sigma^{\mu\nu} = \frac{i}{4}(\sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu)$$

$$\exp[-i(\omega_k - i\varsigma_k)T_k^L] = \exp\left[-\tfrac{i}{2}\left(-i(\omega_{0k} - \omega_{k0}) + \epsilon_{kij}\omega^{ij}\right)\tfrac{\sigma_k}{2}\right]$$

$$\left(\begin{aligned}
\sigma^{0i} &= \frac{i}{4}(-\sigma^i - \sigma^i) = \frac{-i\sigma^i}{2} = \frac{i}{2}\sigma_i \\
\sigma^{ij} &= \frac{-i}{4}(\sigma^i\sigma^j - \sigma^j\sigma^i) = \frac{i}{4}[\sigma^j, \sigma^i] = -\tfrac{1}{2}\epsilon_{jik}\sigma^k = -\tfrac{1}{2}\epsilon_{kij}\sigma_k
\end{aligned} \right.$$

$$= \exp\left[-\tfrac{i}{2}(\omega_{0k}\sigma^{0k} + \omega_{k0}\sigma^{k0} + \omega_{ij}\sigma^{ij})\right]$$

$$= \exp(-\tfrac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu})$$

in spinor space

analogous for Ψ_R

d) $\Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \longrightarrow \Psi' = \begin{pmatrix} \exp(-\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu})\,\psi_L \\ \exp(-\frac{i}{2}\omega_{\mu\nu}\bar{\sigma}^{\mu\nu})\,\psi_R \end{pmatrix}$

$$\Sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu] \underset{\uparrow}{=} \frac{i}{4}\left[\begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \begin{pmatrix} 0 & \sigma^\nu \\ \bar{\sigma}^\nu & 0 \end{pmatrix}\right]$$

only first order

Chiral rep.

$$= \frac{i}{4}\left[\begin{pmatrix} \sigma^\mu\bar{\sigma}^\nu & 0 \\ 0 & \bar{\sigma}^\mu\sigma^\nu \end{pmatrix} - \begin{pmatrix} \sigma^\nu\bar{\sigma}^\mu & 0 \\ 0 & \bar{\sigma}^\nu\sigma^\mu \end{pmatrix}\right]$$

$$= \frac{i}{4}\begin{pmatrix} \sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu & 0 \\ 0 & \bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu \end{pmatrix}$$

$$= \frac{i}{4}\begin{pmatrix} \sigma^{\mu\nu} & 0 \\ 0 & \bar{\sigma}^{\mu\nu} \end{pmatrix}$$

$$= \exp(-\tfrac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu})\,\Psi$$

e) $\Psi \to \Lambda_{1/2}\Psi$,

$\Psi^\dagger \longrightarrow \Psi^\dagger \Lambda_{1/2}^\dagger$

$\bar{\Psi} \longrightarrow \Psi^\dagger \Lambda_{1/2}^\dagger \gamma^0 = \bar{\Psi}\Lambda_{\frac{1}{2}}$

$$\Lambda_{1/2}^\dagger = \exp\Big(\tfrac{i}{2}\omega_{\mu\nu}\underbrace{(\Sigma^{\mu\nu})^\dagger}\Big) \longrightarrow [\gamma^\mu, \gamma^\nu]^\dagger = [\gamma^{\nu\dagger}, \gamma^{\mu\dagger}]$$

$$= +\frac{i}{4}[\gamma^{\mu\dagger}, \gamma^{\nu\dagger}]$$

$$= +\frac{i}{4}\gamma^0[\gamma^\mu, \gamma^\nu]\gamma^0$$

$$= +\gamma^0\Sigma^{\mu\nu}\gamma^0$$

$$= \gamma^0\exp(+\tfrac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu})\gamma^0$$

$$= \gamma^0\Lambda_{\frac{1}{2}}^{-1}\gamma^0$$

f) $[\gamma^\mu, \Sigma^{\nu\sigma}] = \frac{i}{4}[\gamma^\mu, [\gamma^\nu, \gamma^\sigma]] = \frac{i}{4}[\gamma^\mu, \gamma^\nu\gamma^\sigma - \gamma^\sigma\gamma^\nu]$

$= \frac{i}{4}(\gamma^\mu\gamma^\nu\gamma^\sigma - \gamma^\nu\gamma^\sigma\gamma^\mu - \gamma^\mu\gamma^\sigma\gamma^\nu + \gamma^\sigma\gamma^\nu\gamma^\mu)$

$[AB, C] = A\{B, C\} - \{A, C\}B$

$= \frac{i}{4}(-\gamma^\nu\gamma^\mu\gamma^\sigma + 2g^{\mu\nu}\gamma^\sigma - \gamma^\nu\gamma^\sigma\gamma^\mu$

$+ \gamma^\sigma\gamma^\mu\gamma^\nu - 2g^{\mu\sigma}\gamma^\nu + \gamma^\sigma\gamma^\nu\gamma^\mu)$

$= \frac{i}{4}(-\gamma^\nu \cdot 2g^{\mu\sigma} + 2g^{\mu\nu}\gamma^\sigma + \gamma^\sigma 2g^{\nu\mu} - 2g^{\mu\sigma}\gamma^\nu)$

$= i(-g^{\mu\sigma}\gamma^\nu + g^{\mu\nu}\gamma^\sigma)$

$(\mathcal{M}^{\nu\sigma})^\mu{}_\rho \gamma^\rho = i(\eta^\sigma{}_\rho \delta^{\nu\mu} - \eta^\nu{}_\rho \delta^{\sigma\mu})\gamma^\rho$

$= i(\gamma^\sigma \delta^{\nu\mu} - \gamma^\nu \delta^{\sigma\mu})$

$= [\gamma^\mu, \Sigma^{\nu\sigma}]$

$\Lambda_{1/2}^{-1/2}\gamma^\mu\Lambda_{1/2} = \gamma^\mu + \Lambda_{1/2}^{-\frac{1}{2}}[\gamma^\mu, \Lambda_{\frac{1}{2}}]$

$= [\gamma^\mu, \exp(-\frac{i}{2}\omega_{\rho\sigma}\Sigma^{\rho\sigma})]$

$= [\gamma^\mu, 1 - \frac{i}{2}\omega_{\rho\sigma}\Sigma^{\rho\sigma}]$

$= -\frac{i}{2}\omega_{\rho\sigma}[\gamma^\mu, \Sigma^{\rho\sigma}]$

$= -\frac{i}{2}\omega_{\rho\sigma}(\mathcal{M}^{\rho\sigma})^\mu{}_\nu \gamma^\nu$

$= \gamma^\mu - \frac{i}{2}\omega_{\rho\sigma}(\mathcal{M}^{\rho\sigma})^\mu{}_\nu \gamma^\nu$

$= \exp(-\frac{i}{2}\omega_{\rho\sigma}\mathcal{M}^{\rho\sigma})^\mu{}_\nu \gamma^\nu = \Lambda^\mu{}_\nu \gamma^\nu$

g) $[\Sigma^{\mu\nu}, \gamma_5] = \frac{i}{4}[[\gamma^\mu, \gamma^\nu], \gamma_5]$

$= \frac{i}{4}[\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu, \gamma_5]$

$= \frac{i}{4}(\gamma^\mu\gamma^\nu\gamma_5 - \gamma_5\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\gamma_5 + \gamma_5\gamma^\nu\gamma^\mu)$

$= \frac{i}{4}(\gamma^\mu\gamma^\nu\gamma_5 + \gamma^\mu\gamma_5\gamma^\nu - \gamma^\nu\gamma^\mu\gamma_5 - \gamma^\nu\gamma_5\gamma^\mu)$

$= \frac{i}{4}(\gamma^\mu\{\gamma^\nu, \gamma_5\} - \gamma^\nu\{\gamma^\mu, \gamma_5\})$

$$\Lambda_{\frac{1}{2}}^{-1} \gamma^5 \Lambda_{\frac{1}{2}} = \overset{=0}{\gamma^5} + \Lambda_{\frac{1}{2}}^{-1} \underbrace{[\gamma^5, \Lambda_{\frac{1}{2}}]}_{=0}$$

h) $$\Lambda_p^{-1} \gamma_5 \Lambda_p = (\gamma_0)^{-1} \gamma_5 \gamma_0 = -(\gamma_0)^{-1} \gamma_0 \gamma_5 = -\gamma_5$$

i) $\bar{\Psi}\Psi \xrightarrow{\frac{1}{2}} \bar{\Psi} \Lambda_{\frac{1}{2}}^{-1} \Lambda_{\frac{1}{2}} \Psi = \bar{\Psi}\Psi$ → scalar under Λ

$\xrightarrow{P} \Psi^{+}\gamma^0\gamma^0\gamma^0\Psi = \bar{\Psi}\Psi$ → scalar under P

$\bar{\Psi}\gamma_5\Psi \xrightarrow{\frac{1}{2}} \bar{\Psi}\gamma_5\Psi$ → scalar und Λ

$\xrightarrow{P} -\bar{\Psi}\gamma_5\Psi$ → pseudoscalar und P

$\bar{\Psi}\gamma_\mu\Psi \xrightarrow{\frac{1}{2}} \bar{\Psi}\Lambda_\mu^{\ \nu}\gamma_\nu\Psi$ → vector und P

$\xrightarrow{P} \bar{\Psi}\gamma^0\gamma_\mu\gamma^0\Psi = \bar{\Psi}(\gamma_\mu)^{+}\Psi = \begin{cases} \bar{\Psi}\gamma^0\Psi, & \mu=0 \\ -\bar{\Psi}\gamma^i\Psi, & \mu=i \end{cases}$ → vector

$\bar{\Psi}\gamma_5\gamma_\mu\Psi \xrightarrow{\frac{1}{2}} \bar{\Psi}\Lambda_{\frac{1}{2}}\gamma_5\gamma_\mu\Lambda_{\frac{1}{2}}\Psi = \bar{\Psi}\gamma_5\Lambda_\mu^{\ \nu}\gamma_\nu\Psi$ → vector

$\xrightarrow{P} \bar{\Psi}\Lambda_p\gamma_5\gamma_\mu\Lambda_p\Psi = -\bar{\Psi}\gamma_5(\gamma_\mu)^{+}\Psi$

$= \begin{cases} -(\ldots), & \mu=0 \\ +(\ldots), & \mu=i \end{cases}$ → axial vector

$(i\not{\partial}-m)\psi=0 \longrightarrow \psi:\ 4$ DOF ; $m=0. \to 4$ DOF

H3.2

a) $H\psi=\beta m\psi \quad , \qquad \beta=\begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}$

$$=\begin{pmatrix} m\mathbb{1}_2 & 0 \\ 0 & -m\mathbb{1}_2 \end{pmatrix}\psi$$

$\left(\psi=N_0\begin{pmatrix}\chi^{(s)}\\0\end{pmatrix}\right) \qquad \left(\psi=N_0\begin{pmatrix}0\\\chi^{(s)}\end{pmatrix}\right)$

$H\psi=N_0\cdot m\cdot\begin{pmatrix}\chi^{(s)}\\0\end{pmatrix}$ ✓ $\qquad H\psi=-N_0 m\begin{pmatrix}0\\\chi^{(s)}\end{pmatrix}$ ✗

a) $H\psi=\beta m\psi \quad , \quad \psi=\begin{bmatrix}\chi\\\phi\end{bmatrix}$

$$\Rightarrow \quad \begin{aligned} E\chi&=m\chi \\ E\phi&=-m\phi \end{aligned}$$

$E=m,\ \phi\overset{!}{=}0 \qquad E=-m,\ \chi\overset{!}{=}0$

$\psi=\begin{pmatrix}\chi\\0\end{pmatrix} \qquad \psi=\begin{pmatrix}0\\\phi\end{pmatrix}$

$\Rightarrow u_0^s=N_0\begin{pmatrix}\chi^{(s)}\\0\end{pmatrix} \qquad u_0^{s+2}=N_0\begin{pmatrix}0\\\chi^{(s)}\end{pmatrix}$

b) $\gamma^\mu_D=U\gamma^\mu_W U^+ , \qquad \psi_D=U\psi_W$

$$\psi_W \longrightarrow \exp\left(-\tfrac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}_{WB}\right)\psi_W$$

$$\psi_D=U\psi_W \to U\exp\left(-\tfrac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}_{WB}\right)\psi_W$$

$= \cancel{U\exp(--)U^+}\psi_D$

$\exp\left(-\tfrac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}_{DB}\right)\underbrace{U\psi_{WB}}_{\psi_D}$

c) A boost ς in x-direction $\longrightarrow K_1=-M_{01}=\varsigma$

$$\exp\left(-\tfrac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}\right)=\exp\left(+i\varsigma\Sigma^{01}\right)$$ ✓

$$\left(\begin{aligned}\Sigma^{01}&=\tfrac{i}{4}[\gamma^0,\gamma^1]=\tfrac{i}{4}\left(\gamma^0\gamma^1-\gamma^1\gamma^0\right)\\ &=\tfrac{i}{4}\left(\gamma^0\gamma^1+\gamma^0\gamma^1\right)=\tfrac{i}{2}\gamma^0\gamma^1\end{aligned}\right.$$

$$=\exp\left(\varsigma\alpha^1/2\right)$$ ✓

d) $\exp(\varsigma\alpha^1/2) = 1 - \frac{\varsigma}{2}\alpha^1 - \frac{1}{2}\left(\frac{\varsigma}{2}\right)^2\underbrace{(\alpha^1)^2}_{=\mathbb{1}} + \cdots$

$$= 1 - \frac{1}{2}\left(\frac{\varsigma}{2}\right)^2 + \ -\frac{\varsigma}{2}\alpha^1 + \cdots$$

$$= \cosh(\varsigma/2)\mathbb{1}_4 + \sinh(\varsigma/2)\alpha^1 \quad \checkmark$$

$$\psi_D \to U\left(\cosh(\varsigma/2)\mathbb{1}_4 + \sinh(\varsigma/2)\alpha_D^1\right)U^\dagger \psi_D$$

$$= \left(\cosh(\varsigma/2) + \sinh(\varsigma/2)\, U\alpha_D^1 U^\dagger\right)\psi_D$$

$$u^s = \left(\cosh(\varsigma/2)\mathbb{1}_4 + \sinh(\varsigma/2)\alpha^1\right)\cdot N_0 \begin{pmatrix}\chi^{(s)} \\ 0\end{pmatrix}$$

$$= N_0 \begin{pmatrix}\cosh(\varsigma/2)\chi^{(s)} \\ \sinh(\varsigma/2)\sigma^1\chi^{(s)}\end{pmatrix}, \qquad E>0$$

e) $\cosh\varsigma = \gamma = \frac{|E|}{m}, \qquad \sinh\varsigma = \beta\gamma$

$$\cosh(\varsigma/2) = \sqrt{\frac{1+\cosh\varsigma}{2}} = \sqrt{\frac{|E|+m}{2m}}$$

$$\sinh(\varsigma/2) = \sqrt{\frac{\cosh\varsigma - 1}{2}} = \sqrt{\frac{|E|-m}{2m}} = \frac{p_x}{\sqrt{2m(|E|+m)}}$$

$$\Rightarrow u^s = N_0 \begin{pmatrix}\sqrt{\frac{E+m}{2m}}\chi^{(s)} \\ \frac{p_x\sigma^1}{\sqrt{2m(E+m)}}\chi^{(s)}\end{pmatrix} = \left(N_0\sqrt{\frac{E+m}{2m}}\right)\begin{pmatrix}\chi^{(s)} \\ \frac{p_x\sigma^1}{E+m}\chi^{(s)}\end{pmatrix}$$

f) $u^{(s+2)} = N_0 \begin{pmatrix}\sinh(\varsigma/2)\sigma^1\chi^{(s)} \\ \cosh(\varsigma/2)\chi^{(s)}\end{pmatrix}$

$$= \left(N_0\sqrt{\frac{-E+m}{2m}}\right)\begin{pmatrix}\frac{p_x\sigma^1}{-E+m}\chi^{(s)} \\ \chi^{(s)}\end{pmatrix}$$

Assign momentum $-p_x$ to anti-particle

g) $N = N_0\sqrt{\frac{E+m}{2m}} = \sqrt{E+m}$

$$\Rightarrow N_0 = \sqrt{2m}$$