H4.1

a) $\phi = (\phi_1 + i\phi_2)/\sqrt{2}, \qquad \phi^* = (\phi_1 - i\phi_2)/\sqrt{2}$

$\Rightarrow \qquad (\phi + \phi^*) = \sqrt{2}\,\phi_1 \iff \phi_1 = \frac{1}{\sqrt{2}}(\phi + \phi^*)$

$\phi_2 = -\frac{i}{\sqrt{2}}(\phi - \phi^*)$

$$\Rightarrow \mathcal{L} = \frac{1}{2}\cdot\frac{1}{2}\,\partial^\mu(\phi+\phi^*)\partial_\mu(\phi+\phi^*) - \frac{m^2}{2}\cdot\frac{1}{2}(\phi+\phi^*)^2$$

$$+ \frac{1}{2}\left(-\frac{1}{2}\right)\partial^\mu(\phi-\phi^*)\partial_\mu(\phi-\phi^*) - \frac{m^2}{2}\cdot\left(-\frac{1}{2}\right)(\phi-\phi^*)^2$$

$$= \frac{1}{4}(\cancel{\partial^\mu\phi\partial_\mu\phi} + \cancel{\partial^\mu\phi^*\partial_\mu\phi^*} + \partial^\mu\phi\,\partial_\mu\phi^* + \partial^\mu\phi^*\partial_\mu\phi)$$

$$- \frac{m^2}{4}(\phi^2 + \phi^{*2} + 2\phi\phi^*)$$

$$- \frac{1}{4}(\cancel{\partial^\mu\phi\partial_\mu\phi} + \cancel{\partial^\mu\phi^*\partial_\mu\phi^*} - \partial^\mu\phi\,\partial_\mu\phi^* - \partial^\mu\phi^*\partial_\mu\phi)$$

$$+ \frac{m^2}{4}(\phi^2 - 2\phi\phi^* + \phi^{*2})$$

$$= \frac{1}{2}(\partial^\mu\phi^*\partial_\mu\phi + \partial^\mu\phi\,\partial_\mu\phi^*) - m^2\phi\phi^*$$

$$= \partial_\mu\phi^*\partial_\nu\phi\, g^{\mu\nu} - m^2\phi\phi^*$$

ELE for ϕ:

$$\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi)} = \partial_\mu\phi^* g^{\mu\nu}\,\delta_{\alpha\nu} = \partial_\mu\phi^* g^{\mu\alpha} = \partial^\alpha\phi^*$$

$$\frac{\partial\mathcal{L}}{\partial\phi} = -m^2\phi^*$$

$$\Rightarrow \quad \partial_\alpha\partial^\alpha\phi^* + m^2\phi^* = 0$$

ELE for ϕ^*:

$$\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi^*)} = \delta_{\alpha\mu}\,\partial_\nu\phi\, g^{\mu\nu} = \partial^\alpha\phi$$

$$\Rightarrow \quad \partial_\alpha\partial^\alpha\phi + m^2\phi = 0$$

b) $U(1)$: $\phi \to e^{iq\alpha}\phi$

$\phi^* \to e^{-iq\alpha}\phi$

Since $\mathcal{L}$ only contains terms with structure $\phi\phi^*$

$$\phi\phi^* \to e^{iq\alpha}\phi\, e^{-iq\alpha}\phi^* = \phi\phi^*$$

$\Rightarrow \mathcal{L}$ invariant

$$j_\mu = \frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi)}\Delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi^*)}\Delta\phi^* - X_\mu \quad , \quad \partial_\mu X^\mu = 0$$

$$= \partial_\mu \phi^* (iq\alpha\phi) + \partial_\mu\phi(-iq\alpha\phi) - X_\mu$$

$$\begin{pmatrix} \Delta\phi = iq\alpha\phi \\ \Delta\phi^* = -iq\alpha\phi \end{pmatrix}$$

$$= iq\alpha\left(\partial_\mu\phi^* \cdot \phi - \partial_\mu\phi\cdot\phi^*\right) - X_\mu$$

c) $\phi \to e^{-i\alpha(x)}\phi$, $\quad \partial_\mu\phi \to \partial_\mu\left(e^{-i\alpha(x)}\phi\right)$

$\alpha(x) \in \mathbb{R}$

$$= -i\partial_\mu\alpha(x)\cdot e^{-i\alpha(x)}\phi + e^{-i\alpha(x)}\partial_\mu\phi$$

$$= e^{-i\alpha(x)}\left(-i\partial_\mu\alpha(x) + \partial_\mu\right)\phi$$

$$\Rightarrow \mathcal{L} \to \mathcal{L}' = \cancel{e^{-i\alpha(x)}}\left(-i\partial_\mu\alpha(x) + \partial_\mu\right)\phi \cdot \cancel{e^{i\alpha(x)}}\left(i\partial^\mu\alpha(x) + \partial^\mu\right)\phi^*$$

$$- m^2\phi\phi^*$$

$$= \partial_\mu\phi\,\partial^\mu\phi^* - i\partial_\mu\alpha(x)\cdot\phi\cdot\partial^\mu\phi^* + \partial_\mu\phi\cdot i\partial^\mu\alpha(x)$$

$$+ \partial_\mu\alpha\,\partial^\mu\alpha\cdot\phi\phi^* \quad - m^2\phi\phi^*$$

extra terms

$\to$ not invariant !

d) $D_\mu = \partial_\mu + ieA_\mu$

$$D_\mu\phi \longrightarrow D'_\mu\phi' = (\partial_\mu + ieA'_\mu)\, e^{-i\alpha(x)}\phi$$
$$= ieA'_\mu e^{-i\alpha(x)}\phi + e^{-i\alpha(x)}(-i\partial_\mu\alpha(x)\cdot + \partial_\mu)\,\phi$$
$$\stackrel{!}{=} e^{-i\alpha(x)}(\partial_\mu + ieA_\mu)\phi$$

$$\Rightarrow \quad ieA'_\mu e^{-i\alpha(x)}\phi - ie^{-i\alpha(x)}\partial_\mu\alpha(x)\cdot\phi = e^{-i\alpha(x)}\cdot ieA_\mu\phi$$
$$ieA'_\mu e^{-i\alpha(x)}\phi = e^{-i\alpha(x)}\phi\,(ieA_\mu + i\partial_\mu\alpha(x))$$
$$\Rightarrow \quad A'_\mu = \frac{1}{ie}(ieA_\mu + i\partial_\mu\alpha(x))$$
$$= A_\mu + \frac{1}{e}\partial_\mu\alpha(x)$$

e) $D_\mu\phi \to e^{-i\alpha(x)}D_\mu\phi$

$(D_\mu\phi)^* \to e^{i\alpha(x)}(D_\mu\phi)^*$

$$\mathcal{L} = D_\mu\phi(D^\mu\phi)^* + m^2\phi\phi^*$$
$$\longrightarrow \mathcal{L}' = \mathcal{L}$$

$$\mathcal{L} = (\partial_\mu + ieA_\mu)\phi\,(\partial^\mu - ieA^\mu)\phi^* - m^2\phi\phi^*$$
$$= \partial_\mu\phi\partial^\mu\phi^* + e^2A_\mu A^\mu\phi\phi^* - ieA^\mu\phi^*\partial_\mu\phi + ieA_\mu\phi\partial^\mu\phi^* - m^2\phi\phi^*$$

$$j_\mu = \frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi)}\Delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi^*)}\Delta\phi^* - X_\mu$$

$$= (\partial_\mu\phi^* - ieA_\mu\phi^*)(-i\alpha(x)\phi) + (\partial_\mu\phi + ieA_\mu\phi)(i\alpha(x)\phi^*) - X_\mu$$

$$= i\alpha(x)\left(-\phi\partial_\mu\phi^* + \underline{ieA_\mu\phi^*\phi} + \phi^*\partial_\mu\phi + \underline{ieA_\mu\phi\phi^*}\right) - X_\mu$$

$$= i\alpha(x)\left(\phi^*\partial_\mu\phi - \phi\partial_\mu\phi^* + 2ieA_\mu\phi^*\phi\right)$$

one can set $\alpha(x) = 1$, since it doesn't affect physics.

f) $F_{\mu\nu} F^{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)$

$$F_{\mu\nu} \to F'_{\mu\nu} = \partial_\mu (A_\nu + \tfrac{1}{e}\partial_\nu \alpha) - \partial_\nu (A_\mu + \tfrac{1}{e}\partial_\mu \alpha)$$
$$= F_{\mu\nu} + \tfrac{1}{e}\partial_\mu \partial_\nu \alpha - \tfrac{1}{e}\partial_\mu \partial_\nu \alpha$$
$$= F_{\mu\nu}$$

$\to F_{\mu\nu} F^{\mu\nu}$ must be invariant

$[F] = [\partial] + [A] = 1 + 1 = 2$

$\Rightarrow [F^{\mu\nu} F_{\mu\nu}] = 4$

Mass term for A_μ:

$$m^2 A_\mu A^\mu \longrightarrow m^2 (A_\mu + \tfrac{1}{e}\partial_\mu \alpha)(A^\mu + \tfrac{1}{e}\partial^\mu \alpha)$$
$$= m^2 A_\mu A^\mu + m^2 \left(\tfrac{1}{e^2}\partial_\mu \alpha \partial^\mu \alpha + \tfrac{1}{e}\partial^\mu \alpha A_\mu + \tfrac{1}{e}\partial_\mu \alpha A^\mu\right)$$

break the symmetry

g) $\mathcal{L} = \partial_\mu \phi \partial^\mu \phi^* + e^2 A_\mu A^\mu \phi \phi^* - ieA^\mu \phi^* \partial_\mu \phi + ieA_\mu \phi \partial^\mu \phi^* - m^2 \phi \phi^*$
$- \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$

ELE w.r.t. ϕ

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} = 0$$

$$\partial_\mu (\partial^\mu \phi^* - ieA^\mu \phi^*) - (e^2 A_\mu A^\mu \phi^* + ieA_\mu \partial^\mu \phi^* - m^2 \phi^*) = 0$$

$$\Rightarrow \quad \partial^2 \phi^* - ie(\partial_\mu A^\mu \cdot \phi^* + \underline{\partial_\mu \phi^* \cdot A^\mu}) - e^2 A_\mu A^\mu \phi^* - \underline{ieA_\mu \partial^\mu \phi^*}$$
$$+ m^2 \phi^* = 0$$

$$\Rightarrow \quad (\partial^2 + m^2)\phi^* - 2ie\partial_\mu \phi^* \cdot A^\mu - ie\partial_\mu A^\mu \cdot \phi^* - e^2 A_\mu A^\mu \phi^* = 0$$

w.r.t. ϕ^*: $\quad (\partial^2 + m^2)\phi + 2ie\partial_\mu \phi \cdot A^\mu + ie\partial_\mu A^\mu \cdot \phi - e^2 A_\mu A^\mu \phi = 0$

w.r.t. A

$$\partial_\alpha \frac{\partial \mathcal{L}}{\partial(\partial^\alpha A_\beta)} - \frac{\partial \mathcal{L}}{\partial A_\beta} = 0$$

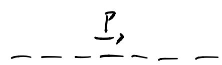
$$-\frac{1}{4}\partial_\alpha \underbrace{\frac{\partial}{\partial(\partial_\alpha A_\beta)}\left((\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)\right)} - \left(e^2 A^\beta \phi\phi^* - ie\phi^*\partial^\beta\phi + ie\phi\partial^\beta\phi^*\right) = 0$$

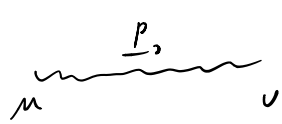
$$= (\delta_{\alpha\mu}\delta_{\beta\nu} - \delta_{\alpha\nu}\delta_{\beta\mu})F^{\mu\nu} + F_{\mu\nu}(\delta^{\alpha\mu}\delta^{\beta\nu} - \delta^{\alpha\nu}\delta^{\beta\mu})$$
$$= F^{\alpha\beta} - F^{\beta\alpha} + F^{\alpha\beta} - F^{\beta\alpha}$$
$$= 4F^{\alpha\beta}$$

$$-\cancel{\tfrac{1}{4}}\,\partial_\alpha \cancel{4} F^{\alpha\beta} - (2e^2 A^\beta\phi\phi^* - ie\phi^*\partial^\beta\phi + ie\phi\partial^\beta\phi^*) = 0$$
$$\Rightarrow -\partial_\alpha F^{\alpha\beta} + j^\beta = 0$$
$$\Rightarrow \partial_\alpha F^{\alpha\beta} = J^\beta$$

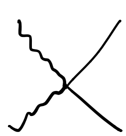
h) $\mathcal{L} = \partial_\mu\phi\partial^\mu\phi^* + e^2 A_\mu A^\mu \phi\phi^* - ieA^\mu\phi^*\partial_\mu\phi + ieA_\mu\phi\partial^\mu\phi^* - m^2\phi\phi^*$
$\quad - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$

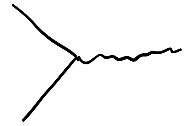
Propagators

 $\sim \dfrac{i}{p^2 - m^2 + i\varepsilon}$

 $\sim -i\dfrac{g^{\mu\nu}}{p^2}$

Vertices

 $\sim -ie^2$

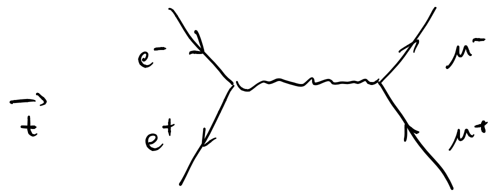
 $\sim -ie$

H4.2

$$\mathcal{L}_{QED} = -\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} + \sum_{f=e,\mu} \overline{\Psi}_f \left(i D_\alpha \gamma^\alpha - m_f \right) \Psi_f$$

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha \quad , \quad D_\alpha = \partial_\alpha + ie\gamma_\alpha$$

a) $e^- (p_1, s_1)\, e^+ (p_2, s_2) \longrightarrow \mu^- (p_3, s_3)\, \mu^+ (p_4, s_4)$



b) $$i\mathcal{M} = \bar{v}(p_2)^{s_2} (-ie\gamma^\mu)\, u(p_1)^{s_1} \frac{-i g_{\mu\nu}}{(p_1+p_2)^2 + i\varepsilon} \bar{u}(p_3)^{s_3} (-ie\gamma^\nu)\, v(p_4)^{s_4}$$

$$= \frac{ie^2}{s} \bar{v}^{s_2}(p_2)\gamma^\mu u^{s_1}(p_1)\, \bar{u}^{s_3}(p_3)\gamma_\mu v^{s_4}(p_4)$$

c) $$\left[\bar{v}^{s_i}(p_i)\gamma^\mu u^{s_j}(p_j)\right]^* = \left[\bar{v}\gamma^\mu u\right]^\dagger$$

$$= u^\dagger (\gamma^\mu)^\dagger \bar{v}^\dagger$$

$$= \bar{u}\gamma^0 (\gamma^\mu)^\dagger \gamma^0 v$$

$$= \bar{u}\gamma^\mu v$$

d) $$\frac{1}{4}\sum_s |\mathcal{M}|^2$$

$$= \frac{1}{4}\sum_s \mathcal{M}\mathcal{M}^*$$

$$= \frac{1}{4}\frac{e^4}{s^2}\sum_s \bar{v}^{s_2}(p_2)\gamma^\mu u^{s_1}(p_1)\bar{u}^{s_3}(p_3)\gamma_\mu v^{s_4}(p_4)$$
$$\times \bar{v}^{s_4}(p_4)\gamma_\nu u^{s_3}(p_3)\bar{u}^{s_1}(p_1)\gamma^\nu v^{s_2}(p_2)$$

$$= \frac{e^4}{4s^2} \mathrm{tr}\left[\sum_s \bar{v}^{s_2}(p_2)\gamma^\mu u^{s_1}(p_1)\bar{u}^{s_3}(p_3)\gamma_\mu v^{s_4}(p_4)\bar{v}^{s_4}(p_4)\gamma_\nu u^{s_3}(p_3)\bar{u}^{s_1}(p_1)\gamma^\nu v^{s_2}(p_2)\right]$$

$$= \frac{e^4}{4s^2} \mathrm{tr}\left[(\not{p}_2 - m_e^2)\gamma^\mu (\not{p}_1 + m_e^2)\gamma_\mu (\not{p}_4 - m_\mu^2)\gamma_\nu (\not{p}_3 + m_\mu^2)\gamma^\nu \right]$$

$$\approx \frac{e^4}{4s^2} \mathrm{tr}\left[\not{p}_2 \gamma^\mu \not{p}_1 \gamma_\mu (\not{p}_4 - m_\mu)\gamma^\nu (\not{p}_3 + m_\mu)\gamma_\nu \right]$$

e) $= \frac{e^4}{4s^2}\, tr[\, \underbrace{\not{p}_2 \gamma^\mu \not{p}_1 \gamma_\mu}_{= -2\not{p}_2 \not{p}_1} \big(\underbrace{\not{p}_4 \gamma^\nu \not{p}_3 \gamma_\nu} - m_\mu^2 \underbrace{\gamma^\nu \gamma_\nu}_{4\mathbb{1}} \big)] \leftarrow tr[\text{odd \# of } \gamma^\mu] = 0$

$$= p_4^\alpha p_3^\beta \gamma^\alpha \underbrace{\gamma^\nu \gamma_\beta \gamma_\nu}$$
$$= p_4^\alpha p_3^\beta \gamma^\alpha (-2\gamma_\beta)$$
$$= -2 \not{p}_4 \not{p}_3$$

$$= \frac{e^4}{4s^2}\, tr[-2\not{p}_2 \not{p}_1 (-2\not{p}_4 \not{p}_3 - 4m_\mu^2)]$$

$$= \frac{e^4}{4s^2} \Big\{ 16\big[(p_2\cdot p_1)(p_4\cdot p_3) + (p_2\cdot p_4)(p_1\cdot p_3) + (p_2\cdot p_3)(p_4\cdot p_1)\big] + 32 m_\mu^2 (p_2\cdot p_1) \Big\}$$

$$= \frac{4e^4}{s^2} \Big\{ (p_2\cdot p_1)(p_4\cdot p_3) + (p_2\cdot p_4)(p_1\cdot p_3) + (p_2\cdot p_3)(p_4\cdot p_1) + 2m_\mu^2 (p_2\cdot p_1) \Big\}$$

f)

$$(p_1\cdot p_2) = \begin{pmatrix} E \\ E\vec{e}_z \end{pmatrix} \cdot \begin{pmatrix} E \\ -E\vec{e}_z \end{pmatrix} = E^2 - (E\vec{e}_z)(-E\vec{e}_z) = 2E^2$$

$$(p_3\cdot p_4) = E^2 + |\vec{p}_\mu|^2 = E^2 + E^2 - m_\mu^2 = 2E^2 - m_\mu^2$$

$$(p_1\cdot p_3) = \begin{pmatrix} E \\ E\vec{e}_z \end{pmatrix} \cdot \begin{pmatrix} E \\ \vec{p}_\mu \end{pmatrix} = E^2 - \begin{pmatrix} 0 \\ 0 \\ E \end{pmatrix} \begin{pmatrix} 0 \\ \sin\theta\, p_\mu \\ \cos\theta\, p_\mu \end{pmatrix} = E^2 - E p_\mu \cos\theta$$

$$(p_2\cdot p_4) = \begin{pmatrix} E \\ -E\vec{e}_z \end{pmatrix} \cdot \begin{pmatrix} E \\ -\vec{p}_\mu \end{pmatrix} = E^2 - E p_\mu \cos\theta$$

$$(p_1\cdot p_4) = \begin{pmatrix} E \\ E\vec{e}_z \end{pmatrix} \begin{pmatrix} E \\ -\vec{p}_\mu \end{pmatrix} = E^2 + \begin{pmatrix} 0 \\ 0 \\ E \end{pmatrix} \begin{pmatrix} 0 \\ \sin\theta\, p_\mu \\ \cos\theta\, p_\mu \end{pmatrix} = E^2 + E p_\mu \cos\theta$$

$$s = (p_1 + p_2)^2 = (2E)^2 = 4E^2$$

$$\Rightarrow \overline{|M|^2} = \frac{e^4}{4E^4} \Big\{ (2E^2)(2E^2 - m_\mu^2) + (E^2 - E p_\mu \cos\theta)^2 + (E^2 + E p_\mu \cos\theta)^2 + 2m_\mu^2 \cdot 2E^2 \Big\}$$

$$= \frac{e^4}{4E^{\cancel{4}\,2}} \cancel{E^2} \Big\{ 4E^2 - 2m_\mu^2 + (E - p_\mu \cos\theta)^2 + (E + p_\mu \cos\theta)^2 + 4m_\mu^2 \Big\}$$

$$= \frac{e^4}{4E^2}\left\{4E^2 + 2m_\mu^2 + (E^2 + P_\mu^2 \cos^2\theta)\cdot 2\right\}$$

$$= \frac{e^4}{4E^2}\left(6E^2 + 2m_\mu^2 + 2P_\mu^2 \cos^4\theta\right)$$

g) $$\frac{d\sigma}{dt} = \frac{1}{16\pi s^2}\,\frac{1}{4}\sum_{\text{spins}} |M|^2 = \frac{1}{16\pi}\,\frac{1}{16E^4}\,\frac{e^4}{4E^2}\left(6E^2 + 2m_\mu^2 + 2P_\mu^2 \cos^2\theta\right)$$

$$\frac{d\sigma}{dt} = \frac{d\sigma}{d\cos\theta}\,\underbrace{\frac{d\cos\theta}{dt}}$$

$$= \left[\frac{d}{d\cos\theta}\left(m_e^2 + m_\mu^2 - 2\,P_1\cdot P_3\right)\right]^{-1}$$

$$\cong \left[\frac{d}{d\cos\theta}\left(m_\mu^2 - 2(E^2 - E P_\mu \cos\theta)\right)\right]^{-1}$$

$$= \frac{1}{2EP_\mu}$$

$$\Rightarrow \sigma = \int d\cos\theta\,\frac{d\sigma}{d\cos\theta}$$

$$= \int_{-1}^{1} d\cos\theta \cdot 2EP_\mu \cdot \frac{e^4}{16\pi\cdot 16E^4\cdot 4E^2}\left(6E^2 + 2m_\mu^2 + 2P_\mu^2\cos^2\theta\right)$$

$$= \frac{2EP_\mu e^4}{16\pi\cdot 16E^4\cdot 4E^2}\cdot 2P_\mu^2 \cdot \underbrace{\left[\frac{1}{3}\cos^3\theta\right]\Bigg|_{\cos\theta=-1}^{\cos\theta=1}}_{=\frac{1}{3}(1+1) = \frac{2}{3}}$$

$$= \frac{4EP_\mu \overbrace{e^4}^{=\alpha^2\pi} P_\mu^2}{16\pi\cdot \underbrace{16E^4\cdot 4E^2}_{E_{com}^6}}\cdot\frac{2}{3}$$

$$= \frac{4\alpha^2\pi}{3E_{com}^6}\,E P_\mu^3 = \frac{4\pi\alpha^2}{3E_{com}^2}\,\underbrace{\frac{EP_\mu^3}{E_{com}^4}}_{=\frac{E}{16E^4}\left(\sqrt{E^2 - m_\mu^2}\right)^3}$$

$$= \frac{E}{16 E^4} \left(1 - \frac{m_\mu^2}{E^2}\right)^{3/2} \cdot E^3$$

$$= \frac{1}{16} \left(1 - \frac{m_\mu^2}{E^2}\right)^{3/2}$$