Introduction to integrability

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0 Preface

0.1 What is integrability?

There is no universal definition and depends on context and model. However, the harmonic oscillator, which can be approximation of some complete model, is integrable.

0.2 FPUT paradox

In year 1954, Los Almos, USA, there was a brand new computer MANIAC I. Fermi had published an article in 1923 "Beweis dass mechanisches Normalsystem im Allgemein quasiergodisch ist". Here ergodic means equipartion of energy among different degrees of freedom. With this new machine, he did numerical experiment on how long it takes until equipartition is reached. A simple model would be a discrete vibrating string. The Hamiltonian is

$$H(p,q) = \frac{1}{2}\sum_{i=1}^{L} p_i^2 + \frac{1}{2}\sum_{i=1}^{L}(q_i - q_{i-1})^2 - 2\alpha\sum_{i=1}^{L}(q_i - q_{i-1})^3$$

with the fixed boundary condition $q_0 = q_L = 0$. By defining the normal mode coordinates

$$Q_k = \sqrt{\frac{2}{L+1}}\sum_{i=1}^{L}\sin\left(\frac{ik\pi}{L+1}\right)q_i,$$
$$P_k = \sqrt{\frac{2}{L+1}}\sum_{i=1}^{L}\sin\left(\frac{ik\pi}{L+1}\right)p_i,$$

the Hamiltonian becomes

$$H(P,Q) = \frac{1}{2}\sum\left(P_k^2 + \omega_k^2 Q_k^2\right) + \alpha V_e(Q), \tag{0.1}$$

with

$$\omega_k = 2\sin\left(\frac{k\pi}{2(L+1)}\right).$$

The first term in the Hamiltonian is a good approximation for energy per site for small α and the last term is the non-linear term. One can define the average energy per site

$$\bar{E}_k = \frac{1}{t_0}\int_t^{t_0} E_k^0(t)\,\mathrm{d}t \tag{0.2}$$

The expectation (from ergodicity or equipartion) is $\lim_{t\to\infty}\bar{E}_k(t_0) = \epsilon$ for all k.

In the FPUT test, some initial energy is given to mode $k = 1$ and wait until equilibrium is reached. The results are shown in figure

System shows periodicity instead of equipartition of energy! Why? Are there some hidden symmetries? Still today, there is no satisfactory explanation for FPUT paradox. The FPUT system is close to a class

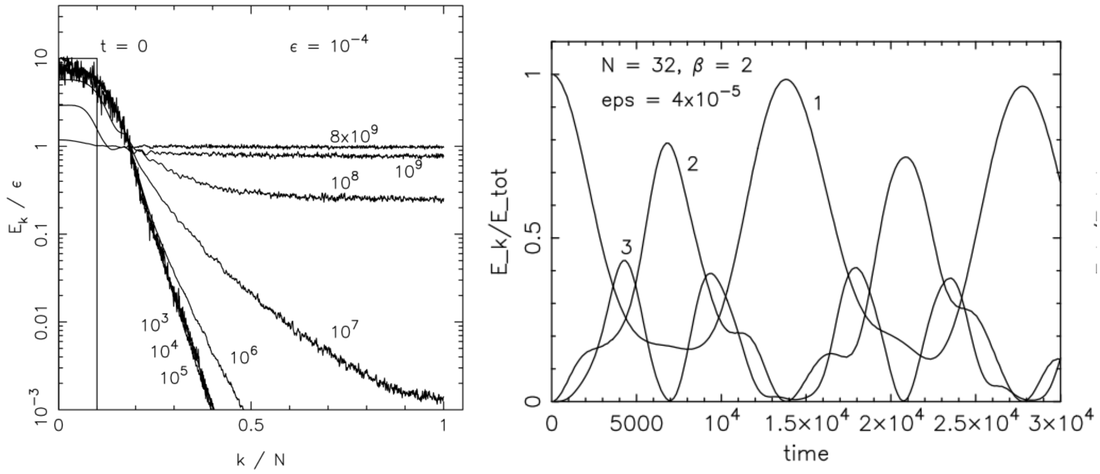

Figure 0.1: Averaged energy spectrum of normal modes $\bar{E}_k(T)$ plotted against k/N at selected times T and instantaneous values $E_k(t)$ for modes $k = 1, 2, 3$ [1].

of so-called *integrable models*, whose a number of (hidden) symmetries, roughly speaking, is the same as the degrees of freedom.

One obtains FPUT model by introducing perturbation to harmonic oscillator (which is integrable). By truncating FPUT model, one has Toda chain. Taking the continuum limit, FPUT model leads to Korteweg-de Vries (KdV) equation, which is integrable. This lecture is about these integrable models with "large" number of symmetries, about mathematical formulation and implications.

About literacy "integrable": There is possibility to integrate equation of motion to obtain a solution in a form as closed as possible (may require some extra steps).

1 Integrability in classical mechanics

1.1 Hamiltonian Formalism

Motion of a system with n degrees of freedom described by trajectory in 2 dimensional phase space $\mathcal{M}$ (manifold) with **local** coordinates (p_j, q_j), $j = 1, \ldots, n$.

Dynamical variables are some function $f : \mathcal{M} \times \mathbb{R} \to \mathbb{R}$, $f = f(p, q, t)$.

Poisson brackets

$$\{f, g\} := \sum_{i=1}^{k} \frac{\partial f}{\partial q_k}\frac{\partial g}{\partial p_k} - \frac{\partial f}{\partial p_k}\frac{\partial g}{\partial q_k} \tag{1.1}$$

with the properties

$$\{f, g\} = -\{f, g\}$$
$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$$

and the canonical "commutation" relation

$$\left\{p_j, p_k\right\} = \left\{q_j, q_k\right\} = 0, \quad \left\{p_j, q_k\right\} = \delta_{jk}$$

Given a Hamiltonian $H = H(p, q, t)$, the dynamics of a dynamical variable is determined by

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \{f, H\}$$

for any $f = f(p, q)$.

Setting $f = p_j$ or $f = q_j$ yields the Hamilton's equation of motion

$$\dot{p}_j = -\frac{\partial H}{\partial q_j}, \quad \dot{q}_j = \frac{\partial H}{\partial p_j} \tag{1.2}$$

The system (1.2) of $2n$ ordinary differential equations (ODEs) is deterministic, meaning the $(p_j(t), q_j(t))$ are uniquely determined by $2n$ initial conditions.

Definition 1. *A function $f = f(p_j, q_j, t)$ which $\dot{f} = 0$, when equation of motion* (1.2) *hold, is called a* ***first integral****, a* ***constant of motion****, or a* ***conserved charge****.*

Equivalently, $f(p(t), q(t), t) = \text{const}$, if $p(t)$ and $q(t)$ satisfy (1.2).

Hamilton's equations will be solvable, if there are "sufficiently" many constants of motion.

Example System with one degree of freedom with $\mathcal{M} = \mathbb{R}^2$ and Hamiltonian $H = \frac{1}{2}p^2 + V(q)$. The Hamilton's equations are

$$\dot{q} = p, \quad \dot{p} = -\frac{\partial V}{\partial q}.$$

The Hamiltonian H is a first integral ($\frac{\mathrm{d}}{\mathrm{d}H} = 0$). Thus,

$$\frac{1}{2}p^2 + V(q) = E = \text{const},$$
$$\dot{q} = p, p = \pm\sqrt{2(E - V(q))},$$
$$\Rightarrow t = \pm\int \frac{\mathrm{d}q}{\sqrt{2(E - V(q))}}$$

Explicit solution could be found if the integral can be performed and the relation $t = t(q)$ can be inverted to get $q(t)$. These two steps are not always possible, but still it is called **integrable**.

One can also look at the systems **geometrically**. First integrals defines $f(p, q) =$ const. in $\mathcal{M}$. Two hypersurfaces corresponding to two first integrals generically intersect in surface of dimension 2 in $\mathcal{M}$. In general, trajectory lies on a surface of dimension $(2n - L)$ with L the number of independent first integrals. If $L = 2n - 1$, this "surface" is a curve, i.e. a solution to Hamilton's equations.

The questions now is how to find first integrals? If two first integrals are given, their Poisson bracket is another first integral. Noether's theorem gives first integrals (translations, rotations and so on). Energy is always a first integral in Hamilton formalism.

1.2 Integrability and action-angle variables

Definition 2. *Consider a Hamiltonian system with $2n$ dimensional phase space $\mathcal{M}$. We call this system (completely) Louville integrable, if n functions $f_1, \ldots, f_n : \mathcal{M} \to \mathbb{R}$ exists such that*

- $\left\{f_j, f_k\right\} = 0, j, k = 1, \ldots, n$
- $\left\{H, f_j\right\} = 0, j = 1, \ldots, n$
- *The functions $f_1, \ldots, f_n$ are independent, i.e. the $\vec{\nabla} f_j$ are linearly independent vectors on a tangent space to any point in $\mathcal{M}$.*

If condition (1) is satisfied, the f_j are in **involution**. Integrability in the above sense leads to solvability of equation of motion.

Coordinate transformations What freedom is there in Hamiltonian structure?

Definition 3. *A transformation $Q_k = Q_k(p, q), P_k = P_k(p, q)$ is canonical, if it preserves the Poisson brackets*

$$\{f, g\}_{p,q} = \{f, g\}_{P,Q}, \forall f, g : \mathcal{M} \to \mathbb{R}.$$

Canonical transformation preserves Hamilton's equations. In $2n$ dimensional phase space, only $2n$ of the coordinates p, q, P, Q are independent. Given a generating function $S(q, P, t)$ with

$$\det\left(\frac{\partial^2 S}{\partial q_i \partial p_k}\right) = 0$$

we can construct a canonical transformation by setting

$$p_k = \frac{\partial S}{\partial q_k}, \quad Q_k = \frac{\partial S}{\partial P_k}, \quad H = H + \frac{\partial S}{\partial t} \tag{1.3}$$

There are other possibilities with

$$\begin{aligned} S(q,Q) &: p = \frac{\partial S}{\partial q}, P = -\frac{\partial S}{\partial Q}, \\ S(p,Q) &: P = -\frac{\partial S}{\partial Q}, q = -\frac{\partial S}{\partial Q}, \\ S(p,P) &: q = -\frac{\partial S}{\partial p}, Q = \frac{\partial S}{\partial P} \end{aligned}$$

Can we find canonical transformation that manifests integrability such that $P_k(t) = P_k(0) = \text{const}$ n constant of motion and $Q_k(t) = Q_k(0) + t\frac{\partial H}{\partial p_k}$ with linear time dependence. To find such a transformation is in general hard. Deciding whether a given H is integrable is still unsolved problem.

Theorem 1. *(Arnold and Liouville) Let $(\mathcal{M}, f_1, \ldots, f_n)$ be an integrable system with a Hamiltonian $H = f_1$ and let*

$$\mathcal{M}_f = \{(p,q) \in \mathcal{M} \,|\, f_k(p,q) = c_k = const,\ k = 1, \ldots, n\}$$

be a so-called n dimensional level set of first integrals f_n.

1. *if M_f is compact and connected, then it is diffeomorphism to torus $T^n = S^1 \times \cdots \times S^1$.*
2. *One introduces (in the neighborhood of this torus in M) the action angle variables*

   $$I_1, \ldots, I_n, \quad \phi_1, \ldots, \phi_n, \quad 0 \leq \phi_n \leq 2\pi$$

   such that the angles ϕ_k are coordinates on M_f and the action (variable) $I_k = I_k(f_1, \ldots, f_n)$ are first integrals.
3. *The canonical equations of motion* (1.2) *becomes*

   $$\dot{I}_k = 0, \quad \dot{\phi}_k = \omega_k(I_1, \ldots, I_n), \quad k = 1, \ldots, n \tag{1.4}$$

   *and the integrable system is solved by **quadratures** (finite number of algebraic equations and integrations of know functions).*

Proof (not to prove (1) here). On (2) and (3)

Motion takes place on surface of dimension $2n - n = n$

$$f_1(p,q) = c_1, \ \ldots, \ f_n(p,q) = c_n. \tag{1.5}$$

From (1), this surface is a torus. Assume $\det\left(\frac{\partial f_i}{\partial p_k}\right) \neq 0$ such that (1.5) can be solved for the momenta

$p_i = p_i(q, c)$ with $f_i(q, p(q, c)) = c_i$

$$\begin{aligned}
\frac{\partial}{\partial q_j} &\Rightarrow \frac{\partial f_i}{\partial q_j} + \sum_{k=0}^{n} \frac{\partial f_i}{\partial p_k}\frac{\partial p_k}{\partial q_j} = 0,\\
\sum_j \cdot \frac{\partial f_i}{\partial p_j} &\Rightarrow \sum_j \frac{\partial f_m}{\partial p_j}\frac{\partial f_i}{\partial q_j} + \sum_{j,k} \frac{\partial f_m}{\partial p_j}\frac{\partial f_i}{\partial p_k}\frac{\partial p_k}{\partial q_j} = 0,\\
(mi) - (im) &\Rightarrow \{f_i, f_m\} + \sum_{j,k}\left(\frac{\partial f_m}{\partial p_j}\frac{\partial f_i}{\partial p_k}\frac{\partial p_k}{\partial q_j} - \frac{\partial f_i}{\partial p_j}\frac{\partial f_m}{\partial p_k}\frac{\partial p_k}{\partial q_j}\right) = 0,\\
&\Rightarrow \sum_{j,k} \frac{\partial f_i}{\partial p_k}\frac{\partial f_m}{\partial p_j}\left(\frac{\partial p_k}{\partial q_j} - \frac{\partial p_j}{\partial q_k}\right) = 0,\\
\left(\frac{\partial f_i}{\partial p_k}\right) \text{ invertible} &\Rightarrow \frac{\partial p_k}{\partial q_j} - \frac{\partial p_j}{\partial q_k} = 0,\\
\text{Stockes' theorem} &\Rightarrow \oint_{\mathcal{G}} \sum_{j=1}^{n} p_j \,\mathrm{d}q_j = 0,
\end{aligned}$$

for any closed curve on torus T^n such are contractible to a point. On T^n there are n closed curves that cannot be contracted to a point, such that the corresponding integrals do not vanish.

Is compactness a condition for Stockes' theorem?

Definition 4. ***action variable***

$$I_k := \frac{1}{2\pi}\oint_{\Gamma_k} \sum_{i=1}^{n} p_j \,\mathrm{d}q_j\,, \quad k = 1, \dots, n \tag{1.6}$$

where the curve Γ_k *is the k-th basic cycle on the torus* T^n

$$\Gamma_k = \left\{(\tilde{\phi}_1, \dots \tilde{\phi}_n) \in T^n;\ 0 \le \tilde{\phi}_k \le 2\pi, \tilde{\phi}_j = const, for j \ne k\right\}.$$

$\tilde{\phi}_k$ denotes some coordinates on T^n. To find these coordinates is non-trivial, in practice it is not clear how to describe a torus explicitly. Arnold-Liouville theorem has character of existence theorem.

why is it non-trivial? In 2d case, they can be parametrised easily.

Stockes' theorem implies the action variables (1.6) are independent of choice of Γ_k. The action variable (1.6) are first integrals since $\oint p(q, c)\,\mathrm{d}q$ only depend on $c_k = f_k$ and f_k's are first integrals.

We have all the action variable in involution, since

$$\begin{aligned}
\left\{I_i, I_j\right\} &= \sum_{r,s,k}\left(\frac{\partial I_i}{\partial f_r}\frac{\partial f_r}{\partial q_k}\frac{\partial I_j}{\partial f_s}\frac{\partial f_s}{\partial p_k} - \frac{\partial I_i}{\partial f_r}\frac{\partial f_r}{\partial p_k}\frac{\partial I_j}{\partial f_s}\frac{\partial f_s}{\partial q_k}\right),\\
&= \sum_{r,s} \frac{\partial I_j}{\partial f_r}\frac{\partial I_j}{\partial f_s}\{f_r, f_s\},\\
&= 0.
\end{aligned}$$

In particular $\{I_k, H\} = 0$.

The torus $\mathcal{M}_f$ can be equivalently defined ny

$$I_1 = \tilde{c}_c, \dots, I_n = \tilde{c}_n$$

One may ask why is I_k (as coordinate) better than f_k. If one defines $I_k = f_k$, the transformation $(p, q) \to (I, \phi)$ would not be canonical.

Canonical angle coordinates ϕ_k, which are the canonically conjugates to the actions via the generating functions

$$S(q, I) = \int_{q_0}^{q} \sum_j p_j \, \mathrm{d}q_j \,, \tag{1.7}$$

with q_0 some point on the torus. Modifying q_0 just adds a constant to S. The angle coordinates are

$$\phi_i = \frac{\partial S}{\partial I_i}.$$

The angles are periodic. Consider two paths C and $C \cup C_k$ (with $C_k = \Gamma_k$ the k-th cycle) between q_0 and q, see figure 1.1. Then

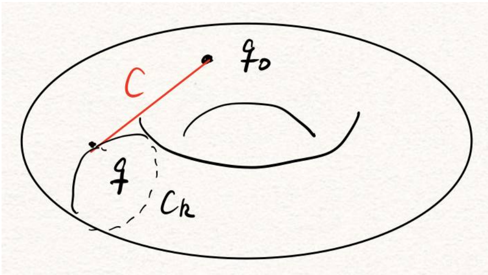

Figure 1.1: Torus with two paths C and $C \cup C_k$

$$\begin{aligned} S(q, I) &= \int_{C \cup C_k} \sum_j p_j \, \mathrm{d}q_j \\ &= \int_C \sum_j p_j \, \mathrm{d}q_j + \int_{C_k = \Gamma_k} \sum_j p_j \, \mathrm{d}q_j \\ &= S(q, I) + 2\pi I_k \\ \Rightarrow \phi_k &= \frac{\partial S}{\partial I_k} = \phi_k + 2\pi \end{aligned}$$

The transformation

$$q = q(\phi, I), \quad p = p(\phi, I)$$

and

$$\phi = \phi(p, q), \quad I = I(p, q)$$

are canonical transformations (defined by the generating function S) and invertible. The Poisson structures are unchanged

$$\left\{I_j, I_k\right\} = 0, \quad \left\{\phi_j, \phi_k\right\} = 0, \quad \left\{\phi_j, I_k\right\} = \delta_{jk}$$

The dynamics are given by

$$\dot{\phi}_k = \left\{\phi_k, \tilde{H}\right\}, \quad \dot{I}_k = \left\{I_k, \tilde{H}\right\}$$

with $\tilde{H} = \tilde{H}(\phi, I) = H(q(\phi, I), p(\phi, I))$. Since I_k's are first integrals,

$$0 = \dot{I}_k = \frac{\partial \tilde{H}}{\partial \phi_k},$$

in other word $\tilde{H} = \tilde{H}(I)$. The derivatives of angle variable

$$\dot{\phi}_k = \frac{\partial \tilde{H}}{\partial I_k} = \omega_k(I)$$

are first integrals as well.

Integration ("integrable" model) yields

$$\begin{aligned} \phi_k(t) &= \omega_k(I)t + \phi_k(0), \\ I_k(t) &= I_k(0). \end{aligned} \tag{1.8}$$

The system is in a circular motion with constant angular velocity.

Geometric picture The phase space of an integrable system is foliated into an n-parameter (c_j) family of invariant tori on which flow is linear with constant frequency ω_k. The trajectory (1.8) may be closed on the torus or it may cover it densely. For $n = 2$, the trajectory is closed if ω_1/ω_2 is rational and dense otherwise.

Degeneracy The periodicity in ϕ means that every function $F(p, q)$ of the state of system is periodic in ϕ. Expand the function in Fourier series, e.g. $n = 2$

$$\begin{aligned} F &= \sum_{l_1=-\infty}^{\infty} \sum_{l_2=-\infty}^{\infty} B_{l_1,l_2} \exp(i(l_1\phi_1 + l_2\phi_2)), \\ &= \sum_{l_1,l_2} B_{l_1,l_2} \exp(it(l_1\omega_1 + l_2\omega_2)). \end{aligned}$$

Every summand is period with frequency $l_1\omega_1 + l_2\omega_2$. Sum of functions is not necessarily periodic. The whole sum is only periodic for rational ω_1/ω_2. If $a_j\omega_j = a_k\omega_k$ for $a_{j,k} \in \mathbb{Z}$ for some j, k, one speaks of **degeneracy**. If $a_1\omega_1 = \cdots = a_n\omega_n$, the system is **maximally** degenerate.

Example All time-indepedent Hamiltonian systems with 2 dimensional phase-space are integrable ($H = f_{1=n}$).

Consider a harmonic oscillator ($n = 1$) with the Hamiltonian

$$H = \frac{1}{2}(p^2 + \omega^2 q^2)$$

Different choices of energy $c_1 = E$ give foliation of $\mathcal{M}$ by ellipses

$$\frac{1}{2}\left(p^2 + \omega^2 q^2\right) = E$$

with two axes $a = \sqrt{2E}$ and $b = \frac{\sqrt{2E}}{\omega}$ and surface $ab\pi$. For fixed E, take $\Gamma = \mathcal{M}_H$

$$I = \frac{1}{2\pi} \oint d\, \mathrm{d}q \overset{\text{Stockes'}}{=} \frac{1}{2\pi} \int_S \mathrm{d}p\, \mathrm{d}q = \frac{E}{\omega}$$

The Hamiltonian in the new variable $\tilde{H} = \omega I$ and $\dot{\phi} = \frac{\partial \tilde{H}}{\partial I} = \omega$, $\phi = \omega t + \phi_0$.

To obtain the transformation $(p, q) \to (I, \phi)$, first the action variable is

$$I(p, q) = \frac{1}{\omega} H(p, q) = \frac{1}{2}\left(\frac{1}{\omega} p^2 + \omega q^2\right).$$

The generating function is

$$S(q,I) = \int_{q_0}^{q} p\,\mathrm{d}\tilde{q} = \pm \int_{q_0}^{q} \sqrt{2I\omega - \omega^2 \tilde{q}}\,\mathrm{d}\tilde{q}$$

and the angle variable

$$\phi = \frac{\partial S}{\partial I} = \int \frac{\omega\ \mathrm{d}\tilde{q}}{\sqrt{2I\omega - \omega^2\tilde{q}^2}} = \arcsin\left(q\sqrt{\frac{\omega}{2I}}\right) - \phi_0.$$

Thus

$$\begin{aligned} q &= \sqrt{\frac{2E}{\omega}}\sin(\omega t + \phi_0) \\ p &= \frac{\partial H}{\partial p} = \dot{q} = \sqrt{2E}\cos(\omega t + \phi_0) \end{aligned}$$

Example The Kepler Problem ($n = 2$)

Consider the Motion in two-dimensional phase space (reduced from three-dimensional to two-dimensional using angular momentum conservation). Then we have four dimensional phase space $q_1 = \phi, q_2 = r, p_1 = p_\phi, p_2 = p_r$. The Hamiltonian is

$$H = \frac{p_\phi^2}{2r^2} + \frac{p_r^2}{2} - \frac{\alpha}{r}$$

with a positive constant α. We have $\left\{H, p_\phi\right\} = 0$, the system is (Liouville) integrable (2 constants of motion).

Level set $\mathcal{M}_f : H = E; p_\phi = \mu$. Then we can solve for p_r

$$p_r = \pm\sqrt{2E - \frac{\mu^2}{r^2} + \frac{2\alpha}{r}}.$$

ϕ is arbitrary, one constraint on (r, p_r). Parametrize $\mathcal{M}_f$ by ϕ and function of (r, p_r). Vary ϕ and fix other coordinate, consider one cycle $\Gamma_\phi \subset \mathcal{M}_f$

$$\begin{aligned} I_\phi &= \frac{1}{2\pi}\oint_{\Gamma_\phi}\left(p_r\,\mathrm{d}r + p_\phi\,\mathrm{d}\phi\right), \\ &= \frac{1}{2\pi}\int_0^{2\pi} p_\phi\,\mathrm{d}\phi = \mu, \end{aligned}$$

To find the second action, fix ϕ

$$\begin{aligned} I_r &= \frac{1}{2\pi}\oint_{\Gamma_r} p_r\,\mathrm{d}r, \\ &= 2\cdot\frac{1}{2\pi}\int_{r_-}^{r_+}\sqrt{2E - \frac{\mu^2}{r^2} + \frac{2\alpha}{r}}\,\mathrm{d}r, \end{aligned}$$

where we have taken the positive and negative roots and integrate $r_- \to r_+$ and backwards. Turning points $r_\pm$ are solutions of $2E - \frac{\mu^2}{r^2} + \frac{2\alpha}{r} = 0$ ($p_r \in \mathbb{R}$). Integral can be done using residual calculus

$$I_r = \alpha\sqrt{\frac{1}{2|E|}} - \mu = \alpha\sqrt{\frac{1}{2|E|}} - I_\phi.$$

Thus, the Hamiltonian written in terms of actions is

$$\tilde{H} = -\frac{\alpha^2}{2(I_r + I_\phi)^2},$$
$$\Rightarrow \frac{\partial \tilde{H}}{\partial I_r} = \frac{\partial \tilde{H}}{\partial I_\phi} = \frac{\alpha^2}{(I_r + I_\phi)^3}.$$

This is a particular case with $\omega_r = \omega_\phi$, and therefore closed orbits.

Superintegrability One may wonder why $\tilde{H} = \tilde{H}(I_r + I_\phi)$. Is there some special property in the system unexplored? In general, an integrable system admits n independent actions I_k, that can be uniquely expressed as functions of the system's state. We may write $(n-1)$ additional constants of motion as

$$A_{ik} := \phi_i \frac{\partial H}{\partial I_k} - \phi_k \frac{\partial H}{\partial I_i}$$

remember $\dot{\phi}_k = \frac{\partial H}{\partial I_k} = \omega_k$. Since $\phi_k = \phi_k + 2\pi$, the A_{ik}'s are not unique functions.

Suppose we have a degenerate system, e.g.

$$a_1 \frac{\partial H}{\partial I_2} = a_2 \frac{\partial H}{\partial I_2} \tag{1.9}$$

for $a_1, a_2 \in \mathbb{Z}$. Then

$$B_{12} := a_1\phi_1 - a_2\phi_2,$$

is a constant of motion with $B_{12} = B_{12} + 2\pi n, n \in \mathbb{Z}$. Any trigonometric function of B_{12} is unique constant of motion. Here (1.9) implies $H = H(a_2 I_1 + a_1 I_2)$. For the Kepler problem, the additional symmetry is the well-known **Laplace-Runge-Lenz vector**.

Definition 5. *A Hamiltonian system with $2n$-dimensional phase space and more than n independent constants of motion is called* ***superintegrable****. If the system has $2n-1$ independent constants of motion, it is* ***maximally superintegrable****.*

1.3 Poisson Structures

Consider phase space $\mathcal{M}$ of dimension m with local coordinates $(\xi^1, \dots, \xi^n)$, where we make no distinction between coordinates and momenta.

Definition 6. *A skew-symmetric matrix $\omega^{ab} = \omega^{ab}(\xi)$ is called a* ***Poisson structure****, if the Poisson bracket defined by*

$$\{f, g\} = \sum_{a,b=1}^{m} \omega^{ab}(\xi) \frac{\partial f}{\partial \xi^a} \frac{\partial g}{\partial \xi^b}$$

satisfies $\{f, g\} = -\{g, f\}$ and the Jacobi identity.

The Jacobi identity puts restrictions on $\omega^{ab}(\xi) = \left\{\xi^a, \xi^b\right\}$

$$\sum_{d=1}^{m} \left(\omega^{dc} \frac{\partial \omega^{ab}}{\partial \xi^d} + \omega^{db} \frac{\partial \omega^{ca}}{\partial \xi^d} + \omega^{da} \frac{\partial \omega^{bc}}{\partial \xi^d} \right) = 0$$

Given a Hamiltonian $H : \mathcal{M} \times \mathbb{R} \to \mathbb{R}$, the dynamics is given by

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \{f, H\}$$

and the Hamilton's equations generalizing (1.2)

$$\dot{\xi}^a = \sum_{b=1}^{m} \omega^{ab}(\xi) \frac{\partial H}{\partial \xi^b}$$

Example $\mathcal{M} = \mathbb{R}^3, \omega^{ab} = \sum_{c=1}^{3} \epsilon^{abc} \xi^c$, then

$$\left\{\xi^a, \xi^b\right\} = \epsilon^{abc} \xi^c = \sum \epsilon^{abc} \xi^c$$

This Poisson structure admits a Casimir, namaly any function $f(r)$, where

$$r = \sqrt{(\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2},$$

and Poisson-commutes with the coordinate function $\{f(r), \xi^a\} = 0$.

Symplectic structures Assume $m = 2n$ even and ω invertible with $W := \omega^{-1}$. Jacobi identity implies

$$\partial_a W_{bc} + \partial_c W_{ab} + \partial_b W_{ca} = 0, \quad \forall a, b, c = 1, \ldots, m$$

In this case we call W a **symplectic structure**.

The **Darboux theorem** states that there exists locally coordinate system with

$$\xi^1 = q_1, \ldots, \xi^n = q_n, \xi^{n+1} = p_1, \ldots, \xi^{2n} = p_n$$

such that

$$\omega = \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix}$$

and the Poisson bracket reduces to the standard form.

Example (Spinning Euler Top)

The coordinates are just the angular momentum $\xi^{1,2,3} = S^{x,y,z}$. The Hamiltonian is

$$H = \frac{1}{2}\left[\frac{(S^x)^2}{\Omega_x} + \frac{(S^y)^2}{\Omega_y} + \frac{(S^z)^2}{\Omega_z}\right]$$

for angular momentum vector $\vec{S}$ of a rigid spinning body fixed at center of mass. Ω_i is the diagonal entry of moment of inertia matrix Ω, such that $\vec{S} = \Omega\vec{\omega}$ with $\vec{\omega}$ the angular velocities.

Hamilton's equations (Euler equations) are

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{S} = -\left\{H, \vec{S}\right\} = \left(\Omega^{-1}\vec{S}\right) \times \vec{S}.$$

(Sometimes written as three decoupled differential equations.)

There are two conserved charges H and $|\vec{S}|$. Use $H = E$ and $|\vec{S}| = l$ to write the equation of motion as

$$\frac{\mathrm{d}}{\mathrm{d}t} S_x = \sqrt{A + BS_x^2 + CS_x^4}$$

with A, B, C functions of Ω, l, E. Extinguish three cases

solution	rational	trigonometric	elliptic
name	xxx	xxz	xyz
$\Omega_{x,y,z}$	$\Omega_x, \Omega_x, \Omega_x$	$\Omega_x, \Omega_x, \Omega_z$	$\Omega_x, \Omega_y, \Omega_z$
symmetry	$\mathbf{SO}(3)$	$\mathbf{SO}(2)$	–

Table 1.1:

1.4 Classical chains and Fields

We can align elementary mechanical models on a one dimensional lattice and it yields a chain model. Examples are FPUT, Toda chain, classical spin chain. Infinite chains have infinitely many degrees of freedom. The question is how many (conserved) charges do we need for integrability? Precise meaning of integrability is not clear, won't discuss classical field here.

Continuum limits Field theories are naturally understood as continuum limits of lattice models (chain). A well-behaved continuum limit of an integrable lattice model should be integrable.

General idea is to consider one dimensional classical chain model of variable ξ_j. Sites are labelled by j at position $x_j = x + j \cdot a$ with a constant lattice spacing $a = x_{j+1} - x_j$. Continuum limit $a \to 0$. Fix limiting continuous field $\phi(x)$ via

$$\xi_j = X_j(a, \phi(x_j))$$

with X_j some function specifies the limit prescription (There is no well-defined continuous limit!).

The Simplest example is $\xi_j = \phi(x_j)$. In the limit $a \to 0$ for instance

$$\phi'(x) = \lim_{a\to 0} \frac{\phi(x_{j+1}) - \phi(x_j)}{a},$$
$$\phi''(x) = \lim_{a\to 0} \frac{\phi(x_{j+1}) - 2\phi(x_j) + \phi(x_{j-1})}{a^2}, \ldots$$

Lattice sums turn into integrals

$$\lim_{a\to 0} a \sum_j (\ldots) = \int \mathrm{d}x \ldots.$$

Delta function becomes kronecker delta

$$\delta(x-y) = \lim_{a\to 0} \frac{1}{a}\delta_{jk}$$

May have expressions including different points $x_j = x + ja$ and $x_k = y + ka$, such that

$$x = \lim_{a\to 0}(x_0 + ja), y = \lim_{a\to 0}(x_0 + ka)$$

Definition of integrability for field theory is even worse than chain models.

Hamiltonian formalism for fields Formally, replace coordinates $\xi(t)$ by field variable $\phi(t, x)$, replace phase space $\mathcal{M}(= \mathbb{R}^m)$ by space of smooth functions on a line ($= \mathbb{R}$) with some boundary conditions (e.g. decay, open, periodic).

Functionals given by integrals

$$F[\phi] = \int_{\mathbb{R}} f(\phi, \phi_x, \phi_{xx}, \ldots)\, \mathrm{d}x$$

elementary mechanics	fields
$\xi^a(t), a = 1, \dots, m$	$\phi(t, x), x \in \mathbb{R}$
$\sum_a$	$\int_{\mathbb{R}} \mathrm{d}x$
function $f(\xi)$	functional $F[\phi]$
$\frac{\partial}{\partial \xi^a}$	$\frac{\delta}{\delta \phi}$
ODEs (t)	PDEs (t, x)

Table 1.2:

Recall
$$\frac{\delta F}{\delta \phi(x)} = \frac{\partial f}{\partial \phi} - \frac{\partial}{\partial x}\frac{\partial f}{\partial \phi_x} + \frac{\partial^2}{\partial x^2}\frac{\partial f}{\partial \phi_{xx}}$$
and
$$\frac{\delta \phi(y)}{\delta \phi(x)} = \delta(x-y)$$
with $\int_{\mathbb{R}} \delta(x)\,\mathrm{d}x = 1$.

Definition 7. *Poisson bracket in this case can be defined as*
$$\{F, G\} = \int_{\mathbb{R}} \omega(x, y, \phi)\frac{\delta F}{\delta \phi(x)}\frac{\delta G}{\delta \phi(y)}\,\mathrm{d}x\,\mathrm{d}y$$
with ω such that the Poisson bracket is anti-symmetric and obeys Jacobi-identity.

Canonical choice
$$\delta(x, y, \phi) = \frac{1}{2}\partial_x \delta(x-y) - \frac{1}{2}\partial_y \delta(x-y)$$
It is analogous to Darboux form, where ω is constant and anti-symmetric. Antisymmetry is analogous to $\frac{\partial}{\partial x}$ being anti-self-dual with respect to inner product $\langle \phi, \psi \rangle = \int_{\mathbb{R}} \phi(x)\psi(x)\,\mathrm{d}x$.

Hence, the canonical bracket is
$$\{F, G\} = \int_{\mathbb{R}} \frac{\delta F}{\delta \phi(x)}\frac{\partial}{\partial x}\frac{\delta G}{\delta \phi(x)}\,\mathrm{d}x$$
with Hamilton's equations
$$\frac{\partial \phi}{\partial t} = \{\phi, H[\phi]\}. \tag{1.10}$$

2 Inverse scattering method and solitons

Our previous definition of (Liouville) integrability works for ODEs. There is no universal definition of integrability for PDEs. One of the problems is that the phase space is infinitely dimensional but having infinitely many first integrals may not be enough (need to compare these infinities). Focus on properties of solutions and solution techniques.

2.1 The KdV equation

John Scott Russell (1808-1882) made experiments and find efficient design canal boats. His famous quote from Russell's "Report on Waves" (1844). Wave in shallow water described by Korteweg-de Vries (KdV) equations

$$\phi_t - 6\phi\phi_x + \phi_{xxx} = 0, \quad \phi = \phi(t,x) \tag{2.1}$$

which is written down and solved by simplest case in 1895 by KdV to explain Russell's observation.

Physical motivation for KdV equation Start with linear wave equation

$$\psi_{xx} - \frac{1}{v^2}\psi_{tt} = 0$$

with the velocity v. One can make three assumptions

1. invariance in $t \to -t$
2. small amplitude; omit terms of order ψ^2
3. constant group velocity; no dispersion

One can relax assumptions to arrive at KdV equation (2.1).
Consider general solutions of wave equation

$$\psi(t,x) = f(x - vt) + g(x + vt)$$

where f and g can be arbitrary functions. These functions are each characterised by first order PDE, for example

$$\psi_x + \frac{1}{v}\psi_t = 0$$

with $\psi = f(x - vt)$.

Introduce dispersion Consider complex wave $\psi = e^{i(kx-\omega(k)t)}$ with $\omega(k) = vk$. The group velocity $\frac{d\omega}{dk}$ equals phase velocity v.
Modify relation to introduce the dispersion

$$\omega(k) = v(k - \beta k^3 + \dots)$$

and higher order terms in k are negligible for small dispersion. Quadratic term leads to complex solution, hence it is undesirable.

The function $\psi = e^{i(kx - v(kt - \beta k^3)t)}$ satisfies the differential equation

$$\psi_x + \beta\psi_{xxx} + \frac{1}{v}\psi_t = 0.$$

It can be rewritten as a conservation law

$$\rho_t + j_x = 0,$$

if we identify the current density and the flux

$$\rho = \frac{1}{v}\psi, \quad j = \psi + \beta\psi_{xx}.$$

Introduce non-linearity Modify current with a non-linear term

$$j = \psi + \beta\psi_{xx} + \frac{\alpha}{2}\psi^2.$$

Then we have

$$\frac{1}{v}\psi_t + \psi_x + \beta\psi_{xxx} + \alpha\psi\psi_x = 0$$

The constants (v, β, α) can be eliminated by change of variables (e.g. linear combination of x and t) and rescaling and one obtains the KdV equation (2.1).

The simplest one-soliton solution found by KdV (1895) is *

$$\phi(t, x) = -\frac{2\chi^2}{\cosh^2[\chi(x - 4\chi^2 t - \phi_0)]} \tag{2.2}$$

where ϕ_0 location of extremum at $t = 0$ and $\chi \in \mathbb{R}$ a free parameter. Note the function $c \cdot \phi$ with $c = \text{const}$ is not a solution due to the non-linearity.

Numerical evidence for special properties of KdV Until 1965 equation (2.2) was the only regular solution ($\phi, \phi_x \overset{|x|\to\infty}{\to} 0$). Zabusky and Krusal (1965) observed numerically that two waves scatter without changing their shape. This is particle-like behaviour, thus the name "soliton", i.e. solitary-ons (like electrons and so on). The existence of stable solitary wave is a consequence of cancellation between dispersion and non-linearity.

Without dispersion $\phi_t - 6\phi\phi_x = 0$ has the solution with discontinuity of first derivative at some $t_0 > 0$. Without non-linearity $\phi_t + \phi_{xxx} = 0$, then the wave will disperse. Only with both terms, we would have stable solutions.

2.2 Inverse Scattering Method (ISM)

The ISM to solve classical soliton equations comes from quantum mechanics.

*Note that since the equation is non-linear, superposition principle doesn't apply, i.e. there is a fundamental difference between one-soliton solution and many-soliton solution.

Mathematical framework for QM Infinite-dimensional complex vector space $\mathcal{H}$ of functions. Wave functions $\psi \in \mathcal{H}, \psi : \mathbb{R} \to \mathbb{C}, \psi = \psi(x)$ (time independent). Inner product defined as

$$\langle \psi_1, \psi_2 \rangle = \int_{\mathbb{R}} \bar{\psi}_1(x)\psi_2(t)\,\mathrm{d}x \tag{2.3}$$

Bound states are functions with $\langle \psi, \psi \rangle < \infty$, e.g. e^{-x^2}. Scattering states not square integrable, e.g. e^{-ix}.

Given a (real) potential $\phi = \phi(x)$, the time-indepdent Schrödinger equation (SE) reads

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + \phi\psi = E\psi,$$

and it represents eigenvalue problem. Given $\phi(x)$ one can solve the SE.

Physical needs are typically the opposite: first measure scattering process/data, i.e. reflection and transmission coefficients and try to recover the potential from it. Now the problem is to recover potential from scattering data.

In 1950s, solved by Delfandm, Levitan, Marchenko (GLM) using algorithm. 1967 Gardener, Greene, Krusal, Miura used that algorithm to solve the Cauchy problem for KdV.

In scattering theory, determine reflection (R) and transimission (T) coefficients with continuous energies. Bound state has discrete energy levels (E).

GLM method knowledge of (E, T, R) allows to relate the scattering data to the potential. Cauchy problem for KdV with some initial condition $\phi(0, x) = \phi_0(x)$, in order to get $\phi(t, x)$. Instead, using Schrödinger equation, input scattering data at t_0, one get scattering data at $t > 0$. Using GLM integral equation, $\phi(t, x)$ can be computed. This is **inverse scattering method**.

The time evolution follows from KdV equation (no time-dependent Schrödinger equation).

Direct scattering One dimensional QM of particle in a potential $\phi(x)$. The Schrödinger equation is

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \phi(x)\right] f = k^2 f = Ef,$$

with the operator in the bracket the Schrödinger operator L and potential $\phi(x)$ such that $|\phi(x)| \to 0$ at $|x| \to 0$ and $\int_{\mathbb{R}}(1 + |x|)|\phi(x)|\,\mathrm{d}x < \infty$. This requirement implies that only finite number of energy levels exist (we don't prove it here).

For $x \to \pm\infty$, we have free particle

$$f_{xx} + k^2 f = 0$$

with the general solution

$$f = c_1 e^{ikx} + c_2 e^{-ikx} \tag{2.4}$$

For each $k \neq 0$, the set of solutions forms two-dimensional vector space $\mathcal{G}_k$. Since ϕ is real (for physical reasons), for f being solution, also $\bar{f}$ is also a solution.

Consider two solution bases $(\psi_1, \bar{\psi}_1)$ and $(\psi_2, \bar{\psi}_2)$ of $\mathcal{G}_k$ determined by asymptotics

$$\psi_1(x, k) \simeq e^{-ikx},\ \bar{\psi}_1(x, k) \simeq e^{ikx},\ \text{for } x \to \infty$$
$$\psi_2(x, k) \simeq e^{-ikx},\ \bar{\psi}_2(x, k) \simeq e^{ikx},\ \text{for } x \to \infty$$

Any solution can be expanded in first basis, so

$$\psi_2(x, k) = a(k)\psi_1(x, k) + b(k)\bar{\psi}_1(x, k)$$

Hence, if $a \neq 0$, consider particle coming from $+\infty$ with wave function e^{-ikx}.

$$\frac{\psi_2(x,k)}{a(k)} = \begin{cases} \frac{e^{-ikx}}{a(k)}, & \text{for } x \to -\infty \\ e^{-ikx} + \frac{b(k)}{a(k)} e^{ikx}, & \text{for } x \to +\infty \end{cases} \tag{2.5}$$

The transimission coeffcient

$$t(k) = \frac{1}{a(k)}$$

and the reflection coefficient

$$r(k) = \frac{b(k)}{a(k)}$$

with $|t(k)|^2 + |r(k)|^2 = 1$.

Properties of scattering data For $k \in \mathbb{C}$, one can prove that

- $a(k)$ is holomorphic in the upper half plane ($\mathrm{Im}(k) > 0$).
- $\{\mathrm{Im}(k) \geq 0, |k| \to \infty\} \Rightarrow |a(k)| \to 1$.
- The zeroes of $a(k)$ lie on the imaginary axis and number of zeroes is finite if $\int_{\mathbb{R}} (1+|x|)|\phi(x)| < \infty$. Then, $a(i\chi_1) = \cdots = a(i\chi_N) = 0$, where $\chi_n \in \mathbb{R}$ can be ordered such that $\chi_1 > \cdots > \chi_N$. We want bound states.
- The asymptotics of ψ_2 at the zeroes follows from (2.5)

  $$\psi_2(x, i\chi_n) = \begin{cases} e^{-i(i\chi_n)x}, & x \to -\infty \\ a(i\chi_n) e^{-i(i\chi_n)x} + b(i\chi_n) e^{i(i\chi_n)x}, & x \to +\infty \end{cases}$$

  and we have

  $$\left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \phi(x)\right] \psi_2(x, i\chi_n) = -\chi_n^2 \psi_2(x, i\chi_n)$$

  with $-\chi_n^2$ the energy,
- Set $b_n = b(i\chi_n)$, then $b_n \in \mathbb{R}$ and $b_n = (-1)^n |b_n|$ and $ia'(i\chi_n)$ has the same sign as b_n.

Inverse scattering Want to recover potential $\phi(x)$ from scattering data, which consists of transmission and reflection coefficients and energy levels

$$t(k), r(k), \{\chi_1, \ldots, \chi_n\}$$

with $E_n = -\chi_n^2$ and

$$\psi_2(x, i\chi_n) = \begin{cases} e^{\chi_n x}, & x \to -\infty \\ b_n e^{-\chi_n x}, & x \to +\infty \end{cases}$$

The inverse scattering method consists of the following steps

1. Set

$$F(x) = \sum_{n=1}^{N} \frac{b_n e^{-\chi_n x}}{ia'(i\chi_n)} + \frac{1}{2\pi} \int_{-\infty}^{\infty} r(k) e^{ikx} \, \mathrm{d}k \tag{2.6}$$

2. Solve the GLM integral equation for K:

$$K(x,y) + F(x+y) + \int_x^\infty K(x,z)F(z+y)\,\mathrm{d}z = 0 \tag{2.7}$$

3. Then the potential in the Schrödinger equation is

$$\phi(x) = -2\frac{\mathrm{d}}{\mathrm{d}x}K(x,x) \tag{2.8}$$

The time can be introduced as an additional parameter, if the time dependence of the scattering data is known. Then we would have $K = K(t,x,z)$ and $\phi = \phi(t,x)$.

2.3 Lax Formulation and Soliton solutions

In general, the potential $\phi(x)$ depends on t, which in general implies that energies in the Schrödinger equation are time-dependent. The ISM is an example of an isospectral problem where this does **not** happen.

Proposition 1. *If a differential operator M exists, such that*

$$\dot{L} = [L,M] \tag{2.9}$$

with $L = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \phi(t,x)$*, then the spectrum of L does not depend on t.*

Proof. Consider eigenvalue problem

$$\begin{gathered} Lf = Ef \\ L_t f + L f_t = E_t f + E f_t \end{gathered}$$

Use $MLf = EMf$ and equation (2.9) (for the first term) to find

$$(L-E)(f_t + Mf) = E_t f \tag{2.10}$$

Take inner product (2.3) of this equation with f and use that L is self-adjoint

$$E_t\|f\| = \langle f, (L-E)(f_t+Mf)\rangle = \langle (L-E)f, f_t + Mf\rangle = 0$$

Since we have the eigenvalue equation $(L-E)f = 0$, thus $E_t = 0$. □

Equation (2.10) also implies that if $f(t,x)$ is an eigenfunction of L with eigenvalue E, then so is $(f_t + Mf)$.

Lax formulation of KdV

$$L = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \phi(t,x), \quad M = 4\frac{\mathrm{d}^3}{\mathrm{d}x^3} - 3\left(\phi\frac{\mathrm{d}}{\mathrm{d}x} + \frac{\mathrm{d}}{\mathrm{d}x}\phi\right) \tag{2.11}$$

Such representation underlies the integrability of PDEs and ODEs.

Evolution of the Scattering Data Assume that $\phi(t,x)$ satisfies KdV equation. Let $Lf = k^2 f$ with asymptotics

$$\lim_{x\to\infty} f = \lim_{x\to\infty} \psi_2(x,k) = e^{-ikx}.$$

Remember from equation (2.10) , $(f_t + Mf)$ is also an eigenfunction of L with eigenvalue k^2 and we have

$$\lim_{|x|\to\infty,\phi\to 0} (\dot{\psi}_2 + M\psi_2) = 4\frac{\mathrm{d}^3}{\mathrm{d}x^3} e^{-ikx} = 4ik^3 e^{-ikx}.$$

Hence, $A = 4ik^3\psi_2$ and $B = (\dot{\psi}_2 + M\psi_2)$ are eigenfunctions of L with the same asymptotics. Furthermore, $(A - B)$ is a linear combination of ψ_1 and $\bar{\psi}_1$ which vanishes at $-\infty$. Since ψ_1 and $\bar{\psi}_1$ are linear-independent, $(A - B)$ must vanish everywhere.

Thus, the ODE

$$\dot{\psi}_2 + M\psi_2 = 4ik^3\psi_2(t)$$

holds for all $x \in \mathbb{R}$. We want to find now ODEs for $a(k)$ and $b(k)$. Recall that

$$\lim_{x\to+\infty} \psi_2(x,k) = a(k,t)e^{-ikx} + b(k,t)e^{ikx}.$$

Plug in the previous ODE,

$$\dot{a}e^{-ikx} + \dot{b}e^{ikx} = \left(-4\frac{\mathrm{d}^3}{\mathrm{d}x^3} + 4ik^3\right)\left(ae^{-ikx} + be^{ikx}\right) = 8ik^3 be^{ikx}$$

Equating the exponentials gives

$$\dot{a} = 0, \quad \dot{b} = 8ik^3 b$$

Thus $a(k,t) = a(k)$ and $b(k,t) = b(k,0)e^{8ik^3t}$. k doesn't depend on t and so zeroes $i\chi_n$ of a are constant.

The evolution of the scattering data is thus given by

$$\begin{aligned}
a(k,t) &= a(k,0) \\
b(k,t) &= b(k,0)e^{8ik^3t} \\
r(k,t) &= \frac{b(k,t)}{a(k,t)} = r(k,0)e^{8ik^3t} \\
\chi_n(t) &= \chi_n(0) \\
b_n(t) &= b_n(0)e^{8\chi_n^3 t} \\
a_n(t) &= 0 \\
\beta_n(t) &= \frac{b_n(t)}{ia'(i\chi_n)} = \beta_n(0)e^{8\chi_n^3 t}
\end{aligned} \tag{2.12}$$

2.4 Solitons

Assume $r(k,0) = 0$, then $r(k,t) = 0$ (reflectionless potential). ISM can be performed explicitly. One-soliton solution $N = 1$

$$\begin{aligned}
(2.6) &\Rightarrow F(t,x) = \beta(t)e^{-\chi x} \\
(2.5) &\Rightarrow K(x,y) + \beta e^{-\chi(x,y)} + \int_x^\infty K(x,z)\beta e^{-\chi(z+y)}\,\mathrm{d}z = 0
\end{aligned}$$

Look for solutions of the form $K(x,y) = K(x)e^{-\chi y}$

$$\begin{aligned}
K(x) + \beta e^{-\chi x} + K(x)\beta \int_x^\infty e^{-2\chi z}\,\mathrm{d}z = 0 \\
K(x) = -\frac{\beta e^{\chi x}}{1 + \frac{\beta}{2\chi}e^{-2\chi x}} \\
K(x,y) = -\frac{\beta e^{-\chi(x+y)}}{1 + \frac{\beta}{2\chi}e^{-2\chi x}}
\end{aligned}$$

with $\beta = \beta(t)$. Finally, (2.6) gives

$$\begin{aligned}
\phi(t,x) &= -2\frac{\partial}{\partial x}K(x,x) \\
&= -4\beta\frac{\chi e^{-2\chi x}}{(1 + \frac{\beta}{2\chi}e^{-2\chi x})^2} \\
&= -\frac{2\chi^2}{\cosh[\chi(x - 4\chi^2 t - \phi_0)]}
\end{aligned}$$

with $\phi_0 = \frac{1}{2\chi}\log\left(\frac{\beta_0}{2\chi}\right)$ and $\beta(t) = \beta_0 e^{8\chi^3 t}$. The energy of the corresponding solution to the Schrödinger equation ($\to -\chi^2$) determines the amplitude and the velocity of the soliton. The solution is of the form $\phi = \phi(x - 4\chi^2 t)$, i.e. it represents a wave moving to the right with velocity $4\chi^2$ and phase ϕ_0. $N = 2$ (or general N) is on the exercise sheet.

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