

Introduction to integrability

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0 Preface

0.1 What is integrability?

There is no universal definition and depends on context and model. However, the harmonic oscillator, which can be approximation of some complete model, is integrable.

0.2 FPUT paradox

In year 1954, Los Alamos, USA, there was a brand new computer MANIAC I. Fermi had published an article in 1923 "Beweis dass mechanisches Normalsystem im Allgemein quasiergodisch ist". Here ergodic means equipartition of energy among different degrees of freedom. With this new machine, he did numerical experiment on how long it takes until equipartition is reached. A simple model would be a discrete vibrating string. The Hamiltonian is

$$H(p, q) = \frac{1}{2} \sum_{i=1}^L p_i^2 + \frac{1}{2} \sum_{i=1}^L (q_i - q_{i-1})^2 - 2\alpha \sum_{i=1}^L (q_i - q_{i-1})^3$$

with the fixed boundary condition $q_0 = q_L = 0$. By defining the normal mode coordinates

$$Q_k = \sqrt{\frac{2}{L+1}} \sum_{i=1}^L \sin\left(\frac{ik\pi}{L+1}\right) q_i,$$
$$P_k = \sqrt{\frac{2}{L+1}} \sum_{i=1}^L \sin\left(\frac{ik\pi}{L+1}\right) p_i,$$

the Hamiltonian becomes

$$H(P, Q) = \frac{1}{2} \sum (P_k^2 + \omega_k^2 Q_k^2) + \alpha V_e(Q), \quad (0.2.1)$$

with

$$\omega_k = 2 \sin\left(\frac{k\pi}{2(L+1)}\right).$$

The first term in the Hamiltonian is a good approximation for energy per site for small α and the last term is the non-linear term. One can define the average energy per site

$$\bar{E}_k = \frac{1}{t_0} \int_{t_0}^{t_0+t_0} E_k^0(t) dt \quad (0.2.2)$$

The expectation (from ergodicity or equipartition) is $\lim_{t \rightarrow \infty} \bar{E}_k(t_0) = \epsilon$ for all k .

In the FPUT test, some initial energy is given to mode $k = 1$ and wait until equilibrium is reached. The results are shown in figure

System shows periodicity instead of equipartition of energy! Why? Are there some hidden symmetries? Still today, there is no satisfactory explanation for FPUT paradox. The FPUT system is close to a class

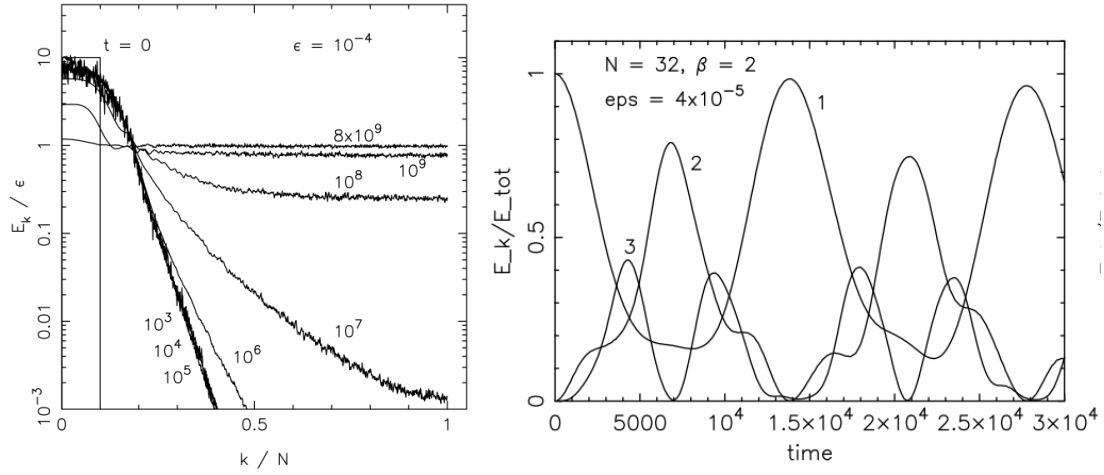


Figure 0.1: Averaged energy spectrum of normal modes $\bar{E}_k(T)$ plotted against k/N at selected times T and instantaneous values $E_k(t)$ for modes $k = 1, 2, 3$ [1].

of so-called *integrable models*, whose a number of (hidden) symmetries, roughly speaking, is the same as the degrees of freedom.

One obtains FPUT model by introducing perturbation to harmonic oscillator (which is integrable). By truncating FPUT model, one has Toda chain. Taking the continuum limit, FPUT model leads to Korteweg-de Vries (KdV) equation, which is integrable. This lecture is about these integrable models with “large” number of symmetries, about mathematical formulation and implications.

About literacy “integrable”: There is possibility to integrate equation of motion to obtain a solution in a form as closed as possible (may require some extra steps).

1 Integrability in classical mechanics

1.1 Hamiltonian Formalism

Motion of a system with n degrees of freedom described by trajectory in 2 dimensional phase space \mathcal{M} (manifold) with **local** coordinates (p_j, q_j) , $j = 1, \dots, n$.

Dynamical variables are some function $f : \mathcal{M} \times \mathbb{R} \rightarrow \mathbb{R}$, $f = f(p, q, t)$.

Poisson brackets

$$\{f, g\} := \sum_{i=1}^k \frac{\partial f}{\partial q_k} \frac{\partial g}{\partial p_k} - \frac{\partial f}{\partial p_k} \frac{\partial g}{\partial q_k} \quad (1.1.1)$$

with the properties

$$\begin{aligned} \{f, g\} &= -\{g, f\} \\ \{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} &= 0 \end{aligned}$$

and the canonical “commutation” relation

$$\{p_j, p_k\} = \{q_j, q_k\} = 0, \quad \{p_j, q_k\} = \delta_{jk}$$

Given a Hamiltonian $H = H(p, q, t)$, the dynamics of a dynamical variable is determined by

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \{f, H\}$$

for any $f = f(p, q)$.

Setting $f = p_j$ or $f = q_j$ yields the Hamilton’s equation of motion

$$\dot{p}_j = -\frac{\partial H}{\partial q_j}, \quad \dot{q}_j = \frac{\partial H}{\partial p_j} \quad (1.1.2)$$

The system (1.1.2) of $2n$ ordinary differential equations (ODEs) is deterministic, meaning the $(p_j(t), q_j(t))$ are uniquely determined by $2n$ initial conditions.

Definition 1 A function $f = f(p_j, q_j, t)$ which $\dot{f} = 0$, when equation of motion (1.1.2) hold, is called a **first integral**, a **constant of motion**, or a **conserved charge**.

Equivalently, $f(p(t), q(t), t) = \text{const}$, if $p(t)$ and $q(t)$ satisfy (1.1.2).

Hamilton’s equations will be solvable, if there are “sufficiently” many constants of motion.

Example 1 System with one degree of freedom with $\mathcal{M} = \mathbb{R}^2$ and Hamiltonian $H = \frac{1}{2}p^2 + V(q)$. The Hamilton’s equations are

$$\dot{q} = p, \quad \dot{p} = -\frac{\partial V}{\partial q}.$$

The Hamiltonian H is a first integral ($\frac{d}{dt}H = 0$). Thus,

$$\begin{aligned}\frac{1}{2}p^2 + V(q) &= E = \text{const}, \\ \dot{q} &= p, p = \pm \sqrt{2(E - V(q))}, \\ \Rightarrow t &= \pm \int \frac{dq}{\sqrt{2(E - V(q))}}\end{aligned}$$

Explicit solution could be found if the integral can be performed and the relation $t = t(q)$ can be inverted to get $q(t)$. These two steps are not always possible, but still it is called **integrable**.

One can also look at the systems **geometrically**. First integrals defines $f(p, q) = \text{const.}$ in \mathcal{M} . Two hypersurfaces corresponding to two first integrals generically intersect in surface of dimension 2 in \mathcal{M} . In general, trajectory lies on a surface of dimension $(2n - L)$ with L the number of independent first integrals. If $L = 2n - 1$, this “surface” is a curve, i.e. a solution to Hamilton’s equations.

The questions now is how to find first integrals? If two first integrals are given, their Poisson bracket is another first integral. Noether’s theorem gives first integrals (translations, rotations and so on). Energy is always a first integral in Hamilton formalism.

1.2 Integrability and action-angle variables

Definition 2 Consider a Hamiltonian system with $2n$ dimensional phase space \mathcal{M} . We call this system (completely) *Louville integrable*, if n functions $f_1, \dots, f_n : \mathcal{M} \rightarrow \mathbb{R}$ exists such that

- $\{f_j, f_k\} = 0, j, k = 1, \dots, n$
- $\{H, f_j\} = 0, j = 1, \dots, n$
- The functions f_1, \dots, f_n are independent, i.e. the $\vec{\nabla} f_j$ are linearly independent vectors on a tangent space to any point in \mathcal{M} .

If condition (1) is satisfied, the f_j are in **involution**. Integrability in the above sense leads to solvability of equation of motion.

Coordinate transformations What freedom is there in Hamiltonian structure?

Definition 3 A transformation $Q_k = Q_k(p, q), P_k = P_k(p, q)$ is *canonical*, if it preserves the Poisson brackets

$$\{f, g\}_{p,q} = \{f, g\}_{P,Q}, \forall f, g : \mathcal{M} \rightarrow \mathbb{R}.$$

Canonical transformation preserves Hamilton’s equations. In $2n$ dimensional phase space, only $2n$ of the coordinates p, q, P, Q are independent. Given a generating function $S(q, P, t)$ with

$$\det \left(\frac{\partial^2 S}{\partial q_i \partial p_k} \right) \neq 0$$

we can construct a canonical transformation by setting

$$p_k = \frac{\partial S}{\partial q_k}, \quad Q_k = \frac{\partial S}{\partial P_k}, \quad H = H + \frac{\partial S}{\partial t} \quad (1.2.1)$$

There are other possibilities with

$$\begin{aligned} S(q, Q) : p &= \frac{\partial S}{\partial q}, P = -\frac{\partial S}{\partial Q}, \\ S(p, Q) : P &= -\frac{\partial S}{\partial Q}, q = -\frac{\partial S}{\partial p}, \\ S(p, P) : q &= -\frac{\partial S}{\partial P}, Q = \frac{\partial S}{\partial p} \end{aligned}$$

Can we find canonical transformation that manifests integrability such that $P_k(t) = P_k(0) = \text{const}$ n constant of motion and $Q_k(t) = Q_k(0) + t \frac{\partial H}{\partial p_k}$ with linear time dependence. To find such a transformation is in general hard. Deciding whether a given H is integrable is still unsolved problem.

Theorem 1 (Arnold and Liouville) Let (M, f_1, \dots, f_n) be integrable system with a Hamiltonian $H = f_1$ and let

$$M_f = \{(p, q) \in M \mid f_k(p, q) = c_k = \text{const}, k = 1, \dots, n\}$$

be a so-called n dimensional level set of first integral f_n .

- if M_f is compact and connected, then it is diffeomorphism to torus $T^n = S^1 \times \dots \times S^1$.
- One introduces (neighborhood of this torus in M) the action angle variables

$$I_1, \dots, I_n, \quad \phi_1, \dots, \phi_n, \quad 0 \leq \phi_n \leq 2\pi$$

such that the angles ϕ_k are coordinates on M_f and the action (variable) $I_k = I_k(f_1, \dots, f_n)$ are first integrals.

- The canonical equations of motion (1.1.2) becomes

$$\dot{I}_k = 0, \quad \dot{\phi}_k = \omega_k(I_1, \dots, I_n), \quad k = 1, \dots, n \quad (1.2.2)$$

and the integrable system is solved by **quadratures** (finite number of algebraic equations and integrations of known functions).

Proof (not to prove (1) here). On (2) and (3)

Motion takes place on surface of dimension $2n - n = n$

$$f_1(p, q) = c_1, \dots, f_n(p, q) = c_n. \quad (1.2.3)$$

From (1), this surface is a torus. Assume $\det\left(\frac{\partial f_i}{\partial p_k}\right) \neq 0$ such that (1.2.3) can be solved for the momenta

$p_i = p_i(q, c)$ with $f_i(q, p(q, c)) = c_i$

$$\begin{aligned} \frac{\partial}{\partial q_j} &\Rightarrow \frac{\partial f_i}{\partial q_j} + \sum_{k=0}^n \frac{\partial f_i}{\partial p_k} \frac{\partial p_k}{\partial q_j} = 0, \\ \sum_j \frac{\partial f_i}{\partial p_j} &\Rightarrow \sum_j \frac{\partial f_m}{\partial p_j} \frac{\partial f_i}{\partial q_j} + \sum_{j,k} \frac{\partial f_m}{\partial p_j} \frac{\partial f_i}{\partial p_k} \frac{\partial p_k}{\partial q_j} = 0, \\ (mi) - (im) &\Rightarrow \{f_i, f_m\} + \sum_{j,k} \left(\frac{\partial f_m}{\partial p_j} \frac{\partial f_i}{\partial p_k} \frac{\partial p_k}{\partial q_j} - \frac{\partial f_i}{\partial p_j} \frac{\partial f_m}{\partial p_k} \frac{\partial p_k}{\partial q_j} \right) = 0, \\ &\Rightarrow \sum_{j,k} \frac{\partial f_i}{\partial p_k} \frac{\partial f_m}{\partial p_j} \left(\frac{\partial p_k}{\partial q_j} - \frac{\partial p_j}{\partial q_k} \right) = 0, \\ \left(\frac{\partial f_i}{\partial p_k} \right) \text{ invertible} &\Rightarrow \frac{\partial p_k}{\partial q_j} - \frac{\partial p_j}{\partial q_k} = 0, \\ \text{Stockes' theorem} &\Rightarrow \oint_{\mathcal{G}} \sum_{j=1}^n p_j dq_j = 0, \end{aligned}$$

for any closed curve on torus T^n such are contractible to a point.

On T^n there are n closed curves that cannot be contracted to a point, such that the corresponding integrals do not vanish.

Definition 4 action variable

$$I_k := \frac{1}{2\pi} \oint_{\Gamma_k} \sum_{i=1}^n p_i dq_i, \quad k = 1, \dots, n \quad (1.2.4)$$

where the curve Γ_k is the k -th basic cycle on the torus T^n

$$\Gamma_k = \{(\tilde{\phi}_1, \dots, \tilde{\phi}_n) \in T^n; 0 \leq \tilde{\phi}_k \leq 2\pi, \tilde{\phi}_j = \text{const, for } j \neq k\}.$$

$\tilde{\phi}_k$ denotes some coordinates on T^n . This is non-trivial, in practice it is not clear how to describe a torus explicitly. Arnold-Liouville theorem has character of existence theorem.

Stockes theorem implies the action (1.2.4) are independent of choice of Γ_k . The action (1.2.4) are first integrals since $\oint p(q, c) dq$ only depend on $c_k = f_k$ and f_k 's are first integrals.

We have all the action variable in involution, since

$$\begin{aligned} \{I_i, I_j\} &= \sum_{r,s,k} \left(\frac{\partial I_i}{\partial f_r} \frac{\partial f_r}{\partial q_k} \frac{\partial I_j}{\partial f_s} \frac{\partial f_s}{\partial p_k} - \frac{\partial I_i}{\partial f_r} \frac{\partial f_r}{\partial p_k} \frac{\partial I_j}{\partial f_s} \frac{\partial f_s}{\partial q_k} \right), \\ &= \sum_{r,s} \frac{\partial I_j}{\partial f_r} \frac{\partial I_j}{\partial f_s} \{f_r, f_s\}, \\ &= 0. \end{aligned}$$

In particular $\{I_k, H\} = 0$. The torus \mathcal{M}_f can be equivalently defined ny

$$I_1 = \tilde{c}_1, \dots, I_n = \tilde{c}_n$$

Why is this integral not zero? Compactness?

why is it non-trivial? In 2d case, they can be parametrised easily.

One may ask why is I_k (as coordinate) better than f_k . If one defines $I_k = f_k$, the transformation $(p, q) \rightarrow (I, \phi)$ would not be canonical.

Canonical **angle** coordinates ϕ_k , which is the canonically conjugate to the actions via the generating functions

$$S(q, I) = \int_{q_0}^q \sum_j p_j dq_j \quad (1.2.5)$$

with q_0 some point on the torus. Modifying q_0 just adds a constant to S . The angle coordinates are

$$\phi_i = \frac{\partial S}{\partial I_i}.$$

The angles are periodic. Consider two paths C and $C \cup C_k$ (with $C_k = \Gamma_k$ the k -th cycle) between q_0 and q , see figure 1.1. Then

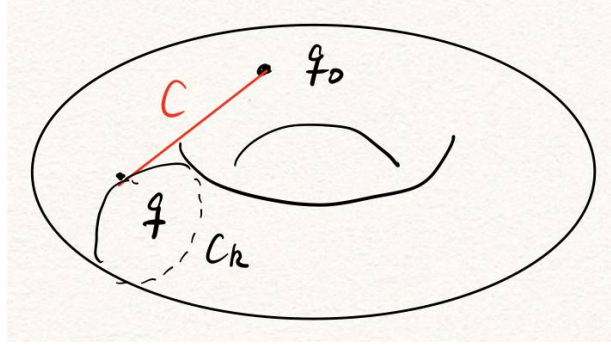


Figure 1.1: Torus with two paths C and $C \cup C_k$

$$\begin{aligned} S(q, I) &= \int_{C \cup C_k} \sum_j p_j dq_j \\ &= \int_C \sum_j p_j dq_j + \int_{C_k = \Gamma_k} \sum_j p_j dq_j \\ &= S(q, I) + 2\pi I_k \\ \Rightarrow \phi_k &= \frac{\partial S}{\partial I_k} = \phi_k + 2\pi \end{aligned}$$

The transformation

$$q = q(\phi, I), \quad p = p(\phi, I)$$

and

$$\phi = \phi(p, q), \quad I = I(p, q)$$

are canonical transformations (defined by the generating function S) and invertible. The Poisson structures are unchanged

$$\{I_j, I_k\} = 0, \quad \{\phi_j, \phi_k\} = 0, \quad \{\phi_j, I_k\} = \delta_{jk}$$

The dynamics are given by

$$\dot{\phi}_k = \{\phi_k, \tilde{H}\}, \quad \dot{I}_k = \{I_k, \tilde{H}\}$$

with $\tilde{H} = \tilde{H}(\phi, I) = H(q(\phi, I), p(\phi, I))$. Since I_k 's are first integrals,

$$0 = \dot{I}_k = \frac{\partial \tilde{H}}{\partial \phi_k},$$

in other word $\tilde{H} = \tilde{H}(I)$. The derivatives of angle variable

$$\dot{\phi}_k = \frac{\partial \tilde{H}}{\partial I_k} = \omega_k(I)$$

are first integrals as well.

Integration ("integrable" model) yields

$$\begin{aligned}\phi_k(t) &= \omega_k(I)t + \phi_k(0), \\ I_k(t) &= I_k(0).\end{aligned}\tag{1.2.6}$$

The system is in a circular motion with constant angular velocity.

Geometric picture The phase space of an integrable system is foliated into an n -parameter (c_j) family of invariant tori on which flow is linear with constant frequency ω_k . The trajectory (1.2.6) may be closed on the torus or it may cover it densely. For $n = 2$, the trajectory is closed if ω_1/ω_2 is rational and dense otherwise.

Degeneracy The periodicity in ϕ means that every function $F(p, q)$ of the state of system is periodic in ϕ . Expand the function in Fourier series, e.g. $n = 2$

$$\begin{aligned}F &= \sum_{l_1=-\infty}^{\infty} \sum_{l_2=-\infty}^{\infty} B_{l_1, l_2} \exp(i(l_1\phi_1 + l_2\phi_2)), \\ &= \sum_{l_1, l_2} B_{l_1, l_2} \exp(it(l_1\omega_1 + l_2\omega_2)).\end{aligned}$$

Every summand is period with frequency $l_1\omega_1 + l_2\omega_2$. Sum of functions is not necessarily periodic. The whole sum is only periodic for rational ω_1/ω_2 . If $a_j\omega_j = a_k\omega_k$ for $a_{j,k} \in \mathbb{Z}$ for some j, k , one speaks of **degeneracy**. If $a_1\omega_1 = \dots = a_n\omega_n$, the system is **maximally** degenerate.

Example All time-indepent Hamiltonian systems with 2 dimensional phase-space are integrable ($H = f_{1=n}$).

Consider a harmonic oscillator ($n = 1$) with the Hamiltonian

$$H = \frac{1}{2}(p^2 + \omega^2 q^2)$$

Different choices of energy $c_1 = E$ give foliation of \mathcal{M} by ellipses

$$\frac{1}{2}(p^2 + \omega^2 q^2) = E$$

with two axes $a = \sqrt{2E}$ and $b = \frac{\sqrt{2E}}{\omega}$ and surface $ab\pi$. For fixed E , take $\Gamma = \mathcal{M}_H$

$$I = \frac{1}{2\pi} \oint d\phi \stackrel{\text{Stokes'}}{=} \frac{1}{2\pi} \int_S dp dq = \frac{E}{\omega}$$

1 Integrability in classical mechanics

The Hamiltonian in the new variable $\tilde{H} = \omega I$ and $\dot{\phi} = \frac{\partial \tilde{H}}{\partial I} = \omega$, $\phi = \omega t + \phi_0$.

To obtain the transformation $(p, q) \rightarrow (I, \phi)$, first the action variable is

$$I(p, q) = \frac{1}{\omega} H(p, q) = \frac{1}{2} \left(\frac{1}{\omega} p^2 + \omega q^2 \right).$$

The generating function is

$$S(q, I) = \int_{q_0}^q p \, d\tilde{q} = \pm \int_{q_0}^q \sqrt{2I\omega - \omega^2 \tilde{q}^2} \, d\tilde{q}$$

and the angle variable

$$\phi = \frac{\partial S}{\partial I} = \int \frac{\omega \, d\tilde{q}}{\sqrt{2I\omega - \omega^2 \tilde{q}^2}} = \arcsin\left(q \sqrt{\frac{\omega}{2I}}\right) - \phi_0.$$

Thus

$$q = \sqrt{\frac{2E}{\omega}} \sin(\omega t + \phi_0)$$
$$p = \frac{\partial H}{\partial p} = \dot{q} = \sqrt{2E} \cos(\omega t + \phi_0)$$

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