An Algebraic Characterisation of First-Order Logic with Neighbour

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FO with Neighbour

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Define the following predicates,

- $\mathsf{N}(x, y) :=$ x and y are neighbouring positions, e.g., $x = y + 1$ or $y = x + 1$.
- min, is the left-most position.
- max, is the right-most position.

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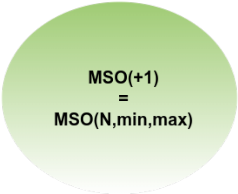
$w \models \mathrm{N}(3,4) \wedge \mathrm{N}(4,3)$ $\quad$ $w \models (\mathsf{min} = 1)$ $\quad$ $w \models (\mathsf{max} = 7)$

FO(N, min, max)

$$\phi = P_a(\min) \wedge P_b(\max) \wedge (\forall x, y.((P_a(x) \wedge \mathrm{N}(x, y) \rightarrow P_b(y)) \wedge (P_b(x) \wedge \mathrm{N}(x, y) \rightarrow P_a(y))))$$

$$L(\phi) = (ab)^+$$

Why FO(N, min, max)?

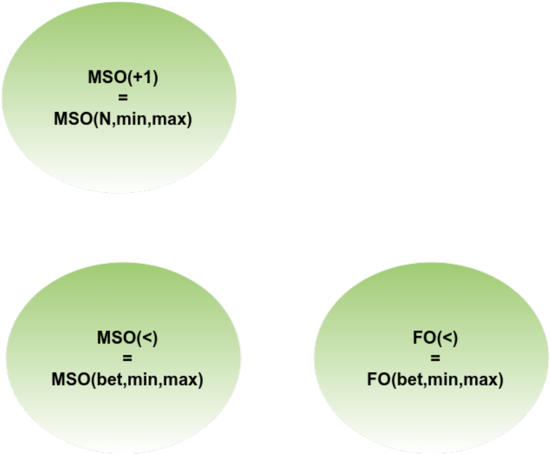


Why FO(N, min, max)?

MSO(+1)
=
MSO(N,min,max)

MSO(<)
=
MSO(bet,min,max)

Why FO(N, min, max)?



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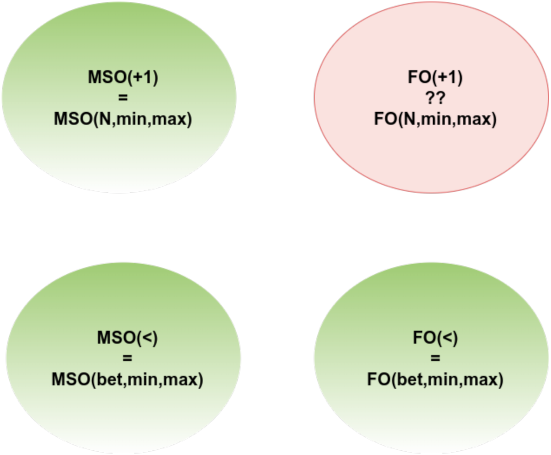

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$$\mathrm{FO}(\mathrm{N}, \min, \max) \subsetneq \mathrm{FO}(+1)$$

$$L = c^* abc^*$$

Use Ehrenfeucht-Fraïssé argument

$$L \notin \mathrm{FO}(\mathrm{N}, \min, \max)$$

Why $\mathrm{FO}(\mathrm{N}, \min, \max)$?

- $\mathrm{FO}(\mathrm{N}, \min, \max)$ is a natural fragment of logic
- It would be interesting to characterize precisely the $\mathrm{FO}(\mathrm{N}, \min, \max)$ definable languages!

Characterization of FO(+1)

Let $t, k > 0$.

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Define congruence on the free monoid, Σ^*.

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- $|w| < k$
  - $w = w'$
- $|w| \geq k$,
  - w' is of length at least k
  - number of times v appears in w is the same as the number of times v appears in w' *upto the threshold* t, for all $v \in A^{\leq k}$
  - both have same suffix of length $k - 1$
  - both have same prefix of length $k - 1$

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Using Hanf's theorem,

$$\mathrm{FO}(+1) = \mathrm{LTT}$$

Characterization of $FO(N, \min, \max)$

Define a new congruence on the free monoid, Σ^*.

$w \overset{r\ t}{\approx}_k w'$

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- $|w| \geq k$,
  - w' is of length at least k
  - number of times v or v^r appears in w is the same as the number of times v or v^r appears in w' *upto the threshold* t, for all $v \in A^{\leq k}$
  - both have same suffix of length $k-1$
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Characterization of FO(N, min, max)

A language is *weakly locally reversible threshold testable* (wLRTT) if it is a union of $\overset{r}{\approx}{}^{t}_{k}$-equivalence classes, for some $t, k > 0$.

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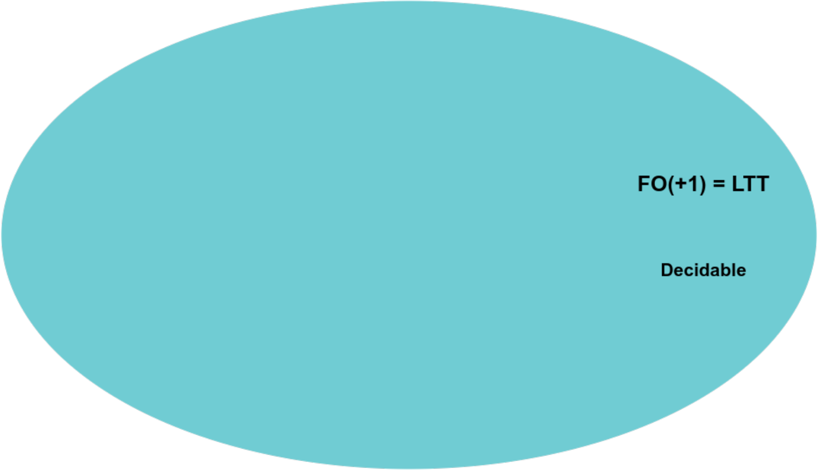
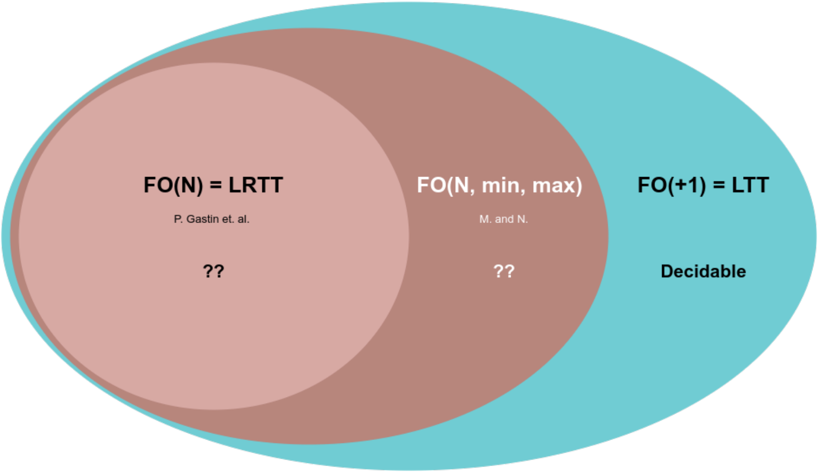
$$\mathsf{FO}(\mathsf{N}, \min, \max) = \mathsf{wLRTT}$$

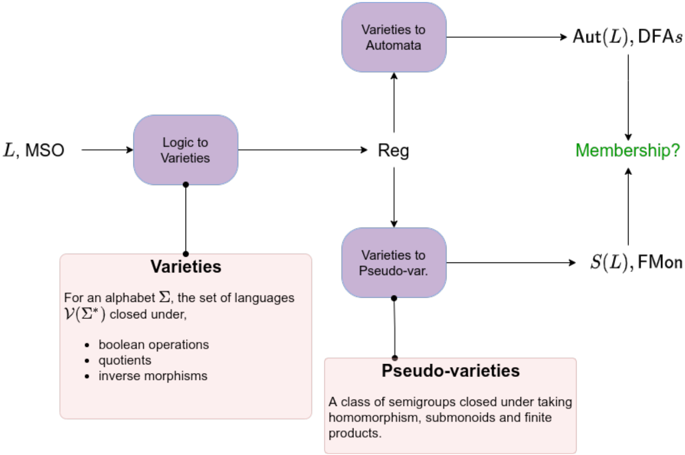
Membership problem



Membership problem

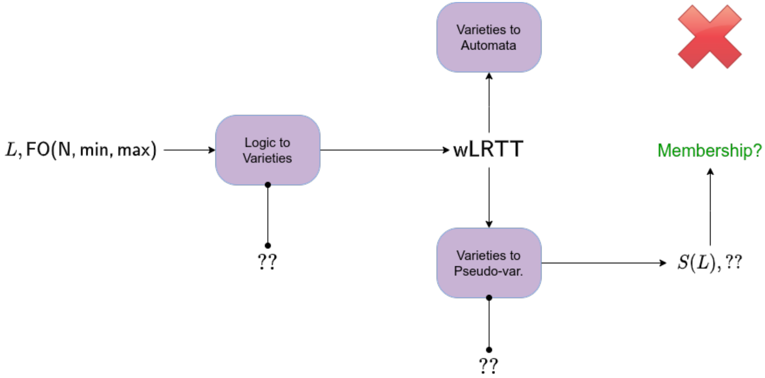


Approach



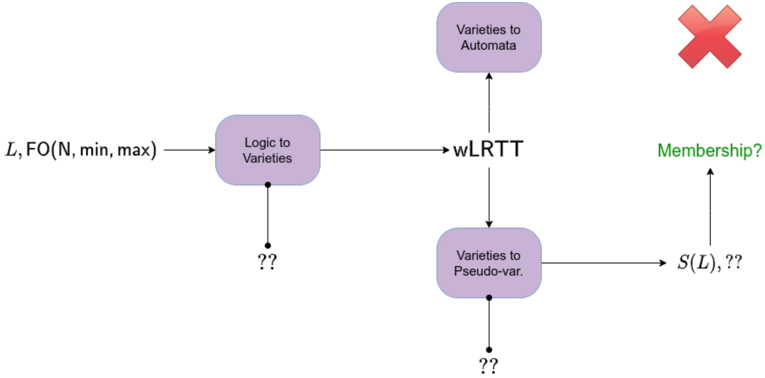
Varieties to Automata
Aut(L), DFAs
L, MSO
Logic to Varieties
Reg
Membership?
Varieties to Pseudo-var.
$S(L)$, FMon
Varieties
For an alphabet Σ, the set of languages $\mathcal{V}(\Sigma^*)$ closed under,
• boolean operations
• quotients
• inverse morphisms
Pseudo-varieties
A class of semigroups closed under taking homomorphism, submonoids and finite products.

Approach



$L, \mathsf{FO}(\mathsf{N}, \mathsf{min}, \mathsf{max})$
Logic to Varieties
??
wLRTT
Varieties to Automata
Varieties to Pseudo-var.
??
Membership?
$S(L), ??$

Approach



Since we enrich our languages with the reverse operation, it's only natural to enrish our recognizers with a similar notion as well.

Definitions

Involution semigroup

$$(.)^{\star} : S \to S$$

$$a^{\star\star} = a$$

$$(ab)^{\star} = b^{\star}a^{\star}$$

The *free* involution monoid over $(\Sigma, \star)$ modulo the involution axioms will replace Σ^{*}

Involutory Varieties

For an *involutory* alphabet $(\Sigma, \star)$, the set of languages $\mathcal{V}((\Sigma, \star)^{*})$ over this alphabet are closed under,

- boolean operations
- quotients
- inverse morphisms
- if $L \in \mathcal{V}((\Sigma, \star)^{*})$ then $L^{\star} \in \mathcal{V}((\Sigma, \star)^{*})$.

Involutory pseudo-varieties

Closed under finite direct products, sub-$\star$-semigroups and quotients.

Eilenberg-type correspondence

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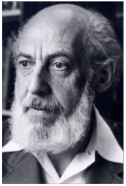

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Semidirect product

A language is in LTT iff it recognized by a monoid which is the semidirect product of an aperiodic-commutative semigroup and a locally trivial semigroup.

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$$\boxed{\mathsf{LTT} = \mathsf{Acom} * \mathsf{LI}}$$

Is there a similar algebraic characterisation for wLRTT?

Locally hermitian semidirect product

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A two-sided action of T on S is involutory if $(tst')^\star = t'^\dagger s^\star t^\dagger$.

An involutory action of T on S is locally hermitian if for all idempotents e, e' of T, $ese^\dagger = es^\star e^\dagger$.

A bilateral semidirect product of S and T with respect to a locally hermitian involutory action is called a locally hermitian semidirect product.

Main result

$$\boxed{\mathsf{LTT} = \mathsf{Acom} * \mathsf{LI}}$$

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$$\boxed{\mathsf{wLRTT} = \mathsf{Acom}^\star \star\star \mathsf{LI}^\star}$$

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$$\boxed{\mathsf{wLRTT} = \mathsf{Acom}^{\star} \star\star \mathsf{LI}^{\star}}$$

Generalizes to **any** involution on languages, not just reverse.

Decidability of membership

S

$\in$

$\boxed{\mathsf{LTT} = \mathsf{Acom} * \mathsf{LI}}$

Decidability of membership

S_E

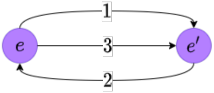
$\mathsf{Obj}(S_E) = E(S)$

$\mathsf{Hom}(e, e') = eSe'$

$$S \iff S_E$$

$$\in \qquad \qquad \in$$

$$\boxed{\mathsf{LTT} = \mathsf{Acom} * \mathsf{LI}} \iff$$



$$1 \cdot 2 \cdot 3 = 3 \cdot 2 \cdot 1$$

Decidability of membership

S $\Longleftrightarrow$ S_E

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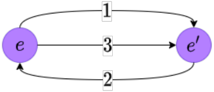
$\mathsf{Hom}(e, e') = eSe'$

$\in$ $\qquad$ $\in$

$\boxed{\mathsf{LTT} = \mathsf{Acom} * \mathsf{LI}}$ $\Longleftrightarrow$



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$\Downarrow$

$\boxed{\mathsf{wLRTT} = \mathsf{Acom}^\star \star\star \mathsf{LI}^\star}$ $\Longleftrightarrow$??

Thank you!