Midterm

ISC 4232

October 28, 2020, 12:00PM

Due date: October 30, 2020 at 12:00PM

Submit online through Canvas

Extensions are not possible and late submissions will not be accepted

- **The midterm is to be done individually.** The midterm will be graded only if you sign the contract on the first page. Any detection of plagiarism will be investigated.
- The midterm is open book, open internet, open everything. Cite all sources you use beyond the class materials.
- All your work must be shown to receive full credit. None of the calculations are to be done with electronic devices.
- The submission should be a single PDF file. Scanned and legible hand-written solutions are permitted.

Question	1	2	3	4	5	TOTAL
Points						
Max	20	20	15	10	10	75

Plagiarism Contract

As described in the Florida State University Academic Honor Policy, I attest that I completed the midterm without discussing my work with or showing my work to anyone except for the course instructor. This includes discussions that are face-to-face and using electronic devices including cell phones, email, and social media. I understand that any indication of plagiarism will be investigated and may result in a grade of zero or a failing grade in the course, and the incident will be reported to applicable offices within FSU.

Abelardo Riipas

Student Name (printed)

[illegible]

Student Signature

10/28/20

Date

1. **[20 points]** Consider the finite difference stencil

$$f''(x) \approx \frac{2f(x) - 5f(x+h) + 4f(x+2h) - f(x+3h)}{h^2}. \tag{1}$$

(a) **[7 points]** Write the Taylor expansions centered at x for $f(x+h)$, $f(x+2h)$, and $f(x+3h)$. Write all fifth- and higher-order terms with big-O notation.

(b) **[7 points]** Substitute these Taylor expansions into the approximation for the second derivative in equation (1). Write the error of this approximation as $O(h^p)$ for some positive integer p.

(c) **[3 points]** Show that equation (1) is exact for the monomials $f(x) = 1$, $f(x) = x$, and $f(x) = x^2$.

(d) **[3 points]** Explain how a convergence study would be used to numerically calculate the rate of convergence. Specifically state what you would plot and what kind of curve you would expect to see, but do not perform an actual convergence study. You may use pictures in your response.

a) $F(x+h) = F(x) + \frac{F'(x)}{1}h + \frac{F''(x)}{2}h^2 + \frac{F'''(x)}{6}h^3 + \frac{F''''(x)}{24}h^4 + O(h^5)$

$F(x+2h) = F(x) + F'(x)2h + \frac{F''(x)4h^2}{2} + \frac{F'''(x)8h^3}{6} + \frac{F^4(x)16}{24}h^4 + O(h^5)$

$F(x+3h) = F(x) + F'(x)3h + \frac{F''(x)9h^2}{2} + \frac{F'''(x)27}{6}h^3 + \frac{F^{(4)}(x)81h^4}{24} + O(h^5)$

Parts B and on next page.

D) Plot the result of the approximation on a log log graph of $h = \frac{h}{1}, \frac{h}{2}, \frac{h}{4}, \frac{h}{8}$, vs the result.

You're looking for a graph with a slope of 2. Where if h is halved, the error decreases by $\frac{1}{4}$.

b

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{6}f'''(x) + \frac{h^4}{24}f^4(x) + O(h^5)$$

$$f(x+2h) = f(x) + 2hf'(x) + 2h^2f''(x) + \frac{8}{6}h^3f'''(x) + \frac{16}{24}f^4(x) + O(h^5)$$

$$f(x+3h) = f(x) + 3hf'(x) + \frac{9}{2}h^2f''(x) + \frac{27}{6}h^3f'''(x) + \frac{81}{24}f^4(x) + O(h^5)$$

$$f''(x) \approx 2\cancel{f(x)} - 5\cancel{f(x)} - 5\cancel{hf'(x)} - \frac{5}{2}h^2f''(x) - \frac{5}{6}f'''(x) - \frac{5}{24}h^4f^4(x) + O(h^5)$$
$$+ 4\cancel{f(x)} + 8\cancel{hf'(x)} + 8h^2f''(x) + 6h^3f'''(x) + \frac{64}{24}h^4f^4(x) + O(h^5)$$
$$- \cancel{f(x)} - 3h\cancel{f'(x)} - \frac{9}{2}h^2f''(x) - \frac{27}{6}h^3f'''(x) - \frac{81}{24}f^4(x) + O(h^5)$$

$$f''(x) \approx h^2f''(x) + \frac{2}{3}h^3f'''(x) - \frac{11}{12}h^4f^4(x)$$

$-\frac{14}{2} + 8 = 1$

$-\frac{32}{6} + \frac{36}{6} = \frac{2}{3}$

$\frac{64 - 81 - 5}{24} = -\frac{22}{24}$

$$\boxed{\text{Error} = O(h^2)}$$

c) Let $f(x) = 1 \Rightarrow f''(x) = 0$ Let $h = .01$

(or whatever you want) I realized it doesn't matter by Part A, but I don't want to rewrite this.

$$\frac{2 - 5 + 4 - 1}{(.01)^2} = 0$$

Let $f(x) = x$ $h = .01$ $\Rightarrow f''(x) = 0$

$$\boxed{\frac{2x - 5x - .05 + 4x + .08 - x - .03}{(.01)^2} = 0}$$

Let $f(x) = x^2 \Rightarrow f''(x) = 2$ $h = .01$

$$\frac{2x^2 - 5x^2 - 10xh - 5h^2 + 4x^2 + 8xh - x^2 - 2xh - h^2}{h^2}$$

$$\boxed{\frac{2h^2}{h^2} = 2}$$

2. **[20 points]** Indicate whether the following statements are true or false. No explanation is necessary.

(a) A consistent method can have a local truncation error that is $\mathcal{O}(\Delta t)$.

(b) If applying a convergence study to a problem with small parameter h, and halving h asymptotically results in the relative error halving, then the method is second-order accurate.

(c) A second-order implicit Taylor series method applied to an IVP requires partial derivatives of the right-hand side $f(y, t)$.

(d) The initial value problem $y'(t) = t^2 y(t)$ is linear.

(e) Crank-Nicolson is a 2-step multistep method.

(f) The order of an initial value problem is the highest-order derivative in the differential equation.

(g) The relative error due to using finite precision is known as discretization error.

(h) If $y(0) = 1$, then the oscillations of the solution of $y'(t) = (2 + \underline{8i})y(t)$ are faster than the oscillations of the solution of $y'(t) = (4 + \underline{2i})y(t)$.

(i) If Backward Euler is applied to a linear initial value problem, then it is possible to write Y_{n+1} explicitly in terms of Y_n.

(j) For certain Δt, backward Euler applied to the initial value problem

$$\begin{aligned} y'(t) &= y(t), \quad t \in (0, 1], \\ y(0) &= 1, \end{aligned}$$

results in a decaying numerical solution.

(k) Adams-Bashforth methods can be implicit or explicit.

(l) Quadrature is another word for numerical differentiation.

(m) There exist A-stable explicit methods.

(n) The midpoint rule is an implicit method.

(o) Applying a quadrature method with the trapezoid rule results in the Crank-Nicolson method.

(p) The b_i coefficients of a consistent Runge-Kutta method sum to zero.

(q) An initial value problem can be a partial differential equation (PDE).

(r) The coefficients c_i, $i = 1, \ldots, s$, of a Runge-Kutta method are not needed when solving the initial value problem $y'(t) = f(y(t))$.

(s) A first-order initial value problem can have both an initial condition and a terminal condition.

(t) It is possible to construct a one-stage Runge-Kutta method with a global discretization error that is second-order accurate.

A) F B) F C) T D) T E) T F) T G) F H) T I) F
J) T K) F L) F M) F N) F O) T P) T
Q) F R) T S) F T) F

3. **[15 points]** For this problem use the Runge-Kutta method with the Butcher tableau:

$$\begin{array}{cc|cc} & & a_{11} & \\ c_1 & \frac{1}{4} & \frac{1}{4} & 0 \\ c_2 & \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ \hline & & \frac{1}{2} & \frac{1}{2} \\ & & b_1 & b_2 \end{array} \qquad \cancel{a_{31}, a_{32}} \quad a_{21}, a_{22}$$

(a) **[1 point]** Is this method explicit, implicit, or diagonally implicit? No explanation is required.

(b) **[1 point]** How many stages does this method have? No explanation is required.

(c) **[3 points]** Write out the formulas for all the stages when solving the IVP y'(t) = f(y(t),t).

(d) **[3 points]** Write out the formula relating Y_{n+1} to Y_n and the stages.

(e) **[7 points]** Calculate the amplification factor of the method. You can use either technique we learnt in class.

(f) **[5 points (Bonus)]** Show that the method is A-stable. This must be done by hand and not using any computational resources.

a) Explicit b) 2 stages

c) $\cancel{K_1 = \Delta t\, f(y_n, t_n)}$

$\cancel{K_2 = \Delta t\, f(Y_n + a_{21} K_1, t_n + c_2 \Delta t)}$

$\cancel{K_3 = \Delta t\, f(Y_n + a_{31} K_1 + a_{32} K_2, t_n + c_3 \Delta t)}$

$\cancel{K_2 = \Delta t\, f\left(Y_n + \frac{1}{4} \Delta t\, f(y_n, t_n), t_n + \frac{1}{4} \Delta t\right)}$

$\cancel{K_3 = \Delta t\, f\left(Y_n + \frac{1}{2} \Delta t\, f(y_n, t_n) + \frac{1}{4}\left(\Delta t\, f\left(Y_n + \frac{1}{4} \Delta t\, f(y_n, t_n), t_n + \frac{1}{4} \Delta t\right)\right), t_n + \frac{3}{4} \Delta t\right)}$

d) $\cancel{Y_{n+1} = Y_n +}$

See Back page for parts C, D, E, F

For grader ↑

c)

$$k_1 = \Delta t\, f(y_n, t_n)$$

$$k_2 = \Delta t\, f\left(y_n + \frac{1}{2}\Delta t\, f(y_n, t_n),\ t_n + \frac{3}{4}\Delta t\right)$$

d)

$$y_{n+1} = y_n + \frac{1}{2}\Delta t\, f(y_n, t_n) + \frac{1}{2}\left(\Delta t\, f\left(y_n + \frac{1}{2}\Delta t\, f(y_n, t_n),\ t_n + \frac{3}{4}\Delta t\right)\right)$$

e) $f(y,t) = \lambda y$

$$z = \Delta t \lambda$$

$$k_1 = \Delta t \lambda y = z y$$

$$k_2 = \cancel{\Delta t \lambda}\ z(y + a_{21} z y)$$

$$y_{n+1} = y_n + \frac{1}{2} z y_n + \frac{1}{2} z y_n + \frac{1}{4} z^2 y_n$$

$$\boxed{A(z) = 1 + z + \frac{z^2}{4}}$$

4. **[10 points]** When solving the IVP $y'(t) = f(y(t))$, an example of a multistep method is

$$Y_{n+1} = Y_n + \frac{3}{2}\Delta t f(Y_n) - \frac{1}{2}\Delta t f(Y_{n-1}).$$

Note that to make the Taylor series expansions less painful, I have assumed that f does not depend on time.

(a) **[1 point]** This is an m-step multistep method. What is m? No explanation is required.

(b) **[1 point]** Is this an Adams-Bashforth, Adams-Moulton, or Backward Difference Formula? No explanation is required.

(c) **[8 points]** Recall that the local truncation error is computed by moving all the terms in the scheme to one side, substituting the exact solution into this equation, and using Taylor series to move the center of each of the terms to a common time. For this problem, it is easiest to center each of the Taylor expansions at time t_n. Show that the local truncation error is third-order accurate.

a) 2 b) Adams Bashforth

$\cancel{LTE = y_{n+1} - y_n + \frac{3}{2}\Delta t f(Y_n) - \frac{1}{2}\Delta t f(Y_{n-1})}$

$\cancel{LTE = y_{n+1} = y_n - \frac{3}{2}\Delta t f(Y_n) + \frac{1}{2}\Delta t f(Y_{n-1})}$

$\cancel{LTE = y_{n+1} = y_{(n+1)} \frac{3}{2}\Delta t f(y(t_n)) + \frac{\Delta t}{2} f(y(t_{n-1}))}$

See Next page for 4c.

$$LTE = y(t_{n+1}) - y(t_n) - \frac{3}{3}\Delta t\, y'(t_n) + \frac{1}{2}\Delta t\, y'(t_{n-1})$$ 4C

Taylor expanding $y(t_{n+1})$ at t_n

$$y(t_{n+1}) = y(t_n) + \Delta t\, y'(t_n) + \frac{\Delta t^2}{2} y''(t_n) + O(\Delta t^3)$$

Tailor expanding $y'(t_{n-1})$ at t_n and multiplying by $\frac{1}{2}\Delta t$

$$\cancel{y'(t_{n+1})} = \cancel{y'(t_n)}$$

$$\frac{1}{2}\Delta t\, y'(t_{n-1}) = +\frac{1}{2}\Delta t\, y'(t_n) - \frac{\Delta t^2}{2} y''(t_n) + \frac{\Delta t^3}{4} y'''(t_n) + O(\Delta t^4)$$

$$LTE = \cancel{y(t_n)} + \cancel{\Delta t\, y'(t_n)} + \cancel{\frac{\Delta t^2}{2} y''(t_n)} + O(\Delta t^3) - \cancel{y(t_n)}$$
$$- \cancel{\frac{3}{2}\Delta t\, y'(t_n)} + \cancel{\frac{1}{2}\Delta t\, y'(t_n)} - \cancel{\frac{\Delta t^2}{2} y''(t_n)} + \frac{\Delta t^3}{4} y'''(t_n) + O(\Delta t^4)$$

$$LTE = \frac{\Delta t^3}{4} y'''(\xi) \qquad \xi \in [t_{n+1}, t_n]$$

$$LTE = O(\Delta t^3)$$ third order.

5. [10 points] Consider the system of IVPs

$$\mathbf{w}' = A\mathbf{w}, \quad \text{where} \quad A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{2}$$

(a) [8 point] We showed in class that the amplification factor of Crank-Nicolson applied to a linear system of IVPs is

$$\Lambda(A\Delta t) = \left(I - \frac{\Delta t}{2}A\right)^{-1}\left(I + \frac{\Delta t}{2}A\right).$$

Compute the amplification factor for the system of IVPs in equation (2).

(b) [2 points] Explain how you would determine if a value of Δt would result in a numerical simulation with $\|\mathbf{W}_n\| \to 0$. Do not do the calculation.

(c) [5 points (Bonus)] Find an expression that Δt must satisify to guarantee that $\|\mathbf{W}_n\| \to 0$.

a) $\Lambda(A\Delta t) = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & \frac{\Delta t}{2} \\ -\frac{\Delta t}{2} & 0 \end{bmatrix}\right)^{-1} \cdot \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & \frac{\Delta t}{2} \\ -\frac{\Delta t}{2} & 0 \end{bmatrix}\right)$

$= \begin{bmatrix} 1 & -\frac{\Delta t}{2} \\ +\frac{\Delta t}{2} & 1 \end{bmatrix}^{-1} \times \begin{bmatrix} 1 & \frac{\Delta t}{2} \\ -\frac{\Delta t}{2} & 1 \end{bmatrix}$

$= \frac{1}{1 + \frac{\Delta t^2}{4}} \begin{bmatrix} 1 & \frac{\Delta t}{2} \\ -\frac{\Delta t}{2} & 1 \end{bmatrix} \times \begin{bmatrix} 1 & \frac{\Delta t}{2} \\ -\frac{\Delta t}{2} & 1 \end{bmatrix}$

$= \begin{bmatrix} \left(\frac{1}{1+\frac{\Delta t^2}{4}} + \frac{-\Delta t^2}{4}\right) & \left(\frac{\Delta t}{2 + \frac{\Delta t^2}{2}} + \frac{\Delta t}{2}\right) \\ \left(\frac{-\Delta t}{2 + \frac{\Delta t^2}{2}} + \frac{-\Delta t}{2}\right) & \left(-\frac{\Delta t^2}{4} + 1\right) \end{bmatrix}$

$\uparrow$

b) Find the eigenvalues of this matrix and choose a Δt which makes the eigenvalue of the largest magnitude have a magnitude less than 1.