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ESC 4232 Final Fall 2020

1. A)

$$\Delta^2 U(x,y) = f(x,y) \quad (x,y) \in \Omega$$
$$\Delta U = 0 \quad (x,y) \in \partial\Omega$$
$$\Delta U = 0 \quad (x,y) \in \partial\Omega$$

$$W = \Delta U$$

System of two BVPs

$$\underset{①}{\Delta W = f} \qquad \underset{②}{\Delta U = W}$$

B) $h = \frac{1}{N+1}$ ~~$\Delta x = ih$~~ $x_i = ih$
$y_i = ih \quad i = 1 \ldots N$

C)

$$W_{i+1,j} + W_{i-1,j} + W_{i,j+1} + W_{i,j-1} - 4W_{i,j} = \Delta x^2 f(x_i, y_j)$$
$$U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1} - 4U_{i,j} = \Delta x^2 W_{i,j}.$$

subtract.

Size $2N^2 \times 2N^2$

Size $2N^2 \times 1$

Size $2N^2 \times 1$

$$\begin{bmatrix} \begin{matrix} S & I & & 0 \\ I & \ddots & \ddots & \\ & \ddots & \ddots & I \\ 0 & & I & S \end{matrix} & \begin{matrix} D & & 0 \\ & \ddots & \\ 0 & & D \end{matrix} \\ 0 & \begin{matrix} S & I & & 0 \\ I & \ddots & \ddots & \\ & \ddots & \ddots & I \\ 0 & & I & S \end{matrix} \end{bmatrix} \begin{bmatrix} \vec{U} \\ \vec{W} \end{bmatrix} = \Delta x^2 \begin{bmatrix} \vec{0} \\ \vec{f} \end{bmatrix}$$

$\vec{0}$ zero vector

$$\vec{U} = \begin{bmatrix} U_{1,1} \\ U_{2,1} \\ \vdots \\ U_{N,1} \\ U_{1,2} \\ \vdots \\ U_{N,N} \end{bmatrix} \quad \vec{W} = \begin{bmatrix} W_{1,1} \\ W_{1,2} \\ \vdots \\ W_{N,1} \\ W_{1,2} \\ \vdots \\ W_{N,N} \end{bmatrix}$$

Lexicographic ordering.

$$S = \begin{bmatrix} -4 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & -4 \end{bmatrix} \quad D = \begin{bmatrix} -\Delta x^2 & & \\ & \ddots & \\ & & -\Delta x^2 \end{bmatrix} \quad I = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$$

ALL $N \times N$

2.

$U_t = U_{xx} \quad x \in (0,\pi) \quad t \in (0,T]$ Diff eq

$U(0,t) = U(\pi,t) = 0$ Boundary Conditions

$U(x,0) = x\pi - x^2$ initial condition

a)

$\phi_1(x,t) = e^{-t}\sin(x)$

$\phi_1(0,t) = e^{-t}\sin(0) = 0 \quad \phi_1(\pi,t) = e^{-t}\sin(\pi) = 0$ ✓

~~$\phi_1(x,0) = x\pi - x^2 = e^{-0}\sin(x)$~~

$\phi_2 = e^{-4t}\sin(2x)$

$\phi_2(0,t) = e^{-4t}\sin(0) = 0 \quad \phi_1(\pi,t) = e^{-4t}\sin(2\pi) = 0$ ✓

$\phi_3(x,t) = e^{-9t}\sin(3x) \quad \phi_3(0,t) = e^{-9t}\sin(0) = 0$

$\phi_3(\pi,t) = e^{-9t}\sin(3\pi) = 0$ ✓

if $U_h(x) = C_1 e^{-t}\sin(x) + C_2 e^{-4t}\sin(2x) + C_3 e^{-9t}\sin(3x)$

then

$$\frac{\partial}{\partial t} U_h(x) = \frac{\partial^2}{\partial x^2} U_h(x)$$ ✓

$\frac{\partial}{\partial t} U_h(x) =$ ~~$-C_1 e^{-t} - 4C_2$~~ $-C_1 e^{-t}\sin(x) - 4C_2 e^{-4t}\sin(2x) - 9C_3 e^{-9t}\sin(x).$

Chain rule

$\frac{\partial^2}{\partial x^2} U_h(x) = -C_1 e^{-t}\sin(x) - 4C_2 e^{-4t}\sin(2x) - 9C_3 e^{-9t}\sin(3x)$

chain rule $\sin \to \cos \to -\sin$

$\boxed{\text{True!}}$

b. I skipped a step here.

If ϕ_1 was a solution to the PDE, then

$$\frac{\partial}{\partial t}\phi_1(x,t) = \frac{\partial^2}{\partial x^2}\phi_1(x,t) \qquad \phi_1(x,t) = e^{-t}\sin(x)$$

$$\frac{\partial}{\partial t}\phi_1 = -e^{-t}\sin(x) \quad \frac{\partial^2}{\partial x^2}\phi_1 = -e^{t}\sin(x)$$

~~chain rule~~ ~~cos~~ $\sin \xrightarrow{\frac{d}{dx}} \cos \xrightarrow{\frac{d}{dx}} -\sin$

And we already proved the boundary condition.

Same for ϕ_2 and ϕ_3

$$\frac{\partial}{\partial t}\phi_2(x,t) = -4e^{-4t}\sin(2x) = -4e^{-4t}\sin(2x) = \frac{\partial^2}{\partial x^2}\phi_2(x,t)$$

$$\frac{\partial}{\partial t}\phi_3(x,t) = -9e^{-9t}\sin(3x) = -9e^{-9t}\sin(3x) = \frac{\partial^2}{\partial x^2}\phi_3(x,t)$$

Since $U_h(x)$ is a linear combination of ϕ_1, ϕ_2, and ϕ_3, if one satisfies the PDE, and boundary conditions, $U_h(x)$ does too, just think about setting the ~~one~~ basis function that works to it's calculated c_i value and setting all other c_i values to zero.

C

Using forward euler

$$\frac{U(x,t_{m+1}) - U(x,t_m)}{\Delta t} = U_{xx}(x,t_m) \Rightarrow$$

$$U(x,t_{m+1}) = U(x,t_m) + U_{xx}(x,t_m)$$

Using $U^h(x,0)$ to approximate initial condition.

$$R(x) = (C_1 \sin(x) + C_2 \sin(2x) + C_3 \sin(3x)) +$$
$$\Delta t(-C_1 \sin(x) - 4C_2 \sin(2x) - 9C_3 \sin(3x)) -$$
$$(C_1 e^{-t_{m+1}} \sin(x) + C_2 e^{-4t_{m+1}} \sin(2x) + C_3 e^{-9t_{m+1}} \sin(3x))$$

d) $W_i(x) = \frac{\partial R}{\partial c_i}$ Least squares

$$W_1(x) = \sin(x) - \Delta t \sin(x) - e^{-t_{m+1}} \sin(x) = (1 - \Delta t - e^{-t_{m+1}}) \sin(x)$$

$$W_2(x) = \sin(2x) - 4\Delta t \sin(2x) - e^{-4t_{m+1}} \sin(x) = (1 - 4\Delta t - e^{-4t_{m+1}}) \sin(2x)$$

$$W_3(x) = (1 - 9\Delta t - e^{-9t_{m+1}}) \sin(3x)$$

Letting $M = 9$ $\Delta t = \frac{1}{10} = .1$ $W_1(x) = -.6048 \sin(x)$

~~$W_1(x) = -.205 \sin(x)$~~ ~~$W_2 =$~~ $W_2(x) = -.0703 \sin(2x)$

~~$W_2(x) = -.305 \sin(x)$~~ $W_3(x) = -.3066 \sin(3x)$

—1

$$e \int_0^{\pi} R(X)\, W_i(X) = 0$$

$$R(x) = C_1 \sin(x) + C_2 \sin(2x) + C_3 \sin(3x) +$$
$$\Delta t(-C_1 \sin(x) - 4C_2 \sin(2x) - 9C_3 \sin(3x)) -$$
$$(C_1 e^{-.1} \sin(x) + C_2 e^{-.4} \sin(2x) + C_3 e^{-.9} \sin(3x))$$

For $i = 1$

$$W_1(x) = -.0048 \sin(x)$$

Using;

$$\int_0^{\pi} \sin(nx) \sin(mx) = \begin{cases} 0 & n \neq m \\ \frac{\pi}{2} & n = m \end{cases}$$

$\cancel{-.0048 C_1 \frac{\pi}{2} - \Delta t\, .0048}$

$$-.0048\, C_1 \frac{\pi}{2} + \Delta t\, .0048\, C_1 \frac{\pi}{2} + e^{-.1} C_1\, .0048 \frac{\pi}{2} = 0$$

$$-.0703\, C_2 \frac{\pi}{2} + 4\Delta t\, .0703\, C_2 \frac{\pi}{2} + e^{-.4} C_2\, .0703 \frac{\pi}{2} = 0$$

$$-.3066\, C_3 \frac{\pi}{2} + 9\Delta t\, .3066\, C_3 \frac{\pi}{2} + e^{-.9} C_3\, .3066 \frac{\pi}{2} = 0$$

$C_1, C_2, C_3 = 0.$

3

$$\Delta U = F \qquad x \in \Omega$$
$$U = 0 \qquad x \in \partial\Omega$$

$$\Delta U(x,y) \approx \frac{1}{h^2}\Big(\frac{1}{6}U_{i-1,j-1} + \frac{2}{3}U_{i,j-1} + \frac{1}{6}U_{i+1,j-1} + \frac{2}{3}U_{i-1,j} - \frac{10}{3}U_{i,j}$$
$$+ \frac{2}{3}U_{i+1,j} + \frac{1}{6}U_{i-1,j+1} + \frac{2}{3}U_{i,j+1} + \frac{1}{6}U_{i+1,j+1}\Big)$$

For $i=1$, $j=1$ (Lexicographic ordering).

$$\frac{1}{6}\cancel{U_{0,0}}^{0} + \frac{2}{3}\cancel{U_{1,0}}^{0} + \frac{1}{6}\cancel{U_{2,0}}^{0} + \frac{2}{3}\cancel{U_{0,1}}^{0} - \cancel{\frac{10}{3}U_{1,1}} + \cancel{\frac{2}{3}U_{2,1}} + \frac{1}{6}\cancel{U_{0,2}}^{0}$$
$$+ \frac{2}{3}\cancel{U_{1,2}} + \frac{1}{6}\cancel{U_{2,2}}$$

For $i=2$, $j=1$

$$\frac{1}{6}U_{1,0}^{0} + \frac{2}{3}\cancel{U_{2,0}}^{0} + \frac{1}{6}\cancel{U_{3,0}}^{0} + \cancel{\frac{2}{3}U_{1,1}} - \cancel{\frac{10}{3}U_{2,1}} + \cancel{\frac{2}{3}U_{3,1}} + \cancel{\frac{1}{6}U_{1,2}}$$
$$+ \frac{2}{3}\cancel{U_{2,2}} + \frac{1}{6}\cancel{U_{3,2}}$$

For $i=3$, $j=1$

$$\frac{1}{6}U_{2,0}^{0} + \frac{2}{3}\cancel{U_{3,0}}^{0} + \frac{1}{6}\cancel{U_{4,0}}^{0} + \cancel{\frac{2}{3}U_{2,1}} - \cancel{\frac{10}{3}U_{3,1}} + \cancel{\frac{2}{3}U_{4,1}} + \cancel{\frac{1}{6}U_{2,2}}$$
$$+ \frac{2}{3}U_{3,2} + \frac{1}{6}\cancel{U_{4,2}}$$

For $i=4$ $j=1$

$$\frac{1}{6}\cancel{U_{3,0}}^{0} + \frac{2}{3}\cancel{U_{4,0}}^{0} + \cancel{\frac{1}{6}U_{5,0}}^{0} + \frac{2}{3}U_{3,1} - \frac{10}{3}U_{4,1} + \frac{2}{3}\cancel{U_{5,1}}^{0} + \frac{1}{6}U_{3,2}$$
$$+ \frac{2}{3}U_{4,2} + \frac{1}{6}\cancel{U_{5,2}}^{0}$$

-10/3	2/3	0	0	2/3	1/6	0	0	0	0	0	0	0	0	0	0
2/3	-10/3	2/3	0	1/6	2/3	1/6	0	0	0	0	0	0	0	0	0
0	2/3	-10/3	2/3	0	1/6	2/3	1/6	0	0	0	0	0	0	0	0
0	0	2/3	-10/3	0	0	1/6	2/3	0	0	0	0	0	0	0	0
2/3	1/6	0	0	-10/3	2/3	0	0	2/3	1/6	0	0	0	0	0	0
1/6	2/3	1/6	0	2/3	-10/3	2/3	0	1/6	2/3	1/6	0	0	0	0	0
0	1/6	2/3	1/6	0	2/3	-10/3	2/3	0	1/6	2/3	1/6	0	0	0	0
0	0	1/6	2/3	0	0	2/3	-10/3	0	0	1/6	2/3	0	0	0	0
0	0	0	0	2/3	1/6	0	0	-10/3	2/3	0	0	2/3	1/6	0	
0	0	0	0	1/6	2/3	1/6	0	2/3	-10/3	2/3	0	1/6	2/3	1/6	
0	0	0	0	0	1/6	2/3	1/6	0	2/3	-10/3	2/3	0	1/6	2/3	
0	0	0	0	0	0	1/6	2/3	0	0	2/3	-10/3	0	0	1/6	
0	0	0	0	0	0	0	0	2/3	1/6	0	0	-10/3	2/3	0	
0	0	0	0	0	0	0	0	1/6	2/3	1/6	0	2/3	-10/3	2/3	
0	0	0	0	0	0	0	0	0	1/6	2/3	1/6	0	2/3	-10/3	
0	0	0	0	0	0	0	0	0	0	1/6	2/3	0	0	2/3	

Read this way

I put grid lines in my matrix because it was too messy to read / interpret without them.

Multiply this matrix by the lexicographically ordered $U_{i,j}$ vector to get the linear system.
And solve with f vector.

[illegible]

-10/3	2/3	0	0	2/3	1/6	0	0	0	0	0	0	0	0	0	0
2/3	-10/3	2/3	0	1/6	2/3	1/6	0	0	0	0	0	0	0	0	0
0	2/3	-10/3	2/3	0	1/6	2/3	1/6	0	0	0	0	0	0	0	0
0	0	2/3	-10/3	0	0	1/6	2/3	0	0	0	0	0	0	0	0
2/3	1/6	0	0	-10/3	2/3	0	0	2/3	1/6	0	0	0	0	0	
1/6	2/3	1/6	0	2/3	-10/3	2/3	0	1/6	2/3	1/6	0	0	0	0	
0	1/6	2/3	1/6	0	2/3	-10/3	2/3	0	1/6	2/3	1/6	0	0	0	
0	0	1/6	2/3	0	0	2/3	-10/3	0	0	1/6	2/3	0	0	0	
0	0	0	0	2/3	1/6	0	0	-10/3	2/3	0	0	2/3	1/6	0	
0	0	0	0	1/6	2/3	1/6	0	2/3	-10/3	2/3	0	1/6	2/3	1/6	
0	0	0	0	0	1/6	2/3	1/6	0	2/3	-10/3	2/3	0	1/6	2/3	
0	0	0	0	0	0	1/6	2/3	0	0	2/3	-10/3	0	0	1/6	
0	0	0	0	0	0	0	0	2/3	1/6	0	0	-10/3	2/3	0	
0	0	0	0	0	0	0	0	1/6	2/3	1/6	0	2/3	-10/3	2/3	
0	0	0	0	0	0	0	0	0	1/6	2/3	1/6	0	2/3	-10/3	
0	0	0	0	0	0	0	0	0	0	1/6	2/3	0	0	2/3	

Read this way

I put grid lines in my matrix because it was too messy to read / interpret without them.

Multiply this matrix by the lexicographically ordered $U_{i,j}$ vector to get the linear system.

And solve with f vector.

LIVE!

B)

$$S = \begin{bmatrix} -\frac{10}{3} & 2/3 & & 0 \\ 2/3 & -\frac{10}{3} & \ddots & \\ & 2/3 & \ddots & 2/3 \\ 0 & & \ddots & -\frac{10}{3} \end{bmatrix}$$

$-10/3$ main diagonal
$2/3$ subdiagonal
$2/3$ superdiagonal
N by N

$$K = \begin{bmatrix} 2/3 & 1/6 & & \\ 1/6 & 2/3 & \ddots & \\ & 1/6 & \ddots & 1/6 \\ & & 1/6 & 2/3 \end{bmatrix}$$

$2/3$ main diagonal
$1/6$ subdiagonal
$1/6$ superdiagonal
N by N

$0 = N \times N$ zero matrix

$$L = \begin{bmatrix} S & K & & 0 \\ K & \ddots & \ddots & \\ & \ddots & \ddots & K \\ 0 & & K & S \end{bmatrix}$$

$L = N^2$ by N^2 in size

$$\boxed{L\vec{U} = \vec{F}}$$

$$\vec{U} = \begin{bmatrix} U_{1,1} \\ U_{2,1} \\ \vdots \\ U_{N,1} \\ U_{1,2} \\ \vdots \\ U_{N,2} \\ \vdots \\ U_{N,N} \end{bmatrix} \quad N^2 \times 1 \qquad \vec{F} = h^2 * \begin{bmatrix} F_{1,1} \\ F_{2,1} \\ F_{3,1} \\ \vdots \\ F_{N,1} \\ \vdots \\ F_{N,N} \end{bmatrix} \quad N^2 \times 1$$

2)

4.

a)

$$\underbrace{U(x,t_{m+1})}_{\text{Unknown}} = \frac{18}{11}U(x,t_m) - \frac{9}{11}U(x,t_{m-1}) + \frac{2}{11}U(x,t_{m-2}) + \underbrace{\frac{6}{11}\Delta t\, D\, U_{xx}(x,t_{m+1})}_{\text{Unknown}}$$

$$\cancel{U(x,t_{m+1})}\; U_i^{m+1} - \frac{6}{11}\frac{\Delta t\, D}{\Delta x^2}\left(U_{i+1}^{m+1} - 2U_i^{m+1} + U_{i-1}^{m+1}\right) = \frac{18}{11}U_i^{m} - \frac{9}{11}U_i^{m-1} + \frac{2}{11}U_i^{m-2}$$

$$\lambda = \frac{D\Delta t}{\Delta x^2}$$

B)

$$-\frac{6}{11}\lambda U_{i+1}^{m+1} + \left(1 + \frac{12}{11}\lambda\right)U_i^{m+1} - \frac{6}{11}\lambda U_{i-1}^{m+1} = \frac{18}{11}U_i^{m} - \frac{9}{11}U_i^{m-1} + \frac{2}{11}U_i^{m-2}$$

Multiply everything by 11 to make it cleaner on the matrix, $(1+\frac{12}{11}\lambda) \Rightarrow (11+12\lambda)$, though.

$$\cancel{\begin{bmatrix} 11+12\lambda & -6\lambda & & \\ -6\lambda & \ddots & \ddots & \\ & \ddots & \ddots & -6\lambda \\ & & -6\lambda & 11+12\lambda \end{bmatrix}}$$

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B. Marix A

U^{m+1} vector

b vector

$$\begin{bmatrix} 11+12\lambda & -6\lambda & & \\ -6\lambda & \ddots & \ddots & \\ & \ddots & \ddots & -6\lambda \\ & & -6\lambda & 11+12\lambda \end{bmatrix} \begin{bmatrix} U_1^{m+1} \\ \vdots \\ \vdots \\ U_N^{m+1} \end{bmatrix} = \begin{bmatrix} 18U_1^m - 9U_1^{m-1} + 2U_1^{m-2} \\ 18U_2^m - 9U_2^{m-1} + 2U_2^{m-2} \\ \vdots \\ 18U_N^m - 9U_N^{m-1} + 2U_N^{m-2} \end{bmatrix}$$

Expression for U_1

$$11+12\lambda U_1^{m+1} - 6\lambda U_2^{m+1} = 18U_1^m - 9U_1^{m-1} + 2U_1^{m-2}$$

C.

Check if eigenvalues of A are ~~less than 1~~. in the interval $(0,1)$. to satisfy the Maximum principal

$$|U_i^{m+1}| \le \max(U_0(x_i)) \text{ for all } i, m.$$

Use

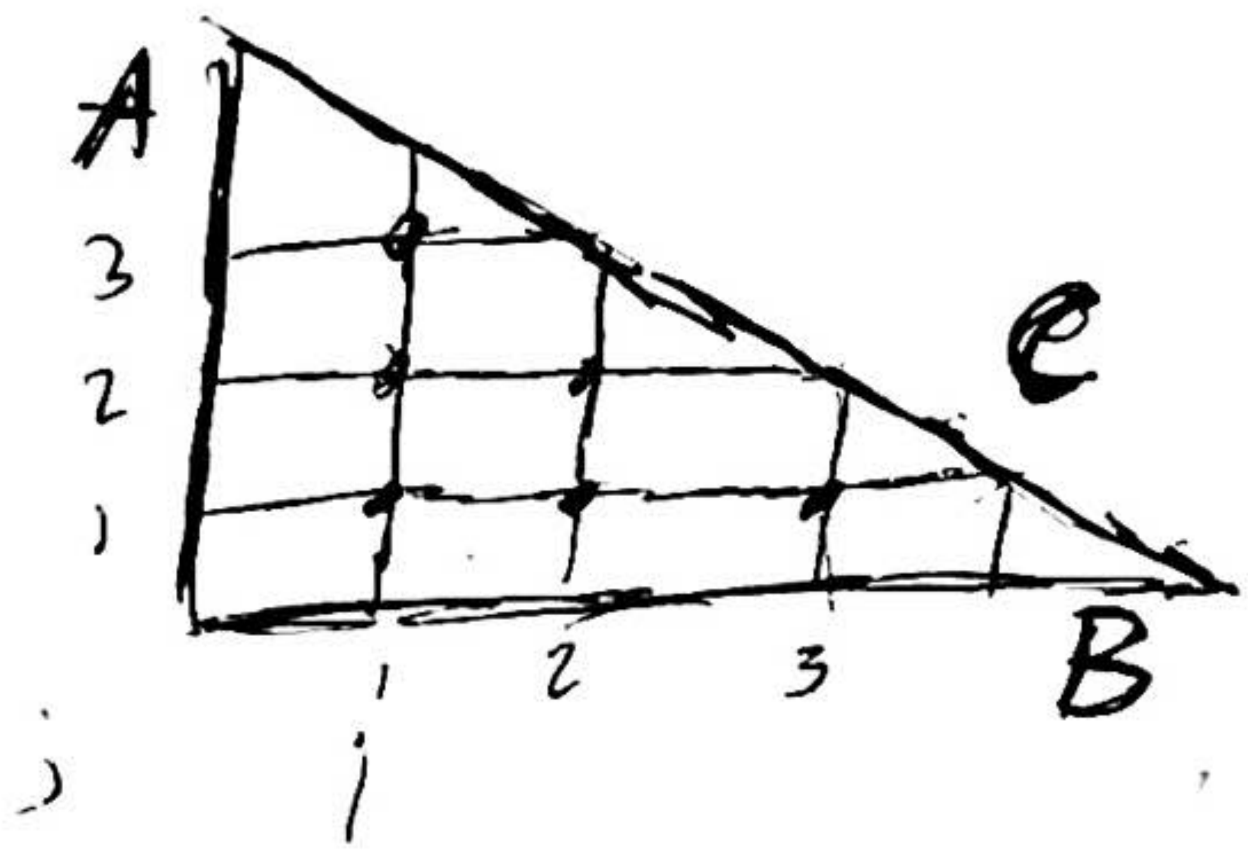
$$\mu_k = a + 2\sqrt{bc}\cos\left(\frac{k\pi}{N+1}\right) \quad k = 1, \dots N$$

$a = 11+12\lambda \quad b = -6\lambda \quad c = -6\lambda$

to find eigenvalues of a tridiagonal matrix.

$\frac{1}{3}$ to $\frac{1}{3}$

5



$\Delta U = 0$

$U = f$

$\Delta x^2 = \Delta y^2$

$$U_{xx} = \frac{U_{i+1,j} - 2U_{i,j} + U_{i-1}}{\Delta x^2}$$

$$U_{yy} = \frac{U_{i,j+1} - 2U_{i,j} + U_{i,j-1}}{\Delta y^2}$$

$$U_{i+1,j} + U_{i,j+1} - 4U_{i,j} + U_{i-1,j} + U_{i,j-1} = 0 \cdot \Delta x^2$$

Using lexicographic ordering

$i=1 \quad j=1$

$$U_{2,1} + U_{1,2} - 4U_{1,1} + \overset{A}{\cancel{U_{0,1}}} + \overset{B}{\cancel{U_{1,0}}} = 0$$

$i=2, \ j=1$

$$U_{3,1} + U_{2,2} - 4U_{2,1} + \cancel{U_{1,1}} + \overset{B}{\cancel{U_{2,0}}} = 0$$

$i=3 \quad j=1$

$$\overset{C}{U_{4,1}} + \overset{C}{\cancel{U_{3,2}}} - 4U_{3,1} + U_{2,1} + \overset{B}{\cancel{U_{3,0}}} = 0$$

$i=1 \quad j=2$

$$U_{2,2} + U_{1,3} - 4\cancel{U_{1,2}} + \overset{A}{\cancel{U_{0,2}}} + \cancel{U_{1,1}} = 0$$

$i=2 \quad j=2$

$$\overset{C}{U_{3,2}} + \overset{C}{U_{2,3}} - 4U_{2,2} + U_{1,2} + U_{2,1} = 0$$

$i=1 \quad j=3$

$$\overset{C}{U_{2,3}} + \overset{C}{U_{1,4}} - 4U_{1,3} + \overset{A}{U_{0,3}} + U_{1,2} = 0$$

$$\begin{bmatrix} -4 & 1 & 0 & 1 & 0 & 0 \\ 1 & -4 & 1 & 0 & 1 & 0 \\ 0 & 1 & -4 & 0 & 0 & 0 \\ 1 & 0 & 0 & -4 & 0 & 1 \\ 0 & 1 & 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 1 & 0 & -4 \end{bmatrix} \begin{bmatrix} U_{1,1} \\ U_{2,1} \\ U_{3,1} \\ U_{1,2} \\ U_{2,2} \\ U_{1,3} \end{bmatrix} = \begin{bmatrix} -A-B \\ -B \\ -2C-B \\ -A \\ -2C \\ -2C-A \end{bmatrix}$$

Neater version

$$\begin{bmatrix} -4 & 1 & 0 & 1 & 0 & 0 \\ 1 & -4 & 1 & 0 & 1 & 0 \\ 0 & 1 & -4 & 0 & 0 & 0 \\ 1 & 0 & 0 & -4 & 1 & 1 \\ 0 & 1 & 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 1 & 0 & -4 \end{bmatrix} \begin{bmatrix} U_{1,1} \\ U_{2,1} \\ U_{3,1} \\ U_{1,2} \\ U_{2,2} \\ U_{1,3} \end{bmatrix} = \begin{bmatrix} -A-B \\ -B \\ -2C-B \\ -A \\ -2C \\ -2C-A \end{bmatrix}$$