STAT2203: Probability Models and Data Analysis for Engineering
Assignment 1

Due by 14:00 on Tuesday the 6^{th} of October, 2020. Submission via Blackboard.

The marks for each question are indicate by the number in square brackets. There are a total of 15 marks for this assignment. Your submission to blackboard should be a single pdf file. You may prepare this however you wish, provided the result is legible.

1. Suppose the random variable X has probability density function

$$f_X(x) = \begin{cases} \frac{2}{x^2}, & x > 2 \\ 0, & x \leq 2. \end{cases} \tag{1}$$

   (a) Let $Y = \log(X - 1)$, where log denotes the natural logarithm. Find the probability density function of Y. [1 mark]

   (b) Determine the quantile function of X. [1 mark]

   (c) Hence, write a small function in `R` or `MATLAB` to simulate random variables from the probability density function (1). Supply your code. [1 mark]

2. Consider a system comprising three components. The system requires at least two out of the three components to be working for the system to work. Components fail independently and the time to failure for each component has an exponential distribution with a mean of one year.

   (a) Determine an expression for the probability that the system is working at time t. Hence, give the probability density function for the time to failure of the system. [1 mark]

   (b) Determine the mean and variance of the time to failure for the system. [2 marks]

   (c) Determine the probability that component one in the system is still working at time t given the system is working at time t. What is the limiting value as $t \to \infty$? [2 marks]

   (d) Suppose now that the system has one spare component which can be used to replace a component in the failed system. The time to failure of the spare component is also an exponential distribution with a mean of one year. Consider the following repair regime. As soon as the system fails, the one of the failed components in the system is replaced by the spare and the system continues to function as long as there are at least two components are working. Simulate the processes in `R` or `MATLAB`. You may wish to use the function `sort` in `MATLAB` or `R`. Provide a histogram of the failure times for the repaired system and an estimate of the mean time to failure for the repaired system. Supply your code. [2 marks]

3. A pair of random variables (X, Y) has a joint probability distribution in which the marginal distribution of X is $\mathsf{Exp}(1)$ and the conditional distribution of Y given $X = x$ is $\mathsf{Exp(x)}$.

   (a) Determine the marginal probability density function of Y. [1 mark]

   (b) Determine the conditional expectation $\mathbb{E}[X|Y = y]$. [1 mark]

4. Suppose $\mathbf{Z}$ has a bivariate normal distribution with mean $\mathbf{0}$ and covariance matrix equal to the identity matrix. Determine a vector $\mathbf{a}$ and matrix B such that $\mathbf{Y} = \mathbf{a} + B\mathbf{Z}$ has a bivariate normal distribution with

$$\mu = \begin{bmatrix} 3 \\ -2 \end{bmatrix} \quad \text{and} \quad \Sigma = \begin{bmatrix} 5 & 13 \\ 13 & 41 \end{bmatrix}$$

   Note that there may be more than one solution. Hint: You may wish to set $B_{12} = 0$, though this is not the only way to solve this problem. [3 marks]

Total [15 marks]