Question 1

1a) The probability of $f_Y(y) = P(Y \leq y)$ is and since $Y = log(X-1)$.
The rule $ln(N) = x \equiv N = e^x$ is also important to note.

$$\begin{aligned} f_Y(y) &= P(Y \leq y) \\ &= P(log(X-1) \leq y) \\ &= P(log(X-1) \leq y) \\ &= P(X - 1 \leq e^y) \\ &= P(X \leq e^y + 1) \\ \implies f_Y(y) &= P(X \leq e^y + 1) \end{aligned} \tag{1}$$

Sub this found X value into the existing x function to get:

$$\frac{2}{(e^y+1)^2}$$

Then multiply this function by the derivative of X which is $\frac{d}{dy}(e^y + 1) = e^y$ to get:

$$\frac{2e^y}{(e^y+1)^2}$$

To add bounds to the cases, sub in $x = 2$ into $Y > log(X-1)$ which gives $Y > ln(1), Y > 0$

$$f_y(y) = \begin{cases} \frac{2e^y}{(e^y+1)^2} & y > 0 \\ 0 & otherwise \end{cases} \tag{2}$$

1b) To find the quartile function. We need to find the we need to integrate the pdf we just got to get Fx(x).

$$\begin{aligned} \int_x^2 F_X(x) &= \int \frac{2}{x^2} \\ \int_x^2 \frac{2}{x^2} &= \left[-\frac{2}{x}\right]_x^2 \\ &= -\frac{2}{x} - -\frac{2}{2} \\ &= 1 - \frac{2}{x} \\ &\text{Switch q and x to get the} \\ x &= -\frac{2}{q} + 1 \\ x - 1 &= -\frac{2}{q} \\ q(x-1) &= -2 \\ q &= -\frac{2}{x-1} \\ \implies qx(x) &= -\frac{2}{x-1} \end{aligned} \tag{3}$$

1c) Randomised code for the pdf function of $f_X(x)$:

```r
pdfFunc <- function() {
    n = runif(1, 0, 10)
    if (n <= 2) {
        print(0)
    } else {
        print(2/(n^2))
    }
}
```

Question 2

2a) - One component $= e^{-t}$
- cdf of the system (failure) $P(T \leq t) = 1 - P(T > t)$
- pdf $= \frac{dy}{dx} F_T(t)$

$$\begin{aligned} P(T > t) &= X_1 X_2 (1 - X_3) 3 + X_1 X_2 X_3 \\ P(T > t) &= 3e^{-2t} - 2e^{-3t} \\ P(T \leq t) = 1 - P(T > t) &= 1 - (3e^{-2t} - 2e^{-3t} \\ P(T \leq t) &= 1 - P(T > t) \\ P(T \leq t) &= 1 + 2e^{-3t} - 3e^{-2t} \\ F_T(t) &= 1 + 2e^{-3t} - 3e^{-2t} \\ f_t(t) &= -6e^{-3t} + 6e^{-2t} \\ \implies \text{pdf of the failure of the system } &= -6e^{-3t} + 6e^{-2t} \end{aligned} \tag{4}$$

(Scroll down for next part LaTeX hates me)

2b) To find the mean or expected value use the following formula: $\int_0^\infty t\left(-6e^{-3t}+6e^{-2t}\right)$

$$\begin{aligned}
\int_0^\infty t\left(-6e^{-3t}+6e^{-2t}\right) &= \int_0^\infty -6e^{-3t}t+6e^{-2t}t \\
&- \int 6e^{-3t}tdt + \int 6e^{-2t}tdt \\
&\text{Integral of first half} \\
&\int 6te^{-3t}dt \\
&= 6\int te^{-3t}dt \\
&u=-3t \\
&= 6\int \frac{e^u u}{9}du \\
&= \frac{6}{9}\int e^u u\, du \\
&u=u,\ v'=e^u \\
=\frac{6}{9}\left(e^u u - \int e^u du\right) &= \frac{6}{9}\left(e^u u - e^u\right) \\
&u=-3t \\
\frac{6}{9}\left(e^{-3t}(-3t)-e^{-3t}\right) = -2e^{-3t}t-\frac{2}{3}e^{-3t} &\text{Integral of second half} \\
&\int 6e^{-2t}tdt \\
&= 6\int e^{-2t}tdt \\
&u=-2t \\
&= 6\int \frac{e^u u}{4}du \\
&= \frac{6}{4}\int e^u udu \\
&u=u,\ v'=e^u \\
&= \frac{6}{4}\left(e^u u-\int e^u du\right) \\
&= \frac{6}{4}\left(e^u u - e^u\right) \\
&u=-2t \\
&= \frac{6}{4}\left(e^{-2t}(-2t)-e^{-2t}\right) \\
&= -3e^{-2t}t-\frac{3}{2}e^{-2t} \\
&\text{Putting the two integrals together we get:} \\
&\left[2e^{-3t}t+\frac{2}{3}e^{-3t}-3e^{-2t}t-\frac{3}{2}e^{-2t}\right]_0^\infty \\
&\text{Subbing in the bounds we end up with:} \\
&0-\left(-\frac{5}{6}\right) \\
&\text{Mean: } = \frac{5}{6}
\end{aligned} \tag{5}$$

I then did something similar to get $E[X^2]$ where I got $\frac{19}{18}$.
I then used these values to calculate variance which can be calculated with the formula $E[X^2]-(E[X])^2$

Variance $= \frac{19}{18} - \left(\frac{5}{6}\right)^2$

Variance $= \frac{13}{36} \approx 0.3611$

2c) To find where $t \to \infty$ find:

$$\lim_{t\to\infty} \frac{X_1 \cap P(T > t)}{P(T > t)}$$

- We know that the System Working can be given by $P(T > t) = 3e^{-2t} - 2e^{-3t}$
- The equation of the System Working and Component 1 Working can be given by $2e^{-2t} - e^{-3t}$

$$\begin{aligned}
&\lim_{t\to\infty} \frac{2e^{-2t} - e^{-3t}}{3e^{-2t} - 2e^{-3t}} \\
&\text{Divide by the largest denominator factor:} \\
&\lim_{t\to\infty} \frac{2 - \frac{1}{e^t}}{3 - \frac{2}{e^t}} \\
&\frac{\lim_{t\to\infty} \left(2 - \frac{1}{e^t}\right)}{\lim_{t\to\infty} \left(3 - \frac{2}{e^t}\right)} \\
&\implies \lim_{t\to\infty} \left(2 - \frac{1}{e^t}\right) = 2 \\
&\implies \lim_{t\to\infty} \left(3 - \frac{2}{e^t}\right) = 3 \\
&\frac{\lim_{t\to\infty} \left(2 - \frac{1}{e^t}\right)}{\lim_{t\to\infty} \left(3 - \frac{2}{e^t}\right)} = \frac{2}{3} \\
&\implies \text{The limiting value as } \lim_{t\to\infty} \frac{2e^{-2t} - e^{-3t}}{3e^{-2t} - 2e^{-3t}} = \frac{2}{3}
\end{aligned} \tag{6}$$

2d) Randomised variables in R were used to generate

Question 3

3a) The conditional distribution of Y can be expressed as:

$$P(Y = y|X = x) = \frac{f_{xy}(x, y)}{f_x(X = X)}$$

$$P(Y = y|X = x) = \frac{f_{xy}(x, y)}{f_x(X = X)} = Exp(x)$$

Given P(X = x) = Exp(1)

$$P(Y = y|X = x) = \frac{f_{xy}}{Exp(1)} = Exp(x)$$

$$f_{xy}(X, Y) = Exp(x)Exp(1) = e^{-x}\left(xe^{-yx}\right)$$

Now that we have Fxy(X, Y), we can find the marginal distribution of Y by $F_y(y) = \int_{-\infty}^{\infty} fxy(x, y,)dx$

$$\begin{aligned} f_y(y) &= \int_{-\infty}^{\infty} fxy(x, y,)dx \\ &= \int_{-\infty}^{\infty} e^{-x}\left(xe^{-yx}\right)dx \end{aligned}$$

Because -infinity approaches 0, we can change the bounds to that

$$\begin{aligned} &= \int_{0}^{\infty} e^{-x}\left(xe^{-yx}\right)dx \\ &= \int e^{-yx-x} x dx \\ u &= x, v' = e^{-yx-x} \\ &= e^{-yx-x}\frac{1}{-y-1}x - \int e^{-yx-x}\frac{1}{-y-1}dx \\ &= e^{-yx-x}\frac{1}{-y-1}x - e^{-yx-x}\frac{1}{(-y-1)^2} \\ &= \left[e^{-yx-x}\frac{1}{-y-1}x - e^{-yx-x}\frac{1}{(-y-1)^2}\right]_0^{\infty} \\ &= 0 - \left(-\frac{1}{(-y-1)^2}\right) \\ F_y(Y) &= \frac{1}{(-y-1)^2} \end{aligned}$$

$$\implies \text{The marginal probability density function of Y} = \frac{1}{(-y-1)^2} \tag{7}$$

3b) Now that we have the marginal pdf of Y, we can calculate $E[X|Y = y]$:

$$P(X = x|Y = y) = \frac{f_{xy}(x, y)}{f_y(Y = y)}$$

$$\begin{aligned} P(X=x|Y=y) &= \frac{f_{xy}(x,y)}{f_y(Y=y)} \\ &= \frac{e^{-x}\left(xe^{-yx}\right)}{\frac{1}{(-y-1)^2}} \\ &= e^{-x-xy}x\left(-y-1\right)^2 \\ \implies E[X|Y=y] &= e^{-x-xy}x\left(-y-1\right)^2 \end{aligned} \tag{8}$$

Question 4

- Since we know that

$$\begin{aligned} cov(Z) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \text{mean} = E(Z) = 0 \\ \text{Find vector a and B such that:} \\ Y = a + BZ \text{ where:} \\ \mu = \begin{bmatrix} 3 \\ -2 \end{bmatrix}, \Sigma = \begin{bmatrix} 5 & 13 \\ 13 & 41 \end{bmatrix} \\ E(Y) = \begin{bmatrix} 3 \\ -2 \end{bmatrix} \\ E(Y) = a + BE(Z) \\ \begin{bmatrix} 3 \\ -2 \end{bmatrix} = a + B\begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \implies a = \begin{bmatrix} 3 \\ -2 \end{bmatrix} \end{aligned} \tag{9}$$

Now that we have a, we can find B using the covariance

$$
\begin{aligned}
cov(Y) &= E((Y - E(Y))(Y - E(Y))) \\
&= E((a + BZ - a)(a + BZ - a)') \\
&= E((BZ)(BZ)') \\
&= BB'E(ZZ') = BB' \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
cov(Z) &= E((Z - E(Z))(Z - E(Z)') \\
cov(Z) &= E((Z - 0)(Z - 0)') \\
&= E(ZZ') \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
\text{Let B } &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\
cov(Y) &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}' \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\
&= \begin{bmatrix} a^2 + b^2 & ac + bd \\ ca + db & c^2 + d^2 \end{bmatrix} \\
&= \begin{bmatrix} 5 & 13 \\ 13 & 41 \end{bmatrix} \implies a^2 + b^2 = 5 \\
ac + bd &= 13 \\
c^2 + d^2 &= 41 \\
&\text{Chosen values that will satisfy these equations are:} \\
a = 0, b &= \sqrt{5}, c = \frac{6}{\sqrt{5}}, d = \frac{13}{\sqrt{5}} \\
\implies B &= \begin{bmatrix} 0 & \sqrt{5} \\ \frac{6}{\sqrt{5}} & \frac{13}{\sqrt{5}} \end{bmatrix}
\end{aligned}
\tag{10}
$$