# Question 1

**1a)** The probability of $f_Y(y) = P(Y \leq y)$ is and since $Y = log(X-1)$.

The rule $ln(N) = x \equiv N = e^x$ is also important to note.

$$\begin{aligned} f_Y(y) &= P(Y \leq y) \\ &= P(log(X-1) \leq y) \\ &= P(log(X-1) \leq y) \\ &= P(X-1 \leq e^y) \\ &= P(X \leq e^y + 1) \\ \implies f_Y(y) &= P(X \leq e^y + 1) \end{aligned} \tag{1}$$

Sub this found X value into the existing x function to get:

$$\frac{2}{(e^y+1)^2}$$

Then multiply this function by the derivative of X which is $\frac{d}{dy}(e^y + 1) = e^y$ to get:

$$\frac{2e^y}{(e^y+1)^2}$$

To add bounds to the cases, sub in $x = 2$ into $Y > log(X-1)$ which gives $Y > ln(1), Y > 0$

$$f_y(y) = \begin{cases} \frac{2e^y}{(e^y+1)^2} & y > 0 \\ 0 & otherwise \end{cases} \tag{2}$$

**1b)** To find the quartile function. We need to find the we need to integrate the pdf we just got to get Fx(x).

$$\begin{aligned} \int_x^2 F_X(x) &= \int \frac{2}{x^2} \\ \int_x^2 \frac{2}{x^2} &= \left[-\frac{2}{x}\right]_x^2 \\ &= -\frac{2}{x} - -\frac{2}{2} \\ &= 1 - \frac{2}{x} \\ &\text{Switch q and x to get the} \\ x &= -\frac{2}{q} + 1 \\ x - 1 &= -\frac{2}{q} \\ q(x-1) &= -2 \\ q &= -\frac{2}{x-1} \\ \implies qx(x) &= -\frac{2}{x-1} \end{aligned} \tag{3}$$

**1c)** Randomised code for the pdf function of $f_X(x)$:

```
pdfFunc <- function() {
    n = runif(1, 0, 10)
    if (n <= 2) {
        print(0)
    } else {
        print(2/(n^2))
    }
}
```

# Question 2

**2a)** Expression that the system is working at time $t$:
- Probability density function of components can be expressed as $X_i \sim Exp(1)$ for $i = 1, 2, 3$
- $1 - (1 - P(X \leq t))$ = probability that the system is not working

$$\begin{aligned} P(X_i \leq t) = (P(X_1) \cap P(X_2) \cup (P(X_1) \cap P(X_2)) \cup (P(X_2) \cap P(X_3)) - (P(X_1) \cap P(X_2) \cap P(X_3)) \\ \text{Due to independence} \\ P(X_i \leq t) = (e^{-t} * e^{-t}) - (e^{-t} * e^{-t}) + (e^{-t} * e^{-t}) + (e^{-t} * e^{-t} * e^{-t}) \\ P(X_i \leq t) = e^{-2t} + e^{-2t} + e^{-2t} - e^{-3t} \\ P(X_i \leq t) = 3e^{-2t} - e^{-3t} \\ P(\text{System not working}) = 1 - (1 - (3e^{-2t} - e^{-3t})) \\ P(\text{System not working}) = 3e^{-2t} - e^{-3t} \end{aligned} \tag{4}$$

**2b)** To find the mean/expected value, integrate the value like so $\int_0^\infty t(3e^{-2t} - e^{-3t})dt$.

$$\begin{aligned}
\int_0^\infty t(3e^{-2t} - e^{-3t})dt \\
\text{Can split up two functions} \\
\int_0^\infty 3te^{-2t}dt - \int_0^2 te^{-3t}dt \\
\text{First integral} \\
\int_0^\infty 3te^{-2t}dt \\
u = -2t \\
3\int \frac{e^u u}{4}du \\
3\frac{1}{4}\int e^u u du \\
\text{Apply integration by parts} \\
u = u, v' = e^u \\
\frac{3}{4}\left(e^u u - \int e^u du\right) \\
\frac{3}{4}\left(e^u u - e^u\right), \text{Sub back u} \\
\frac{3}{4}\left(e^{-2t}(-2t) - e^{-2t}\right) \\
\int 3te^{-2t}dt = \frac{3}{4}\left(-2e^{-2t}t - e^{-2t}\right) \\
\text{Second integral} \\
-\int te^{-3t}dt \\
-\int \frac{e^u u}{9}du \\
-\frac{1}{9}\cdot\int e^u u du \\
\text{Apply integration by parts} \\
u = u,\ v' = e^u \\
-\frac{1}{9}\left(e^u u - \int e^u du\right) \\
-\frac{1}{9}\left(e^u u - e^u\right) \\
\text{Sub back u} \\
-\frac{1}{9}\left(e^{-3t}(-3t) - e^{-3t}\right) \\
-\int te^{-3t}dt = -\frac{1}{9}\left(-3e^{-3t}t - e^{-3t}\right) \\
\text{Applying the Sum Rule:} \\
\left[\frac{3}{4}\left(-2e^{-2t}t - e^{-2t}\right) - \frac{1}{9}\left(-3e^{-3t}t - e^{-3t}\right)\right]_0^\infty \\
= 0 - \left(\frac{3}{4}\left(-2e^{-2\,0}\,0 - e^{-2\,0}\right) - \frac{1}{9}\left(-3e^{-3\,0}\cdot\,0 - e^{-3\,0}\right)\right) \\
= 0 - \frac{3}{4}\left(-2e^{-2\,0}\,0 - e^{-2\,0}\right) - \frac{1}{9}\left(-3e^{-3\,0}\,0 - e^{-3\,0}\right) \\
= -\frac{3}{4} + \frac{1}{9} = \frac{23}{36} = \text{Mean}
\end{aligned} \tag{5}$$

To calculate the variance, I took similar steps to calculate $E[X^2]$.
$E[X^2]$ was calculated to be: $\frac{73}{108}$.

To find the variance I used the formula $E[X^2] - (E[X])^2$
$\frac{73}{108} - (\frac{23}{36})^2 = \frac{347}{1296} = 0.26774 = \text{Variance}$

**2c)** Find $P(X_1|P(X_i \geq t))$

$P(A|B) = \frac{A \cap B}{B}$ where A is component 1 working and B is the system is working.

$$= \frac{1 - \left(2e^{-2t} - e^{-3t}\right)}{1 - (3e^{-2t} - e^{-3t})}$$

Then find the limit of $t \to \infty$

$$= \lim_{t \to \infty} \left( \frac{1 - \left(2e^{-2t} - e^{-3t}\right)}{1 - (3e^{-2t} - e^{-3t})} \right)$$

First remove the constants

$$= \frac{2e^{-2t} + e^{-3t}}{-3e^{-2t} + e^{-3t}}$$

Divide by the highest common denominator which is $e^{-2t}$ and gives:

$$\lim_{t \to \infty} \left( \frac{2 + \frac{1}{e^t}}{3 + \frac{1}{e^t}} \right)$$

Substituting the limit, and the fact that $\lim_{t \to \infty} (e^t) \to \infty$ and $\frac{n}{\infty} = 0$

$$= \frac{2}{3}$$

$$\implies \lim_{t \to \infty} \left( \frac{1 - \left(2e^{-2t} - e^{-3t}\right)}{1 - (3e^{-2t} - e^{-3t})} \right) = \frac{2}{3}$$

**2d)**

# Question 3

**3a)** The conditional distribution of Y can be expressed as:

$$P(Y = y|X = x) = \frac{f_{xy}(x, y)}{f_x(X = X)}$$

$$P(Y = y|X = x) = \frac{f_{xy}(x, y)}{f_x(X = X)} = Exp(x)$$

Given P(X = x) = Exp(1)

$$P(Y = y|X = x) = \frac{f_{xy}}{Exp(1)} = Exp(x)$$

$$f_{xy}(X, Y) = Exp(x)Exp(1) = e^{-x}\left(xe^{-yx}\right)$$

Now that we have Fxy(X, Y), we can find the marginal distribution of Y by $F_y(y) = \int_{-\infty}^{\infty} fxy(x, y, )dx$

$$f_y(y) = \int_{-\infty}^{\infty} fxy(x, y, )dx$$
$$= \int_{-\infty}^{\infty} e^{-x}\left(xe^{-yx}\right)dx$$

Because -infinity approaches 0, we can change the bounds to that

$$= \int_{0}^{\infty} e^{-x}\left(xe^{-yx}\right)dx$$
$$= \int e^{-yx-x}xdx$$
$$u = x, v' = e^{-yx-x}$$
$$= e^{-yx-x}\frac{1}{-y-1}x - \int e^{-yx-x}\frac{1}{-y-1}dx$$
$$= e^{-yx-x}\frac{1}{-y-1}x - e^{-yx-x}\frac{1}{(-y-1)^2}$$
$$= \left[e^{-yx-x}\frac{1}{-y-1}x - e^{-yx-x}\frac{1}{(-y-1)^2}\right]_0^{\infty}$$
$$= 0 - \left(-\frac{1}{(-y-1)^2}\right)$$
$$F_y(Y) = \frac{1}{(-y-1)^2}$$

$\implies$ The marginal probability density function of Y = $\frac{1}{(-y-1)^2}$

(6)

**3b)** Now that we have the marginal pdf of Y, we can calculate $E[X|Y = y]$:

$$P(X = x|Y = y) = \frac{f_{xy}(x, y)}{f_y(Y = y)}$$

$$\begin{aligned}
P(X = x|Y = y) &= \frac{f_{xy}(x,y)}{f_y(Y = y)} \\
&= \frac{e^{-x}\left(xe^{-yx}\right)}{\frac{1}{(-y-1)^2}} \\
&= e^{-x-xy}x\left(-y-1\right)^2 \\
\implies E[X|Y = y] &= e^{-x-xy}x\left(-y-1\right)^2
\end{aligned} \tag{7}$$

# Question 4

- Since we know that

$$\begin{aligned}
cov(Z) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \text{mean} = E(Z) = 0 \\
\text{Find vector a and B such that:} \\
Y = a + BZ \text{ where:} \\
\mu = \begin{bmatrix} 3 \\ -2 \end{bmatrix}, \Sigma = \begin{bmatrix} 5 & 13 \\ 13 & 41 \end{bmatrix} \\
E(Y) = \begin{bmatrix} 3 \\ -2 \end{bmatrix} \\
E(Y) = a + BE(Z) \\
\begin{bmatrix} 3 \\ -2 \end{bmatrix} = a + B\begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\implies a = \begin{bmatrix} 3 \\ -2 \end{bmatrix}
\end{aligned} \tag{8}$$

Now that we have a, we can find B using the covariance

$$\begin{aligned}
cov(Y) &= E((Y - E(Y))(Y - E(Y))) \\
&= E((a + BZ - a)(a + BZ - a)') \\
&= E((BZ)(BZ)') \\
&= BB'E(ZZ') = BB' \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
cov(Z) &= E((Z - E(Z))(Z - E(Z)') \\
cov(Z) &= E((Z - 0)(Z - 0)') \\
&= E(ZZ') \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
\text{Let B } &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\
cov(Y) &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}' \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\
&= \begin{bmatrix} a^2 + b^2 & ac + bd \\ ca + db & c^2 + d^2 \end{bmatrix} \\
&= \begin{bmatrix} 5 & 13 \\ 13 & 41 \end{bmatrix} \implies a^2 + b^2 = 5 \\
ac + bd &= 13 \\
c^2 + d^2 &= 41 \\
&\text{Chosen values that will satisfy these equations are:} \\
a = 0, b = \sqrt{5}, c &= \frac{6}{\sqrt{5}}, d = \frac{13}{\sqrt{5}} \\
\implies B &= \begin{bmatrix} 0 & \sqrt{5} \\ \frac{6}{\sqrt{5}} & \frac{13}{\sqrt{5}} \end{bmatrix}
\end{aligned} \tag{9}$$