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1 HW 1

1.1 Problem 1. (5 points). Soft deadline: Oct 19

Task: Choose any function but sphere from the list. Prove that:

- global minimum from the forth column is at least a local minimum by applying the sufficient condition on optimality;
- pick some other point that does not look like a local minimum and prove that the point is not a local minimum by applying the necessary condition.

Solution: Let's consider McCormick function: $f(x, y) = \sin(x + y) + (x - y)^2 - 1.5x + 2.5y + 1$. The search domain is: $-1.5 \leqslant x \leqslant 4$, $-3 \leqslant y \leqslant 4$. The expected minimum is $f(-0.54719, -1.54719) = -1.9133$.

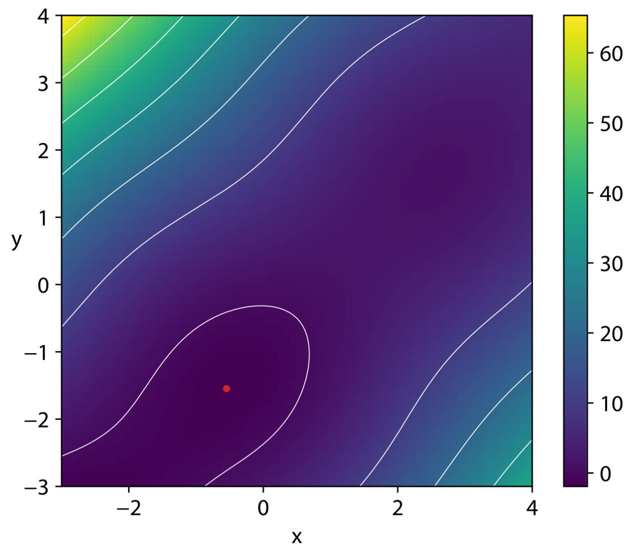

Figure 1: McCormick Function plot

Finding the gradient:

$$\nabla f(x, y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)^T = \begin{pmatrix} cos(x + y) + 2(x - y) - 1.5 \\ cos(x + y) - 2(x - y) + 2.5 \end{pmatrix}$$

Finding the Hessian:

$$H_f(x, y) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} = \begin{pmatrix} -sin(x + y) + 2 & -sin(x + y) - 2 \\ -sin(x + y) - 2 & -sin(x + y) + 2 \end{pmatrix}$$

Part 1: Let's check the sufficient condition on optimality:

1. $\nabla f(x, y) = 0$:

$$\nabla f(x_{min}, y_{min}) = \begin{pmatrix} cos(-0.54719 - 1.54719) + 2(-0.54719 + 1.54719) - 1.5 \\ cos(-0.54719 - 1.54719) - 2(-0.54719 + 1.54719) + 2.5 \end{pmatrix} =$$
$$= \begin{pmatrix} -0.499987 + 2 - 1.5 \\ -0.499987 - 2 + 2.5 \end{pmatrix} = [assuming\ -0.54719 \sim -(2\pi/3 - 1)/2] = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The condition is satisfied.

2. $H_f(x, y) > 0$ (positive definite)

$$H_f(x, y) = \begin{pmatrix} -sin(-0.54719 - 1.54719) + 2 & -sin(-0.54719 - 1.54719) - 2 \\ -sin(-0.54719 - 1.54719) - 2 & -sin(-0.54719 - 1.54719) + 2 \end{pmatrix} =$$
$$= \begin{pmatrix} \frac{\sqrt{3}}{2} + 2 & \frac{\sqrt{3}}{2} - 2 \\ \frac{\sqrt{3}}{2} - 2 & \frac{\sqrt{3}}{2} + 2 \end{pmatrix}$$

Let's find eigenvalues:

$$|H_f - \lambda I| = 0$$
$$\begin{pmatrix} \frac{\sqrt{3}}{2} + 2 - \lambda & \frac{\sqrt{3}}{2} - 2 \\ \frac{\sqrt{3}}{2} - 2 & \frac{\sqrt{3}}{2} + 2 - \lambda \end{pmatrix} = 0$$
$$\left(\frac{\sqrt{3}}{2} + 2 - \lambda\right)^2 - \left(\frac{\sqrt{3}}{2} - 2\right)^2 = 0$$
$$(4 - \lambda)(\sqrt{3} - \lambda) = 0$$
$$\lambda = 4,\ \lambda_2 = \sqrt{3}$$

All the eigenvalues are positive, the matrix is positive definite, so the condition is satisfied.

Part 2: Let's test that some point, e.g. $(x_0,\ y_0) = (\pi,\ \pi)$, doesn't satisfy the necessary condition.

$$\nabla f(x_0, y_0) = \begin{pmatrix} cos(\pi + \pi) + 2(\pi - \pi) - 1.5 \\ cos(\pi + \pi) - 2(\pi - \pi) + 2.5 \end{pmatrix} = \begin{pmatrix} -0.5 \\ 3.5 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The first part of the criterion is not met – then the point is not a local minimum.

1.2 Problem 2 (5 points). Soft deadline: Oct 19

Task: Let's consider the problem:

$$\begin{cases} f(x, y) = 100(y - x^2)^2 + (x - 1)^2 \to \max \\ \text{s.t. } y - x = 3 \end{cases}$$

Find all local minimas by constructing Lagrange function and setting all its partial derivatives to zero.

Solution: Lagrange function is: $L(x, y, \lambda) = 100(y - x^2)^2 + (x - 1)^2 + \lambda(y - x - 3)$
Let's find the derivatives:

$$\frac{\partial L}{\partial x} = 100 \cdot 2(y - x^2) \cdot (-2x) + 2(x - 1) - \lambda = 400x^3 - 400xy + 2x - 2 - \lambda$$
$$\frac{\partial L}{\partial y} = 100 \cdot 2(y - x^2) + \lambda = -200x^2 + 200y + \lambda$$

Thus, we get the following system:

$$\begin{cases} 400x^3 - 400xy + 2x - 2 - \lambda = 0 & (1) \\ -200x^2 + 200y + \lambda & (2) \\ y - x - 3 = 0 & (3) \end{cases}$$

Solving it. Let's substitute (3) to (1) and (2):

$$\begin{cases} 400x^3 - 400x(x+3) + 2x - 2 - \lambda = 0 \\ -200x^2 + 200(x+3) + \lambda = 0 \\ y - x - 3 = 0 \end{cases}$$

$$\begin{cases} 400x^3 - 400x^2 - 1198x - 2 - \lambda = 0 \\ -200x^2 + 200x + 600 + \lambda = 0 \\ y - x - 3 = 0 \end{cases}$$

The sum of (4) and (5):

$$400x^3 - 400x^2 - 1198x - 2 - \lambda - 200x^2 + 200x + 600 + \lambda = 0$$
$$200x^3 - 600x^2 - 998x + 598 = 0$$

[*It seems to be some kind of issue here. If instead of* $(x-1)^2$ *in the original equation there had been* $(x-0.5)^2$*, one of the roots here would have been* 0.5 *which makes the equation solvable analytically.*]

Approximate roots are: $x_{1,2,3} = (-1.3010, 0.49923, 2.3018)$. Corresponding y values are $y_{1,2,3} = (1.6990, 3.49923, 5.3018)$. Let's test, which of them are local maximums and which are minimums. The Hessian is:

$$H_f(x,y) = \begin{pmatrix} 1200x^2 - 400y + 2 & -400x \\ -400x & 200y \end{pmatrix}$$

Finding eigenvalues, solving $|H_f - \theta I| = 0$:

$$(1200x^2 - 400y + 2\theta) \cdot (200y - \theta) - (-400x)^2 = 0$$
$$240000x^2y - 80000y + 400y\theta - 1200x^2\theta + 400y\theta - 2\theta^2 - 160000x^2 = 0$$
$$120000x^2y - 40000y + 400y\theta - 600x^2\theta - \theta^2 - 80000x^2 = 0$$
$$(-1) \cdot \theta^2 + (400y - 600x^2)\theta + 120000x^2y - 40000y - 80000x^2 = 0$$

Substituting extreme points:

1. $(x_1, y_1) = (-1.3010, 1.6990)$:
   The equation is: $\theta^2 + 335.961\theta - 141719.0 = 0$, the roots are $\theta_1 = -580.214$, $\theta_2 = 244.254$

2. $(x_2, y_2) = (0.49923, 3.49923)$:
   The equation is: $\theta^2 - 1250.15\theta - 55253.8 = 0$, the roots are $\theta_1 = 45.8815$, $\theta_2 = 1204.27$

3. $(x_3, y_3) = (2.3018, 5.3018)$:
   The equation is: $\theta^2 + 1058.25\theta - 2734917.9 = 0$, the roots are $\theta_1 = -2265.47$, $\theta_2 = 1207.22$

Thus, the only valid extremal point here is $(x_2, y_2) = (0.49923, 3.49923)$, which corresponds to the local minimum.