Section 1.4

- We found $A_1 = \begin{bmatrix} \frac{5}{2} & \frac{1}{2} \\ \frac{3}{2} & \frac{3}{2} \end{bmatrix}$. We apply the QR factorization to this to find A_2 as follows:

$$R_1 = \begin{bmatrix} \frac{2}{\sqrt{34}} & \cdot \\ \cdot & \cdot \end{bmatrix}$$

$A_1 = [a_1 \ a_2]$

Using the Gram schmidt process:

$$v_1 = a_1 = \begin{bmatrix} \frac{5}{2} \\ \frac{3}{2} \end{bmatrix}. \qquad q_1 = \frac{1}{\sqrt{\left(\frac{5}{2}\right)^2 + \left(\frac{3}{2}\right)^2}} \begin{bmatrix} \frac{5}{2} \\ \frac{3}{2} \end{bmatrix} = \frac{2}{\sqrt{34}} \begin{bmatrix} \frac{5}{2} \\ \frac{3}{2} \end{bmatrix} = \begin{bmatrix} \frac{5}{\sqrt{34}} \\ \frac{3}{\sqrt{34}} \end{bmatrix}$$

$$v_2 = a_2 - (a_2 \cdot q_1) \cdot \begin{bmatrix} \frac{5}{2} \\ \frac{3}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} \\ \frac{3}{2} \end{bmatrix} - \left(\begin{bmatrix} \frac{1}{2} \\ \frac{3}{2} \end{bmatrix} \cdot \begin{bmatrix} \frac{5}{\sqrt{34}} \\ \frac{3}{\sqrt{34}} \end{bmatrix} \right) \cdot \begin{bmatrix} \frac{5}{\sqrt{34}} \\ \frac{3}{\sqrt{34}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{3}{2} \end{bmatrix} - \left(\frac{5}{2\sqrt{34}} + \frac{9}{2\sqrt{34}} \right) \cdot \begin{bmatrix} \frac{5}{\sqrt{34}} \\ \frac{3}{\sqrt{34}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} \\ \frac{3}{2} \end{bmatrix} - \frac{7\sqrt{34}}{34} \begin{bmatrix} \frac{5}{\sqrt{34}} \\ \frac{3}{\sqrt{34}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{3}{2} \end{bmatrix} - \begin{bmatrix} \frac{35}{34} \\ \frac{21}{34} \end{bmatrix} = \begin{bmatrix} -\frac{9}{17} \\ \frac{15}{17} \end{bmatrix}$$

$$q_2 = \frac{1}{\sqrt{\left(-\frac{9}{17}\right)^2 + \left(\frac{15}{17}\right)^2}} \begin{bmatrix} -\frac{9}{17} \\ \frac{15}{17} \end{bmatrix} = \frac{\sqrt{34}}{6} \begin{bmatrix} -\frac{9}{17} \\ \frac{15}{17} \end{bmatrix} = \begin{bmatrix} -\frac{3}{\sqrt{34}} \\ \frac{5}{\sqrt{34}} \end{bmatrix}$$

$$R_1 = \begin{bmatrix} \frac{2}{\sqrt{34}} & \frac{7}{\sqrt{34}} \\ 0 & \frac{\sqrt{34}}{6} \end{bmatrix}$$

$$a_1 \cdot q_2 = \begin{bmatrix} \frac{5}{2} \\ \frac{3}{2} \end{bmatrix} \cdot \begin{bmatrix} -\frac{3}{\sqrt{34}} \\ \frac{5}{\sqrt{34}} \end{bmatrix}$$

$$= \frac{-15}{2\sqrt{34}} + \frac{15}{2\sqrt{34}} = 0$$

$$Q_1 = \begin{bmatrix} \frac{5}{\sqrt{34}} & -\frac{3}{\sqrt{34}} \\ \frac{3}{\sqrt{34}} & \frac{5}{\sqrt{34}} \end{bmatrix}$$

$$A_2 = R_1 Q_1 = \begin{bmatrix} \frac{2}{\sqrt{34}} & \frac{7}{\sqrt{34}} \\ 0 & \frac{\sqrt{34}}{6} \end{bmatrix} \cdot \begin{bmatrix} \frac{2}{\sqrt{34}} & \frac{-3}{\sqrt{34}} \\ \frac{3}{\sqrt{34}} & \frac{5}{\sqrt{34}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{10}{34} + \frac{21}{34} & -\frac{6}{34} + \frac{35}{34} \\ 0 + \frac{3}{6} & \frac{5}{6} + 0 \end{bmatrix} = \begin{bmatrix} \frac{31}{34} & \frac{29}{34} \\ \frac{1}{2} & \frac{5}{6} \end{bmatrix}$$

- Applying QR factorization to A_2:

$$A_2 = \begin{bmatrix} \frac{31}{34} & \frac{29}{34} \\ \frac{1}{2} & \frac{5}{6} \end{bmatrix}$$

$$v_1 = \begin{bmatrix} \frac{31}{34} \\ \frac{1}{2} \end{bmatrix}, \quad q_1 = \frac{1}{\sqrt{(\frac{31}{34})^2 + (\frac{1}{2})^2}} \begin{bmatrix} \frac{31}{34} \\ \frac{1}{2} \end{bmatrix} = \frac{17\sqrt{2}}{25} \begin{bmatrix} \frac{31}{34} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{31\sqrt{2}}{50} \\ \frac{17\sqrt{2}}{50} \end{bmatrix}$$

$$v_2 = a_2 - \frac{a_2 \cdot v_1}{\|v_1\|^2} v_1 = \begin{bmatrix} \frac{29}{34} \\ \frac{5}{6} \end{bmatrix} - \frac{\begin{bmatrix} \frac{29}{34} \\ \frac{5}{6} \end{bmatrix} \cdot \begin{bmatrix} \frac{31}{34} \\ \frac{1}{2} \end{bmatrix}}{\left(\frac{25}{17\sqrt{2}}\right)^2} \cdot \begin{bmatrix} \frac{31}{34} \\ \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{29}{34} \\ \frac{5}{6} \end{bmatrix} - \frac{2071}{1875} \begin{bmatrix} \frac{31}{34} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{29}{34} \\ \frac{5}{6} \end{bmatrix} - \begin{bmatrix} 1.007 \\ 0.552 \end{bmatrix}$$

$$= \begin{bmatrix} -0.154 \\ 0.281 \end{bmatrix}$$

$$q_2 = \frac{1}{\sqrt{(0.154)^2 + (0.281)^2}} \cdot \begin{bmatrix} -0.154 \\ 0.281 \end{bmatrix} = 3.121 \begin{bmatrix} -0.154 \\ 0.281 \end{bmatrix} = \begin{bmatrix} -0.481 \\ 0.877 \end{bmatrix}$$

$$Q_2 = \begin{bmatrix} \frac{31\sqrt{2}}{50} & -\frac{17\sqrt{2}}{50} \\ \frac{17\sqrt{2}}{50} & \frac{31\sqrt{2}}{50} \end{bmatrix} \qquad R_2 = \begin{bmatrix} \frac{25}{17\sqrt{2}} & 1.15 \\ 0 & 3.12 \end{bmatrix}$$

$$a_2 \cdot q_1 = \begin{pmatrix} \frac{29}{34} \\ \frac{5}{6} \end{pmatrix} \cdot \begin{pmatrix} \frac{31\sqrt{2}}{50} \\ \frac{17\sqrt{2}}{50} \end{pmatrix} = 1.149$$

$$a_1 \cdot q_2 = \begin{bmatrix} \frac{31}{34} \\ \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} -0.481 \\ 0.877 \end{bmatrix} \approx 0$$

$$A_3 = R_2 Q_2 = \begin{bmatrix} \frac{25}{17\sqrt{2}} & 1.15 \\ 0 & 3.12 \end{bmatrix} \begin{bmatrix} \frac{31\sqrt{2}}{50} & -\frac{17\sqrt{2}}{50} \\ \frac{17\sqrt{2}}{50} & \frac{31\sqrt{2}}{50} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{31}{34} + 0.553 & -\frac{1}{2} + 1.01 \\ 0 + 1.50 & 0 + 2.736 \end{bmatrix} = \begin{bmatrix} 1.46 & 0.51 \\ 1.50 & 2.736 \end{bmatrix}$$

$$\therefore A_3 = \begin{bmatrix} 1.46 & 0.51 \\ 1.50 & 2.736 \end{bmatrix}$$

• Applying QR-factorization to A_3:

$$v_1 = \begin{bmatrix} 1.46 \\ 1.50 \end{bmatrix} \qquad q_1 = \frac{1}{\sqrt{(1.46)^2+(1.50)^2}} \begin{bmatrix} 1.46 \\ 1.50 \end{bmatrix} = 0.478 \begin{bmatrix} 1.46 \\ 1.50 \end{bmatrix} = \begin{bmatrix} 0.697 \\ 0.717 \end{bmatrix}$$

$$v_2 = a_2 - \frac{a_2 \cdot v_1}{\|v_1\|^2} \cdot v_1$$

$$= \begin{bmatrix} 0.51 \\ 2.736 \end{bmatrix} - \frac{\begin{bmatrix} 0.51 \\ 2.736 \end{bmatrix} \cdot \begin{bmatrix} 1.46 \\ 1.50 \end{bmatrix}}{\left(\frac{1}{0.478}\right)^2} \cdot \begin{bmatrix} 1.46 \\ 1.50 \end{bmatrix}$$

$$= \begin{bmatrix} 0.51 \\ 2.736 \end{bmatrix} - 1.11 \begin{bmatrix} 1.46 \\ 1.50 \end{bmatrix} = \begin{bmatrix} 0.51 - 1.621 \\ 2.736 - 1.665 \end{bmatrix} = \begin{bmatrix} -1.111 \\ 1.071 \end{bmatrix}$$

$$q_2 = \frac{1}{\sqrt{(1.111)^2+(1.071)^2}} \begin{bmatrix} -1.111 \\ 1.071 \end{bmatrix} = 0.648 \begin{bmatrix} -1.111 \\ 1.071 \end{bmatrix} = \begin{bmatrix} -0.7199 \\ 0.694 \end{bmatrix}$$

$$Q_3 = \begin{bmatrix} 0.697 & -0.7196 \\ 0.717 & 0.694 \end{bmatrix} \qquad R_3 = \begin{bmatrix} 2.09 & 2.317 \\ 0 & 1.543 \end{bmatrix}$$

$$a_2 \cdot q_1 = \begin{bmatrix} 0.51 \\ 2.736 \end{bmatrix} \cdot \begin{bmatrix} 0.697 \\ 0.717 \end{bmatrix} = 2.317$$

$$a_1 \cdot q_2 = \begin{bmatrix} 1.46 \\ 1.50 \end{bmatrix} \cdot \begin{bmatrix} -0.7196 \\ 0.694 \end{bmatrix} \approx 0$$

$$A_4 = R_3 Q_3 = \begin{bmatrix} 2.09 & 2.32 \\ 0 & 1.54 \end{bmatrix} \begin{bmatrix} 0.70 & -0.72 \\ 0.72 & 0.70 \end{bmatrix}$$

$$\therefore A_4 = \begin{bmatrix} 3.13 & 0.1192 \\ 1.11 & 1.08 \end{bmatrix}$$

- Lastly, to find R_4 & Q_4, we apply the QR factorization to A_4:

$$v_1 = \begin{bmatrix} 3.13 \\ 1.11 \end{bmatrix} \qquad q_1 = \frac{1}{\sqrt{(3.13)^2+(1.11)^2}} \begin{bmatrix} 3.13 \\ 1.11 \end{bmatrix} = 0.30 \begin{bmatrix} 3.13 \\ 1.11 \end{bmatrix} = \begin{bmatrix} 0.942 \\ 0.333 \end{bmatrix}$$

$$v_2 = a_2 - \frac{a_2 \cdot v_1}{\|v_1\|^2} v_1 = \begin{bmatrix} 0.1192 \\ 1.08 \end{bmatrix} - \frac{\begin{bmatrix} 3.13 \\ 1.11 \end{bmatrix} \cdot \begin{bmatrix} 0.1192 \\ 1.08 \end{bmatrix}}{(3.13)^2+(1.11)^2} \begin{bmatrix} 3.13 \\ 1.11 \end{bmatrix}$$

$$= \begin{bmatrix} 0.1192 \\ 1.08 \end{bmatrix} - 0.143 \begin{bmatrix} 3.13 \\ 1.11 \end{bmatrix} = \begin{bmatrix} -0.328 \\ 0.921 \end{bmatrix}$$

$$q_2 = \frac{1}{\sqrt{(0.328)^2+(0.921)^2}} \begin{bmatrix} -0.328 \\ 0.921 \end{bmatrix} = 1.022 \begin{bmatrix} -0.328 \\ 0.921 \end{bmatrix} = \begin{bmatrix} -0.335 \\ 0.941 \end{bmatrix}$$

$$Q_4 = \begin{bmatrix} 0.94 & -0.33 \\ 0.33 & 0.94 \end{bmatrix} \qquad R_4 = \begin{bmatrix} 3.33 & 0.469 \\ 0 & 0.978 \end{bmatrix}$$

$$a_2 \cdot q_1 = \begin{bmatrix} 0.1192 \\ 1.08 \end{bmatrix} \cdot \begin{bmatrix} 0.942 \\ 0.33 \end{bmatrix} = 0.469$$

$$A_5 = \begin{bmatrix} 3.33 & 0.469 \\ 0 & 0.978 \end{bmatrix} \begin{bmatrix} 0.94 & -0.33 \\ 0.33 & 0.94 \end{bmatrix} = \begin{bmatrix} 3.28 & -0.658 \\ 0.323 & 0.919 \end{bmatrix}$$

We notice that with increasing value for k, the diagonals of A_k converge to the eigenvalues of A. In this case, the diagonals are approaching $\lambda_1 = 3$ and $\lambda_2 = 1$.