# 1.8 Using a Shift

Shift the eigenvalues of the matrix in problem 1.7 by replacing A with $B = A + 0.9I$. Apply the QR algorithm to $B = A + 0.9I$ and then shift back by subtracting 0.9 from the (approximate) eigenvalues of B.

Prove that if $B = A + \alpha I$ then $\lambda$ is an eigenvalue of A if and only if $\lambda + \alpha$ is an eigenvalue of B.

Step 1. Find eigenvalues

$$A = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix} \quad \det(A - \lambda I) = 0$$

$$\det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 3 \\ -1 & -2-\lambda \end{bmatrix}$$

$$= (2-\lambda)(-2-\lambda) + 3$$
$$= (\lambda^2 - 4) + 3$$
$$= \lambda^2 - 1$$

$$\lambda^2 - 1 = 0$$
$$\lambda_1 = 1$$
$$\lambda_2 = -1$$

Step 2. Replace A with $B = A + 0.9I$

$$B = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix} + \begin{bmatrix} 0.9 & 0 \\ 0 & 0.9 \end{bmatrix} = \begin{bmatrix} 2.9 & 3 \\ -1 & -1.1 \end{bmatrix}$$

Step 3. Find eigenvalues of B

$\lambda_1 = 1$ $\quad 1 + 0.9 = 1.9$ $\quad$ so eigenvalues of B $\quad \lambda_1 = 1.9$

$\lambda_2 = -1$ $\quad -1 + 0.9 = -0.1$ $\quad \lambda_2 = -0.1$

Step 4. Apply the QR algorithm to $B = A + 0.9I$

$$B_0 = \begin{bmatrix} 2.9 & 3 \\ -1 & -1.1 \end{bmatrix} \quad b_1 = \begin{bmatrix} 2.9 \\ -1 \end{bmatrix} \quad b_2 = \begin{bmatrix} 3 \\ -1.1 \end{bmatrix}$$

$B_1 = Q_1 R_1$

$$v_1 = b_1 = \begin{bmatrix} 2.9 \\ -1 \end{bmatrix}$$

$$q_1 = \frac{1}{\sqrt{(2.9)^2 + (-1)^2}} \begin{bmatrix} 2.9 \\ -1 \end{bmatrix}$$

$$q_1 = \frac{1}{3.068} \begin{bmatrix} 2.9 \\ -1 \end{bmatrix}$$

$$q_1 = \begin{bmatrix} 0.9452 \\ 0.3259 \end{bmatrix}$$

Continued...

$v_2 = b_2 - (b_2 \cdot q_1) q_1 \longrightarrow (b_2 \cdot q_1) = \begin{bmatrix} 3 \\ -1.1 \end{bmatrix} \cdot \begin{bmatrix} 0.9452 \\ 0.3259 \end{bmatrix}$

$v_2 = \begin{bmatrix} 3 \\ -1.1 \end{bmatrix} - (2.478) \begin{bmatrix} 0.9452 \\ 0.3259 \end{bmatrix}$

$= (2.836 - 0.358)$

$= 2.478$

$v_2 = \begin{bmatrix} 3 \\ -1.1 \end{bmatrix} - \begin{bmatrix} 2.34 \\ 0.81 \end{bmatrix}$

$Q_1 = \begin{bmatrix} 0.95 & 0.92 \\ 0.33 & -0.4 \end{bmatrix}$

$v_2 = \begin{bmatrix} 0.66 \\ -0.29 \end{bmatrix}$

$R_1 = \begin{bmatrix} 3.07 & 2.478 \\ 0 & 0.721 \end{bmatrix}$

$q_2 = \dfrac{1}{\sqrt{(0.66)^2 + (-0.29)^2}} \begin{bmatrix} 0.66 \\ -0.29 \end{bmatrix}$

$q_2 = \dfrac{1}{\sqrt{0.5197}} \begin{bmatrix} 0.66 \\ -0.29 \end{bmatrix}$

$A_2 = R_1 Q_1$

$A_2 = \begin{bmatrix} 3.73 & 1.88 \\ 0.24 & -0.29 \end{bmatrix}$

$q_2 = \dfrac{1}{0.721} \begin{bmatrix} 0.66 \\ -0.29 \end{bmatrix}$

$q_2 = \begin{bmatrix} 0.92 \\ -0.4 \end{bmatrix}$

$\hookrightarrow$ approximate eigenvalues of $A_2$ $\quad \lambda_1 = 3.73$

$\lambda_2 = -0.29$

Step 5. Shift back - subtract 0.9 from approximate eigenvalues of B

$\hookrightarrow$ lots of rounding

$3.73 - 0.9 = 2.83 \approx 2.9$

$-0.29 - 0.9 = -1.19 \approx -1.1$

$B = \begin{bmatrix} 2.9 & 3 \\ -1 & -1.1 \end{bmatrix}$

$\hookrightarrow$ matches with diagonal elements of B

Step 6. proof

Continued...

Suppose $v$ is an eigenvector of $A$ with respect to the eigenvalue $\lambda$

$$Av = \lambda v$$

$\rightarrow$ consider: $B = A + \alpha I$

$$Bv = (A + \alpha I)v = Av + \alpha Iv = \lambda v + \alpha Iv = (\lambda + \alpha)v$$

$\therefore$ This proves that $v$ is an eigenvector of $B$ corresponding to eigenvalue $\lambda + \alpha$

$$Bv = (\lambda + \alpha)v$$

$$(A + \alpha I)v = (\lambda + \alpha)v$$

$$Av + \alpha v = (\lambda + \alpha)v$$

$$Av = \lambda v$$

$\therefore$ $v$ is an eigenvector of $A$ related with an eigenvalue of $\lambda$

This proves the proposition