1.7 Another Example

Apply the QR algorithm to the matrix $A = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$ What happens, why?

$$A_0 = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$$

$$a_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \quad a_2 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

$$A_1 = Q_1 R_1$$

$$v_1 = a_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$q_1 = \frac{1}{\sqrt{(2)^2 + (-1)^2}} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$q_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$q_1 = \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix}$$

$$(a_2 \cdot q_1) = \begin{bmatrix} 3 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix}$$

$$= (6/\sqrt{5} + 2/\sqrt{5})$$

$$= (8/\sqrt{5})$$

$$v_2 = a_2 - (a_2 \cdot q_1) q_1$$

$$v_2 = \begin{bmatrix} 3 \\ -2 \end{bmatrix} - (8/\sqrt{5}) \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 3 \\ -2 \end{bmatrix} - \begin{bmatrix} 16/5 \\ -8/5 \end{bmatrix}$$

$$\frac{15}{5} - \frac{16}{5} = -\frac{1}{5}$$

$$-\frac{10}{5} + \frac{8}{5} = -\frac{2}{5}$$

$$v_2 = \begin{bmatrix} -1/5 \\ -2/5 \end{bmatrix}$$

$$Q_1 = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ -1/\sqrt{5} & -2/\sqrt{5} \end{bmatrix}$$

$$q_2 = \frac{1}{\sqrt{(-1/5)^2 + (-2/5)^2}} \begin{bmatrix} -1/5 \\ -2/5 \end{bmatrix}$$

$$R_1 = \begin{bmatrix} \sqrt{5} & 8/\sqrt{5} \\ 0 & \sqrt{5}/5 \end{bmatrix}$$

$$q_2 = \frac{5}{\sqrt{5}} \begin{bmatrix} -1/5 \\ -2/5 \end{bmatrix}$$

$$q_2 = \begin{bmatrix} -1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix}$$

Continued...

$$A_2 = R_1 Q_1$$

$$A_2 = \begin{bmatrix} \sqrt{5} & 8/\sqrt{5} \\ 0 & \sqrt{5}/5 \end{bmatrix} \cdot \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ -1/\sqrt{5} & -2/\sqrt{5} \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0.4 & -4.2 \\ -0.2 & -0.4 \end{bmatrix}$$

$$A_2 = Q_2 R_2 \qquad a_1 = \begin{bmatrix} 0.4 \\ -0.2 \end{bmatrix} \qquad a_2 = \begin{bmatrix} -4.2 \\ -0.4 \end{bmatrix}$$

$$v_1 = a_1 = \begin{bmatrix} 0.4 \\ -0.2 \end{bmatrix}$$

$$q_1 = \frac{1}{\sqrt{(0.4)^2 + (-0.2)^2}} \begin{bmatrix} 0.4 \\ -0.2 \end{bmatrix}$$

$$q_1 = \frac{1}{0.447} \begin{bmatrix} 0.4 \\ -0.2 \end{bmatrix}$$

$$q_1 = \begin{bmatrix} 0.895 \\ -0.447 \end{bmatrix}$$

$$(a_2 \cdot q_1) = \begin{bmatrix} -4.2 \\ -0.4 \end{bmatrix} \cdot \begin{bmatrix} 0.895 \\ -0.447 \end{bmatrix}$$

$$= (-3.759 + 0.179)$$

$$= (-3.58)$$

$$v_2 = a_2 - (a_2 \cdot q_1) q_1$$

$$v_2 = \begin{bmatrix} -4.2 \\ -0.4 \end{bmatrix} - (-3.58) \begin{bmatrix} 0.895 \\ -0.447 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} -4.2 \\ -0.4 \end{bmatrix} - \begin{bmatrix} -3.204 \\ 1.6003 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} -0.996 \\ -2 \end{bmatrix}$$

Continued...

$$q_2 = \frac{1}{\sqrt{(-0.996)^2 + (-2)^2}} \begin{bmatrix} -0.996 \\ -2 \end{bmatrix}$$

$$q_2 = \frac{1}{\sqrt{1+4}} \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

$$q_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

$$q_2 = \begin{bmatrix} -1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix}$$

$$q_2 = \begin{bmatrix} -0.447 \\ -0.894 \end{bmatrix}$$

$$Q_2 = \begin{bmatrix} 0.895 & -0.447 \\ -0.447 & -0.894 \end{bmatrix}$$

$$R_2 = \begin{bmatrix} 0.447 & -3.58 \\ 0 & 2.236 \end{bmatrix}$$

$$A_3 = R_2 Q_2$$

$$A_3 = \begin{bmatrix} 0.447 & -3.58 \\ 0 & 2.236 \end{bmatrix} \cdot \begin{bmatrix} 0.895 & -0.447 \\ -0.447 & -0.894 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$$

... continued iterations in python and kept repeating back and forth

∴ What happened: The iterates alternate between two matrices without converging to a triangular or diagonal form.

Why: The eigenvalues are 1 and -1 which means they have the same absolute value but different signs. The QR algorithm can struggle because these eigenvalues may cancel out rotational progress leading to a cyclic pattern. As well as, for non-symmetric matrices the QR algorithm is not guaranteed to converge without additional modifications, (e.g. shifts or Hessenburg reduction).