2.1) a) $A = U\Sigma V^T$

$U = \begin{bmatrix} u_1 & u_2 & \cdots & u_r & u_{r+1} & \cdots & u_m \end{bmatrix}$

$\Sigma = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$

$V = \begin{bmatrix} v_1 & v_2 & \cdots & v_r & v_{r+1} & \cdots & v_n \end{bmatrix}$

$D = \text{diag}(\sigma_1, \ldots, \sigma_r)$

rewriting Σ with D, and A being an $m \times n$ matrix,

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & \cdots & & 0 \\ 0 & \sigma_2 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & & \vdots \\ \vdots & \vdots & & \sigma_r & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} u_1 & u_2 & \cdots & u_r & u_{r+1} & \cdots & u_m \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & \cdots & & 0 \\ 0 & \sigma_2 & \cdots & & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ & & & \sigma_r & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_r \\ v_{r+1} \\ \vdots \\ v_n \end{bmatrix}$$

$$= \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \cdots + \sigma_r u_r v_r^T$$

2.1) b)

$$A = \begin{bmatrix} -2 & 2 \\ -1 & 1 \\ 2 & -2 \end{bmatrix} \quad A^TA = \begin{bmatrix} -2 & -1 & 2 \\ 2 & 1 & -2 \end{bmatrix}\begin{bmatrix} -2 & 2 \\ -1 & 1 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix}$$

$$\det(A^TA - \lambda I) = \det\begin{bmatrix} 9-\lambda & -9 \\ -9 & 9-\lambda \end{bmatrix} = (9-\lambda)(9-\lambda) - 81$$

$$= \lambda^2 - 18\lambda \qquad \lambda_1 = 18, \ \lambda_2 = 0$$

$$(A^TA - 18I)v = 0 \Rightarrow \begin{bmatrix} -9 & -9 \\ -9 & -9 \end{bmatrix} v = 0 \Rightarrow v_1 + v_2 = 0 \quad v_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$(A^TA)v = 0 \Rightarrow \begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix} v = 0 \Rightarrow v_1 - v_2 = 0 \quad v_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$V = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \quad \Sigma = \begin{bmatrix} \sqrt{18} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{\sqrt{18}}\begin{bmatrix} -2 & 2 \\ -1 & 1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} = \frac{1}{\sqrt{18}}\begin{bmatrix} -4/\sqrt{2} \\ -2/\sqrt{2} \\ 4/\sqrt{2} \end{bmatrix} = \begin{bmatrix} -4/\sqrt{36} \\ -2/\sqrt{36} \\ 4/\sqrt{36} \end{bmatrix}$$

$$x \cdot u_1 = 0 \Rightarrow -4x_1 - 2x_2 + 4x_3 = 0 \qquad \text{Let } x_2 = 0 \Rightarrow u_2 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

$$x \cdot u_1 = 0 \ \& \ x \cdot u_2 = 0 \Rightarrow u_3 \qquad -4x_1 - 2x_2 + 4x_3 = 0$$

$$1x_1 + x_3 = 0$$

2.1) b)

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ -4 & -2 & 4 & 0 \end{array}\right] \quad R_3 \leftarrow R_3 + 4R_1 \quad \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & -2 & 8 & 0 \end{array}\right]$$

$x_1 = -x_3$

$x_2 = 4x_3$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} = w_3 \qquad u_3 = \frac{w_3}{\|w_3\|} = \begin{bmatrix} -1/\sqrt{18} \\ 4/\sqrt{18} \\ 1/\sqrt{18} \end{bmatrix}$$

$A = U \Sigma V^T$

$$A = \begin{bmatrix} -4/6 & 1/\sqrt{2} & -1/\sqrt{18} \\ -2/6 & 0 & 4/\sqrt{18} \\ 4/6 & 1/\sqrt{2} & 1/\sqrt{18} \end{bmatrix} \begin{bmatrix} \sqrt{18} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$A = \sigma_1 u_1 v_1^T + \dots + \sigma_r u_r v_r^T = \sigma_1 u_1 v_1^T$$

$$= \sqrt{18} \begin{bmatrix} -4/6 \\ -2/6 \\ 4/6 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ -1 & 1 \\ 2 & -2 \end{bmatrix}$$