Section 1.2

$A = \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix}$ Let $a_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $a_2 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$

We use Gramschmidt to find $q_1, q_2$:

- $v_1 = a_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $W = \text{span}\{q_1\}$

$q_1 = \frac{1}{\sqrt{1^2+1^2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$

- $R = \begin{bmatrix} \sqrt{2} & \cdot \\ \cdot & \cdot \end{bmatrix}$

- $v_2 = a_2 - \text{proj}_W a_2$

$= \begin{bmatrix} 0 \\ 3 \end{bmatrix} - \frac{\begin{bmatrix} 0 \\ 3 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$a_2 \cdot q_1 = \begin{bmatrix} 0 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{3}{\sqrt{2}}$

- $R = \begin{bmatrix} \sqrt{2} & \frac{3}{\sqrt{2}} \\ \cdot & \cdot \end{bmatrix}$

$= \begin{bmatrix} 0 \\ 3 \end{bmatrix} - \frac{3}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} 0 - \frac{3}{2} \\ 3 - \frac{3}{2} \end{bmatrix} = \begin{bmatrix} -\frac{3}{2} \\ \frac{3}{2} \end{bmatrix}$

$q_2 = \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{(\frac{3}{2})^2 + (\frac{3}{2})^2}} \begin{bmatrix} -\frac{3}{2} \\ \frac{3}{2} \end{bmatrix}$

- $R = \begin{bmatrix} \sqrt{2} & \frac{3}{\sqrt{2}} \\ \cdot & \frac{3}{\sqrt{2}} \end{bmatrix}$

$= \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$

$a_1 \cdot q_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$

$= -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = 0$

$A_1 = RQ$

$= \begin{bmatrix} \sqrt{2} & \frac{3}{\sqrt{2}} \\ 0 & \frac{3}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$

- $R = \begin{bmatrix} \sqrt{2} & \frac{3}{\sqrt{2}} \\ 0 & \frac{3}{\sqrt{2}} \end{bmatrix}$

$= \begin{bmatrix} 1 + \frac{3}{2} & -1 + \frac{3}{2} \\ \frac{3}{2} & \frac{3}{2} \end{bmatrix} = \begin{bmatrix} \frac{5}{2} & \frac{1}{2} \\ \frac{3}{2} & \frac{3}{2} \end{bmatrix}$ $\therefore A_1 = \begin{bmatrix} \frac{5}{2} & \frac{1}{2} \\ \frac{3}{2} & \frac{3}{2} \end{bmatrix}$

- $A = \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix}$

$$\det \begin{vmatrix} 1-\lambda & 0 \\ 1 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) = 0$$

$\lambda_2 = 1$ and $\lambda_1 = 3$ $\rightarrow$ eigenvalues of $A$

- $\det(A_1 - \lambda I) = \begin{vmatrix} \frac{5}{2}-\lambda & \frac{1}{2} \\ \frac{3}{2} & \frac{3}{2}-\lambda \end{vmatrix} = \left(\frac{5}{2}-\lambda\right)\left(\frac{3}{2}-\lambda\right) - \frac{3}{4} = 0$

$\Rightarrow 4\left(\frac{5}{2}-\lambda\right)\left(\frac{3}{2}-\lambda\right) - 3 = 0$

$\Rightarrow (10 - 4\lambda)\left(\frac{3}{2}-\lambda\right) - 3 = 0$

$\Rightarrow 15 - 10\lambda - 6\lambda + 4\lambda^2 - 3 = 0$

$\Rightarrow 4\lambda^2 - 16\lambda + 12 = 0$ $\quad \Rightarrow \lambda_1 = 3$ and $\lambda_2 = 1$ $\rightarrow$ eigenvalues of $A_1$

$\therefore$ Hence, $A$ and $A_1$ have the same eigenvalues.