1.10.1 The 2x2 Matrix Q

Showing that the standard matrix Q of the reflection in the line through the origin in the direction of $u^\perp$ is $Q = \begin{bmatrix} 1-2d_1^2 & -2d_1d_2 \\ -2d_1d_2 & 1-2d_2^2 \end{bmatrix} = I - 2uu^T$

- The reflection of x in the line perpendicular to x−y: $u = \frac{x-y}{\|x-y\|}$, where u is a unit vector in the direction of x−y.

... reflection of x across the line: $Q(x) = x - 2\,proj_u(x)$

... $[proj_u(x) = (u^T x)u] \rightarrow Q(x) = x - 2(u^T x)u$

(since u is a unit vector $\|u_i\| = 1$ for all $i = 1 \dots n$)

$Q(x) = x - 2(u^T x)u$ must be true for all x

$\therefore Q = I - 2uu^T$ where I is 2x2.

... if $u = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \rightarrow uu^T = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \begin{bmatrix} d_1 & d_2 \end{bmatrix} = \begin{bmatrix} d_1^2 & d_1d_2 \\ d_1d_2 & d_2^2 \end{bmatrix}$

↳ $Q = I - 2uu^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 2\begin{bmatrix} d_1^2 & d_1d_2 \\ d_1d_2 & d_2^2 \end{bmatrix}$

↳ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2d_1^2 & 2d_1d_2 \\ 2d_1d_2 & 2d_2^2 \end{bmatrix} = \begin{bmatrix} 1-d_1^2 & -2d_1d_2 \\ -2d_1d_2 & 1-d_2^2 \end{bmatrix}$ //

Q reflects x across the line perpendicular to x−y $\therefore$ Q is orthogonal

$Q^TQ = (I-2uu^T)^T(I-2uu^T) = I^T - 2(u^T)^T u^T (I-2uu^T)$

$= I - 2uu^T(I-2uu^T) = I - 4uu^T + 4(uu^T)(uu^T)$

... A unit vector multiplied by itself does not change the product $\therefore (uu^T)^2 = uu^T$ & uu^T is symmetric $(uu^T = I)$, thus Q is orthogonal

↳ $Qx = x - 2(u^Tx)u$, which reflects x

$\therefore \begin{bmatrix} 1-2d_1^2 & -2d_1d_2 \\ -2d_1d_2 & 1-2d_2^2 \end{bmatrix} = I - uu^T = Q$ $\boxed{///}$

1.10.2 An example of Q

Computing Q for a.) $u = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$, b.) $x = \begin{bmatrix} 5 \\ 5 \end{bmatrix}, y = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$

Using $Q = I - 2uu^T$ (the Householder matrix)

a.) $\|u\| = \sqrt{(3/5)^2 + (4/5)^2} = \sqrt{\frac{9+16}{25}} = \sqrt{\frac{25}{25}} = \sqrt{1} = 1$

$\therefore$ u is a unit vector ... $Q = I - 2uu^T$

$\hookrightarrow uu^T = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix} \begin{bmatrix} 3/5 & 4/5 \end{bmatrix} = \begin{bmatrix} 9/25 & 12/25 \\ 12/25 & 16/25 \end{bmatrix}$

$Q = I - 2uu^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 2 \begin{bmatrix} 9/25 & 12/25 \\ 12/25 & 16/25 \end{bmatrix}$

$\hookrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 18/25 & 24/25 \\ 24/25 & 32/25 \end{bmatrix} = \begin{bmatrix} 25 - 18/25 & -24/25 \\ -24/25 & 25 - 32/25 \end{bmatrix} = \boxed{\begin{bmatrix} 7/25 & -24/25 \\ -24/25 & -7/25 \end{bmatrix}} \to Q$

b.) $x = \begin{bmatrix} 5 \\ 5 \end{bmatrix}, y = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$ ... $u = \frac{x-y}{\|x-y\|} = \frac{1}{\|x-y\|} x-y$

$\hookrightarrow \frac{1}{\sqrt{20}} \begin{bmatrix} 4 \\ -2 \end{bmatrix} = \frac{1}{2\sqrt{5}} \begin{bmatrix} 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix}$ & $\|u\| = 1$

$\therefore Q = I - 2uu^T \to uu^T = \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 4/5 & -2/5 \\ -2/5 & 1/5 \end{bmatrix}$

$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 8/5 & -4/5 \\ -4/5 & 2/5 \end{bmatrix} = \boxed{\begin{bmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{bmatrix} = Q}$

1.10.3 Properties of Householder matrices

Proving every householder matrix Q satisfies:

a.) Q is symmetric ... $Q^T = Q$ //

$\hookrightarrow Q^T = (I - 2uu^T)^T = I^T - (2uu^T)^T$ (2 is just a scalar & $I^T = I$ is symm.

$\hookrightarrow = I - 2(u^T)^T u^T = I - 2uu^T$ // $\therefore Q^T = Q$ ///

b.) Q is orthogonal ... $Q^T Q = I$ //

$\hookrightarrow Q^T Q = (I - 2uu^T)^T (I - 2uu^T)$ ... we proved in (a.) that $Q^T = I - 2uu^T \therefore Q^T Q = (I - 2uu^T)(I - 2uu^T)$

$= I^2 - 4uu^T + 4(uu^T)(uu^T)$ & in 1.10.1, we showed $(uu^T)^2 = uu^T \therefore Q^T Q = I - \cancel{4uu^T} + \cancel{4uu^T} = I$ ///

c.) $Q^2 = I$ ... $Q^2 = (I - 2uu^T)^2 = I^2 - (2uu^T)^2$

$= I^2 - 4uu^T + 4(uu^T)(uu^T)$ & as in (b.), $(uu^T)^2 = uu^T$, so $Q^2 = I - 4uu^T + 4uu^T = I$ // $= Q^2$ ///

1.10.4 computing Qv for some vectors v
Proving if Q is a Householder matrix corresponding to the unit vector u, then $Qv = \begin{cases} -v & \text{if } v \text{ is in span}\{u\} \\ v & \text{if } v \cdot u = 0 \end{cases}$

- $Q = I - 2uu^T$ where u is a unit vector ($\|u\| = 1$)
- $v \in \mathbb{R}^n \rightarrow$ proving that v is in the span $\{u\}$

↳ Let $v = cu$ for some scalar c; $v = cu$ in Qv

↳ $Qv = Q(cu) = cQ(v)$ (By Associative Law)

--- $Q(u) = (I - 2uu^T)u = u - 2uu^Tu$ ($u^Tu = \|u\|^2 = 1$)

↳ $Q(u) = u - 2u = -u$ ___ $Qv = \underbrace{c(-u)}_{-v} = -v$

so if $v \in$ span$\{u\}$, $Qv = -v$ ▨

- $v \cdot u = 0$ — $v \cdot u = 0$ means v & u are orthogonal to each other.

so it must be true that $u^Tv = 0$, so v times the orthogonal matrix Q is equal to v ---

$Qv = (I - 2uu^T)v = v - 2uu^Tv$ ___ ($u^Tv = 0 \therefore uu^Tv = 0$)

↳ $Qv = v - 2(0) = v$ $\therefore$ if $v \cdot u = 0$, then $Qv = v$ ▨

1.10.5 Proving that $Qx = y$; $x \neq y$ w/ $\|x\| = \|y\|$ & $u = \frac{x-y}{\|x-y\|}$

$\therefore$ Householder matrix Q satisfies $Qx = y$ for $x = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ & $y = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$

↳ $Qx = (I - 2uu^T)x = \left[I - 2\left(\frac{x-y}{\|x-y\|}\right)\left(\frac{x-y}{\|x-y\|}\right)^T\right]x$

↳ $Qx = I - 2u\left(\frac{x^T - y^T}{\|x-y\|}\right)$ ($\|x-y\|$ is just a scalar; transpose will not alter it)

↳ $Qx = x - 2u\frac{x^Tx - y^Tx}{\|x-y\|} = x - 2u\frac{x \cdot x - y \cdot x}{\|x-y\|}$ (y^Tx is a scalar)

↳ $Qx = x - u\frac{2x^2 - 2y \cdot x}{\|x-y\|} \rightarrow$ (factored) $\rightarrow = x - u\frac{\|x\|^2 + \|x\|^2 - 2y \cdot x}{\|x-y\|}$
($2x^2 = \|x\|^2 + \|x\|^2$)

($\|x\| = \|y\|$) $\rightarrow Qx = x - u\frac{\|x\|^2 + \|y\|^2 - 2yx}{\|x-y\|} \rightarrow$ ($\|x-y\|^2$)

↳ $Qx = x - u\frac{\|x-y\|^2}{\|x-y\|} \rightarrow Qx = x - u\frac{\|x-y\| \cdot \cancel{\|x-y\|}}{\cancel{\|x-y\|}}$

↳ $Qx = x - u\|x-y\| \rightarrow Qx = x - \frac{x-y}{\cancel{\|x-y\|}}\cancel{\|x-y\|}$

↳ $Qx = x - (x-y) \rightarrow Qx = \cancel{x} - \cancel{x} + y \rightarrow \boxed{Qx = y}$ ▨

1.10.5 continued ~ verifying for x & y

$$Qx = \left(I - 2\left(\frac{x-y}{\|x-y\|}\right)\left(\frac{x-y}{\|x-y\|}\right)^T\right)x$$

- $x - y = \begin{bmatrix}1\\2\\2\end{bmatrix} - \begin{bmatrix}3\\0\\0\end{bmatrix} = \begin{bmatrix}-2\\2\\2\end{bmatrix}$ ... $\|x-y\| = \sqrt{4+4+4} = \sqrt{12} = 2\sqrt{3}$
- $\frac{x-y}{\|x-y\|} = \frac{1}{2\sqrt{3}}\begin{bmatrix}-2\\2\\2\end{bmatrix} = \begin{bmatrix}-1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{bmatrix}$

↳ $Qx = \left(\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} - 2\left(\begin{bmatrix}-1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{bmatrix}\begin{bmatrix}-1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3}\end{bmatrix}\right)\right)\begin{bmatrix}1\\2\\2\end{bmatrix}$

↳ $Qx = \left(\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} - 2\left(\begin{bmatrix}1/3 & -1/3 & -1/3\\-1/3 & 1/3 & 1/3\\-1/3 & 1/3 & 1/3\end{bmatrix}\right)\right)\begin{bmatrix}1\\2\\2\end{bmatrix}$

$Qx = \left(\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} + \begin{bmatrix}-2/3 & 2/3 & 2/3\\2/3 & -2/3 & -2/3\\2/3 & -2/3 & -2/3\end{bmatrix}\right)\begin{bmatrix}1\\2\\2\end{bmatrix}$

$Qx = \begin{bmatrix}1/3 & 2/3 & 2/3\\2/3 & 1/3 & -2/3\\2/3 & -2/3 & 1/3\end{bmatrix}\begin{bmatrix}1\\2\\2\end{bmatrix} = \begin{bmatrix}1/3 + 4/3 + 4/3\\2/3 + 2/3 - 4/3\\2/3 - 4/3 + 2/3\end{bmatrix}$

$= \begin{bmatrix}9/3\\0\\0\end{bmatrix} = \begin{bmatrix}3\\0\\0\end{bmatrix} = y = Qx$

1.10.6 Reduction to Upper Hessenberg Form

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$ → Transforming into an upper Hessenberg matrix

a) why H_1 is an orthogonal, symmetric matrix;

∘ $H_1 = \begin{bmatrix} 1 & 0 \\ 0 & Q_1 \end{bmatrix}$, where $Q_1 = \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{bmatrix}$ ∴ $H_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & q_{11} & q_{12} & q_{13} \\ 0 & q_{21} & q_{22} & q_{23} \\ 0 & q_{31} & q_{32} & q_{33} \end{bmatrix}$

Also, $Q_1 = I - 2uu^T$, where u is a unit vector & Q_1 is orthogonal ($Q_1^T Q_1 = I$, as proved in 1.10.3), thus H_1 is orthogonal---

$$H_1^T H_1 = \begin{bmatrix} 1 & 0 \\ 0 & Q_1^T Q_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & I \end{bmatrix} = I$$

And to prove symmetry, we proved $Q_1 = I - 2uu^T$ is also symmetric in 1.10.3, such that $Q_1 = Q_1^T$, thus H_1 must also be symmetric---

$$H_1^T = \begin{bmatrix} 1 & 0 \\ 0 & Q_1^T \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & Q_1 \end{bmatrix} = H_1$$

Thus H_1 is orthogonal & symmetric.

b.) Because orthogonal transformations preserve eigenvalues, if H_1 is orthogonal as proved in part a.), then the eigenvalues of A & A_1 must be the same --- $H_1 A H_1 = A_1 \Rightarrow H_1^{-1} A H_1 = A$ --- since H_1 is orthogonal where $H_1^{-1} = H_1^T$, A_1 & A are similar matrices (as proved in 1.1) --- so the eigenvalues of A_1 are equal to those of A.

c.) Showing $A_1 = H_1 A H_1$ is a matrix of the form

$$A_1 = \begin{bmatrix} a_{11} & b_{12} & b_{13} & b_{14} \\ \pm\|x\| & b_{22} & b_{23} & b_{24} \\ 0 & b_{32} & b_{33} & b_{34} \\ 0 & b_{42} & b_{43} & b_{44} \end{bmatrix} \text{ --- } H_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & q_{11} & q_{12} & q_{13} \\ 0 & q_{21} & q_{22} & q_{23} \\ 0 & q_{31} & q_{32} & q_{33} \end{bmatrix}$$

$$x = \begin{bmatrix} a_{21} \\ a_{31} \\ a_{41} \end{bmatrix} \text{ --- } y = \begin{bmatrix} \pm\|x\| \\ 0 \\ 0 \end{bmatrix}, \quad \|x\| = \sqrt{a_{21}^2 + a_{31}^2 + a_{41}^2}$$

--- $Q_1 = I - 2uu^T$, where

$$u = \frac{x - y}{\|x - y\|}$$

1.10.6 part c.) continued

$\therefore x-y = \begin{bmatrix} a_{21} \mp \|x\| \\ a_{31} \\ a_{41} \end{bmatrix}$ — $\|x-y\| = \sqrt{(a_{21} \mp \|x\|)^2 + a_{31}^2 + a_{41}^2}$

$u = \frac{1}{\|x-y\|} \begin{bmatrix} a_{21} \mp \|x\| \\ a_{31} \\ a_{41} \end{bmatrix} \rightarrow Q = I - 2 \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}$

— $H_1 A = \begin{bmatrix} 1 & 0 \\ 0 & Q_1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$ by multiplying these, the first row stays as $[a_{11}\ a_{12}\ a_{13}\ a_{14}]$

— because the only values not multiplied by zero are those of the first row ($\times 1$). For the other rows, we have $[Q_1] \begin{bmatrix} a_{21} \\ a_{31} \\ a_{41} \end{bmatrix} = y = \begin{bmatrix} \pm\|x\| \\ 0 \\ 0 \end{bmatrix}$

and thus, the first column of $H_1 A$ will be $\begin{bmatrix} a_{11} \\ \pm\|x\| \\ 0 \\ 0 \end{bmatrix}$. When we conduct $H_1 A H_1 = A_1$, we will still get the entries $\pm\|x\|$ still equal to zero by convention of the multiplication,

and thus $A_1 = \begin{bmatrix} a_{11} & b_{12} & b_{13} & b_{14} \\ \pm\|x\| & b_{22} & b_{23} & b_{24} \\ 0 & b_{32} & b_{33} & b_{34} \\ 0 & b_{42} & b_{43} & b_{44} \end{bmatrix}$, which reduces A to upper-Hessenberg form.

Upper Hessenberg Form of A

d.) $A = \begin{bmatrix} 1 & 1 & -1 & 4 \\ -2 & -1 & -5 & -1 \\ 4 & -3 & 0 & 2 \\ 4 & 2 & 3 & 1 \end{bmatrix}$; $x = \begin{bmatrix} -2 \\ 4 \\ 4 \end{bmatrix}$, $y = \begin{bmatrix} \pm\|x\| \\ 0 \\ 0 \end{bmatrix}$

$\|x\| = \sqrt{-2^2 + 4^2 + 4^2} = \sqrt{36} = 6 \quad \therefore y = \pm \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$

$u = \frac{x-y}{\|x-y\|} \rightarrow x - y = \begin{bmatrix} -2 \\ 4 \\ 4 \end{bmatrix} - \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -8 \\ 4 \\ 4 \end{bmatrix}$

$\|x-y\| = \sqrt{-8^2 + 4^2 + 4^2} = \sqrt{98} = \sqrt{7 \cdot 7 \cdot 2} = 7\sqrt{2}$

$\therefore u = \frac{\langle -8, 4, 4 \rangle}{7\sqrt{2}}$ — $Q_1 = I - 2uu^T \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2 \begin{bmatrix} -8/7\sqrt{2} \\ 4/7\sqrt{2} \\ 4/7\sqrt{2} \end{bmatrix} \begin{bmatrix} -8/7\sqrt{2} & 4/7\sqrt{2} & 4/7\sqrt{2} \end{bmatrix}$

$Q_1 \hookrightarrow I - \begin{bmatrix} 64/49 & -32/49 & -32/49 \\ -32/49 & 16/49 & 16/49 \\ -32/49 & 16/49 & 16/49 \end{bmatrix} \rightarrow \begin{bmatrix} -15/49 & 32/49 & 32/49 \\ 32/49 & 33/49 & -16/49 \\ 32/49 & -16/49 & 33/49 \end{bmatrix}$

$\therefore H_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -15/49 & 32/49 & 32/49 \\ 0 & 32/49 & 33/49 & -16/49 \\ 0 & 32/49 & -16/49 & 33/49 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & Q_1 \end{bmatrix}$

.10.6 Case of Symmetric Matrices; tridiagonal matrix

e.) Why a symmetric matrix A has an upper Hessenberg form that is tridiagonal:

For the 5x5 tridiagonal matrix...

$$A = \begin{bmatrix} * & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 \\ 0 & * & * & * & 0 \\ 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * \end{bmatrix}$$

- All symmetric matrices satisfy $A = A^T$, meaning every entry $a_{ij} = a_{ji}$ (which is mirrored across the diagonal of that matrix)
- An upper Hessenberg form matrix has all zeros below the subdiagonal entry for each column (that is, for every $i > j+1$, $a_{ij} = 0$)
- Given this, we can infer that a symmetric matrix A will have an Upper Hessenberg form that reflects the symmetry in it's initial form, across the diagonal of the Upper Hessenberg form
  - ↳ Specifically, the entries on the main diagonal (every a_{ii}) will remain nonzero, the entries on the first upper diagonal will be nonzero ($a_{i,i+1}$, $i = 1, 2 \ldots n-1$), & the entries on the first subdiagonal will be nonzero ($a_{i,i-1}$), & due to the symmetry, $a_{i,i-1} = a_{i-1,i}$
  - ↳ Thus we can expect for entries above & below each first upper & subdiagonal will all be zero; for 5x5 symmetric matrices, this will result in the form of A as seen above

Overall, the only nonzero entries for a symmetric matrix's upper Hessenberg form lie on, directly above, and directly below the diagonal entries, and by convention, this creates a tridiagonal matrix.

∴ Any Upper Hessenberg form of a symmetric matrix is a tridiagonal matrix