## 1.9 The QR Factorization of $A^{K+1}$

Let $Q_0 = Q$ and $R_0 = R$ First show that

$$Q_0 Q_1 \cdots Q_{K-1} A_K = A Q_0 Q_1 \cdots Q_{K-1}$$

for all $K \geq 1$. Then show that

$$(Q_0 Q_1 \cdots Q_K)(R_K \cdots R_1 R_0) = A(Q_0 Q_1 \cdots Q_{K-1})(R_{K-1} \cdots R_1 R_0)$$

Finally, deduce that $(Q_0 Q_1 \cdots Q_K)(R_K \cdots R_1 R_0)$ is the QR factorization of $A^{K+1}$.

Step 1. proof by induction - base case $K=1$

$A = Q_0 R_0$

$Q_0 A_1 = Q_0 R_0 = A \Rightarrow A_1 = R_0$

$Q_0 A_1 = A Q_0$

assume property holds for K

→ show it holds for K+1 - use QR factorization of $A_K$

$A = Q_K R_K$

where Q is orthogonal and $R_K$ is upper triangular

$Q_0 Q_1 \cdots Q_{K-1} A_K = Q_0 Q_1 \cdots Q_{K-1}(Q_K R_K)$

since $Q_K$ is orthogonal, regroup:

$Q_0 Q_1 \cdots Q_{K-1} A_K = (Q_0 Q_1 \cdots Q_K) R_K$

The inductive hypothesis:

$Q_0 Q_1 \cdots Q_{K-1} A_K = A Q_0 Q_1 \cdots Q_{K-1}$

substitute: $A_K = Q_K R_K$

$Q_0 Q_1 \cdots Q_{K-1} Q_K R_K = A Q_0 Q_1 \cdots Q_K$

Continued...

so property holds for $k+1$

$\therefore Q_0 Q_1 \dots Q_{k-1} A_k = A Q_0 Q_1 \dots Q_{k-1}$ , $\forall k \geq 1$

step 2. use result from step 1.

$Q_0 Q_1 \dots Q_{k-1} A_k = A Q_0 Q_1 \dots Q_{k-1}$

Multiply both sides by $Q_k R_k$ (the QR decomposition of $A_k$)

$Q_0 Q_1 \dots Q_{k-1} Q_k R_k = A Q_0 Q_1 \dots Q_{k-1}$

regroup terms

$(Q_0 Q_1 \dots Q_k)(R_k \dots R_1 R_0) = A (Q_0 Q_1 \dots Q_{k-1})(R_{k-1} \dots R_1 R_0)$

$\therefore$ second part is completed

step 3. use inductive structure - $Q_i$ is orthogonal, $R_i$ upper $\Delta$

↳ this means the product $Q_0 Q_1 \dots Q_k$ is orthogonal and $R_k \dots R_1 R_0$ is upper triangular

this proves: $(Q_0 Q_1 \dots Q_k)(R_k \dots R_1 R_0) = A^{k+1}$

product represents the QR factorization of $A^{k+1}$, where

$Q = Q_0 Q_1 \dots Q_k$ and $R = R_k \dots R_1 R_0$

$\therefore$ QR factorization of $A^{k+1}$ is given by

$(Q_0 Q_1 \dots Q_k)(R_k \dots R_1 R_0)$