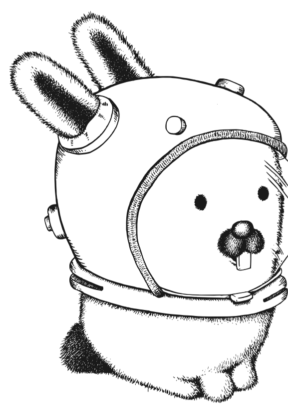
Evan's
Latex Template



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Chapter 1

1.1 Random Examples

Definition 1.1.1: Limit of Sequence in $\mathbb{R}$

Let $\{s_n\}$ be a sequence in $\mathbb{R}$. We say

$$\lim_{n\to\infty} s_n = s$$

where $s \in \mathbb{R}$ if $\forall$ real numbers $\varepsilon > 0$ $\exists$ natural number N such that for $n > N$

$$s - \varepsilon < s_n < s + \varepsilon \text{ i.e. } |s - s_n| < \varepsilon$$

Question 1

Is the set $x-$axis$\setminus\{$Origin$\}$ a closed set

Solution: We have to take its complement and check whether that set is a open set i.e. if it is a union of open balls

Note

We will do topology in Normed Linear Space (Mainly $\mathbb{R}^n$ and occasionally $\mathbb{C}^n$)using the language of Metric Space

Claim 1.1.1 Topology

Topology is cool

A

Example 1.1.1 (Open Set and Close Set)

Open Set:
- φ
- $\bigcup\limits_{x\in X} B_r(x)$ (Any $r > 0$ will do)
- $B_r(x)$ is open

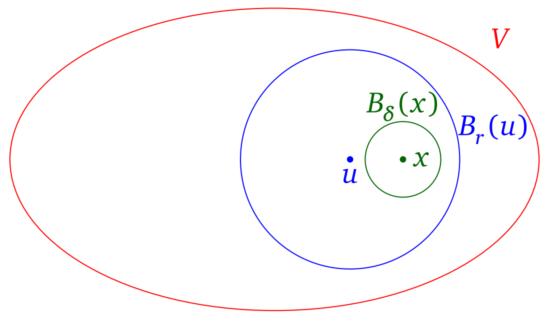
Closed Set:
- $X,\ \varphi$
- $\overline{B_r(x)}$

$x-$axis $\cup$ $y-$axis

Theorem 1.1.1

If $x \in$ open set V then $\exists\, \delta > 0$ such that $B_\delta(x) \subset V$

Proof: By openness of V, $x \in B_r(u) \subset V$



Given $x \in B_r(u) \subset V$, we want $\delta > 0$ such that $x \in B_\delta(x) \subset B_r(u) \subset V$. Let $d = d(u,x)$. Choose δ such that $d + \delta < r$ (e.g. $\delta < \frac{r-d}{2}$)

If $y \in B_\delta(x)$ we will be done by showing that $d(u,y) < r$ but

$$d(u,y) \leqslant d(u,x) + d(x,y) < d + \delta < r$$

Corollary 1.1.1

By the result of the proof, we can then show...

Lemma 1.1.1

Suppose $\vec{v}_1, \dots, \vec{v}_n \in \mathbb{R}^n$ is subspace of $\mathbb{R}^n$.

Proposition 1.1.1

$1 + 1 = 2$.

1.2 Random

Definition 1.2.1: Normed Linear Space and Norm $\|\cdot\|$

Let V be a vector space over $\mathbb{R}$ (or $\mathbb{C}$). A norm on V is function $\|\cdot\|\ V \to \mathbb{R}_{\geq 0}$ satisfying

1. $\|x\| = 0 \iff x = 0\ \forall\, x \in V$
2. $\|\lambda x\| = |\lambda|\,\|x\|\ \forall\, \lambda \in \mathbb{R}(\text{or } \mathbb{C}), x \in V$
3. $\|x + y\| \leqslant \|x\| + \|y\|\ \forall\, x, y \in V$ (Triangle Inequality/Subadditivity)

And V is called a normed linear space.

• Same definition works with V a vector space over $\mathbb{C}$ (again $\|\cdot\| \to \mathbb{R}_{\geq 0}$) where 2 becomes $\|\lambda x\| = |\lambda|\,\|x\|\ \forall\, \lambda \in \mathbb{C}, x \in V$, where for $\lambda = a + ib$, $|\lambda| = \sqrt{a^2 + b^2}$

Example 1.2.1 ($p-$Norm)

$V = \mathbb{R}^m$, $p \in \mathbb{R}_{\geq 0}$. Define for $x = (x_1, x_2, \cdots, x_m) \in \mathbb{R}^m$

$$\|x\|_p = \left(|x_1|^p + |x_2|^p + \cdots + |x_m|^p\right)^{\frac{1}{p}}$$

(In school $p = 2$)

Special Case $p = 1$: $\|x\|_1 = |x_1| + |x_2| + \cdots + |x_m|$ is clearly a norm by usual triangle inequality.

Special Case $p \to \infty$ ($\mathbb{R}^m$ with $\|\cdot\|_\infty$): $\|x\|_\infty = \max\{|x_1|, |x_2|, \cdots, |x_m|\}$
For $m = 1$ these $p-$norms are nothing but $|x|$. Now exercise

Question 2

Prove that triangle inequality is true if $p \geqslant 1$ for $p-$norms. (What goes wrong for $p < 1$?)

***Solution*: For Property 3 for norm-2**

When field is $\mathbb{R}$:

We have to show

$$\begin{aligned}
&\sum_i (x_i + y_i)^2 \leqslant \left(\sqrt{\sum_i x_i^2} + \sqrt{\sum_i y_i^2}\right)^2 \\
\implies &\sum_i (x_i^2 + 2x_i y_i + y_i^2) \leqslant \sum_i x_i^2 + 2\sqrt{\left[\sum_i x_i^2\right]\left[\sum_i y_i^2\right]} + \sum_i y_i^2 \\
\implies &\left[\sum_i x_i y_i\right]^2 \leqslant \left[\sum_i x_i^2\right]\left[\sum_i y_i^2\right]
\end{aligned}$$

So in other words prove $\langle x, y\rangle^2 \leqslant \langle x, x\rangle\langle y, y\rangle$ where

$$\langle x, y\rangle = \sum_i x_i y_i$$

Note

- $\|x\|^2 = \langle x, x\rangle$
- $\langle x, y\rangle = \langle y, x\rangle$
- $\langle \cdot, \cdot\rangle$ is $\mathbb{R}-$linear in each slot i.e.

$$\langle rx + x', y\rangle = r\langle x, y\rangle + \langle x', y\rangle \text{ and similarly for second slot}$$

Here in $\langle x, y\rangle$ x is in first slot and y is in second slot.

Now the statement is just the Cauchy-Schwartz Inequality. For proof

$$\langle x, y\rangle^2 \leqslant \langle x, x\rangle\langle y, y\rangle$$

expand everything of $\langle x - \lambda y, x - \lambda y\rangle$ which is going to give a quadratic equation in variable λ

$$\begin{aligned}
\langle x - \lambda y, x - \lambda y\rangle &= \langle x, x - \lambda y\rangle - \lambda\langle y, x - \lambda y\rangle \\
&= \langle x, x\rangle - \lambda\langle x, y\rangle - \lambda\langle y, x\rangle + \lambda^2\langle y, y\rangle \\
&= \langle x, x\rangle - 2\lambda\langle x, y\rangle + \lambda^2\langle y, y\rangle
\end{aligned}$$

Now unless $x = \lambda y$ we have $\langle x - \lambda y, x - \lambda y\rangle > 0$ Hence the quadratic equation has no root therefore the discriminant is greater than zero.

When field is $\mathbb{C}$:

Modify the definition by

$$\langle x, y\rangle = \sum_i \overline{x_i} y_i$$

Then we still have $\langle x, x\rangle \geqslant 0$

1.3 Algorithms

Algorithm 1: what

Input: This is some input
Output: This is some output

```
/* This is a comment */

some code here;
x ← 0;
y ← 0;
if x > 5 then
    x is greater than 5 ;                    // This is also a comment
else
    x is less than or equal to 5;
end
foreach y in 0..5 do
    y ← y + 1;
end
for y in 0..5 do
    y ← y − 1;
end
while x > 5 do
    x ← x − 1;
end
return Return something here;
```

Theorem 1.3.1

高斯公式（Gauss's Divergence Theorem）

$$\begin{aligned}\iint_{\partial\Omega^+} P\mathrm{d}y\mathrm{d}z + Q\mathrm{d}z\mathrm{d}x + R\mathrm{d}x\mathrm{d}y &\overset{\text{Gauss}'}{=\!=} \iiint_{\Omega}\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)\mathrm{d}v \\ &=\!= \iint_{\partial\Omega^+}\big(P\cos\alpha + Q\cos\beta + R\cos\gamma\big)\,\mathrm{d}S\end{aligned}$$

其中 $\partial\Omega^+$ 是 Ω 的外侧边界，α,β,γ 是 $\partial\Omega^+$ 在 (x,y,z) 处的法向量与 x,y,z 轴的夹角。

1.4 Regional Subsets

This *text* is *written* in **English**
这段文字，以**「简体中文」**写成。
這段文字，以**「繁體中文」**寫成。
このテキストは、**「日本語」**で書かれています。

1.5 混排

漢文訓讀形成的日語文體稱為漢文訓讀體（漢文訓読体），是文語體的一種。這種文體相當於漢文的直譯，因貼近漢文脈而有強烈的漢文口調，和口語體差別很大。不過漢文訓讀體後來在明治時代成為了官方文體，稱作「標準文」。漢文訓讀體中夾雜的假名一般以片假名表記，原因是片假名較方正，和漢字相類。以《教育敕語》為例：

1.5.1 教育勅語（舊字舊假名，標點係編者所加）

朕惟フニ、我カ皇祖皇宗、國ヲ肇ムルコト宏遠ニ、德ヲ樹ツルコト深厚ナリ。我カ臣民、克ク忠ニ克ク孝ニ、億兆心ヲ一ニシテ、世世厥ノ美ヲ濟セルハ、此レ我カ國體ノ精華ニシテ、教育ノ淵源、亦實ニ此ニ存ス。爾臣民、父母ニ孝ニ、兄弟ニ友ニ、夫婦相和シ、朋友相信シ、恭儉己レヲ持シ、博愛衆ニ及ホシ、學ヲ修メ、業ヲ習ヒ、以テ智能ヲ啓發シ、德器ヲ成就シ、進テ公益ヲ廣メ、世務ヲ開キ、常ニ國憲ヲ重シ、國法ニ遵ヒ、一旦緩急アレハ、義勇公ニ奉シ、以テ天壤無窮ノ皇運ヲ扶翼スヘシ。是ノ如キハ、獨リ朕カ忠良ノ臣民タルノミナラス、又以テ爾祖先ノ遺風ヲ顯彰スルニ足ラン。斯ノ道ハ、實ニ我カ皇祖皇宗ノ遺訓ニシテ、子孫臣民ノ俱ニ遵守スヘキ所、之ヲ古今ニ通シテ謬ラス、之ヲ中外ニ施シテ悖ラス。朕爾臣民ト俱ニ拳々服膺シテ、咸其德ヲ一ニセンコトヲ庶幾フ。

1.5.2 文部省標準漢譯（標點係編者所加）

朕惟我皇祖皇宗、肇國宏遠、樹德深厚。我臣民、克忠克孝、億兆一心、世濟其美、此我國體之精華、而教育之淵源、亦實存乎此。爾臣民、孝于父母、友于兄弟、夫婦相和、朋友相信、恭儉持己、博愛及衆、修學習業、以啓發智能、成就德器、進廣公益、開世務、常重國憲、遵國法、一旦緩急、則義勇奉公、以扶翼天壤無窮之皇運。如是者、不獨爲朕忠良臣民、又足以顯彰爾祖先之遺風矣。斯道也、實我皇祖皇宗之遺訓、而子孫臣民之所當遵守、通諸古今而不謬、施諸中外而不悖。朕庶幾與爾臣民、俱拳拳服膺、咸一其德。