5. & 8. Bernoulli-Energy Methods

5.1 General Procedure

1. There are 2 equations that are generally useful for these types of problems:
   i) Bernoulli's equation. Valid in regions of steady, incompressible flow where net frictional forces are negligible.
   ii) Mass flow rate: $\dot{m} = \rho A \dot{x} = \rho V$
2. Identify the assumptions so the appropriate equations can be used.
3. Try and cancel out as many terms as possible from the Bernoulli equation. Use mass flow rate to determine $\dot{x}$.
4. Use energy methods to determine the pressure head if necessary.

5.2 Variable Definitions

- P: Pressure
- $\dot{x}$: Velocity
- z: Elevation
- V: Volume
- C_d: Discharge coefficient
- β: Ratio of throat diameter to pipe diameter d/D

5.3 Formulas

Classic Bernoulli Equations:

Bernoulli's Equation: $\frac{P_1}{\rho} + \frac{\dot{x}_1^2}{2} + gz_1 = \frac{P_2}{\rho} + \frac{\dot{x}_2^2}{2} + gz_2$

Mass Conservation: $\Delta m_{\text{CV}} = \dot{m}_{\text{in}} - \dot{m}_{\text{out}}$

Mass Flow Rate: $\dot{m} = \rho A \dot{x} = \rho V$

Obstruction flowmeter:

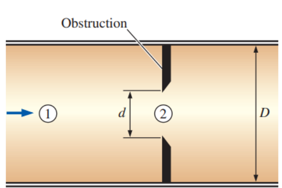

Figure 1: Obstruction flowmeter

Obstruction flowmeter: $\dot{V} = A_0 C_d \sqrt{\frac{2(P_1 - P_2)}{\rho(1-\beta^4)}}$

Mass Balance : $\implies \dot{x}_1 = (d/D)^2 \dot{x}_2$

Head Loss: $h_L = \frac{P_1}{\rho g} + \frac{\dot{x}_1^2}{2g} + z_1 - \frac{P_2}{\rho g} - \frac{\dot{x}_2^2}{2g} - z_2$

6. Momentum Analysis of Flow Systems

6.1 General Procedure

1. Utilize the Bernoulli equation to obtain the $P_{1,\text{gage}}$
2. $\sum \vec{F}$ represents external forces acting on the system. Some examples are:
   i) Pressure: $P_{1,\text{gage}} A_1$
   ii) Reaction force: F_R
3. Use momentum equation to obtain forces. For uniform flow, $\beta = 1$. If not given, it is expected to assume uniform flow.

6.2 Variable Definitions

- β: Momentum-flux correction factor. It's a correction factor for the surface integral.

6.3 Formulas

Momentum Equation: $\sum \vec{F} = \sum_{\text{out}} \beta \dot{m} \vec{V} - \sum_{\text{in}} \beta \dot{m} \vec{V}$

Momentum Correction Factor: $\beta = \frac{1}{A_c} \int_{A_c} \left(\frac{V}{V_{\text{avg}}}\right)^2 dA_c$

9. Differential Analysis of Fluid Flow

9.1 General Procedure

1. Most problems will be simplifiable to 2D or 1D because full form Navier-Stokes equations are too difficult to solve. The art of these problems is to simplify the equations to a form that can be solved.
2. Common assumptions are: steady, laminar, incompressible, constant viscosity, constant pressure, constant temperature, and parallel flow (velocity only in one direction). Gravity typically acts in the negative z-direction (unless it's like an inclined plane where you'd set your coordinates to be tangential and normal to the plane).
3. Check the problem statement for these key words:
   i) Steady: All $\frac{\partial}{\partial t} = 0$
   ii) Laminar: Generally implies parallel flow, flow in one direction only.
   iii) Incompressible: $\text{div}(\vec{V}) = \nabla \cdot \vec{V} = 0$, $\frac{\partial \rho}{\partial t} = 0$
   iv) Pressure acts in only one-direction: $\frac{\partial P}{\partial x} = 0$, $\frac{\partial P}{\partial y} = 0$, or $\frac{\partial P}{\partial z} = 0$
   v) Parallel flow: Velocity in the direction of motion is non-zero, velocity in the other directions is zero.
   vi) Gravity only in z-direction: $\vec{g} = -g\hat{k}$
4. Boundary conditions:

i) No-slip: $\vec{V}_{\text{fluid}} = \vec{V}_{\text{boundary}}$ at an interface boundary.

ii) No-shear at a : $\tau_{\text{fluid}} = \tau_{\text{boundary}} \approx 0$ at a free surface boundary with small surface tension like air.

5. Try to simplify the continuity equation first. Use the results in simplifying the Navier-Stokes equation.
6. Solve for whatever is asked for in the problem statement.

9.2. Operator Definitions

- ∇: The gradient operator, $\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$
- $\frac{\partial \vec{V}}{\partial x}$: The vector partial derivative, $\frac{\partial \vec{V}}{\partial x} = \frac{\partial u}{\partial x}\hat{i} + \frac{\partial v}{\partial x}\hat{j} + \frac{\partial w}{\partial x}\hat{k}$
- $\frac{D}{Dt}$: The material derivative, $\frac{D\vec{T}}{Dt} = \frac{\partial \vec{T}}{\partial t} + (\vec{V} \cdot \nabla)\vec{T}$ [1]
- $(\vec{V} \cdot \nabla)$: The convective derivative, $(\vec{V} \cdot \nabla)\vec{T} = u\frac{\partial \vec{T}}{\partial x} + v\frac{\partial \vec{T}}{\partial y} + w\frac{\partial \vec{T}}{\partial z}$
- ∇^2: The Laplacian operator, $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

9.3 Variable Definitions

- $\vec{V}$: Velocity vector, $\vec{V} = u\hat{i} + v\hat{j} + w\hat{k}$
- ρ: Density
- μ: Viscosity
- P: Pressure

9.4 Formulas

Continuity Equation: $\dfrac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = 0$

Special Case 1: Steady Compressible Flow: $\nabla \cdot (\rho \vec{V}) = 0$

Special Case 2: Incompressible Flow: $\nabla \cdot \vec{V} = 0$

Incompressible flow, Newtonian, Navier-Stokes Equation:

$$\rho \frac{D\vec{V}}{Dt} = -\nabla P + \rho \vec{g} + \mu \nabla^2 \vec{V}$$

For example in x-direction:

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial P}{\partial x} + \rho g_x + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

9.5 General Terms

- Control volume analysis: A method of analysis in which a volume in space is selected and the conservation of mass, momentum, and energy are applied to the volume
- Differential analysis: involves application of differential equations of fluid motion to any and every point in the flow field over a region called the flow domain.

[1] $(\vec{V} \cdot \nabla)$ is the convective derivative operator, not $\text{div}(\vec{V})$

10. Boundary Layer Approximation

10.1 General Procedure

1. Identify the type of flow using the Reynolds number. If the Re $> 5 \times 10^5$, the flow is turbulent. If the Re $< 5 \times 10^5$, the flow is laminar.
2. Use Table 1 to determine whatever you need.

10.2 Variable Definitions

10.3 Formulas

Reynolds Number: $\text{Re}_x = \dfrac{\rho V x}{\mu} = \dfrac{Vx}{\nu}$

Boundary Layer Thickness: $\dfrac{\delta}{x} = 4.91\sqrt{\text{Re}_x}$

Wall Shear Stress: $\tau_w = \dfrac{0.332\rho U^2}{\sqrt{\text{Re}_x}}$

Local Friction Coefficient: $C_f = \dfrac{\tau_w}{\frac{1}{2}\rho U^2} = \dfrac{0.664}{\sqrt{\text{Re}_x}}$

Displacement Thickness: $\delta^* = \displaystyle\int_0^\infty \left(1 - \frac{u}{U}\right) dy = \frac{1.72x}{\sqrt{\text{Re}_x}}$

Momentum Thickness: $\theta = \displaystyle\int_0^\infty \frac{u}{U}\left(1 - \frac{u}{U}\right) dy = \frac{0.664x}{\sqrt{\text{Re}_x}}$

Drag Force: $F_D = \displaystyle\int_A \tau_w dA = \int_0^L \tau_w w dx$

Continuity Equation: $\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0$

Momentum Equation: $u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} = U\dfrac{dU}{dx} + \nu\dfrac{\partial^2 u}{\partial y^2}$

Table 1: Boundary Layer Approximation

Property	Laminar	Turbulent
Boundary Layer Thickness	$\frac{\delta}{x} = 4.91\sqrt{\text{Re}_x}$	$\frac{\delta}{x} = \frac{0.16}{(\text{Re}_x)^{1/7}}$
Displacement Thickness	$\frac{\delta^*}{x} = \frac{1.72}{\sqrt{\text{Re}_x}}$	$\frac{\delta^*}{x} = \frac{0.020}{(\text{Re}_x)^{1/7}}$
Momentum Thickness	$\frac{\theta}{x} = \frac{0.664}{\sqrt{\text{Re}_x}}$	$\frac{\theta}{x} = \frac{0.016}{(\text{Re}_x)^{1/7}}$
Local Friction Coefficient	$C_f = \frac{0.664}{\sqrt{\text{Re}_x}}$	$C_f = \frac{0.027}{(\text{Re}_x)^{1/7}}$
Wall Shear Stress	$\tau_w = \frac{0.332\rho U^2}{\sqrt{\text{Re}_x}}$	$\tau_w = \frac{0.013\rho U^2}{(\text{Re}_x)^{1/7}}$