MEC E 331 Midterm 2 Formula Sheet

5. & 8. Bernoulli-Energy Methods

5.1 General Procedure

1. There are 2 equations that are generally useful for these types of problems:
   i) Bernoulli's equation. Valid in regions of steady, incompressible flow where net frictional forces are negligible.
   ii) Mass flow rate: $\dot{m} = \rho A\dot{x} = \rho V$
2. Identify the assumptions so the appropriate equations can be used.
3. Try and cancel out as many terms as possible from the Bernoulli equation. Use mass flow rate to determine $\dot{x}$.
4. Use energy methods to determine the pressure head if necessary.

5.2 Variable Definitions

- P: Pressure
- $\dot{x}$: Velocity
- z: Elevation
- V: Volume
- C_d: Discharge coefficient
- β: Ratio of throat diameter to pipe diameter d/D

5.3 Formulas

Classic Bernoulli Equations:

Bernoulli's Equation: $\frac{P_1}{\rho} + \frac{\dot{x}_1^2}{2} + gz_1 = \frac{P_2}{\rho} + \frac{\dot{x}_2^2}{2} + gz_2$

Mass Conservation: $\Delta m_{\text{CV}} = \dot{m}_{\text{in}} - \dot{m}_{\text{out}}$

Mass Flow Rate: $\dot{m} = \rho A\dot{x} = \rho V$

Obstruction flowmeter:

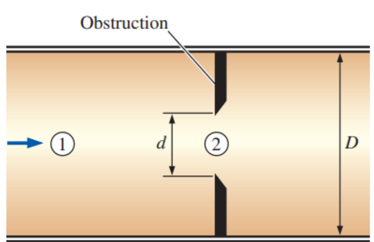

Figure 1: Obstruction flowmeter

Obstruction flowmeter: $\dot{V} = A_0 C_d \sqrt{\frac{2(P_1 - P_2)}{\rho(1-\beta^4)}}$

Mass Balance : $\implies \dot{x}_1 = (d/D)^2 \dot{x}_2$

Head Loss: $h_L = \frac{P_1}{\rho g} + \frac{\dot{x}_1^2}{2g} + z_1 - \frac{P_2}{\rho g} - \frac{\dot{x}_2^2}{2g} - z_2$

6. Momentum Analysis of Flow Systems

6.1 General Procedure

1. Utilize the Bernoulli equation to obtain the $P_{1,\text{gage}}$
2. $\sum \vec{F}$ represents external forces acting on the system. Some examples are:
   i) Pressure: $P_{1,\text{gage}} A_1$
   ii) Reaction force: F_R
3. Use momentum equation to obtain forces. For uniform flow, $\beta = 1$. If not given, it is expected to assume uniform flow.

6.2 Variable Definitions

- β: Momentum-flux correction factor. It's a correction factor for the surface integral.

6.3 Formulas

Momentum Equation: $\sum \vec{F} = \sum_{\text{out}} \beta \dot{m} \vec{V} - \sum_{\text{in}} \beta \dot{m} \vec{V}$

Momentum Correction Factor: $\beta = \frac{1}{A_c} \int_{A_c} \left(\frac{V}{V_{\text{avg}}} \right)^2 dA_c$

9. Differential Analysis of Fluid Flow

9.1 General Procedure

1. Most problems will be simplifiable to 2D or 1D because full form Navier-Stokes equations are too difficult to solve. The art of these problems is to simplify the equations to a form that can be solved.
2. Common assumptions are: steady, laminar, incompressible, constant viscosity, constant pressure, constant temperature, and parallel flow (velocity only in one direction). Gravity typically acts in the negative z-direction (unless it's like an inclined plane where you'd set your coordinates to be tangential and normal to the plane).
3. Check the problem statement for these key words:
   i) Steady: All $\frac{\partial}{\partial t} = 0$
   ii) Laminar: Generally implies parallel flow, flow in one direction only.
   iii) Incompressible: $\text{div}(\vec{V}) = \nabla \cdot \vec{V} = 0$, $\frac{\partial \rho}{\partial t} = 0$
   iv) Pressure acts in only one-direction: $\frac{\partial P}{\partial x} = 0$, $\frac{\partial P}{\partial y} = 0$, or $\frac{\partial P}{\partial z} = 0$
   v) Parallel flow: Velocity in the direction of motion is non-zero, velocity in the other directions is zero.
   vi) Gravity only in z-direction: $\vec{g} = -g\hat{k}$
4. Boundary conditions:
   i) No-slip: $\vec{V}_{\text{fluid}} = \vec{V}_{\text{boundary}}$ at an interface boundary.
   ii) No-shear at a : $\tau_{\text{fluid}} = \tau_{\text{boundary}} \approx 0$ at a free surface boundary with small surface tension like air.
5. Try to simplify the continuity equation first. Use the results in simplifying the Navier-Stokes equation.
6. Solve for whatever is asked for in the problem statement.

9.2. Operator Definitions

- ∇: The gradient operator, $\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$
- $\frac{\partial \vec{V}}{\partial x}$: The vector partial derivative, $\frac{\partial \vec{V}}{\partial x} = \frac{\partial u}{\partial x}\hat{i} + \frac{\partial v}{\partial x}\hat{j} + \frac{\partial w}{\partial x}\hat{k}$
- $\frac{D}{Dt}$: The material derivative, $\frac{D\vec{T}}{Dt} = \frac{\partial \vec{T}}{\partial t} + (\vec{V} \cdot \nabla)\vec{T}$ [1]
- $(\vec{V} \cdot \nabla)$: The convective derivative, $(\vec{V} \cdot \nabla)\vec{T} = u\frac{\partial \vec{T}}{\partial x} + v\frac{\partial \vec{T}}{\partial y} + w\frac{\partial \vec{T}}{\partial z}$
- ∇^2: The Laplacian operator, $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

9.3 Variable Definitions

- $\vec{V}$: Velocity vector, $\vec{V} = u\hat{i} + v\hat{j} + w\hat{k}$
- ρ: Density
- μ: Viscosity
- P: Pressure

9.4 Formulas

Continuity Equation: $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = 0$

Special Case 1: Steady Compressible Flow: $\nabla \cdot (\rho \vec{V}) = 0$

Special Case 2: Incompressible Flow: $\nabla \cdot \vec{V} = 0$

Incompressible flow, Newtonian, Navier-Stokes Equation:

[1] $(\vec{V} \cdot \nabla)$ is the convective derivative operator, not $\text{div}(\vec{V})$

$$\rho \frac{D\vec{V}}{Dt} = -\nabla P + \rho \vec{g} + \mu \nabla^2 \vec{V}$$

For example in x-direction:

$$\rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = -\frac{\partial P}{\partial x} + \rho g_x + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

10. Boundary Layer Approximation

10.1 General Procedure

1. Identify the type of flow using the Reynolds number. If the Re $> 5 \times 10^5$, the flow is turbulent. If the Re $< 5 \times 10^5$, the flow is laminar.
2. If you need to use the Navier-stokes equations, the boundary conditions are
   i) No slip at wall ($u = v = 0$ at $y = 0$)
   ii) Known outer flow $u \to U$ as $y \to \infty$.
   iii) Starting profile $u = u_{\text{starting}}(y)$ at $x = x_{\text{starting}}$.
3. Use Table 1 to determine whatever you need.

10.2 Variable Definitions

- ν: Kinematic viscosity
- μ: Dynamic viscosity
- U: Velocity of the free stream

10.3 Formulas

Reynolds Number: $\mathrm{Re}_x = \dfrac{\rho V x}{\mu} = \dfrac{Vx}{\nu}$

Boundary Layer Thickness: $\dfrac{\delta}{x} = 4.91\sqrt{\mathrm{Re}_x}$

Wall Shear Stress: $\tau_w = \dfrac{0.332\rho U^2}{\sqrt{\mathrm{Re}_x}}$

Local Friction Coefficient: $C_f = \dfrac{\tau_w}{\frac{1}{2}\rho U^2} = \dfrac{0.664}{\sqrt{\mathrm{Re}_x}}$

Displacement Thickness: $\delta^* = \displaystyle\int_0^\infty \left(1 - \frac{u}{U}\right) dy = \frac{1.72x}{\sqrt{\mathrm{Re}_x}}$

Momentum Thickness: $\theta = \displaystyle\int_0^\infty \frac{u}{U}\left(1 - \frac{u}{U}\right) dy = \frac{0.664x}{\sqrt{\mathrm{Re}_x}}$

Drag Force: $F_D = \displaystyle\int_A \tau_w dA = \int_0^L \tau_w w dx$

Continuity Equation: $\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0$

Momentum Equation: $u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} = U\dfrac{dU}{dx} + \nu\dfrac{\partial^2 u}{\partial y^2}$

Table 1: Boundary Layer Approximation

Property	Laminar	Turbulent
Boundary Layer Thickness	$\frac{\delta}{x} = 4.91\sqrt{\mathrm{Re}_x}$	$\frac{\delta}{x} = \frac{0.16}{(\mathrm{Re}_x)^{1/7}}$
Displacement Thickness	$\frac{\delta^*}{x} = \frac{1.72}{\sqrt{\mathrm{Re}_x}}$	$\frac{\delta^*}{x} = \frac{0.020}{(\mathrm{Re}_x)^{1/7}}$
Momentum Thickness	$\frac{\theta}{x} = \frac{0.664}{\sqrt{\mathrm{Re}_x}}$	$\frac{\theta}{x} = \frac{0.016}{(\mathrm{Re}_x)^{1/7}}$
Local Friction Coefficient	$C_f = \frac{0.664}{\sqrt{\mathrm{Re}_x}}$	$C_f = \frac{0.027}{(\mathrm{Re}_x)^{1/7}}$
Wall Shear Stress	$\tau_w = \frac{0.332\rho U^2}{\sqrt{\mathrm{Re}_x}}$	$\tau_w = \frac{0.013\rho U^2}{(\mathrm{Re}_x)^{1/7}}$

Terms

- Static Pressure: the actual thermodynamic pressure
- Dynamic Pressure: represents the pressure rise when fluid motion is brought to a stop isentropically
- Hydrostatic pressure: value depends on the reference level selected where z=0. A 'potential' pressure.
- Total Pressure = static pressure + dynamic pressure hydrostatic pressure.
- Stagnation pressure= static pressure+dynamic pressure.
- Pressure head: height of fluid needed to produce a static pressure P
- Velocity head: height of fluid needed to produce velocity V during a free, frictionless vertical fall.
- Elevation head: height relative to some reference plate z=0.
- Hydraulic Grade Line(HGL): the line that represents pressure and elevation heads.
- Energy Grade Line (EGL): the line that represents the total head.
- Obstruction flow meters: obstruction flow meters intentionally constrict the flow, and measure the drop in static pressure at the constriction. This pressure drop is proportional to the flow velocity through the constriction.
- Control volume analysis: very useful tool to engineering for flow analysis. Gives 'engineering analysis' answer, sometimes crude approximation, but always useful.
- Differential (small-scale) analysis: in principle can be used for any problems. In practice, limited cases where exact analytical solutions exist. Nowadays, CFD (computational fluid dynamics) simulations are widely performed based on differential analysis.
- Experimental (dimensional) analysis: based on the results of experiments. Technique to derive the most use out of the fewest number of experiments (which cost time and money).
- Displacement thickness: the distance that a streamline just outside the boundary layer is deflected away from the wall due to the effect of the boundary layer. The imaginary increase in the thickness of the wall, as seen by the outer flow due to the effect of the growing boundary layer
- Momentum thicknes: defined as the loss of momentum flux per unit width divided by ρU^2 due to the presence of the growing boundary layer. $\rho U^2 \theta$ the loss of momentum flux per unit width due to the growing boundary layer.