9. Differential Analysis of Fluid Flow

9.1

1. Most problems will be simplifiable to 2D or 1D because full form Navier-Stokes equations are too difficult to solve. The art of these problems is to simplify the equations to a form that can be solved.
2. Common assumptions are: steady, laminar, incompressible, constant viscosity, constant pressure, constant temperature, and parallel flow (velocity only in one direction). Gravity typically acts in the negative z-direction (unless it's like an inclined plane where you'd set your coordinates to be tangential and normal to the plane).
3. Check the problem statement for these key words:
   i) Steady: All $\frac{\partial}{\partial t} = 0$
   ii) Laminar: Generally implies parallel flow, flow in one direction only.
   iii) Incompressible: $\text{div}(\vec{V}) = \nabla \cdot \vec{V} = 0$, $\frac{\partial \rho}{\partial t} = 0$
   iv) Pressure acts in only one-direction: $\frac{\partial P}{\partial x} = 0$, $\frac{\partial P}{\partial y} = 0$, or $\frac{\partial P}{\partial z} = 0$
   v) Parallel flow: Velocity in the direction of motion is non-zero, velocity in the other directions is zero.
   vi) Gravity only in z-direction: $\vec{g} = -g\hat{k}$
4. Boundary conditions:
   i) No-slip: $\vec{V}_{\text{fluid}} = \vec{V}_{\text{boundary}}$ at an interface boundary.
   ii) No-shear at a : $\tau_{\text{fluid}} = \tau_{\text{boundary}} \approx 0$ at a free surface boundary with small surface tension like air.
5. Try to simplify the continuity equation first. Use the results in simplifying the Navier-Stokes equation.
6. Solve for whatever is asked for in the problem statement.

9.2. Operator Definitions

- ∇: The gradient operator, $\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$
- $\frac{\partial \vec{V}}{\partial x}$: The vector partial derivative, $\frac{\partial \vec{V}}{\partial x} = \frac{\partial u}{\partial x}\hat{i} + \frac{\partial v}{\partial x}\hat{j} + \frac{\partial w}{\partial x}\hat{k}$
- $\frac{D}{Dt}$: The material derivative, $\frac{D\vec{T}}{Dt} = \frac{\partial \vec{T}}{\partial t} + (\vec{V} \cdot \nabla)\vec{T}$ [1]
- $(\vec{V} \cdot \nabla)$: The convective derivative, $(\vec{V} \cdot \nabla)\vec{T} = u\frac{\partial \vec{T}}{\partial x} + v\frac{\partial \vec{T}}{\partial y} + w\frac{\partial \vec{T}}{\partial z}$
- ∇^2: The Laplacian operator, $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

9.3 Variable Definitions

- $\vec{V}$: Velocity vector, $\vec{V} = u\hat{i} + v\hat{j} + w\hat{k}$
- ρ: Density
- μ: Viscosity
- P: Pressure

9.4 Formulas

Continuity Equation: $\dfrac{\partial \rho}{\partial t} + \nabla \cdot (\rho\vec{V}) = 0$

Special Case 1: Steady Compressible Flow: $\nabla \cdot (\rho\vec{V}) = 0$

Special Case 2: Incompressible Flow: $\nabla \cdot \vec{V} = 0$

Incompressible flow, Newtonian, Navier-Stokes Equation:

$$\rho\frac{D\vec{V}}{Dt} = -\nabla P + \rho\vec{g} + \mu\nabla^2\vec{V}$$

For example in x-direction:

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial P}{\partial x} + \rho g_x + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

9.5 General Terms

- Control volume analysis: A method of analysis in which a volume in space is selected and the conservation of mass, momentum, and energy are applied to the volume
- Differential analysis: involves application of differential equations of fluid motion to any and every point in the flow field over a region called the flow domain.

[1] $(\vec{V} \cdot \nabla)$ is the convective derivative operator, not $\text{div}(\vec{V})$