# Question 1

A tube acts as a water siphon. Determine the speed of jet and the minimum pressure of water in the bend (at the point A).

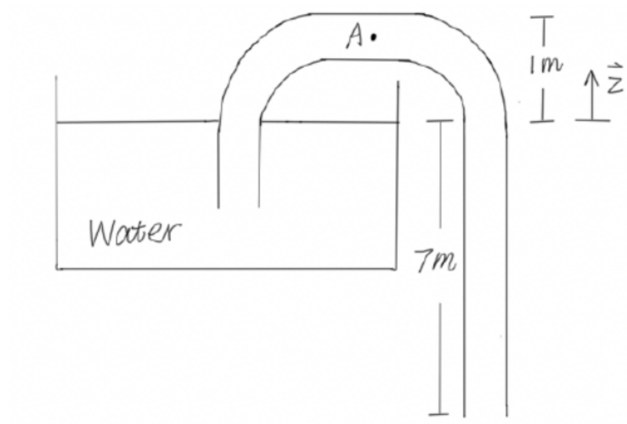


Figure 1: Tube with water siphon

**Solution**
Assumptions:

- Steady flow
- Incompressible flow
- Negligible viscous effects
- Negligible diameter change
- Reservoir is large enough to be considered infinite

By the Bernoulli equation, the speed of jet is given by:

$$\underbrace{\cancel{\frac{P_1}{\rho} - \frac{P_2}{\rho}}}_{\text{Both exposed to atm}} + \frac{v_2^2}{2} - \underbrace{\cancel{\frac{v_1^2}{2}}}_{\text{large reservoir}} + g(z_2 - z_1) = 0$$

$$\frac{v_2^2}{2} + g(z_2 - z_1) = 0$$

Which results in

$$\begin{aligned} v_2 &= \sqrt{2g(z_1 - z_2)} \\ &= \sqrt{2(9.81)(7)} \\ &= \boxed{11.7\,\mathrm{m\,s^{-1}}} \end{aligned}$$

In the pipe, by the assumptions, the velocity is constant. Therefore taking the Bernoulli equation between the pipe exit and the bend,

$$\begin{aligned}\frac{P_A}{\rho} - \frac{P_2}{\rho} + \cancel{\frac{v_A^2}{2} - \frac{v_2^2}{2}} + g(z_A - z_2) &= 0\\ \frac{P_A}{\rho} - \frac{P_2}{\rho} + g(z_A - z_2) &= 0\end{aligned}$$

Rearranging for $P_A$,

$$\begin{aligned}P_A &= \rho g(z_2 - z_A) + P_2\\ &= (1000)(9.81)(-7 - 1) + 101325\\ &= \boxed{22.84\,\mathrm{kPa}}\end{aligned}$$

# Question 2

A pitot-static probe is used to measure the speed of an aircraft flying at 3000 m. If the differential pressure reading is 3 kPa, determine the speed of the aircraft. (The density of the atmosphere at an elevation of 3000 m is $\rho = 0.909\,\mathrm{kg\,m^{-3}}$!)

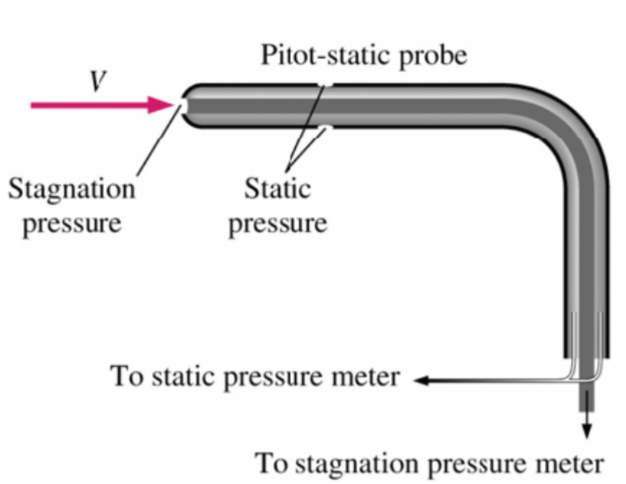


Figure 2: Pitot-static probe

**Solution**
Assumptions:

- Steady flow
- Incompressible flow
- Negligible viscous effects

At the stagnation point, the velocity is zero. By the Bernoulli equation,

$$\frac{P_1}{\rho} + \frac{P_2}{\rho} + \frac{v_1^2}{2} - \cancel{\frac{v_2^2}{2}} + \cancel{g(z_1 - z_2)} = 0$$

$$\frac{\Delta P}{\rho} + \frac{v_1^2}{2} = 0$$

Rearranging for $v_1$,

$$\begin{aligned} v_1 &= \sqrt{2\frac{\Delta P}{\rho}} \\ &= \sqrt{2\frac{3000}{0.909}} \\ &= \boxed{81.2\,\mathrm{m\,s^{-1}}} \end{aligned}$$

# Question 3

Water enters a tank of diameter 1 m steadily at a mass flow rate of $\dot{m}$. An orifice at the bottom with diameter $D_0$ allows water to escape. The orifice has a rounded entrance, so the frictional losses are negligible. If the tank is initially empty, (a) determine the maximum height that the water will reach in the tank and (b) obtain a relation for water height $z$

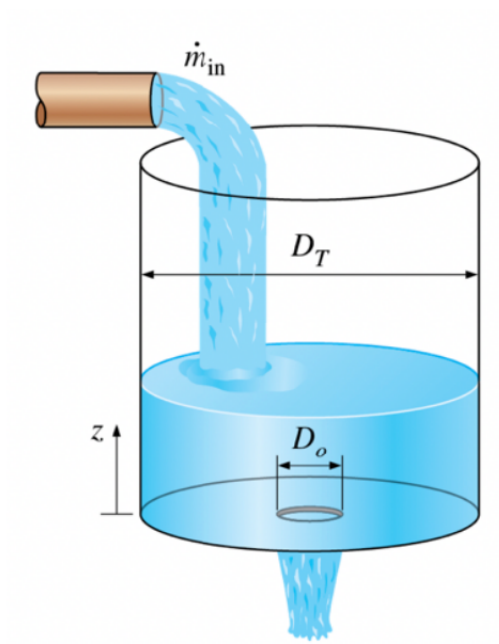


Figure 3: Water tank

**Solution**

## (a)

Let the height of the water at the maximum height be $h$. At the maximum height, equilibrium is reached, so the mass flow rate in is equal to the mass flow rate out.

By the Bernoulli equation,

$$\underbrace{\cancel{\frac{P_1}{\rho} - \frac{P_2}{\rho}}}_{\text{Both at atm}} + \underbrace{\cancel{\frac{v_1^2}{2}}}_{\text{Sufficiently large}} - \frac{v_2^2}{2} + g(z_1 - z_2) = 0$$

$$h = \frac{v_2^2}{2g}$$

By the continuity equation,

$$\begin{aligned} \dot{m} &= \rho A_0 v_2 \\ \implies v_2 &= \frac{\dot{m}}{\rho A_0} \\ &= \frac{\dot{m}}{\rho \pi D_0^2/4} \end{aligned}$$

Substituting into the Bernoulli equation,

$$\begin{aligned} h &= \frac{v_2^2}{2g} \\ &= \frac{\left(\frac{\dot{m}}{\rho \pi D_0^2/4}\right)^2}{2g} \\ &= \boxed{\frac{8\dot{m}^2}{\rho^2 \pi^2 g D_0^4}} \end{aligned}$$

## (b)

Let the height of the water at time $t$ be $z(t)$. At a given time, the exit velocity is given by the Bernoulli equation,

$$\underbrace{\cancel{\frac{P_1}{\rho} - \frac{P_2}{\rho}}}_{\text{Both at atm}} + \underbrace{\cancel{\frac{v_1^2}{2}}}_{\text{Sufficiently large}} - \frac{v_2^2}{2} + g(z_1 - z_2) = 0$$

$$\implies v_2 = \sqrt{2gz(t)}$$

Accumulation of mass in the tank is given by the continuity equation,

$$\dot{V} = \dot{m} - A_0 v_2$$

Substituting expressions for $V$, $A_0$, and $v_2$,

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\pi D_T^2}{4} z(t)\right) &= \dot{m} - \frac{\pi D_0^2}{4}\sqrt{2gz(t)} \\
\frac{\pi D_T^2}{4}\dot{z}(t) &= \dot{m} - \frac{\pi D_0^2}{4}\sqrt{2gz(t)}
\end{aligned}$$

Observe this is a constant coefficient, first order, non-homogeneous, non-linear differential equation. I do not know how to solve this, nor does it seem particularly useful to solve this with tools such as `ODE45`. As such, I will just box the implicit relationship.

$$\boxed{\frac{\pi D_T^2}{4}\dot{z}(t) = \dot{m} - \frac{\pi D_0^2}{4}\sqrt{2gz(t)}}$$

$$\begin{aligned}
\theta_{wall} &= A_1 \exp(-\lambda_1^2 \tau) \\
\implies \tau &= \frac{-\ln\left(\frac{\theta_{wall}}{A_1}\right)}{\lambda_1^2} \\
\implies t &= \frac{L^2}{\alpha}\frac{-\ln\left(\frac{\theta_{wall}}{A_1}\right)}{\lambda_1^2}
\end{aligned}$$