Question 1

A wood cantilever beam with cross section as shown in Fig. 1 is subjected to an inclined load P at its free end. Determine, (a) the orientation of the neutral axis, (b) the maximum bending stress. Given $P = 1$ kN, $\alpha = 30°$, $b = 80$ mm, $h = 150$ mm, $L = 1.2$ m.

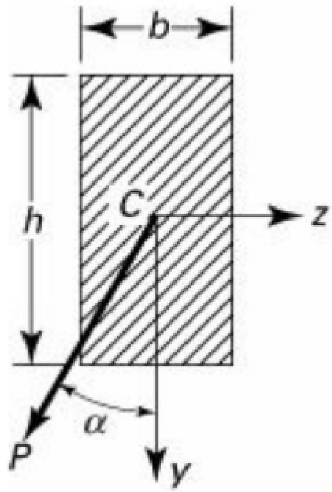

Figure 1: Problem Diagram

Finding the moments,

$$M_y = -PL\sin(\alpha) = 1 \times 1.2 \times \sin(30°) = -0.6 \text{ kNm}$$
$$M_z = PL\cos(\alpha) = 1 \times 1.2 \times \cos(30°) = 1.04 \text{ kNm}$$

Moment of inertias

$$I_y = \frac{1}{12}b^3h = \frac{1}{12} \times 80^3 \times 150 = 6.4 \times 10^6 \text{ mm}^4$$
$$I_z = \frac{1}{12}h^3b = \frac{1}{12} \times 150^3 \times 80 = 2.25 \times 10^7 \text{ mm}^4$$

Since the y and z axes are axes of symmetry, $I_{yz} = 0$.

The neutral axis is given by

$$\begin{aligned}\tan\phi &= \frac{M_y I_z}{M_z I_y} \\ &= \frac{-0.6 \times 10^6 \times 2.25 \times 10^7}{1.04 \times 10^6 \times 6.4 \times 10^6} \\ &= -2.03 \\ \implies \phi &= \arctan(-2.03) = \boxed{-63.77°}\end{aligned}$$

(a)

The maximum bending stress is at the furthest point from the neutral axis, which is at the top left or bottom right corner of the cross section. Using the bottom right point,

$$\begin{aligned}\sigma_x &= \frac{M_{y'}z'}{I_y} - \frac{M_{z'}y'}{I_z} \\ &= \frac{-0.6\times10^6\times80/2}{6.4\times10^6} - \frac{1.04\times10^6\times150/2}{2.25\times10^7} \\ &= \boxed{-7.22 \text{ MPa}}\end{aligned}$$

Question 2

A concentrated load P acts on a cantilever, as shown in Fig. 2. The beam is constructed of a 2024- T4 aluminum alloy having a yield strength $\sigma_{yp} = 290$ MPa, $L = 1.5$ m, $t = 20$ mm, $c = 60$ mm, and $b = 80$ mm. Based on a factor of safety $n = 1.2$ against initiation of yielding, calculate the magnitude of P for $\alpha = 30°$. Neglect the effect of shear in bending and assume that beam twisting is prevented.

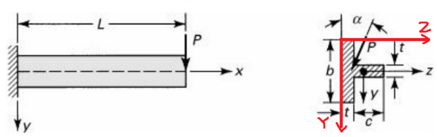

Figure 2: Problem diagram for Question 2.

(a)

First find the applied moments,

$$\begin{aligned}M_y &= P_z L = -P\sin(\alpha)L = -0.75P \\ M_z &= P_y L = P\cos(\alpha)L = 1.30P\end{aligned}$$

Next, find the centroid from the reference Z and Y axes,

$$\bar{z} = \frac{\sum \bar{z}_i A_i}{\sum A_i} = \frac{A_1\bar{z}_1 + A_2\bar{z}_2}{A_1 + A_2}$$

where A_1 is the vertical rectangle and A_2 is the horizontal rectangle. Proceeding,

$$\begin{aligned}\bar{z} &= \frac{(bt)(t/2) + (ct)(t + c/2)}{bt + ct} \\ &= \frac{(80\times20)(20/2) + (60\times20)(20 + 60/2)}{80\times20 + 60\times20} \\ &= 27.14 \text{ mm}\end{aligned}$$

Since the there is an axis of symmetry in the Y direction, $\bar{y} = b/2 = 40$ mm.

Next, find the moments of inertia,

$$\begin{aligned} I_z = \sum (I_{\bar{z},i} + A_i d_{y,i}^2) &= I_{\bar{z},1} + A_1 d_{y,1}^2 + I_{\bar{z},2} + A_2 d_{y,2}^2 \\ &= \frac{b^3 t}{12} + (bt)(0)^2 + \frac{ct^3}{12} + (ct)(0)^2 \\ &= \frac{80^3 \times 20}{12} + \frac{60 \times 20^3}{12} \\ &= 8.93 \times 10^5 \text{ mm}^4 \end{aligned}$$

and

$$\begin{aligned} I_y = \sum (I_{\bar{y},i} + A_i d_{z,i}^2) &= I_{\bar{y},1} + A_1 d_{z,1}^2 + I_{\bar{y},2} + A_2 d_{z,2}^2 \\ &= \frac{bt^3}{12} + (bt)(\bar{z} - t/2)^2 + \frac{c^3 t}{12} + (ct)(\bar{z} - (t + c/2))^2 \\ &= \frac{80 \times 20^3}{12} + (80 \times 20)(27.14 - 20/2)^2 + \frac{60^3 \times 20}{12} + (60 \times 20)(27.14 - (20 + 60/2)) \\ &= 1.51 \times 10^6 \text{ mm}^4 \end{aligned}$$

Since y is an axis of symmetry, $I_{yz} = 0$. Then,

$$\begin{aligned} \tan\phi &= \frac{M_y I_z}{M_z I_y} \\ &= \frac{-0.75P \times 8.93 \times 10^5}{1.30P \times 1.51 \times 10^6} \\ &= -0.3412 \\ \implies \phi &= \arctan(-0.3412) = \boxed{-18.84^\circ} \end{aligned}$$

(b)

The maximum stress occurs at the top left corner or bottom left corner of the cross section. Using the top left corner,

$$\begin{aligned} \sigma_x &= \frac{(M_y I_z + M_z I_{yz})z - (M_y I_{yz} + M_z I_y)y}{I_y I_z - I_{yz}^2} \\ &= \frac{(M_y I_z \bar{z} - M_z I_y \bar{y})}{I_y I_z} \\ &= \frac{(-0.75P \times 8.93 \times 10^5 \times (-27.14) - 1.30P \times 1.51 \times 10^6 \times (-40))}{1.51 \times 10^6 \times 8.93 \times 10^5} \\ &= -7.171 \times 10^{-5} P \end{aligned}$$

By yield stress and factor of safety,

$$\begin{aligned} \sigma_x &= \frac{-7.171 \times 10^{-5} P}{1.2} = \sigma_{yp} \\ \implies P &= \frac{290 \times 1.2}{-7.171 \times 10^{-5} \times 10^3} = \boxed{-4.85 \text{ kN}} \end{aligned}$$

Question 3

Question 4

For a thin cantilever, the stress function is given by

$$\Phi = -c_1xy + \frac{c_2x^3}{6} - \frac{c_3x^3y}{6} - \frac{c_4xy^3}{6} - \frac{c_5x^3y^3}{9} - \frac{c_6xy^5}{20}$$

Determine the stresses σ_x, σ_y, and τ_{xy} by using the elasticity method.

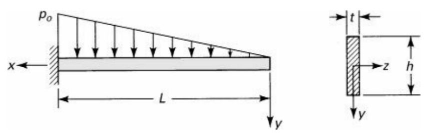

Figure 3: Problem diagram for Question 4.

I am not doing this by hand.

With Matlab

```
clc; clear; close all;
syms x y c1 c2 c3 c4 c5 c6 h p0 L t real
Phi = -c1*x*y + c2*x^3/6 - c3*x^3*y/6 - c4*x*y^3/6 - c5*x^3*y^3/9 -
   c6*x*y^5/20;
% Define the stress components
sigma_x = diff(Phi, y, 2);
sigma_y = diff(Phi, x, 2);
tau_xy = -diff(Phi, x, y);

% biharmonic equation
eqn1 = diff(sigma_x, x, 2) + 2*diff(tau_xy, x, y) + diff(sigma_y, y,
    2) == 0;

% boundary conditions
eqn2 = subs(sigma_y, y, h/2) == 0; % bottom
eqn3 = subs(tau_xy, y, h/2) == 0; % bottom
eqn4 = subs(sigma_y, y, -h/2) == -p0*x/(L*t); % top
eqn5 = subs(tau_xy, y, -h/2) == 0; % top
eqn6 = int(subs(tau_xy, x, 0)*t, y, -h/2, h/2) == 0; % no force at
   free end

% solve the system of equations
sol = solve([eqn1, eqn2, eqn3, eqn4, eqn5, eqn6], [c1, c2, c3, c4,
   c5, c6]);

% print stresses
sigma_x
```

```
sigma_y
tau_xy
```

This solves for the constants as

```
>> sol

sol =

  struct with fields:

    c1: -(p0*x^2)/(4*L*h*t)
    c2: -p0/(2*L*t)
    c3: -p0/(L*h*t)
    c4: (6*p0*x^2)/(L*h^3*t)
    c5: 0
    c6: 0

>>
```

and the stresses as

```
sigma_x =

- (2*c5*x^3*y)/3 - c6*x*y^3 - c4*x*y


sigma_y =

c2*x - (2*c5*x*y^3)/3 - c3*x*y


tau_xy =

c5*x^2*y^2 + (c3*x^2)/2 + (c6*y^4)/4 + (c4*y^2)/2 + c1
```

Question 5

A wooden, simply supported beam of length L is subjected to a uniform load p. Determine the beam length and the loading necessary to develop simultaneously $\sigma_{max} = 8.4$ MPa and $\tau_{max} = 0.7$ MPa. Take thickness $t = 0.05$ m and depth $h = 0.15$ m.

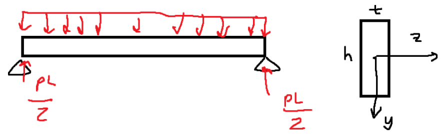

Figure 4: Problem diagram for Question 5.

Using singularity functions,

$$V = \frac{pL}{2}\langle x-0\rangle^0 - p\langle x-0\rangle^1$$
$$M = \frac{pL}{2}\langle x-0\rangle^1 - \frac{p}{2}\langle x-0\rangle^2$$

By symmetry, the maximum bending moment occurs at the center of the beam. From inspection, the max shear force is at the supports. Then,

$$V_{\max} = \frac{pL}{2}$$
$$M_{\max} = \frac{pL^2}{8}$$

Moment of inertia,

$$I_z = \frac{th^3}{12} = \frac{(0.05)(0.15^3)}{12} = 1.4063\times10^{-5}\ \text{m}^4$$

From bending,

$$\begin{aligned}
\sigma_{\max} &= \frac{M_{\max}y}{I_z} \\
&= \frac{pL^2y}{8I_z} \\
\implies pL^2 &= \frac{16I_z\sigma_{\max}}{h} \\
&= \frac{16(1.4063\times10^{-5})(8.4\times10^6)}{0.15} \\
&= 12600\ \text{N}
\end{aligned}$$

From shear on a rectangular cross section,

$$\begin{aligned}
\tau_{\max} &= \frac{3V_{\max}}{2A} \\
&= \frac{3pL}{4(0.05)(0.15)} \\
&= 100pL \\
\implies pL &= 7000\ \text{N}
\end{aligned}$$

Solving with Matlab,

```
syms p L
eqn1 = p*L^2 == 12600;
eqn2 = p*L == 7000;
sol = vpasolve([eqn1, eqn2], [p, L])

p: 3888.888888888888888888888888889
L: 1.8
```

Question 6

A simple wooden beam is under a uniform load of intensity p, as illustrated in Fig. 5.

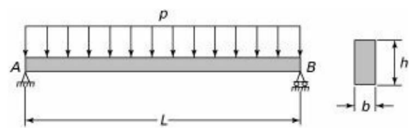

Figure 5: Problem diagram for Question 6.

(a) Find the ratio of the maximum shearing stress to the largest bending stress in terms of the depth h and length L of the beam.

(b) Using $\sigma_{all} = 9$ MPa, $\tau_{all} = 1.4$ MPa, $b = 50$ mm, and $h = 160$ mm, calculate the maximum permissible length L and the largest permissible distributed load of intensity p.

Like in Question 5, the shear force and bending moment are given by

$$V = \frac{pL}{2}\langle x-0\rangle^0 - p\langle x-0\rangle^1$$
$$M = \frac{pL}{2}\langle x-0\rangle^1 - \frac{p}{2}\langle x-0\rangle^2$$

By symmetry, the maximum bending moment occurs at the center of the beam. From inspection, the max shear force is at the supports. Then,

$$V_{\max} = \frac{pL}{2}$$
$$M_{\max} = \frac{pL^2}{8}$$

Then,

$$\begin{aligned}
\sigma_{\max} &= \frac{M_{\max} y}{I_z} = \frac{pL^2 h/2}{8bh^3/12} \\
&= \frac{3pL^2}{4bh^2} \\
\tau_{\max} &= \frac{3V_{\max}}{2A} = \frac{3pL}{4bh} \\
\frac{\tau_{\max}}{\sigma_{\max}} &= \boxed{\frac{h}{L}}
\end{aligned}$$

```
syms p L b h
sigma_max = 3*p*L^2/(4*b*h^2);
tau_max = 3*p*L/(4*b*h);
ratio = tau_max/sigma_max;
```