

### 3. Problems in Elasticity

#### 8.2. Formulas

##### Plane Strain

On the plane  $x$ - $y$ , the equilibrium and compatibility equations are

$$\begin{aligned}\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} &= 0 \\ \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} &= 0 \\ \frac{\partial^2 \sigma_x}{\partial y^2} + \frac{\partial^2 \sigma_y}{\partial x^2} &= \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \\ \implies \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) (\sigma_x + \sigma_y) &= 0\end{aligned}$$

Strain-stress relations are

$$\begin{aligned}\epsilon_x &= \frac{1}{E} (\sigma_x - \nu \sigma_y) \\ \epsilon_y &= \frac{1}{E} (\sigma_y - \nu \sigma_x) \\ \epsilon_z &= -\frac{\nu}{E} (\sigma_x + \sigma_y) \\ \gamma_{xy} &= \frac{\tau_{xy}}{G} \\ \gamma_{xz} &= \gamma_{yz} = 0\end{aligned}$$

Stress-strain relations are

$$\begin{aligned}\sigma_x &= \frac{E}{1-\nu^2} (\epsilon_x + \nu \epsilon_y) \\ \sigma_y &= \frac{E}{1-\nu^2} (\epsilon_y + \nu \epsilon_x) \\ \tau_{xy} &= G \gamma_{xy} \\ \sigma_z &= -\frac{\nu}{1-\nu} (\epsilon_x + \epsilon_y)\end{aligned}$$

Airy's stress function  $\Phi$  relations

$$\begin{aligned}\nabla^4 \Phi &= \frac{\partial^4 \Phi}{\partial x^4} + 2 \frac{\partial^4 \Phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \Phi}{\partial y^4} = 0 \\ \sigma_x &= \frac{\partial^2 \Phi}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 \Phi}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2 \Phi}{\partial x \partial y}\end{aligned}$$

##### Thermalelasticity

Thermal strain,  $\epsilon_t = \alpha T$ , relations by superposition,

$$\begin{aligned}\epsilon_x &= \frac{1}{E} (\sigma_x - \nu \sigma_y) + \alpha T \\ \epsilon_y &= \frac{1}{E} (\sigma_y - \nu \sigma_x) + \alpha T \\ \epsilon_z &= -\frac{\nu}{E} (\sigma_x + \sigma_y) + \alpha T \\ \gamma_{xy} &= \frac{\tau_{xy}}{G}\end{aligned}$$

Thermal stress relations,

$$\begin{aligned}\sigma_x &= \frac{E}{1-\nu^2} (\epsilon_x + \nu \epsilon_y) - \frac{E \alpha T}{1-\nu} \\ \sigma_y &= \frac{E}{1-\nu^2} (\epsilon_y + \nu \epsilon_x) - \frac{E \alpha T}{1-\nu} \\ \sigma_z &= -\frac{\nu}{1-\nu} (\epsilon_x + \epsilon_y) - \frac{E \alpha T}{1-\nu} \\ \tau_{xy} &= G \gamma_{xy}\end{aligned}$$

Stress function  $\Phi$  relations,

$$\begin{aligned}\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) (\sigma_x + \sigma_y + \alpha E T) &= 0 \\ \implies \nabla^4 \Phi + \alpha E \nabla^2 T &= 0\end{aligned}$$

##### Polar Coordinates

Displacement-strain relations,

$$\begin{aligned}\epsilon_r &= \frac{\partial u}{\partial r}, \quad \epsilon_\theta = \frac{u}{r} + \frac{1}{r} \frac{\partial v}{\partial \theta} \\ 2\epsilon_{r\theta} &= \gamma_{r\theta} = \frac{\partial v}{\partial r} - \frac{v}{r} + \frac{1}{r} \frac{\partial u}{\partial \theta}\end{aligned}$$

Strain-stress relations for plane stress,

$$\begin{aligned}\epsilon_r &= \frac{1}{E} (\sigma_r - \nu \sigma_\theta), \quad \epsilon_\theta = \frac{1}{E} (\sigma_\theta - \nu \sigma_r) \\ \epsilon_{r\theta} &= \frac{1}{2G} \tau_{r\theta}\end{aligned}$$

Airy's stress function  $\Phi$  relations,

$$\begin{aligned}\sigma_r &= \frac{1}{r} \frac{\partial \Phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2}, \quad \sigma_\theta = \frac{\partial^2 \Phi}{\partial r^2} \\ \tau_{r\theta} &= \frac{1}{r^2} \frac{\partial \Phi}{\partial \theta} - \frac{1}{r} \frac{\partial^2 \Phi}{\partial r \partial \theta} = -\frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \Phi}{\partial \theta} \right)\end{aligned}$$

Compatibility,

$$\begin{aligned}\nabla^2 \Phi &= \frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r} \frac{\partial \Phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2} \\ \nabla^4 \Phi &= \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) \nabla^2 \Phi = 0\end{aligned}$$

Transformation equations,

$$\begin{aligned}\sigma_r &= \frac{1}{2} (\sigma_x + \sigma_y) + \frac{1}{2} (\sigma_x - \sigma_y) \cos 2\theta + \tau_{xy} \sin 2\theta \\ \tau_{r\theta} &= -\frac{1}{2} (\sigma_x - \sigma_y) \sin 2\theta + \tau_{xy} \cos 2\theta \\ \sigma_\theta &= \frac{1}{2} (\sigma_x + \sigma_y) - \frac{1}{2} (\sigma_x - \sigma_y) \cos 2\theta - \tau_{xy} \sin 2\theta\end{aligned}$$

or

$$\begin{aligned}\sigma_x &= \frac{1}{2} (\sigma_\theta + \sigma_r) + \frac{1}{2} (\sigma_r - \sigma_\theta) \cos 2\theta - \tau_{r\theta} \sin 2\theta \\ \tau_{xy} &= -\frac{1}{2} (\sigma_r - \sigma_\theta) \sin 2\theta + \tau_{r\theta} \cos 2\theta \\ \sigma_y &= \frac{1}{2} (\sigma_\theta + \sigma_r) - \frac{1}{2} (\sigma_r - \sigma_\theta) \cos 2\theta + \tau_{r\theta} \sin 2\theta\end{aligned}$$

## Concentrated Loads

Wedge of unit thickness, under load  $P$ , and angle  $\alpha$ ,

$$\sigma_r = -\frac{P \cos \theta}{r(\alpha + \frac{1}{2} \sin 2\alpha)}, \quad \sigma_\theta = 0, \quad \tau_{r\theta} = 0$$

$$\sigma_x = \sigma_r \cos^2 \theta = -\frac{P \cos^4 \theta}{L(\alpha + \frac{1}{2} \sin 2\alpha)}$$

$$\tau_{xy} = \frac{P \sin \theta \cos^3 \theta}{L(\alpha + \frac{1}{2} \sin 2\alpha)}$$

$$(\sigma_x)_{\text{elem}} = -\frac{P}{2L \tan \alpha}$$

Note that the normal stress is maximum at  $\theta = 0$  and minimum at  $\theta = \alpha$ . Shear stress is maximum at  $\theta = \alpha$  if  $\alpha < 30^\circ$  and at  $\theta = 30^\circ$  if  $\alpha \geq 30^\circ$ .

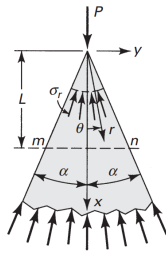


Figure 1: Wedge of unit thickness under load  $P$  per unit thickness

## Stress Concentrations

For stress concentration factor  $K$ ,

$$K = \frac{\sigma_{\max}}{\sigma_{\text{nom}}}$$

For circular hole in a large plate in tension stress  $\sigma_o$ ,

$$(\sigma_\theta)_{\max} = 3\sigma_o, \quad \theta = \pm\pi/2$$

$$(\sigma_\theta)_{\min} = -\sigma_o, \quad \theta = 0, \pm\pi$$

For tension  $\sigma_{ox}$  and  $\sigma_{oy}$ ,

$$(\sigma_\theta)_{\max, x} = 3\sigma_{ox} - \sigma_{oy}, \quad \theta = \pm\pi/2$$

$$(\sigma_\theta)_{\min, y} = 3\sigma_{oy} - \sigma_{ox}, \quad \theta = 0, \pm\pi$$

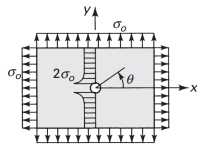


Figure 2: Stress concentration factor for circular hole in a large plate (10/10 figure)