

## Question 1

At a point in a loaded body, the stress relative to an x, y, and z coordinate system are

$$\sigma = \begin{bmatrix} 40 & 40 & 30 \\ 40 & 20 & 0 \\ 30 & 0 & 20 \end{bmatrix} \text{ MPa}$$

Determine the normal stress  $\sigma$  and the shearing stress  $\tau$  on a plane whose outward normal is oriented at angles of  $40^\circ$ ,  $75^\circ$ , and  $54^\circ$  with the x, y, and z axes, respectively.

First calculate the normal vector:

$$\hat{n} = \begin{bmatrix} \cos(40^\circ) \\ \cos(75^\circ) \\ \cos(54^\circ) \end{bmatrix}$$

Then calculate the normal stress:

$$\begin{aligned} \sigma_n &= \hat{n}^T \sigma \hat{n} \\ &= \begin{bmatrix} \cos(40^\circ) & \cos(75^\circ) & \cos(54^\circ) \end{bmatrix} \begin{bmatrix} 40 & 40 & 30 \\ 40 & 20 & 0 \\ 30 & 0 & 20 \end{bmatrix} \begin{bmatrix} \cos(40^\circ) \\ \cos(75^\circ) \\ \cos(54^\circ) \end{bmatrix} \\ &= \boxed{74.6 \text{ MPa}} \end{aligned}$$

To determine shearing stress magnitude, the following equation is used:

$$\begin{aligned} \tau &= \sqrt{p_x^2 + p_y^2 + p_z^2 - \sigma_n^2} \\ &= ((\sigma \hat{n})^T (\sigma \hat{n}) - \sigma_n^2)^{1/2} \\ &= (5926.856 - 5565.16)^{1/2} \\ &= \boxed{19.02 \text{ MPa}} \end{aligned}$$

## Question 2

The state of stress at a point in a member relative to an x, y, and z coordinate system is given by

$$\sigma = \begin{bmatrix} -100 & 0 & -80 \\ 0 & 20 & 0 \\ -80 & 0 & 20 \end{bmatrix} \text{ MPa}$$

- (a) The principal stresses by expansion of the characteristic stress determinant.
- (b) The octahedral stresses and the maximum shearing stress.

**(a)**

The characteristic polynomial is given by:

$$\sigma_p^3 - I_1\sigma_p^2 + I_2\sigma_p - I_3 = 0$$

where  $I_1$ ,  $I_2$ , and  $I_3$  are the first, second, and third invariants, respectively. The invariants are given by:

$$I_1 = \sigma_x + \sigma_y + \sigma_z = -60$$

$$I_2 = \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2 = \text{tr}(\det(\sigma)\sigma^{-1}) = -10000$$

$$I_3 = \det(\sigma) = -168000$$

Plugging into calculator, the eigenvalues are:

$$\sigma_p = \begin{bmatrix} -140 & 0 & 0 \\ 0 & 60 & 0 \\ 0 & 0 & 20 \end{bmatrix} \text{ MPa}$$

**(b)**

The octahedral stresses are given by:

$$\begin{aligned} \sigma_{oct} &= \frac{1}{3}(\sigma_{p,1} + \sigma_{p,2} + \sigma_{p,3}) \\ &= \frac{1}{3}(-140 + 60 + 20) \\ &= \boxed{-20 \text{ MPa}} \end{aligned}$$

The magnitude of the octohedral shear stress is given by:

$$\begin{aligned} \tau_{oct} &= \sqrt{\frac{(\sigma_{p,1} - \sigma_{p,2})^2 + (\sigma_{p,2} - \sigma_{p,3})^2 + (\sigma_{p,3} - \sigma_{p,1})^2}{9}} \\ &= \sqrt{\frac{(-140 - 60)^2 + (60 - 20)^2 + (20 + 140)^2}{9}} \\ &= \boxed{86.41 \text{ MPa}} \end{aligned}$$

The maximum shearing stress is given by:

$$\begin{aligned}\tau_{max} &= \frac{\sigma_{p,2} - \sigma_{p,1}}{2} \\ &= \frac{60 - (-140)}{2} \\ &= \boxed{100 \text{ MPa}}\end{aligned}$$

### Question 3

Find the normal and shearing stresses on an oblique plane defined by

$$\left\{ l = \sqrt{\frac{3}{13}}, m = \sqrt{\frac{1}{13}}, n = \sqrt{\frac{9}{13}} \right\}$$

The principal stresses are  $\sigma_1 = 40 \text{ MPa}$ ,  $\sigma_2 = 15 \text{ MPa}$ , and  $\sigma_3 = 25 \text{ MPa}$ .

Since the principal stresses are given directly from the problem statement, the octohedral stresses and the maximum shearing stress can be found directly:

$$\begin{aligned}\sigma_{oct} &= \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) \\ &= \frac{1}{3}(40 + 15 + 25) \\ &= \boxed{26.67 \text{ MPa}} \\ \tau_{oct} &= \sqrt{\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}{9}} \\ &= \sqrt{\frac{(40 - 15)^2 + (15 - 25)^2 + (25 - 40)^2}{9}} \\ &= \boxed{10.27 \text{ MPa}} \\ \tau_{max} &= \frac{\sigma_1 - \sigma_2}{2} \\ &= \frac{40 - 15}{2} \\ &= \boxed{12.5 \text{ MPa}}\end{aligned}$$

### Question 4

A displacement field in a body is given by

$$\begin{aligned}u &= c(x^2 + 10) \\ v &= 2cxyz \\ w &= c(-xy + z^2)\end{aligned}$$

where  $c = 10^{-4}$ . Determine the state of strain on an element position at  $(0, 2, 1)$ .

Calculate all 6 unique entries of the strain tensor:

$$\begin{aligned}
 \epsilon_x|_{(0,2,1)} &= \frac{\partial u}{\partial x} = 2cx = 2(10^{-4})(0) = 0 \\
 \epsilon_y|_{(0,2,1)} &= \frac{\partial v}{\partial y} = 2cz = 2(10^{-4})(1) = 2(10^{-4}) \\
 \epsilon_z|_{(0,2,1)} &= \frac{\partial w}{\partial z} = 2cz = 2(10^{-4})(1) = 2(10^{-4}) \\
 \epsilon_{xy}|_{(0,2,1)} &= \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) = \frac{1}{2}(0 + 0) = 0 \\
 \epsilon_{xz}|_{(0,2,1)} &= \frac{1}{2} \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) = \frac{1}{2}(0 + -cy) = \frac{1}{2}(0 + -2(10^{-4})) = -10^{-4} \\
 \epsilon_{yz}|_{(0,2,1)} &= \frac{1}{2} \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) = \frac{1}{2}(2cy + -cx) = \frac{1}{2}(2(10^{-4})(1) + 0) = 10^{-4}
 \end{aligned}$$

Therefore, the strain tensor is

$$\epsilon = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix} \times 10^{-4}$$

## Question 5

The displacement field and strain distribution in a member have the form

$$\begin{aligned}
 \epsilon_x &= a_0 + a_1 y^2 + y^4 \\
 \epsilon_y &= b_0 + b_1 x^2 + x^4 \\
 \gamma_{xy} &= c_0 + c_1 xy(x^2 + y^2 + c_2) = c_0 + c_1 x^3 y + c_1 xy^3 + c_1 c_2 xy
 \end{aligned}$$

What relationships connecting the constants (a's, b's, and c's) make the foregoing expressions possible?

There are 6 compatibility equations that must be satisfied since 6 strains cannot be arbitrarily chosen. Assuming the strains involving  $z$  are zero, there is only one compatibility equation that must be satisfied:

$$2 \frac{\partial^2 \epsilon_{xy}}{\partial x \partial y} = \frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} \quad (1)$$

Evaluating the partial derivatives:

$$\begin{aligned}\frac{\partial^2 \epsilon_{xy}}{\partial x \partial y} &= \frac{1}{2} \frac{\partial}{\partial x} (c_1 x^3 + 3c_1 x y^2 + c_1 c_2 x) \\ &= \frac{1}{2} (3c_1 x^2 + 3c_1 y^2 + c_1 c_2) \\ \frac{\partial^2 \epsilon_x}{\partial y^2} &= 2a_1 + 12y^2 \\ \frac{\partial^2 \epsilon_y}{\partial x^2} &= 2b_1 + 12x^2\end{aligned}$$

Substituting into (1):

$$\begin{aligned}3c_1 x^2 + 3c_1 y^2 + c_1 c_2 &= 2a_1 + 12y^2 + 2b_1 + 12x^2 \\ 3c_1 x^2 + 3c_1 y^2 + c_1 c_2 - 12x^2 - 12y^2 &= 2a_1 + 2b_1 \\ 3c_1 x^2 - 12x^2 + 3c_1 y^2 - 12y^2 + c_1 c_2 &= 2a_1 + 2b_1 \\ (3c_1 - 12)x^2 + (3c_1 - 12)y^2 + c_1 c_2 &= 2a_1 + 2b_1 \\ \underbrace{(3c_1 - 12)(x^2 + y^2)}_{\text{function of x, y}} &= \underbrace{2a_1 + 2b_1 - c_1 c_2}_{\text{constant}} \stackrel{\text{set}}{=} 0 \\ \implies 3c_1 - 12 &= 0 \\ \implies c_2 &= \frac{2}{c_1} (a_1 + b_1)\end{aligned}$$

Therefore, the relationship between the constants is

$$\boxed{\begin{aligned}c_1 &= 4 \\ c_2 &= \frac{1}{2} (a_1 + b_1)\end{aligned}}$$

## Question 6

Find the normal strain in the members  $\overline{AB}$  and  $\overline{BC}$  of the pin-connected plane structure shown in Fig.(1) if point B is moved leftward 2.5 mm. Assume that axial deformation is uniform throughout the length of each member.

Convert the displacement of point B with respect to point A to a strain:

$$\epsilon_{AB} = \frac{\Delta L}{L} = \frac{-2.5e - 3}{1.8} = \boxed{-1.39 \times 10^{-3}}$$

Find the displacement of point B with respect to point C:

$$\begin{aligned}l_{BC,f} &= \sqrt{2.4^2 + (1.8 - 2.5 \times 10^{-3})^2} = 2.9985 \text{ m} \\ \Delta l_{BC} &= l_{BC,f} - l_{BC,i} = 2.9985 - 3.0 = -1.50 \times 10^{-3} \text{ m}\end{aligned}$$

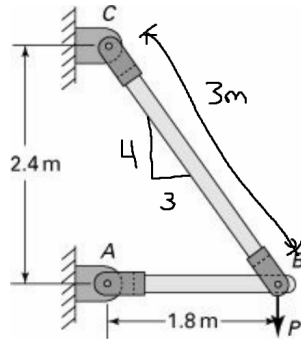


Figure 1: Pin connected plane structure

Convert the displacement of point B with respect to point C to a strain:

$$\epsilon_{BC} = \frac{\Delta L}{L} = \frac{-1.5e-3}{3.0} = \boxed{-5.00 \times 10^{-4}}$$