Question 1

At a point in a loaded body, the stress relative to an x, y, and z coordinate system are

$$\sigma = \begin{bmatrix} 40 & 40 & 30 \\ 40 & 20 & 0 \\ 30 & 0 & 20 \end{bmatrix} \text{MPa}$$

Determine the normal stress σ and the shearing stress τ on a plane whose outward normal is oriented at angles of 40°, 75°, and 54° with the x, y, and z axes, respectively.

First calculate the normal vector:

$$\hat{n} = \begin{bmatrix} \cos(40°) \\ \cos(75°) \\ \cos(54°) \end{bmatrix}$$

Then calculate the normal stress:

$$\begin{aligned} \sigma_n &= \hat{n}^T \sigma \hat{n} \\ &= \begin{bmatrix} \cos(40°) & \cos(75°) & \cos(54°) \end{bmatrix} \begin{bmatrix} 40 & 40 & 30 \\ 40 & 20 & 0 \\ 30 & 0 & 20 \end{bmatrix} \begin{bmatrix} \cos(40°) \\ \cos(75°) \\ \cos(54°) \end{bmatrix} \\ &= \boxed{74.6\,\text{MPa}} \end{aligned}$$

To determine shearing stress magnitude, the following equation is used:

$$\begin{aligned} \tau &= \sqrt{p_x^2 + p_y^2 + p_z^2 - \sigma_n^2} \\ &= ((\sigma\hat{n})^T(\sigma\hat{n}) - \sigma_n^2)^{1/2} \\ &= (5926.856 - 5565.16)^{1/2} \\ &= \boxed{19.02\,\text{MPa}} \end{aligned}$$

Question 2

The state of stress at a point in a member relative to an x, y, and z coordinate system is given by

$$\sigma = \begin{bmatrix} -100 & 0 & -80 \\ 0 & 20 & 0 \\ -80 & 0 & 20 \end{bmatrix} \text{MPa}$$

(a) The principal stresses by expansion of the characteristic stress determinant.

(b) The octahedral stresses and the maximum shearing stress.

(a)

The characteristic polynomial is given by:

$$\sigma_p^3 - I_1\sigma_p^2 + I_2\sigma_p - I_3 = 0$$

where I_1, I_2, and I_3 are the first, second, and third invariants, respectively. The invariants are given by:

$$\begin{aligned}
I_1 &= \sigma_x + \sigma_y + \sigma_z = -60 \\
I_2 &= \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2 = \mathrm{tr}(\det(\sigma)\sigma^{-1}) = -10000 \\
I_3 &= \det(\sigma) = -168000
\end{aligned}$$

Plugging into calculator, the eigenvalues are:

$$\boxed{\sigma_p = \begin{bmatrix} -140 & 0 & 0 \\ 0 & 60 & 0 \\ 0 & 0 & 20 \end{bmatrix} \mathrm{MPa}}$$

(b)

The octahedral stresses are given by:

$$\begin{aligned}
\sigma_{oct} &= \frac{1}{3}(\sigma_{p,1} + \sigma_{p,2} + \sigma_{p,3}) \\
&= \frac{1}{3}(-140 + 60 + 20) \\
&= \boxed{-20\,\mathrm{MPa}}
\end{aligned}$$

The magnitude of the octohedral shear stress is given by:

$$\begin{aligned}
\tau_{oct} &= \sqrt{\frac{(\sigma_{p,1} - \sigma_{p,2})^2 + (\sigma_{p,2} - \sigma_{p,3})^2 + (\sigma_{p,3} - \sigma_{p,1})^2}{9}} \\
&= \sqrt{\frac{(-140 - 60)^2 + (60 - 20)^2 + (20 + 140)^2}{9}} \\
&= \boxed{86.41\,\mathrm{MPa}}
\end{aligned}$$

The maximum shearing stress is given by:

$$\begin{aligned}
\tau_{max} &= \frac{\sigma_{p,2} - \sigma_{p,1}}{2} \\
&= \frac{60 - (-140)}{2} \\
&= \boxed{100\,\text{MPa}}
\end{aligned}$$

Question 3

Find the normal and shearing stresses on an oblique plane defined by

$$\left\{ l = \sqrt{\frac{3}{13}},\ m = \sqrt{\frac{1}{13}},\ n = \sqrt{\frac{9}{13}} \right\}$$

The principal stresses are $\sigma_1 = 40\,\text{MPa}$, $\sigma_2 = 15\,\text{MPa}$, and $\sigma_3 = 25\,\text{MPa}$.

Since the principal stresses are given directly from the problem statement, the octohedral stresses and the maximum shearing stress can be found directly:

$$\begin{aligned}
\sigma_{oct} &= \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) \\
&= \frac{1}{3}(40 + 15 + 25) \\
&= \boxed{26.67\,\text{MPa}} \\
\tau_{oct} &= \sqrt{\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}{9}} \\
&= \sqrt{\frac{(40 - 15)^2 + (15 - 25)^2 + (25 - 40)^2}{9}} \\
&= \boxed{10.27\,\text{MPa}} \\
\tau_{max} &= \frac{\sigma_1 - \sigma_2}{2} \\
&= \frac{40 - 15}{2} \\
&= \boxed{12.5\,\text{MPa}}
\end{aligned}$$

Question 4

A displacement field in a body is given by

$$\begin{aligned}
u &= c(x^2 + 10) \\
v &= 2cyz \\
w &= c(-xy + z^2)
\end{aligned}$$

where $c = 10^{-4}$. Determine the state of strain on an element position at (0, 2, 1).

Calculate all 6 unique entries of the strain tensor:

$$\begin{aligned}
\epsilon_x|_{(0,2,1)} &= \frac{\partial u}{\partial x} = 2cx = 2(10^{-4})(0) = 0 \\
\epsilon_y|_{(0,2,1)} &= \frac{\partial v}{\partial y} = 2cz = 2(10^{-4})(1) = 2(10^{-4}) \\
\epsilon_z|_{(0,2,1)} &= \frac{\partial w}{\partial z} = 2cz = 2(10^{-4})(1) = 2(10^{-4}) \\
\epsilon_{xy}|_{(0,2,1)} &= \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \frac{1}{2}(0+0) = 0 \\
\epsilon_{xz}|_{(0,2,1)} &= \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = \frac{1}{2}(0+-cy) = \frac{1}{2}(0+-2(10^{-4}) = -10^{-4} \\
\epsilon_{yz}|_{(0,2,1)} &= \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = \frac{1}{2}(2cy+-cx) = \frac{1}{2}(2(10^{-4})(1)+0) = 10^{-4}
\end{aligned}$$

Therefore, the strain tensor is

$$\boxed{\epsilon = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix} \times 10^{-4}}$$

Question 5

The displacement field and strain distribution in a member have the form

$$\begin{aligned}
\epsilon_x &= a_0 + a_1 y^2 + y^4 \\
\epsilon_y &= b_0 + b_1 x^2 + x^4 \\
\gamma_{xy} = c_0 + c_1 xy(x^2 + y^2 + c_2) &= c_0 + c_1 x^3 y + c_1 x y^3 + c_1 c_2 xy
\end{aligned}$$

What relationships connecting the constants (a's, b's, and c's) make the foregoing expressions possible?

There are 6 compatibility equations that must be satisfied since 6 strains cannot be arbitrarily chosen. Assuming the strains involving z are zero, there is only one compatibility equation that must be satisfied:

$$2\frac{\partial^2 \epsilon_{xy}}{\partial x \partial y} = \frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} \tag{1}$$

Evaluating the partial derivatives:

$$\begin{aligned}
\frac{\partial^2 \epsilon_{xy}}{\partial x \partial y} &= \frac{1}{2}\frac{\partial}{\partial x}(c_1x^3 + 3c_1xy^2 + c_1c_2x) \\
&= \frac{1}{2}(3c_1x^2 + 3c_1y^2 + c_1c_2) \\
\frac{\partial^2 \epsilon_x}{\partial y^2} &= 2a_1 + 12y^2 \\
\frac{\partial^2 \epsilon_y}{\partial x^2} &= 2b_1 + 12x^2
\end{aligned}$$

Substituting into (1):

$$\begin{aligned}
3c_1x^2 + 3c_1y^2 + c_1c_2 &= 2a_1 + 12y^2 + 2b_1 + 12x^2 \\
3c_1x^2 + 3c_1y^2 + c_1c_2 - 12x^2 - 12y^2 &= 2a_1 + 2b_1 \\
3c_1x^2 - 12x^2 + 3c_1y^2 - 12y^2 + c_1c_2 &= 2a_1 + 2b_1 \\
(3c_1 - 12)x^2 + (3c_1 - 12)y^2 + c_1c_2 &= 2a_1 + 2b_1 \\
\underbrace{(3c_1 - 12)(x^2 + y^2)}_{\text{function of x, y}} &= \underbrace{2a_1 + 2b_1 - c_1c_2}_{\text{constant}} \overset{\text{set}}{=} 0
\end{aligned}$$

$$\begin{aligned}
&\implies 3c_1 - 12 = 0 \\
&\implies c_2 = \frac{2}{c_1}(a_1 + b_1)
\end{aligned}$$

Therefore, the relationship between the constants is

$$\boxed{\begin{aligned} c_1 &= 4 \\ c_2 &= \frac{1}{2}(a_1 + b_1) \end{aligned}}$$

Question 6

Find the normal strain in the members $\overline{AB}$ and $\overline{BC}$ of the pin-connected plane structure shown in Fig.(1) if point B is moved leftward 2.5 mm. Assume that axial deformation is uniform throughout the length of each member.

Convert the displacement of point B with respect to point A to a strain:

$$\epsilon_{AB} = \frac{\Delta L}{L} = \frac{-2.5e-3}{1.8} = \boxed{-1.39 \times 10^{-3}}$$

Find the displacement of point B with respect to point C:

$$\begin{aligned}
l_{BC,f} &= \sqrt{2.4^2 + (1.8 - 2.5 \times 10^{-3})^2} = 2.9985\,\text{m} \\
\Delta l_{BC} &= l_{BC,f} - l_{BC,i} = 2.9985 - 3.0 = -1.50 \times 10^{-3}\,\text{m}
\end{aligned}$$

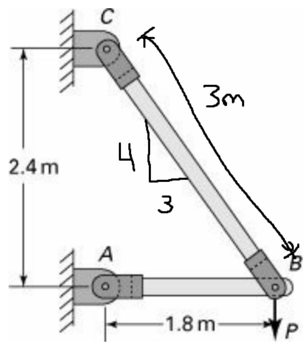

Figure 1: Pin connected plane structure

Convert the displacement of point B with respect to point C to a strain:

$$\epsilon_{BC} = \frac{\Delta L}{L} = \frac{-1.5e-3}{3.0} = \boxed{-5.00 \times 10^{-4}}$$