Question 1

A thin rectangular plate $a = 30\,\text{mm} \times b = 15\,\text{mm}$ is acted upon by a stress distribution (Fig. 1) resulting in the uniform strain $\epsilon_x = 400\,\mu$, $\epsilon_y = 200\,\mu$, and $\gamma_{xy} = -300\,\mu$. Determine the changes in length of diagonals $\overline{QB}$ and $\overline{AC}$.

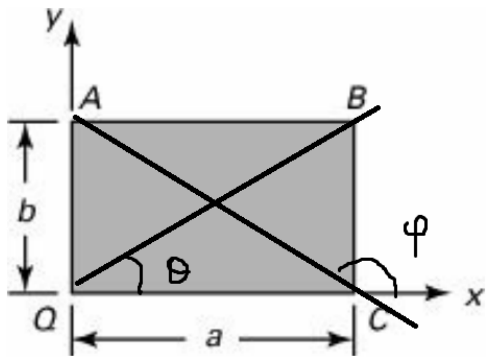

Figure 1: Stress distribution on a thin rectangular plate.

First, the expression for the strain along $\overline{QB}$, $\epsilon_{x'}$, is governed by the following equation:

$$\epsilon_{x'} = \epsilon_x \cos^2\theta + \epsilon_y \sin^2\theta + \gamma_{xy} \sin\theta\cos\theta \tag{1}$$

The angle θ is given by:

$$\begin{aligned} \theta &= \arctan\left(\frac{b}{a}\right) \\ &= \arctan\left(\frac{15}{30}\right) \\ &= 26.57^\circ \end{aligned}$$

By the rectangular geometry,

$$\phi = 180 - \theta = 153.4^\circ$$

The length of the diagonals are given by:

$$\begin{aligned} \overline{QB} = \overline{AC} &= \sqrt{a^2 + b^2} \\ &= \sqrt{30^2 + 15^2} \\ &= 33.54\,\text{mm} \end{aligned}$$

Using strain-displacement relations, the change in length of $\overline{QB}$ is given by:

$$\begin{aligned}\Delta\overline{QB} &= \overline{QB}\epsilon_{x'} \\ &= 33.54(400\times10^{-6}\cos^2(26.57) + 200\times10^{-6}\sin^2(26.57) - 300\times10^{-6}\sin(26.57)\cos(26.57)) \\ &= \boxed{0.00805\,\text{mm}}\end{aligned}$$

Similarly, $\Delta\overline{AC}$ is given by:

$$\begin{aligned}\Delta\overline{AC} &= \overline{AC}\epsilon_{x'} \\ &= 33.54(400\times10^{-6}\cos^2(153.4) + 200\times10^{-6}\sin^2(153.4) - 300\times10^{-6}\sin(153.4)\cos(153.4)) \\ &= \boxed{0.0161\,\text{mm}}\end{aligned}$$

Question 2

A $3\,\text{m}\times 2\,\text{m}$ thin rectangular plate is deformed by the movement of point B to B' as shown by the dashed lines in Fig. 2. Assuming a displacement field of the form $u = c_1xy$ and $v = c_2xy$ where c_1 and c_2 are constants, determine,

(a) Expressions for displacements u and v.

(b) Strain components ϵ_x, ϵ_y, and γ_{xy} at point B.

(c) The normal strain $\epsilon_{x'}$ in the direction of $\overline{QB}$.

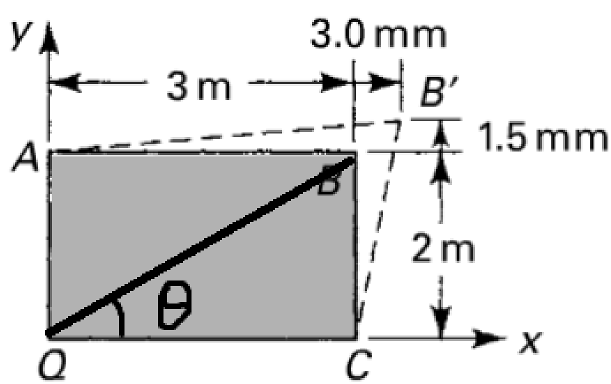

Figure 2: Stress distribution on a thin rectangular plate.

(a)

Using the geometry of the problem, the constants c_1 and c_2 can be determined. First, c_1:

$$\begin{aligned}
u &= c_1 xy \\
c_1 &= \frac{u}{xy} \\
&= \frac{3}{2 \times 3 \times 10^6} \\
&= 5 \times 10^{-7}\,\mathrm{mm}^{-1}
\end{aligned}$$

Similarly, c_2 is given by:

$$\begin{aligned}
v &= c_2 xy \\
c_2 &= \frac{v}{xy} \\
&= \frac{1.5}{2 \times 3 \times 10^6} \\
&= 2.5 \times 10^{-7}\,\mathrm{mm}^{-1}
\end{aligned}$$

Therefore the expressions for u and v are given by:

$$\boxed{\begin{aligned}
u &= 5 \times 10^{-7} xy\ [\mathrm{mm}] \\
v &= 2.5 \times 10^{-7} xy\ [\mathrm{mm}]
\end{aligned}}$$

(b)

The strain components ϵ_x, ϵ_y, and γ_{xy} at point B are given by:

$$\begin{aligned}
\epsilon_x|_B &= \frac{\partial u}{\partial x}|_B = 5 \times 10^{-7} y|_B = 5 \times 10^{-7} \times 2000 = \boxed{1.00 \times 10^{-3}} \\
\epsilon_y|_B &= \frac{\partial v}{\partial y}|_B = 2.5 \times 10^{-7} x|_B = 2.5 \times 10^{-7} \times 3000 = \boxed{7.50 \times 10^{-4}} \\
\gamma_{xy}|_B &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}|_B = 5 \times 10^{-7} x + 2.5 \times 10^{-7} y|_B \\
&= 5 \times 10^{-7} \times 3000 + 2.5 \times 10^{-7} \times 2000 = \boxed{2.00 \times 10^{-3}}
\end{aligned}$$

(c)

Recall that the normal strain $\epsilon_{x'}$ in the direction of $\overline{QB}$ is given by:

$$\epsilon_{x'} = \epsilon_x \cos^2\theta + \epsilon_y \sin^2\theta + 2\gamma_{xy} \sin\theta\cos\theta$$

$$\begin{aligned}\theta &= \tan^{-1}\left(\frac{\overline{BC}}{\overline{QC}}\right)\\ &= \tan^{-1}\left(\frac{2000}{3000}\right)\\ &= 33.69^\circ\end{aligned}$$

Therefore, by substitution,

$$\begin{aligned}\epsilon_{x'} &= \epsilon_x \cos^2\theta + \epsilon_y \sin^2\theta + 2\gamma_{xy}\sin\theta\cos\theta\\ &= 1.00\times10^{-3}\cos^2(33.69) + 7.50\times10^{-4}\sin^2(33.69) + 22.00\times10^{-3}\sin(33.69)\cos(33.69)\\ &= \boxed{1.85\times10^{-3}}\end{aligned}$$

(d)

The relevant compatibility equation for field is given by:

$$2\frac{\partial^2\epsilon_{xy}}{\partial x\partial y} = \frac{\partial^2\epsilon_x}{\partial y^2} + \frac{\partial^2\epsilon_y}{\partial x^2}$$

The left hand side evaluates to:

$$\begin{aligned}\frac{\partial^2\gamma_{xy}}{\partial x\partial y} &= \frac{\partial^2}{\partial x\partial y}(5\times10^{-7}x + 2.5\times10^{-7}y)\\ &= 0\end{aligned}$$

The right hand side evaluates to:

$$\begin{aligned}\frac{\partial^2\epsilon_x}{\partial y^2} + \frac{\partial^2\epsilon_y}{\partial x^2} &= \frac{\partial}{\partial y}(5\times10^{-7}y) + \frac{\partial}{\partial x}(2.5\times10^{-7}x)\\ &= 0 + 0\end{aligned}$$

$\boxed{\text{Therefore, the compatibility equation is satisfied and the strain field is possible.}}$

Question 3

At a point in a stressed body, the strains, related to the coordinate set xyz, are given by:

$$\epsilon = \begin{bmatrix}\epsilon_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz}\\ \frac{1}{2}\gamma_{xy} & \epsilon_y & \frac{1}{2}\gamma_{yz}\\ \frac{1}{2}\gamma_{xz} & \frac{1}{2}\gamma_{yz} & \epsilon_z\end{bmatrix} = \begin{bmatrix}400 & 100 & 0\\ 100 & 0 & -200\\ 0 & -200 & 600\end{bmatrix}\times10^{-6} \tag{2}$$

Determine,

(a) the strain invariants.

(b) the normal strain in the x' direction, which is directed at an angle 30° from the x-axis.

(c) the principal strains ϵ_1, ϵ_2, and ϵ_3.

(d) the maximum shear strain.

(a)

Determine I_1, I_2, and I_3.

$$\begin{aligned}
I_1 &= \epsilon_x + \epsilon_y + \epsilon_z \\
&= 400 + 0 + 600 \\
&= \boxed{1.00 \times 10^{-3}} \\
I_2 &= \epsilon_x\epsilon_y + \epsilon_y\epsilon_z + \epsilon_z\epsilon_x - \gamma_{xy}^2 - \gamma_{yz}^2 - \gamma_{xz}^2 \\
&= 400(0) + 0(-200) + 600(400) - 100^2 - (-200)^2 - 0^2 \\
&= \boxed{1.90 \times 10^{-7}} \\
I_3 &= det(\epsilon) \\
&= \boxed{-2.2 \times 10^{-11}}
\end{aligned}$$

(b)

The normal strain $\epsilon_{x'}$ in the direction of $\theta = 30°$ is given by:

$$\begin{aligned}
\epsilon_{x'} &= \epsilon_x \cos^2\theta + \epsilon_y \sin^2\theta + \gamma_{xy} \sin\theta\cos\theta \\
&= 400\cos^2(30°) + 0\sin^2(30°) + 2(100)\sin(30°)\cos(30°) \\
&= \boxed{386 \times 10^{-6}}
\end{aligned}$$

(c)

The characteristic equation for the principal strains is given by:

$$\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0$$

Plugging these into polynomial solver yields:

$$\boxed{\epsilon = 6.64 \times 10^{-4},\ 4.16 \times 10^{-4},\ -7.97 \times 10^{-5}}$$

(d)

The maximum shear strain is given by the difference between the maximum and minimum principal strains:

$$\begin{aligned}\gamma_{max} &= \epsilon_1 - \epsilon_3 \\ &= \boxed{7.44 \times 10^{-4}}\end{aligned}$$

Question 4

A 40 mm diameter bar ABC is composed of an aluminum part AB and a steel part BC as shown in Fig. 3. After axial force P is applied, a strain gage attached to the steel measures normal strain at the longitudinal direction as $\epsilon_s = 600\mu$. Determine,

(a) the magnitude of the applied force P.

(b) the total elongation of the bar if each material behaves elastically. Take $E_{\text{aluminum}} = 70\,\text{GPa}$ and $E_{\text{steel}} = 210\,\text{GPa}$.

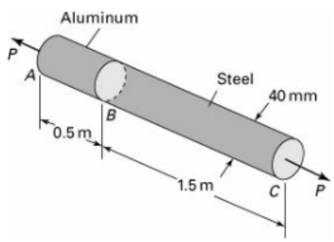

Figure 3: A composite steel rod

(a)

By Hooke's Law, the strain in the steel part is given by:

$$\begin{aligned}\sigma_s &= E_{\text{steel}}\epsilon_s \\ \frac{P}{A_{\text{steel}}} &= E_{\text{steel}}\epsilon_s \\ P &= E_{\text{steel}}A_{\text{steel}}\epsilon_s \\ &= 210 \times \frac{\pi}{4} \times (40)^2 \times 600 \times 10^{-6} \\ &= \boxed{158\,\text{kN}}\end{aligned}$$

(b)

The total elongation of the bar is given by the addition of the elongation of the aluminum and steel parts. First we find the strain in the aluminum part:

$$\begin{aligned}\epsilon_{\text{aluminum}} &= \frac{P}{A_{\text{aluminum}}E_{\text{aluminum}}}\\ &= \frac{158}{\frac{\pi}{4}(40)^2(70)}\\ &= 1.796\times10^{-3}\end{aligned}$$

Next,

$$\begin{aligned}\Delta L &= \epsilon_{\text{aluminum}}L_{\text{aluminum}} + \epsilon_{\text{steel}}L_{\text{steel}}\\ &= 1.796\times10^{-3}(0.5\times10^3) + 600\times10^{-6}(1.5\times10^3)\\ &= \boxed{1.80\,\text{mm}}\end{aligned}$$

Question 5

A solid sphere of diameter d experiences a uniform pressure of p. Given: $d = 250\,\text{mm}$, $p = 160\,\text{MPa}$, $E = 70\,\text{GPa}$, and $\nu = 0.3$. Determine,

(a) the decrease in volume of the sphere ΔV.

(b) the decrease in circumference of the sphere.

Note: Volume of a sphere is $V = \frac{4}{3}\pi r^3$, where $r = \frac{d}{2}$.

Bulk modulus will be used to find the change in volume of the sphere. The bulk modulus is given by:

$$\begin{aligned}K &= \frac{E}{3(1-2\nu)}\\ &= \frac{70}{3(1-2(0.3))}\\ &= 58.333\,\text{GPa}\end{aligned}$$

By definition of bulk modulus, we have:

$$\begin{aligned}\frac{\Delta V}{V} &= -\frac{p}{K}\\ \Delta V &= -\frac{p}{K}V\\ &= -\frac{160}{58.333\times10^3}\times\frac{4}{3}\pi(250/2)^3\\ &= -22439.9\,\text{mm}^3\\ &= \boxed{-2.24\times10^4\,\text{mm}^3}\end{aligned}$$

The new volume is:

$$\begin{aligned} V_{\text{new}} &= V + \Delta V \\ &= \frac{4}{3}\pi(250/2)^3 - 2.24e4 \\ &= \boxed{8.158790 \times 10^6\,\text{mm}^3} \end{aligned}$$

The new radius is:

$$\begin{aligned} r_{\text{new}} &= \sqrt[3]{\frac{3V_{\text{new}}}{4\pi}} \\ &= \boxed{124.8856\,\text{mm}} \end{aligned}$$

The decrease in circumference is:

$$\begin{aligned} \Delta C &= 2\pi(r_{\text{new}} - r) \\ &= 2\pi(124.8856 - 125) \\ &= \boxed{-0.719\,\text{mm}} \end{aligned}$$

Question 6

A 50 mm square plate is subjected to the stresses as shown in Fig. 4. What deformation is experienced by diagonal $\overline{BD}$? Determine the stress on planes perpendicular and parallel to $\overline{BD}$ and then employ generalized Hooke's law. Express the solution in terms of E for $\nu = 0.3$.

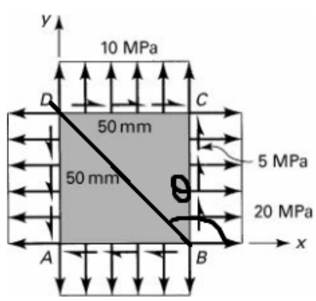

Figure 4: A square plate

Generalized hooke's law

$$\begin{aligned} \epsilon_x &= \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)] \\ \epsilon_y &= \frac{1}{E}[\sigma_y - \nu(\sigma_x + \sigma_z)] \\ \epsilon_z &= \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)] \end{aligned}$$

From the Fig. 4,

$$\begin{aligned}
\sigma_x &= 20\,\text{MPa} \\
\sigma_y &= 10\,\text{MPa} \\
\sigma_z &= 0\,\text{MPa} \\
\tau_{xy} &= 5\,\text{MPa} \\
\nu &= 0.3
\end{aligned}$$

The strain in the x and $y$directions are:

$$\begin{aligned}
\epsilon_x &= \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)] \\
&= \frac{1}{E}[20 - 0.3(10 + 0)] \\
&= \frac{17\,\text{MPa}}{E} \\
\epsilon_y &= \frac{1}{E}[\sigma_y - \nu(\sigma_x + \sigma_z)] \\
&= \frac{1}{E}[10 - 0.3(20 + 0)] \\
&= \frac{4\,\text{MPa}}{E} \\
\epsilon_z &= \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)] \\
&= \frac{1}{E}[0 - 0.3(20 + 10)] \\
&= \frac{-9\,\text{MPa}}{E}
\end{aligned}$$

The shear strain is:

$$\begin{aligned}
\gamma_{xy} &= \frac{\tau_{xy}}{G} \\
&= \frac{5}{\frac{E}{2(1+\nu)}} \\
&= \frac{5}{\frac{E}{2(1+0.3)}} \\
&= \frac{13\,\text{MPa}}{E}
\end{aligned}$$

From the Figure, θ:

$$\begin{aligned}
\theta &= 90 + \tan^{-1}\left(\frac{50}{50}\right) \\
&= 135^\circ
\end{aligned}$$

The length of the diagonal is:

$$\begin{aligned}\overline{BD} &= \sqrt{50^2 + 50^2} \\ &= 70.71\,\text{mm}\end{aligned}$$

The strain in the diagonal is:

$$\begin{aligned}\epsilon_{BD} &= \epsilon_x \cos^2\theta + \epsilon_y \sin^2\theta + \gamma_{xy}\sin\theta\cos\theta \\ &= \frac{17\,\text{MPa}}{E}\cos^2(135^\circ) + \frac{4\,\text{MPa}}{E}\sin^2(135^\circ) + \frac{13\,\text{MPa}}{E}\sin(135^\circ)\cos(135^\circ) \\ &= \frac{1}{E}(17\cos^2(135^\circ) + 4\sin^2(135^\circ) + 13\sin(135^\circ)\cos(135^\circ)) \\ &= \frac{4\,\text{MPa}}{E}\end{aligned}$$

The change in length of the diagonal is:

$$\begin{aligned}\Delta L_{BD} &= \epsilon_{BD}\overline{BD} \\ &= \frac{4\,\text{MPa}}{E}70.71\,\text{mm} \\ &= \boxed{\frac{283\,\text{mm}\,\text{MPa}}{E}}\end{aligned}$$