Question 1

A propped cantilever beam AB is supported at one end by a spring of constant stiffness k and subjected to a uniform load of intensity p, as shown in Fig. 1. Use the unit-load method to determine the deflection of the beam at its free end B.

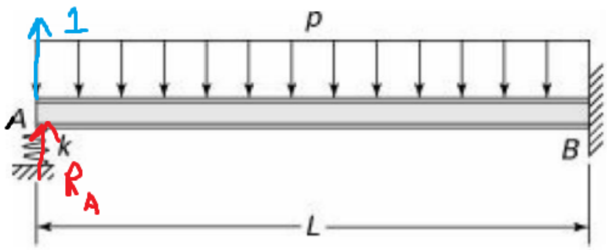

Figure 1: Propped cantilever beam AB

First, the moment from A to B is found:

$$M = \frac{-px^2}{2}$$

Apply a virtual unit load at A. The moment equation is:

$$m = (1)x$$

The virtual work equation is:

$$\begin{aligned}\delta_A &= \int_0^L \frac{Mm}{EI} dx \\ &= \frac{1}{EI} \int_0^L \left(\frac{-px^2}{2} \right) (1x) dx \\ &= \frac{-pL^4}{8EI}\end{aligned}$$

By Hooke's Law, the reaction force is $R_A = -k\delta_A$. Therefore,

$$\boxed{R_A = \frac{kpL^4}{8EI}}$$

Question 2

A beam is supported and loaded as shown in Fig. 2. Apply Castigliano's theorem to determine the reactions.

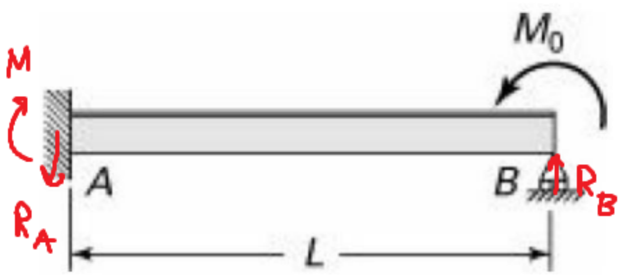

Figure 2: Beam supported and loaded as shown.

The moment equation from B to A is:

$$\begin{aligned} M &= R_B x + M_0 \\ \frac{\partial M}{\partial R_B} &= x \end{aligned}$$

By Castigliano's theorem,

$$\begin{aligned} \delta_B &= \int_0^L \frac{M}{EI}\frac{\partial M}{\partial R_B} dx \\ &= \int_0^L \frac{(R_B x + M_0)}{EI}(x) dx \\ &= \frac{R_B L^3}{3EI} + \frac{M_0 L^2}{2EI} \end{aligned}$$

There is a pin at B. It cannot carry and deflection. Therefore, $\delta_B = 0$. Solving for R_B,

$$\begin{aligned} 0 &= \frac{R_B L^3}{3EI} + \frac{M_0 L^2}{2EI} \\ \implies R_B &= \frac{-3M_0}{2L} \end{aligned}$$

The reaction at A has equal magnitude and opposite direction to R_B. Therefore,

$$\boxed{R_A = \frac{-3M_0}{2L}}$$

The moment at A is found by summing the moment about A:

$$\begin{aligned} \sum M_A &= 0 \\ &= -M + R_B L + M_0 \\ \implies M &= R_B L + M_0 \\ &= \frac{-3M_0}{2} + M_0 \\ &= \boxed{\frac{-M_0}{2}} \end{aligned}$$

Question 3

A steel rod of constant flexural rigidity is described by Fig. 3. For force P applied at the simply supported end, derive a formula for roller reaction Q. Apply Castigliano's theorem.

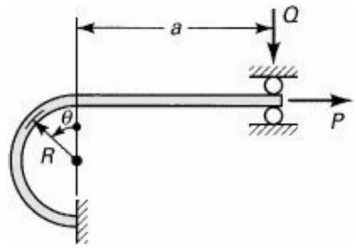

Figure 3: Problem diagram for Question 3.

The moment equation of the straight section is:

$$\begin{aligned} M &= -Qx \\ \implies \frac{\partial M}{\partial Q} &= -x \end{aligned}$$

For the curved section, a cut is made. The FBD is shown in Fig. 4.

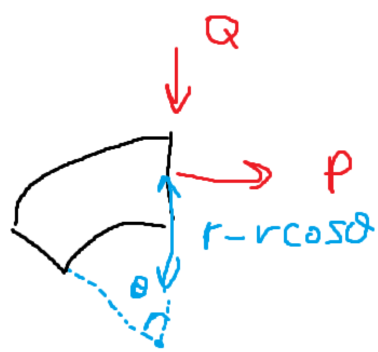

Figure 4: Free body diagram of the curved section.

The equation of the moment is:

$$\begin{aligned} M &= M_a - QR\sin\theta - PR(1-\cos\theta) \\ &= -Q(a + R\sin\theta) - PR(1-\cos\theta) \\ \implies \frac{\partial M}{\partial Q} &= -a - R\sin\theta \end{aligned}$$

The roller cannot carry a deflection. By Castigliano's theorem,

$$\delta_Q = \frac{1}{EI}\left[\int_0^a M\frac{\partial M}{\partial Q}dx + \int_0^\pi M\frac{\partial M}{\partial Q}Rd\theta\right] \overset{\text{set}}{=} 0$$

$$\implies 0 = \int_0^a Qx^2dx + \int_0^\pi QR(a+R\sin\theta)^2 - PR^2(1-\cos\theta)(a+R\sin\theta)d\theta$$

By CAS software (Matlab Symbolic Toolbox), the integral is:

$$0 = \frac{Qa^3}{3} + 2PR^3 + \frac{Q\pi R^3}{2} + 4QR^2a + \pi PR^2a + \pi QRa^2$$

$$\implies Q = \boxed{\frac{-PR^2(2R+a\pi)}{\frac{a^3}{3} + R(4Ra + \frac{\pi R^2}{2} + \pi a^2)}}$$

Question 4

Using Castigliano's theorem, find the slope of the deflection curve at midlength C of a beam due to applied couple moment M_0 Fig. 5.

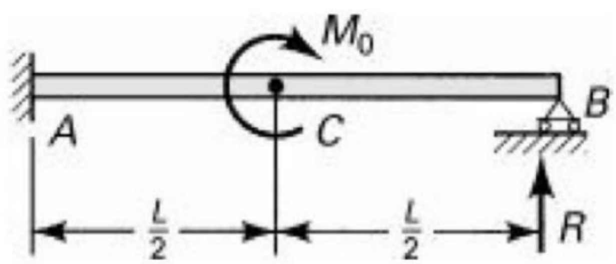

Figure 5: Propped cantilever beam AB

First, the reaction force R needs to be determined. The moment equation from B to A is:

$$M = Rx - M_0\langle x - \frac{L}{2}\rangle^0, \quad \langle x - \frac{L}{2}\rangle^0 = H\left(x - \frac{L}{2}\right)$$

$$\implies \frac{\partial M}{\partial R} = x$$

$$\implies \frac{\partial M}{\partial M_0} = -\langle x - \frac{L}{2}\rangle^0$$

By Castigliano's theorem, the deflection at B is:

$$\delta_B = \frac{1}{EI}\left[\int_0^{L/2} Rx^2dx - \int_{L/2}^{L} Rx^2 - M_0xdx\right]$$

$$= \frac{1}{EI}\left[\frac{RL^3}{24} - \frac{L^2(9M_0 - 7LR)}{24}\right]$$

Since the pin at B cannot carry deflection, $\delta_B = 0$. Therefore,

$$\delta_B \overset{\text{set}}{=} 0 = \frac{1}{EI}\left[\frac{RL^3}{24} - \frac{L^2(9M_0 - 7LR)}{24}\right]$$
$$\implies R = \boxed{\frac{9M_0L}{8L}}$$

To find the slope at C, apply Castigliano's theorem again:

$$\theta_C = \frac{1}{EI}\left[\cancel{\int_0^{L/2} M\left(\frac{\partial M}{\partial M_0}\right)} dx + \int_{L/2}^{L} M\left(\frac{\partial M}{\partial M_0}\right) dx\right]$$
$$= \frac{1}{EI}\int_{L/2}^{L} M_0 - Rx dx$$
$$= \frac{4M_0L - 3RL^2}{8EI}$$
$$= \boxed{\frac{5LM_0}{64EI}}$$

Question 5

The symmetrical frame shown in Fig. 6 supports a uniform loading of p per unit length. Assume that each horizontal and vertical member has the modulus of rigidity E_1I_1 and E_2I_2, respectively. Determine the resultant R_A at the left support, employing Castigliano's theorem.

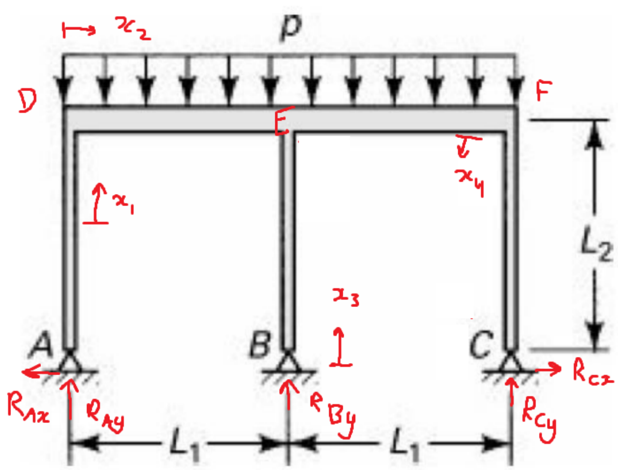

Figure 6: Symmetrical frame

From A to D, the moment equation is:

$$M_{AD} = -R_{Ax}x$$
$$\implies \frac{\partial M_{AD}}{\partial R_{Ax}} = -x \implies \frac{\partial M_{AD}}{\partial R_{Ay}} = 0$$

From D to F, the moment equation is:

$$\begin{aligned} M_{DF} &= M_{AD}|_{x=L_2} + R_{Ay}x - \frac{px^2}{2} \\ &= -R_{Ax}L_2 + R_{Ay}x - \frac{px^2}{2} \\ \implies \frac{\partial M_{DF}}{\partial R_{Ax}} &= -L_2 \\ \implies \frac{\partial M_{DF}}{\partial R_{Ay}} &= x \end{aligned}$$

From B to E, the moment equation is:

$$\begin{aligned} M_{BE} &= 0 \\ \implies \frac{\partial M_{EB}}{\partial R_{Ax}} &= 0 \\ \implies \frac{\partial M_{EB}}{\partial R_{Ay}} &= 0 \end{aligned}$$

From C to F, the moment equation is:

$$\begin{aligned} M_{CF} &= M_{DF}|_{x=2L_1} \\ &= -R_{Ax}L_2 + 2R_{Ay}L_1 - 2pL_1^2 \\ \implies \frac{\partial M_{CF}}{\partial R_{Ax}} &= -L_2 \\ \implies \frac{\partial M_{CF}}{\partial R_{Ay}} &= 2L_1 \end{aligned}$$

By Castigliano's theorem, the horizontal deflection at A is:

$$\begin{aligned} \delta_{A,x} =& \frac{1}{E_1I_1}\left[\int_0^{L_2} M_{AD}\left(\frac{\partial M_{AD}}{\partial R_{Ax}}\right)dx + \cancel{\int_0^{L_2} M_{BE}\left(\frac{\partial M_{BE}}{\partial R_{Ax}}\right)dx} + \int_0^{L_2} M_{CF}\left(\frac{\partial M_{CF}}{\partial R_{Ax}}\right)dx\right] \\ &+ \frac{1}{E_2I_2}\left[\int_0^{2L_1} M_{DF}\left(\frac{\partial M_{DF}}{\partial R_{Ay}}\right)dx\right] \\ &= \frac{1}{E_1I_1}\left[\int_0^{L_2} R_{Ax}x^2dx + \int_0^{L_2}(-R_{Ax}L_2 + 2R_{Ay}L_1 - 2pL_1^2)(-L_2)dx\right] \\ &+ \frac{1}{E_2I_2}\left[\int_0^{2L_1}(-R_{Ax}L_2 + R_{Ay}x - \frac{px^2}{2})xdx\right] \\ &= \frac{1}{E_1I_1}\left[\frac{L_2^3R_{Ax}}{3} + L_2^2(2pL_1^2 - 2R_{Ay}L_1 + L_2R_{Ax})\right] - \frac{1}{E_2I_2}\left[\frac{2pL_1^4}{3} - \frac{2R_{Ay}L_1^3}{3} + \frac{L_2R_{Ax}L_1^2}{2}\right] \end{aligned}$$

Since the pin at A cannot carry deflection, $\delta_{A,x} = 0$. Therefore,

$$\delta_{A,x} \overset{\text{set}}{=} 0$$
$$\implies R_{Ax} = -\frac{\frac{L_2^2(2L_1^2 - 2L_1 R_{Ay})}{E_1 I_1} + \frac{\frac{8L_1^3 R_{Ay}}{3} - \frac{2L_1^4 p}{3}}{E_2 I_2}}{\frac{4L_2^3}{3E_1 I_1} - \frac{2L_1^2 L_2}{E_2 I_2}}$$

By Castigliano's theorem, the vertical deflection at A is:

$$\begin{aligned}\delta_{A,y} =& \frac{1}{E_1 I_1}\left[\cancel{\int_0^{L_2} M_{AD}\left(\frac{\partial M_{AD}}{\partial R_{Ay}}\right)} dx + \cancel{\int_0^{L_2} M_{BE}\left(\frac{\partial M_{BE}}{\partial R_{Ay}}\right)} dx + \int_0^{L_2} M_{CF}\left(\frac{\partial M_{CF}}{\partial R_{Ay}}\right) dx\right] \\ &+ \frac{1}{E_2 I_2}\left[\int_0^{2L_1} M_{DF}\left(\frac{\partial M_{DF}}{\partial R_{Ay}}\right) dx\right] \\ &= \frac{1}{E_1 I_1}\left[\int_0^{L_2} (-R_{Ax}L_2 + 2R_{Ay}L_1 - 2pL_1^2)(2L_1) dx\right] + \frac{1}{E_2 I_2}\left[\int_0^{2L_1} (-R_{Ax}L_2 + R_{Ay}x - \frac{px^2}{2})(x) dx\right]\end{aligned}$$

Too much algebra, by Matlab Symbolic Toolbox:

```
ans =

struct with fields:

Rax: (3*E1*I1*L1^2*p*(2*L1^2 + L2*L1))/(L2*(6*E1*I1*L1^2 + E1*I1*L1*L2 - 3*E2*I2*L2^2))
Ray: (3*L1*p*(3*E1*I1*L1^2 + E1*I1*L1*L2 - E2*I2*L2^2))/...
(6*E1*I1*L1^2 + E1*I1*L1*L2 - 3*E2*I2*L2^2)
```

The script was used to aid in this solution:

```
clc; clear; close all;
syms x L1 L2 p Rax Ray E1 I1 E2 I2
delta_Ax = (1/(E1*I1)) * (int(Rax*x^2, x, 0, L2) + int((-Rax*L2 + 2*Ray*L1 - 2*p*L1^2)*(
    , x, 0, L2)) + (1/(E2*I2)) * (int((-Rax*L2 + Ray*x - (p*x^2)/2)*x, x, 0, 2*L1))

delta_Ay = (1/(E1*I1)) * (int((-Rax*L2 + 2*Ray*L1 - 2*p*L1^2)*(2*L1), x, 0, L2)) ...
    + (1/(E2*I2)) * (int((-Rax*L2 + Ray*x - (p*x^2)/2)*(x), x, 0, 2*L1))

eqn1 = delta_Ax == 0;
eqn2 = delta_Ay == 0;
solve(eqn1, Rax)
solve([eqn1, eqn2], [Rax, Ray])
```

Question 6

A frame of constant flexural rigidity EI carries a concentrated load P at point E (Fig. 7). Determine the reaction R at support A using Castigliano's theorem.

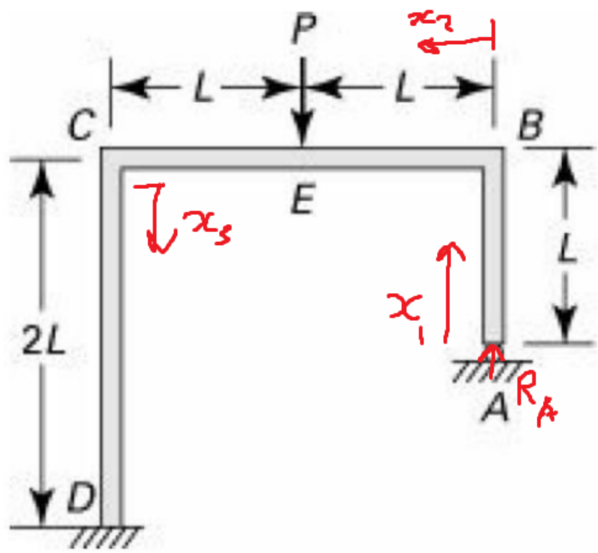

Figure 7: Frame with pinned connection at A

The moment equation from A to B is:

$$M_{AB} = 0$$
$$\implies \frac{\partial M_{AB}}{\partial R_A} = 0$$

The moment equation from B to C is:

$$M_{BC} = R_A x - P\langle x - L\rangle^1, \quad \langle x - L\rangle^1 = (x - L)H(x - L)$$
$$\implies \frac{\partial M_{BC}}{\partial R_A} = x$$

The moment equation from C to D is:

$$M_{CD} = M_{BC}|_{x=2L} = 2LR_A - PL$$
$$\implies \frac{\partial M_{CD}}{\partial R_A} = 2L$$

By Castigliano's theorem, the deflection at A is:

$$\delta_A = \frac{1}{EI}\left[\cancel{\int_0^L M_{AB}\left(\frac{\partial M_{AB}}{\partial R_A}\right)} dx + \int_0^{2L} M_{BC}\left(\frac{\partial M_{BC}}{\partial R_A}\right) dx + \int_0^{2L} M_{CD}\left(\frac{\partial M_{CD}}{\partial R_A}\right) dx\right]$$
$$= \frac{1}{EI}\left[\int_0^L R_A x^2 dx + \int_L^{2L} R_A x^2 - Px(x - L)dx + \int_0^{2L} (2LR_A - PL)(2L)dx\right]$$
$$= \frac{1}{EI}\left[\frac{L^3 R_A}{3} - \frac{4L^3(P - 2R_A)}{3} - \frac{L^3(5P - 14R_A)}{6}\right]$$

Since the pin at A cannot carry deflection, $\delta_A = 0$. Therefore,

$$\delta_A \overset{\text{set}}{=} 0 = \frac{1}{EI}\left[\frac{L^3 R_A}{3} - \frac{4L^3(P - 2R_A)}{3} - \frac{L^3(5P - 14R_A)}{6}\right]$$
$$\implies R_A = \boxed{\frac{29P}{64}}$$