Question 1

Fig. 1 shows a long, thing steel plate of thickness t, width $2h$, and length $2a$. The plate is subjected to loads that produce the uniform stress σ_o at the ends. The edges at $y = \pm h$ are placed between two rigid walls. Show that, by using an inverse method, the displacements are expressed by

$$u = -\frac{1-\nu^2}{E}\sigma_o x, \quad v = 0, \quad w = \frac{\nu(1+\nu)}{E}\sigma_o z$$

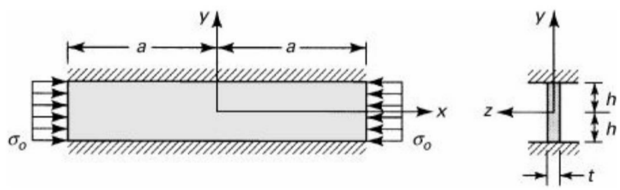

Figure 1: Steel plate subjected to uniform stress σ_o at the ends.

From the figure, $\sigma_x = -\sigma_o$ and $\sigma_z = 0$. The plane strain equations are

$$\begin{aligned}
\epsilon_x &= \frac{1}{E}\left(\sigma_x - \nu\sigma_y\right) \\
\epsilon_y &= \frac{1}{E}\left(\sigma_y - \nu\sigma_x\right) \\
\epsilon_z &= -\frac{1}{E}\left(\sigma_x + \sigma_y\right)
\end{aligned}$$

Since there is a rigid wall at $y = \pm h$, $\epsilon_y = 0$. Therefore,

$$\begin{aligned}
\epsilon_y &= \frac{1}{E}\left(\sigma_y - \nu\sigma_x\right) \overset{\text{set}}{=} 0 \\
\implies \sigma_y &= \nu\sigma_x = -\nu\sigma_o
\end{aligned}$$

Also,

$$\epsilon_y = \frac{\partial v}{\partial y} = 0 \implies \boxed{v = 0}$$

From the ϵ_x equation,

$$\begin{aligned}
\epsilon_x &= \frac{1}{E}\left(\sigma_x - \nu\sigma_y\right) \\
&= \frac{1}{E}\left(-\sigma_o - \nu(-\nu\sigma_o)\right) \\
&= \frac{1}{E}\left(-\sigma_o + \nu^2\sigma_o\right) \\
&= \frac{1}{E}\left(\nu^2 - 1\right)\sigma_o
\end{aligned}$$

Since $\epsilon_x = \frac{\partial u}{\partial x}$, we can integrate to find u

$$\begin{aligned}
\epsilon_x &= \frac{\partial u}{\partial x} \\
\implies u &= \boxed{\frac{1}{E}\left(\nu^2 - 1\right)\sigma_o x}
\end{aligned}$$

From the ϵ_z equation,

$$\begin{aligned}
\epsilon_z &= -\frac{1}{E}\left(\sigma_x + \sigma_y\right) \\
&= -\frac{1}{E}\left(-\sigma_o - \nu\sigma_o\right) \\
&= \frac{1}{E}\left(1 + \nu\right)\sigma_o
\end{aligned}$$

Since $\epsilon_z = \frac{\partial w}{\partial z}$, we can integrate to find w

$$\begin{aligned}
\epsilon_z &= \frac{\partial w}{\partial z} \\
\implies w &= \boxed{\frac{1}{E}\left(1 + \nu\right)\sigma_o z}
\end{aligned}$$

Question 2

Determine whether the following stress distribution is a valid solution for a two-dimensional problem:

$$\sigma_x = -ax^2y, \quad \sigma_y = -\frac{1}{3}ay^3, \quad \tau_{xy} = -axy^2$$

where a is a constant. Body forces may be neglected.

The compatibility equation is

$$\nabla^4\Phi = \frac{\partial^4\Phi}{\partial x^4} + 2\frac{\partial^4\Phi}{\partial x^2\partial y^2} + \frac{\partial^4\Phi}{\partial y^4} = 0$$

Term by term,

$$\begin{aligned}
\frac{\partial^4\Phi}{\partial x^4} &= \frac{\partial^2}{\partial x^2}\sigma_y \\
&= \frac{\partial^2}{\partial x^2}\left(-\frac{1}{3}ay^3\right) \\
&= 0 \\
\frac{\partial^4\Phi}{\partial y^4} &= \frac{\partial^2}{\partial y^2}\sigma_x \\
&= \frac{\partial^2}{\partial y^2}\left(-ax^2y\right) \\
&= 0 \\
\frac{\partial^4\Phi}{\partial x^2\partial y^2} &= \frac{\partial^2}{\partial x^2}\sigma_x \\
&= \frac{\partial^2}{\partial x^2}\left(-ax^2y\right) \\
&= -2ay
\end{aligned}$$

Therefore,

$$\begin{aligned}
\nabla^4\Phi &= \frac{\partial^4\Phi}{\partial x^4} + 2\frac{\partial^4\Phi}{\partial x^2\partial y^2} + \frac{\partial^4\Phi}{\partial y^4} \\
&= 0 + 2(-2ay) + 0 \\
&= -4ay
\end{aligned}$$

Since $\nabla^4\Phi \neq 0$, the stress distribution is **not** a valid solution for a two-dimensional problem.

Question 3

Figure 2 shows a thin cantilever beam of unit thickness carrying a uniform load of intensity p per unit length. Assume that the stress function is expressed by

$$\Phi = ax^2 + bx^2y + cy^3 + dy^5 + ex^2y^3$$

in which $a, ..., e$ are constants. Determine (a) the required values of $a, ..., e$ so that Φ is biharmonic; (b) the stresses σ_x, σ_y, and τ_{xy}

(a)

The biharmonic equation is

$$\nabla^4\Phi = \frac{\partial^4\Phi}{\partial x^4} + 2\frac{\partial^4\Phi}{\partial x^2\partial y^2} + \frac{\partial^4\Phi}{\partial y^4} = 0$$

Figure P3.15.

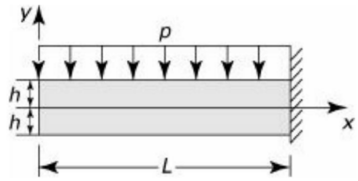

Figure 2: Problem diagram for Question 3.

Substituting Φ into the biharmonic equation,

$$\begin{aligned} \frac{\partial^4\Phi}{\partial x^4} &= 0 \\ \frac{\partial^4\Phi}{\partial y^4} &= 120dy \\ \frac{\partial^4\Phi}{\partial x^2\partial y^2} &= \frac{\partial^2}{\partial x^2}\left(6cy + 20dy^3 + 6ex^2y\right) = 12ey \\ \implies \nabla^4\Phi &= 0 + 2(12ey) + 120dy \overset{\text{set}}{=} 0 \\ \implies e &= -5d \end{aligned}$$

Therefore, Φ is biharmonic when $\boxed{e = -5d}$.

(b)

The stress function can now be expressed as

$$\Phi = ax^2 + bx^2y + cy^3 + dy^5 - 5dx^2y^3 = ax^2 + bx^2y + cy^3 + d(y^5 - 5x^2y^3)$$

The boundary conditions are

$$\begin{aligned} \tau_{xy}\big|_{y=\pm h} &= 0 \\ \sigma_y\big|_{y=h} &= \frac{-pL}{Lt} = -\frac{p}{t} \\ \sigma_y\big|_{y=-h} &= 0 \end{aligned}$$

Since there is no axial load,

$$\int_{-h}^{h} \sigma_x y dy = 0$$

Finding expressions for σ_x, σ_y, and τ_{xy},

$$\begin{aligned} \sigma_x &= \frac{\partial^2\Phi}{\partial y^2} = 6cy + 20dy^2 - 30dx^2y \\ \sigma_y &= -\frac{\partial^2\Phi}{\partial x^2} = 2a + 2by - 10dy^3 \\ \tau_{xy} &= -\frac{\partial^2\Phi}{\partial x\partial y} = -2bx + 30dxy^2 \end{aligned}$$

Applying the boundary conditions, first at $\tau_{xy}|_{y=h}$,

$$\begin{aligned}
\tau_{xy}|_{y=h} &= 0 \\
\implies -2bx + 30dxh^2 &= 0 \\
\implies b &= 15dh^2
\end{aligned}$$

second at $\sigma_y|_{y=-h}$,

$$\begin{aligned}
\sigma_y|_{y=-h} &= 0 \\
\implies 2a + 2(15dh^2)(-h) - 10d(-h)^3 &= 0 \\
\implies a &= 10dh^3
\end{aligned}$$

lastly at $\sigma_y|_{y=h}$,

$$\begin{aligned}
\sigma_y|_{y=h} &= -\frac{p}{t} \\
\implies 2(10dh^3) + 2(15dh^2)(h) - 10d(h)^3 &= -\frac{p}{t} \\
\implies d &= \frac{p}{40h^3t}
\end{aligned}$$

Question 4

A prismatic bar is restrained in the x (axial) and y direction but free to expand in the z direction. Determine the stresses and strains in the bar for a temperature rise of T_1 degrees.

Since the bar is restrained, $\epsilon_y = \epsilon_z = 0$. From strain relations,

$$\begin{aligned}
\epsilon_x &= \frac{1}{E}\left(\sigma_x - \nu\sigma_y\right) + \alpha T_1 \overset{\text{set}}{=} 0 \\
\implies \sigma_x &= \nu\sigma_y - E\alpha T_1
\end{aligned}$$

In the y direction,

$$\begin{aligned}
\epsilon_y &= \frac{1}{E}\left(\sigma_y - \nu\sigma_x\right) + \alpha T_1 \overset{\text{set}}{=} 0 \\
\implies \sigma_y &= \nu\sigma_x - E\alpha T_1
\end{aligned}$$

Substituting σ_y into the σ_x equation,

$$\begin{aligned}
\sigma_x &= -\nu\sigma_y - E\alpha T_1 \\
&= -\nu(\nu\sigma_x - E\alpha T_1) - E\alpha T_1 \\
&= -\nu^2\sigma_x + \nu E\alpha T_1 - E\alpha T_1 \\
\implies \sigma_x &= \frac{\nu E\alpha T_1 - E\alpha T}{1+\nu^2} \\
&= \frac{E\alpha T}{\nu - 1}
\end{aligned}$$

Since the equations for σ_x and σ_y are linear, by symmetry, $\sigma_x = \sigma_y$. In the z direction,

$$\begin{aligned} \epsilon_z &= -\frac{\nu}{E}\left(\sigma_x + \sigma_y\right) + \alpha T_1 & &= -\frac{\nu}{E}\frac{2E\alpha T}{\nu - 1} + \alpha T_1 \\ &= -\frac{2\nu\alpha T}{\nu - 1} + \alpha T_1 \end{aligned}$$

Additionally,

$$\begin{aligned} \tau_{xy} &= \tau_{xz} = \tau_{yz} = 0 \\ \sigma_z &= 0 \end{aligned}$$

Question 5

The symmetrical frame shown in Fig. 3 supports a uniform loading of p per unit length. Assume that each horizontal and vertical member has the modulus of rigidity $E_1 I_1$ and $E_2 I_2$, respectively. Determine the resultant R_A at the left support, employing Castigliano's theorem.

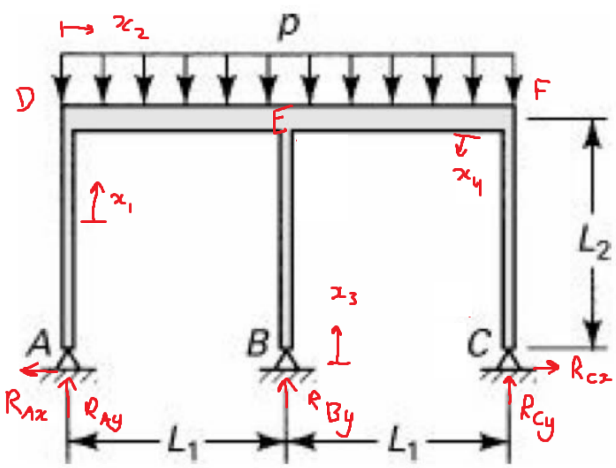

Figure 3: Symmetrical frame

From A to D, the moment equation is:

$$\begin{aligned} M_{AD} &= -R_{Ax} x \\ \implies \frac{\partial M_{AD}}{\partial R_{Ax}} &= -x \implies \frac{\partial M_{AD}}{\partial R_{Ay}} = 0 \end{aligned}$$

From D to F, the moment equation is:

$$\begin{aligned}
M_{DF} &= M_{AD}|_{x=L_2} + R_{Ay}x - \frac{px^2}{2} \\
&= -R_{Ax}L_2 + R_{Ay}x - \frac{px^2}{2} \\
\implies \frac{\partial M_{DF}}{\partial R_{Ax}} &= -L_2 \\
\implies \frac{\partial M_{DF}}{\partial R_{Ay}} &= x
\end{aligned}$$

From B to E, the moment equation is:

$$\begin{aligned}
M_{BE} &= 0 \\
\implies \frac{\partial M_{EB}}{\partial R_{Ax}} &= 0 \\
\implies \frac{\partial M_{EB}}{\partial R_{Ay}} &= 0
\end{aligned}$$

From C to F, the moment equation is:

$$\begin{aligned}
M_{CF} &= M_{DF}|_{x=2L_1} \\
&= -R_{Ax}L_2 + 2R_{Ay}L_1 - 2pL_1^2 \\
\implies \frac{\partial M_{CF}}{\partial R_{Ax}} &= -L_2 \\
\implies \frac{\partial M_{CF}}{\partial R_{Ay}} &= 2L_1
\end{aligned}$$

By Castigliano's theorem, the horizontal deflection at A is:

$$\begin{aligned}
\delta_{A,x} =& \frac{1}{E_1I_1}\left[\int_0^{L_2} M_{AD}\left(\frac{\partial M_{AD}}{\partial R_{Ax}}\right)dx + \cancel{\int_0^{L_2} M_{BE}\left(\frac{\partial M_{BE}}{\partial R_{Ax}}\right)dx} + \int_0^{L_2} M_{CF}\left(\frac{\partial M_{CF}}{\partial R_{Ax}}\right)dx\right] \\
&+ \frac{1}{E_2I_2}\left[\int_0^{2L_1} M_{DF}\left(\frac{\partial M_{DF}}{\partial R_{Ay}}\right)dx\right] \\
&= \frac{1}{E_1I_1}\left[\int_0^{L_2} R_{Ax}x^2dx + \int_0^{L_2}(-R_{Ax}L_2 + 2R_{Ay}L_1 - 2pL_1^2)(-L_2)dx\right] \\
&+ \frac{1}{E_2I_2}\left[\int_0^{2L_1}(-R_{Ax}L_2 + R_{Ay}x - \frac{px^2}{2})xdx\right] \\
&= \frac{1}{E_1I_1}\left[\frac{L_2^3R_{Ax}}{3} + L_2^2(2pL_1^2 - 2R_{Ay}L_1 + L_2R_{Ax})\right] - \frac{1}{E_2I_2}\left[\frac{2pL_1^4}{3} - \frac{2R_{Ay}L_1^3}{3} + \frac{L_2R_{Ax}L_1^2}{2}\right]
\end{aligned}$$

Since the pin at A cannot carry deflection, $\delta_{A,x} = 0$. Therefore,

$$
\begin{aligned}
&\delta_{A,x} \overset{\text{set}}{=} 0 \\
\implies\ & R_{Ax} = -\frac{\frac{L_2^2(2L_1^2 - 2L_1 R_{Ay})}{E_1 I_1} + \frac{\frac{8L_1^3 R_{Ay}}{3} - \frac{2L_1^4 p}{3}}{E_2 I_2}}{\frac{4L_2^3}{3E_1 I_1} - \frac{2L_1^2 L_2}{E_2 I_2}}
\end{aligned}
$$

By Castigliano's theorem, the vertical deflection at A is:

$$
\begin{aligned}
\delta_{A,y} =& \frac{1}{E_1 I_1}\left[\cancel{\int_0^{L_2} M_{AD}\left(\frac{\partial M_{AD}}{\partial R_{Ay}}\right)} dx + \cancel{\int_0^{L_2} M_{BE}\left(\frac{\partial M_{BE}}{\partial R_{Ay}}\right)} dx + \int_0^{L_2} M_{CF}\left(\frac{\partial M_{CF}}{\partial R_{Ay}}\right) dx\right] \\
&+ \frac{1}{E_2 I_2}\left[\int_0^{2L_1} M_{DF}\left(\frac{\partial M_{DF}}{\partial R_{Ay}}\right) dx\right] \\
&= \frac{1}{E_1 I_1}\left[\int_0^{L_2} (-R_{Ax}L_2 + 2R_{Ay}L_1 - 2pL_1^2)(2L_1) dx\right] + \frac{1}{E_2 I_2}\left[\int_0^{2L_1} (-R_{Ax}L_2 + R_{Ay}x - \frac{px^2}{2})(x) dx\right]
\end{aligned}
$$

Too much algebra, by Matlab Symbolic Toolbox:

```
ans =

struct with fields:

Rax: (3*E1*I1*L1^2*p*(2*L1^2 + L2*L1))/(L2*(6*E1*I1*L1^2 + E1*I1*L1*L2 - 3*E2*I2*L2^2))
Ray: (3*L1*p*(3*E1*I1*L1^2 + E1*I1*L1*L2 - E2*I2*L2^2))/...
(6*E1*I1*L1^2 + E1*I1*L1*L2 - 3*E2*I2*L2^2)
```

The script was used to aid in this solution:

```
clc; clear; close all;
syms x L1 L2 p Rax Ray E1 I1 E2 I2
delta_Ax = (1/(E1*I1)) * (int(Rax*x^2, x, 0, L2) + int((-Rax*L2 + 2*Ray*L1 - 2*p*L1^2)*(
    , x, 0, L2)) + (1/(E2*I2)) * (int((-Rax*L2 + Ray*x - (p*x^2)/2)*x, x, 0, 2*L1))

delta_Ay = (1/(E1*I1)) * (int((-Rax*L2 + 2*Ray*L1 - 2*p*L1^2)*(2*L1), x, 0, L2)) ...
    + (1/(E2*I2)) * (int((-Rax*L2 + Ray*x - (p*x^2)/2)*(x), x, 0, 2*L1))

eqn1 = delta_Ax == 0;
eqn2 = delta_Ay == 0;
solve(eqn1, Rax)
solve([eqn1, eqn2], [Rax, Ray])
```

Question 6

A frame of constant flexural rigidity EI carries a concentrated load P at point E (Fig. 4). Determine the reaction R at support A using Castigliano's theorem.

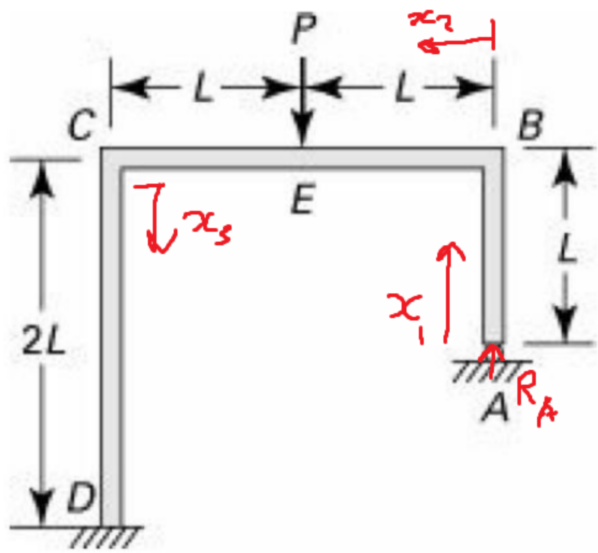

Figure 4: Frame with pinned connection at A

The moment equation from A to B is:

$$\begin{aligned} M_{AB} &= 0 \\ \implies \frac{\partial M_{AB}}{\partial R_A} &= 0 \end{aligned}$$

The moment equation from B to C is:

$$\begin{aligned} M_{BC} &= R_A x - P\langle x - L\rangle^1, \quad \langle x - L\rangle^1 = (x - L)H\left(x - L\right) \\ \implies \frac{\partial M_{BC}}{\partial R_A} &= x \end{aligned}$$

The moment equation from C to D is:

$$\begin{aligned} M_{CD} &= M_{BC}|_{x=2L} = 2LR_A - PL \\ \implies \frac{\partial M_{CD}}{\partial R_A} &= 2L \end{aligned}$$

By Castigliano's theorem, the deflection at A is:

$$\begin{aligned} \delta_A &= \frac{1}{EI}\left[\cancel{\int_0^L M_{AB}\left(\frac{\partial M_{AB}}{\partial R_A}\right)} dx + \int_0^{2L} M_{BC}\left(\frac{\partial M_{BC}}{\partial R_A}\right) dx + \int_0^{2L} M_{CD}\left(\frac{\partial M_{CD}}{\partial R_A}\right) dx\right] \\ &= \frac{1}{EI}\left[\int_0^L R_A x^2 dx + \int_L^{2L} R_A x^2 - Px(x - L)dx + \int_0^{2L} (2LR_A - PL)(2L)dx\right] \\ &= \frac{1}{EI}\left[\frac{L^3 R_A}{3} - \frac{4L^3(P - 2R_A)}{3} - \frac{L^3(5P - 14R_A)}{6}\right] \end{aligned}$$

Since the pin at A cannot carry deflection, $\delta_A = 0$. Therefore,

$$\delta_A \overset{\text{set}}{=} 0 = \frac{1}{EI}\left[\frac{L^3 R_A}{3} - \frac{4L^3(P - 2R_A)}{3} - \frac{L^3(5P - 14R_A)}{6}\right]$$

$$\implies R_A = \boxed{\frac{29P}{64}}$$