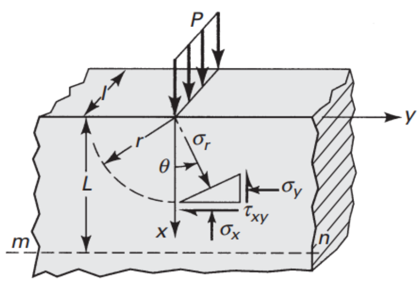
Question 1

Show that the case of a concentrated load on a straight boundary (Figure 1a) is represented by the stress function

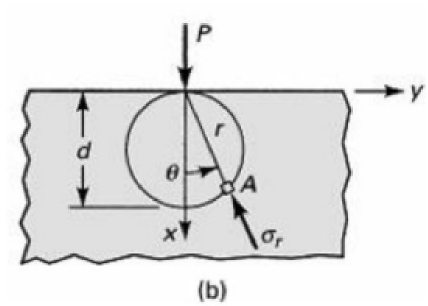
$$\Phi = -\frac{P}{\pi} r\theta \sin\theta$$

and derive

$$\sigma_r = -\frac{2P}{\pi}\frac{\cos\theta}{r}, \quad \sigma_\theta = 0, \quad \tau_{r\theta} = 0$$



(a) Concentrated load on a straight boundary of a large plate



(b) Circle of constant radial stress

The stress function must satisfy the biharmonic equation

$$\nabla^4\Phi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) = 0$$

First we calculate partials for Φ,

$$\begin{aligned}
\frac{\partial \Phi}{\partial r} &= -\frac{P}{\pi}\theta\sin\theta \\
\frac{\partial^2 \Phi}{\partial r^2} &= 0 \\
\frac{\partial \Phi}{\partial \theta} &= -\frac{P}{\pi} r\sin\theta - \frac{P}{\pi} r\theta\cos\theta \\
\frac{\partial^2 \Phi}{\partial \theta^2} &= \frac{P}{\pi} r\theta\sin\theta - \frac{2P}{\pi} r\cos\theta
\end{aligned}$$

Next, $\nabla^2\Phi$ is found by

$$\begin{aligned}
\nabla^2\Phi &= \frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} \\
&= 0 + \frac{1}{r}\left(-\frac{P}{\pi}\theta\sin\theta\right) + \frac{1}{r^2}\left(\frac{P}{\pi}r\theta\sin(\theta) - \frac{2P}{\pi}r\cos(\theta)\right) \\
&= -\frac{P}{\pi}\frac{\theta\sin\theta}{r} + \frac{P}{\pi}\frac{\theta\sin\theta}{r} - \frac{2P}{\pi}\frac{\cos\theta}{r} \\
&= -\frac{2P}{\pi r}\cos\theta
\end{aligned}$$

Using the result for $\nabla^2\Phi$ to find partials for $\nabla^4\Phi$,

$$\begin{aligned}
\frac{\partial\nabla^2\Phi}{\partial r} &= \frac{2P\cos\theta}{\pi r^2} \\
\frac{\partial^2\nabla^2\Phi}{\partial r^2} &= -\frac{4P\cos\theta}{\pi r^3} \\
\frac{\partial^2\nabla^2\Phi}{\partial\theta^2} &= \frac{2P\cos\theta}{\pi r}
\end{aligned}$$

Finally, for $\nabla^4\Phi$,

$$\begin{aligned}
\nabla^4\Phi &= \left(-\frac{4P\cos\theta}{\pi r^3}\right) + \frac{1}{r}\left(\frac{2P\cos\theta}{\pi r^2}\right) + \frac{1}{r^2}\left(\frac{2P\cos\theta}{\pi r}\right) \\
&= -\frac{4P\cos\theta}{\pi r^3} + \frac{2P\cos\theta}{\pi r^3} + \frac{2P\cos\theta}{\pi r^3} \\
&= \boxed{0}
\end{aligned}$$

Therefore, Φ satisfies the biharmonic equation.

Next, σ_r is found by

$$\begin{aligned}
\sigma_r &= \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} \\
&= \frac{1}{r}\left(-\frac{P}{\pi}\theta\sin\theta\right) + \frac{1}{r^2}\left(\frac{P}{\pi}r\theta\sin\theta - \frac{2P}{\pi}r\cos\theta\right) \\
&= -\frac{P\theta}{\pi r}\sin\theta + \frac{P\theta}{\pi r}\sin\theta - \frac{2P}{\pi r}\cos\theta \\
&= \boxed{-\frac{2P}{\pi r}\cos\theta}
\end{aligned}$$

σ_θ is found by

$$\begin{aligned}
\sigma_\theta &= \frac{\partial^2\Phi}{\partial r^2} \\
&= \boxed{0}
\end{aligned}$$

$\tau_{r\theta}$ is found by

$$\begin{aligned}
\tau_{r\theta} &= -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \Phi}{\partial \theta}\right) \\
&= -\frac{\partial}{\partial r}\left(\frac{1}{r}\left(-\frac{P}{\pi} r \sin\theta - \frac{P}{\pi} r\theta\cos\theta\right)\right) \\
&= -\frac{\partial}{\partial r}\left(-\frac{P}{\pi}\sin\theta - \frac{P}{\pi}\theta\cos\theta\right) \\
&= \boxed{0}
\end{aligned}$$

Question 2

Referring to Figure 2, verify the results

$$\int_0^{\pi/2} (\sigma_r \sin\theta) r d\theta = \int_0^{\pi/2} \frac{2P}{\pi}\sin\theta\cos\theta d\theta = \frac{P}{\pi}$$
$$\int_{-\pi/2}^{\pi/2} (\sigma_\theta \cos\theta) r d\theta = \int_{-\pi/2}^{\pi/2} \frac{2P}{\pi}\cos^2\theta d\theta = P$$

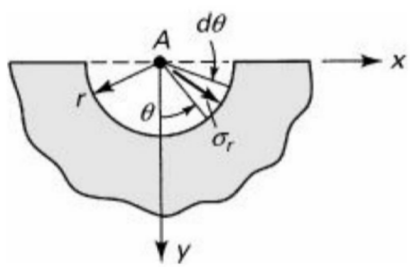


Figure 2: Problem diagram for Question 2.

For radial stress distribution in a very large plate (semi-infinite solid) under normal load at its horizontal surface, (Eq. 3.48 in the textbook)

$$\sigma_r = \frac{2P}{\pi r}\cos\theta$$

Verifying the first integral,

$$\begin{aligned}
\int_0^{\pi/2} (\sigma_r \sin\theta) r d\theta &= \int_0^{\pi/2} \frac{2P}{\pi} \sin\theta\cos\theta d\theta \\
&= \frac{2P}{\pi} \int_0^{\pi/2} \sin\theta\cos\theta d\theta \\
&= \frac{2P}{\pi} \int_0^{\pi/2} \frac{1}{2}\sin(2\theta) d\theta \\
&= \frac{2P}{\pi} \left[-\frac{1}{4}\cos(2\theta) \right]_0^{\pi/2} \\
&= \left(\frac{2P}{\pi}\right) \left[-\left(\frac{1}{4}(-1) - \frac{1}{4}(1) \right) \right] \\
&= \boxed{\frac{P}{\pi}}
\end{aligned}$$

Verifying the second integral,

$$\begin{aligned}
\int_{-\pi/2}^{\pi/2} (\sigma_\theta \cos\theta) r d\theta &= \int_{-\pi/2}^{\pi/2} \frac{2P}{\pi} \cos^2\theta d\theta \\
&= \frac{2P}{\pi} \int_{-\pi/2}^{\pi/2} \frac{1}{2}(1+\cos(2\theta)) d\theta \\
&= \frac{P}{\pi} \left[\theta + \frac{1}{2}\sin(2\theta) \right]_{-\pi/2}^{\pi/2} \\
&= \frac{P}{\pi} \left(\frac{\pi}{2} + \frac{1}{2}(-1) + \frac{\pi}{2} + \frac{1}{2}(1) \right) \\
&= \frac{P}{\pi}\pi \\
&= \boxed{P}
\end{aligned}$$

Question 3

Consider the pivot of unit thickness subject to force P per unit thickness at its vertex (Figure 3). Determine the maximum values of σ_x and τ_{xy} on a plane a distance L from the apex through the use of σ_r given by Eq. (3.43) and the formulas of the elementary theory. Take $\alpha = 30°$.

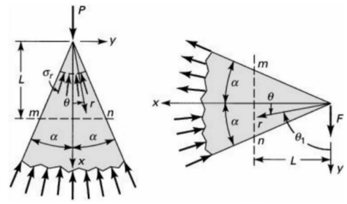


Figure 3: Problem diagram for Question 3.

From the textbook the maximum stress occurs at $\theta = 0$. $\alpha = 30^\circ = \frac{\pi}{6}$

$$\begin{aligned}(\sigma_x)_{\text{elast}} &= -\frac{P}{L(\alpha + \frac{1}{2}\sin 2\alpha)} \\ &= -\frac{P}{L(\pi/6 + \frac{1}{2}\sin \pi/3)} \\ &= \boxed{-\frac{P}{0.9566L}}\end{aligned}$$

Using elementary theory,

$$\begin{aligned}(\sigma_x)_{\text{elem}} &= -\frac{P}{A} = -\frac{P}{2L\tan\alpha} \\ &= -\frac{P}{2L\tan\pi/6} \\ &= \boxed{-\frac{P}{1.1547L}}\end{aligned}$$

From the textbook,

$$(\tau_{xy})_{\text{elast}} = \frac{P\sin\theta\cos\theta}{L}(\alpha + \frac{1}{2}\sin 2\alpha)$$

Since $\alpha \geq 30^\circ$, the maximum shear stress occurs at $\theta = 30^\circ$.

$$\begin{aligned}(\tau_{xy})_{\text{elast}} &= \frac{P\sin\theta\cos^3\theta}{L(\alpha + \frac{1}{2}\sin 2\alpha)} \\ &= \frac{P\sin(30^\circ)\cos^3(30^\circ)}{L(\pi/6 + \frac{1}{2}\sin\pi/3)} \\ &= \frac{0.324759526419P}{0.956611477491L} \\ &= \frac{0.3395P}{L} \\ &= \boxed{\frac{P}{2.946L}}\end{aligned}$$

From elementary theory,

$$(\tau_{xy})_{\text{elem}} = \frac{Tr}{J} = \boxed{0}$$

Question 4

Verify the results in Fig. 4 by employing

$$\sigma_\theta = \frac{1}{2}\sigma_o\left[\left(1+\frac{a^2}{r^2}\right)-\left(1+\frac{3a^4}{r^4}\cos 2\theta\right)\right]$$

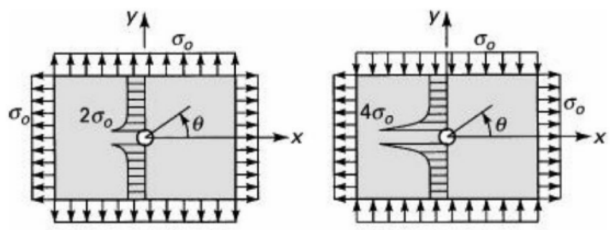


Figure 4: Problem diagram for Question 4.

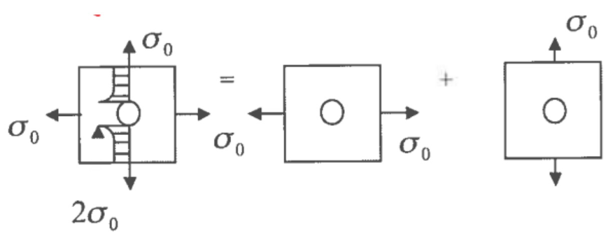


Figure 5: Superposition of stresses for Question 4.

(a)

First the axial load in the x direction will be considered. The maximum is

$$(\sigma_\theta)_{\max} = 3\sigma_o, \quad \theta = \pm\pi/2$$

Secondly, in the y direction,

$$(\sigma_\theta)_{\min} = -\sigma_o, \quad \theta = \pm\pi/2$$

Since x and y are offset by 90°, the maximum of x adds to the minimum of y. Therefore, we can verify

$$\boxed{\begin{aligned}&\sigma_r = \tau_{r\theta}0, \text{by boundary conditions}\\ &\sigma_\theta = (3+(-1))\sigma_o = 2\sigma_o\end{aligned}}$$

(b)

In the y direction, the direction of σ_o is reversed. The maximum and minimum become

$$(\sigma_\theta)_{\min} = \sigma_o, \quad \theta = 0, \pi$$

Again, since x and y are offset by 90°, so

$$\boxed{\begin{aligned}&\sigma_r = \tau_{r\theta}0, \text{by boundary conditions}\\ &\sigma_\theta = (3+1)\sigma_o = 4\sigma_o\end{aligned}}$$

Question 5

A 20-mm-thick steel bar with a slot (25-mm radii at ends) is subjected to an axial load P, as shown in Figure 4. What is the maximum stress for $P = 180\,\text{kN}$? Use Fig. D.8B to estimate the value of K.

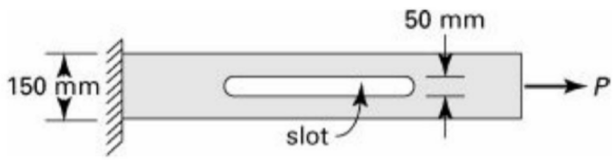


Figure 6: Problem diagram for Question 5.

Finding the stress at the slot ends (circular hole), $d = 50\,\text{mm}$, $D = 150\,\text{mm}$, $t = 20\,\text{mm}$. Then, $d/D = 1/3$ which, from Fig. D.8B, $K = 2.3$. Finding the nominal stress using

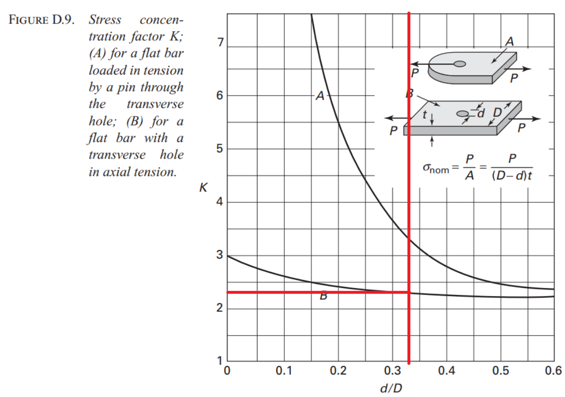


Figure 7: Chart for Question 5.

$$
\begin{aligned}
\sigma_{\text{nom}} &= \frac{P}{(D-d)t} \\
&= \frac{180 \times 10^3}{(150-50)(20) \times 10^{-6}} \\
&= 90\,\text{MPa}
\end{aligned}
$$

Finding the maximum stress using K,

$$
\begin{aligned}
\sigma_{\text{max}} &= K\sigma_{\text{nom}} \\
&= (2.3)(90) \\
&= \boxed{207\,\text{MPa}}
\end{aligned}
$$

Question 6

A thin-walled circular cylindrical vessel of diameter d and wall thickness t is subjected to internal pressure p (see Table 1.1). Given a small circular hole in the vessel wall, show that the maximum tangential and axial stress at the hole are $\sigma_\theta = \frac{5pd}{4t}$ and $\sigma_a = \frac{pd}{4t}$, respectively.

From the Table 1.1

$$\sigma_t = \frac{pr}{t} = \frac{pd}{2t}, \quad \sigma_a = \frac{pr}{2t} = \frac{pd}{4t}$$

From the textbook,

$$\begin{aligned}(\sigma_\theta)_{\max} &= 3\sigma_o, \quad \theta = \pm\pi/2 \\ (\sigma_\theta)_{\min} &= -\sigma_o, \quad \theta = \pm\pi/2\end{aligned}$$

In the a direction, the maximum and minimum are

$$\begin{aligned}(\sigma_\theta)_{a,\max} &= 3\sigma_a \\ &= \frac{3pd}{4t} \\ (\sigma_\theta)_{a,\min} &= -\sigma_a \\ &= -\frac{pd}{4t}\end{aligned}$$

In the t direction, the maximum and minimum are

$$\begin{aligned}(\sigma_\theta)_{t,\max} &= 3\sigma_t \\ &= \frac{3pd}{2t} \\ (\sigma_\theta)_{t,\min} &= -\sigma_t \\ &= -\frac{pd}{2t}\end{aligned}$$

Since a and t are offset by 90°, the maximum of a adds to the minimum of θ. In the a direction,

$$\begin{aligned}(\sigma_a)_{\max} &= 3\sigma_a + (-\sigma_t) \\ &= \frac{3pd}{4t} - \frac{pd}{2t} \\ &= \boxed{\frac{pd}{4t}}\end{aligned}$$

In the t direction,

$$\begin{aligned}(\sigma_t)_{\max} &= 3\sigma_t + (-\sigma_a) \\ &= \frac{3pd}{2t} - \frac{pd}{4t} \\ &= \boxed{\frac{5pd}{4t}}\end{aligned}$$