MEC E 380 Quiz 4 Formula Sheet

10. Energy Methods

Castigliano's Theorem: Displacement

$$\delta_i = \frac{1}{EI}\int M_i \frac{\partial M_i}{\partial P_i}dx$$

where P_i is a (dummy) concentrated load.
Angle

$$\delta_i = \frac{1}{EI}\int M_i \frac{\partial V_i}{\partial C_i}dx$$

where C_i is a (dummy) concentrated moment.
For polar coordinates, recall

$$\delta_i = \frac{1}{EI}\int M_i \frac{\partial M_i}{\partial P_i}r dr d\theta$$

3. Problems in Elasticity

3.2. Formulas

Plane Strain

On the plane x-y, the equilibrium and compatibility equations are

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} = 0 \frac{\partial^2 \sigma_x}{\partial y^2} + \frac{\partial^2 \sigma_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

$$\implies \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_x + \sigma_y) = 0$$

Strain-stress relations are

$$\begin{aligned}
\epsilon_x &= \frac{1}{E}(\sigma_x - \nu\sigma_y) \\
\epsilon_y &= \frac{1}{E}(\sigma_y - \nu\sigma_x) \\
\epsilon_z &= -\frac{\nu}{E}(\sigma_x + \sigma_y) \\
\gamma_{xy} &= \frac{\tau_{xy}}{G} \\
\gamma_{xz} &= \gamma_{yz} = 0
\end{aligned}$$

Stress-strain relations are

$$\begin{aligned}
\sigma_x &= \frac{E}{1-\nu^2}(\epsilon_x + \nu\epsilon_y) \\
\sigma_y &= \frac{E}{1-\nu^2}(\epsilon_y + \nu\epsilon_x) \\
\tau_{xy} &= G\gamma_{xy} \\
\sigma_z &= -\frac{\nu}{1-\nu}(\epsilon_x + \epsilon_y)
\end{aligned}$$

Airy's stress function Φ relations

$$\nabla^4\Phi = \frac{\partial^4\Phi}{\partial x^4} + 2\frac{\partial^4\Phi}{\partial x^2 \partial y^2} + \frac{\partial^4\Phi}{\partial y^4} = 0$$

$$\sigma_x = \frac{\partial^2\Phi}{\partial y^2}, \quad \sigma_y = \frac{\partial^2\Phi}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2\Phi}{\partial x \partial y}$$

Thermalelasticity

Thermal strain, $\epsilon t = \alpha T$, relations by superposition,

$$\begin{aligned}
\epsilon_x &= \frac{1}{E}(\sigma_x - \nu\sigma_y) + \alpha T \\
\epsilon_y &= \frac{1}{E}(\sigma_y - \nu\sigma_x) + \alpha T \\
\epsilon_z &= -\frac{\nu}{E}(\sigma_x + \sigma_y) + \alpha T \\
\gamma_{xy} &= \frac{\tau_{xy}}{G}
\end{aligned}$$

Thermal stress relations,

$$\begin{aligned}
\sigma_x &= \frac{E}{1-\nu^2}(\epsilon_x + \nu\epsilon_y) - \frac{E\alpha T}{1-\nu} \\
\sigma_y &= \frac{E}{1-\nu^2}(\epsilon_y + \nu\epsilon_x) - \frac{E\alpha T}{1-\nu} \\
\sigma_z &= -\frac{\nu}{1-\nu}(\epsilon_x + \epsilon_y) - \frac{E\alpha T}{1-\nu} \\
\tau_{xy} &= G\gamma_{xy}
\end{aligned}$$

Stress function Φ relations,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_x + \sigma_y + \alpha ET) = 0$$

$$\implies \nabla^4\Phi + \alpha E \nabla^2 T = 0$$

Polar Coordinates

Displacement-strain relations,

$$\epsilon_r = \frac{\partial u}{\partial r}, \quad \epsilon_\theta = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta}$$

$$2\epsilon_{r\theta} = \gamma_{r\theta} = \frac{\partial v}{\partial r} - \frac{v}{r} + \frac{1}{r}\frac{\partial u}{\partial \theta}$$

Strain-stress relations for plane stress,

$$\epsilon_r = \frac{1}{E}(\sigma_r - \nu\sigma_\theta), \quad \epsilon_\theta = \frac{1}{E}(\sigma_\theta - \nu\sigma_r)$$

$$\epsilon_{r\theta} = \frac{1}{2G}\tau_{r\theta}$$

Airy's stress function Φ relations,

$$\sigma_r = \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2}, \quad \sigma_\theta = \frac{\partial^2\Phi}{\partial r^2}$$

$$\tau_{r\theta} = \frac{1}{r^2}\frac{\partial\Phi}{\partial\theta} - \frac{1}{r}\frac{\partial^2\Phi}{\partial r \partial\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\Phi}{\partial\theta}\right)$$

Compatibility,

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2}$$

$$\nabla^4\Phi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\nabla^2\Phi = 0$$

Transformation equations,

$$\begin{aligned}
\sigma_r &= \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta + \tau_{xy}\sin 2\theta \\
\tau_{r\theta} &= -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta + \tau_{xy}\cos 2\theta \\
\sigma_\theta &= \frac{1}{2}(\sigma_x + \sigma_y) - \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta - \tau_{xy}\sin 2\theta
\end{aligned}$$

or

$$\begin{aligned}
\sigma_x &= \frac{1}{2}(\sigma_\theta + \sigma_r) + \frac{1}{2}(\sigma_r - \sigma_\theta)\cos 2\theta - \tau_{r\theta}\sin 2\theta \\
\tau_{xy} &= -\frac{1}{2}(\sigma_r - \sigma_\theta)\sin 2\theta + \tau_{r\theta}\cos 2\theta \\
\sigma_y &= \frac{1}{2}(\sigma_\theta + \sigma_r) - \frac{1}{2}(\sigma_r - \sigma_\theta)\cos 2\theta + \tau_{r\theta}\sin 2\theta
\end{aligned}$$

Concentrated Loads

Wedge of unit thickness, under load P, and angle α,

$$\sigma_r = -\frac{P\cos\theta}{r(\alpha + \frac{1}{2}\sin 2\alpha)}, \quad \sigma_\theta = 0, \quad \tau_{r\theta} = 0$$

$$\sigma_x = \sigma_r\cos^2\theta = -\frac{P\cos^4\theta}{L(\alpha + \frac{1}{2}\sin 2\alpha)}$$

$$\tau_{xy} = \frac{P\sin\theta\cos^3\theta}{L(\alpha + \frac{1}{2}\sin 2\alpha)}$$

$$(\sigma_x)_{\text{elem}} = -\frac{P}{2L\tan\alpha}$$

Note that the normal stress is maximum at $\theta = 0$ and minimum at $\theta = \alpha$. Shear stress is maximum at $\theta = \alpha$ if $\alpha < 30°$ and at $\theta = 30°$ if $\alpha \geq 30°$.

If the wedge is a straight boundary, $\alpha = \pi/2$, then

$$\sigma_r = -\frac{2P\cos\theta}{\pi r}, \quad \sigma_\theta = 0, \quad \tau_{r\theta} = 0$$

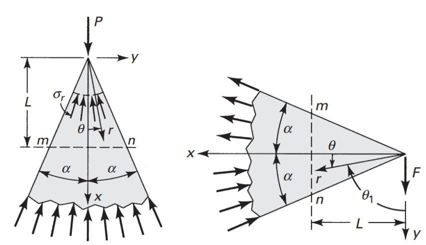

Figure 1: Wedge of unit thickness under load P and F per unit thickness

For bending of a wedge, under load P, and angle α,

$$\begin{aligned}
\sigma_r &= -\frac{F\cos\theta_1}{r(\alpha - 0.5\sin(2\alpha))} = -\frac{F\sin\theta}{r(\alpha + 0.5\sin(2\alpha))} \\
\sigma_\theta &= \tau_{r\theta} = 0 \\
\sigma_x &= \sigma_r\cos^2\theta = -\frac{F\sin\theta\cos^2\theta}{r(\alpha + 0.5\sin(2\alpha))} \\
\sigma_y &= \sigma_r\sin^2\theta = -\frac{F\sin^3\theta}{r(\alpha + 0.5\sin(2\alpha))} \\
\tau_{xy} &= \sigma_r\sin\theta\cos\theta = -\frac{F\sin^2\theta\cos\theta}{r(\alpha + 0.5\sin(2\alpha))}
\end{aligned}$$

$$(\sigma_x)_{\text{elem}} = -\frac{F}{2r\tan\alpha}$$

So for a combined load P and F,

$$\begin{aligned}
\sigma_r &= -\frac{P\cos\theta}{r(\alpha + 0.5\sin(2\alpha))} - \frac{F\sin\theta}{r(\alpha + 0.5\sin(2\alpha))} \\
\sigma_\theta &= \tau_{r\theta} = 0
\end{aligned}$$

Stress Concentrations

For stress concentration factor K,

$$K = \frac{\sigma_{\max}}{\sigma_{\text{nom}}}$$

For circular hole in a large plate in tension stress σ_o,

$$\begin{aligned}
(\sigma_\theta)_{\max} &= 3\sigma_o, \quad \theta = \pm\pi/2 \\
(\sigma_\theta)_{\min} &= -\sigma_o, \quad \theta = 0, \pm\pi
\end{aligned}$$

For tension σ_{ox} and σ_{oy},

$$\begin{aligned}
(\sigma_\theta)_{\max,x} &= 3\sigma_{ox} - \sigma_{oy}, \quad \theta = \pm\pi/2 \\
(\sigma_\theta)_{\min,y} &= 3\sigma_{oy} - \sigma_{ox}, \quad \theta = 0, \pm\pi
\end{aligned}$$

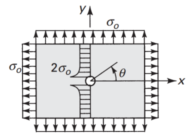

Figure 2: Stress concentration factor for circular hole in a large plate (10/10 figure)

5. Bending of Beams

5.1. General Procedure

General procedure of asymmetric bending problems

1. Identify the location of the centroid of the cross-section, and define it as the origin of the (y, z) coordinate system. If the centroid is unknown, set an arbitrary origin and use parallel axis theorem to find the centroid.
2. Define the orientation of (y, z) axes of the cross-section wisely so that all required moments of inertia I_y, I_z, and I_{yz} can be obtained (from Table) or calculated easily.
3. Determine bending moments M_z and M_y at your cross-section. Use elementary beam theory to find the bending moments if given a load.
4. Use the relations to find the stress σ_x and the neutral axis.

5.2. Formulas

Centroid equations:

$$\bar{x} = \frac{\sum \bar{x}_i A_i}{\sum A_i}$$

where $\bar{x}_i$ is the x-coordinate of the centroid of the i-th area, and A_i is the area of the i-th area.

Moment equations:

$$\begin{aligned}
M_y &= P_z L \\
M_z &= P_y L
\end{aligned}$$

where P_z and P_y are positive in the positive z and y directions, respectively. Parallel axis theorem:

$$\begin{aligned}
\bar{z} &= \frac{\sum \bar{z}_i A_i}{\sum A_i} \\
\bar{y} &= \frac{\sum \bar{y}_i A_i}{\sum A_i} \\
I_z &= \sum(I_{\bar{z},i} + A_i d_{y,i}^2) \\
I_y &= \sum(I_{\bar{y},i} + A_i d_{z,i}^2) \\
I_{yz} &= \sum(I_{\bar{y}\bar{z},i} + A_i d_{y,i} d_{z,i})
\end{aligned}$$

where $I_{\bar{z},i}$, $I_{\bar{y},i}$, and $I_{\bar{y}\bar{z},i}$ are the moments of inertia about the centroidal axes, and $d_{y,i}$ and $d_{z,i}$ are the distances from the centroidal axes to the parallel axes. Note: $I_{yz} = 0$ if there is symmetry about **either** the y or z direction.
Moment to stress:

$$\begin{aligned}
\tau &= \frac{VQ}{Ib} \overset{\text{rect}}{=} \frac{3V}{2A_c} \\
\sigma_x &= \frac{(M_y I_z + M_z I_{yz})d_z - (M_y I_{yz} + M_z I_y)d_y}{I_y I_z - I_{yz}^2} \\
\tan\phi &= \frac{M_y I_z + M_z I_{yz}}{M_z I_y + M_y I_{yz}}
\end{aligned}$$

stress is maximum at the furthest point from the neutral axis on the cross-section. For σ_x, d_y and d_z are the signed displacements ($\pm$) from the centroid to the point of interest in the y and z directions.
Cheesy