3. Problems in Elasticity

8.1. General Procedure

8.2. Formulas

Plane Strain

On the plane x-y, the equilibrium and compatibility equations are

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} = 0 \frac{\partial^2 \sigma_x}{\partial y^2} + \frac{\partial^2 \sigma_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

$$\implies \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_x + \sigma_y) = 0$$

Strain-stress relations are

$$\begin{aligned}
\epsilon_x &= \frac{1}{E}(\sigma_x - \nu\sigma_y) \\
\epsilon_y &= \frac{1}{E}(\sigma_y - \nu\sigma_x) \\
\epsilon_z &= -\frac{\nu}{E}(\sigma_x + \sigma_y) \\
\gamma_{xy} &= \frac{\tau_{xy}}{G} \\
\gamma_{xz} &= \gamma_{yz} = 0
\end{aligned}$$

Stress-strain relations are

$$\begin{aligned}
\sigma_x &= \frac{E}{1-\nu^2}(\epsilon_x + \nu\epsilon_y) \\
\sigma_y &= \frac{E}{1-\nu^2}(\epsilon_y + \nu\epsilon_x) \\
\tau_{xy} &= G\gamma_{xy} \\
\sigma_z &= -\frac{\nu}{1-\nu}(\epsilon_x + \epsilon_y)
\end{aligned}$$

Airy's stress function Φ relations

$$\nabla^4\Phi = \frac{\partial^4\Phi}{\partial x^4} + 2\frac{\partial^4\Phi}{\partial x^2 \partial y^2} + \frac{\partial^4\Phi}{\partial y^4} = 0$$

$$\sigma_x = \frac{\partial^2\Phi}{\partial y^2}, \quad \sigma_y = \frac{\partial^2\Phi}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2\Phi}{\partial x \partial y}$$

Thermalelasticity

Thermal strain, $\epsilon t = \alpha T$, relations by superposition,

$$\begin{aligned}
\epsilon_x &= \frac{1}{E}(\sigma_x - \nu\sigma_y) + \alpha T \\
\epsilon_y &= \frac{1}{E}(\sigma_y - \nu\sigma_x) + \alpha T \\
\epsilon_z &= -\frac{\nu}{E}(\sigma_x + \sigma_y) + \alpha T \\
\gamma_{xy} &= \frac{\tau_{xy}}{G}
\end{aligned}$$

Thermal stress relations,

$$\begin{aligned}
\sigma_x &= \frac{E}{1-\nu^2}(\epsilon_x + \nu\epsilon_y) - \frac{E\alpha T}{1-\nu} \\
\sigma_y &= \frac{E}{1-\nu^2}(\epsilon_y + \nu\epsilon_x) - \frac{E\alpha T}{1-\nu} \\
\sigma_z &= -\frac{\nu}{1-\nu}(\epsilon_x + \epsilon_y) - \frac{E\alpha T}{1-\nu} \\
\tau_{xy} &= G\gamma_{xy}
\end{aligned}$$

Stress function Φ relations,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_x + \sigma_y + \alpha E T) = 0$$

$$\implies \nabla^4\Phi + \alpha E \nabla^2 T = 0$$

Polar Coordinates

Displacement-strain relations,

$$\epsilon_r = \frac{\partial u}{\partial r}, \quad \epsilon_\theta = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta}$$

$$2\epsilon_{r\theta} = \gamma_{r\theta} = \frac{\partial v}{\partial r} - \frac{v}{r} + \frac{1}{r}\frac{\partial u}{\partial \theta}$$

Strain-stress relations for plane stress,

$$\epsilon_r = \frac{1}{E}(\sigma_r - \nu\sigma_\theta), \quad \epsilon_\theta = \frac{1}{E}(\sigma_\theta - \nu\sigma_r)$$

$$\epsilon_{r\theta} = \frac{1}{2G}\tau_{r\theta}$$

Airy's stress function Φ relations,

$$\sigma_r = \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2}, \quad \sigma_\theta = \frac{\partial^2\Phi}{\partial r^2}$$

$$\tau_{r\theta} = \frac{1}{r^2}\frac{\partial\Phi}{\partial\theta} - \frac{1}{r}\frac{\partial^2\Phi}{\partial r\partial\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\Phi}{\partial\theta}\right)$$

Compatibility,

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2}$$

$$\nabla^4\Phi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\nabla^2\Phi = 0$$

Transformation equations,

$$\begin{aligned}
\sigma_r &= \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta + \tau_{xy}\sin 2\theta \\
\tau_{r\theta} &= -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta + \tau_{xy}\cos 2\theta \\
\sigma_\theta &= \frac{1}{2}(\sigma_x + \sigma_y) - \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta - \tau_{xy}\sin 2\theta
\end{aligned}$$

or

$$\begin{aligned}
\sigma_x &= \frac{1}{2}(\sigma_\theta + \sigma_r) + \frac{1}{2}(\sigma_r - \sigma_\theta)\cos 2\theta - \tau_{r\theta}\sin 2\theta \\
\tau_{xy} &= -\frac{1}{2}(\sigma_r - \sigma_\theta)\sin 2\theta + \tau_{r\theta}\cos 2\theta \\
\sigma_y &= \frac{1}{2}(\sigma_\theta + \sigma_r) - \frac{1}{2}(\sigma_r - \sigma_\theta)\cos 2\theta + \tau_{r\theta}\sin 2\theta
\end{aligned}$$

Concentrated Loads

Wedge of unit thickness, under load P, and angle α,

$$\sigma_r = -\frac{P\cos\theta}{r(\alpha + \frac{1}{2}\sin 2\alpha)}, \quad \sigma_\theta = 0, \quad \tau_{r\theta} = 0$$

$$\sigma_x = \sigma_r \cos^2\theta = -\frac{P\cos^4\theta}{L(\alpha + \frac{1}{2}\sin 2\alpha)}$$

$$\tau_{xy} = \frac{P\sin\theta\cos^3\theta}{L(\alpha + \frac{1}{2}\sin 2\alpha)}$$

$$(\sigma_x)_{\text{elem}} = -\frac{P}{2L\tan\alpha}$$

Note that the normal stress is maximum at $\theta = 0$ and minimum at $\theta = \alpha$. Shear stress is maximum at $\theta = \alpha$ if $\alpha < 30°$ and at $\theta = 30°$ if $\alpha \geq 30°$.

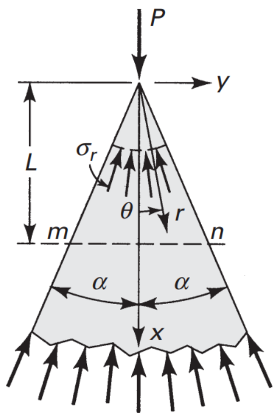

Figure 1: Wedge of unit thickness under load P per unit thickness

Stress Concentrations

For stress concentration factor K,

$$K = \frac{\sigma_{\max}}{\sigma_{\text{nom}}}$$

For circular hole in a large plate in tension stress σ_o,

$$(\sigma_\theta)_{\max} = 3\sigma_o, \quad \theta = \pm\pi/2$$
$$(\sigma_\theta)_{\min} = -\sigma_o, \quad \theta = 0, \pm\pi$$

For tension σ_{ox} and σ_{oy},

$$(\sigma_\theta)_{\max,x} = 3\sigma_{ox} - \sigma_{oy}, \quad \theta = \pm\pi/2$$
$$(\sigma_\theta)_{\min,y} = 3\sigma_{oy} - \sigma_{ox}, \quad \theta = 0, \pm\pi$$

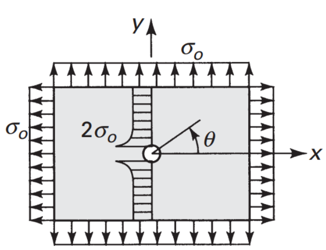

Figure 2: Stress concentration factor for circular hole in a large plate (10/10 figure)