Question 1

The distribution of stress in an aluminum machine component is given by:

$$\sigma = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} = \begin{bmatrix} y+2z^2 & 3z^2 & 2y^2 \\ 3z^2 & x+z & x^2 \\ 2y^2 & x^2 & 3x+y \end{bmatrix} \text{MPa}$$

Calculate the state of strain of a point positioned at $(1, 2, 4)$. Use $E = 70\,\text{GPa}$ and $\nu = 0.3$.

Solution

From the generalized Hooke's law, the stress-strain relation is given by:

$$\begin{aligned} \epsilon_x &= \frac{1}{E}(\sigma_x - \nu(\sigma_y + \sigma_z)) \\ \epsilon_y &= \frac{1}{E}(\sigma_y - \nu(\sigma_x + \sigma_z)) \\ \epsilon_z &= \frac{1}{E}(\sigma_z - \nu(\sigma_x + \sigma_y)) \\ \gamma_{xy} &= \frac{1}{2G}\tau_{xy} \\ \gamma_{yz} &= \frac{1}{2G}\tau_{yz} \\ \gamma_{xz} &= \frac{1}{2G}\tau_{xz} \end{aligned}$$

Where

$$G = \frac{E}{2(1+\nu)} = \frac{70}{2(1+0.3)} = 26.923\,\text{GPa}$$

Evaluate the stress state at $(1, 2, 4)$:

$$\sigma = \begin{bmatrix} 2+2(4)^2 & 3(4)^2 & 2(2)^2 \\ 3(4)^2 & 1+4 & 1^2 \\ 2(2)^2 & 1^2 & 3(1)+2 \end{bmatrix} = \begin{bmatrix} 34 & 48 & 8 \\ 48 & 5 & 1 \\ 8 & 1 & 5 \end{bmatrix} \text{MPa}$$

Using the stress-strain relations, the strain state is given by:

$$\epsilon = \begin{bmatrix} \frac{1}{E}(\sigma_x - \nu(\sigma_y + \sigma_z)) & \frac{1}{2G}\tau_{xy} & \frac{1}{2G}\tau_{xz} \\ \frac{1}{2G}\tau_{yx} & \frac{1}{E}(\sigma_y - \nu(\sigma_x + \sigma_z)) & \frac{1}{2G}\tau_{yz} \\ \frac{1}{2G}\tau_{zx} & \frac{1}{2G}\tau_{zy} & \frac{1}{E}(\sigma_z - \nu(\sigma_x + \sigma_y)) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{70\times10^3}(34 - 0.3(5+5)) & \frac{1}{2(26.923)\times10^3}48 & \frac{1}{2(26.923)\times10^3}8 \\ \frac{1}{2(26.923)\times10^3}48 & \frac{1}{70\times10^3}(5 - 0.3(34+5)) & \frac{1}{2(26.923)\times10^3}1 \\ \frac{1}{2(26.923)\times10^3}8 & \frac{1}{2(26.923)\times10^3}1 & \frac{1}{70\times10^3}(5 - 0.3(34+5)) \end{bmatrix}$$

$$= \begin{bmatrix} 4.43\times10^{-4} & 8.91\times10^{-4} & 1.49\times10^{-4} \\ 8.91\times10^{-4} & -9.57\times10^{-5} & 1.86\times10^{-5} \\ 1.49\times10^{-4} & 1.86\times10^{-5} & -9.57\times10^{-5} \end{bmatrix}$$

Question 2

The aluminum rectangular parallelepiped ($E = 70$ GPa and $\nu = \frac{1}{3}$) shown in Fig. 1 has dimensions $a = 150\,\text{mm}$, $b = 100\,\text{mm}$, and $c = 75\,\text{mm}$. It is subjected to tri-axial stresses $\sigma_x = 70\,\text{MPa}$, $\sigma_y = -30\,\text{MPa}$, and $\sigma_z = -15\,\text{MPa}$ acting on the x, y, and z faces, respectively. Determine,

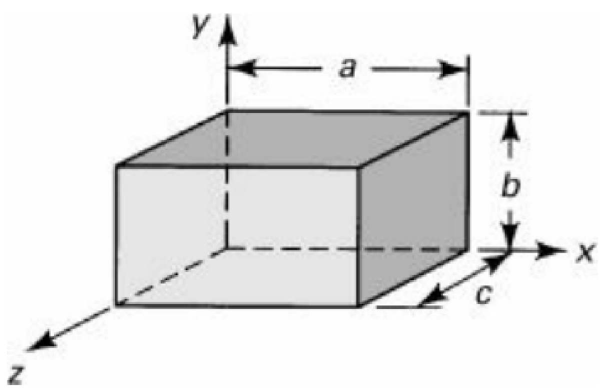

Figure 1: Rectangular parallelepiped subjected to tri-axial stresses.

(a) the changes Δa, Δb, and Δc in the dimensions of the block.

(b) the change ΔV in the volume.

Solution

(a)

From the generalized Hooke's law, the stress-strain relation is given by:

$$\begin{aligned}
\epsilon_x &= \frac{1}{E}(\sigma_x - \nu(\sigma_y + \sigma_z)) \\
\epsilon_y &= \frac{1}{E}(\sigma_y - \nu(\sigma_x + \sigma_z)) \\
\epsilon_z &= \frac{1}{E}(\sigma_z - \nu(\sigma_x + \sigma_y))
\end{aligned}$$

From the definition of strain,

$$\begin{aligned}
\Delta x &= \epsilon_x a \\
\Delta y &= \epsilon_y b \\
\Delta z &= \epsilon_z c
\end{aligned}$$

By direct substitution,

$$\begin{aligned}
\Delta x &= \frac{1}{E}(\sigma_x - \nu(\sigma_y + \sigma_z))a \\
&= \frac{1}{70 \times 10^3}(70 - \frac{1}{3}(-30 - 15))150 \\
&= \boxed{0.182\,\mathrm{mm}} \\
\Delta b &= \frac{1}{E}(\sigma_y - \nu(\sigma_x + \sigma_z))b \\
&= \frac{1}{70 \times 10^3}(-30 - \frac{1}{3}(70 - 15))100 \\
&= \boxed{-0.0690\,\mathrm{mm}} \\
\Delta c &= \frac{1}{E}(\sigma_z - \nu(\sigma_x + \sigma_y))c \\
&= \frac{1}{70 \times 10^3}(-15 - \frac{1}{3}(70 - 30))75 \\
&= \boxed{-0.0304\,\mathrm{mm}}
\end{aligned}$$

(b)

The change in volume is given by:

$$\Delta V = eV_0$$

Where dilation e is given by:

$$\begin{aligned}
e = \epsilon_x + \epsilon_y + \epsilon_z = &= \frac{1 - 2\nu}{E}(\sigma_x + \sigma_y + \sigma_z) \\
&= \frac{1 - \frac{2}{3}}{70 \times 10^3}(70 - 30 - 15) \\
&= 1.1905 \times 10^{-4}
\end{aligned}$$

Therefore,

$$\begin{aligned}\Delta V &= eV_0 \\ &= (1.1905 \times 10^{-4})(150)(100)(75) \\ &= \boxed{134\,\mathrm{mm}^3}\end{aligned}$$

Question 3

A rectangular plate is subjected to uniform tensile stress σ along its upper and lower edges as shown in Fig. 2. Determine the displacements u and v in terms of x, y, and material properties (E, ν) using Eqns. (1) and (2) and the appropriate conditions at the origin.

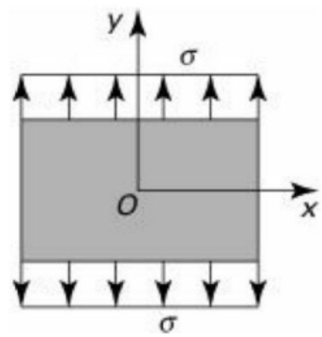

Figure 2: Rectangular plate subjected to uniform tensile stress.

$$\epsilon_x = \frac{\partial u}{\partial x}, \quad \epsilon_y = \frac{\partial v}{\partial y} \tag{1}$$

$$\gamma_{xy} = \alpha_x - \alpha_y = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \tag{2}$$

Solution

(a)

First we derive expressions for ϵ_x and ϵ_y in terms of E, ν, and σ. By generalized Hooke's law,

$$\begin{aligned}\epsilon_x &= \frac{1}{E}(\cancel{\sigma_x} - \nu(\sigma_y)) &= \frac{-\nu\sigma_y}{E} \\ \epsilon_y &= \frac{1}{E}(\sigma_y - \nu(\cancel{\sigma_x})) &= \frac{\sigma_y}{E}\end{aligned}$$

By direct substitution,

$$\begin{aligned}\epsilon_x &= \frac{-\nu\sigma}{E} = \frac{\partial u}{\partial x} \\ \epsilon_y &= \frac{\sigma}{E} = \frac{\partial v}{\partial y}\end{aligned}$$

Integrating,

$$
\begin{aligned}
\int \partial u &= \int \frac{-\nu\sigma}{E}\partial x \\
u &= \frac{-\nu\sigma}{E}x + g(y) \\
\int \partial v &= \int \frac{\sigma}{E}\partial y \\
v &= \frac{\sigma}{E}y + h(x)
\end{aligned}
$$

Where $g(y)$ and $h(x)$ are arbitrary functions of y and x, respectively. We can determine the functions $g(y)$ and $h(x)$ by applying the boundary conditions. Define the origin such that $(x, y) = (0,\, 0)$, $(u, v) = (0,\, 0)$. As a consequence, shear strain γ_{xy} is zero at the origin. By Eqn. (2),

$$
\begin{aligned}
\gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\
0 &= \frac{\partial}{\partial y}\left(\frac{-\nu\sigma}{E}x + g(y)\right) + \frac{\partial}{\partial x}\left(\frac{\sigma}{E}y + h(x)\right) \\
0 &= g'(y) + h'(x) \\
g'(y) &= -h'(x) = -\lambda
\end{aligned}
$$

Where λ is a constant. Integrating gives us $g(y) = c_1$ and $h(x) = c_2$. By the boundary conditions, $(u, v) = (0,\, 0)$ at $(x, y) = (0,\, 0)$.

$$
\begin{aligned}
u|_{(0,0)} &= \frac{-\nu\sigma}{E}(0) + c_1 \overset{\text{set}}{=} 0 \\
\implies c_1 &= 0 \\
v|_{(0,0)} &= \frac{\sigma}{E}(0) + c_2 \overset{\text{set}}{=} 0 \\
\implies c_2 &= 0
\end{aligned}
$$

Therefore, the final expressions for u and v are given by:

$$
\boxed{\begin{aligned}
u &= \frac{-\nu\sigma}{E}x \\
v &= \frac{\sigma}{E}y
\end{aligned}}
$$

Question 4

The stress field in an elastic body is given by

$$
\sigma = \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{bmatrix} = \begin{bmatrix} cy^2 & 0 \\ 0 & -cx^2 \end{bmatrix}
$$

Where c is a constant. Derive expressions for the displacement components $u(x, y)$ and $v(x, y)$ in the body.

Solution

(a)

First convert the stresses into strains using the generalized Hooke's law:

$$\begin{aligned}
\epsilon_x &= \frac{1}{E}(\sigma_x - \nu(\sigma_y)) \\
&= \frac{1}{E}(cy^2 - \nu(-cx^2)) \\
&= \frac{cy^2 + \nu cx^2}{E} \\
\epsilon_y &= \frac{1}{E}(\sigma_y - \nu(\sigma_x)) \\
&= \frac{1}{E}(-cx^2 - \nu(cy^2)) \\
&= \frac{-cx^2 - \nu cy^2}{E}
\end{aligned}$$

Where E is the Young's modulus and ν is the Poisson's ratio. The strains are related to the displacements by:

$$\begin{aligned}
\epsilon_x = \frac{\partial u}{\partial x} \implies u &= \int \epsilon_x dx \\
&= \int \frac{cy^2 + \nu cx^2}{E} dx \\
&= \boxed{\frac{1}{E}\left(cxy^2 + \frac{\nu}{3}cx^3\right) + g(y)} \\
\epsilon_y = \frac{\partial v}{\partial y} \implies v &= \int \epsilon_y dy \\
&= \int \frac{-cx^2 - \nu cy^2}{E} dy \\
&= \boxed{\frac{1}{E}\left(-cxy^2 - \frac{\nu}{3}cy^3\right) + h(x)}
\end{aligned}$$

Question 5

A uniform bar of rectangular cross section $2h \times b$ and specific weight γ hangs in the vertical plane as shown in Fig. 3. Its weight results in displacements shown below. Demonstrate whether this solution satisfies the 15 equations of elasticity and the boundary conditions.

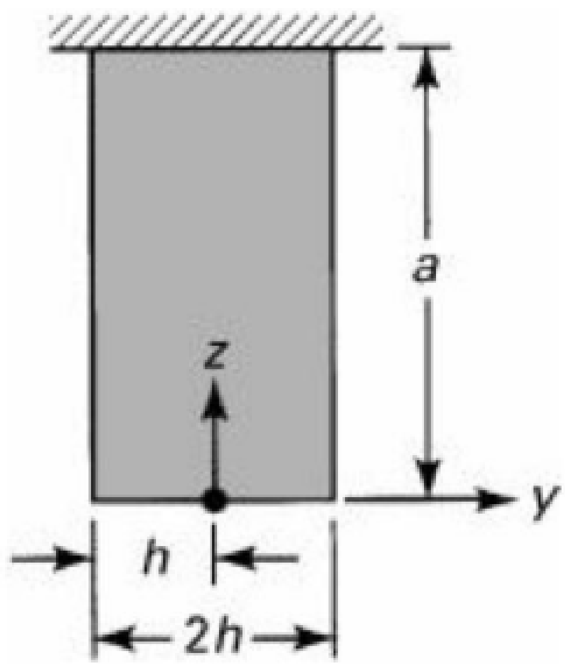

Figure 3: Uniform rectangular bar.

$$u = \frac{-\nu\gamma}{E}xz$$
$$v = \frac{-\nu\gamma}{E}yz$$
$$w = \frac{\gamma}{2E}\left[\frac{z^2 - a^2) + \nu(x^2 + y^2)}{2}\right]$$

Solution First calculate the strains from the displacements (6 equations):

$$\epsilon_x = \frac{\partial u}{\partial x} = \frac{-\nu\gamma}{E}z$$
$$\epsilon_y = \frac{\partial v}{\partial y} = \frac{-\nu\gamma}{E}z$$
$$\epsilon_z = \frac{\partial w}{\partial z} = \frac{\gamma}{E}z$$
$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0$$
$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = 0$$
$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = 0$$

Next, check compatibility (6 equations):

$$
\begin{aligned}
\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} &= \frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} \\
\frac{\partial^2}{\partial x \partial y} 0 &= \frac{\partial^2}{\partial y^2} \frac{-\nu\gamma}{E} z + \frac{\partial^2}{\partial x^2} \frac{-\nu\gamma}{E} z \\
0 &= 0 + 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 \gamma_{xz}}{\partial x \partial z} &= \frac{\partial^2 \epsilon_x}{\partial z^2} + \frac{\partial^2 \epsilon_z}{\partial x^2} \\
\frac{\partial^2}{\partial x \partial z} 0 &= \frac{\partial^2}{\partial z^2} \frac{-\nu\gamma}{E} z + \frac{\partial^2}{\partial x^2} \frac{\gamma}{E} z \\
0 &= 0 + 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 \gamma_{yz}}{\partial y \partial z} &= \frac{\partial^2 \epsilon_y}{\partial z^2} + \frac{\partial^2 \epsilon_z}{\partial y^2} \\
\frac{\partial^2}{\partial y \partial z} 0 &= \frac{\partial^2}{\partial z^2} \frac{-\nu\gamma}{E} z + \frac{\partial^2}{\partial y^2} \frac{\gamma}{E} z \\
0 &= 0 + 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial x} \left(\frac{\partial \epsilon_x y}{\partial z} + \frac{\partial \epsilon_x z}{\partial y} - \frac{\partial \epsilon_y z}{\partial x} \right) &= \frac{\partial^2 \epsilon_x}{\partial y \partial z} \\
\frac{\partial}{\partial x} \left(\frac{\partial}{\partial z} 0 + \frac{\partial}{\partial y} 0 - \frac{\partial}{\partial x} 0 \right) &= \frac{\partial^2}{\partial y \partial z} \frac{-\nu\gamma}{E} z \\
0 &= 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial y} \left(\frac{\partial \epsilon_x y}{\partial z} + \frac{\partial \epsilon_x z}{\partial y} - \frac{\partial \epsilon_y z}{\partial x} \right) &= \frac{\partial^2 \epsilon_y}{\partial x \partial z} \\
\frac{\partial}{\partial y} \left(\frac{\partial}{\partial z} 0 + \frac{\partial}{\partial y} 0 - \frac{\partial}{\partial x} 0 \right) &= \frac{\partial^2}{\partial x \partial z} \frac{-\nu\gamma}{E} z \\
0 &= 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial z} \left(\frac{\partial \epsilon_x y}{\partial z} + \frac{\partial \epsilon_x z}{\partial y} - \frac{\partial \epsilon_y z}{\partial x} \right) &= \frac{\partial^2 \epsilon_z}{\partial x \partial y} \\
\frac{\partial}{\partial z} \left(\frac{\partial}{\partial z} 0 + \frac{\partial}{\partial y} 0 - \frac{\partial}{\partial x} 0 \right) &= \frac{\partial^2}{\partial x \partial y} \frac{\gamma}{E} z \\
0 &= 0
\end{aligned}
$$

Finally, check equilibrium (3 equations):

$$\begin{aligned}
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + F_x = 0 \\
\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + F_y = 0 \\
\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + F_z = 0
\end{aligned}$$

Using Lame's equations,

$$\begin{aligned}
\lambda &= \frac{E\nu}{(1+\nu)(1-2\nu)} \\
e &= \epsilon_x + \epsilon_y + \epsilon_z \\
&= \frac{-\nu\gamma}{E}z + \frac{-\nu\gamma}{E}z + \frac{\gamma}{E}z \\
&= \frac{z\gamma}{E}(1-2\nu)
\end{aligned}$$

Also note since shear strains are zero, $\tau_{xy} = \tau_{xz} = \tau_{yz} = 0$ by the relation $\tau = G\gamma$. Also there are no body forces, $F_x = F_y = F_z = 0$. The equilibrium equations reduce:

$$\begin{aligned}
\frac{\partial \sigma_x}{\partial x} &= \\
0 &= \frac{\partial}{\partial x}(2G\epsilon_x + \lambda e) \\
0 &= \frac{\partial}{\partial x}(2G\frac{-\nu\gamma}{E}z + \lambda\frac{z\gamma}{E}(1-2\nu)) \\
0 &= 0
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \sigma_y}{\partial y} &= \\
0 &= \frac{\partial}{\partial y}(2G\epsilon_y + \lambda e) \\
0 &= \frac{\partial}{\partial y}(2G\frac{-\nu\gamma}{E}z + \lambda\frac{z\gamma}{E}(1-2\nu)) \\
0 &= 0
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \sigma_z}{\partial z} &= \\
0 &= \frac{\partial}{\partial z}(2G\epsilon_z + \lambda e) \\
0 &= \frac{\partial}{\partial z}(2G\frac{\gamma}{E}z + \frac{z\gamma}{E}(1-2\nu)) \\
0 &= 0
\end{aligned}$$

Question 6

A round bar is composed of three segments of the same material as shown in Fig. 4. The diameter is d for the lengths BC and DE and $n \times d$ for length CD, where n is the ratio of the two diameters. Neglecting the stress concentrations, verify that the strain energy of the bar when subjected to axial load P is

$$U = \frac{1+3n^2}{4n^2}\frac{P^2L}{2AE}$$

where $A = \frac{\pi d^2}{4}$. Compare the results for $n = 1$ with those for $n = \frac{1}{2}$ and $n = 2$.

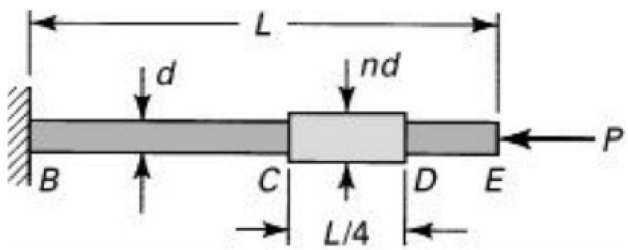

Figure 4: Round bar composed of three segments.

Solution

The strain energy of the bar segment is given by the integral

$$U = \int_{L_i}^{L_{i+1}} \frac{P^2}{2AE} dx$$

The strain energy of the bar is the sum of the strain energies of the three segments:

$$U = \int_{\mathrm{B}}^{\mathrm{C}} \frac{P^2}{2A_{\mathrm{BC}}} dx + \int_{\mathrm{C}}^{\mathrm{D}} \frac{P^2}{2A_{\mathrm{CD}}} dx + \int_{\mathrm{D}}^{\mathrm{E}} \frac{P^2}{2A_{\mathrm{DE}}} dx$$

Since the rod is of the same material, E is constant. For segment CD, $A_{\mathrm{CD}} = n^2 A_{\mathrm{BC}} = n^2 A_{\mathrm{DE}}$. The larger area segment comprises of $L/4$ of the total length. The rest of the rod of cross-sectional area A.

$$\begin{aligned} U &= \int_0^{3L/4} \frac{P^2}{2EA} dx + \int_0^{L/4} \frac{P^2}{2En^2A} dx \\ &= \left.\frac{P^2L}{2EA}\right|_0^{3L/4} + \left.\frac{P^2L}{2En^2A}\right|_0^{L/4} \\ &= \left(\frac{3}{4} + \frac{1}{4n^2}\right)\frac{P^2L}{2EA} \\ &= \boxed{\frac{1+3n^2}{4n^2}\frac{P^2L}{2AE}} \end{aligned}$$

For $n = 1, \frac{1}{2}, 2$,

$$U|_{n=1} = \frac{1+3(1)^2}{4(1)^2}\frac{P^2L}{2AE} = \frac{4}{4}\frac{P^2L}{2AE} = \boxed{\frac{P^2L}{2AE}}$$

$$U|_{n=\frac{1}{2}} = \frac{1+3(\frac{1}{2})^2}{4(\frac{1}{2})^2}\frac{P^2L}{2AE} = \boxed{\frac{7}{4}\frac{P^2L}{2AE}}$$

$$U|_{n=2} = \frac{1+3(2)^2}{4(2)^2}\frac{P^2L}{2AE} = \frac{1+12}{16}\frac{P^2L}{2AE} = \boxed{\frac{13}{16}\frac{P^2L}{2AE}}$$