3. Problems in Elasticity

8.1. General Procedure

8.2. Formulas

Plane Strain

On the plane x-y, the equilibrium and compatibility equations are

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} = 0 \frac{\partial^2 \sigma_x}{\partial y^2} + \frac{\partial^2 \sigma_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

$$\implies \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_x + \sigma_y) = 0$$

Strain-stress relations are

$$\epsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y)$$
$$\epsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x)$$
$$\epsilon_z = -\frac{\nu}{E}(\sigma_x + \sigma_y)$$
$$\gamma_{xy} = \frac{\tau_{xy}}{G}$$
$$\gamma_{xz} = \gamma_{yz} = 0$$

Stress-strain relations are

$$\sigma_x = \frac{E}{1-\nu^2}(\epsilon_x + \nu\epsilon_y)$$
$$\sigma_y = \frac{E}{1-\nu^2}(\epsilon_y + \nu\epsilon_x)$$
$$\tau_{xy} = G\gamma_{xy}$$
$$\sigma_z = -\frac{\nu}{1-\nu}(\epsilon_x + \epsilon_y)$$

Airy's stress function Φ relations

$$\nabla^4\Phi = \frac{\partial^4\Phi}{\partial x^4} + 2\frac{\partial^4\Phi}{\partial x^2\partial y^2} + \frac{\partial^4\Phi}{\partial y^4} = 0$$

$$\sigma_x = \frac{\partial^2\Phi}{\partial y^2}, \quad \sigma_y = \frac{\partial^2\Phi}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2\Phi}{\partial x\partial y}$$

Thermalelasticity

Thermal strain, $\epsilon t = \alpha T$, relations by superposition,

$$\epsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) + \alpha T$$
$$\epsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) + \alpha T$$
$$\epsilon_z = -\frac{\nu}{E}(\sigma_x + \sigma_y) + \alpha T$$
$$\gamma_{xy} = \frac{\tau_{xy}}{G}$$

Thermal stress relations,

$$\sigma_x = \frac{E}{1-\nu^2}(\epsilon_x + \nu\epsilon_y) - \frac{E\alpha T}{1-\nu}$$
$$\sigma_y = \frac{E}{1-\nu^2}(\epsilon_y + \nu\epsilon_x) - \frac{E\alpha T}{1-\nu}$$
$$\sigma_z = -\frac{\nu}{1-\nu}(\epsilon_x + \epsilon_y) - \frac{E\alpha T}{1-\nu}$$
$$\tau_{xy} = G\gamma_{xy}$$

Stress function Φ relations,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_x + \sigma_y + \alpha ET) = 0$$
$$\implies \nabla^4\Phi + \alpha E\nabla^2 T = 0$$

Polar Coordinates

Displacement-strain relations,

$$\epsilon_r = \frac{\partial u}{\partial r}, \quad \epsilon_\theta = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial\theta}$$
$$2\epsilon_{r\theta} = \gamma_{r\theta} = \frac{\partial v}{\partial r} - \frac{v}{r} + \frac{1}{r}\frac{\partial u}{\partial\theta}$$

Strain-stress relations for plane stress,

$$\epsilon_r = \frac{1}{E}(\sigma_r - \nu\sigma_\theta), \quad \epsilon_\theta = \frac{1}{E}(\sigma_\theta - \nu\sigma_r)\, \epsilon_{r\theta} = \frac{1}{2G}\tau_{r\theta}$$

Airy's stress function Φ relations,

$$\sigma_r = \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2}, \quad \sigma_\theta = \frac{\partial^2\Phi}{\partial r^2}$$
$$\tau_{r\theta} = \frac{1}{r^2}\frac{\partial\Phi}{\partial\theta} - \frac{1}{r}\frac{\partial^2\Phi}{\partial r\partial\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\Phi}{\partial\theta}\right)$$

Compatibility,

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2}$$
$$\nabla^4\Phi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\nabla^2\Phi = 0$$

Transformation equations,

$$\sigma_r = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta + \tau_{xy}\sin 2\theta$$
$$\tau_{r\theta} = -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta + \tau_{xy}\cos 2\theta$$
$$\sigma_\theta = \frac{1}{2}(\sigma_x + \sigma_y) - \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta - \tau_{xy}\sin 2\theta$$
$$\sigma_x = \frac{1}{2}(\sigma_\theta + \sigma_r) + \frac{1}{2}(\sigma_r - \sigma_\theta)\cos 2\theta - \tau_{r\theta}\sin 2\theta$$
$$\tau_{xy} = -\frac{1}{2}(\sigma_r - \sigma_\theta)\sin 2\theta + \tau_{r\theta}\cos 2\theta$$
$$\sigma_y = \frac{1}{2}(\sigma_\theta + \sigma_r) - \frac{1}{2}(\sigma_r - \sigma_\theta)\cos 2\theta + \tau_{r\theta}\sin 2\theta$$

Concentrated Loads

Wedge of unit thickness, under load P, and angle α,

$$\sigma_r = -\frac{P\cos\theta}{r(\alpha + \frac{1}{2}\sin 2\alpha)}, \quad \sigma_\theta = 0, \quad \tau_{r\theta} = 0$$

$$\sigma_x = \sigma_r \cos^2\theta = -\frac{P\cos^4\theta}{L(\alpha + \frac{1}{2}\sin 2\alpha)}$$

$$\tau_{xy} = \frac{P\sin\theta\cos^3\theta}{L}(\alpha + \frac{1}{2}\sin 2\alpha)$$

$$(\sigma_x)_{\text{elem}} = -\frac{P}{2L\tan\alpha}$$

Note that the normal stress is maximum at $\theta = 0$ and minimum at $\theta = \alpha$. Shear stress is maximum at $\theta = \alpha$ if $\alpha < 30^\circ$ and at $\theta = 30^\circ$ if $\alpha \geq 30^\circ$.

Stress Concentrations

For stress concentration factor K,

$$K = \frac{\sigma_{\text{max}}}{\sigma_{\text{nom}}}$$