An Investigation of Simple Harmonic Oscillations of a Damped System

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Abstract

This will be a summary of the lab. It will include the purpose of the lab, the methods used, and the results obtained. This will be completed after the lab is finished.

Contents

List of Figures

List of Tables

1 Nomenclature

will do this after the lab is done because I don't know what all the variables used will be yet.

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2 Introduction

2.1 Background

TO DO: Reword and find sources for information.

3 Procedure

4 Theory

4.1 Free Vibrations

The experimental setup for free vibrations is modelled in Figure 1. The system consists of a mass m_e attached to a spring with stiffness k_e.

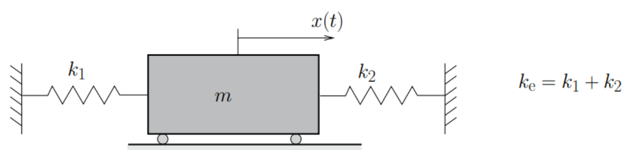

Figure 1: Spring-Mass System

If x is the displacement of the mass from its equilibrium position, the equation of motion is given by

$$m_e\ddot{x} + k_e x = 0 \tag{1}$$

The solution to Equation 1 is given by

$$x(t) = \frac{v_0}{p}\sin(pt) + x_0\cos(pt) \tag{2}$$

where v_0 is the initial velocity, x_0 is the initial displacement, and $p = \sqrt{\frac{k_e}{m_e}}$ is the natural frequency of the system. The natural frequency is the frequency at which the system will oscillate if it is displaced and released. The period of the system is given by

$$\tau = \frac{2\pi}{p} \tag{3}$$

4.2 Forced Vibrations

The experimental setup for forced vibrations is modelled in Figure 2. The system consists of a mass m_e attached to a spring with stiffness k_e. The force is $F(t) = kY_0\sin(\omega t)$.

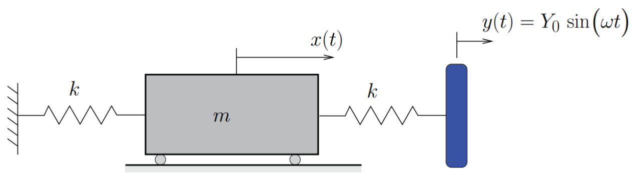

Figure 2: Forced Damped Vibrations System

The equation of motion for the system is given by

$$m_e\ddot{x} + k_e x = F_0 \cos(\omega t) \tag{4}$$

where $F_0 = kY_0$. The time-dependent solution to Equation 4 is

$$x(t) = \frac{Y_0}{2}\left[\frac{1}{1-\left(\frac{\omega}{p}\right)^2}\right]\sin(\omega t) \tag{5}$$

$$\tag{6}$$

where $p = \sqrt{\frac{k_e}{m_e}}$ is the natural frequency of the system. The DMF is given by

$$\text{DMF} = \frac{1}{\left|1-\left(\frac{\omega}{p}\right)^2\right|} \tag{7}$$

Plotting the DMF against ω/p will give the frequency response of the system, as shown in Figure 3. At $\omega/p < \sqrt{2}$, the DMF > 1, which means the system amplifies the input force. At $\omega/p > \sqrt{2}$, the DMF < 1, which means the system attenuates the input force. The system is in resonance at $\omega/p = 1$. Defining static deflection as

$$\mathbb{X}_0 = \frac{F_0}{k_e} \tag{8}$$

we can see that the $Y_0/2$ term in Eq. 5 is static deflection.

4.3 Damped Spring Mass System

An energy dissipation method is added to the system to model the energy loss in the system. The most common approach is to add viscous damping, which is proportional to the velocity of the mass. The equation of motion from Eq. 1 is modified to include damping as

$$m_e\ddot{x} + c_e\dot{x} + k_e x = 0 \tag{9}$$

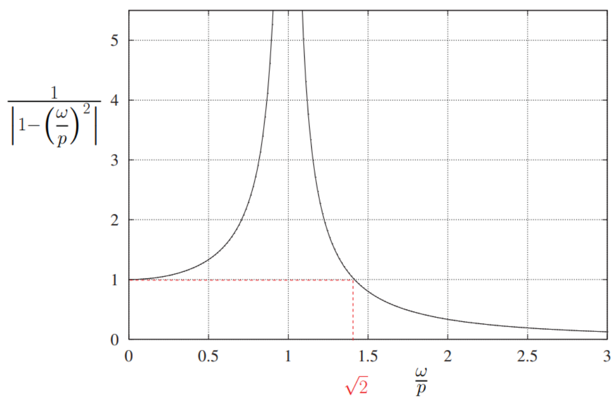

Figure 3: DMF vs. ω/p

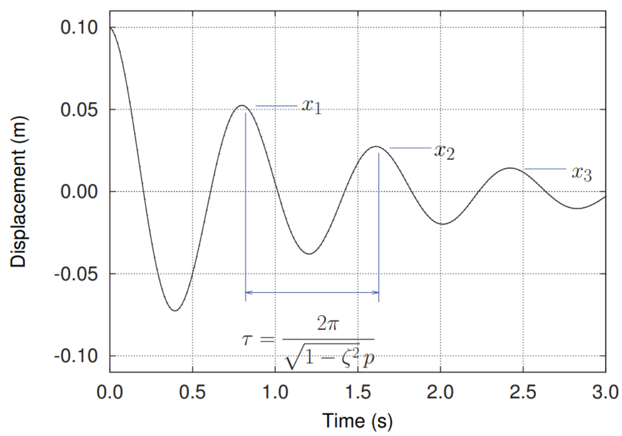

Figure 4: Damped Response of a Spring-Mass System

where c_e is the damping coefficient. Assuming the mass is given an initial displacement and zero initial velocity, the solution to Eq. 9 is given by

$$x(t) = Ae^{\zeta t} \cos\left(\sqrt{1-\zeta^2}t\right) \tag{10}$$

where,

$$\zeta = \frac{c_e}{2m_e p} = \frac{c_e}{2\sqrt{k_e m_e}} \tag{11}$$

The solution to Eq. 9 is a decaying sinusoidal function, plotted in Figure 4. It can be shown that the peaks can be related by

$$\delta = \ln\left(\frac{x_n}{x_{n+1}}\right) = \frac{2\pi}{\sqrt{1-\zeta^2}} \tag{12}$$

From which, the damping ratio can be determined as

$$\zeta = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}} \tag{13}$$

4.4 Equivalent Mass of Measurement System

A schematic of the measurement system is shown in Figure 5. The equivalent mass of the system is given by

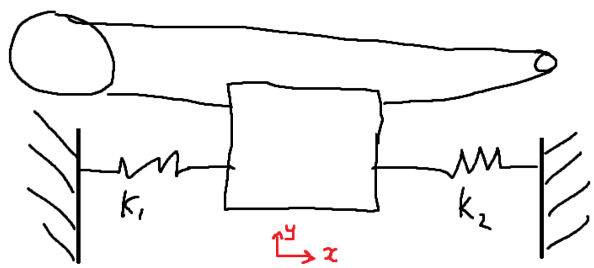

Figure 5: Cart and Pulley Measurement System

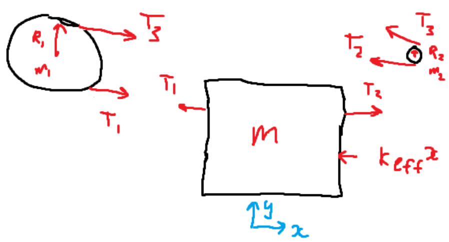

Figure 6: Free Body Diagram of the Cart and Pulleys

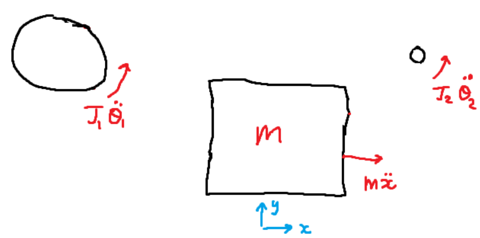

Figure 7: Mass Acceleration Diagram of the Cart and Pulleys

Taking the sum of forces in x, the moment about pulley 1's and pulley's 2 mass center,

$$\begin{aligned} \rightarrow \sum F_x &:= m_{\text{cart}}\ddot{x} \\ &= T_2 - T_1 - k_{\text{eff}}x \qquad (14) \\ \circlearrowleft \sum M_{\text{pulley 1}} &:= J_1\ddot{\theta}_1 \\ &= R_1(T_1 - T_3) \qquad (15) \\ \circlearrowleft \sum M_{\text{pulley 2}} &:= J_2\ddot{\theta}_2 \\ &= R_2(T_3 - T_2) \qquad (16) \end{aligned}$$

Assuming the cable does not slip,

$$\theta_1 = \frac{\ddot{x}}{R_1}, \quad \theta_2 = \frac{\ddot{x}}{R_2} \tag{17}$$

Since the disks are uniform disks,

$$J_1 = \frac{1}{2}m_1 R_1^2, \quad J_2 = \frac{1}{2}m_2 R_2^2 \tag{18}$$

Combining Eq. (14), (15), (16) and using Eq. (17) and (18), we get

$$\underbrace{\left(m_{\text{cart}} + \frac{1}{2}m_1 + \frac{1}{2}m_2\right)}_{m_e}\ddot{x} + k_{\text{eff}}x = 0 \tag{19}$$

where m_e is the equivalent mass of the system. A full derivation is shown in Appendix E.

5 Results and Discussion

5.1 Effective Stiffness From Load-Deflection Data

The effective stiffness was determined to be 13.5 ± 0.2 N/m. The effective stiffness was determined by measuring the deflection of the spring under a various known masses under gravitational force. The results of the trial as well as the linear regression forced through the origin can be seen in Figure 8. The plot was forced through the origin since zero force should be observed at zero deflection. The coefficient of determination was found to be $R^2 = 0.9998$, which indicates a strong linear relationship between force and deflection.

From Hooke's, law, the slope of Figure 8 is the effective stiffness of the spring, which was found to be 13.5 N/m. The spring stiffness should be independent of the cart masses as well as the measurement system.

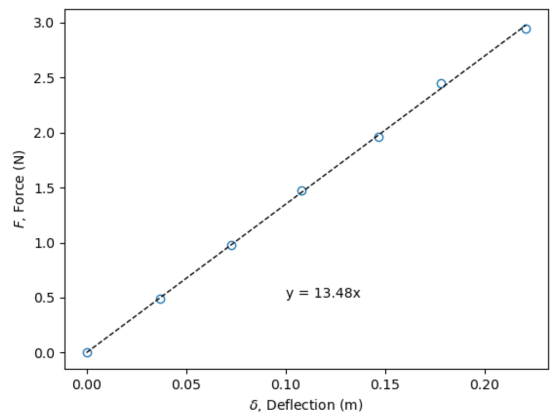

Figure 8: Force vs. Deflection

5.2 Experimental Resonance

Table 1: Natural Frequency Data

	Small		Big	
	f_{sys} (Hz)	$f_{\text{no sys}}$ (Hz)	f_{sys} (Hz)	$f_{\text{no sys}}$ (Hz)
Derived from period	1.1317 ± 0.013	0.9096 ± 0.009	0.8576 ± 0.012	0.7513 ± 0.016
Derived from using k_e and m_e	1.1355	0.9221	0.8744	0.7651
Relative error (%)	0.3286	1.3536	1.9203	1.7989

The natural frequencies of the mass-cart with and without measurement system for experimental and theoretical values are shown in Table 1.The period of 10 cycles was determined experimentally by displacing the mass and measuring the time it took to complete 10 cycles. Details of the analysis can be found in Appendix C. The natural frequency of the system was found to be 1.1317 ± 0.013 Hz for the small cart with the measurement system, 0.9096 ± 0.009 Hz for the small cart without the measurement system, 0.8576 ± 0.012 Hz for the big cart with the measurement system, and 0.7513 ± 0.016 Hz for the big cart without the measurement system.

The theoretical natural frequencies were calculated using the effective stiffness and effective mass. Effective stiffness was found to be k_e = 13.5 ± 0.2 N/m, and the effective mass was found by using Eq. (19), as summarized in Table 3. The theoretical natural frequencies were found to be 1.1355 Hz for the small cart with the measurement system, 0.9221 Hz for the small cart without the measurement system, 0.8744 Hz for the big cart with the measurement system, and 0.7651 Hz for the big cart without the measurement system.

The experimental natural frequencies were found to be within 0.3286% to 1.9203% of the theoretical values. The theoretical values were slightly higher than the experimental values, which could be due to neglecting the mass of the spring and the mass of the cable. In addition, the experimental period calculations neglected the effects of damping, which would would impact the calculation of the natural frequency.

5.3 Damping Ratio

The time response of the amplitude of the small and big carts can be seen in Figures 9, 10, 11, and 12. The damping ratio was calculated using the logarithmic decrement method (Eq. 13) and shown in Appendix B. The average damping ratio was taken from the linear region (first 6 cycles) as shown in Figures 13 and 14. The average damping ratios were found to be $\zeta_{\text{small},1,2} = 0.021, 0.016$ and $\zeta_{\text{big},1,2} = 0.027, 0.023$.

The discrepancy between the two damping ratios obtained for each trial is due to the non-linear nature of the damping in the system as it approached small amplitudes. The theory suggests, for a linear viscous damping model, that the damping ratio would remain constant, independent of the cycle. From Figures 13 and 14, the damping ratio was found to change nearly every cycle, increasing exponential-like. This is not consistent with the linear viscous damping model. The linear viscous damping model does not accurately represent the energy dissipation in the air track system at small amplitudes.

Further work could be done in displacing the cart by a larger amplitude to see if the damping ratio remains constant initially. Other effects that could be considered are the effects of friction from the air track, torsional friction in the pins of the pulleys, and the mass of the springs and cables.

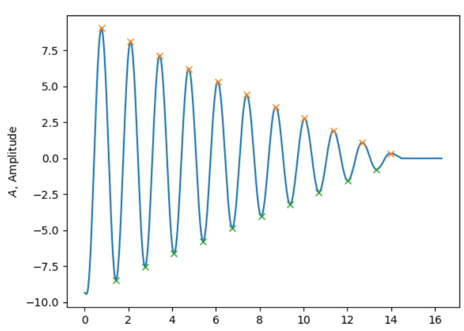

Figure 9: Big Cart Amplitude vs. Time for Trial 1

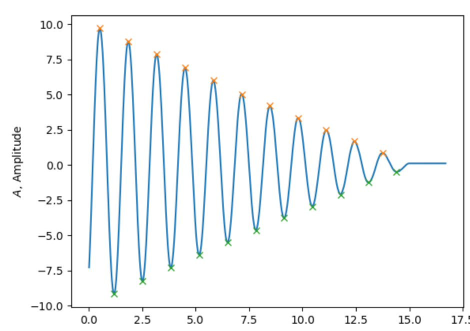

Figure 10: Big Cart Amplitude vs. Time for Trial 2

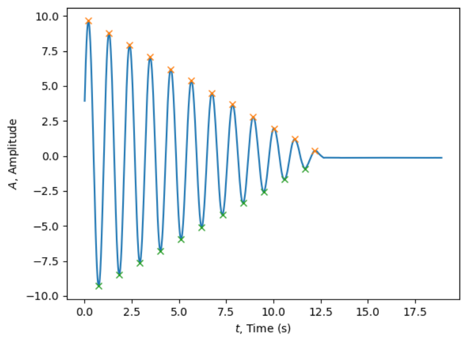

Figure 11: Small Cart Amplitude vs. Time for Trial 1

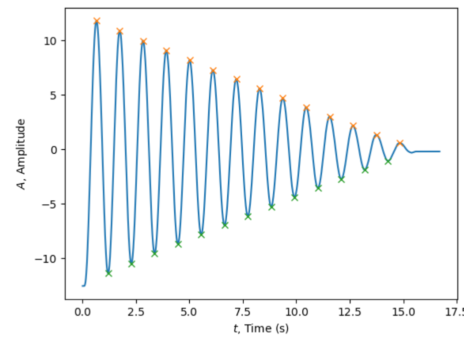

Figure 12: Small Cart Amplitude vs. Time for Trial 2

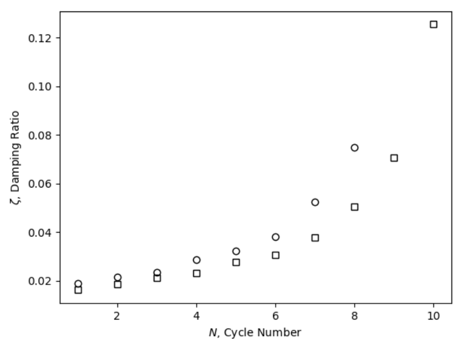

Figure 13: Big Cart Damping Ratio Vs. Cycle Number

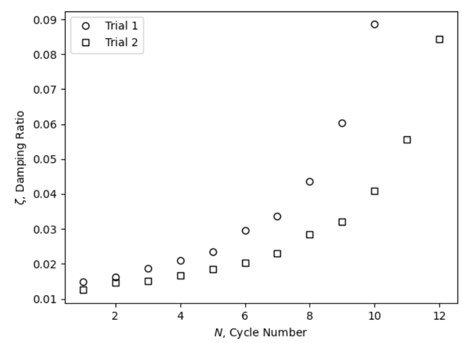

Figure 14: Small Cart Damping Ratio Vs. Cycle Number

5.4 Dynamic Magnification Factor

The dynamic magnification factor was calculated using the measured displacements for both masses and plotted against the frequency ratio ω/p as shown in Figures 15 and 16. This plot was made from the forced response of the system experiment, which tested the response at four forcing frequencies and constant amplitude, $Y_0 = 3.55$ cm.

The DMF follows the theoretical curve closely, with the in-phase region being very close to the theory. The out-of-phase region was slightly off, with the amplitude being smaller than expected. This could be due to neglecting some inertial terms, such as the mass of the springs and cables as well as the torsional friction in the pins of the pulleys or the friction from the air track. It also can be noted that the DMF is less than one after $\omega/p = \sqrt{2}$, which follows the theory. More details on the analysis can be found in Appendix D.

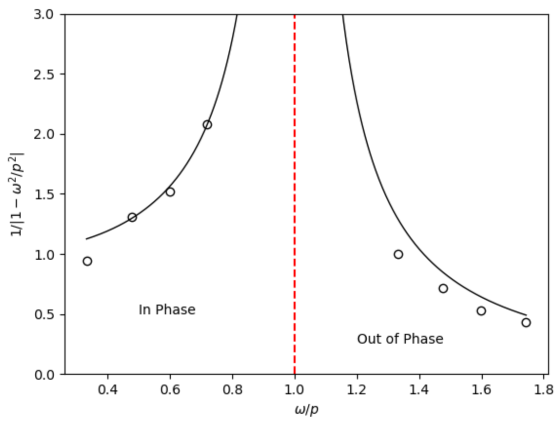

Figure 15: Big Cart DMF vs. ω/p

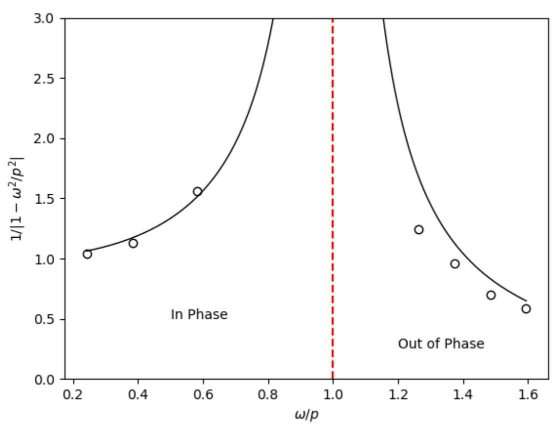

Figure 16: Small Cart DMF vs. ω/p

5.5 Effects of Pulley Measurement System

The diagram of the pulley system was shown in the theory, Section 4.4. Assuming the pulley is a uniform disk, the effective mass, m_e, was found to be $m_e = m_{\text{cart}} + 0.5m_{\text{pulley 1}} + 0.5m_{\text{pulley 2}}$. The partial derivation was shown in Section 4.4, with the full derivation shown in Appendix E.

Table 2: Effective Mass Data

	Small (kg)	Big (kg)
Cart	0.2648	0.4465
Pulley	0.2599	0.0136

Table 3: Effective Mass Data

	System (kg)	No System (kg)
m_e, Small	0.40155	0.2648
m_e, Big	0.58325	0.4465

The pulleys and carts were measured by a mass balance. The masses are shown in Table 2. The effective mass of the system was found to be $m_e = 0.40155$ kg for the small cart with the measurement system, 0.2648 kg for the small cart without the measurement system, 0.58325 kg for the big cart with the measurement system, and 0.4465 kg for the big cart without the measurement system.

The effective mass of the system was found to be higher with the measurement system, which is expected since the pulleys add mass to the system. The inertial effects of the pulley system adds 0.1368 kg to the effective mass.

5.6 Effective Mass Derived From Free Vibration

From the free vibration data, as shown in Table 1, the effective mass was derived using the natural frequencies of the system. The effective mass was found to be 0.4099 kg for the small cart and 0.5818 kg for the big cart. The theoretical effective mass was found to be 0.4016 kg for the small cart and 0.5833 kg for the big cart. The relative error was found to be 2.09% for the small cart and 0.25% for the big cart. The effective mass was found to be within 2.09% of the theoretical value for the small cart and 0.25% for the big cart. The effective mass was found to be higher than the theoretical value, which could be due to neglecting the mass of the springs and cables, and other inertial effects such as the friction from the air track and the torsional friction in the pins of the pulleys.

Table 4: Effective Mass Data

	Small Cart (kg)	Big Cart (kg)
Effective Mass from Trials	0.4099	0.5818
Effective Mass from Theory	0.4016	0.5833
Relative Error	2.09%	0.25%

6 Conclusion

- Summarize the results of the lab
- Discuss the significance of the results
- Discuss the sources of error
- Discuss the limitations of the experiment
- Discuss the implications of the results
- Discuss the future work

6.1 Technical Recommendations

7 References

[1] A. J. Wheeler and A. R. Ganji, *Introduction to engineering experimentation*, 3rd ed. Upper Saddle River, N.J: Pearson Higher Education, 2010, oCLC: ocn459211853.

A Appendix: Effective Stiffness from Load-Deflection Data

This appendix outlines the calculations used to determine the experimental effective stiffness of the system. The effective stiffness is determined by measuring the deflection of the spring under a known load. The stiffness was determined by a linear regression through the origin of the load-deflection data. The data is shown in Table A.5.

A.1 Effective Stiffness Calculation

Table A.5: Load-Deflection Data

Mass (g)	Force (N)	Initial Position, a (cm)	Final Position, b (cm)	Deflection, x (m)
0	0.000	91.1	91.1	0.00
50	0.491	91.1	87.4	0.0370
100	0.981	91.1	83.8	0.0730
150	1.47	91.1	80.3	0.108
200	1.96	91.1	76.4	0.147
250	2.45	91.1	73.3	0.178
300	2.94	91.1	69.0	0.221

Sample calculations for the effective stiffness are shown for the 50 g mass. The deflection was found by

The spring was subject to gravitational force from the applied mass. The force was found by

$$\begin{aligned} F &= mg \\ &= 50 \times 10^{-3}\,\mathrm{kg} \times 9.81\,\mathrm{m\,s^{-2}} \\ &= 0.491\,\mathrm{N} \end{aligned}$$

The deflection was found by

$$\begin{aligned} x &= a - b \\ &= 91.1\,\mathrm{cm} - 87.4\,\mathrm{cm} \\ &= 0.0370\,\mathrm{m} \end{aligned}$$

Next, a linear regression was applied to the data to determine the effective stiffness. From Excel,

Table A.6: Linear Regression Data

Parameter	Value
Spring constant, k_e	$13.5\,\mathrm{N\,m^{-1}}$
Standard error, S_k	$0.082\,\mathrm{N\,m^{-1}}$
R^2	0.9998

So, the effective stiffness of the spring is

$$\boxed{k_e = 13.5\,\mathrm{N\,m^{-1}}}$$

A.2 Effective Stiffness Error

A.2.1 Error Propagation Derivation

To be thorough, the error propagation formula will be derived for completeness.

If we know how a quantity of interest depends on other, directly measurable quantizes, it is posable to estimate the uncertainty of this ”output” quantity based on the uncertainties in the measured quantizes. For example, we can calculate the uncertainty associated to a volume based on the uncertainty of the measurement of the individual dimensions.

Consider as results, R, which is a function of n variables, $x_1, \ldots, x_n$:

$$R = f(x_1, \ldots, x_n)$$

If the individual measurands, x_i, have an associated uncertainty w_{x_i}, what is teh uncertainty of w_R of the result R?

Defining $x := (x_1, \ldots, x_n)$, and $x_m := (x_{m,1}, \ldots, x_{m,n})$, perform the Taylor series expansion of $R = f(x)$ about the point $x = x_m$, taking $x_i - x_{m,i} = w_{x_i}$:

$$R = f(x_m) + \left.\frac{\partial f}{\partial x_1}\right|_{x=x_m} \underbrace{(x_1 - x_{m,1})}_{w_{x_1}} + \ldots + \left.\frac{\partial f}{\partial x_n}\right|_{x=x_m} \underbrace{(x_n - x_{m,n})}_{w_{x_n}} + \text{H.O.T.}$$

$$\underbrace{R - f(x_m)}_{w_R} = \left.\frac{\partial f}{\partial x_1}\right|_{x=x_m} w_{x_1} + \ldots + \left.\frac{\partial f}{\partial x_n}\right|_{x=x_m} w_{x_n} + \text{H.O.T.}$$

The higher-order terms contain quadratic terms $w_{x_i} w_{x_j}$, cubic terms $w_{x_i} w_{x_j} w_{x_k}$, and so on. Assuming the individual uncertainties w_{x_i} are small, we can take these higher-order terms as zero, giving

$$w_R = \sum_{i=1}^{n} \left| \left. \frac{\partial f}{\partial x_i} \right|_{x=x_m} w_{x_i} \right|$$

However, this is the worst-case uncertainty, and is an overly conservative estimate. A better estimate is to use the root of sum of squares

$$w_R = \sqrt{\sum_{i=1}^{n} \left[\left. \frac{\partial f}{\partial x_i} \right|_{x=x_m} w_{x_i} \right]^2} \tag{20}$$

If the confidence levels associated to the individual uncertainties w_{x_i} are all identical (for instance 95%), the confidence level of the result w_R will be the same.

The key assumption behind RSS is that the set of measured variables $x_1, \ldots, x_n$ are **statistically indecent**. If this is not the case, a different formulation needs to be used. Also note that all uncertainties w_{x_i} need to be small such that the first-order Taylor series approximation holds.

Consider the case where the result R is dependent only on the product of the measured variables, $x_1, \ldots, x_n$ with associated uncertainties $w_{x_1}, \ldots, w_{x_n}$ as

$$R = C x_1^{c_1} x_2^{c_2} \ldots x_n^{c_n}$$

where C and $c_1, \ldots, c_n$ are constants. In this case, the RSS formula gives

$$\begin{aligned} w_R &= \sqrt{\left(C c_1 x_1^{c_1 - 1} w_{x_1}\right)^2 + \ldots + \left(C c_n x_1^{c_1} x_2^{c_2} \ldots c_n x_n^{c_n - 1} w_{x_n}\right)^2} \\ \implies \frac{w_R}{|R|} &= \sqrt{\left(\frac{c_1 w_{x_1}}{x_1}\right)^2 + \ldots + \left(\frac{c_n w_{x_n}}{x_n}\right)^2} \end{aligned} \tag{21}$$

Recall from MEC E 300 that the error for the coefficients for the form $f(x) = ax + b$ can be found by finding standard error of a and b by first finding

$$S_{y,x} = \sqrt{\frac{\sum (y_i - a x_i - b)^2}{n - 2}}$$

then,

$$S_a = S_{y,x}\sqrt{\frac{1}{\sum(x_i - \bar{x})^2}}$$
$$S_b = S_{y,x}\sqrt{\frac{\sum x_i^2}{n\sum(x_i - \bar{x})^2}}$$

then with a given confidence of $1 - \alpha$, the uncertainties are

$$a \pm t_{\alpha/2,n-2}S_a$$
$$b \pm t_{\alpha/2,n-2}S_b$$

The uncertainty of the slope is then [1]

$$w_a = t_{\alpha/2,n-2}S_a \tag{22}$$

A.2.2 Error Propagation for Effective Stiffness

Given a confidence level of 95%, the t-distribution value was determined by

$$\begin{aligned} \alpha/2 &= \frac{1 - 0.95}{2} = 0.025 \\ n - 2 &= 7 \\ t_{\alpha/2,n-2} &= 2.5706 \end{aligned}$$

By Eq. (22), the error in the effective stiffness with a confidence level of 95% is

$$\begin{aligned} \delta_{k_e} &= t_{\alpha/2,n-2}S_k \\ &= 2.5706 \times 0.082\,\mathrm{N\,m^{-1}} \\ &= \boxed{0.21\,\mathrm{N\,m^{-1}}} \end{aligned}$$

B Appendix: Damping Ratio

B.1 Damping Ratio Calculation

Table B.7: Damping Ratio Data

Dataset	Peak Number	Amplitude, x (cm)	δ	ζ
Big 1	1	8.2775	0.119385043	0.018997291
Big 1	2	7.346	0.134348497	0.021377341
Big 1	3	6.4225	0.146730503	0.02334652
Big 1	4	5.546	0.179764425	0.028598694
Big 1	5	4.6335	0.202002469	0.032133089
Big 1	6	3.786	0.240058119	0.038178581
Big 1	7	2.978	0.328811927	0.052260531
Big 1	8	2.1435	0.471264045	0.074793917
Big 1	9	1.338	-	-
Big 2	1	9.4465	0.101586118	0.01616582
Big 2	2	8.534	0.116963123	0.018612035
Big 2	3	7.592	0.132252668	0.021044005
Big 2	4	6.6515	0.145034028	0.023076735
Big 2	5	5.7535	0.17320361	0.027555743
Big 2	6	4.8385	0.192437655	0.030613049
Big 2	7	3.9915	0.23835321	0.037907825
Big 2	8	3.145	0.317698849	0.05049883

Dataset	Peak Number	Amplitude, x (cm)	δ	ζ
Big 2	9	2.289	0.444895542	0.070630487
Big 2	10	1.467	0.79495422	0.125520247
Big 2	11	0.6625	-	-
Small 1	1	9.473	0.093317023	0.014850228
Small 1	2	8.629	0.102608739	0.016328511
Small 1	3	7.7875	0.118031863	0.018782041
Small 1	4	6.9205	0.131541363	0.020930872
Small 1	5	6.0675	0.147293322	0.023436022
Small 1	6	5.2365	0.185362555	0.029488537
Small 1	7	4.3505	0.211971847	0.033717185
Small 1	8	3.5195	0.27362217	0.043507086
Small 1	9	2.677	0.379561461	0.060299159
Small 1	10	1.8315	0.55872893	0.088574955
Small 1	11	1.0475	-	-
Small 2	1	23.182	0.078630971	0.012513528
Small 2	2	21.429	0.091411866	0.014547111
Small 2	3	19.557	0.095540303	0.015203954
Small 2	4	17.775	0.10451699	0.016632095
Small 2	5	16.011	0.1158877	0.018440964
Small 2	6	14.259	0.126950746	0.020200716
Small 2	7	12.559	0.144982969	0.023068616

Dataset	Peak Number	Amplitude, x (cm)	δ	ζ
Small 2	8	10.864	0.178279663	0.028362675
Small 2	9	9.09	0.201245362	0.032012778
Small 2	10	7.433	0.257773714	0.040991478
Small 2	11	5.744	0.350674281	0.055724823
Small 2	12	4.045	0.532063149	0.084378491
Small 2	13	2.376	-	-

The raw data for the damping ratio calculation is shown in Table B.7. Sample calculations for the damping ratio are shown for the Big 1 dataset. First, δ was found by

$$\begin{aligned}\delta &= \ln\left(\frac{x_n}{x_{n+1}}\right)\\ &= \ln\left(\frac{8.2775}{7.346}\right)\\ &= 0.119385043\end{aligned}$$

Then, ζ was found by

$$\begin{aligned}\zeta &= \frac{\delta}{\sqrt{4\pi^2+\delta^2}}\\ &= \frac{0.119385043}{\sqrt{4\pi^2+0.119385043^2}}\\ &= \boxed{0.018997}\end{aligned}$$

C Appendix: Natural Frequency

C.1 Period Calculation

Table C.8: Natural Frequency Trial Data

	Small		Big	
Trial	$10\tau_{\text{no sys}}$ (s)	$10\tau_{\text{sys}}$ (s)	$10\tau_{\text{no sys}}$ (s)	$10\tau_{\text{sys}}$ (s)
1	8.81	11.13	11.49	13.7
2	8.71	10.98	11.67	13.14
3	8.85	11.01	11.69	13.31
4	8.88	10.90	11.84	13.19
5	8.93	10.95	11.61	13.21
Average	8.836	10.994	11.66	13.31
τ	0.8836	1.0994	1.166	1.331

The experimental data for the period of the system is shown in Table C.8. Sample calculations for the period are shown for the small no system trial. The average was found by Excel,

$$\begin{aligned}(10\tau_{\text{no sys}})_{\text{avg}} &= \frac{1}{5}\sum_{i=1}^{5}(10\tau_{\text{no sys}})_i \\ &= \frac{1}{5}\times(8.81+8.71+8.85+8.88+8.93) \\ &= 8.836\end{aligned}$$

The period was found by

$$\begin{aligned}\tau_{\text{no sys}} &= \frac{(10\tau_{\text{no sys}})_{\text{avg}}}{10} \\ &= \frac{8.836}{10} \\ &= \boxed{0.8836}\end{aligned}$$

C.2 Natural Frequency Calculation

Table C.9: Natural Frequency Data

	Small		Big	
	f_{sys} (Hz)	$f_{\text{no sys}}$ (Hz)	f_{sys} (Hz)	$f_{\text{no sys}}$ (Hz)
Derived from period	1.1317	0.9096	0.8576	0.7513
Derived from using k_e and m_e	1.1355	0.9221	0.8744	0.7651

Table C.10: Effective Mass Data

	System (kg)	No System (kg)
m_e, Small	0.40155	0.2648
m_e, Big	0.58325	0.4465

The natural frequency of the system was calculated using the period data in Table C.9. Sample calculations for the natural frequency are shown for the small system trial. The natural frequency was found by

$$\begin{aligned} f_{\text{sys}} &= \frac{1}{\tau_{\text{sys}}} \\ &= \frac{1}{1.0994} \\ &= \boxed{0.9096} \end{aligned}$$

The natural frequency was also calculated using the effective stiffness and mass data. The effective mass data is shown in Table C.10. The effective mass for the system was found by

$$\begin{aligned} m_{e,\text{sys}} &= m_{\text{cart}} + \frac{1}{2} m_{\text{big pulley}} + \frac{1}{2} m_{\text{small pulley}} \\ &= 0.2648\,\text{kg} + \frac{1}{2} 0.2599\,\text{kg} + \frac{1}{2} 0.0136\,\text{kg} \\ &= 0.40155\,\text{kg} \end{aligned}$$

The natural frequency was found by

$$\begin{aligned} f_{\text{sys}} &= \frac{1}{2\pi}\sqrt{\frac{k_e}{m_e}} \\ &= \frac{1}{2\pi}\sqrt{\frac{13.5\,\mathrm{N\,m^{-1}}}{0.40155\,\mathrm{kg}}} \\ &= \boxed{1.1355} \end{aligned}$$

D Appendix: Dynamic Magnification Factor

Table D.11: Dynamic Magnification Factor Data

Dataset	Frequency (Hz)	DMF	$\frac{f}{p}$	Mean Amplitude, A
Small 0.22 Hz	0.22	1.0396	0.2419	1.8454
Small 0.35 Hz	0.35	1.1287	0.3848	2.0034
Small 0.53 Hz	0.53	1.5598	0.5827	2.7686
Small 1.15 Hz	1.15	1.2427	1.2643	2.2057
Small 1.25 Hz	1.25	0.9588	1.3743	1.7019
Small 1.35 Hz	1.35	0.7020	1.4842	1.2460
Small 1.45 Hz	1.45	0.5868	1.5941	1.0417
Big 0.25 Hz	0.25	0.9437	0.3328	1.6751
Big 0.36 Hz	0.36	1.3088	0.4792	2.3231
Big 0.45 Hz	0.45	1.5176	0.5990	2.6937
Big 0.54 Hz	0.54	2.0809	0.7187	3.6937
Big 1.00 Hz	1.00	1.0029	1.3310	1.7801
Big 1.11 Hz	1.11	0.7129	1.4774	1.2654
Big 1.20 Hz	1.20	0.5327	1.5972	0.9455
Big 1.31 Hz	1.31	0.4284	1.7436	0.7605

Sample calculations for the dynamic magnification factor are shown for the small 0.22 Hz dataset. The mean amplitude was found by Excel by averaging the amplitudes across the number of cycles. The dynamic magnification factor was found by

$$\begin{aligned} \text{DMF} &= \frac{2 \times A}{Y_0} \\ &= \frac{2 \times 1.8454\,\text{cm}}{3.55\,\text{cm}} \\ &= \boxed{1.0396} \end{aligned}$$

The ratio of the frequency to the experimental natural frequency was found by

$$\begin{aligned}\frac{f}{p} &= \frac{0.22\,\text{Hz}}{0.909587047\,\text{Hz}} \\ &= \boxed{0.2419}\end{aligned}$$

where Y_0 is 3.55 cm

E Appendix: Equivalent Mass

This appendix outlines the calculations used to determine the equivalent mass of the measurement system. The equivalent mass is determined by analyzing the forces and moments acting on the system. The system is shown in Figure 5.

A schematic of the measurement system is shown in Figure 5. The equivalent mass of the system is given by

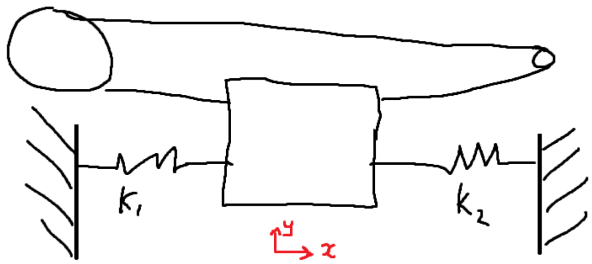

Figure E.17: Cart and Pulley Measurement System

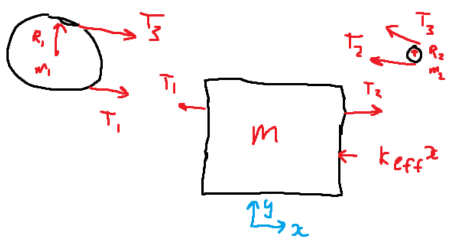

Figure E.18: Free Body Diagram of the Cart and Pulleys

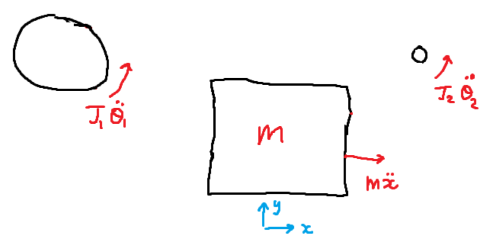

Figure E.19: Mass Acceleration Diagram of the Cart and Pulleys

Taking the sum of forces in x, the moment about pulley 1's and pulley's 2 mass center,

$$\rightarrow \sum F_x := m_{\text{cart}}\ddot{x}$$

$$= T_2 - T_1 - k_{\text{eff}}x \tag{23}$$

$$\circlearrowleft \sum M_{\text{pulley 1}} := J_1\ddot{\theta}_1$$

$$= r_1(T_1 - T_3) \tag{24}$$

$$\circlearrowleft \sum M_{\text{pulley 2}} := J_2\ddot{\theta}_2$$

$$= r_2(T_3 - T_2) \tag{25}$$

Assuming the cable does not slip in the the grooves of the pulleys at radius r_1 and r_2,

$$\theta_1 = \frac{\ddot{x}}{r_1}, \quad \theta_2 = \frac{\ddot{x}}{r_2} \tag{26}$$

Next, combining Eq. (23), (24), (25) and using Eq. (26), we get

$$\begin{aligned} J_1 \frac{\ddot{x}}{r_1} &= r_1(T_1 - T_3) \\ \implies \frac{J_1}{r_1^2}\ddot{x} &= T_1 - T_3 \end{aligned} \tag{27}$$

$$\begin{aligned} J_2 \frac{\ddot{x}}{r_2} &= r_2(T_3 - T_2) \\ \implies \frac{J_2}{r_2^2}\ddot{x} &= T_3 - T_2 \end{aligned} \tag{28}$$

Performing Eq. (27) + Eq. (28) gives

$$\begin{aligned} \frac{J_1}{r_1^2}\ddot{x} + \frac{J_2}{r_2^2}\ddot{x} &= T_1 - T_3 + T_3 - T_2 \\ \frac{J_1}{r_1^2}\ddot{x} + \frac{J_2}{r_2^2}\ddot{x} &= T_1 - T_2 \end{aligned} \tag{29}$$

And Eq. (29) + Eq. (23) gives

$$\left(m + \frac{J_1}{r_1^2} + \frac{J_2}{r_2^2}\right)\ddot{x} = -k_{\text{eff}}x \tag{30}$$

Finally,

$$\boxed{\underbrace{\left(m_{\text{cart}} + \frac{J_1}{r_1^2} + \frac{J_2}{r_2^2}\right)}_{m_e}\ddot{x} + k_{\text{eff}}x = 0} \tag{31}$$

If the pulleys are assumed to be uniform disks,

$$J_1 = \frac{1}{2}m_1 r_1^2, \quad J_2 = \frac{1}{2}m_2 r_2^2 \tag{32}$$

Substituting Eq. (32) into Eq. (31) gives

$$\boxed{\underbrace{\left(m_{\text{cart}} + \frac{1}{2}m_1 + \frac{1}{2}m_2\right)}_{m_e}\ddot{x} + k_{\text{eff}}x = 0} \tag{33}$$

In actuality, the equivalent mass in (33) should be larger than (31) since the pulleys aren't uniform disks due to the groove.

F Effective Mass Calculation

This section outlines the calculations used to determine the experimental effective mass of the system. The effective mass is determined by comparing the natural frequency of the no-measurement-system (whose mass is known) to the natural frequency of the measurement system (whose mass is unknown).

Suppose the natural frequency of the no-measurement-system for the small cart is $p_{\text{small, no sys}}$ and the natural frequency of the measurement system for the small cart is $p_{\text{small, sys}}$. Using the definition of resonance,

$$p_{\text{small, no sys}} = \sqrt{\frac{k_e}{m_{e_{\text{small, no sys}}}}}, \quad p_{\text{small, sys}} = \sqrt{\frac{k_e}{m_{e_{\text{small, sys}}}}}$$
$$\implies \frac{p_{\text{small, no sys}}}{p_{\text{small, sys}}} = \sqrt{\frac{m_{e_{\text{small, sys}}}}{m_{e_{\text{small, no sys}}}}}$$
$$\implies m_{e_{\text{small, sys}}} = \left(\frac{p_{\text{small, no sys}}}{p_{\text{small, sys}}}\right)^2 m_{e_{\text{small, no sys}}}$$

Table F.12: Natural Frequency Data from Free Vibration Trials

	Small Cart		Big Cart	
	w/o measuring system (Hz)	w/ measuring system (Hz)	w/o measuring system (Hz)	w/ measuring system (Hz)
Natural Frequency from Trials	1.1317	0.9096	0.8576	0.7513

Table F.13: Effective Mass Data

	Small Cart (kg)	Big Cart (kg)
Effective Mass from Trials	0.4099	0.5818
Effective Mass from Theory	0.4016	0.5833
Relative Error	2.09%	0.25%

Sample calculations will be shown for Table F.13. The effective mass of the small cart was found by using the natural frequency data from Table F.12. Then, knowing tha mass of the small cart is 0.2648 kg,

$$\begin{aligned} m_{e_{\text{small, sys}}} &= \left(\frac{1.1317}{0.9096}\right)^2 \times 0.2648 \\ &= \boxed{0.4099} \end{aligned}$$

The theoretical effective mass of the small cart is 0.4016 kg. The percent error is

$$\begin{aligned} \text{Percent Error} &= \frac{0.4099 - 0.4016}{0.4016} \times 100\% \\ &= \boxed{2.09\%} \end{aligned}$$

G Appendix: Table Dump

Will format later, formatting tables is aids

```
Mass Force Intitial Position, $a$ Final Position, $b$ Deflection
(g) (N) (cm) (cm) (m)
0 0 91.1 91.1 0
50 0.4905 91.1 87.4 0.037
100 0.981 91.1 83.8 0.073
150 1.4715 91.1 80.3 0.108
200 1.962 91.1 76.4 0.147
250 2.4525 91.1 73.3 0.178
300 2.943 91.1 69 0.221

Small mass (kg) 0.2648 Big mass (kg) 0.4465
Trial w/o measuring system w/ measuring system w/o measuring system w/ measuring syste
1 8.81 11.13 11.49 13.7
2 8.71 10.98 11.67 13.14
3 8.85 11.01 11.69 13.31
4 8.88 10.9 11.84 13.19
5 8.93 10.95 11.61 13.21
avg, 10 tau 8.836 10.994 11.66 13.31
tau 0.8836 1.0994 1.166 1.331
natural freq, p, derived from period (rad/s) 7.110893286 5.715103972 5.388666644
4.720650118
natural freq, p, derived from period (Hz) 1.131733816 0.909587047 0.857632933
0.751314801
natural freq, p, derived from using k_e (rad/s) 7.134339805 5.494169447
natural freq, p, derived from using k_e (Hz) 1.135465446 0.874424226

Pulley Diamater (mm) Mass Groove Depth (mm)
Big 195.98 259.9 1.5
Small 52.61 13.6 1.5

dataset peak number amplitude delta \zeta
big 1 1 8.2775 0.119385043 0.018997291
big 1 2 7.346 0.134348497 0.021377341
```

```
big 1 3 6.4225 0.146730503 0.02334652
big 1 4 5.546 0.179764425 0.028598694
big 1 5 4.6335 0.202002469 0.032133089
big 1 6 3.786 0.240058119 0.038178581
big 1 7 2.978 0.328811927 0.052260531
big 1 8 2.1435 0.471264045 0.074793917
big 1 9 1.338
big 2 1 9.4465 0.101586118 0.01616582
big 2 2 8.534 0.116963123 0.018612035
big 2 3 7.592 0.132252668 0.021044005
big 2 4 6.6515 0.145034028 0.023076735
big 2 5 5.7535 0.17320361 0.027555743
big 2 6 4.8385 0.192437655 0.030613049
big 2 7 3.9915 0.23835321 0.037907825
big 2 8 3.145 0.317698849 0.05049883
big 2 9 2.289 0.444895542 0.070630487
big 2 10 1.467 0.79495422 0.125520247
big 2 11 0.6625
small 1 1 9.473 0.093317023 0.014850228
small 1 2 8.629 0.102608739 0.016328511
small 1 3 7.7875 0.118031863 0.018782041
small 1 4 6.9205 0.131541363 0.020930872
small 1 5 6.0675 0.147293322 0.023436022
small 1 6 5.2365 0.185362555 0.029488537
small 1 7 4.3505 0.211971847 0.033717185
small 1 8 3.5195 0.27362217 0.043507086
small 1 9 2.677 0.379561461 0.060299159
small 1 10 1.8315 0.55872893 0.088574955
small 1 11 1.0475
small 2 1 23.182 0.078630971 0.012513528
small 2 2 21.429 0.091411866 0.014547111
small 2 3 19.557 0.095540303 0.015203954
small 2 4 17.775 0.10451699 0.016632095
small 2 5 16.011 0.1158877 0.018440964
small 2 6 14.259 0.126950746 0.020200716
small 2 7 12.559 0.144982969 0.023068616
```

```
small 2 8 10.864 0.178279663 0.028362675
small 2 9 9.09 0.201245362 0.032012778
small 2 10 7.433 0.257773714 0.040991478
small 2 11 5.744 0.350674281 0.055724823
small 2 12 4.045 0.532063149 0.084378491
small 2 13 2.376

DMF stuff peak number amplitude doing analysis in pandas cause this is aids

dataset frequency dmf freq/p
small 0.22hz 0.22 1.039647887 0.241868
small 0.35hz 0.35 1.128692153 0.38479
small 0.53hz 0.53 1.559769526 0.582682
small 1.15hz 1.15 1.242669933 1.26431
small 1.25hz 1.25 0.958841941 1.37425
small 1.35hz 1.35 0.701959584 1.48419
small 1.45hz 1.45 0.586846295 1.59413
big 0.25hz 0.25 0.94371831 0.33275
big 0.36hz 0.36 1.308779343 0.47916
big 0.45hz 0.45 1.517589984 0.59895
big 0.54hz 0.54 2.080947503 0.71874
big 1.00hz 1 1.002852113 1.331
big 1.11hz 1.11 0.712910798 1.47741
big 1.20hz 1.2 0.532661231 1.5972
big 1.31hz 1.31 0.42843729 1.74361
```