

1 State Models

Consider a system with

1. n , the order of the ODE
2. m , the number of inputs
3. p , the number of outputs

General procedure:

1. Create n state variables
2. Create n first order ODEs
3. Write $\dot{x} = f(x, u)$
4. Write $y = h(x, u)$

If the system is linear, then the state model is

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du\end{aligned}$$

2 Numerical Simulation with MATLAB

Example, some random order 4 system with $x_1(0) = 1$, $x_2(0) = 2$, $u_1 = \cos(t)$, $u_2 = \sin(t)$.

```
function x_dot = f(t, x)
x_dot = [
    x(3);
    x(4);
    -10*x(1) + 10*x(2) + cos(t)
    10*x(1) - 10*x(2) - sin(t)
];
end

[t, x] = ode45(@f, [0, 10], [1, 2, 0, 0]);
```

3 Linearization

Select a point (x_0, u_0) and to be the equilibrium point. That is,

$$f(x_0, u_0) = 0$$

From the set of equilibrium points, choose the appropriate one to linearize about. Then,

$$A = \left. \frac{\partial f}{\partial x} \right|_{x=x_0, u=u_0}$$

$$B = \left. \frac{\partial f}{\partial u} \right|_{x=x_0, u=u_0}$$

$$C = \left. \frac{\partial h}{\partial x} \right|_{x=x_0, u=u_0}$$

$$D = \left. \frac{\partial h}{\partial u} \right|_{x=x_0, u=u_0}$$

Example for inverted pendulum on a cart with equilibrium point $x_0 = (x_1 0, 0, 0, 0)$ and $u_0 = 0$.

```
% Declare symbolic variables
```

```
syms x1 x2 x3 x4 u
```

```
% Define the system
```

```
f = [
    x3;
    x4;
    (4*x4^2*sin(x2) - 3*cos(x2)*sin(x2) + 4*u)/(4 - 3*cos(x2)^2);
    (-3*x4^2*sin(x2)*cos(x2) + 3*sin(x2) - 3*u*cos(x2))/(4 - 3*cos(x2)^2);
];
h = [x1; x2];
```

```
% Compute the Jacobian
```

```
dfdx = jacobian(f, [x1, x2, x3, x4])
dfdu = jacobian(f, u)
dhdx = jacobian(h, [x1, x2, x3, x4])
dhdu = jacobian(h, u)
```

```
A = subs(dfdx, [x1, x2, x3, x4, u], [x10, 0, 0, 0, 0])
B = subs(dfdu, [x1, x2, x3, x4, u], [x10, 0, 0, 0, 0])
C = subs(dhdx, [x1, x2, x3, x4, u], [x10, 0, 0, 0, 0])
D = subs(dhdu, [x1, x2, x3, x4, u], [x10, 0, 0, 0, 0])
```

4 Solutions to Linear Systems

Split the system into two parts: the zero-input response and the zero-state response.

$$x(t) = x_{z-i}(t) + x_{z-s}(t)$$

$$y(t) = Cx_{z-i}(t) + Cx_{z-s}(t) + Du(t)$$

4.1 Zero-Input Response

The zero-input problem is:

$$\begin{aligned}\dot{x} &= Ax \\ y &= Cx \\ x(0) &= x_0 \\ u &= 0\end{aligned}$$

The solution is

$$\begin{aligned}x_{z-i}(t) &= e^{At}x_0 \\ y_{z-i}(t) &= Ce^{At}x_0\end{aligned}$$

4.2 Matrix exponential properties

$$\begin{aligned}e^{At} &= \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} \\ e^{At}|_{t=0} &= I \\ e^{At_1}e^{At_2} &= e^{A(t_1+t_2)} \\ e^{A_1t}e^{A_2t} &= e^{(A_1+A_2)t} \iff A_1A_2 = A_2A_1 \\ (e^{At})^{-1} &= e^{-At} \\ Ae^{At} &= e^{At}A \\ \frac{d}{dt}e^{At} &= Ae^{At} = e^{At}A \\ e^{At} &= Ve^{Dt}V^{-1} = \mathcal{L}^{-1}\{sI - A\}^{-1}\end{aligned}$$

4.3 Zero-State Response

The zero-state problem is:

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du \\ x(0) &= 0 \\ u &= u(t)\end{aligned}$$

By integrating factor method, the solution is:

$$\begin{aligned}x_{z-s}(t) &= \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \\ y_{z-s}(t) &= \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)\end{aligned}$$

4.4 Total Response

The total trajectory and response respectively are:

$$x(t) = \underbrace{e^{At}x_0}_{x_{z-i}(t)} + \underbrace{\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau}_{x_{z-s}(t)}$$

$$y(t) = \underbrace{Ce^{At}x_0}_{y_{z-i}(t)} + \underbrace{\int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau}_{y_{z-s}(t)} + Du(t)$$

5 Laplace Transform Method

5.1 Laplace Transform and Properties

$$\mathcal{L}f(t) = F(s) = \int_0^\infty f(t)e^{-st}dt$$

$$\mathcal{L}\dot{f}(t) = sF(s) - f(0)$$

$$\mathcal{L}\ddot{f}(t) = s^2F(s) - sf(0) - \dot{f}(0)$$

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{F(s)}{s}$$

$$\mathcal{L}f(t - t_d) = e^{-st_d}F(s)$$

5.2 Poles and Convergence

- In general, the right most pole determines the region of convergence.
- Use the analogy that the real part of poles correspond to the exponential decay rate of the system and the imaginary part corresponds to the oscillation frequency.
- Repeated poles is correspond to te^{-at} or $t\sin(at)$.
- idk i might add fvt later

5.3 Solution to State Space Model

The trajectory and response respectively are:

$$x(t) = \underbrace{\mathcal{L}^{-1}\{(sI - A)^{-1}x_0\}}_{x_{z-i}(t)} + \underbrace{\mathcal{L}^{-1}\{(sI - A)^{-1}BU(s)\}}_{x_{z-s}(t)}$$

$$y(t) = \underbrace{\mathcal{L}^{-1}\{C(sI - A)^{-1}x_0\}}_{y_{z-i}(t)} + \underbrace{\mathcal{L}^{-1}\{[C(sI - A)^{-1}B + D]U(s)\}}_{y_{z-s}(t)}$$

5.4 Transfer Function

The transfer function is defined from:

$$Y(s) = \underbrace{C(sI - A)^{-1}x_0}_{Y_{z-i}(s)} + \underbrace{[C(sI - A)^{-1}B + D]U(s)}_{Y_{z-s}(s)}^{G(s)}$$

$$G(s) = C(sI - A)^{-1}B + D$$

6 Step Response

6.1 First Order System

The standard first-order system is described by the following transfer function:

$$G(s) = \frac{K}{\tau s + 1}$$

For a step response $u = u_0(t) \implies U = 1/s$, the output is:

$$Y(s) = \frac{K}{\tau s + 1} \frac{1}{s}$$

$$y(t) = K(1 - e^{-t/\tau})$$

To find the time constant τ and the gain K , we can use some properties:

$$\lim_{t \rightarrow \infty} y(t) = K(1 - e^{-t/\tau}) = K$$

$$y(\tau) = K(1 - e^{-1}) = 0.632K = 63.2\%K$$

6.2 Second Order System

The standard second-order system is described by the following transfer function:

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

For a step response $u = \alpha u_0(t) \implies U = 1/s$, the response can be obtained by using $\omega_d = \omega_n \sqrt{1 - \zeta^2}$:

$$y(t) = \alpha - \alpha e^{-\zeta\omega_n t} \left[\cos(\omega_d t) + \frac{\zeta\omega_n}{\omega_d} \sin(\omega_d t) \right]$$

To determine ζ and ω_n , define overshoot $M_p = (y_{\max} - y_{\infty})/y_{\infty}$ and peak time $t_p \implies y(t_p) = y_{\max}$.

$$\zeta = \frac{(\ln(M_p))^2}{\pi^2 + (\ln(M_p))^2}$$

$$\omega_n = \frac{\pi}{t_p \sqrt{1 - \zeta^2}}$$

7 Impulse Response

Recall the Dirac delta function $\delta(t)$ is defined as:

$$\begin{aligned}\int_{-\infty}^{\infty} \delta(t) dt &= 1 \\ \delta(t) &= 0 \quad \forall t \neq 0 \\ \int_{-\infty}^{\infty} f(t) \delta(t - t_0) dt &= f(t_0)\end{aligned}$$

The zero-state response to a impulse input $u = \delta(t)$ is:

$$\begin{aligned}g(t) &= \int_0^t C e^{A(t-\tau)} B \delta(\tau) d\tau + D \delta(t) \\ &= C e^{At} B + D \delta(t)\end{aligned}$$

The Laplace transform of the impulse response is:

$$G(s) = C(sI - A)^{-1}B + D$$

This is no coincidence. The transfer function is the Laplace transform of the impulse response.

By convolution,

$$\begin{aligned}y_{z-s}(t) &= (g * u)(t) \\ &= \int_0^t g(t - \tau) u(\tau) d\tau \\ &= \int_0^t C e^{A(t-\tau)} B u(\tau) d\tau + D u(t)\end{aligned}$$

Which is exactly the zero-state response obtained using the time-domain integrating factor method.

8 Realization

to do later cause lazy

9 Stability

9.1 Internal Stability

Internal stability is concerned with unforced ($u = 0$) systems. This corresponds to the trajectory $x_{z-i}(t)$ and the response $y_{z-i}(t)$.

For $x(0)$, the system is internally **stable** about an equilibrium point x_e if and only if for every $\epsilon > 0$, there exists a $\delta > 0$ such that:

$$\|x(0) - x_e\| < \delta \implies \|x(t) - x_e\| < \epsilon, \quad \forall t \geq 0$$

A system is said to be **asymptotically stable** if it is stable and $\lim_{t \rightarrow \infty} x(t) = x_e$.

For linear systems, the system is internally stable if and only if every eigenvalue of A has a negative real part. $\text{Re}(\lambda_i) < 0, \forall i = 1, \dots, n$.

9.2 BIBO Stability

BIBO (bounded-input bounded-output) stability is concerned with zero state systems. This corresponds to the trajectory $x_{z-s}(t)$ and the response $y_{z-s}(t)$.

Assume $G(s)$ is a rational and proper transfer function. Then, the system is BIBO stable if and only if every pole of $G(s)$ has a negative real part. $\text{Re}(\text{pole}_i) < 0, \forall i = 1, \dots, n$.

If the system is not stable, then there exists a bounded input $u(t)$ such that the output $y(t)$ is unbounded. Not all bounded inputs will cause an unstable BIBO system to be unbounded.

9.3 Connection between Internal and BIBO Stability

For a linear system, internal stability implies BIBO stability. However, BIBO stability does not imply internal stability. This is because the poles of $G(s)$ are a subset of the eigenvalues of A .

asymptotic stability \implies BIBO stability

9.4 Closed Loop Systems

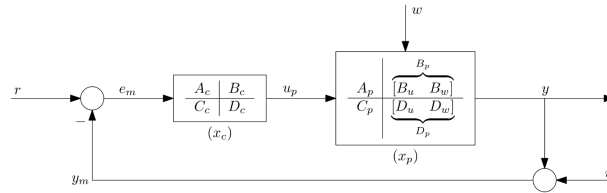


Figure 1: Closed loop system

Assume:

- Both controller (A_c, B_c, C_c, D_c) and plant (A_p, B_p, C_p, D_p) are minimal realizations.
- $D_c = 0$ or $D_p = [D_u, D_w] = 0$, such that $D_c D_u = D_u D_c = 0$ and $D_c D_w = D_w D_c = 0$.

Then, the state-space form of the closed loop system is:

$$\begin{aligned} \dot{x}_{cl} = \begin{bmatrix} \dot{x}_c \\ \dot{x}_p \end{bmatrix} &= \underbrace{\begin{bmatrix} A_c - B_c D_u C_c & -B_c C_p \\ B_u C_c & A_p - B_u D_c C_p \end{bmatrix}}_{A_{cl}} \begin{bmatrix} x_c \\ x_p \end{bmatrix} + \underbrace{\begin{bmatrix} B_c & -B_c D_w & -B_c \\ B_u D_c & B_w & -B_u D_c \end{bmatrix}}_{B_{cl}} \begin{bmatrix} r \\ w \\ v \end{bmatrix} \\ y_{cl} = \begin{bmatrix} e_m \\ u_p \\ y \\ y_m \end{bmatrix} &= \underbrace{\begin{bmatrix} -D_u C_c & -C_p \\ C_c & -D_c C_p \\ D_u C_c & C_p \\ D_u C_c & C_p \end{bmatrix}}_{C_{cl}} \begin{bmatrix} x_c \\ x_p \end{bmatrix} + \underbrace{\begin{bmatrix} 1 & -D_w & -1 \\ D_c & 0 & -D_c \\ 0 & D_w & 0 \\ 0 & D_w & 1 \end{bmatrix}}_{D_{cl}} \begin{bmatrix} r \\ w \\ v \end{bmatrix} \end{aligned}$$

Casting to transfer functions where

$$\begin{aligned} G_c(s) &:= C_c(sI - A_c)^{-1}B_c + D_c \\ G(s) &:= C_p(sI - A_p)^{-1}B_p + D_p := \begin{bmatrix} G_p(s) & G_d(s) \end{bmatrix} \end{aligned}$$

Then the system becomes:

$$\begin{bmatrix} E_m \\ U_p \\ Y \\ Y_m \end{bmatrix} = \begin{bmatrix} \frac{1}{1+G_p G_c} & \frac{G_d}{1+G_p G_c} & \frac{-1}{1+G_p G_c} \\ \frac{G_c}{1+G_p G_c} & \frac{-G_c G_d}{1+G_p G_c} & \frac{-G_c}{1+G_p G_c} \\ \frac{G_p G_c}{1+G_p G_c} & \frac{G_d}{1+G_p G_c} & \frac{-G_p G_c}{1+G_p G_c} \\ \frac{G_p G_c}{1+G_p G_c} & \frac{G_d}{1+G_p G_c} & \frac{-1}{1+G_p G_c} \end{bmatrix} \begin{bmatrix} R \\ W \\ V \end{bmatrix}$$

9.5 Unmodeled Disturbance Input

Most of the time we just assume that disturbance is added to the input u_p . In the transfer function matrix, $G_d = G_p$.

9.6 Internal Stability of Closed Loop Systems

The closed loop system is internally stable if and only if every eigenvalue of A_{cl} has a negative real part.

9.7 I/O Stability of Closed Loop Systems

The closed loop system is I/O stable if and only if every entry of the transfer function matrix is BIBO stable. There are 5 unique transfer functions to be considered:

$$\begin{aligned} & \frac{1}{1 + G_p G_c} \\ & \frac{G_c}{1 + G_p G_c} \\ & \frac{G_p G_c}{1 + G_p G_c} \\ & \frac{G_d}{1 + G_p G_c} \\ & \frac{G_c G_d}{1 + G_p G_c} \end{aligned}$$

If the disturbance is added to the input, then $G_d = G_p$ and then we only need to check:

$$\begin{aligned} & \frac{1}{1 + G_p G_c} \\ & \frac{G_c}{1 + G_p G_c} \\ & \frac{G_p G_c}{1 + G_p G_c} \\ & \frac{G_p}{1 + G_p G_c} \end{aligned}$$

9.7.1 Characteristic Polynomial for Closed Loop Systems

Define the numerator and denominator of the transfer functions:

$$G_c := \frac{n_c}{d_c}, \quad G := [G_p, G_d] := \left[\frac{n_p}{d_p}, \frac{n_d}{d_d} \right]$$

The characteristic polynomial is:

$$P = n_p n_c + d_p d_c$$

By Theorem 4.4.1, the closed loop system is I/O stable if and only if all roots of P have a negative real part.

9.7.2 Theorem 4.4.2

The closed loop system is I/O stable if and only if

1. $1 + G_p G_c$ has only negative real part roots
2. $G_p G_c$ has no pole-zero cancellations