Question 1

Using the Inverse Laplace Transform method, compute the matrix exponential e^{At} for

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

Note the solution should match Assignment #2.

Solution
The equation linking the matrix exponential and the Laplace transform is

$$e^{At} = \mathcal{L}^{-1}\left\{(sI - A)^{-1}\right\}$$

By Matlab,

```
ans =

[exp(3*t)/2 + exp(t)/2, exp(t)/2 - exp(3*t)/2]
[exp(t)/2 - exp(3*t)/2, exp(3*t)/2 + exp(t)/2]
```

Written nicely,

$$\boxed{e^{At} = \begin{bmatrix} \frac{e^{3t}}{2} + \frac{e^t}{2} & \frac{e^t}{2} - \frac{e^{3t}}{2} \\ \frac{e^t}{2} - \frac{e^{3t}}{2} & \frac{e^{3t}}{2} + \frac{e^t}{2} \end{bmatrix}}$$

The code used to generate the above was

```
clc; clear all; close all;
syms s
A = [2 -1; -1 2];
ilaplace(inv(s*eye(2) - A))
```

Question 2

Using the Laplace Transform approach, compute the (total) response $y(t)$ of the state-space system

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u, \qquad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \qquad u(t) = 1_+(t)$$
$$y = \begin{bmatrix} 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Note the solution should match Assignment #3.

Solution
Observe that the system is already in state-space form. We can immediately utlize the Laplace Transform response equation:

$$Y(s) = C(sI - A)^{-1}x_0 + C(sI - A)^{-1}BU(s) + DU(s)$$

Computing the Laplace Transform of the input $u(t)$,

$$\begin{aligned} U(s) &= \mathcal{L}\left\{1_+(t)\right\} \\ &= \frac{1}{s} \end{aligned}$$

By Matlab,

```
Y =


2*exp(3*t) + 3*exp(t) - 2
```

Written nicely,

$$\boxed{y(t) = 2e^{3t} + 3e^{t} - 2}$$

The code used to generate the above is

```
clc; clear all; close all;
syms s
A = [2 -1; -1 2];
B = [2; 0];
C = [0 3];
D = 0;
x0 = [-1; 1];

Y = ilaplace(C*inv(s*eye(2) - A)*x0 + C*inv(s*eye(2) - A)*B*1/s + D
    *1/s)
```

Question 3

Recall the analog circuit from Assignment #1

The input is the voltage u and the response (output) is the voltage y. Find the transfer function of this system

(a) Using the formula $G(s) = C(sI - A)^{-1}B + D$

(b) By applying the Laplace transform to the governing ODEs

Solution

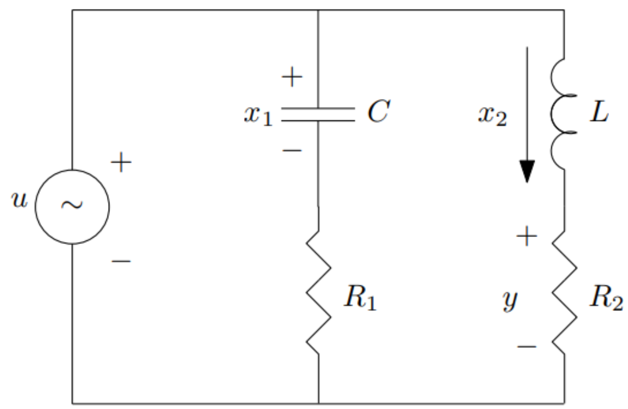

Figure 1: Analog circuit from Assignment #1

(a)

From Assignment #1, we have the state-space form

$$A = \begin{bmatrix} -1/(R_1C) & 0 \\ 0 & -R_2/L \end{bmatrix}, \quad B = \begin{bmatrix} 1/(R_1C) \\ 1/L \end{bmatrix}$$
$$C = [0, R_2], \quad D = 0$$

By Matlab,

```
G =

R2/(R2 + L*s)
```

Written nicely,

$$\boxed{G(s) = \frac{R_2}{R_2 + Ls}}$$

This was given by the Matlab code:

```
clc; clear; close all;
syms s R1 R2 C L

A = [-1/(R1*C), 0; 0, -R2/L];
B = [1/(R1*C); 1/L];
C = [0, R2];
D = 0;

G = C*inv(s*eye(2) - A)*B + D
```

(b)

From the solution of Assignment #1, we have the governing ODEs:

$$\begin{aligned} \dot{x_2} &= -\frac{x_2}{L} + \frac{1}{L}u \\ y &= R_2 x_2 \\ \dot{y} &= R_2 \dot{x_2} \end{aligned}$$

Writing in terms of y,

$$\begin{aligned} \frac{\dot{y}}{R_2} &= -\frac{y}{L} + \frac{1}{L}u \\ \dot{y} &= -\frac{R_2}{L}y + \frac{R_2}{L}u \end{aligned}$$

Applying the Laplace transform to both sides,

$$\begin{aligned} sY(s) - y(0) &= -\frac{R_2}{L}Y(s) + \frac{R_2}{L}U(s) \\ (s + \frac{R_2}{L})Y(s) &= \frac{R_2}{L}U(s) + y(0) \\ Y(s) &= \frac{R_2}{L(s + \frac{R_2}{L})}U(s) + \frac{y(0)}{s + \frac{R_2}{L}} \\ &= \underbrace{\frac{R_2}{R_2 + Ls}}_{G(s)}U(s) + \frac{y(0)}{s + \frac{R_2}{L}} \end{aligned}$$

By the definition of the transfer function,

$$\boxed{G(s) = \frac{R_2}{R_2 + Ls}}$$