

## Question 1

A linear, first-order system with zero initial conditions experiences a unit step input  $u(t) = 1_+(t)$ . The system's response looks as follows:

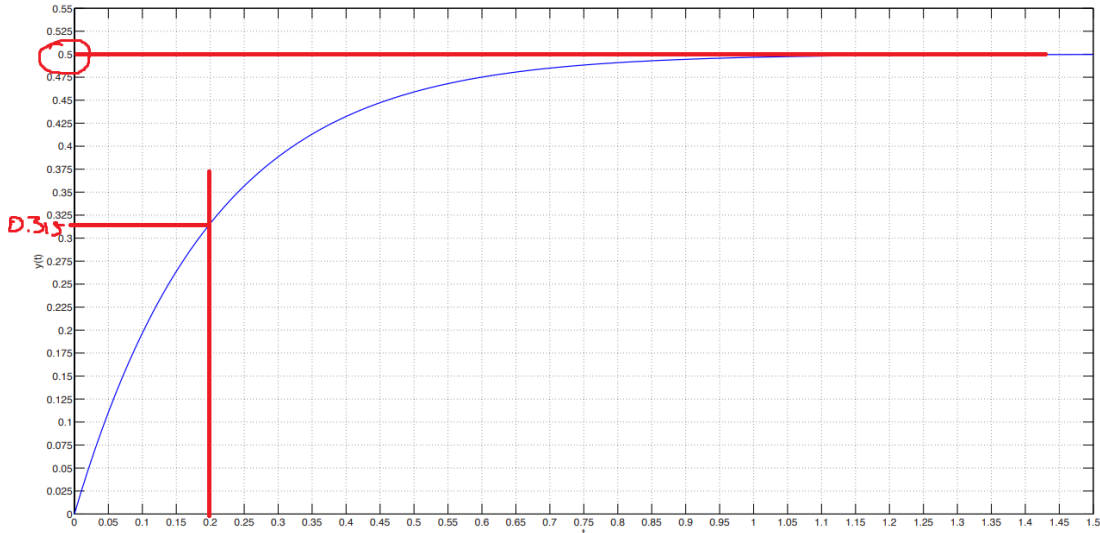


Figure 1: System-response to a unit step input with annotations

(a)

Find the values of the DC gain and time constant of this system

### Solution

The form of a first-order system transfer function is:

$$G(s) = \frac{\alpha}{\tau s + 1}$$

The DC gain is the value of  $y$  as  $t \rightarrow \infty$ . From the graph, we see that the DC gain is

$$\boxed{\begin{aligned} G(0) &= 0.5 \\ \implies \alpha &= 0.5 \end{aligned}}$$

The time constant is the time it takes for the system to reach 63.2% of its final value.

$$\begin{aligned} y(\tau) &= 0.632 \cdot 0.5 \\ \implies y(\tau) &= 0.316 \end{aligned}$$

From the graph,

$$\boxed{\tau = 0.2}$$

(b)

Give the transfer function  $G(s)$  of this system

**Solution**

Since  $\alpha = 0.5$  and  $\tau = 0.2$ ,

$$G(s) = \frac{0.5}{0.2s + 1}$$

## Question 2

The mass is  $M = 1$  kg, but the values of the damper  $D$  and spring  $K$  are unknown. In order to identify these, a 10 N amplitude step input  $u(t) = 10[1_+(t)]$  is applied to the system initially at rest. The resulting (underdamped) response  $y(t)$  is plotted below, showing the peak and steady-state values:

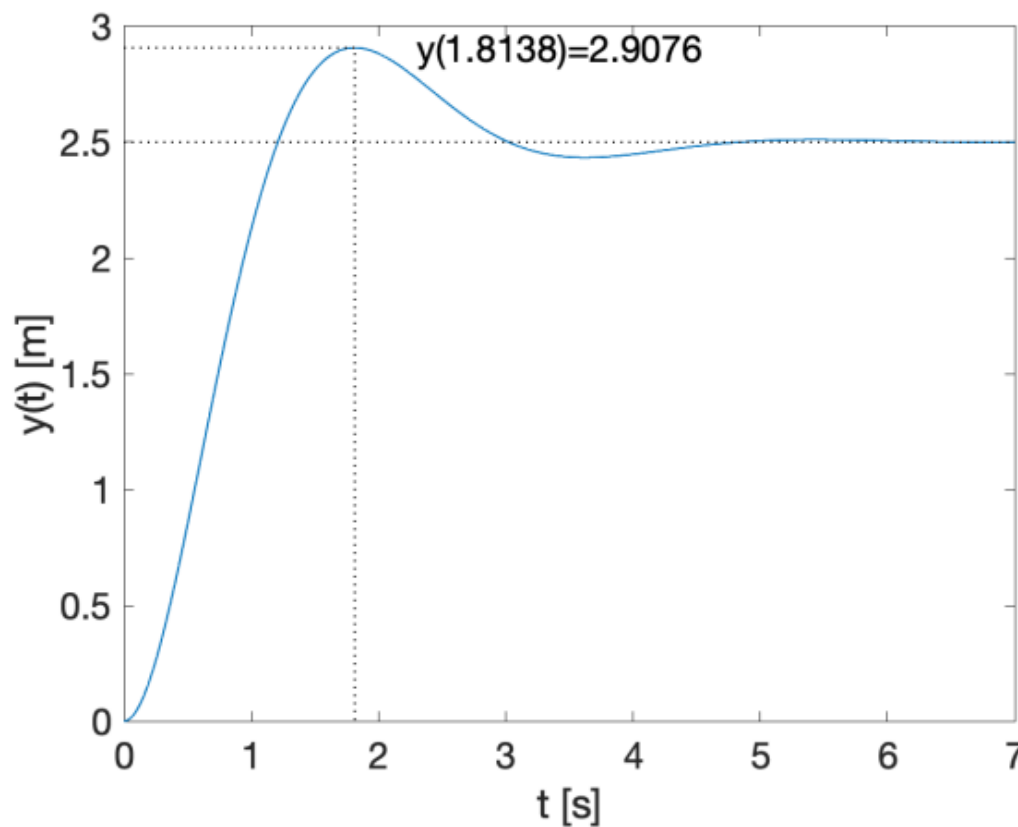


Figure 2: Second-order system response to a unit step input with annotations

**(a)**

Calculate the values of  $\zeta$  and  $\omega_n$  for this system

**Solution**

The transfer function of this system, as given in the notes (p. 53) is:

$$G(s) = \frac{1/M}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\omega_n = \sqrt{\frac{K}{M}} \tag{1}$$

$$\zeta = \frac{D}{2\sqrt{MK}} \tag{2}$$

Converting the transfer function to standard form, we get:

$$G(s) = \frac{1}{K} \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

The response is:

$$Y(s) = \underbrace{\frac{10}{K}}_{\alpha} \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

For a standard second-order system (p. 59),

$$\omega_n = \frac{2\pi}{t_p \sqrt{1 - \zeta^2}}$$

$$\zeta = \sqrt{\frac{\ln^2(M_p)}{\pi^2 + \ln^2(M_p)}}$$

From the graph, we see that  $t_p = 1.8138$ ,  $y(t_p) = 2.9076$ , and  $y_\infty = 2.5$ .

$$\begin{aligned} M_p &= \frac{y(t_p) - y_\infty}{y_\infty} \\ &= \frac{2.9076 - 2.5}{2.5} \\ &= 0.16304 \end{aligned}$$

$$\begin{aligned}
 \zeta &= \sqrt{\frac{\ln^2(0.16304)}{\pi^2 + \ln^2(0.16304)}} \\
 &= 0.4999918 \approx 0.5 \\
 \omega_n &= \frac{2\pi}{1.8138\sqrt{1 - 0.5^2}} \\
 &= 3.99999859794 \approx 4
 \end{aligned}$$

(b)

Using the information from (a), find the values of  $D$  and  $K$

**Solution** From Eq. (1),

$$\begin{aligned}
 K &= M\omega_n^2 \\
 &= 1 \cdot 4^2 \\
 &= \boxed{16}
 \end{aligned}$$

From Eq. (2),

$$\begin{aligned}
 D &= 2\sqrt{MK}\zeta \\
 &= 2\sqrt{1 \cdot 16} \cdot 0.5 \\
 &= \boxed{4}
 \end{aligned}$$

### Question 3

Consider the transfer function

$$G(s) = \frac{s+1}{s^2 + 2s + 3}$$

(a)

By hand, obtain a state-space realization of this transfer function.

**Solution**

From  $Y(s) = G(s)U(s)$ ,

$$Y(s) = (s+1) \underbrace{\frac{1}{s^2 + 2s + 3} U(s)}_{:=V(s)}$$

Define an intermediate signal  $V(s)$ . This leads to:

$$\begin{aligned}
 \ddot{v} + 2\dot{v} + 3v &= u \\
 Y(s) &= (s+1)V(s) \implies y = \dot{v} + v
 \end{aligned}$$

Let  $x_1 = v$  and  $x_2 = \dot{v}$ . Then,

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -3x_1 - 2x_2 + u \\ y &= x_2 + x_1 \end{aligned}$$

In state-space form,

$$\begin{aligned} \dot{x} &= \underbrace{\begin{bmatrix} 0 & 1 \\ -3 & -2 \end{bmatrix}}_A \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_x + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_B u \\ y &= \underbrace{\begin{bmatrix} 1 & 1 \end{bmatrix}}_C \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_x + \underbrace{0}_D \end{aligned}$$

(b)

Use MATLAB to obtain a (different) realization of the system

**Solution**

From Matlab,

A =

$$\begin{bmatrix} -2 & -3 \\ 1 & 0 \end{bmatrix}$$

B =

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

C =

$$\begin{bmatrix} 1 & 1 \end{bmatrix}$$

D =

$$0$$

The code used to generate this is:

```
clc; clear; close all;
[A, B2, C, D] = tf2ss([1 1], [1 2 3])
```

(c)

Demonstrate that the realizations in (a) and (b) both correspond to the same  $G(s)$ . Note you can employ MATLAB's symbolic toolbox for this.

### Solution

Recall the definition of the transfer function:

$$G(s) = C(sI - A)^{-1}B + D$$

Define the hand solution as  $G_1(s)$  and the Matlab solution as  $G_2(s)$ . Then from Matlab,

G1 =

$$s/(s^2 + 2s + 3) + 1/(s^2 + 2s + 3)$$

G2 =

$$s/(s^2 + 2s + 3) + 1/(s^2 + 2s + 3)$$

It is clear that  $G_1(s) = G_2(s)$

The code used to generate this is:

```
clc; clear; close all;
A1 = [0 1; -3 -2];
B1 = [0; 1];
C1 = [1 1];
D1 = 0;
[A2, B2, C2, D2] = tf2ss([1 1], [1 2 3]);
```

```
syms s
G1 = C1*inv(s*eye(2) - A1)*B1 + D1
G2 = C2*inv(s*eye(2) - A2)*B2 + D2
```

## Question 4

Recall the analog circuit from Assignment #1

As we saw in Assignment #1, the state-space form of the system is:

$$A = \begin{bmatrix} -1/(R_1 C) & 0 \\ 0 & -R_2/L \end{bmatrix}, \quad B = \begin{bmatrix} 1/(R_1 C) \\ 1/L \end{bmatrix}$$

$$C = [0, R_2], \quad D = 0$$

Computing the transfer function in Matlab,

G =

R2/(R2 + L\*s)

Since  $\text{size}(A) = 2$ ,  $\deg(G(s)) = 1$ , the system is not minimal as  $2 > 1$

The code used to generate this is:

```
clc; clear; close all;
syms s R1 R2 C L
A = [-1/(R1*C) 0; 0 -R2/L];
B = [1/(R1*C); 1/L];
c = [0 R2];
D = 0;

% Transfer function
G = c*inv(s*eye(2)-A)*B + D
```