1   State Models

Consider a system with

1. n, the order of the ODE

2. m, the number of inputs

3. p, the number of outputs

General procedure:

1. Create n state variables

2. Create n first order ODEs

3. Write $\dot{x} = f(x, u)$

4. Write $y = h(x, u)$

If the system is linear, then the state model is

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

2   Numerical Simulation with MATLAB

Example, some random order 4 system with $x_1(0) = 1$, $x_2(0) = 2$, $u_1 = \cos(t)$, $u_2 = \sin(t)$.

```matlab
function x_dot = f(t, x)
x_dot = [
    x(3);
    x(4);
    -10*x(1) + 10*x(2) + cos(t)
    10*x(1) - 10*x(2) - sin(t)
];
end

[t, x] = ode45(@f, [0, 10], [1, 2, 0, 0]);
```

3   Linearization

Select a point (x_0, u_0) and to be the equilibrium point. That is,

$$f(x_0, u_0) = 0$$

From the set of equilibrium points, choose the appropriate one to linearize about. Then,

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=x_0, u=u_0}$$

$$B = \left.\frac{\partial f}{\partial u}\right|_{x=x_0, u=u_0}$$

$$C = \left.\frac{\partial h}{\partial x}\right|_{x=x_0, u=u_0}$$

$$D = \left.\frac{\partial h}{\partial u}\right|_{x=x_0, u=u_0}$$

Example for inverted pendulum on a cart with equilibrium point $x_0 = (x_1 0, 0, 0, 0)$ and $u_0 = 0$.

```
% Declare symbolic variables
syms x1 x2 x3 x4 u

% Define the system
f = [
    x3;
    x4;
    (4*x4^2*sin(x2) - 3*cos(x2)*sin(x2) + 4*u)/(4 - 3*cos(x2)^2);
    (-3*x4^2*sin(x2)*cos(x2) + 3*sin(x2) -3*u*cos(x2))/(4 - 3*cos(x2
    )^2);
];
h = [x1; x2];

% Compute the Jacobian
dfdx = jacobian(f, [x1, x2, x3, x4])
dfdu = jacobian(f, u)
dhdx = jacobian(h, [x1, x2, x3, x4])
dhdu = jacobian(h, u)

A = subs(dfdx, [x1, x2, x3, x4, u], [x10, 0, 0, 0, 0])
B = subs(dfdu, [x1, x2, x3, x4, u], [x10, 0, 0, 0, 0])
C = subs(dhdx, [x1, x2, x3, x4, u], [x10, 0, 0, 0, 0])
D = subs(dhdu, [x1, x2, x3, x4, u], [x10, 0, 0, 0, 0])
```

4   Solutions to Linear Systems

Split the system into two parts: the zero-input response and the zero-state response.

$$x(t) = x_{\text{z-i}}(t) + x_{\text{z-s}}(t)$$
$$y(t) = Cx_{\text{z-i}}(t) + Cx_{\text{z-s}}(t) + Du(t)$$

4.1   Zero-Input Response

The zero-input problem is:

$$\dot{x} = Ax$$
$$y = Cx$$
$$x(0) = x_0$$
$$u = 0$$

The solution is

$$x_{\text{z-i}}(t) = e^{At}x_0$$
$$y_{\text{z-i}}(t) = Ce^{At}x_0$$

4.2   Matrix exponential properties

$$e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}$$
$$e^{At}\big|_{t=0} = I$$
$$e^{At_1}e^{At_2} = e^{A(t_1+t_2)}$$
$$e^{A_1 t}e^{A_2 t} = e^{(A_1+A_2)t} \iff A_1 A_2 = A_2 A_1$$
$$(e^{At})^{-1} = e^{-At}$$
$$Ae^{At} = e^{At}A$$
$$\frac{d}{dt}e^{At} = Ae^{At} = e^{At}A$$
$$e^{At} = Ve^{Dt}V^{-1} = \mathcal{L}^{-1}\{sI - A\}^{-1}$$

4.3   Zero-State Response

The zero-state problem is:

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$
$$x(0) = 0$$
$$u = u(t)$$

By integrating factor method, the solution is:

$$x_{\text{z-s}}(t) = \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

$$y_{\text{z-s}}(t) = \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$

4.4 Total Response

The total trajectory and response respectively are:

$$x(t) = \underbrace{e^{At}x_0}_{x_{\text{z-i}}(t)} + \underbrace{\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau}_{x_{\text{z-s}}(t)}$$

$$y(t) = \underbrace{Ce^{At}x_0}_{y_{\text{z-i}}(t)} + \underbrace{\int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)}_{y_{\text{z-s}}(t)}$$

5 Laplace Transform Method

5.1 Laplace Transform and Properties

$$\mathcal{L}f(t) = F(s) = \int_0^\infty f(t)e^{-st}dt$$

$$\mathcal{L}\dot{f}(t) = sF(s) - f(0)$$

$$\mathcal{L}\ddot{f}(t) = s^2F(s) - sf(0) - \dot{f}(0)$$

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{F(s)}{s}$$

$$\mathcal{L}f(t - t_d) = e^{-st_d}F(s)$$

5.2 Poles and Convergence

- In general, the right most pole determines the region of convergence.

- Use the analogy that the real part of poles correspond to the expontential decay rate of the system and the imaginary part corresponds to the oscillation frequency.

- Repeated poles is correspond to te^{-at} or $t\sin(at)$.

- idk i might add fvt later

5.3 Solution to State Space Model

The trajectory and response respectively are:

$$x(t) = \underbrace{\mathcal{L}^{-1}\{(sI - A)^{-1}x_0\}}_{x_{\text{z-i}}(t)} + \underbrace{\mathcal{L}^{-1}\{(sI - A)^{-1}BU(s)\}}_{x_{\text{z-s}}(t)}$$

$$y(t) = \underbrace{\mathcal{L}^{-1}\{C(sI - A)^{-1}x_0\}}_{y_{\text{z-i}}(t)} + \underbrace{\mathcal{L}^{-1}\{[C(sI - A)^{-1}B + D]U(s)\}}_{y_{\text{z-s}}(t)}$$

5.4   Transfer Function

The transfer function is defined from:

$$Y(s) = \underbrace{C(sI - A)^{-1}x_0}_{Y_{\text{z-i}}(s)} + \underbrace{\overbrace{\left[C(sI - A)^{-1}B + D\right]}^{G(s)} U(s)}_{Y_{\text{z-s}}(s)}$$

$$G(s) = C(sI - A)^{-1}B + D$$

6   Step Response

6.1   First Order System

The standard first-order system is described by the following transfer function:

$$G(s) = \frac{K}{\tau s + 1}$$

For a step response $u = u_0(t) \implies U = 1/s$, the output is:

$$Y(s) = \frac{K}{\tau s + 1}\frac{1}{s}$$

$$y(t) = K(1 - e^{-t/\tau})$$

To find the time constant τ and the gain K, we can use some properties:

$$\lim_{t \to \infty} y(t) = K(1 - e^{-t/\tau}) = K$$

$$y(\tau) = K(1 - e^{-1}) = 0.632K = 63.2\%K$$

6.2   Second Order System

The standard second-order system is described by the following transfer function:

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

For a step response $u = \alpha u_0(t) \implies U = 1/s$, the response can be obtained by using $\omega_d = \omega_n\sqrt{1 - \zeta^2}$:

$$y(t) = \alpha - \alpha e^{-\zeta\omega_n t}\left[\cos(\omega_d t) + \frac{\zeta\omega_n}{\omega_d}\sin(\omega_d t)\right]$$

To determine ζ and ω_n, define overshoot $M_p = (y_{\max} - y_\infty)/y_\infty$ and peak time $t_p \implies y(t_p) = y_{\max}$.

$$\zeta = \frac{(\ln(M_p))^2}{\pi^2 + (\ln(M_p))^2}$$

$$\omega_n = \frac{\pi}{t_p\sqrt{1 - \zeta^2}}$$

7   Impulse Response

Recall the Dirac delta function $\delta(t)$ is defined as:

$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$

$$\delta(t) = 0 \quad \forall t \neq 0$$

$$\int_{-\infty}^{\infty} f(t)\delta(t - t_0)dt = f(t_0)$$

The zero-state response to a impulse input $u = \delta(t)$ is:

$$g(t) = \int_0^t Ce^{A(t-\tau)}B\delta(\tau)d\tau + D\delta(t)$$

$$= Ce^{At}B + D\delta(t)$$

The Laplace transform of the impulse response is:

$$G(s) = C(sI - A)^{-1}B + D$$

This is no coincidence. The transfer function is the Laplace transform of the impulse response.

By convolution,

$$y_{\text{z-s}}(t) = (g * u)(t)$$

$$= \int_0^t g(t - \tau)u(\tau)d\tau$$

$$= \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$

Which is exactly the zero-state response obtained using the time-domain integrating factor method.

8   Realization

to do later cause lazy

9   Stability

9.1   Internal Stability

$x(0)$