# Question 1

A linear, first-order system with zero initial conditions experiences a unit step input $u(t) = 1_+(t)$. The system's response looks as follows:

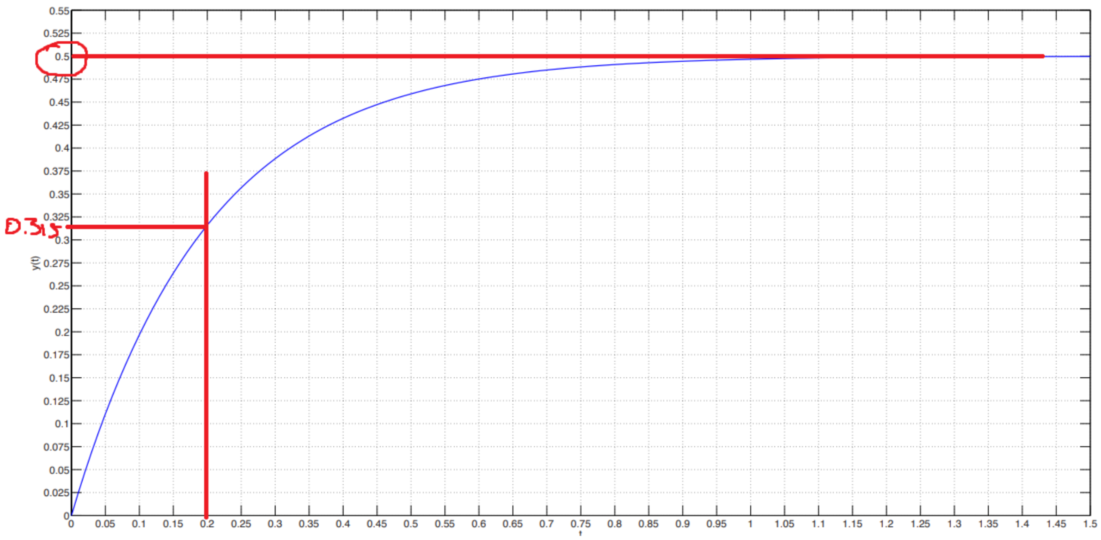


Figure 1: System-response to a unit step input with annotations

## (a)

Find the values of the DC gain and time constant of this system

**Solution**

The form of a first-order system transfer function is:

$$G(s) = \frac{\alpha}{\tau s + 1}$$

The DC gain is the value of $y$ as $t \to \infty$. From the graph, we see that the DC gain is

$$\boxed{\begin{aligned} G(0) &= 0.5 \\ \implies \alpha &= 0.5 \end{aligned}}$$

The time constant is the time it takes for the system to reach 63.2% of its final value.

$$\begin{aligned} y(\tau) &= 0.632 \cdot 0.5 \\ \implies y(\tau) &= 0.316 \end{aligned}$$

From the graph,

$$\boxed{\tau = 0.2}$$

### (b)

Give the transfer function G(s) of this system

**Solution**
Since $\alpha = 0.5$ and $\tau = 0.2$,

$$\boxed{G(s) = \frac{0.5}{0.2s + 1}}$$

## Question 2

The mass is $M = 1$ kg, but the values of the damper $D$ and spring $K$ are unknown. In order to identify these, a 10 N amplitude step input $u(t) = 10[1_+(t)]$ is applied to the system initially at rest. The resulting (underdamped) response $y(t)$ is plotted below, showing the peak and steady-state values:

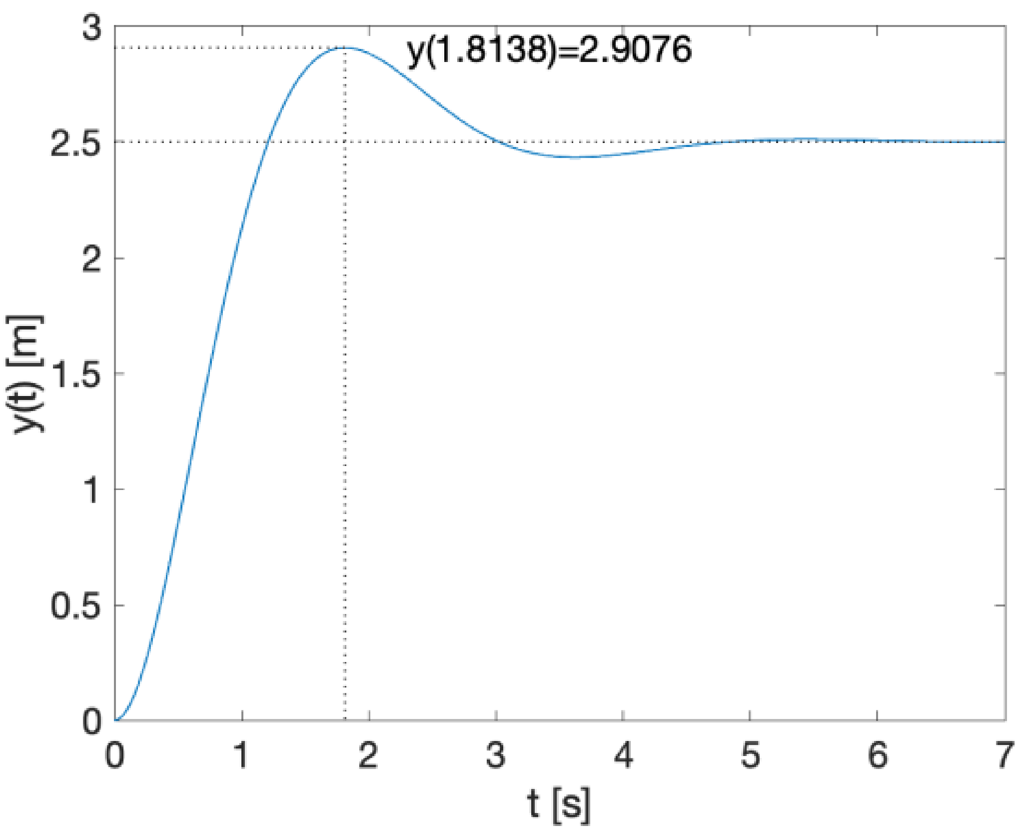


Figure 2: Second-order system response to a unit step input with annotations

## (a)

Calculate the values of $\zeta$ and $\omega_n$ for this system

### Solution

The transfer function of this system, as given in the notes (p. 53) is:

$$
\begin{aligned}
G(s) &= \frac{1/M}{s^2 + 2\zeta\omega_n s + \omega_n^2} \\
\omega_n &= \sqrt{\frac{K}{M}} \qquad (1) \\
\zeta &= \frac{D}{2\sqrt{MK}} \qquad (2)
\end{aligned}
$$

Converting the transfer function to standard form, we get:

$$G(s) = \frac{1}{K}\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

The response is:

$$Y(s) = \underbrace{\frac{10}{K}}_{\alpha}\frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

For a standard second-order system (p. 59),

$$
\begin{aligned}
\omega_n &= \frac{\pi}{t_p\sqrt{1-\zeta^2}} \\
\zeta &= \sqrt{\frac{\ln^2(M_p))}{\pi^2 + \ln^2(M_p)}}
\end{aligned}
$$

From the graph, we see that $t_p = 1.8138$, $y(t_p) = 2.9076$, and $y_\infty = 2.5$.

$$
\begin{aligned}
M_p &= \frac{y(t_p) - y_\infty}{y_\infty} \\
&= \frac{2.9076 - 2.5}{2.5} \\
&= 0.16304
\end{aligned}
$$

$$\boxed{\begin{aligned}\zeta &= \sqrt{\frac{\ln^2(0.16304)}{\pi^2 + \ln^2(0.16304)}} \\ &= 0.4999918 \approx 0.5 \\ \omega_n &= \frac{\pi}{1.8138\sqrt{1-0.5^2}} \\ &= 1.9999993 \approx 2\end{aligned}}$$

## (b)

Using the information from (a), find the values of $D$ and $K$

**Solution** From Eq. (1),

$$\begin{aligned}K &= M\omega_n^2 \\ &= 1 \cdot 2^2 \\ &= \boxed{4}\end{aligned}$$

From Eq. (2),

$$\begin{aligned}D &= 2\sqrt{MK}\zeta \\ &= 2\sqrt{1 \cdot 4} \cdot 0.5 \\ &= \boxed{2}\end{aligned}$$

# Question 3

Consider the transfer function

$$G(s) = \frac{s+1}{s^2 + 2s + 3}$$

## (a)

By hand, obtain a state-space realization of this transfer function.

**Solution**
From $Y(s) = G(s)U(s)$,

$$Y(s) = (s+1)\underbrace{\frac{1}{s^2 + 2s + 3}U(s)}_{:=V(s)}$$

Define an intermediate signal V(s). This leads to:

$$\begin{aligned}\ddot{v} + 2\dot{v} + 3v &= u \\ Y(s) = (s+1)V(s) &\implies y = \dot{v} + v\end{aligned}$$

Let $x_1 = v$ and $x_2 = \dot{v}$. Then,

$$\begin{aligned} \dot{x_1} &= x_2 \\ \dot{x_2} &= -3x_1 - 2x_2 + u \\ y &= x_2 + x_1 \end{aligned}$$

In state-space form,

$$\boxed{\begin{aligned} \dot{x} &= \underbrace{\begin{bmatrix} 0 & 1 \\ -3 & -2 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u \\ y &= \underbrace{\begin{bmatrix} 1 & 1 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{0}_{D} \end{aligned}}$$

## (b)

Use MATLAB to obtain a (different) realization of the system

**Solution**
From Matlab,

```
A =

    -2    -3
     1     0


B =

     1
     0


C =

     1     1


D =

     0
```

The code used to generate this is:

```
clc; clear; close all;
[A, B2, C, D] = tf2ss([1 1], [1 2 3])
```

## (c)

Demonstrate that the realizations in (a) and (b) both correspond to the same $G(s)$. Note you can employ MATLAB's symbolic toolbox for this.

**Solution**
Recall the definition of the transfer function:

$$G(s) = C(sI - A)^{-1}B + D$$

Define the hand solution as $G_1(s)$ and the Matlab solution as $G_2(s)$. Then from Matlab,

```
G1 =

s/(s^2 + 2*s + 3) + 1/(s^2 + 2*s + 3)


G2 =

s/(s^2 + 2*s + 3) + 1/(s^2 + 2*s + 3)
```

$$\boxed{\text{It is clear that } G_1(s) = G_2(s)}$$

The code used to generate this is:

```
clc; clear; close all;
A1 = [0 1; -3 -2];
B1 = [0; 1];
C1 = [1 1];
D1 = 0;
[A2, B2, C2, D2] = tf2ss([1 1], [1 2 3]);


syms s
G1 = C1*inv(s*eye(2) - A1)*B1 + D1
G2 = C2*inv(s*eye(2) - A2)*B2 + D2
```

# Question 4

Recall the analog circuit from Assignment #1

As we saw in Assignment #1, the state-space form of the system is:

$$A = \begin{bmatrix} -1/(R_1C) & 0 \\ 0 & -R_2/L \end{bmatrix}, \qquad B = \begin{bmatrix} 1/(R_1C) \\ 1/L \end{bmatrix}$$
$$C = [0, R_2], \qquad D = 0$$

Computing the transfer function in Matlab,

```
G =

R2/(R2 + L*s)
```

$$\boxed{\text{Since size}(A) = 2, \deg(G(s)) = 1, \text{the system is not minimal as } 2 > 1}$$

The code used to generate this is:

```
clc; clear; close all;
syms s R1 R2 C L
A = [-1/(R1*C) 0; 0 -R2/L];
B = [1/(R1*C); 1/L];
c = [0 R2];
D = 0;

% Transfer function
G = c*inv(s*eye(2)-A)*B + D
```