

# 1 State Models

Consider a system with

1.  $n$ , the order of the ODE
2.  $m$ , the number of inputs
3.  $p$ , the number of outputs

General procedure:

1. Create  $n$  state variables
2. Create  $n$  first order ODEs
3. Write  $\dot{x} = f(x, u)$
4. Write  $y = h(x, u)$

If the system is linear, then the state model is

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du\end{aligned}$$

# 2 Numerical Simulation with MATLAB

Example, some random order 4 system with  $x_1(0) = 1$ ,  $x_2(0) = 2$ ,  $u_1 = \cos(t)$ ,  $u_2 = \sin(t)$ .

```
function x_dot = f(t, x)
x_dot = [
    x(3);
    x(4);
    -10*x(1) + 10*x(2) + cos(t)
    10*x(1) - 10*x(2) - sin(t)
];
end

[t, x] = ode45(@f, [0, 10], [1, 2, 0, 0]);
```

# 3 Linearization

Select a point  $(x_0, u_0)$  and to be the equilibrium point. That is,

$$f(x_0, u_0) = 0$$

From the set of equilibrium points, choose the appropriate one to linearize about. Then,

$$A = \left. \frac{\partial f}{\partial x} \right|_{x=x_0, u=u_0}$$

$$B = \left. \frac{\partial f}{\partial u} \right|_{x=x_0, u=u_0}$$

$$C = \left. \frac{\partial h}{\partial x} \right|_{x=x_0, u=u_0}$$

$$D = \left. \frac{\partial h}{\partial u} \right|_{x=x_0, u=u_0}$$

Example for inverted pendulum on a cart with equilibrium point  $x_0 = (x_1 0, 0, 0, 0)$  and  $u_0 = 0$ .

```
% Declare symbolic variables
```

```
syms x1 x2 x3 x4 u
```

```
% Define the system
```

```
f = [
    x3;
    x4;
    (4*x4^2*sin(x2) - 3*cos(x2)*sin(x2) + 4*u)/(4 - 3*cos(x2)^2);
    (-3*x4^2*sin(x2)*cos(x2) + 3*sin(x2) - 3*u*cos(x2))/(4 - 3*cos(x2)^2);
];
h = [x1; x2];
```

```
% Compute the Jacobian
```

```
dfdx = jacobian(f, [x1, x2, x3, x4])
dfdu = jacobian(f, u)
dhdx = jacobian(h, [x1, x2, x3, x4])
dhdu = jacobian(h, u)
```

```
A = subs(dfdx, [x1, x2, x3, x4, u], [x10, 0, 0, 0, 0])
B = subs(dfdu, [x1, x2, x3, x4, u], [x10, 0, 0, 0, 0])
C = subs(dhdx, [x1, x2, x3, x4, u], [x10, 0, 0, 0, 0])
D = subs(dhdu, [x1, x2, x3, x4, u], [x10, 0, 0, 0, 0])
```

## 4 Solutions to Linear Systems

Split the system into two parts: the zero-input response and the zero-state response.

$$x(t) = x_{z-i}(t) + x_{z-s}(t)$$

$$y(t) = Cx_{z-i}(t) + Cx_{z-s}(t) + Du(t)$$

## 4.1 Zero-Input Response

The zero-input problem is:

$$\begin{aligned}\dot{x} &= Ax \\ y &= Cx \\ x(0) &= x_0 \\ u &= 0\end{aligned}$$

The solution is

$$\begin{aligned}x_{z-i}(t) &= e^{At}x_0 \\ y_{z-i}(t) &= Ce^{At}x_0\end{aligned}$$

## 4.2 Matrix exponential properties

$$\begin{aligned}e^{At} &= \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} \\ e^{At}|_{t=0} &= I \\ e^{At_1}e^{At_2} &= e^{A(t_1+t_2)} \\ e^{A_1t}e^{A_2t} &= e^{(A_1+A_2)t} \iff A_1A_2 = A_2A_1 \\ (e^{At})^{-1} &= e^{-At} \\ Ae^{At} &= e^{At}A \\ \frac{d}{dt}e^{At} &= Ae^{At} = e^{At}A \\ e^{At} &= Ve^{Dt}V^{-1} = \mathcal{L}^{-1}\{sI - A\}^{-1}\end{aligned}$$

## 4.3 Zero-State Response

The zero-state problem is:

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du \\ x(0) &= 0 \\ u &= u(t)\end{aligned}$$

By integrating factor method, the solution is:

$$\begin{aligned}x_{z-s}(t) &= \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \\ y_{z-s}(t) &= \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)\end{aligned}$$

## 5 Laplace Transform Method

### 5.1 Laplace Transform and Properties

$$\begin{aligned}
 \mathcal{L}f(t) &= F(s) = \int_0^\infty f(t)e^{-st}dt \\
 \mathcal{L}\dot{f}(t) &= sF(s) - f(0) \\
 \mathcal{L}\ddot{f}(t) &= s^2F(s) - sf(0) - \dot{f}(0) \\
 \mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} &= \frac{F(s)}{s} \\
 \mathcal{L}f(t - t_d) &= e^{-st_d}F(s)
 \end{aligned}$$

### 5.2 Poles and Convergence

- In general, the right most pole determines the region of convergence.
- Use the analogy that the real part of poles correspond to the exponential decay rate of the system and the imaginary part corresponds to the oscillation frequency.
- Repeated poles is correspond to  $te^{-at}$  or  $t\sin(at)$ .
- idk i might add fvt later

### 5.3 Transfer Function

The transfer function is defined from:

$$\begin{aligned}
 Y(s) &= \underbrace{C(sI - A)^{-1}x_0}_{Y_{z-i}(s)} + \overbrace{[C(sI - A)^{-1}B + D]U(s)}^{G(s)} \\
 &\quad \underbrace{\hspace{10em}}_{Y_{z-s}(s)} \\
 G(s) &= C(sI - A)^{-1}B + D
 \end{aligned}$$