

Question 1

Using the Inverse Laplace Transform method, compute the matrix exponential e^{At} for

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

Note the solution should match Assignment #2.

Solution

The equation linking the matrix exponential and the Laplace transform is

$$e^{At} = \mathcal{L}^{-1} \{ (sI - A)^{-1} \}$$

By Matlab,

ans =

```
[exp(3*t)/2 + exp(t)/2, exp(t)/2 - exp(3*t)/2]
[exp(t)/2 - exp(3*t)/2, exp(3*t)/2 + exp(t)/2]
```

Written nicely,

$$e^{At} = \begin{bmatrix} \frac{e^{3t}}{2} + \frac{e^t}{2} & \frac{e^t}{2} - \frac{e^{3t}}{2} \\ \frac{e^t}{2} - \frac{e^{3t}}{2} & \frac{e^{3t}}{2} + \frac{e^t}{2} \end{bmatrix}$$

The code used to generate the above was

```
clc; clear all; close all;
syms s
A = [2 -1; -1 2];
ilaplace(inv(s*eye(2) - A))
```

Question 2

Using the Laplace Transform approach, compute the (total) response $y(t)$ of the state-space system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad u(t) = 1_+(t)$$

$$y = \begin{bmatrix} 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Note the solution should match Assignment #3.

Solution

Observe that the system is already in state-space form. We can immediately utilize the Laplace Transform response equation:

$$Y(s) = C(sI - A)^{-1}x_0 + C(sI - A)^{-1}BU(s) + DU(s)$$

Computing the Laplace Transform of the input $u(t)$,

$$\begin{aligned} U(s) &= \mathcal{L}\{1_+(t)\} \\ &= \frac{1}{s} \end{aligned}$$

By Matlab,

Y =

$$2\exp(3t) + 3\exp(t) - 2$$

Written nicely,

$$y(t) = 2e^{3t} + 3e^t - 2$$

The code used to generate the above is

```
clc; clear all; close all;
syms s
A = [2 -1; -1 2];
B = [2; 0];
C = [0 3];
D = 0;
x0 = [-1; 1];

Y = ilaplace(C*inv(s*eye(2) - A)*x0 + C*inv(s*eye(2) - A)*B*1/s + D
*1/s)
```

Question 3

Recall the analog circuit from Assignment #1

The input is the voltage u and the response (output) is the voltage y . Find the transfer function of this system

- Using the formula $G(s) = C(sI - A)^{-1}B + D$
- By applying the Laplace transform to the governing ODEs

Solution

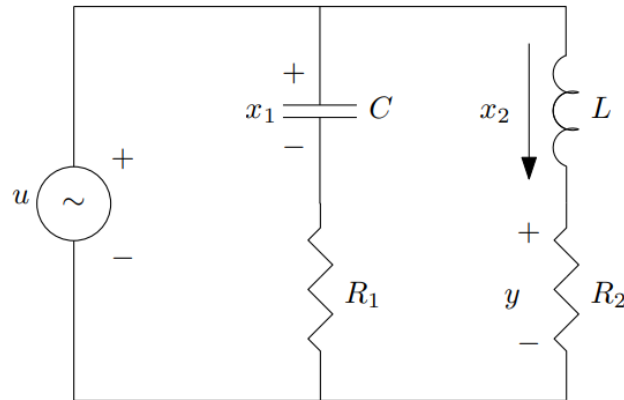


Figure 1: Analog circuit from Assignment #1

(a)

From Assignment #1, we have the state-space form

$$A = \begin{bmatrix} -1/(R_1 C) & 0 \\ 0 & -R_2/L \end{bmatrix}, \quad B = \begin{bmatrix} 1/(R_1 C) \\ 1/L \end{bmatrix}$$

$$C = [0, R_2], \quad D = 0$$

By Matlab,

G =

$R_2/(R_2 + Ls)$

Written nicely,

$$G(s) = \frac{R_2}{R_2 + Ls}$$

This was given by the Matlab code:

```
clc; clear; close all;
syms s R1 R2 C L
```

```
A = [-1/(R1*C), 0; 0, -R2/L];
B = [1/(R1*C); 1/L];
C = [0, R2];
D = 0;
```

```
G = C*inv(s*eye(2) - A)*B + D
```

(b)

From the solution of Assignment #1, we have the governing ODEs:

$$\begin{aligned}\dot{x}_2 &= -\frac{x_2}{L} + \frac{1}{L}u \\ y &= R_2 x_2 \\ \dot{y} &= R_2 \dot{x}_2\end{aligned}$$

Writing in terms of y ,

$$\begin{aligned}\frac{\dot{y}}{R_2} &= -\frac{y}{L} + \frac{1}{L}u \\ \dot{y} &= -\frac{R_2}{L}y + \frac{R_2}{L}u\end{aligned}$$

Applying the Laplace transform to both sides,

$$+ y(0) \frac{R_2}{s + \frac{R_2}{L}}$$

By the definition of the transfer function,

$$G(s) = \frac{R_2}{R_2 + Ls}$$