Question 1

Consider the state-space form:

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} 2-1 \\ -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

With the the initial conditions:

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$
$$u(t) = 1_+(t)$$

In Assignment #2, you found that

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \implies e^{At} = \frac{1}{2} \begin{bmatrix} e^t + e^{3t} & e^t - e^{3t} \\ e^t - e^{3t} & e^t + e^{3t} \end{bmatrix}$$

(a) Find the zero-input trajectory $x_{z-i}(t)$ and response $y_{z-i}(t)$ using hand calculations

(b) Find the zero-state trajectory $x_{z-s}(t)$ and response $y_{z-s}(t)$ using hand calculations

(c) Find the trajectory $x(t)$ and response $y(t)$ of the system using hand calculations

(d) Redo (a)-(c) using only MATLAB's symbolic toolbox, and ensure your solutions match. Provide the input commands that you used.

(a)

The zero-input trajectory problem is given by:

$$\dot{x} = Ax$$
$$x(0) = x_0$$

The solution to this matrix differential equation is similar to the scalar case. The solution is given by $x(t) = e^{At}x_0$. Bashing out the calculations gives:

$$\begin{aligned} x(t) &= e^{At}x_0 \\ &= \frac{1}{2} \begin{bmatrix} e^t + e^{3t} & e^t - e^{3t} \\ e^t - e^{3t} & e^t + e^{3t} \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -e^{3t} \\ e^{3t} \end{bmatrix} \end{aligned}$$

The zero-input response is given by:

$$\begin{aligned} y(t) &= Cx(t) \\ &= \begin{bmatrix} 0 & 3 \end{bmatrix} \begin{bmatrix} -e^{3t} \\ e^{3t} \end{bmatrix} \\ &= 3e^{3t} \end{aligned}$$

The zero-input trajectory is therefore

$$\boxed{\begin{aligned} x_{z-i}(t) &= \begin{bmatrix} -e^{3t} \\ e^{3t} \end{bmatrix} \\ y_{z-i}(t) &= 3e^{3t} \end{aligned}}$$

(b)

The zero-state trajectory problem is given by:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ x(0) &= 0 \end{aligned}$$

The solution to this matrix differential equation is similar to the scalar case where an integrating factor approach is used. The solution is given by $x(t) = \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau$.

First step to evaluating this *garbage* is to evaluate the integrand:

$$e^{A(t-\tau)} Bu(\tau) = \frac{1}{2} \begin{bmatrix} e^{t-\tau} + e^{3(t-\tau)} & e^{t-\tau} - e^{3(t-\tau)} \\ e^{t-\tau} - e^{3(t-\tau)} & e^{t-\tau} + e^{3(t-\tau)} \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} u(\tau)$$

Conveniently, $u(\tau) = 1_+(\tau)$, so the integrand becomes:

$$\begin{aligned} e^{A(t-\tau)} Bu(\tau) &= \frac{1}{2} \begin{bmatrix} e^{t-\tau} + e^{3(t-\tau)} & e^{t-\tau} - e^{3(t-\tau)} \\ e^{t-\tau} - e^{3(t-\tau)} & e^{t-\tau} + e^{3(t-\tau)} \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} e^{t-\tau} + e^{3(t-\tau)} \\ e^{t-\tau} - e^{3(t-\tau)} \end{bmatrix} \end{aligned}$$

Taking the integral term-by-term gives:

$$\begin{aligned}
\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau &= \begin{bmatrix} \int_0^t e^{t-\tau} + e^{3(t-\tau)}d\tau \\ \int_0^t e^{t-\tau} - e^{3(t-\tau)}d\tau \end{bmatrix} \\
&= \begin{bmatrix} \int_0^t e^{t-\tau}d\tau + \int_0^t e^{3(t-\tau)}d\tau \\ \int_0^t e^{t-\tau}d\tau - \int_0^t e^{3(t-\tau)}d\tau \end{bmatrix} \\
1.51.5 \qquad &= -\begin{bmatrix} e^{t-\tau}\Big|_0^t + \frac{1}{3}e^{3(t-\tau)}\Big|_0^t \\ e^{t-\tau}\Big|_0^t - \frac{1}{3}e^{3(t-\tau)}\Big|_0^t \end{bmatrix} \\
&= -\begin{bmatrix} (1-e^t) + \left(\frac{1}{3} - \frac{1}{3}e^{3t}\right) \\ (1-e^t) - \left(\frac{1}{3} - \frac{1}{3}e^{3t}\right) \end{bmatrix} \\
&= \begin{bmatrix} e^t + \frac{e^{3t}}{3} - \frac{4}{3} \\ e^t - \frac{e^{3t}}{3} - \frac{2}{3} \end{bmatrix}
\end{aligned}$$

The zero-state response is given by $y_{z-s} = Cx_{z-s} + Du$. Since $D = 0$, the output is simply:

$$\begin{aligned}
y_{z-s}(t) &= Cx_{z-s}(t) \\
&= \begin{bmatrix} 0 & 3 \end{bmatrix} \begin{bmatrix} e^t + \frac{e^{3t}}{3} - \frac{4}{3} \\ e^t - \frac{e^{3t}}{3} - \frac{2}{3} \end{bmatrix} \\
&= 3e^t + e^{3t} - 2
\end{aligned}$$

The zero-state trajectory is therefore

$$\boxed{\begin{aligned}
&x_{z-s}(t) = \begin{bmatrix} e^t + \frac{e^{3t}}{3} - \frac{4}{3} \\ e^t - \frac{e^{3t}}{3} - \frac{2}{3} \end{bmatrix} \\
&y_{z-s}(t) = 3e^t + e^{3t} - 2
\end{aligned}}$$

(c)

Lastly, summing the zero-input and zero-state trajectories gives the full trajectory:

$$\boxed{\begin{aligned}
x(t) &= x_{z-i}(t) + x_{z-s}(t) \\
&= \begin{bmatrix} -e^{3t} \\ e^{3t} \end{bmatrix} + \begin{bmatrix} e^t + \frac{e^{3t}}{3} - \frac{4}{3} \\ e^t - \frac{e^{3t}}{3} - \frac{2}{3} \end{bmatrix} \\
&= \begin{bmatrix} e^t - \frac{2e^{3t}}{3} - \frac{4}{3} \\ e^t + \frac{2e^{3t}}{3} - \frac{2}{3} \end{bmatrix}
\end{aligned}}$$

The full response is given by $y(t) = Cx(t) + Du(t)$. Since $D = 0$, the output is simply:

$$\boxed{\begin{aligned} y(t) &= Cx(t) \\ &= \begin{bmatrix} 0 & 3 \end{bmatrix} \begin{bmatrix} e^t - \frac{2e^{3t}}{3} - \frac{4}{3} \\ e^t + \frac{2e^{3t}}{3} - \frac{2}{3} \end{bmatrix} \\ &= 3e^t + 2e^{3t} - 2 \end{aligned}}$$

(d)

The code is given by:

```
clc; clear all; close all;

A = [2 -1; -1 2];
B = [2; 0];
C = [0 3];
D = [0];
x0 = [-1; 1];

syms t tau

% Part (a)
x_zi = expm(A*t)*x0;
y_zi = C*x_zi;

disp('Part (a)')
disp('x_zi(t) = ')
disp(x_zi)
disp('y_zi(t) = ')
disp(y_zi)

% Part (b)
x_zs = int(expm(A*(t-tau))*B, tau, 0, t);
y_zs = C*x_zs;

disp('Part (b)')
disp('x_zs(t) = ')
disp(expand(x_zs))
disp('y_zs(t) = ')
disp(expand(y_zs))

% Part (c)
x = x_zi + x_zs;
y = C*x;

disp('Part (c)')
```

```
disp('x(t) = ')
disp(expand(x))
disp('y(t) = ')
disp(expand(y))
```

Question 2

Using the Laplace transform integral, show that the transform of $f(t) = \cos \Omega t$ is $F(s) = s/(s^2 + \Omega^2)$, and that the associated region of convergence (the values of s for which the integral converges) is $\Re\{s\} > 0$. Hint: start by using the identity

$$\cos \Omega t = \frac{1}{2}\left(e^{j\Omega t} + e^{-j\Omega t}\right)$$

(a)

Using the exponential form of the cosine function and the definition of the Laplace transform, we have:

$$\begin{aligned}
\mathcal{L}(\cos \Omega t) = \int_0^\infty e^{-st} \cos \Omega t dt &= \int_0^\infty e^{-st} \frac{1}{2}\left(e^{j\Omega t} + e^{-j\Omega t}\right) dt \\
&= \frac{1}{2}\int_0^\infty e^{-(s-j\Omega)t} + e^{-(s+j\Omega)t} dt \\
&= \frac{1}{2}\left(\frac{-1}{s-j\Omega}e^{-(s-j\Omega)t} + \frac{-1}{s+j\Omega}e^{-(s+j\Omega)t}\right)\Bigg|_0^\infty \\
&= \frac{1}{2}\left(\frac{-1}{s-j\Omega}(0-1) + \frac{-1}{s+j\Omega}(0-1)\right) \\
&= \frac{1}{2}\left(\frac{1}{s+j\Omega} + \frac{1}{s-j\Omega}\right) \\
&= \frac{1}{2}\left(\frac{s-j\Omega}{s^2+\Omega^2} + \frac{s+j\Omega}{s^2+\Omega^2}\right) \\
&= \frac{1}{2}\left(\frac{s-j\Omega+s+j\Omega}{s^2+\Omega^2}\right) \\
&= \frac{1}{2}\left(\frac{2s}{s^2+\Omega^2}\right) \\
&= \frac{s}{s^2+\Omega^2}
\end{aligned}$$

Question 3

Consider the Laplace Transform of $f(t)$ given by

$$F(s) = \frac{16s^2 + 23s + 13}{(s+1)^2(s+2)}$$

(a) What are the poles of $F(s)$?

(b) Using the Final Value Theorem, predict the behaviour of $f(t)$ as $t \to \infty$

(c) By hand, obtain the partial fraction expansion of $F(s)$

(d) Use MATLAB's `partfrac` to confirm your result; provide the input commands you used.

(e) Obtain $f(t)$, and confirm the prediction from (b) holds. Hint: $\lim_{t\to\infty} te^{-t} = 0$

(a)

The poles of $F(s)$ are $s = -1$ and $s = -2$. The pole at $s = -1$ has a multiplicity of 2.

(b)

Using the Final Value Theorem, we can see the contribution of each pole.

1. $s = -1$ has a multiplicity of 2, so the contribution will be e^{-t} and te^{-t}.
2. $s = -2$ has a multiplicity of 1, so the contribution will be e^{-2t}.

Therefore, as $t \to \infty$, $f(t) \to 0$. This is statement 3 of the Final Value Theorem in the course notes.

(c)

By the Heaviside Cover-up Method, we can obtain the partial fraction expansion of $F(s)$.

$$
\begin{aligned}
F(s) &= \frac{16s^2 + 23s + 13}{(s+1)^2(s+2)} \\
&= \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{s+2} \\
&= \frac{A(s+1)(s+2) + B(s+2) + C(s+1)^2}{(s+1)^2(s+2)} \\
&= \frac{(A+C)s^2 + (3A+2B+2C)s + (2A+B+C)}{(s+1)^2(s+2)}
\end{aligned}
$$

By cover-up,

$$
\begin{aligned}
B &= \left.\frac{16s^2 + 23s + 13}{(s+2)}\right|_{s=-1} \\
&= \frac{16(-1)^2 + 23(-1) + 13}{(-1+2)} \\
&= 6 \\
C &= \left.\frac{16s^2 + 23s + 13}{(s+1)^2}\right|_{s=-2} \\
&= \frac{16(-2)^2 + 23(-2) + 13}{(-2+1)^2} \\
&= 31 \\
A &= \left.\frac{d}{ds}\left[\frac{16s^2 + 23s + 13}{(s+2)}\right]\right|_{s=-1} \\
&= \left.\frac{(s+2)(32s+23) - (16s^2 + 23s + 13)(1)}{(s+2)^2}\right|_{s=-1} \\
&= \frac{(-1+2)(-32+23) - (16-23+13)}{(-1+2)^2} \\
&= -15
\end{aligned}
$$

Solving this system of equations, we get $A = -15$, $B = 6$, and $C = 31$.

The partial fraction expansion of $F(s)$ is therefore:

$$
\boxed{F(s) = \frac{16s^2 + 23s + 13}{(s+1)^2(s+2)} = \frac{-15}{s+1} + \frac{6}{(s+1)^2} + \frac{31}{s+2}}
$$

(d)

Using MATLAB's `partfrac` command, we get the following result.

```
>> syms s
>> partfrac(16*s^2 + 23*s + 13, (s + 1)^2*(s + 2))

ans =

    6/(s + 1)^2 - 15/(s + 1) + 31/(s + 2)
```

(e)

Using the inverse Laplace Transform, we get the following result.

$$\begin{aligned}
f(t) &= \mathcal{L}^{-1}\left\{\frac{16s^2 + 23s + 13}{(s+1)^2(s+2)}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{-15}{s+1} + \frac{6}{(s+1)^2} + \frac{31}{s+2}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{-15}{s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{6}{(s+1)^2}\right\} + \mathcal{L}^{-1}\left\{\frac{31}{s+2}\right\}
\end{aligned}$$

From the table of Laplace Transforms, the following Laplace relations are of use:

$$\begin{aligned}
\mathcal{L}\{te^{at}\} &= \frac{1}{(s-a)^2} \\
\mathcal{L}\{e^{at}\} &= \frac{1}{s-a}
\end{aligned}$$

Therefore,

$$\boxed{f(t) = -15e^{-t} + 6te^{-t} + 31e^{-2t}}$$

Again this can be verified by Matlab using the following commands.

```
>> syms t
>> ilaplace(partfrac(16*s^2 + 23*s + 13, (s + 1)^2*(s + 2)))

ans =
    6/(s + 1)^2 - 15/(s + 1) + 31/(s + 2)
```

Taking the limit as $t \to \infty$, we get the following result.

$$\boxed{\begin{aligned}
\lim_{t\to\infty} f(t) &= \lim_{t\to\infty}\left(-15e^{-t} + 6te^{-t} + 31e^{-2t}\right) \\
&= 0 + 0 + 0 \\
&= 0
\end{aligned}}$$