1 State Models

Consider a system with

1. n, the order of the ODE
2. m, the number of inputs
3. p, the number of outputs

General procedure:

1. Create n state variables
2. Create n first order ODEs
3. Write $\dot{x} = f(x, u)$
4. Write $y = h(x, u)$

If the system is linear, then the state model is

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

2 Numerical Simulation with MATLAB

Example, some random order 4 system with $x_1(0) = 1$, $x_2(0) = 2$, $u_1 = \cos(t)$, $u_2 = \sin(t)$.

```
    function x_dot = f(t, x)
    x_dot = [
        x(3);
        x(4);
        -10*x(1) + 10*x(2) + cos(t)
        10*x(1) - 10*x(2) - sin(t)
    ];
end

[t, x] = ode45(@f, [0, 10], [1, 2, 0, 0]);
```

3 Linearization

Select a point (x_0, u_0) and to be the equilibrium point. That is,

$$f(x_0, u_0) = 0$$

From the set of equilibrium points, choose the appropriate one to linearize about. Then,

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=x_0,u=u_0}$$
$$B = \left.\frac{\partial f}{\partial u}\right|_{x=x_0,u=u_0}$$
$$C = \left.\frac{\partial h}{\partial x}\right|_{x=x_0,u=u_0}$$
$$D = \left.\frac{\partial h}{\partial u}\right|_{x=x_0,u=u_0}$$

Example for inverted pendulum on a cart with equilibrium point $x_0 = (x_10, 0, 0, 0)$ and $u_0 = 0$.

```
% Declare symbolic variables
syms x1 x2 x3 x4 u

% Define the system
f = [
    x3;
    x4;
    (4*x4^2*sin(x2) - 3*cos(x2)*sin(x2) + 4*u)/(4 - 3*cos(x2)^2);
    (-3*x4^2*sin(x2)*cos(x2) + 3*sin(x2) -3*u*cos(x2))/(4 - 3*cos(x2
   )^2);
];
h = [x1; x2];

% Compute the Jacobian
dfdx = jacobian(f, [x1, x2, x3, x4])
dfdu = jacobian(f, u)
dhdx = jacobian(h, [x1, x2, x3, x4])
dhdu = jacobian(h, u)

A = subs(dfdx, [x1, x2, x3, x4, u], [x10, 0, 0, 0, 0])
B = subs(dfdu, [x1, x2, x3, x4, u], [x10, 0, 0, 0, 0])
C = subs(dhdx, [x1, x2, x3, x4, u], [x10, 0, 0, 0, 0])
D = subs(dhdu, [x1, x2, x3, x4, u], [x10, 0, 0, 0, 0])
```

4 Solutions to Linear Systems

Split the system into two parts: the zero-input response and the zero-state response.

$$x(t) = x_{\text{z-i}}(t) + x_{\text{z-s}}(t)$$
$$y(t) = Cx_{\text{z-i}}(t) + Cx_{\text{z-s}}(t) + Du(t)$$

4.1 Zero-Input Response

The zero-input problem is:

$$\begin{aligned} \dot{x} &= Ax \\ y &= Cx \\ x(0) &= x_0 \\ u &= 0 \end{aligned}$$

The solution is

$$\begin{aligned} x_{\text{z-i}}(t) &= e^{At}x_0 \\ y_{\text{z-i}}(t) &= Ce^{At}x_0 \end{aligned}$$

4.2 Matrix exponential properties

$$\begin{aligned} e^{At} &= \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} \\ e^{At}\big|_{t=0} &= I \\ e^{At_1}e^{At_2} &= e^{A(t_1+t_2)} \\ e^{A_1 t}e^{A_2 t} = e^{(A_1+A_2)t} &\iff A_1A_2 = A_2A_1 \\ (e^{At})^{-1} &= e^{-At} \\ Ae^{At} &= e^{At}A \\ \frac{d}{dt}e^{At} &= Ae^{At} = e^{At}A \\ e^{At} &= Ve^{Dt}V^{-1} = \mathcal{L}^{-1}\{sI - A\}^{-1} \end{aligned}$$

4.3 Zero-State Response

The zero-state problem is:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \\ x(0) &= 0 \\ u &= u(t) \end{aligned}$$

By integrating factor method, the solution is:

$$\begin{aligned} x_{\text{z-s}}(t) &= \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \\ y_{\text{z-s}}(t) &= \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t) \end{aligned}$$

4.4 Total Response

The total trajectory and response respectively are:

$$x(t) = \underbrace{e^{At}x_0}_{x_{\text{z-i}}(t)} + \underbrace{\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau}_{x_{\text{z-s}}(t)}$$

$$y(t) = \underbrace{Ce^{At}x_0}_{y_{\text{z-i}}(t)} + \underbrace{\int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)}_{y_{\text{z-s}}(t)}$$

5 Laplace Transform Method

5.1 Laplace Transform and Properties

$$\begin{aligned}
\mathcal{L}f(t) = F(s) &= \int_0^\infty f(t)e^{-st}dt \\
\mathcal{L}\dot{f}(t) &= sF(s) - f(0) \\
\mathcal{L}\ddot{f}(t) &= s^2F(s) - sf(0) - \dot{f}(0) \\
\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} &= \frac{F(s)}{s} \\
\mathcal{L}f(t-t_d) &= e^{-st_d}F(s)
\end{aligned}$$

5.2 Poles and Convergence

- In general, the right most pole determines the region of convergence.
- Use the analogy that the real part of poles correspond to the expontential decay rate of the system and the imaginary part corresponds to the oscillation frequency.
- Repeated poles is correspond to te^{-at} or $t\sin(at)$.
- idk i might add fvt later

5.3 Solution to State Space Model

The trajectory and response respectively are:

$$x(t) = \underbrace{\mathcal{L}^{-1}\{(sI-A)^{-1}x_0\}}_{x_{\text{z-i}}(t)} + \underbrace{\mathcal{L}^{-1}\{(sI-A)^{-1}BU(s)\}}_{x_{\text{z-s}}(t)}$$

$$y(t) = \underbrace{\mathcal{L}^{-1}\{C(sI-A)^{-1}x_0\}}_{y_{\text{z-i}}(t)} + \underbrace{\mathcal{L}^{-1}\{[C(sI-A)^{-1}B+D]U(s)\}}_{y_{\text{z-s}}(t)}$$

5.4 Transfer Function

The transfer function is defined from:

$$\begin{aligned}
Y(s) &= \underbrace{C(sI-A)^{-1}x_0}_{Y_{\text{z-i}}(s)} + \underbrace{\overbrace{[C(sI-A)^{-1}B+D]}^{G(s)}U(s)}_{Y_{\text{z-s}}(s)} \\
G(s) &= C(sI-A)^{-1}B+D
\end{aligned}$$

6 Step Response

6.1 First Order System

The standard first-order system is described by the following transfer function:

$$G(s) = \frac{K}{\tau s + 1}$$

For a step response $u = u_0(t) \implies U = 1/s$, the output is:

$$\begin{aligned}
Y(s) &= \frac{K}{\tau s+1}\frac{1}{s} \\
y(t) &= K(1-e^{-t/\tau})
\end{aligned}$$

To find the time constant τ and the gain K, we can use some properties:

$$\begin{aligned}
\lim_{t\to\infty} y(t) = K(1-e^{-t/\tau}) &= K \\
y(\tau) = K(1-e^{-1}) = 0.632K &= 63.2\%K
\end{aligned}$$

6.2 Second Order System

The standard second-order system is described by the following transfer function:

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

For a step response $u = \alpha u_0(t) \implies U = 1/s$, the response can be obtained by using $\omega_d = \omega_n\sqrt{1-\zeta^2}$:

$$y(t) = \alpha - \alpha e^{-\zeta\omega_n t}\left[\cos(\omega_d t) + \frac{\zeta\omega_n}{\omega_d}\sin(\omega_d t)\right]$$

To determine ζ and ω_n, define overshoot $M_p = (y_{\max} - y_\infty)/y_\infty$ and peak time $t_p \implies y(t_p) = y_{\max}$.

$$\begin{aligned}
\zeta &= \frac{(\ln(M_p))^2}{\pi^2 + (\ln(M_p))^2} \\
\omega_n &= \frac{\pi}{t_p\sqrt{1-\zeta^2}}
\end{aligned}$$

7 Impulse Response

Recall the Dirac delta function $\delta(t)$ is defined as:

$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$
$$\delta(t) = 0 \quad \forall t \neq 0$$
$$\int_{-\infty}^{\infty} f(t)\delta(t-t_0)dt = f(t_0)$$

The zero-state response to a impulse input $u = \delta(t)$ is:

$$g(t) = \int_0^t Ce^{A(t-\tau)}B\delta(\tau)d\tau + D\delta(t)$$
$$= Ce^{At}B + D\delta(t)$$

The Laplace transform of the impulse response is:

$$G(s) = C(sI - A)^{-1}B + D$$

This is no coincidence. The transfer function is the Laplace transform of the impulse response.

By convolution,

$$\begin{aligned} y_{\text{z-s}}(t) &= (g * u)(t) \\ &= \int_0^t g(t-\tau)u(\tau)d\tau \\ &= \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t) \end{aligned}$$

Which is exactly the zero-state response obtained using the time-domain integrating factor method.

8 Realization

to do later cause lazy

9 Stability

9.1 Internal Stability

Internal stability is concerned with unforced ($u = 0$) systems. This corresponds to the trajectory $x_{\text{z-i}}(t)$ and the response $y_{\text{z-i}}(t)$.

For $x(0)$, the system is internally **stable** about an equilibrium point x_e if and only if for every $\epsilon > 0$, there exists a $\delta > 0$ such that:

$$\|x(0) - x_e\| < \delta \implies \|x(t) - x_e\| < \epsilon, \quad \forall t \geq 0$$

A system is said to be **asymptotically stable** if it is stable and $\lim_{t\to\infty} x(t) = x_e$.

For linear systems, the system is internally stable if and only if every eigenvalue of A has a negative real part. $\text{Re}(\lambda_i) < 0, \forall i = 1, \dots n$.

9.2 BIBO Stability

BIBO (bounded-input bounded-output) stability is concerned with zero state systems. This corresponds to the trajectory $x_{\text{z-s}}(t)$ and the response $y_{\text{z-s}}(t)$.

Assume $G(s)$ is a rational and proper transfer function. Then, the system is BIBO stable if and only if every pole of $G(s)$ has a negative real part. $\text{Re}(\text{pole}_i) < 0, \forall i = 1, \dots n$.

If the system is not stable, then there exists a bounded input $u(t)$ such that the output $y(t)$ is unbounded. Not all bounded inputs will cause an unstable BIBO system to be unbounded.

9.3 Connection between Internal and BIBO Stability

For a linear system, internal stability implies BIBO stability. However, BIBO stability does not imply internal stability. This is because the poles of $G(s)$ are a subset of the eigenvalues of A.

asymptotic stability $\implies$ BIBO stability

9.4 Closed Loop Systems

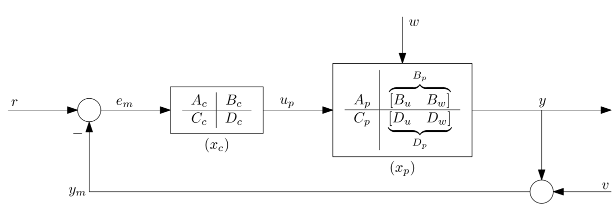

Figure 1: Closed loop system

Assume:

- Both controller (A_c, B_c, C_c, D_c) and plant (A_p, B_p, C_p, D_p) are minimal realizations.
- $D_c = 0$ or $D_p = [D_u, D_w] = 0$, such that $D_c D_u = D_u D_c = 0$ and $D_c D_w = D_w D_c = 0$.

Then, the state-space form of the closed loop system is:

$$\dot{x}_c l = \begin{bmatrix} \dot{x}_c \\ \dot{x}_p \end{bmatrix} = \underbrace{\begin{bmatrix} A_c - B_c D_u C_c & -B_c C_p \\ B_u C_c & A_p - B_u D_c C_p \end{bmatrix}}_{A_{cl}} \begin{bmatrix} x_c \\ x_p \end{bmatrix} + \underbrace{\begin{bmatrix} B_c & -B_c D_w & -B_c \\ B_u D_c & B_w & -B_u D_c \end{bmatrix}}_{B_{cl}} \begin{bmatrix} r \\ w \\ v \end{bmatrix}$$

$$y_c l = \begin{bmatrix} e_m \\ u_p \\ y \\ y_m \end{bmatrix} = \underbrace{\begin{bmatrix} -D_u C_c & -C_p \\ C_c & -D_c C_p \\ D_u C_c & C_p \\ D_u C_c & C_p \end{bmatrix}}_{C_{cl}} \begin{bmatrix} x_c \\ x_p \end{bmatrix} + \underbrace{\begin{bmatrix} 1 & -D_w & -1 \\ D_c & 0 & -D_c \\ 0 & D_w & 0 \\ 0 & D_w & 1 \end{bmatrix}}_{D_{cl}} \begin{bmatrix} r \\ w \\ v \end{bmatrix}$$

Casting to transfer functions where

$$\begin{aligned} G_c(s) &:= C_c(sI - A_c)^{-1}B_c + D_c \\ G(s) &:= C_p(sI - A_p)^{-1}B_p + D_p := \begin{bmatrix} G_p(s) & G_d(s) \end{bmatrix} \end{aligned}$$

Then the system becomes:

$$\begin{bmatrix} E_m \\ U_p \\ Y \\ Y_m \end{bmatrix} = \begin{bmatrix} \frac{1}{1+G_pG_c} & \frac{G_d}{1+G_pG_c} & \frac{-1}{1+G_pG_c} \\ \frac{G_c}{1+G_pG_c} & \frac{-G_cG_d}{1+G_pG_c} & \frac{-G_c}{1+G_pG_c} \\ \frac{G_pG_c}{1+G_pG_c} & \frac{G_d}{1+G_pG_c} & \frac{-G_pG_c}{1+G_pG_c} \\ \frac{G_pG_c}{1+G_pG_c} & \frac{G_d}{1+G_pG_c} & \frac{-1}{1+G_pG_c} \end{bmatrix} \begin{bmatrix} R \\ W \\ V \end{bmatrix}$$

9.5 Unmodeled Disturbance Input

Most of the time we just assume that disturbance is added to the input u_p. In the transfer function matrix, $G_d = G_p$.

9.6 Internal Stability of Closed Loop Systems

The closed loop system is internally stable if and only if every eigenvalue of A_{cl} has a negative real part.

9.7 I/O Stability of Closed Loop Systems

The closed loop system is I/O stable if and only if every entry of the transfer function matrix is BIBO stable. There are 5 unique transfer functions to be considered:

$$\frac{1}{1+G_pG_c}$$
$$\frac{G_c}{1+G_pG_c}$$
$$\frac{G_pG_c}{1+G_pG_c}$$
$$\frac{G_d}{1+G_pG_c}$$
$$\frac{G_cG_d}{1+G_pG_c}$$

If the disturbance is added to the input, then $G_d = G_p$ and then we only need to check:

$$\frac{1}{1+G_pG_c}$$
$$\frac{G_c}{1+G_pG_c}$$
$$\frac{G_pG_c}{1+G_pG_c}$$
$$\frac{G_p}{1+G_pG_c}$$

9.7.1 Characteristic Polynomial for Closed Loop Systems

Define the numerator and denominator of the transfer functions:

$$G_c := \frac{n_c}{d_c}, \quad G := [G_p, G_d] := [\frac{n_p}{d_p}, \frac{n_d}{d_d}]$$

The characteristic polynomial is:

$$P = n_pn_c + d_pd_c$$

By **Theorem 4.4.1**, the closed loop system is I/O stable if and only if all roots of P have a negative real part.

9.7.2 Theorem 4.4.2

The closed loop system is I/O stable if and only if

1. $1 + G_pG_c$ has only negative real part roots
2. G_pG_c has no pole-zero cancellations

9.8 Connection Between Internal and I/O Stability

9.8.1 Theorem 4.4.3

The closed loop system is internally stable if and only if it is I/O stable.

9.8.2 Routh-Hurwitz Criterion

The characteristic polynomial is

$$p(s) = n_p n_c + d_p d_c = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0, \quad a_i \in \mathbb{R}$$

Again, from **Theorem 4.4.1**, the closed loop system is I/O stable if and only if all roots of P have a negative real part. A quick test is

p(s)	Stability Criterion
$s + a_0$	$a_0 > 0$
$s^2 + a_1 s + a_0$	$(\forall i)a_i > 0$
$s^3 + a_2 s^2 + a_1 s + a_0$	$(\forall i)a_i > 0$ and $a_1 a_2 > a_0$
$s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0$	$(\forall i)a_i > 0$ and $a_2 a_3 > a_1$ and $(a_1 a_2 a_3 - a_1^2)/a_3^2 > a_0$