Question 1

Consider the state-space system

$$x = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & -0.5 & -1 \end{bmatrix} x$$

(a) Is this system internally asymptotically stable? Why or why not?

(b) Obtain the transfer function from input u to output y.

(c) Is this system BIBO stable? Why or why not?

(a)

The system is internally asymptotically stable if all eigenvalues of A have negative real parts. The eigenvalues of A are -0.5, -0.5, 1. **The system is not internally asymptotically stable.**

```
A =

     0     1     0
     0     0     1
     1     0     0

>> [V, D] = eig(A)

V =

   0.5774 + 0.0000i   0.5774 + 0.0000i  -0.5774 + 0.0000i
  -0.2887 + 0.5000i  -0.2887 - 0.5000i  -0.5774 + 0.0000i
  -0.2887 - 0.5000i  -0.2887 + 0.5000i  -0.5774 + 0.0000i

D =

  -0.5000 + 0.8660i   0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i  -0.5000 - 0.8660i   0.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i

>> real(D)

ans =
```

```
   -0.5000         0         0
         0   -0.5000         0
         0         0    1.0000
```

(b)

The transfer function is:

$$G(s) = C(sI - A)^{-1}B + D$$

Evaluating with Matlab:

```
>> A = [0 1 0; 0 0 1; 1 0 0]

A =

     0     1     0
     0     0     1
     1     0     0

>> B = [1; 0; -1]

B =

     1
     0
    -1

>> C = [1 -0.5 -1]

C =

    1.0000   -0.5000   -1.0000

>> D = 0

D =

     0

>> syms s
>> C*inv(s*eye(3)-A)*B+D

ans =

(2*s^2)/(s^3 - 1) - 3/(2*(s^3 - 1)) - s/(2*(s^3 - 1))
```

Then,

$$\boxed{G(s) = \frac{2s^2}{s^3-1} - \frac{3}{2(s^3-1)} - \frac{s}{2(s^3-1)}}$$

(c)

The system is BIBO stable if all poles of $G(s)$ have negative real parts. The poles of $G(s)$ are 1, $-0.5+0.8660i$, $-0.5-0.8660i$. **The system is not BIBO stable.**

Question 2

Check the BIBO stability of each of the following transfer functions:

(a) $\frac{s-1}{s+1}$

(b) $\frac{s+1}{s(s+2)^2}$

(c) $\frac{s}{s^2+4}$

(a)

The pole is at $s=-1$. The system is BIBO stable.

(b)

The poles are at $s=0$ and $s=-2$. The system is not BIBO stable.

(c)

The poles are at $s=0\pm 2j$. The system is not BIBO stable.

Question 3

Consider the following closed-loop system: The controller's state-space matrices are

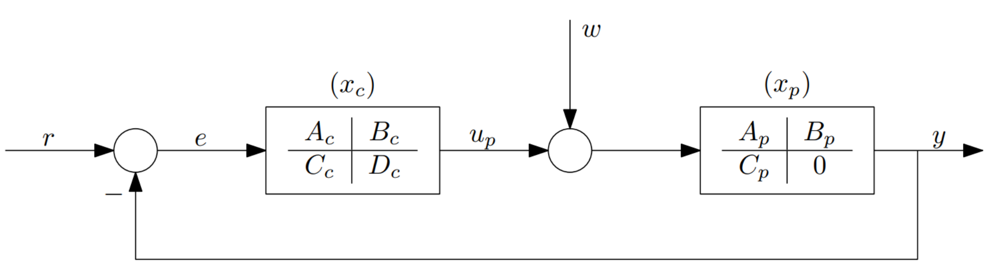

Figure 1: Block diagram of the closed-loop system

$$A_c = -4, \quad B_c = 1, \quad C_c = -5, \quad D_c = 1$$

and the plant's are

$$A_p = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \quad B_p = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C_p = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

(a) Form the closed-loop system dynamics matrix A_{cl}. Verify the internal stability of the system.

(b) Convert the state-space control and plant blocks into transfer functions $G_c(s)$ and $G_p(s)$, then form the characteristic polynomial. Verify the input-output stability of the system (note you can either compute the roots numerically, or use the Routh-Hurwitz criterion)

(c) What's the relationship between the results in (a) and (b)?

(a)

To be able to use the developments in the notes, two assumptions have to be satisfied:

- Both controller (A_c, B_c, C_c, D_c) and plant (A_p, B_p, C_p, D_p) are minimal realizations.
- $D_c = 0$ or $D_p = [D_u, D_w] = 0$, such that $D_c D_u = D_u D_c = 0$ and $D_c D_w = D_w D_c = 0$. This is satisfied in this case, since $D_p = 0$.

Then, the state-space form of the closed loop system is calculated by Matlab:

```
>> Gc

Gc =

1 - 5/(s + 4)

>> Gp

Gp =

(s + 3)/(s^2 + 3*s + 2)
```

Both $G_c(s)$ and $G_p(s)$ are minimal realizations, so the derivations in the notes can be used.

$$A_c l = \begin{bmatrix} A_c & -B_c C_p \\ B_u C_c & A_p - B_u D_c C_p \end{bmatrix}$$

Employing Matlab again,

```
Acl =

    -4    -1     0
    -5    -1     1
     0    -2    -3


V =

   -0.5774    0.2145    0.1345
   -0.5774   -0.7762   -0.1859
   -0.5774    0.5929    0.9733


D =

   -5.0000         0         0
         0   -0.3820         0
         0         0   -2.6180

Real part of eigenvalues of Acl:

ans =

   -5.0000         0         0
         0   -0.3820         0
         0         0   -2.6180
```

Since all the eigenvalues of A_{cl} have negative real parts, the closed-loop system is internally stable.

All Matlab calculations can be found in the script:

```
clc; clear; close all;
Ac = -4;
Bc = 1;
Cc = -5;
Dc = 1;

Ap = [0 1; -2 -3];
Bp = [1; 0];
Cp = [1 0];
Dp = 0;

syms s
Gc = Cc*inv(s*eye(1)-Ac)*Bc + Dc
Gp = Cp*inv(s*eye(2)-Ap)*Bp + Dp
```

```
Acl = [Ac -Bc*Cp; Bp*Cc Ap-Bp*Dc*Cp]
[V, D] = eig(Acl)
disp('Real part of eigenvalues of Acl: ')
real(D)
```

(b)

The transfer functions were obtained earlier in part (a). They are

$$G_c(s) = 1 - \frac{5}{s+4} = \frac{s-1}{s+4}$$
$$G_p(s) = \frac{s+3}{s^2+3s+2}$$

The characteristic polynomial is

$$P = n_p n_c + d_p d_c = (s+3)(s-1) + (s^2+3s+2)(s+4)$$

By Matlab,

```
>> syms s
>> (s+3)*(s-1) + (s^2  + 3*s + 2)*(s + 4)

ans =

(s - 1)*(s + 3) + (s + 4)*(s^2 + 3*s + 2)

>> expand(ans)

ans =

s^3 + 8*s^2 + 16*s + 5

>> roots([1 8 16 5])

ans =

   -5.0000
   -2.6180
   -0.3820
```

The roots of P are -5, -2.618, and -0.382. **Since all the roots have negative real parts, the closed-loop system is I/O stable.**

(c)

Theorem 4.4.3 states that the closed loop is internally stable iff it is I/O stable.

That means the results from (a) and (b) are stating the same thing, that the closed-loop system is stable.

Also the eigenvalues of A_{cl} are the roots of the characteristic polynomial P.

Question 4

Consider the following generic closed-loop system:

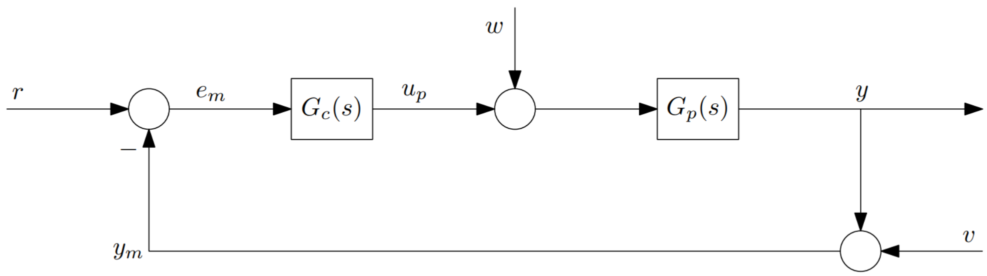

Figure 2: Block diagram of the closed-loop system

For each of the following cases, verify the stability of the system:

(a) $G_c(s) = 1$, $G_p(s) = \frac{1}{s+1}$

(b) $G_c(s) = 5$, $G_p(s) = \frac{1}{s^2+1}$

(c) $G_c(s) = 1 + \frac{1}{s}$, $G_p(s) = \frac{s}{(s+1)(s+2)}$

(d) $G_c(s) = \frac{s-1}{s+1}$, $G_p(s) = \frac{1}{s-2}$

(a)

By direct computation,

$$1 + G_c(s)G_p(s) = 1 + \frac{1}{s+1} = \frac{s+2}{s+1}$$

The zero is at $s = -2$. Since there are no term cancellations, the system is **stable.**

(b)

By direct computation,

$$1 + G_c(s)G_p(s) = 1 + \frac{5}{s^2+1} = \frac{s^2+6}{s^2+1}$$

The zeros are at $s = 0 \pm i\sqrt{6}$. Since the real part is 0, the system is **unstable.**

(c)

By direct computation,

$$1 + G_c(s)G_p(s) = 1 + \frac{\cancel{s}\cancel{(s+1)}}{\cancel{s}\cancel{(s+1)}(s+2)}$$

Since there are term cancellations in $G_c(s)G_p(s)$, the system is **unstable.**

(d)

By direct computation,

$$1 + G_c(s)G_p(s) = 1 + \frac{s-1}{(s+1)(s-2)} = \frac{s^2-3}{(s+1)(s-2)}$$

```
>> syms s

>> simplifyFraction(1 + (s-1)/((s+1)*(s-2)))

ans =

(s^2 - 3)/((s + 1)*(s - 2))
```

The zeros are at $s = \pm\sqrt{3}$. Since the real part is positive, the system is **unstable.**