1 State Models

Consider a system with

1. n, the order of the ODE
2. m, the number of inputs
3. p, the number of outputs

General procedure:

1. Create n state variables
2. Create n first order ODEs
3. Write $\dot{x} = f(x, u)$
4. Write $y = h(x, u)$

If the system is linear, then the state model is

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

2 Numerical Simulation with MATLAB

Example, some random order 4 system with $x_1(0) = 1$, $x_2(0) = 2$, $u_1 = \cos(t)$, $u_2 = \sin(t)$.

```
    function x_dot = f(t, x)
    x_dot = [
        x(3);
        x(4);
        -10*x(1) + 10*x(2) + cos(t)
        10*x(1) - 10*x(2) - sin(t)
    ];
end

[t, x] = ode45(@f, [0, 10], [1, 2, 0, 0]);
```

3 Linearization

Select a point (x_0, u_0) and to be the equilibrium point. That is,

$$f(x_0, u_0) = 0$$

From the set of equilibrium points, choose the appropriate one to linearize about. Then,

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=x_0,u=u_0}$$
$$B = \left.\frac{\partial f}{\partial u}\right|_{x=x_0,u=u_0}$$
$$C = \left.\frac{\partial h}{\partial x}\right|_{x=x_0,u=u_0}$$
$$D = \left.\frac{\partial h}{\partial u}\right|_{x=x_0,u=u_0}$$

Example for inverted pendulum on a cart with equilibrium point $x_0 = (x_1 0, 0, 0, 0)$ and $u_0 = 0$.

```
% Declare symbolic variables
syms x1 x2 x3 x4 u

% Define the system
f = [
    x3;
    x4;
    (4*x4^2*sin(x2) - 3*cos(x2)*sin(x2) + 4*u)/(4 - 3*cos(x2)^2);
    (-3*x4^2*sin(x2)*cos(x2) + 3*sin(x2) -3*u*cos(x2))/(4 - 3*cos(x2
   )^2);
];
h = [x1; x2];

% Compute the Jacobian
dfdx = jacobian(f, [x1, x2, x3, x4])
dfdu = jacobian(f, u)
dhdx = jacobian(h, [x1, x2, x3, x4])
dhdu = jacobian(h, u)

A = subs(dfdx, [x1, x2, x3, x4, u], [x10, 0, 0, 0, 0])
B = subs(dfdu, [x1, x2, x3, x4, u], [x10, 0, 0, 0, 0])
C = subs(dhdx, [x1, x2, x3, x4, u], [x10, 0, 0, 0, 0])
D = subs(dhdu, [x1, x2, x3, x4, u], [x10, 0, 0, 0, 0])
```

4 Solutions to Linear Systems

Split the system into two parts: the zero-input response and the zero-state response.

$$x(t) = x_{\text{z-i}}(t) + x_{\text{z-s}}(t)$$
$$y(t) = Cx_{\text{z-i}}(t) + Cx_{\text{z-s}}(t) + Du(t)$$

4.1 Zero-Input Response

The zero-input problem is:

$$\begin{aligned}
\dot{x} &= Ax \\
y &= Cx \\
x(0) &= x_0 \\
u &= 0
\end{aligned}$$

The solution is

$$\begin{aligned}
x_{\text{z-i}}(t) &= e^{At}x_0 \\
y_{\text{z-i}}(t) &= Ce^{At}x_0
\end{aligned}$$

4.2 Matrix exponential properties

$$\begin{gathered}
e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} \\
e^{At}|_{t=0} = I \\
e^{At_1}e^{At_2} = e^{A(t_1+t_2)} \\
e^{A_1 t}e^{A_2 t} = e^{(A_1+A_2)t} \iff A_1A_2 = A_2A_1 \\
(e^{At})^{-1} = e^{-At} \\
Ae^{At} = e^{At}A \\
\frac{d}{dt}e^{At} = Ae^{At} = e^{At}A \\
e^{At} = Ve^{Dt}V^{-1} = \mathcal{L}^{-1}\{sI-A\}^{-1}
\end{gathered}$$

4.3 Zero-State Response

The zero-state problem is:

$$\begin{aligned}
\dot{x} &= Ax + Bu \\
y &= Cx + Du \\
x(0) &= 0 \\
u &= u(t)
\end{aligned}$$

By integrating factor method, the solution is:

$$\begin{aligned}
x_{\text{z-s}}(t) &= \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \\
y_{\text{z-s}}(t) &= \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)
\end{aligned}$$

4.4 Total Response

The total trajectory and response respectively are:

$$x(t) = \underbrace{e^{At}x_0}_{x_{\text{z-i}}(t)} + \underbrace{\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau}_{x_{\text{z-s}}(t)}$$
$$y(t) = \underbrace{Ce^{At}x_0}_{y_{\text{z-i}}(t)} + \underbrace{\int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)}_{y_{\text{z-s}}(t)}$$

5 Laplace Transform Method

5.1 Laplace Transform and Properties

$$\mathcal{L}f(t) = F(s) = \int_0^\infty f(t)e^{-st}dt$$
$$\mathcal{L}\dot{f}(t) = sF(s) - f(0)$$
$$\mathcal{L}\ddot{f}(t) = s^2F(s) - sf(0) - \dot{f}(0)$$
$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{F(s)}{s}$$
$$\mathcal{L}f(t - t_d) = e^{-st_d}F(s)$$

5.2 Poles and Convergence

- In general, the right most pole determines the region of convergence.
- Use the analogy that the real part of poles correspond to the expontential decay rate of the system and the imaginary part corresponds to the oscillation frequency.
- Repeated poles is correspond to te^{-at} or $t\sin(at)$.
- idk i might add fvt later

5.3 Solution to State Space Model

The trajectory and response respectively are:

$$x(t) = \underbrace{\mathcal{L}^{-1}\{(sI - A)^{-1}x_0\}}_{x_{\text{z-i}}(t)} + \underbrace{\mathcal{L}^{-1}\{(sI - A)^{-1}BU(s)\}}_{x_{\text{z-s}}(t)}$$
$$y(t) = \underbrace{\mathcal{L}^{-1}\{C(sI - A)^{-1}x_0\}}_{y_{\text{z-i}}(t)} + \underbrace{\mathcal{L}^{-1}\{[C(sI - A)^{-1}B + D]U(s)\}}_{y_{\text{z-s}}(t)}$$

5.4 Transfer Function

The transfer function is defined from:

$$
\begin{aligned}
Y(s) &= \underbrace{C(sI - A)^{-1}x_0}_{Y_{\text{z-i}}(s)} + \underbrace{\overbrace{\left[C(sI - A)^{-1}B + D\right]}^{G(s)} U(s)}_{Y_{\text{z-s}}(s)} \\
G(s) &= C(sI - A)^{-1}B + D
\end{aligned}
$$