Question 1

Consider a pendulum consisting of a mass M on a thin (nearly massless) rod with length L. An input torque τ is applied at the pivot point of the pendulum. The angle θ measures the deflection of the pendulum away from the vertical.

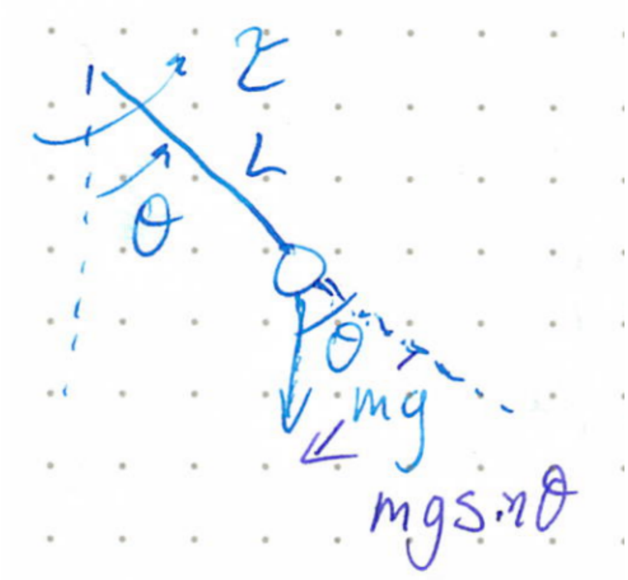

Figure 1: Pendulum free-body diagram.

(a)

Starting from a free-body diagram, obtain the equation of motion of the system. (Note: $J = ML^2$ is the mass moment of inertia of the pendulum about its pivot point).

Taking the moment at the pivot point of the pendulum, we have

$$
\begin{aligned}
\circlearrowleft \sum M &= J\ddot{\theta} = \tau - MgL\sin\theta \\
\implies ML^2\ddot{\theta} &= \tau - MgL\sin\theta \\
\ddot{\theta} &= \boxed{\frac{\tau}{ML^2} - \frac{g}{L}\sin\theta}
\end{aligned}
$$

(b)

Put the ODE into the state dynamics form $\dot{x} = f(x, u)$.

Let $x = [\theta, \dot{\theta}]$, then

$$
\boxed{\dot{x} = \begin{bmatrix} x_2 \\ \frac{\tau}{ML^2} - \frac{g}{L}\sin x_1 \end{bmatrix}}
$$

(c)

Taking the parameter values $M = 0.2$ kg, $L = 0.5$ m, $g = 9.81$ m/s^2 and input $\tau(t) = 0.1 \sin t$, obtain a plot of the response $\theta(t)$ for $0 \le t \le 10$ given initial conditions $\theta(0) = \frac{\pi}{6}$ rad, $\dot{\theta}(0) = -1$ rad/s. Submit your plot plus the .m function you used.

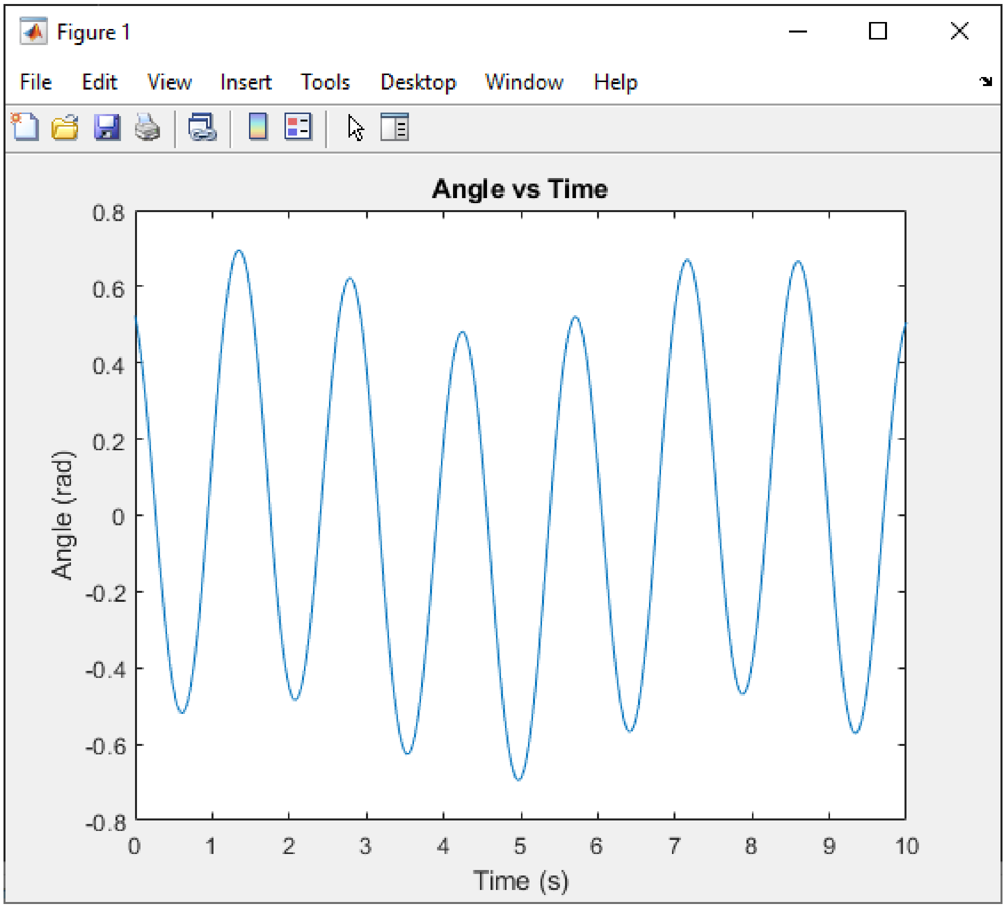

Figure 2: Plot of $\theta(t)$ for $0 \le t \le 10$ given initial conditions

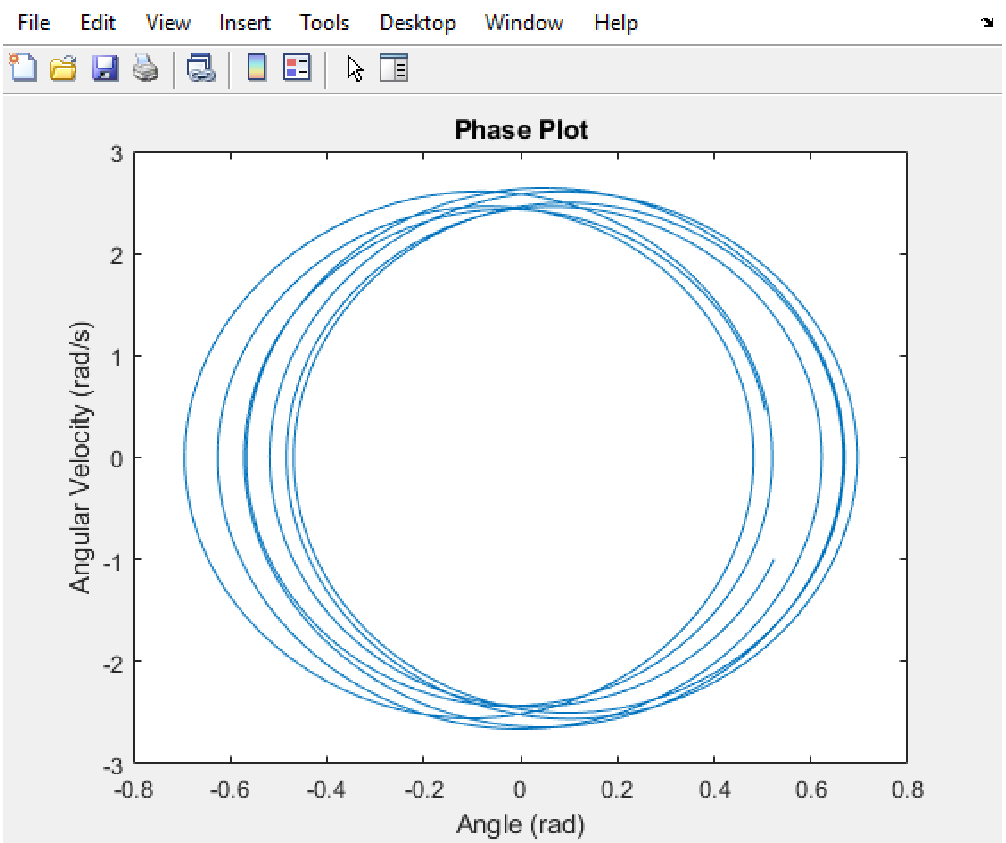

Figure 3: Phase portrait of $\theta(t)$ for $0 \leq t \leq 10$ given initial conditions

The Matlab code used to generate the plots is:

```
clc; clear all
% define initial conditions
x0 = [pi/6, -1];
t = linspace(0, 10, 1000);

[t, x] = ode45(@pendulum, t, x0);

% plot results
figure(1)
plot(t, x(:, 1))
xlabel('Time (s)')
ylabel('Angle (rad)')
title('Angle vs Time')

% phase plot
figure(2)
plot(x(:, 1), x(:, 2))
xlabel('Angle (rad)')
ylabel('Angular Velocity (rad/s)')
```

```
title('Phase Plot')

% define ode function
function x_dot = pendulum(t, x)
    % define constants
    g = 9.81; % m/s^2
    M = 0.2; % kg
    u = @(t) 0.1*sin(t); % N
    L = 0.5; % m

    % define state dynamics
    x_dot = [
        x(2);
        (u(t) - M*g*L*sin(x(1)))/(M*L^2)
    ];
end
```

Question 2

Recall the rolling-cylinder-on-a-board system from Assignment #1:

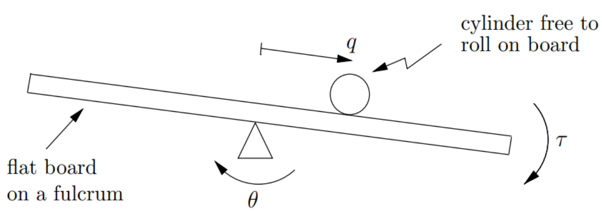

Figure 4: Rolling cylinder on a board.

In Assignment #1, defining the state vector $x = [x_1, x_2, x_3, x_4] = [q, \theta, \dot{q}, \dot{\theta}]$, input $u = \tau$ and output $y = q$, you obtained the state model

$$\dot{x} = \begin{bmatrix} x_3 \\ x_4 \\ \frac{-M_c g \sin x_2 + M_c x_1 x_4^2}{\frac{J_c}{R_c^2} + M_c} \\ \frac{-2M_c x_1 x_3 x_4 - M_c g x_1 \cos x_2 + u}{M_c x_1^2 + J + J_c} \end{bmatrix} = f(x, u)$$

$$y = x_1$$

(a)

Find the two equilibrium points of this system

An equilibrium point is defined as a state $x_0 = [x_{10}, x_{20}, x_{30}, x_{40}]$ and input u_0 that satisfies $f(x_0, u_0) = 0$. Working through some algebra, we get the following the following observations:

$$\begin{aligned}
&x_{30} = 0 \\
&x_{40} = 0 \\
&\frac{-M_c g \sin x_{20} + \cancel{M_c x_{10} x_{40}^2}}{\frac{J_c}{R_c^2} + M_c} = 0 \\
&\frac{\cancel{-2M_c x_{10} x_{30} x_{40}} - M_c g x_{10} \cos x_{20} + u_0}{M_c x_{10}^2 + J + J_c} = 0 \\
&\dot{x_{10}} = 0
\end{aligned}$$

We see that $x_{30} = x_{40} = 0$ and $x_{10} \in \mathbb{R}$. Next, solve for x_{20}:

$$\begin{aligned}
\frac{-M_c g \sin x_{20}}{\frac{J_c}{R_c^2} + M_c} &= 0 \\
\implies \sin x_{20} &= 0 \\
x_{20} &= 0, \pi
\end{aligned}$$

Finally, solve for u_0:

$$\begin{aligned}
\frac{-M_c g x_{10} \cos x_{20} + u_0}{M_c x_{10}^2 + J + J_c} &= 0 \\
\implies u_0 &= M_c g x_{10} \cos x_{20} \\
u_0 &= \pm M_c g x_{10}
\end{aligned}$$

Therefore, the two equilibrium points are:

$$\boxed{\begin{aligned}
x_{0,1} &= [x_{10}, 0, 0, 0] \\
u_{0,1} &= M_c g x_{10} \\
x_{0,2} &= [x_{10}, \pi, 0, 0] \\
u_{0,2} &= -M_c g x_{10}
\end{aligned}}$$

(b)

One of these is mathematically valid but not physically relevant. Which one and why?'

The second equilibrium point is not physically relevant because the fulcrum limits the range of motion of the board. In addition, by the time the board reaches $\theta = \pi$, the cylinder would have rolled off the board.

(c)

Using MATLAB, linearize the system about the relevant equilibrium point and give the corresponding A, B, C and D matrices.

By Matlab,

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{M_c g}{M_c + \frac{J_c}{R_c^2}} & 0 & 0 \\ -\frac{M_c g}{M_c x_{10}^2 + J + J_c} & 0 & 0 & 0 \end{bmatrix}$$
$$B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{M_c x_{10}^2 + J + J_c} \end{bmatrix}$$
$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$
$$D = 0$$

The Matlab code used to generate the above matrices is shown below:

```
% Declare symbolic variables
syms x1 x2 x3 x4 M_c g J_c J u x_10 R_c

% State dynamics
f = [
    x3;
    x4;
    (-M_c*g*sin(x2) + M_c*x1*x4^2)/(J_c/R_c^2 + M_c);
    (-2*M_c*x1*x3*x4 - M_c*g*x1*cos(x2) + u)/(M_c*x1^2 + J + J_c);
];

% State output
h = [x1];

% Compute the Jacobian of f and h
dfdx = jacobian(f, [x1 x2 x3 x4]);
dfdu = jacobian(f, u);
dhdx = jacobian(h, [x1 x2 x3 x4]);
dhdu = jacobian(h, u);

% Substitute in the equilibrium point
A = subs(dfdx, [x1 x2 x3 x4 u], [x_10 0 0 0 M_c*g*x_10]);
B = subs(dfdu, [x1 x2 x3 x4 u], [x_10 0 0 0 M_c*g*x_10]);
C = dhdx;
D = dhdu;
```

```
% Print the result
pretty(A)
pretty(B)
pretty(C)
pretty(D)
```

Question 3

Compute the matrix exponential

$$e^{At},\ A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

(a)

By hand calculations (start by solving for the eigenvalues and eigenvectors of A)

First we solve for the eigenvalues of A:

$$\begin{aligned} \det(A - \lambda I) &= 0 \\ \det \begin{bmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{bmatrix} &= 0 \\ (2-\lambda)^2 - 1 &= 0 \\ \lambda &= 2 \pm 1 \\ \lambda &= 1, 3 \end{aligned}$$

Next, we solve for the eigenvectors of A. First for $\lambda = 1$:

$$\begin{aligned} (A - \lambda I)v &= 0 \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} v &= 0 \end{aligned}$$

Since the system is underdetermined, we can let $v_1 = \alpha$, then

$$\begin{aligned} v_2 &= v_1 \\ v &= \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}$$

Next, for $\lambda = 3$:

$$(A - \lambda I)v = 0$$
$$\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} v = 0$$

Again, since the system is underdetermined, we can let $v_1 = \beta$, then

$$v_2 = -v_1$$
$$v = \beta \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Therefore, the D and V matrices are

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}, \ V = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Next, determine V^{-1}:

$$\begin{aligned} V^{-1} &= \frac{1}{\det(V)} \mathrm{adj}(V) \\ &= \frac{1}{-2} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \end{aligned}$$

Next, evaluating e^{Dt} by the special case of diagonal matrices:

$$\begin{aligned} e^{Dt} &= \begin{bmatrix} e^{D_{11}t} & 0 \\ 0 & e^{D_{22}t} \end{bmatrix} \\ &= \begin{bmatrix} e^{t} & 0 \\ 0 & e^{3t} \end{bmatrix} \end{aligned}$$

Finally, we can solve for e^{At}:

$$
\begin{aligned}
e^{At} &= Ve^{Dt}V^{-1} \\
&= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{3t} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \\
&= \begin{bmatrix} e^t & e^{3t} \\ e^t & -e^{3t} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{e^t+e^{3t}}{2} & \frac{e^t-e^{3t}}{2} \\ \frac{e^t-e^{3t}}{2} & \frac{e^t+e^{3t}}{2} \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} e^t + e^{3t} & e^t - e^{3t} \\ e^t - e^{3t} & e^t + e^{3t} \end{bmatrix}
\end{aligned}
$$

So,

$$
\boxed{e^{At} = \frac{1}{2} \begin{bmatrix} e^t + e^{3t} & e^t - e^{3t} \\ e^t - e^{3t} & e^t + e^{3t} \end{bmatrix}}
$$

(b)

Using MATLAB

Using the Matlab code:

```
clc
clear
% Find eigenvalues and eigenvectors of the matrix
syms t
A =[2, -1; -1, 2];
expm(A*t)
```

The output is

```
>>> ans =
     [exp(3*t)/2 + exp(t)/2, exp(t)/2 - exp(3*t)/2]
     [exp(t)/2 - exp(3*t)/2, exp(3*t)/2 + exp(t)/2]
```

Which is the same as the answer from part (a).