Question 1

Air enters a normal shock at 26 kPa, 230 K, and 815 m/s. Calculate the stagnation pressure and Mach number upstream of the shock, as well as pressure, temperature, velocity, Mach number, and stagnation pressure downstream of the shock.

Solution

- The flow is steady, adiabatic, and one dimensional
- Air is an ideal gas

First, let us calculate the upstream parameters. At room temperature,

$$\begin{aligned} R &= 0.287 \text{ kJ/kg K} \\ c_p &= 1.005 \text{ kJ/kg K} \\ k &= 1.4 \end{aligned}$$

Then stagnation temperature is given by

$$\begin{aligned} T_0 &= T + \frac{V^2}{2c_p} \\ &= 230 + \frac{815^2}{2 \cdot 1.005 \cdot 1000} \\ &= 560.460199 \text{ K} \end{aligned}$$

Then stagnation pressure is given by

$$\begin{aligned} P_0 &= P\left(\frac{T_0}{T}\right)^{k/(k-1)} \\ &= 26\left(\frac{560.460199}{230}\right)^{1.4/0.4} \\ &= \boxed{587.263 \text{ kPa}} \end{aligned}$$

And Mach number is given by

$$\begin{aligned} Ma &= \frac{V}{c} \\ &= \frac{V}{\sqrt{kRT}} \\ &= \frac{815}{\sqrt{1.4 \cdot 0.287 \cdot 230 \cdot 1000}} \\ &= \boxed{2.68095} \end{aligned}$$

Using Table A14 from Cengel and Cimbala, we can interpolate to calculate the fluid properties [1].

Table 1: Table A14

Ma_1	Ma_2	P_2/P_1	T_2/T_1	P_{02}/P_{01}
2.6	0.5039	7.72	2.2383	0.499
2.7	0.4956	8.3383	2.3429	0.4601
2.68095	0.497181145	8.220514236	2.322973765	0.467510426

Where, the last row was calculated using linear interpolation:

$$y = \frac{y_i + (y_{i+1} - y_i) \cdot (x - x_i)}{x_{i+1}}$$

Then downstream of the shock, we have

$$\begin{aligned} P_2 &= 26 \cdot 8.2205 \\ &= \boxed{213.7 \text{ kPa}} \\ T_2 &= 230 \cdot 2.3229 \\ &= \boxed{534.3 \text{ K}} \\ P_{02} &= 587.263 \cdot 0.4675 \\ &= \boxed{274.6 \text{ kPa}} \end{aligned}$$

For velocity, since $\mathrm{Ma}_2 = 0.4972$, we have

$$\begin{aligned} \mathrm{Ma}_2 &= \frac{V_2}{c_2} \\ \implies V_2 &= \sqrt{kRT_2} \cdot \mathrm{Ma}_2 \\ &= \sqrt{1.4 \cdot 0.287 \cdot 534.3 \cdot 1000} \cdot 0.4972 \\ &= \boxed{230.4 \text{ m/s}} \end{aligned}$$

Question 2

Air flowing at 32 kPa, 240 K, and $Ma_1 = 3.6$ is forced to undergo an expansion turn of $10°$. Determine the Mach number, pressure, and temperature of air after the expansion.

Solution

Assume

- The flow is steady, adiabatic, and one dimensional
- Air is an ideal gas
- Boundary layer is very thin
- The flow is isentropic

By the thin boundary layer assumption, $\theta \approx beta \approx 10°$. Then by the Prandtl-Meyer function, we have

$$\nu(\mathrm{Ma}) = \sqrt{\frac{k+1}{k-1}}\tan^{-1}\left(\sqrt{\frac{k-1}{k+1}(\mathrm{Ma}^2-1)}\right) - \tan^{-1}\left(\sqrt{\mathrm{Ma}^2-1}\right)$$

Then,

$$\begin{aligned}\nu(\mathrm{Ma}_1) &= \sqrt{\frac{1.4+1}{1.4-1}}\tan^{-1}\left(\sqrt{\frac{1.4-1}{1.4+1}(3.6^2-1)}\right) - \tan^{-1}\left(\sqrt{3.6^2-1}\right)\\ &= 60.0914555936°\end{aligned}$$

And,

$$\begin{aligned}\theta &= \nu(\mathrm{Ma}_2) - \nu(\mathrm{Ma}_1)\\ \implies \nu(\mathrm{Ma}_2) &= \theta + \nu(\mathrm{Ma}_1)\\ &= 10 + 60.0914555936\\ &= 70.0914555936°\end{aligned}$$

Using MATLAB, (or an approved engineering calculator)

```
syms Ma2 positive
eqn = sqrt(2.4/0.4) * atan(sqrt(0.4/2.4 * (Ma2^2 - 1))) - atan(sqrt(Ma2^2 - 1))...
== deg2rad(70.0914555936);
Ma2 = vpasolve(eqn, Ma2, [1, 10]);
Ma2 = double(Ma2)

>> Ma2 =

    4.3468
```

Therefore,

$$\mathrm{Ma}_2 = \boxed{4.3468}$$

Using the isentropic relations, we have the ratios upstream as

$$\begin{aligned}
\frac{P_0}{P_1} &= \left(1 + \frac{k-1}{2}\mathrm{Ma}_1^2\right)^{k/(k-1)} \\
&= \left(1 + \frac{1.4-1}{2}(3.6)^2\right)^{1.4/0.4} \\
&= 87.8369299384 \\
\frac{T_0}{T_1} &= 1 + \left(\frac{k-1}{2}\right)\mathrm{Ma}_1^2 \\
&= 1 + \left(\frac{1.4-1}{2}\right)(3.6)^2 \\
&= 3.592
\end{aligned}$$

Then, downstream we have

$$\begin{aligned}
\frac{P_0}{P_2} &= \left(1 + \frac{k-1}{2}\mathrm{Ma}_2^2\right)^{k/(k-1)} \\
&= \left(1 + \frac{1.4-1}{2}(4.3468)^2\right)^{1.4/0.4} \\
&= 238.593511903 \\
\frac{T_0}{T_2} &= 1 + \left(\frac{k-1}{2}\right)\mathrm{Ma}_2^2 \\
&= 1 + \left(\frac{1.4-1}{2}\right)(4.3468)^2 \\
&= 4.778934048
\end{aligned}$$

Then,

$$\begin{aligned}
P_2 &= \frac{P_0/P_1}{P_0/P_2}P_1 \\
&= \frac{87.8369299384}{238.593511903} \cdot 32 \\
&= \boxed{11.78 \text{ kPa}} \\
T_2 &= \frac{T_0/T_1}{T_0/T_2}T_1 \\
&= \frac{3.592}{4.778934048} \cdot 240 \\
&= \boxed{180.39 \text{ K}}
\end{aligned}$$

Question 3

Air flowing at 40 kPa, 210 K, and a Mach number of 3.4 impinges on a two-dimensional wedge of half-angle 8°. Determine the two possible oblique shock angles, β_{weak} and β_{strong}, that could be formed by this wedge. For each case, calculate the pressure and Mach number downstream of the oblique shock.

Solution

Assume

- The flow is steady, adiabatic, and one dimensional
- Air is an ideal gas

Air at room temperature,

$$\begin{aligned} R &= 0.287 \text{ kJ/kg K} \\ c_p &= 1.005 \text{ kJ/kg K} \\ k &= 1.4 \end{aligned}$$

The θ-β-Ma relationship is given by

$$\begin{aligned} \tan\theta &= \frac{2\cot(\beta)\left(Ma^2\sin^2\beta - 1\right)}{Ma^2\left(k+\cos(2\beta)\right)+2} \\ \tan(8^\circ) &= \frac{2\cot(\beta)\left(3.4^2\sin^2\beta - 1\right)}{3.4^2\left(1.4+\cos(2\beta)\right)+2} \\ 0.140540834702 &= \frac{2\cot(\beta)\left(11.56\sin^2\beta - 1\right)}{11.56\left(1.4+\cos(2\beta)\right)+2} \end{aligned}$$

Then, using MATLAB, (or an approved engineering calculator)

```
syms beta positive
eqn = 2 * cotd(beta) * (11.56 * sind(beta)^2 - 1) / (11.56 * (1.4 + cosd(2 * beta)) + 2)
== 0.140540834702;
beta1 = vpasolve(eqn, beta, [0, 45]);
beta2 = vpasolve(eqn, beta, [45, 90]);
beta1 = double(beta1)
beta2 = double(beta2)

>> beta1 =

   23.1466


>> beta2 =

   87.4532
```

Therefore,

$$\boxed{\begin{aligned}\beta_{\text{weak}} &= 23.1466^\circ \\ \beta_{\text{strong}} &= 87.4532^\circ\end{aligned}}$$

Calculating the normal Mach number,

$$\begin{aligned}
\text{Ma}_{1,n,w} &= \text{Ma}_1 \sin(\beta_{\text{weak}}) \\
&= 3.4 \sin(23.1466) \\
&= 1.33648933705 \\
\text{Ma}_{1,n,s} &= \text{Ma}_1 \sin(\beta_{\text{strong}}) \\
&= 3.4 \sin(87.4532) \\
&= 3.39664168189
\end{aligned}$$

Then for weak shock,

$$\begin{aligned}
\frac{P_{2,w}}{P_1} &= \frac{2k\text{Ma}_{1,n,w}^2 - k + 1}{k+1} \\
\implies P_{2,w} &= 40 \cdot \frac{2 \cdot 1.4 \cdot (1.33648933705)^2 - 1.4 + 1}{1.4 + 1} \\
&= \boxed{76.69 \text{ kPa}}
\end{aligned}$$

And for strong shock,

$$\begin{aligned}
\frac{P_{2,s}}{P_1} &= \frac{2k\text{Ma}_{1,n,s}^2 - k + 1}{k+1} \\
\implies P_{2,s} &= 40 \cdot \frac{2 \cdot 1.4 \cdot (3.39664168189)^2 - 1.4 + 1}{1.4 + 1} \\
&= \boxed{531.7 \text{ kPa}}
\end{aligned}$$

The normal Mach number downstream of the weak shock is given by

$$\begin{aligned}
\text{Ma}_{2,n,w} &= \sqrt{\frac{(k-1)\text{Ma}_{1,n,w}^2 + 2}{2k\text{Ma}_{1,n,w}^2 - k + 1}} \\
&= \sqrt{\frac{(1.4-1)(1.33648933705)^2 + 2}{2 \cdot 1.4 \cdot (1.33648933705)^2 - 1.4 + 1}} \\
&= 0.768068300414
\end{aligned}$$

And for the strong shock,

$$\begin{aligned}
\text{Ma}_{2,n,s} &= \sqrt{\frac{(k-1)\text{Ma}_{1,n,s}^2 + 2}{2k\text{Ma}_{1,n,s}^2 - k + 1}} \\
&= \sqrt{\frac{(1.4-1)(3.39664168189)^2 + 2}{2 \cdot 1.4 \cdot (3.39664168189)^2 - 1.4 + 1}} \\
&= 0.455341755517
\end{aligned}$$

Therefore, the downstream Mach numbers for the weak shock is

$$\begin{aligned}
\mathrm{Ma}_{2,w} &= \frac{\mathrm{Ma}_{2,n,w}}{\sin(\beta_{\text{weak}} - \theta)} \\
&= \frac{0.768068300414}{\sin(23.1466 - 8)} \\
&= \boxed{2.940}
\end{aligned}$$

And for the strong shock,

$$\begin{aligned}
\mathrm{Ma}_{2,s} &= \frac{\mathrm{Ma}_{2,n,s}}{\sin(\beta_{\text{strong}} - \theta)} \\
&= \frac{0.455341755517}{\sin(87.4532 - 8)} \\
&= \boxed{0.4632}
\end{aligned}$$

References

[1] Y. A. Cengel and J. M. Cimbala, *Fluid mechanics: fundamentals and applications*, 4th ed. New York, NY: McGraw-Hill Education, 2018.