Question 1

Consider the Blasius solution for a laminar flat plate boundary layer. The nondimensional slope at the wall is given by equation 1 below.

$$\left.\frac{d(u/U)}{d\eta}\right|_{\eta=0} = f''(0) = 0.332 \tag{1}$$

Transform this result to physical variables and show that the following equation is correct.

$$\tau_w = 0.332 \cdot \rho U^2 \sqrt{Re_x} \tag{2}$$

Solution

At the wall, we specify that the shear stress is given by

$$\tau_w = \mu \left(\frac{du}{dy}\right)_{y=0} \tag{3}$$

From the Blasius,

$$\begin{aligned} \eta &= y\sqrt{\frac{U}{\nu x}} \\ \implies d\eta &= \sqrt{\frac{U}{\nu x}}dy \\ \eta = 0 &\implies y = 0 \end{aligned}$$

Then,

$$\begin{aligned} \frac{df'}{d\eta} &= \frac{d(u/U)}{d\eta} \\ &= \frac{d(u/U)}{dy\sqrt{\frac{U}{\nu x}}} \\ &= \frac{du}{dy}\frac{1}{U}\sqrt{\frac{\nu x}{U}} \end{aligned}$$

Then from the solution,

$$\begin{aligned} \left.\frac{d(u/U)}{d\eta}\right|_{\eta=0} &= 0.332 \\ &= \left.\frac{du}{dy}\right|_{y=0}\frac{1}{U}\sqrt{\frac{\nu x}{U}} \end{aligned}$$

Solving for $\left.\frac{du}{dy}\right|_{y=0}$,

$$\implies \left.\frac{du}{dy}\right|_{y=0} = 0.332U\sqrt{\frac{U}{\nu x}}$$

Substituting this into (3),

$$\begin{aligned}
\tau_w &= \mu \left.\frac{du}{dy}\right|_{y=0} \\
&= \mu \cdot 0.332U\sqrt{\frac{U}{\nu x}} \\
&= 0.332U\sqrt{\frac{\mu^2 U}{\frac{\mu}{\rho}x}} \\
&= 0.332U\sqrt{\frac{\mu\rho U}{x}\cdot\frac{\rho U}{\rho U}} \\
&= 0.332U\sqrt{\rho^2U^2\cdot\frac{\mu}{\rho U x}} \\
&= \boxed{0.332\frac{\rho U^2}{\sqrt{\mathrm{Re}_x}}}
\end{aligned}$$

Thus, we have shown that equation (2) is correct.

Question 2

In order to avoid boundary layer interference, engineers design a "boundary layer scoop" to skim off the boundary layer in a large wind tunnel (see Figure 1). The scoop is constructed of thin sheet metal. The air is at 20°C and flows at $V = 45.0m/s$. How high (dimension h) should the scoop be at downstream distance $x = 1.45m$?

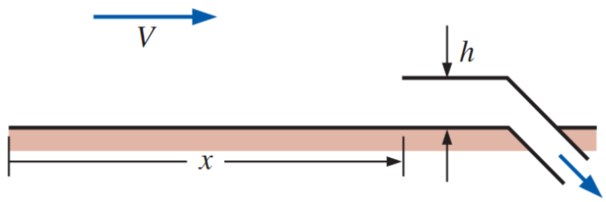

Figure 1: Boundary Layer Scoop

First, calculate the Reynolds number at $x = 1.45m$. The kinematic viscosity of air at $20°C$ is $\nu = 1.516 \times 10^{-5} m^2/s$ [1].

$$\text{Re}_x = \frac{Vx}{\nu} = \frac{45.0 \times 1.45}{1.516 \times 10^{-5}} = 4.30 \times 10^6$$

This is past the $\text{Re}_{\text{engineer}} = 5 \times 10^5$ threshold, so this is turbulent. Using the turbulent boundary layer thickness equation,

$$\begin{aligned}\frac{\delta}{x} &= \frac{0.16}{\text{Re}_x^{1/7}} \\ &= \frac{0.16}{(4.30 \times 10^6)^{1/7}} \\ &= 0.0181 \times 10^{-3}\end{aligned}$$

Then,

$$\begin{aligned}\delta &= 0.0181 \times 10^{-3} \times 1.45 \\ &= 2.62 \times 10^{-5} \\ &= \boxed{26.2 \text{ mm}}\end{aligned}$$

So the scoop should be at least 26.2 mm high at $x = 1.45$m.

Question 3

The streamwise velocity component of a steady, incompressible, laminar, flat plate boundary layer of boundary layer thickness δ is approximated by the simple linear expression, $u = Uy/\delta$ for $y < \delta$, and $u = U$ for $y > \delta$ (see Figure 2). Generate expressions for displacement thickness and momentum thickness as functions of δ, based on this linear approximation. Compare the approximate values of δ^*/δ and θ/δ to the values of δ^*/δ and θ/δ obtained from the Blasius solution.

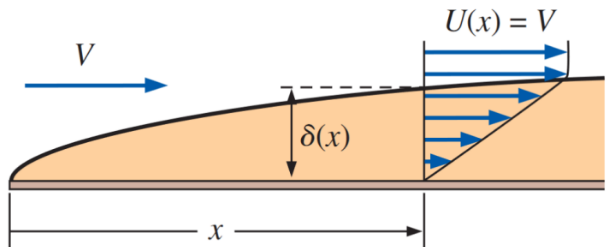

Figure 2: Flat Plate Boundary Layer

By conservation of mass,

$$\begin{aligned}
\delta^* &= \int_0^\delta \left(1 - \frac{y}{\delta}\right) dy \\
&= \left[y - \frac{y^2}{2\delta}\right]_0^\delta \\
&= \delta - \frac{\delta^2}{2\delta} \\
&= \frac{\delta}{2}
\end{aligned}$$

or more conveniently,

$$\boxed{\frac{\delta^*}{\delta} = \frac{1}{2}}$$

By conservation of momentum,

$$\begin{aligned}\theta &= \int_0^\delta \frac{u}{U}\left(1-\frac{u}{U}\right) dy \\ &= \int_0^\delta \frac{y}{\delta}\left(1-\frac{y}{\delta}\right) dy \\ &= \int_0^\delta \left(\frac{y}{\delta}-\frac{y^2}{\delta^2}\right) dy \\ &= \left[\frac{y^2}{2\delta}-\frac{y^3}{3\delta^2}\right]_0^\delta \\ &= \frac{\delta^2}{2\delta}-\frac{\delta^3}{3\delta^2} \\ &= \frac{\delta}{2}-\frac{\delta}{3} \\ &= \frac{\delta}{6}\end{aligned}$$

or more conveniently,

$$\boxed{\frac{\theta}{\delta}=\frac{1}{6}}$$

Recall from the Blasius solution for laminar boundary layers on a flat plate,

$$\begin{aligned}\frac{\delta}{x} &= \frac{4.91}{\sqrt{\mathrm{Re}_x}} \\ \frac{\delta^*}{x} &= \frac{1.72}{\sqrt{\mathrm{Re}_x}} \\ \frac{\theta}{x} &= \frac{0.664}{\sqrt{\mathrm{Re}_x}}\end{aligned}$$

Then,

$$\boxed{\begin{aligned}\frac{\delta^*}{\delta} &= \frac{1.72}{4.91} = 0.350 \\ \frac{\theta}{\delta} &= \frac{0.664}{4.91} = 0.135\end{aligned}}$$

The relative errors are then,

$$\boxed{\begin{aligned}\mathrm{Error}_{\delta^*} &= \frac{0.350-0.5}{0.350}\times 100\% = 42.9\% \\ \mathrm{Error}_{\theta} &= \frac{0.135-0.166}{0.135}\times 100\% = 23.0\%\end{aligned}}$$

The approximation is not very accurate, with high errors in both δ^* and θ.

Question 4

Helium ($k = 1.667$ and $c_P = 5.1926 kJ/(kg \cdot K)$) enters a converging-diverging nozzle at $0.7 MPa$, $800K$, and $100 m/s$. What are the lowest temperature and pressure that can be obtained at the throat of the nozzle?

Solution

Assume

- The flow is steady, adiabatic, and one dimensional
- Isentropic flow
- Helium is an ideal gas

Let us first find the stagnation temperature and pressure at the inlet. Then,

$$\begin{aligned} T_0 &= T + \frac{V^2}{2c_p} \\ &= 800 + \frac{100^2}{2(5.1926) \cdot 1000} \\ &= 800 + 0.0096 \\ &= 800.96 \text{ K} \end{aligned}$$

and,

$$\begin{aligned} P_0 &= P \left(\frac{T_0}{T}\right)^{\frac{k}{k-1}} \\ &= 0.7 \left(\frac{800.96}{800}\right)^{\frac{1.667}{0.667}} \\ &= 0.7021 \text{ MPa} \end{aligned}$$

At the throat, the lowest temperature and pressure is the critical properties, which can be found by

$$\begin{aligned} T^\star &= T_0 \left(\frac{2}{k+1}\right) \\ &= 800.96 \left(\frac{2}{1.667+1}\right) \\ &= \boxed{600.64 \text{ K}} \end{aligned}$$

and,

$$\begin{aligned} P^\star &= P_0 \left(\frac{2}{k+1}\right)^{\frac{k}{k-1}} \\ &= 0.7021 \left(\frac{2}{1.667+1}\right)^{\frac{1.667}{0.667}} \\ &= \boxed{0.3420 \text{ MPa}} \end{aligned}$$

Question 5

An aircraft is designed to cruise at Mach number Ma = 1.1 at 12,000 m where the atmospheric temperature is 236.15 K. Determine the stagnation temperature on the leading edge of the wing.

Solution

Assume

- The flow is steady, adiabatic, and one dimensional
- Isentropic flow
- Air is an ideal gas with properties $c_p = 1.005 kJ/(kg{\cdot}K)$, $k = 1.4$, and $R = 287 J/(kg{\cdot}K)$

First we need to find the speed of air. For an ideal gas,

$$\begin{aligned} c &= \sqrt{kRT} \\ \mathrm{Ma} &= \frac{V}{c} \\ \implies V &= \mathrm{Ma} \cdot c \\ &= \mathrm{Ma}\sqrt{kRT} \end{aligned}$$

Then,

$$\begin{aligned} V &= 1.1\sqrt{1.4 \cdot 287 \cdot 236.15} \\ &= 338.8376 \text{ m/s} \end{aligned}$$

Stagnation temperature is then,

$$\begin{aligned} T_0 &= T + \frac{V^2}{2c_p} \\ &= 236.15 + \frac{(338.8376)^2}{2(1.005) \cdot 1000} \\ &= \boxed{293.27 \text{ K}} \end{aligned}$$

Question 6

An ideal gas with $k = 1.4$ is flowing through a nozzle such that the Mach number is 1.6 where the flow area is $45cm^2$. Approximating the flow as isentropic, determine the flow area at the location where the Mach number is 0.8.

Solution Assume

- The flow is steady, adiabatic, and one dimensional
- Isentropic flow
- Air is an ideal gas

The relation between flow area to throat area is given by

$$\begin{aligned}\frac{A_1}{A^\star} &= \frac{1}{\mathrm{Ma}}\left[\frac{2}{k+1}\left(1+\frac{k-1}{2}\mathrm{Ma}^2\right)\right]^{(k+1)/(2(k-1))} \\ &= \frac{1}{1.6}\left[\frac{2}{1.4+1}\left(1+\frac{1.4-1}{2}(1.6)^2\right)\right]^{(1.4+1)/(2(1.4-1))} \\ &= 1.250235\end{aligned}$$

Then,

$$\begin{aligned}A^\star &= \frac{45}{1.250235} \\ &= 35.993\ \mathrm{cm}^2\end{aligned}$$

Then using the same relation, but with $\mathrm{Ma} = 0.8$,

$$\begin{aligned}\frac{A_2}{A^\star} &= \frac{1}{0.8}\left[\frac{2}{1.4+1}\left(1+\frac{1.4-1}{2}(0.8)^2\right)\right]^{(1.4+1)/(2(1.4-1))} \\ &= 1.03823\end{aligned}$$

Then,

$$\begin{aligned}A_2 &= 35.993 \times 1.03823 \\ &= \boxed{37.376\ \mathrm{cm}^2}\end{aligned}$$

Question 7

Air at $900kPa$ and $400K$ enters a converging nozzle with a negligible velocity. The throat area of the nozzle is $10cm^2$. Approximating the flow as isentropic, calculate and plot the exit pressure, the exit velocity, and the mass flow rate versus the back pressure P_b for $0.9 \geq P_b \geq 0.1MPa$. (Note: Use MATLAB to plot the results)

Solution

Assume

- The flow is steady, adiabatic, and one dimensional
- Isentropic flow
- Air is an ideal gas

First find the critical pressure,

$$\begin{aligned} P^\star &= P\left(\frac{2}{k+1}\right)^{\frac{k}{k-1}} \\ &= 900\left(\frac{2}{1.4+1}\right)^{\frac{1.4}{0.4}} \\ &= 900 \times 0.5283 \\ &= 475.4536 \text{ kPa} \end{aligned}$$

Recall that exit pressure, P_e is described as

$$P_e = \begin{cases} P_b, & P_b \geq P^\star \\ P^\star, & P_b < P^\star \end{cases}$$

so,

$$\boxed{P_e = \begin{cases} P_b, & P_b \geq 475.4536 \\ 475.4536, & P_b < 475.4536 \end{cases}}$$

Assuming air is ideal,

$$\begin{aligned} c_pT_0 &= c_pT + \frac{V^2}{2} \\ \implies V &= \sqrt{2c_p(T_0 - T)} \end{aligned}$$

Let is consider the case where $P_b < P^\star$. Then, $P_e = 475.4536$ kPa. Solving for T_e,

$$\begin{aligned} T_e &= T_0\left(\frac{P_e}{P_0}\right)^{\frac{k-1}{k}} \\ &= 400\left(\frac{475.4536}{900}\right)^{\frac{0.4}{1.4}} \\ &= 333.33 \text{ K} \end{aligned}$$

Then,

$$\begin{aligned} V &= \sqrt{2 \cdot 1.005 \cdot 10^3 \cdot (400 - 333.33)} \\ &= 366.069 \text{ m/s} \end{aligned}$$

For the case where $P_b \geq P^\star$, $P_e = P_b$. Then,

$$\begin{aligned} T_e &= T_0 \left(\frac{P_e}{P_0}\right)^{\frac{k-1}{k}} \\ &= 400 \left(\frac{P_b}{900}\right)^{\frac{0.4}{1.4}} \\ &= 57.278 P_b^{0.286} \end{aligned}$$

Then,

$$\begin{aligned} V &= \sqrt{2 \cdot 1.005 \cdot 10^3 \cdot (400 - 57.278 P_b^{0.286})} \\ &= 44.83\sqrt{400 - 57.278 P_b^{0.286}} \end{aligned}$$

So,

$$\boxed{V = \begin{cases} 366.069, & P_b < 475.4536 \text{ kPa} \\ 44.83\sqrt{400 - 57.278 P_b^{0.286}}, & P_b \geq 475.4536 \text{ kPa} \end{cases}}$$

The mass flow rate is given by

$$\dot{m} = \rho_e A V$$

By ideal gas law,

$$\begin{aligned} \rho_e &= \frac{P_e}{R T_e} \\ &= \frac{P_e}{0.287 \cdot T_e} \end{aligned}$$

Then for $P_b < 475.4536$ kPa,

$$\begin{aligned} \rho_e &= \frac{475.4536}{0.287 \cdot 333.33} \\ &= 4.9699479573 \text{ kg/m}^3 \end{aligned}$$

and for $P_b \geq 475.4536$ kPa,

$$\begin{aligned} \rho_e &= \frac{P_b}{0.287 \cdot 57.278 P_b^{0.286}} \\ &= \frac{1}{0.287 \cdot 57.278 P_b^{0.286-1}} \\ &= \frac{1}{16.439 P_b^{-0.7143}} \end{aligned}$$

Then,

$$\boxed{\dot{m} = \begin{cases} 4.9699479573 \cdot 10 \times 10^{-4} \cdot 366.069, & P_b < 475.4536 \text{ kPa} \\ \frac{1}{16.439 P_b^{-0.7143}} \cdot 10 \times 10^{-4} \cdot 44.83\sqrt{400 - 57.278 P_b^{0.286}}, & P_b \geq 475.4536 \text{ kPa} \end{cases}}$$

This gets way too complicated, so I'll just make a table using Excel.

P_b (kPa)	P_e (kPa)	T_e (K)	V_e (m/s)	ρ_e (kg/m^3)	$\dot{m}$ (kg/s)
100	100	475.4536	366.069	4.9699	1.8193
200	200	475.4536	366.069	4.9699	1.8193
300	300	475.4536	366.069	4.9699	1.8193
400	400	475.4536	366.069	4.9699	1.8193
500	500	333.3333	366.0601	4.9699	1.8193
600	600	356.2445	296.5611	5.8684	1.7403
700	700	372.2853	236.0225	6.5515	1.5463
800	800	386.7631	163.1142	7.2071	1.1756
900	900	400	0	7.8397	0

Plotting the results with Matplotlib [2], Figures 3, 4, and 5 are obtained.

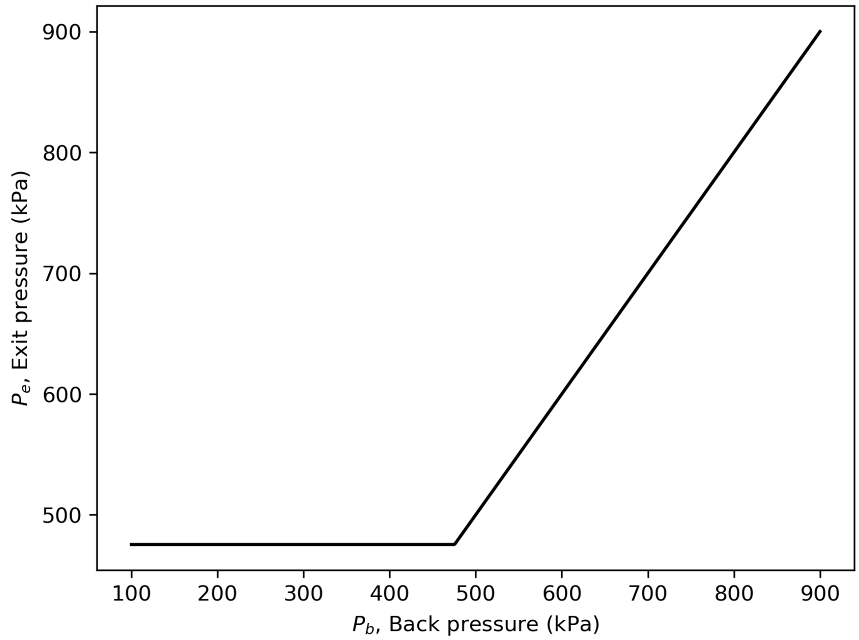

Figure 3: Exit Pressure vs. Back Pressure

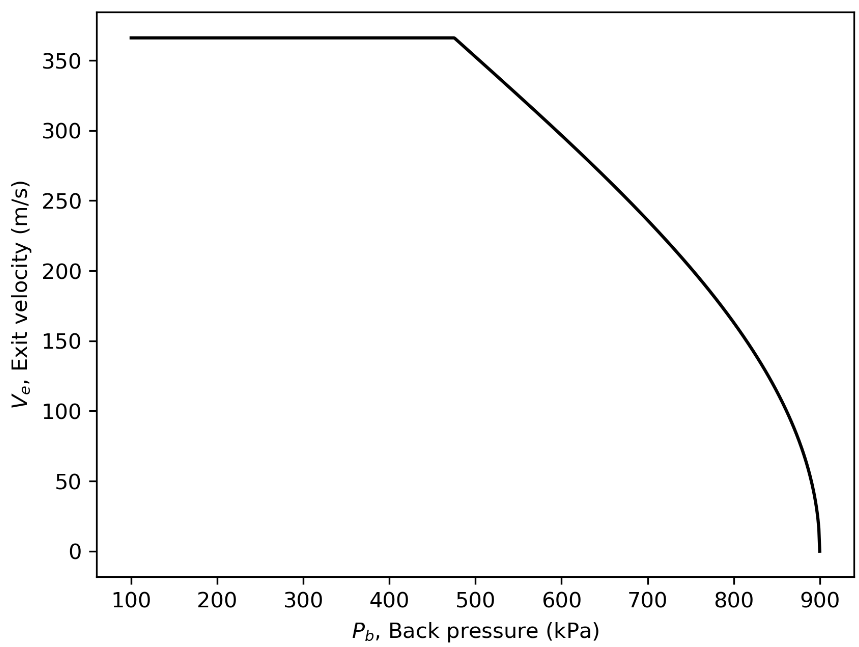

Figure 4: Exit Velocity vs. Back Pressure

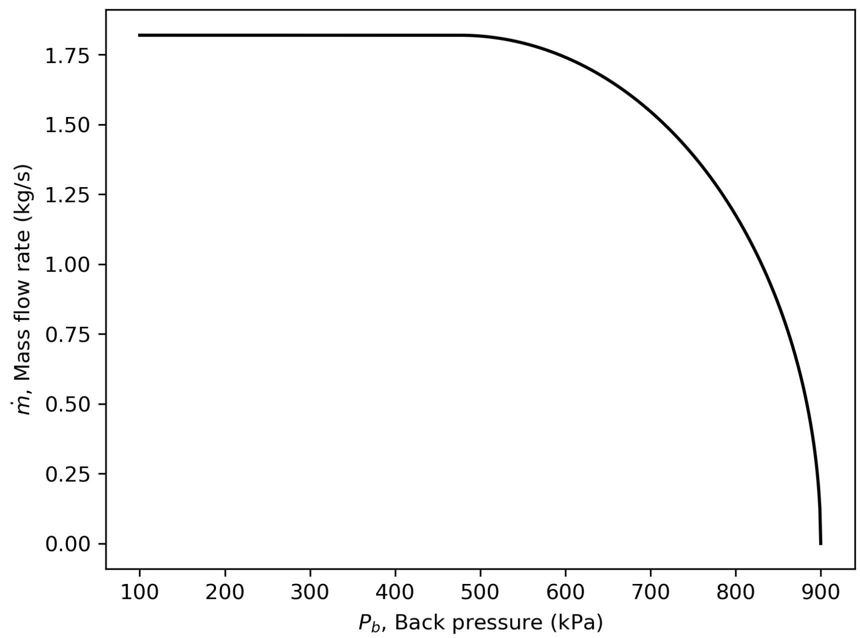

Figure 5: Mass Flow Rate vs. Back Pressure

References

[1] Y. A. Cengel and J. M. Cimbala, *Fluid mechanics: fundamentals and applications*, 4th ed. New York, NY: McGraw-Hill Education, 2018.

[2] J. D. Hunter, "Matplotlib: A 2d graphics environment," *Computing in Science & Engineering*, vol. 9, no. 3, pp. 90–95, 2007.