Question 1

A person drops 3 aluminum balls of diameters 2 mm, 4 mm, and 10 mm into a tank filled with glycerin at 22°C ($\mu = 1\,kg \cdot m/s$), and measured the terminal velocities to be 3.2 mm/s, 12.8 mm/s, and 60.4 mm/s, respectively. The measurements are to be compared with theory using Stokes law for drag force acting on a spherical object of diameter D expressed as $F_D = 3\pi\mu VD$ for $Re \ll 1$. Compare experimental velocities values with those predicted theoretically.

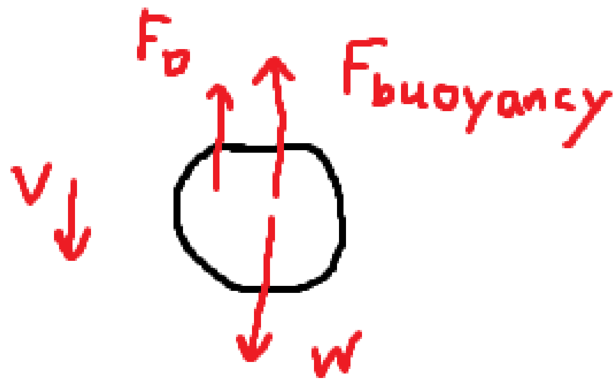

Figure 1: Free body diagram of a sphere in a fluid.

Since at terminal velocity, the acceleration is zero, by Newton's Second Law,

$$w = F_D + F_{\text{buoyancy}}$$

By Stokes' Law,

$$\frac{\pi}{6} D^3 \rho_{\text{aluminum}} g = 3\pi\mu VD + \frac{\pi}{6} D^3 \rho_{\text{glycerin}} g$$

Rearranging for V,

$$V = \frac{D^2}{18\mu} \left(\rho_{\text{aluminum}} - \rho_{\text{glycerin}}\right) g$$

Assuming pure aluminum, $\rho_{\text{aluminum}} = 2702\,\text{kg/m}^3$, and at 22°C $\rho_{\text{glycerin}} = 1263$ [1]. Substituting these values into the equation, we can calculate the theoretical terminal velocities for each diameter, shown in Table 1.

The error is calculated as

$$\text{Error} = \frac{|V_{\text{exp}} - V_{\text{theo}}|}{V_{\text{theo}}} \times 100\%$$

The Reynolds number is calculated as

$$\text{Re} = \frac{\rho_{\text{glycerin}} VD}{\mu}$$

The agreement between the theoretical and experimental terminal velocities is good for the 2 mm and 4 mm diameters, with relative errors of 2.01%. However, the relative error for the 10 mm diameter is 22.98%. This is likely due to the Reynolds number being a magnitude higher than the 4 mm diameter. As diameter increases, the Reynolds number increases, and the assumption that $Re \ll 1$ will become less accurate.

Table 1: Comparison of Theoretical and Experimental Terminal Velocities

Diameter (mm)	Theorerical Terminal Velocity, V (mm/s)	Experimental Terminal Velocity (mm/s)	Relative Error (%)	Reynolds Number
2	3.14	3.2	2.01	0.008
4	12.55	12.8	2.01	0.032
10	78.43	60.4	22.98	0.198

Question 2

Estimate the speed at which you would need to swim in room temperature water to be in the creeping flow regime. (An order-of-magnitude estimate will suffice.) Discuss.

The creeping flow approximation is valid when the Reynolds number is much less than 1. As an estimation,

$$\mathrm{Re} = \frac{\rho V D}{\mu} < 0.1$$

Rearranging

$$V < \frac{0.1\mu}{\rho D}$$

Using a characteristic length of $D = 1\,\mathrm{m}$ (order of magnitude estimation of human height) and evaluating the properties at 20°C, $\mu = 1.002 \times 10^{-3}\,\mathrm{Pa \cdot s}$, $\rho = 998\,\mathrm{kg/m^3}$ [1]. Then,

$$\boxed{V < \frac{0.1(1.002 \times 10^{-3})}{998} \approx 1.0010^{-7}\,\mathrm{m/s}}$$

This is an extremely slow speed, and is not feasible for a human to swim at this speed. This is due to the kinematic viscosity of water being very small, and the density of water being very large. Creeping flow is not a practical regime for human swimming.

Question 3

Consider creeping flow of a sphere of diameter D moving through a fluid at speed V. We gave an expression for drag force, $F_D = 3\pi\mu V D$. The drag coefficient C_D over three-dimensional bodies is typically defined as $C_D = \frac{F_D}{\frac{1}{2}\rho V^2 A}$, where A is the frontal area of the body (the area you "see" when looking at the body from upstream). Generate an expression for C_D in terms of Reynolds number for this flow.

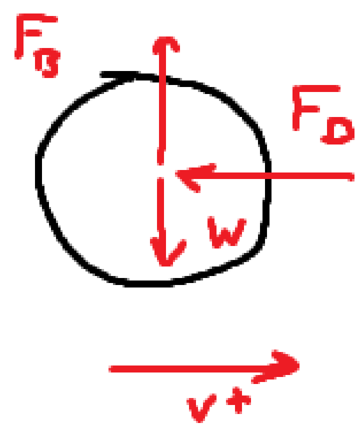

Figure 2: Free body diagram of a sphere in a fluid.

The frontal area of a sphere is $A = \pi D^2/4$. Substituting the expression for F_D into the definition of C_D,

$$\begin{aligned} C_D &= \frac{3\pi\mu V D}{\frac{1}{2}\rho V^2 \frac{\pi}{4} D^2} \\ &= \frac{24\mu}{\rho V D} \qquad = \boxed{\frac{24}{\mathrm{Re}}} \end{aligned}$$

Since for Stokes flow, the Reynolds number is $\mathrm{Re} \ll 1$, the drag coefficient will be very large.

Question 4

Consider the following steady, two-dimensional, incompressible velocity field:

$$\vec{V} = (u, v) = \left(\frac{1}{2}ay^2 + b\right)\hat{i} + (axy + c)\hat{j}$$

Is this flow field irrotational? If so, generate an expression for the velocity potential function.

First, check the vorticity,

$$\vec{\nabla} \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{1}{2}ay^2 + b & axy + c & 0 \end{vmatrix}$$

$$\begin{aligned} &= \left(\frac{\partial}{\partial x}(axy + c) - \frac{\partial}{\partial y}(\frac{1}{2}ay^2 + b) \right) \hat{k} \\ &= (a - a)\hat{k} \\ &= 0\hat{k} \end{aligned}$$

Since the vorticity is zero, the flow field is irrotational. The velocity potential function, ϕ, can be found by integrating the velocity components,

$$\begin{aligned} u &= \frac{\partial \phi}{\partial x} \\ \implies \phi &= \frac{1}{2}axy^2 + bx + f(y) \end{aligned}$$

From the y-component,

$$\begin{aligned} v &= \frac{\partial \phi}{\partial y} \\ \implies \frac{\partial \phi}{\partial y} &= axy + f'(y) \\ \implies f'(y) &= c \\ \implies f(y) &= cy + C \end{aligned}$$

Therefore, the velocity potential function is

$$\boxed{\phi = \frac{1}{2}axy^2 + bx + cy + C}$$

References

[1] Y. A. Çengel, M. A. Boles, and M. Kanoğlu, *Thermodynamics: An Engineering Approach*, tenth edition ed. New York, NY: McGraw Hill, 2024.