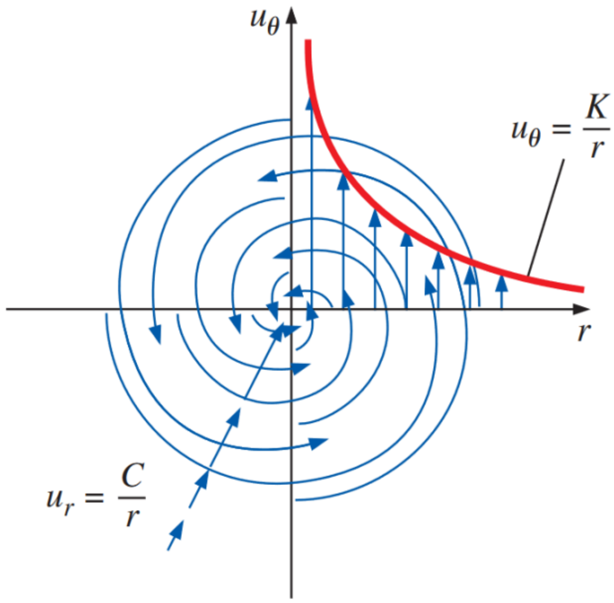
Question 1

Consider steady, two-dimensional, incompressible flow due to a spiraling line vortex/sink flow centered on the z-axis. Streamlines and velocity components are shown in the following figure. The velocity field is $u_r = C/r$ and $u_\theta = K/r$, where C and K are constants. Calculate the pressure as a function of r.



Solution

First we list our assumptions and their consequences: Starting with the continuity equation,

Number	Assumption	Consequence
#1	Steady flow	$\frac{\partial}{\partial t} = 0$
#2	Incompressible flow	$\rho =$ constant
#3	Two-dimensional flow	$u_z = 0$, $\partial_z \vec{v} = 0$
#4	No Gravity on r, θ	$g_r = g_\theta = 0$
#5	u_r has θ Independence	$\frac{\partial u_r}{\partial \theta} = 0$
#6	u_θ has θ Independence	$\frac{\partial u_\theta}{\partial \theta} = 0$

we have

$$\frac{1}{r}\frac{\partial (r u_r)}{\partial r} + \frac{1}{r}\overbrace{\cancel{\frac{\partial u_\theta}{\partial \theta}}}^{\#6} + \overbrace{\cancel{\frac{\partial u_z}{\partial z}}}^{\#3} = 0$$

Now the momentum equation in the r direction is

$$\underbrace{\cancel{\frac{\partial u_r}{\partial t}}}_{\#1} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\underbrace{\cancel{\frac{\partial u_r}{\partial \theta}}}_{\#5} - \frac{u_\theta^2}{r} + u_z\underbrace{\cancel{\frac{\partial u_r}{\partial z}}}_{\#3} = -\frac{1}{\rho}\frac{\partial P}{\partial r} + \underbrace{\cancel{g_r}}_{\#4}$$
$$+\frac{\mu}{\rho}\left[\underbrace{\cancel{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_r)\right)}}_{\text{Cont.}} + \frac{1}{r^2}\underbrace{\cancel{\frac{\partial^2 u_r}{\partial \theta^2}}}_{\#6} - \frac{2}{r^2}\underbrace{\cancel{\frac{\partial u_\theta}{\partial \theta}}}_{\#6} + \underbrace{\cancel{\frac{\partial^2 u_r}{\partial z^2}}}_{\#3}\right]$$

which simplifies to

$$u_r\frac{\partial u_r}{\partial r} - \frac{u_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial P}{\partial r}$$

substituting in $u_r = C/r$ and $u_\theta = K/r$ gives

$$\frac{C}{r}\left(\frac{-C}{r^2}\right) - \frac{K^2}{r^3} = -\frac{1}{\rho}\frac{\partial P}{\partial r}$$
$$\rho\frac{C^2+K^2}{r^3} = \frac{\partial P}{\partial r}$$

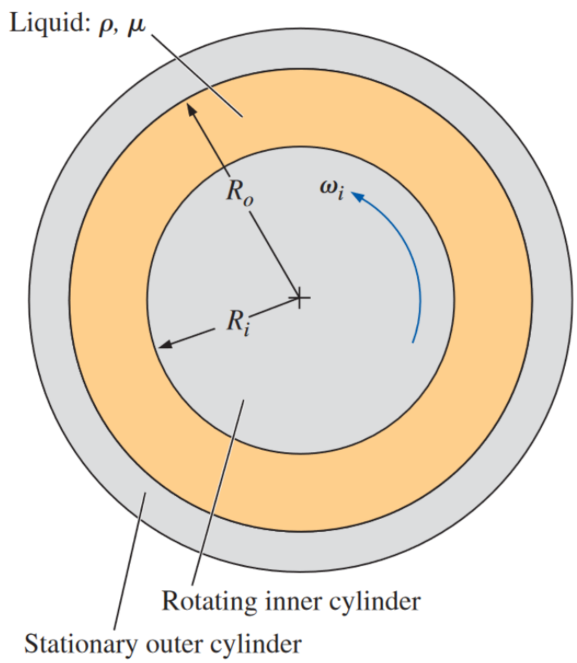
Solving,

$$\boxed{P = -\frac{\rho}{2}\left(\frac{C^2+K^2}{r^2}\right) + C_1}$$

From the Navier-Stokes equation in the θ direction, it can be shown that P is only a function of r and not θ. Thus, C_1 is some constant and not a function of θ.

Question 2

An incompressible Newtonian liquid is confined between two concentric circular cylinders of infinite length – a solid inner cylinder of radius R_i and a hollow, stationary outer cylinder of radius R_o. The inner cylinder rotates at angular velocity ω_i. The flow is steady, laminar, and two-dimensional in the $r\theta$-plane. The flow is also rotationally symmetric, meaning that nothing is a function of coordinate θ (u_θ and P are functions of radius r only). The flow is also circular, meaning that velocity component $u_r = 0$ everywhere. Generate an exact expression for velocity component u_θ as a function of radius r and the other parameters in the problem. You may ignore gravity.



Solution

First we list our assumptions and their consequences: Also note the boundary conditions:

Number	Assumption	Consequence
#1	Steady flow	$\frac{\partial}{\partial t} = 0$
#2	Incompressible flow	$\rho = \text{constant}$
#3	Two-dimensional flow	$u_z = 0,\ \partial_z \vec{v} = 0$
#4	No Gravity on r, θ	$g_r = g_\theta = 0$
#5	Rotationally Symmetric Flow	$u_\theta = u_\theta(r),\ P = P(r)$
#6	Circular Flow	$u_r = 0$

$$u_\theta(R_i) = R_i\omega_i$$
$$u_\theta(R_o) = 0$$

Starting with the continuity equation, we have

$$\frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \overset{\#5}{\cancel{\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}}} + \overset{\#3}{\cancel{\frac{\partial u_z}{\partial z}}} = 0$$

Now the momentum equation in the θ direction is

$$\cancelto{\#1}{\frac{\partial u_\theta}{\partial t}} + \cancelto{\#6}{u_r \frac{\partial u_\theta}{\partial r}} + \frac{u_\theta}{r}\cancelto{\#5}{\frac{\partial u_\theta}{\partial \theta}} + \cancelto{\#6}{\frac{u_r u_\theta}{r}} + u_z \cancelto{\#3}{\frac{\partial u_\theta}{\partial z}} = -\frac{1}{\rho r}\cancelto{\#5}{\frac{\partial P}{\partial \theta}} + \cancelto{\#4}{g_\theta}$$
$$+ \frac{\mu}{\rho}\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r u_\theta)\right) + \frac{1}{r^2}\cancelto{\#5}{\frac{\partial^2 u_\theta}{\partial \theta^2}} + \frac{2}{r^2}\cancelto{\#5}{\frac{\partial u_\theta}{\partial \theta}} + \cancelto{\#3}{\frac{\partial^2 u_\theta}{\partial z^2}}\right]$$

which simplifies to

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r u_\theta)\right) = 0$$

Integrating,

$$\begin{aligned}
\frac{1}{r}\frac{\partial}{\partial r}(r u_\theta) &= C_1 \\
\frac{\partial}{\partial r}(r u_\theta) &= C_1 r \\
r u_\theta &= \frac{1}{2}C_1 r^2 + C_2 \\
u_\theta &= \frac{1}{2}C_1 r + \frac{C_2}{r}
\end{aligned}$$

Applying the boundary conditions,

$$\begin{aligned}
u_\theta(R_o) = 0 = \frac{1}{2}C_1 R_o + \frac{C_2}{R_o} &\implies C_2 = -\frac{1}{2}C_1 R_o^2 \\
u_\theta(R_i) = R_i\omega_i = \frac{1}{2}C_1 R_i + \frac{C_2}{R_i} &\implies R_i\omega_i = \frac{1}{2}C_1 R_i - \frac{1}{2}C_1\frac{R_o^2}{R_i}
\end{aligned}$$

Solving,

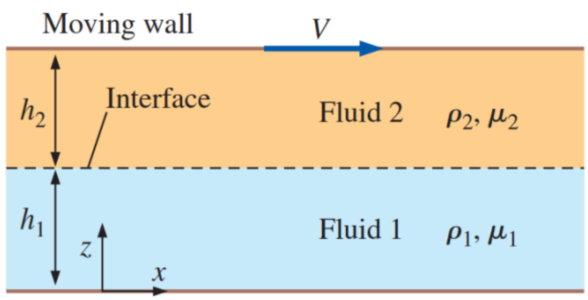
$$\begin{aligned}
&\implies R_i\omega_i = C_1\left(\frac{R_i}{2} - \frac{R_o^2}{2R_i}\right) \\
&\implies C_1 = \frac{2R_i^2\omega_i}{R_i^2 - R_o^2} \\
&\implies C_2 = \frac{R_i^2 R_o^2\omega_i}{R_o^2 - R_i^2}
\end{aligned}$$

Thus,

$$\begin{aligned}
u_\theta &= \frac{R_i^2\omega_i}{R_i^2 - R_o^2}r + \frac{R_i^2 R_o^2\omega_i}{R_o^2 - R_i^2}\frac{1}{r} \\
&\boxed{= \left(\frac{R_i^2\omega_i}{R_o^2 - R_i^2}\right)\left(\frac{R_o^2}{r} - r\right)}
\end{aligned}$$

Question 3

Consider a modified form of Couette flow in which there are two immiscible fluids sandwiched between two infinitely long and wide, parallel flat plates. The flow is steady, incompressible, parallel, and laminar. The top plate moves at velocity V to the right, and the bottom plate is stationary. Gravity acts in the $-z$-direction (downward in the figure). There is no forced pressure gradient pushing the fluids through the channel – the flow is set up solely by viscous effects created by the moving upper plate. You may ignore surface tension effects and assume that the interface is horizontal. The pressure at the bottom of the flow ($z = 0$) is equal to P_0.



a) *List all the appropriate boundary conditions on both velocity and pressure. (Hint: There are six required boundary conditions)*

b) *Solve for the velocity field. (Hint: Split up the solution into two portions, one for each fluid. Generate expressions for u_1 as a function of z and u_2 as a function of z)*

c) *Solve for the pressure field. (Hint: Again split up the solution. Solve for P_1 and P_2)*

d) *Let fluid 1 be water and let fluid 2 be unused engine oil, both at $80°C$. Also, let $h_1 = 5.0\,mm$, $h_2 = 8.0\,mm$, and $V = 10.0\,m/s$. Plot u as a function of z across the entire channel. Discuss the results.*

Solution

(a)

The boundary conditions are

Boundary Condition	Equation
No-slip at bottom plate	$u_1(0) = 0$
Continuity of velocity	$u_1(h_1) = u_2(h_1)$
No-slip at top plate	$u_2(h_1 + h_2) = V$
Continuity of pressure	$P_1(h_1) = P_2(h_1)$
Pressure at bottom	$P_1(0) = P_0$
Interface shear stress	$\tau_{12} = \mu_1 \frac{du_1}{dz} = \mu_2 \frac{du_2}{dz}$

(b)

We begin with the assumptions and their consequences:

Number	Assumption	Consequence
#1	Steady flow	$\partial_t = 0$
#2	Incompressible flow	$\rho =$ constant
#3	2D flow	$u_y = 0$, $\partial_y \vec{v} = 0$
#4	Parallel flow	$w = 0$
#5	Gravity in z	$g_z = -g$
#6	Fully Developed Flow	$\partial_x = 0$

Starting with the continuity equation, we have

$$\frac{\partial u_1}{\partial x} + \cancelto{\#3}{\frac{\partial v_1}{\partial y}} + \cancelto{\#4}{\frac{\partial w_1}{\partial z}} = 0$$

which implies that $u_1 = u_1(z)$. Now the momentum equation in the x direction is

$$\rho \left(\cancelto{\#1}{\frac{\partial u_1}{\partial t}} + u_1 \cancelto{\text{Cont.}}{\frac{\partial u_1}{\partial x}} + \cancelto{\#3}{v_1 \frac{\partial u_1}{\partial y}} + \cancelto{\#4}{w_1 \frac{\partial u_1}{\partial z}} \right) = -\cancelto{\#6}{\frac{\partial P_1}{\partial x}} + \mu_1 \left(\cancelto{\text{Cont.}}{\frac{\partial^2 u_1}{\partial x^2}} + \cancelto{\#3}{\frac{\partial^2 u_1}{\partial y^2}} + \frac{\partial^2 u_1}{\partial z^2} \right) - \cancelto{\#5}{\rho g_x}$$

which simplifies to

$$\begin{aligned} \frac{d^2 u_1}{dz^2} &= 0 \\ \implies \frac{du_1}{dz} &= A \\ \implies u_1 &= Az + B \end{aligned}$$

Since the assumptions hold for both fluid 1 and 2, the Navier-Stokes equation for fluid 2 is the same as for fluid 1. Thus, the velocity field for fluid 2 is

$$\begin{aligned}\frac{du_2}{dz} &= C \\ u_2 &= Cz + D\end{aligned}$$

Applying the bottom plate boundary condition gives

$$\begin{aligned}u_1(0) &= 0 = B \\ \implies u_1 &= Az\end{aligned}$$

Applying the top plate boundary condition gives

$$\begin{aligned}u_2(h_1+h_2) &= V = C(h_1+h_2) + D \\ \implies D &= V - C(h_1+h_2) \\ \implies u_2 &= Cz + V - C(h_1+h_2)\end{aligned}$$

Applying the continuity of velocity gives

$$\begin{aligned}u_1(h_1) &= u_2(h_1) \\ Ah_1 &= Ch_1 + V - C(h_1+h_2) \\ &= Ch_1 + V - Ch_1 - Ch_2 \\ &= V - Ch_2 \\ \implies C &= \frac{V - Ah_1}{h_2}\end{aligned}$$

Lastly, the shear stress condition gives

$$\begin{aligned}\mu_1\frac{du_1}{dz} &= \mu_2\frac{du_2}{dz} \\ \mu_1 A &= \mu_2 C \\ \implies A &= \frac{\mu_2}{\mu_1}C \\ &= \frac{\mu_2}{\mu_1}\frac{V - Ah_1}{h_2} \\ \implies A &= \frac{\mu_2 V}{\mu_1 h_2 + \mu_2 h_1}\end{aligned}$$

Thus, the velocity field is

$$\begin{aligned}u_1 &= \frac{\mu_2 V}{\mu_1 h_2 + \mu_2 h_1}z \\ u_2 &= \frac{V - \frac{\mu_2 V}{\mu_1 h_2+\mu_2 h_1}h_1}{h_2}z + V - \frac{V - \frac{\mu_2 V}{\mu_1 h_2+\mu_2 h_1}h_1}{h_2}(h_1+h_2)\end{aligned}$$

simplifying

```
syms mu1 mu2 V h1 h2 z
u2 = (V - (mu2*V*h1)/(mu1*h2 + mu2*h1))/h2*z ...
        + V - (V - (mu2*V*h1)/(mu1*h2 + mu2*h1))/h2*(h1 + h2);
u2 = simplify(u2)

>> u2 =
        (V*(h1*mu2 - h1*mu1 + mu1*z))/(h1*mu2 + h2*mu1)
```

$$\boxed{\begin{aligned} u_1 &= \frac{\mu_2 V z}{\mu_1 h_2 + \mu_2 h_1} \\ u_2 &= \frac{V \mu_1 z}{\mu_1 h_2 + \mu_2 h_1} + \frac{V h_1 \mu_2 - V h_1 \mu_1}{\mu_1 h_2 + \mu_2 h_1} \end{aligned}}$$

(c)

Now the momentum equation in the z direction is

$$\rho_1 \left(\cancelto{\#1}{\frac{\partial w_1}{\partial t}} + u_1 \cancelto{\text{Cont.}}{\frac{\partial w_1}{\partial x}} + v_1 \cancelto{\#4}{\frac{\partial w_1}{\partial y}} + \cancelto{\#4}{w_1 \frac{\partial w_1}{\partial z}} \right) = -\frac{\partial P_1}{\partial z} + \mu_1 \left(\cancelto{\#4}{\frac{\partial^2 w_1}{\partial x^2}} + \cancelto{\#4}{\frac{\partial^2 w_1}{\partial y^2}} + \cancelto{\#4}{\frac{\partial^2 w_1}{\partial z^2}} \right) - \rho g_z$$

which simplifies to

$$\begin{aligned} \frac{dP_1}{dz} &= -\rho_1 g \\ \implies P_1 &= -\rho_1 g z + E \end{aligned}$$

Since the assumptions hold for both fluid 1 and 2, the pressure field for fluid 2 is the same as for fluid 1. Thus, the pressure field for fluid 2 is

$$P_2 = -\rho_2 g z + F$$

Using the boundary condition $P_1(0) = P_0$ gives

$$\begin{aligned} P_1(0) &= P_0 = E \\ \implies P_1 &= -\rho g z + P_0 \end{aligned}$$

Lastly, the continuity of pressure gives

$$\begin{aligned} P_1(h_1) &= P_2(h_1) \\ -\rho_1 g h_1 + P_0 &= -\rho_2 g h_1 + F \\ \implies F &= (\rho_2 - \rho_1) g h_1 + P_0 \end{aligned}$$

Thus, the pressure field is

$$\boxed{\begin{aligned} P_1 &= -\rho_1 g z + P_0 \\ P_2 &= -\rho_2 g z + (\rho_2 - \rho_1) g h_1 + P_0 \end{aligned}}$$

Engine Oil (unused)								
0	899.0	1797	0.1469	9.097×10^{-8}	3.814	4.242×10^{-3}	46,636	0.00070
20	888.1	1881	0.1450	8.680×10^{-8}	0.8374	9.429×10^{-4}	10,863	0.00070
40	876.0	1964	0.1444	8.391×10^{-8}	0.2177	2.485×10^{-4}	2,962	0.00070
60	863.9	2048	0.1404	7.934×10^{-8}	0.07399	8.565×10^{-5}	1,080	0.00070
80	852.0	2132	0.1380	7.599×10^{-8}	0.03232	3.794×10^{-5}	499.3	0.00070
100	840.0	2220	0.1367	7.330×10^{-8}	0.01718	2.046×10^{-5}	279.1	0.00070
120	828.9	2308	0.1347	7.042×10^{-8}	0.01029	1.241×10^{-5}	176.3	0.00070
140	816.8	2395	0.1330	6.798×10^{-8}	0.006558	8.029×10^{-6}	118.1	0.00070
150	810.3	2441	0.1327	6.708×10^{-8}	0.005344	6.595×10^{-6}	98.31	0.00070

Figure 1: Properties of Engine Oil at 80°C

TABLE A–3

Properties of saturated water

Temp. T, °C	Saturation Pressure P_{sat}, kPa	Density ρ, kg/m³		Enthalpy of Vaporization h_{fg}, kJ/kg	Specific Heat c_p, J/kg·K		Thermal Conductivity k, W/m·K		Dynamic Viscosity μ, kg/m·s		Prandtl Number Pr		Volume Expansion Coefficient β, 1/K	Surface Tension, N/m
		Liquid	Vapor		Liquid	Vapor	Liquid	Vapor	Liquid	Vapor	Liquid	Vapor	Liquid	Liquid
0.01	0.6113	999.8	0.0048	2501	4217	1854	0.561	0.0171	1.792×10^{-3}	0.922×10^{-5}	13.5	1.00	-0.068×10^{-3}	0.0756
5	0.8721	999.9	0.0068	2490	4205	1857	0.571	0.0173	1.519×10^{-3}	0.934×10^{-5}	11.2	1.00	0.015×10^{-3}	0.0749
10	1.2276	999.7	0.0094	2478	4194	1862	0.580	0.0176	1.307×10^{-3}	0.946×10^{-5}	9.45	1.00	0.733×10^{-3}	0.0742
15	1.7051	999.1	0.0128	2466	4186	1863	0.589	0.0179	1.138×10^{-3}	0.959×10^{-5}	8.09	1.00	0.138×10^{-3}	0.0735
20	2.339	998.0	0.0173	2454	4182	1867	0.598	0.0182	1.002×10^{-3}	0.973×10^{-5}	7.01	1.00	0.195×10^{-3}	0.0727
25	3.169	997.0	0.0231	2442	4180	1870	0.607	0.0186	0.891×10^{-3}	0.987×10^{-5}	6.14	1.00	0.247×10^{-3}	0.0720
30	4.246	996.0	0.0304	2431	4178	1875	0.615	0.0189	0.798×10^{-3}	1.001×10^{-5}	5.42	1.00	0.294×10^{-3}	0.0712
35	5.628	994.0	0.0397	2419	4178	1880	0.623	0.0192	0.720×10^{-3}	1.016×10^{-5}	4.83	1.00	0.337×10^{-3}	0.0704
40	7.384	992.1	0.0512	2407	4179	1885	0.631	0.0196	0.653×10^{-3}	1.031×10^{-5}	4.32	1.00	0.377×10^{-3}	0.0696
45	9.593	990.1	0.0655	2395	4180	1892	0.637	0.0200	0.596×10^{-3}	1.046×10^{-5}	3.91	1.00	0.415×10^{-3}	0.0688
50	12.35	988.1	0.0831	2383	4181	1900	0.644	0.0204	0.547×10^{-3}	1.062×10^{-5}	3.55	1.00	0.451×10^{-3}	0.0679
55	15.76	985.2	0.1045	2371	4183	1908	0.649	0.0208	0.504×10^{-3}	1.077×10^{-5}	3.25	1.00	0.484×10^{-3}	0.0671
60	19.94	983.3	0.1304	2359	4185	1916	0.654	0.0212	0.467×10^{-3}	1.093×10^{-5}	2.99	1.00	0.517×10^{-3}	0.0662
65	25.03	980.4	0.1614	2346	4187	1926	0.659	0.0216	0.433×10^{-3}	1.110×10^{-5}	2.75	1.00	0.548×10^{-3}	0.0654
70	31.19	977.5	0.1983	2334	4190	1936	0.663	0.0221	0.404×10^{-3}	1.126×10^{-5}	2.55	1.00	0.578×10^{-3}	0.0645
75	38.58	974.7	0.2421	2321	4193	1948	0.667	0.0225	0.378×10^{-3}	1.142×10^{-5}	2.38	1.00	0.607×10^{-3}	0.0636
80	47.39	971.8	0.2935	2309	4197	1962	0.670	0.0230	0.355×10^{-3}	1.159×10^{-5}	2.22	1.00	0.653×10^{-3}	0.0627
85	57.83	968.1	0.3536	2296	4201	1977	0.673	0.0235	0.333×10^{-3}	1.176×10^{-5}	2.08	1.00	0.670×10^{-3}	0.0617
90	70.14	965.3	0.4235	2283	4206	1993	0.675	0.0240	0.315×10^{-3}	1.193×10^{-5}	1.96	1.00	0.702×10^{-3}	0.0608
95	84.55	961.5	0.5045	2270	4212	2010	0.677	0.0246	0.297×10^{-3}	1.210×10^{-5}	1.85	1.00	0.716×10^{-3}	0.0599

Figure 2: Properties of Water at 80°C

(d)

From the textbook,

From Desmos, The flow is like two Couette flows connected to each other. The slope of the water is larger than the slope of the engine oil, which is expected since the viscosity of water is much lower than the viscosity of engine oil. The water's velocity nearly approaches the speed of the plate as it reaches the interface.

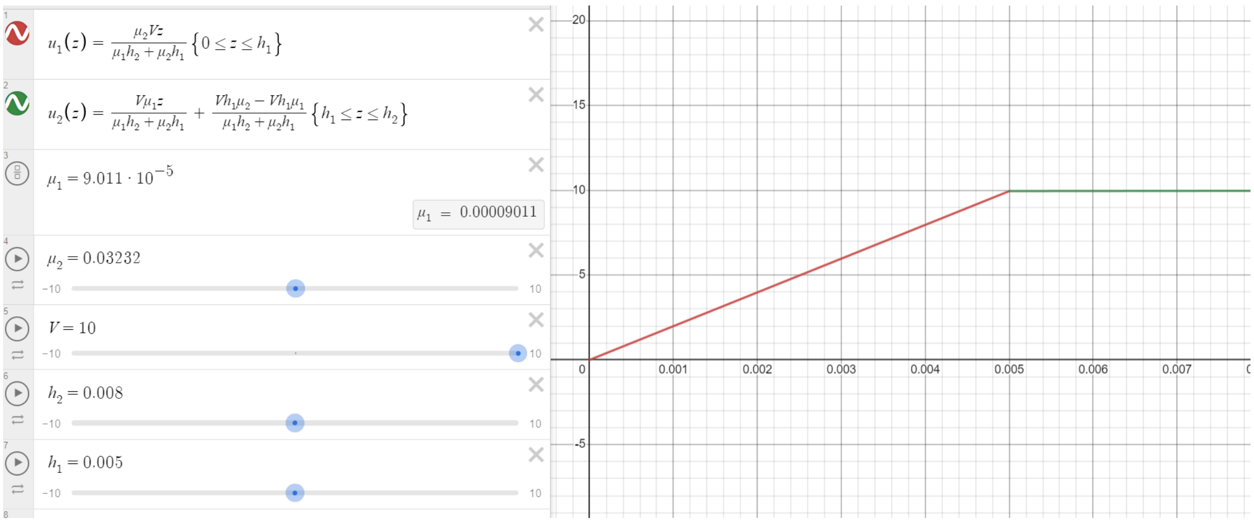


Figure 3: u as a function of z across the entire channel