Question 1

A Taylor series expansion of function $f(x)$ about some x-location x_0 is given as:

$$f(x_0 + \Delta x) = f(x_0) + \left(\frac{df}{dx}\right)_{x=x_0} \Delta x + \frac{1}{2!}\left(\frac{d^2 f}{dx^2}\right)_{x=x_0} (\Delta x)^2 + \frac{1}{3!}\left(\frac{d^3 f}{dx^3}\right)_{x=x_0} (\Delta x)^3 + \cdots$$

Consider the function $f(x) = e^x$. Suppose we know the value of $f(x)$ at $x = x_0$, i.e., we know the value of $f(x_0)$, and we want to estimate the value of this function at some x-location near x_0. Generate the first four terms of the Taylor series expansion for the given function (up to order $(\Delta x)^3$ as in the above equation). For $x_0 = 0$ and $\Delta x = -0.1$, use your truncated Taylor series expansion to estimate $f(x_0 + \Delta x)$. Compare your result with the exact value of $e^{-0.1}$. How many digits of accuracy do you achieve with your truncated Taylor series?

Solution

The first four terms of the Taylor series expansion for the function $f(x) = e^x$ about $x_0 = 0$ (Maclaurin series) are given by:

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

Using $\Delta x = -0.1$, we can estimate $f(x_0 + \Delta x) = f(-0.1)$ as:

$$\begin{aligned} e^{-0.1} &\approx 1 + (-0.1) + \frac{(-0.1)^2}{2!} + \frac{(-0.1)^3}{3!} \\ &\approx 1 - 0.1 + \frac{0.01}{2} - \frac{0.001}{6} \\ &\approx \boxed{0.904833} \end{aligned}$$

The exact value of $e^{-0.1} \approx 0.904837$. The truncated Taylor series expansion is accurate to the first 5 digits.

Question 2

The product rule can be applied to the divergence of scalar f times vector $\vec{G}$ as:

$$\nabla \cdot (f\vec{G}) = \vec{G} \cdot \nabla f + f \nabla \cdot \vec{G}$$

Expand both sides of this equation in Cartesian coordinates and verify that it is correct.

Solution

The LHS of the equation is given by

$$\begin{aligned}
\text{LHS} &= \nabla \cdot (f\vec{G}) \\
&= \frac{\partial}{\partial x}(fG_x) + \frac{\partial}{\partial y}(fG_y) + \frac{\partial}{\partial z}(fG_z) \\
&= \frac{\partial f}{\partial x}G_x + f\frac{\partial G_x}{\partial x} + \frac{\partial f}{\partial y}G_y + f\frac{\partial G_y}{\partial y} + \frac{\partial f}{\partial z}G_z + f\frac{\partial G_z}{\partial z} \\
&\boxed{= f\left(\frac{\partial G_x}{\partial x} + \frac{\partial G_y}{\partial y} + \frac{\partial G_z}{\partial z}\right) + G_x\frac{\partial f}{\partial x} + G_y\frac{\partial f}{\partial y} + G_z\frac{\partial f}{\partial z}}
\end{aligned}$$

The RHS of the equation is given by

$$\begin{aligned}
\text{RHS} &= \vec{G} \cdot \nabla f + f\nabla \cdot \vec{G} \\
&= \boxed{G_x\frac{\partial f}{\partial x} + G_y\frac{\partial f}{\partial y} + G_z\frac{\partial f}{\partial z} + f\left(\frac{\partial G_x}{\partial x} + \frac{\partial G_y}{\partial y} + \frac{\partial G_z}{\partial z}\right)}
\end{aligned}$$

The LHS is equal to the RHS, so the product rule for the divergence of a scalar times a vector is verified to be correct.

Question 3

Consider the steady flow of water through an axisymmetric garden hose nozzle shown below. The axial component of velocity increases linearly from $u_{z,entrance}$ to $u_{z,exit}$ as sketched. Between $z = 0$ and $z = L$, the axial velocity component is given by $u_z = u_{z,entrance} + [(u_{z,exit} - u_{z,entrance})/L]z$. Generate an expression for the radial velocity component u_r between $z = 0$ and $z = L$. You may ignore frictional effects on the walls.

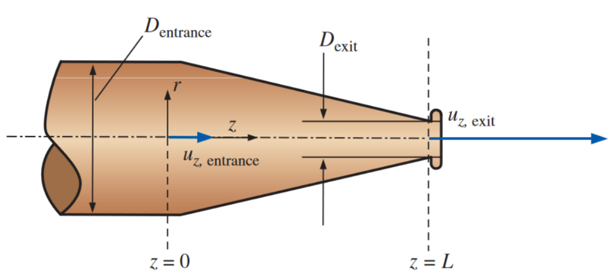

Figure 1: Garden hose nozzle

Solution

First we list assumptions

Table 1: Assumptions

Assumption	Approximation
Steady flow	$\partial_t = 0$
Incompressible flow	$\rho = \text{constant}$
Axisymmetric flow	$\partial_\theta = 0$
No friction	$\tau_w = 0$
No heat transfer	$q = 0$
No body forces	$\mathbf{F} = 0$

The continuity equation for incompressible flow is given by

$$\nabla \cdot \vec{v} = 0$$

In cylindrical coordinates, the continuity equation is given by

$$\frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \cancelto{\text{axisymmetric}}{\frac{1}{r}\frac{\partial}{\partial \theta}}(u_\theta) + \frac{\partial}{\partial z}(u_z) = 0$$

Substituting the given expression for u_z into the continuity equation, we have

$$\begin{aligned} \frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{\partial}{\partial z}\left(u_{z,entrance} + \frac{u_{z,exit} - u_{z,entrance}}{L} z\right) &= 0 \\ \frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{u_{z,exit} - u_{z,entrance}}{L} &= 0 \end{aligned}$$

Solving for u_r, we have

$$\begin{aligned} \frac{\partial}{\partial r}(r u_r) &= -\left(u_{z,exit} - u_{z,entrance}\right)\frac{r}{L} \\ r u_r &= -\frac{r^2}{2L}\left(u_{z,exit} + \frac{r^2}{2L} u_{z,entrance}\right) + g(z) \end{aligned}$$

Notice that at $r = 0$,

$$\begin{aligned} (0)u_r &= -\frac{0^2}{2L}\left(u_{z,exit} + \frac{0^2}{2L} u_{z,entrance}\right) + g(z) \\ \implies g(z) &= 0 \end{aligned}$$

Thus, the radial velocity component is given by

$$\boxed{u_r = -\frac{r}{2L}\left(u_{z,exit} + \frac{r^2}{2L} u_{z,entrance}\right)}$$

Question 4

The u velocity component of a steady, two-dimensional, incompressible flow field is $u = 3ax^2 - 2bxy$, where a and b are constants. The velocity component v is unknown. Generate an expression for v as a function of x and y.

Solution

The continuity equation for incompressible flow is given by

$$\nabla \cdot \vec{v} = 0$$
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Substituting the given expression for u into the continuity equation, we have

$$\frac{\partial}{\partial x}(3ax^2 - 2bxy) + \frac{\partial v}{\partial y} = 0$$
$$6ax - 2by + \frac{\partial v}{\partial y} = 0$$

Solving for v, we have

$$\frac{\partial v}{\partial y} = 2by - 6ax$$
$$\boxed{v = by^2 - 6axy + g(x)}$$

Question 5

A common flow encountered in practice is the crossflow of a fluid approaching a long cylinder of radius R at a freestream speed of U_∞. For incompressible inviscid flow, the velocity field of the flow is given as:

$$u_r = U_\infty \left(1 - \frac{R^2}{r^2}\right) \cos\theta \tag{1}$$
$$u_\theta = -U_\infty \left(1 + \frac{R^2}{r^2}\right) \sin\theta \tag{2}$$

(a) Show that the velocity field satisfies the continuity equation, and

(b) Determine the stream function corresponding to this velocity field.

Solution

(a)

First we list some assumptions:

Table 2: Assumptions

Assumption	Approximation
Steady flow	$\partial_t = 0$
Incompressible flow	$\rho = \text{constant}$
2D flow	$\partial_z = 0,\ u_z = 0$

The continuity equation for incompressible flow is given by

$$\nabla \cdot \vec{v} = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(u_\theta) + \overbrace{\cancel{\frac{\partial}{\partial z}(u_z)}}^{\text{2D}} = 0$$

Substituting the given expressions for u_r and u_θ into the continuity equation, we have

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rU_\infty\left(1 - \frac{R^2}{r^2}\right)\cos\theta\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(-U_\infty\left(1 + \frac{R^2}{r^2}\right)\sin\theta\right) = 0$$

$$\frac{\partial}{\partial r}\left(rU_\infty\left(1 - \frac{R^2}{r^2}\right)\cos\theta\right) - \frac{\partial}{\partial \theta}\left(U_\infty\left(1 + \frac{R^2}{r^2}\right)\sin\theta\right) = 0$$

$$\frac{\partial}{\partial r}\left(rU_\infty\cos\theta - \frac{R^2}{r}U_\infty\cos\theta\right) - \frac{\partial}{\partial \theta}\left(U_\infty\left(1 + \frac{R^2}{r^2}\right)\sin\theta\right) = 0$$

Expanding the derivatives, we have

$$U_\infty\cos\theta + \frac{R^2}{r^2}U_\infty\cos\theta - U_\infty\cos\theta - \frac{R^2}{r^2}U_\infty\cos\theta = 0$$

$$\boxed{0 = 0}$$

Thus, the velocity field satisfies the continuity equation.

(b)

Let us derive the stream function ψ for the velocity field. Observe that if

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \quad u_\theta = -\frac{\partial \psi}{\partial r}$$

Then, the continuity equation for 2D flow is

$$\begin{aligned}
\frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(u_\theta) &= 0 \\
\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{1}{r}\frac{\partial \psi}{\partial \theta}\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(-\frac{\partial \psi}{\partial r}\right) &= 0 \\
\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial \psi}{\partial \theta}\right) - \frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{\partial \psi}{\partial r}\right) &= 0 \\
\frac{\partial \psi}{\partial r \partial \theta} - \frac{\partial \psi}{\partial \theta \partial r} &= 0 \\
0 &= 0
\end{aligned}$$

Thus, the stream function ψ holds. We can solve for ψ by integrating the u_r and u_θ expressions. First, we integrate u_r with respect to θ:

$$\begin{aligned}
\psi &= \int r u_r \, d\theta \\
&= \int r U_\infty \left(1 - \frac{R^2}{r^2}\right) \cos\theta \, d\theta \\
&= r U_\infty \left(1 - \frac{R^2}{r^2}\right) \sin\theta + g(r)
\end{aligned}$$

Then deriving ψ with respect to r, we have

$$\begin{aligned}
\frac{\partial \psi}{\partial r} &= \frac{\partial}{\partial r}\left(U_\infty \left(r - \frac{R^2}{r}\right)\sin\theta + g(r)\right) \\
&= U_\infty \left(1 + \frac{R^2}{r^2}\right)\sin\theta + g'(r)
\end{aligned}$$

Equating this expression to $-u_\theta$, we have

$$\begin{aligned}
U_\infty \left(1 + \frac{R^2}{r^2}\right)\sin\theta + g'(r) &= U_\infty \left(1 + \frac{R^2}{r^2}\right)\sin\theta \\
\implies g'(r) &= 0 \\
\implies g(r) &= C
\end{aligned}$$

Thus, the stream function ψ is given by

$$\boxed{\psi = r U_\infty \left(1 - \frac{R^2}{r^2}\right)\sin\theta + C}$$

Question 6

Flow separates at a sharp corner along a wall and forms a recirculating separation bubble as sketched below (streamlines are shown). The value of the stream function at the wall is zero, and that of the uppermost streamline shown is some positive value ψ_{upper}. Discuss the value of the stream function inside the separation bubble. In particular, is it positive or negative? Why? Where in the flow is ψ a minimum?

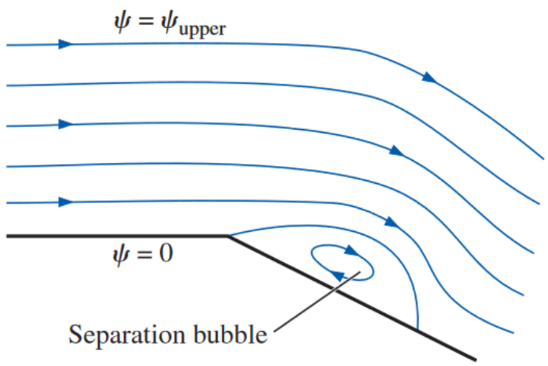

Figure 2: Flow separation

Solution

Notice from the figure that the $\Phi = 0$ streamline forms the top of the separation bubble. Since the difference of stream function between two streamlines is equal to the mass flow rate. Since the mass flow rate between the top of the bubble and the streamline shown is positive, the stream function inside the bubble must be **negative**.

The minimum value of the stream function occurs at the center of the separation bubble, where the mass flow rate is zero. Thus, the stream function will have the most negative value at the **center** of the separation bubble.

Question 7

A steady, two-dimensional, incompressible flow field in the xy-plane has a stream function given by $\psi = ax^2 - by^2 + cx + dxy$, where a, b, c, and d are constants.

(a) Obtain expressions for velocity components u and v,

(b) Verify that the flow field satisfies the incompressible continuity equation.

Solution

(a)

The velocity components are given by

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}$$

So,

$$\begin{aligned} u &= \frac{\partial}{\partial y}(ax^2 - by^2 + cx + dxy) \\ &= \boxed{-2by + dx} \end{aligned}$$

and

$$\begin{aligned} v &= -\frac{\partial}{\partial x}(ax^2 - by^2 + cx + dxy) \\ &= \boxed{-2ax - c + dy} \end{aligned}$$

(b)

The continuity equation for incompressible flow is given by

$$\begin{aligned} \nabla \cdot \vec{v} &= 0 \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \end{aligned}$$

Substituting the given expressions for u and v into the continuity equation, we have

$$\begin{aligned} \frac{\partial}{\partial x}(-2by + dx) + \frac{\partial}{\partial y}(-2ax - c + dy) &= 0 \\ \boxed{d - d = 0} \end{aligned}$$

Thus, the flow field satisfies the incompressible continuity equation.