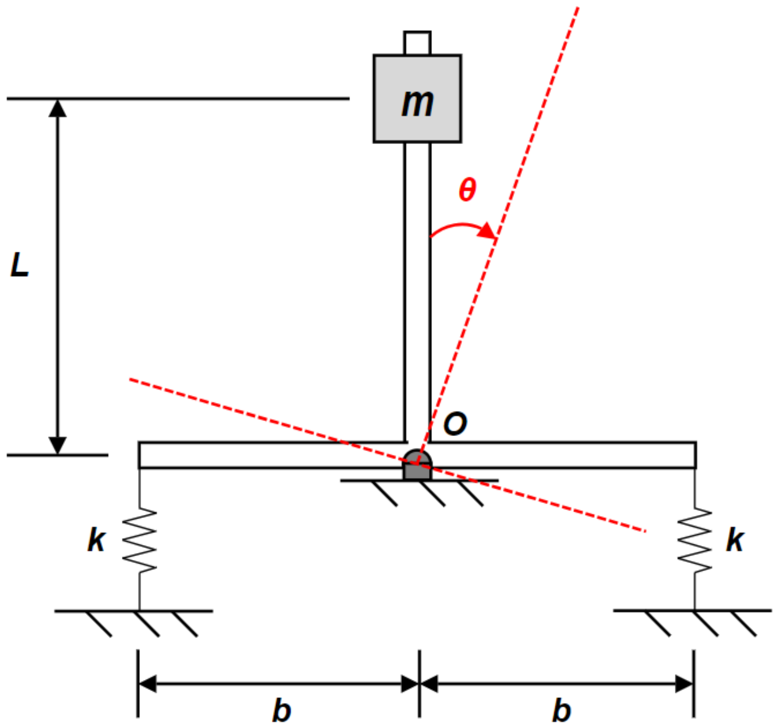
Question 1 (10 points)

The metronome shown is used as a timing device for musicians. The essence of the device is an inverted 'T' mounted on springs mounted on springs. The natural frequency of the device is varied by moving the mass m to different heights L.



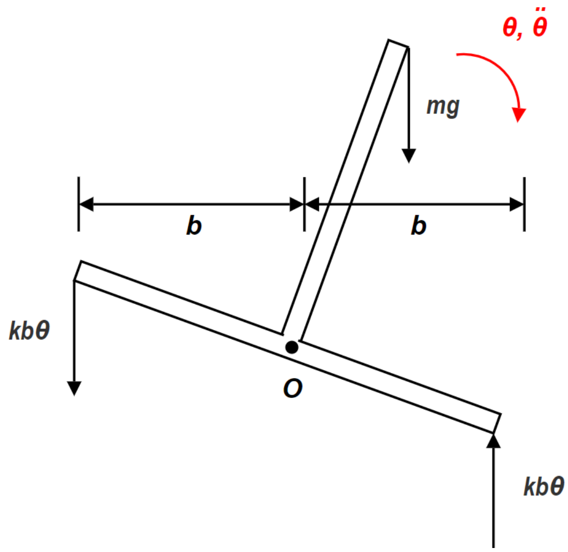
Assuming that the system undergoes only small oscillations about the axis through point O (the static equilibrium configuration is at $\theta = 0$):

a) **(5 pts)** Determine the equation of motion of the system **using θ as your coordinate**. Neglect the mass of the 'T' bar. Be sure to include the force of gravity as there is no static deflection of the springs in the static equilibrium configuration.

b) **(5 pts)** What is the limiting value of L so that the system is **stable**?

QUESTION 1 SOLUTION

PART A (5 pts)



Small oscillations:

$$b\cos(\theta) \cong b, \qquad L\sin(\theta) \cong L$$

$$\circlearrowright +\sum M_O = mL^2\ddot{\theta} = -2kb^2\theta + mgL\,\theta$$

The equation of motion is therefore:

$$mL^2\ddot{\theta} + (2kb^2 - mgL)\theta = 0$$

PART B (5 pts)

For a stable system, $p \geq 0$

$$p = \sqrt{\frac{k_{eff}}{m_{eff}}} = \sqrt{\frac{2kb^2 - mgL}{mL^2}} \geq 0$$

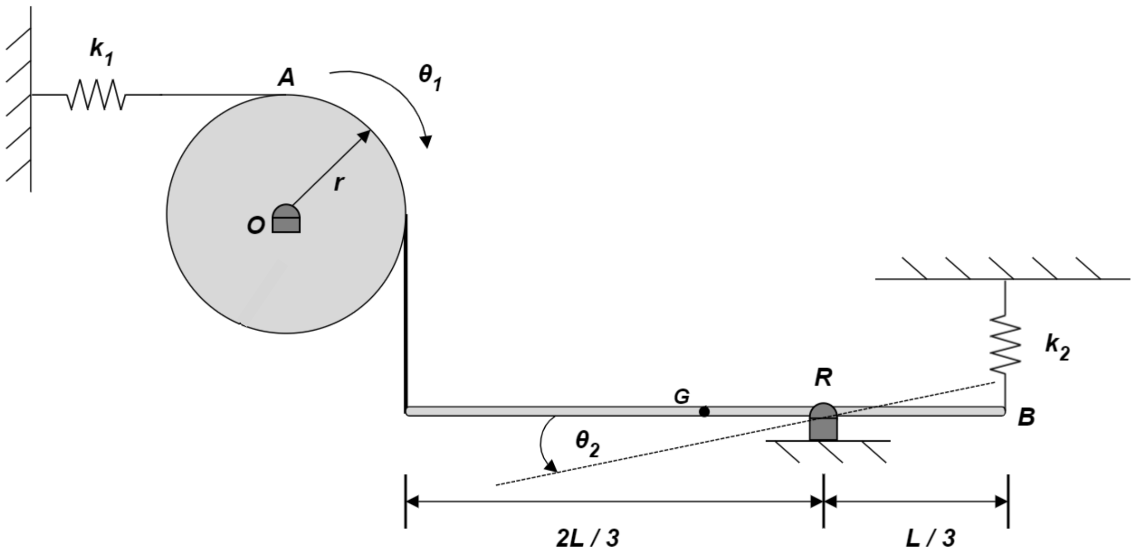
Therefore:

$$2kb^2 \geq mgL$$

$$L \leq \frac{2kb^2}{mg}$$

Question 2 (10 points)

The system shown below consists of a pulley (radius r) pinned about point O, which is connected to a uniform rod (length L) that is pinned about point R. The system is supported by two springs k_1 and k_2, connected to points A and B, respectively. The cable connecting the pulley and rod can be considered approximately inextensible (ie. the pulley and rod are rigidly connected). Assume small oscillations.



a) **(5 pts)** Determine the effective stiffness of the system with respect to θ_2 **using the stiffness approach,** by setting $\theta_2 = 1$.

b) **(5 pts)** Assume that a downward force of 5 N is applied at the rod's centre of mass (point G) while the system is initially as shown above. Using your answer from part a), determine the resulting compression of spring k_2 if $r = 0.5$ m, $L = 3$ m, and $k_1 = k_2 = 100$ N/m.

QUESTION 2 SOLUTION

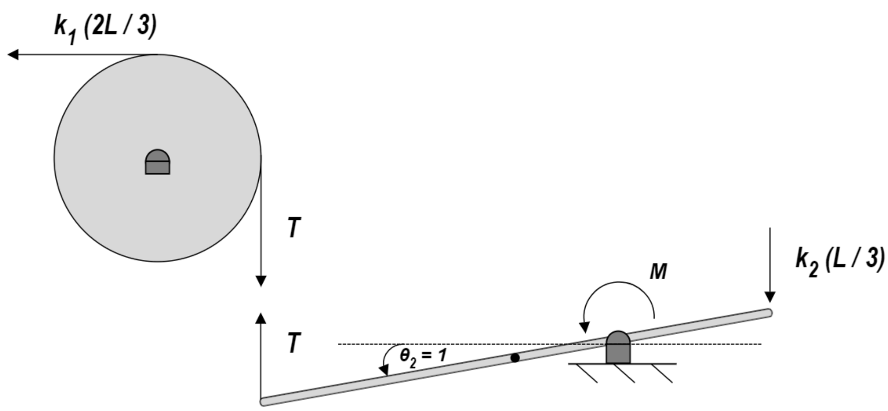
PART A) (5 pts)

For the stiffness approach, determine the moment M required at the coordinate θ_2 to maintain a corresponding applied unit rotation of $\theta_2 = 1$.

$$\delta_A = r\theta_1 = \frac{2L}{3}\theta_2, \qquad \delta_B = \frac{L}{3}\theta_2$$

$$\delta_A = \frac{2L}{3}, \qquad \delta_B = \frac{L}{3}$$

For a positive rotation of θ_2, spring k_1 is elongated (pulley will rotate CW) and spring k_2 is compressed (point B moves upwards).



For the disk:

$$\circlearrowright +\sum M_O = 0 = Tr - k_1 r\left(\frac{2L}{3}\right)$$

$$T = k_1\left(\frac{2L}{3}\right)$$

For the rod:

$$\circlearrowleft +\sum M_R = 0 = M - k_2\left(\frac{L}{3}\right)\left(\frac{L}{3}\right) - T\left(\frac{2L}{3}\right)$$

$$M = T\left(\frac{2L}{3}\right) + k_2\left(\frac{L^2}{9}\right)$$

Substituting in the expression for T:

$$M = k_1\left(\frac{4L^2}{9}\right) + k_2\left(\frac{L^2}{9}\right)$$

With the stiffness method, the effective stiffness for an applied moment M at θ_2 is:

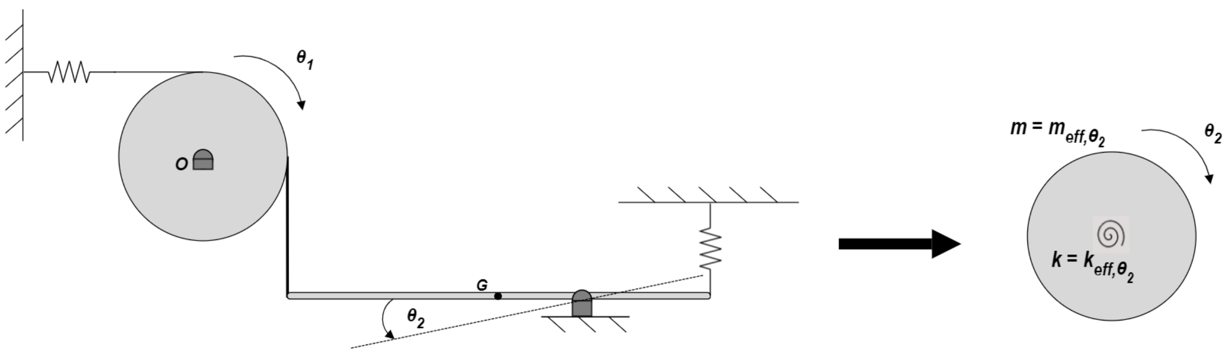
$$M = k_{eff,\theta_2}\theta_2 = k_{eff,\theta_2}(1)$$

$$k_{eff,\theta_2} = \frac{(4k_1 + k_2)L^2}{9}$$

PART B) (5 pts)

For a specific coordinate, any system can be simplified as a single mass-spring system with the effective mass and stiffness of the original system. For a rotational coordinate, this corresponds to a rotating body ($m = m_{eff}$) that is supported by a torsional spring ($k = k_{eff}$).

Using θ_2:



Original System *Equivalent System*

Since the applied force creates a moment that is applied at θ_2 (moment about the pin), the resultant θ_2 from the applied force can be found using the effective stiffness with respect to θ_2 from part a).

The applied force of 5 N at point G creates a moment about the pin equal to:

$$M_R = (5\text{ N})\left(\frac{2L}{3} - \frac{L}{2}\right) = 2.5\text{ Nm}$$

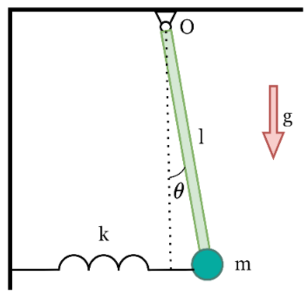
The elongation of spring k_2 can be found from the rotation of the rod resulting from the moment created by the applied force about the pin:

$$k_{eff,\theta_2} = \frac{(4k_1 + k_2)L^2}{9} = \frac{(400\text{ N/m} + 100\text{ N/m})(3\text{ m})^2}{9} = 500\text{ Nm}$$

$$\theta_2 = \frac{M_R}{k_{eff,\theta_2}} = \frac{2.5}{500}\text{ Nm} = 0.005\text{ rad}$$

$$\delta_B = \frac{L}{3}\theta_2 = 0.005\text{ m}$$

Question 3. A machine component is modeled as a system comprising a pendulum and a spring, illustrated in Fig 1. This setup combines gravitational and spring forces, with the pendulum contributing rotational dynamics influenced by gravity, and the spring providing a proportional restoring force. The geometric arrangement and interconnection of these components are depicted in Fig 1. (**Assumption:** the rotation is small enough so that the spring only deflects horizontally.)



Solution:

a) Derive the equation of motion for this system using energy method.

The energies of the system will be calculated as:

$$T = \frac{1}{2}mv^2 = \frac{1}{2}ml^2\dot{\theta}^2, U_{spring} = \frac{1}{2}kl^2\theta^2, U_m = mgl(1 - \cos(\theta)) \tag{1}$$

Therefore, using the energy method:

$$\frac{d}{dt}(T + U_{spring} + U_m) = \frac{d}{dt}\left(\frac{1}{2}ml^2\dot{\theta}^2 + \frac{1}{2}kl^2\theta^2 + mgl(1 - \cos(\theta))\right) = 0 \tag{2}$$

By taking time-derivative of the Eq. (2):

$$\frac{d}{dt}(T + U_{spring} + U_m) = ml^2\ddot{\theta} + kl^2\theta + mgl\sin(\theta) = 0 \tag{3}$$

b) Linearize the equation of motion and calculate the natural frequency of the system as function of k, m, l, g, and θ.

The linearized of the Eq. (3), for the small oscillations, can be written as:

$$ml^2\ddot{\theta} + (kl^2 + mgl)\theta = 0 \tag{4}$$

c) Compare the natural frequency of this system with the case where there is no spring.

From the Eq. (4) it can be seen that for the case where there is a spring the natural frequency of the system will be as follows.

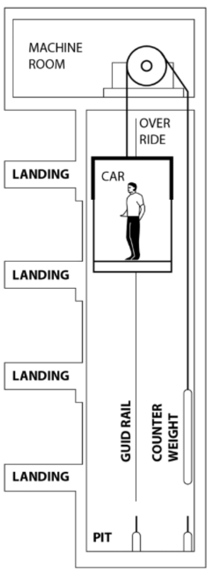
$$p = \sqrt{\frac{(kl^2 + mgl)}{ml^2}} \tag{5}$$

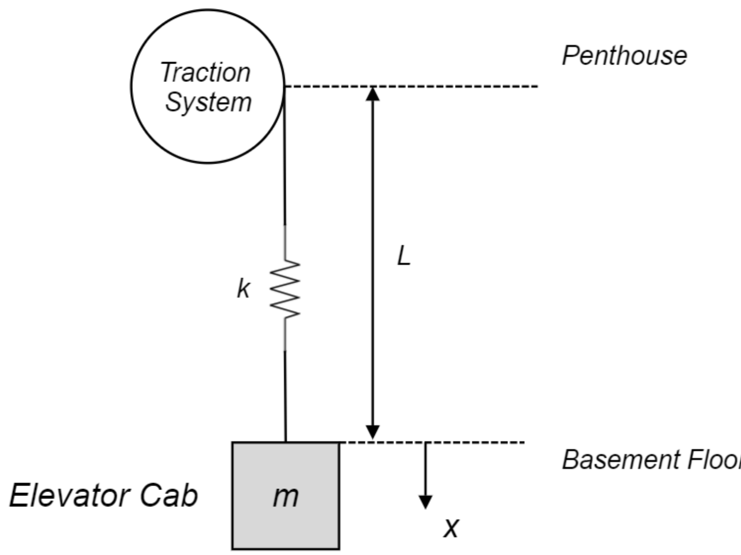
By setting $k = 0$ in the above equation the natural frequency of the system without spring will be obtained as:

$$p = \sqrt{\frac{g}{l}} \tag{6}$$

Question 4 (10 points)

The figure shows a traction elevator system used in high-rise residential buildings. These traction elevators consist of hoist cables connected to the top of the cab operated by a traction machine (electric motor) located in the penthouse. The system is modelled as a simple spring-mass system, where the spring represents the cable stiffness and the mass corresponds to the elevator cab and its occupants (counterweights are neglected).





The elevator provides a rapid ascent/descent, while not causing excessive acceleration to the passengers or stress in the cable system. The situation under consideration is the stop after descent to the basement floor level for a 10-story apartment building (assume 3.5 m per story). Assume that the traction motor stops instantly when reaching the basement floor (acts as a fixed support). The velocity of the cab before stopping is 1.5 m/s. The cables have an equivalent stiffness of a single cable with a radius of 2 cm and an elastic modulus of 100 GPa $\left(k = \frac{EA}{L}\right)$.

a) Consider both cases with an unloaded cab (mass of 1.2 metric tonnes) and with a maximum capacity of 15 people with an average weight of 70 kg each. For each case, determine:

   i. **(1 pt)** The overshoot of the cab past the basement floor level after stopping.

   ii. **(1 pt)** The maximum acceleration felt by the occupants.

   iii. **(1 pt)** The maximum stress in the cables.

b) **(3 pts)** To reduce the maximum tension in the cables and acceleration of the cab, a coil spring ($k = 600$ kN/m) is inserted between the cable attachment and the cab. How does this change the maximum displacement, acceleration, and stress for both the loaded and unloaded cases?

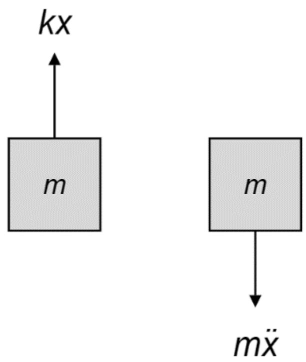
The results for the vibration analysis of the original elevator system **(no coil spring)** was done under the assumption of no damping. However, the system components have an inherent damping. a test was done on an **UNLOADED cab** and it was found that the cab's oscillation amplitude decreased by 50% in two cycles.

c) **(2 pts)** Determine the damping ratio for the **LOADED case** assuming viscous damping.

d) **(2 pts)** For the **LOADED** case, estimate how much time is needed after reaching the ground floor so that the passengers feel virtually no vibration of the elevator cab. Assume that the vibrations essentially stop when the amplitude decreases to 8% of its maximum value. **Hint:** recall that the logarithmic decrement is measured between subsequent peaks, so you must account for the time from $t = 0$ to the first peak.

QUESTION 4 SOLUTION

PART A) (3 pts)



The equation of motion is:

$$m\ddot{x} + kx = 0$$

With initial conditions:

$$x_0 = 0, \qquad v_0 = 1.5 \text{ m/s}$$

The solution to the equation of motion is therefore:

$$x(t) = \frac{v_0}{p}\sin(pt)$$

$$\ddot{x}(t) = -v_0 p \sin(pt)$$

The overshoot of the cab corresponds to the maximum displacement of the cab with respect to ground level.

The effective stiffness of the cables can be calculated:

$$k = \frac{EA}{L}$$

For a 10-floor apartment with 3.5 m per story:

$$L = 10(3.5 \text{ m}) = 35 \text{ m}$$

$$k = \frac{(100 \times 10^9 \text{ GPa})(\pi(0.02 \text{ m})^2)}{35 \text{ m}} = 3{,}590{,}392 \text{ N/m}$$

The maximum stress in the cables is due to both the **cable elongation at the maximum displacement of the cab <u>AND</u> the static equilibrium stretch of the cable which balances the weight of the cab and its occupants.**

$$\sigma_{max} = \frac{kx_{max} + mg}{A}$$

<u>For the unloaded case:</u>

$$m_1 = 1200 \text{ kg}$$

$$p_1 = \sqrt{\frac{k}{m}} = 54.7 \text{ rad/s}$$

$$x_{1,max} = \frac{v_0}{p_1} = \frac{1.5 \text{ m/s}}{54.7 \text{ rad/s}} = 0.0274 \text{ m}$$

$$\ddot{x}_{1,max} = v_0 p_1 = 82.1 \text{ m/s}^2$$

$$\sigma_{1max} = \frac{(3590392 \text{ N/m})(0.0274 \text{ m}) + (1200 \text{ kg})(9.81 \text{ m/s}^2)}{\pi(0.02 \text{ m})^2} = 87.7 \text{ MPa}$$

<u>For the loaded case:</u>

$$m_2 = 1200 \text{ kg} + 15(70 \text{ kg}) = 2250 \text{ kg}$$

$$p_2 = \sqrt{\frac{k}{m}} = 39.95 \text{ rad/s}$$

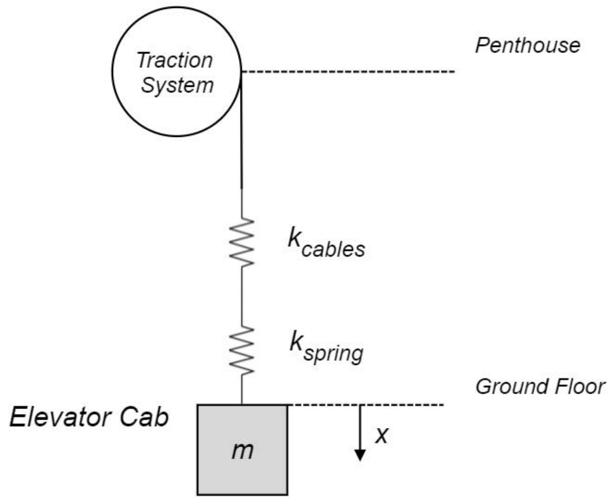
$$x_{2,max} = \frac{v_0}{p_2} = 0.0376 \text{ m}$$

$$\ddot{x}_{2,max} = v_0 p_2 = 59.9 \text{ m/s}^2$$

$$\sigma_{2max} = \frac{(3590392 \text{ N/m})(0.0274 \text{ m}) + (1200 \text{ kg})(9.81 \text{ m/s}^2)}{\pi(0.02 \text{ m})^2} = 124.8 \text{ MPa}$$

	Maximum Overshoot x_{max}	Max Acceleration	Max Cable Stress
Unloaded Case	2.74 cm	82.1 m/s^2	87.7 MPa
Loaded Case	3.76 cm	59.9 m/s^2	124.8 MPa

PART B) (3 pts)



With the addition of the coil spring, the effective stiffness (springs in series) is:

$$k_{eff} = \frac{k_{cables}k_{spring}}{k_{cables} + k_{spring}} = \frac{(3{,}590{,}392\ \mathrm{N/m})(600{,}000\ \mathrm{N/m})}{(3{,}590{,}392 + 600{,}000)\mathrm{N/m}}$$

$$k_{eff} = 514{,}089\ \mathrm{N/m}$$

Then, the calculations in part a) can be repeated with k_{eff} as the new spring stiffness:

	Maximum Overshoot x_{max}	Max Acceleration	Max Cable Stress
Unloaded Case	7.25 cm	$31.05\ \mathrm{m/s^2}$	39.0 MPa
Loaded Case	9.92 cm	$22.7\ \mathrm{m/s^2}$	58.2 MPa

PART C) (2 pts)

Logarithmic decrement:

$$\delta = \frac{1}{n}\ln\left(\frac{x_0}{x_n}\right) = \frac{1}{2}\ln(2) = 0.347$$

$$\delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

Rearranging for ζ:

$$\zeta = \sqrt{\frac{\delta^2}{4\pi^2 + \delta^2}}$$

$$\zeta = \sqrt{\frac{(0.347)^2}{4\pi^2 + (0.347)^2}} = 0.055$$

However, this is the damping ratio **for the UNLOADED case.** Since $m_1 \neq m_2$, $\zeta_1 \neq \zeta_2$.

The stiffness and damping of the system remains the same ($k_1 = k_2,\ c_1 = c_2$).

Therefore:

$$\zeta = \frac{c}{c_c} = \frac{c}{2mp}, \qquad c = 2\zeta mp$$

$$2\zeta_1 m_1 p_1 = 2\zeta_2 m_2 p_2$$

$$\zeta_2 = \left(\frac{m_1 p_1}{m_2 p_2}\right)\zeta_1 = \frac{(1200\ \mathrm{kg})\left(54.7\,\frac{\mathrm{rad}}{\mathrm{s}}\right)}{(2250\ \mathrm{kg})\left(39.95\,\frac{\mathrm{rad}}{\mathrm{s}}\right)}(0.055)$$

$$\zeta_2 = 0.402$$

PART D) (2 pts)

$$\delta = \frac{2\pi\zeta_2}{\sqrt{1-\zeta_2^2}} = \frac{1}{n}\ln\left(\frac{x_0}{x_n}\right)$$

$$n = \frac{\ln\left(\frac{x_0}{x_n}\right)\sqrt{1-\zeta_2^2}}{2\pi\zeta_2} = \frac{\ln\left(\frac{1}{0.08}\right)\sqrt{1-(0.402)^2}}{2\pi(0.402)}$$

$$n = 10 \text{ cycles}$$

Because the cab has only an initial velocity and zero initial displacement, the response is given as:

$$x(t) = \frac{v_0 e^{-\zeta p t}}{p\sqrt{1-\zeta^2}}\sin(\omega_d t)\,, \qquad \omega_d = p\sqrt{1-\zeta^2}$$

The response is simply an underdamped sine function with no phase difference. Therefore, there is a quarter of a cycle between $x(t=0) = 0$ and the first peak.

From when the brakes are applied to when the amplitude has reached 8% of its maximum value, 10.25 (damped) cycles have approximately elapsed.

$$t = (10.25 \text{ cycles}) \times \tau, \qquad \tau = \frac{2\pi}{p\sqrt{1-\zeta^2}}$$

$$t = 10.25\left(\frac{2\pi}{p\sqrt{1-\zeta^2}}\right)$$

$$t = 10.25\left(\frac{2\pi}{39.95\sqrt{1-(0.402)^2}}\right)$$

$$\boldsymbol{t = 1.61 \text{ s}}$$

Question 5. The system shown, represents a floor of mass M supported by springs of stiffness k. A mass m is dropped on the floor from a height of h. Calculate the motion of floor using x, the displacement from equilibrium configuration before mass m impacts. Determine the maximum displacement for the case when h=0. NOTE: Do not neglect the weight of the floor and the weight of mass m.

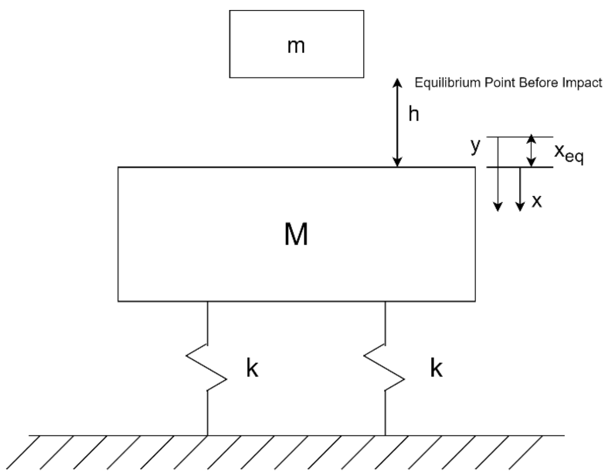


Fig1. The floor and mass system

Solution:

a) The equilibrium point prior to impact can be calculated as follows: (the equivalent stiffness is $2k$)

$$Mg = 2kX_{eq} \rightarrow X_{eq} = \frac{Mg}{2k} \tag{1}$$

b) The energies of the masses will be calculated as:

$$T = \frac{1}{2}(M+m)\dot{y}^2, U_{mass} = -(M+m)gy, U_{spings} = 2\frac{1}{2}ky^2 \tag{2}$$

Where $y = x + X_{eq}$.

By utilizing the energy method follow equation will be obtained.

$$\frac{dE}{dt} = \frac{d}{dt}(T+U) = 0 \rightarrow (M+m)\ddot{y} - (M+m)g + 2ky = 0 \tag{3}$$

Then, replace $y = x + X_{eq}$ and $X_{eq} = \frac{Mg}{2k}$ give the following equation of motion.

$$(M+m)\ddot{x}-(M+m)g+2k\left(x+\frac{Mg}{2k}\right)=0\rightarrow(M+m)\ddot{x}+2kx=mg \tag{4}$$

The solution to this equation can be written in the form of Eq (5):

$$x=Asin(pt)+Bcos(pt)+C \tag{5}$$

Afterward, to find the coefficients of the solution, the initial conditions need to be taken into account:

a) Initial position: at $t=0$, $x=0$, then:

$$A(0)+B(1)+C=0\rightarrow\ B=-C \tag{6}$$

b) Initial velocity: the velocity of m before impact can be calculated as:

$$mgh=\frac{1}{2}mv_0^2\rightarrow v_0=\sqrt{2gh} \tag{7}$$

Then, the velocity of $(M+m)$ immediately after impact will be $\frac{m\sqrt{2gh}}{M+m}$. Also, by taking the time-derivative of Eq (5) we can calculate the velocity as:

$$v=-pBsin(pt)+Apcos(pt)\rightarrow v(0)=Ap=\frac{m\sqrt{2gh}}{M+m}\rightarrow A=\frac{m\sqrt{2gh}}{p(M+m)} \tag{8}$$

Where $p=\sqrt{\frac{2k}{M+m}}$.

c) The particular solution of the Eq (4) gives:

$$C=\frac{mg}{2k}\rightarrow B=-\frac{mg}{2k} \tag{9}$$

Therefore, the solution of the equation of motion can be written as:

$$x=\frac{m\sqrt{2gh}}{p(M+m)}sin(pt)-\frac{mg}{2k}cos(pt)+\frac{mg}{2k} \tag{10}$$

Then, for the case where $h=0$:

$$x=\frac{mg}{2k}(1-cos(pt)) \tag{10}$$

It can be concluded that the maximum amplitude will be $\frac{mg}{k}$.