# 1 System Response for Unforced SDOF Systems

Table 1: Steady State Solutions for Unforced SDOF Systems

| System | Response |
|---|---|
| Undamped Spring Mass | $\frac{v_0}{p}\sin(pt) + x_0\cos(pt)$ |
| Damped Spring Mass | $e^{-\zeta pt}\left[\frac{v_0+\zeta p x_0}{\sqrt{1-\zeta^2}p}\sin(\sqrt{1-\zeta^2}pt) + x_0\cos(\sqrt{1-\zeta^2}pt)\right]$ |

# 2 Steady State Solutions for Forced SDOF Systems

Table 2: Steady State Solutions for Forced SDOF Systems

| System | Steady State | DMF (or Amplitude Response) | Transmissibility |
|---|---|---|---|
| | | Undamped Forced Spring Mass | |
| Forced Spring Mass | $\left(\frac{F_0}{k}\right)\left(\frac{1}{1-\left(\frac{\omega}{p}\right)^2}\right)\sin(\omega t)$ | $\frac{\mathbb{X}}{\delta_{\mathrm{ST}}} = \frac{1}{\left\|1-\left(\frac{\omega}{p}\right)^2\right\|}$ | $\frac{F_{T_{\max}}}{F_0} = \frac{1}{\left\|1-\left(\frac{\omega}{p}\right)^2\right\|}$ |
| Rotating Imbalance | $\frac{F_0/k}{\left\|1-\left(\frac{\omega}{p}\right)^2\right\|}\sin(\omega t-\phi)$ | $\frac{M\mathbb{X}}{\tilde{m}e} = \frac{\left(\frac{\omega}{p}\right)^2}{\left\|1-\left(\frac{\omega}{p}\right)^2\right\|}$ | $\frac{F_{T_{\max}}}{\tilde{m}e\omega^2} = \frac{1}{\left\|1-\left(\frac{\omega}{p}\right)^2\right\|}$ |
| Base Excitation | $a\left(\frac{1}{1-\left(\frac{\omega}{p}\right)^2}\right)\sin(\omega t)$ | $\frac{\mathbb{X}}{a} = \frac{1}{\left\|1-\left(\frac{\omega}{p}\right)^2\right\|}$ | $\frac{F_{T_{\max}}}{ka} = \frac{\left(\frac{\omega}{p}\right)^2}{\left\|1-\left(\frac{\omega}{p}\right)^2\right\|}$ |
| | | Damped Forced Spring Mass | |
| Forced Spring Mass | $\frac{F_0/k}{\sqrt{\left[1-\left(\frac{\omega}{p}\right)^2\right]^2+\left(2\zeta\frac{\omega}{p}\right)^2}}$ | $\frac{\mathbb{X}}{\delta_{\mathrm{ST}}} = \frac{1}{\sqrt{\left[1-\left(\frac{\omega}{p}\right)^2\right]^2+\left(2\zeta\frac{\omega}{p}\right)^2}}$ | $\frac{F_{T_{\max}}}{F_0} = \frac{\sqrt{1+\left(2\zeta\frac{\omega}{p}\right)^2}}{\sqrt{\left[1-\left(\frac{\omega}{p}\right)^2\right]^2+\left(2\zeta\frac{\omega}{p}\right)^2}}$ |
| Rotating Imbalance | $\frac{\tilde{m}e\omega^2/k}{\sqrt{\left[1-\left(\frac{\omega}{p}\right)^2\right]^2+\left(2\zeta\frac{\omega}{p}\right)^2}}$ | $\frac{M\mathbb{X}}{\tilde{m}e} = \frac{\left(\frac{\omega}{p}\right)^2}{\sqrt{\left[1-\left(\frac{\omega}{p}\right)^2\right]^2+\left(2\zeta\frac{\omega}{p}\right)^2}}$ | $\frac{F_{T_{\max}}}{\tilde{m}e\omega^2} = \frac{\sqrt{1+\left(2\zeta\frac{\omega}{p}\right)^2}}{\sqrt{\left[1-\left(\frac{\omega}{p}\right)^2\right]^2+\left(2\zeta\frac{\omega}{p}\right)^2}}$ |
| Base Excitation | $\frac{a\sqrt{k^2+(c\omega)^2}/k}{\sqrt{\left[1-\left(\frac{\omega}{p}\right)^2\right]^2+\left(2\zeta\frac{\omega}{p}\right)^2}}$ | $\frac{\mathbb{X}}{a} = \frac{\sqrt{1+\left(2\zeta\frac{\omega}{p}\right)^2}}{\sqrt{\left[1-\left(\frac{\omega}{p}\right)^2\right]^2+\left(2\zeta\frac{\omega}{p}\right)^2}}$ <br> $\frac{\mathbb{Z}}{a} = \frac{\left(\frac{\omega}{p}\right)^2}{\sqrt{\left[1-\left(\frac{\omega}{p}\right)^2\right]^2+\left(2\zeta\frac{\omega}{p}\right)^2}}$ | $\frac{F_{T_{\max}}}{ka} = \frac{\left(\frac{\omega}{p}\right)^2\sqrt{1+\left(2\zeta\frac{\omega}{p}\right)^2}}{\sqrt{\left[1-\left(\frac{\omega}{p}\right)^2\right]^2+\left(2\zeta\frac{\omega}{p}\right)^2}}$ |

General solution for forced damped SDOF system [1]

$$m_{\mathrm{eff}}\ddot{x} + c_{\mathrm{eff}}\dot{x} + k_{\mathrm{eff}}x = F_0\sin(\omega t-\alpha)$$

is given by

$$x(t) = \mathbb{X}\sin(\omega t-\alpha-\phi)$$

[1] $F_0$ for spring, imbalance, and excitation is $F_0$, $me\omega^2$, and $a\sqrt{k^2+(c\omega)^2}$, respectively.

where

$$\mathbb{X} = \frac{F_0}{\sqrt{(k_{\text{eff}} - m_{\text{eff}}\omega^2)^2 + (c_{\text{eff}}\omega)^2}} = \frac{F_0/k}{\sqrt{\left[1 - \left(\frac{\omega}{p}\right)^2\right]^2 + \left(2\zeta\frac{\omega}{p}\right)^2}}$$

## 3 Steady State Solutions for Forced Non-Harmonic SDOF Systems

The general steady state solution for a periodic ($\tau = 2\pi/\omega$) forced damped SDOF system is given by:

$$\begin{aligned} x(t) = \frac{a_0}{2k} &+ \sum_{j=1}^{\infty} \frac{a_j/k}{\sqrt{\left[1 - \left(\frac{\omega}{p}\right)^2\right]^2 + \left(2\zeta\frac{\omega}{p}\right)^2}} \cos(j\omega t - \phi_j) \\ &+ \sum_{j=1}^{\infty} \frac{b_j/k}{\sqrt{\left[1 - \left(\frac{\omega}{p}\right)^2\right]^2 + \left(2\zeta\frac{\omega}{p}\right)^2}} \sin(j\omega t - \phi_j) \end{aligned}$$

where

$$\begin{aligned} \phi_j &= \tan^{-1}\left[\frac{2\zeta\frac{\omega}{p}}{1 - \left(\frac{j\omega}{p}\right)^2}\right] \\ a_0 &= \frac{2}{\tau}\int_0^{\tau} F(t)dt = 2F_{\text{avg}} \\ a_j &= \frac{2}{\tau}\int_0^{\tau} F(t)\cos(j\omega t)dt, \quad j = 1, 2, 3, \ldots \\ b_j &= \frac{2}{\tau}\int_0^{\tau} F(t)\sin(j\omega t)dt, \quad j = 1, 2, 3, \ldots \end{aligned}$$

Todo: add common solutions such as step, ramp. Ask TA if you can infer some coefficients are zero based on some symmetry.

## 4 Transient Response of Spring-Mass Systems

Table 3: Particular Response of Spring-Mass Systems

| Input | $F_0(t)$ | Particular Response |
|---|---|---|
| Step | $F_0(t) = F_1$ | $x_p(t) = \dfrac{F_1}{k}$ |
| Ramp | $F_0(t) = \beta t$ | $x_p(t) = \dfrac{\beta}{k}t$ |
| Exponential | $F_0(t) = F_1 e^{-at}$ | $x_p(t) = \dfrac{F_1}{ma^2 + k}e^{-at}$ |

## 5 Multiple Degree of Freedom Systems

### 5.1 Unforced Undamped Systems

For an $N$ degree of freedom system,

- $N$ equations of motion are required to describe the system.
- $N$ natural frequencies and mode shapes are obtained from the eigenvalue problem.

The general form of the equation of motion for an $N$ degree of freedom system is given by

$$\mathbf{m_{eff}}\ddot{\mathbf{q}} + \mathbf{k_{eff}}\mathbf{q} = \mathbf{0}$$

where

- $\mathbf{m_{eff}}$ is a $N \times N$ mass matrix,
- $\mathbf{k_{eff}}$ is a $N \times N$ stiffness matrix,
- $\mathbf{q}$ is a $N \times 1$ vector of generalized coordinates,

## 6 Transverse Vibrations in Cable

For a cable along the $x$-axis with deflected shape $y(x,t)$, the partial differential equation governing the transverse vibrations is given by

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

where $c$ is the wave speed in the cable,

$$c = \sqrt{\frac{T}{\rho}}$$

where $T$ is the tension in the cable and $\rho$ is the mass density of the cable. By separation of variables, the general solution to the wave equation with pinned boundary conditions is given by

$$y(x,t) = \sum_{n=1}^{\infty}\left[A_n \sin\left(\frac{nc\pi}{L}t\right) + B_n \cos\left(\frac{nc\pi}{L}t\right)\right]\sin\left(\frac{n\pi}{L}x\right)$$

where $L$ is the length of the cable. The coefficients $A_n$ and $B_n$ can be determined by the initial conditions using inner product. For initial conditions

$$y(x,0) = y_0(x) \quad \text{and} \quad \frac{\partial y}{\partial t}(x,0) = \dot{y}_0(x)$$

the coefficients are given by

$$A_n = \frac{2}{n\pi c}\int_0^L \dot{y}_0(x)\sin\left(\frac{n\pi}{L}x\right)dx$$
$$B_n = \frac{2}{L}\int_0^L y_0(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

The resonant frequencies of the cable are given by

$$p = \frac{nc\pi}{L}, \quad n = 1, 2, 3, \ldots$$