Question 1

For the two degree of freedom system shown, determine:

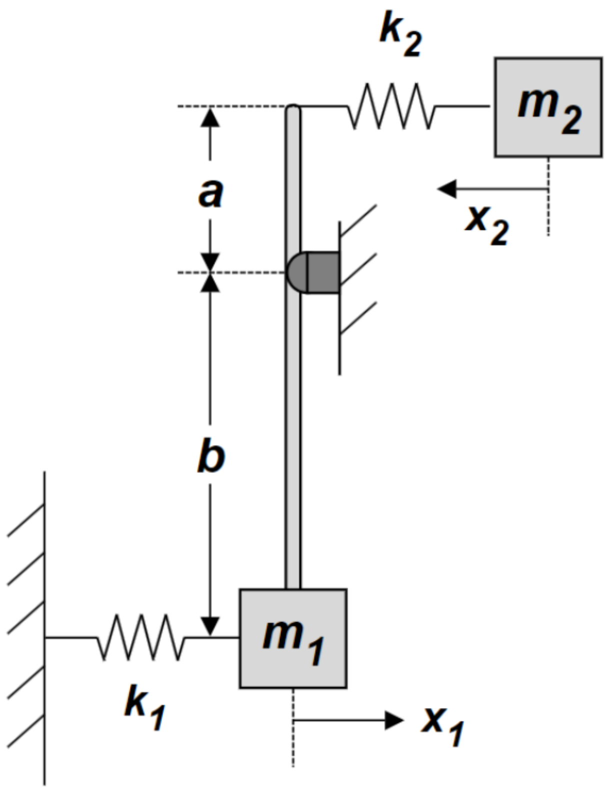

Figure 1: Two degree of freedom system.

(a)

The flexibility influence coefficients, starting from the definition.

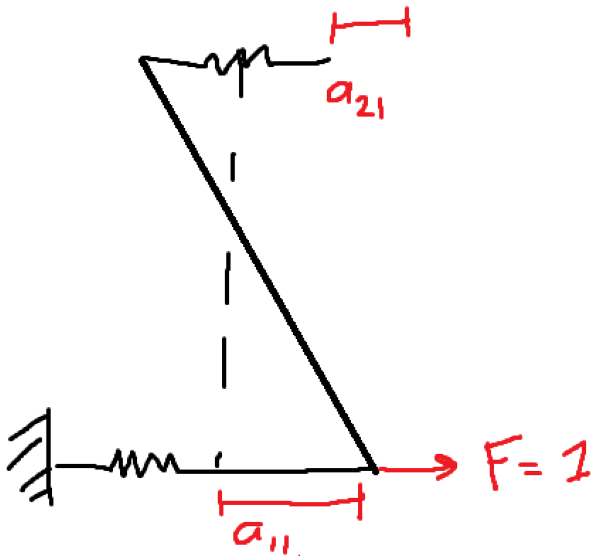

Figure 2: Resulting Displacements from Unit Force at Point 1.

By definition, a_{ij} is the displacement at point i due to a unit force at point j, where all other coordinates are allowed to freely move. Starting with a unit force at point 1, the resulting displacements are shown in Figure 2.

Since all other coordinates are allowed to freely move, the displacement of a_{11} is then

$$a_{11} = \frac{1}{k_1}$$

and the displacement of a_{21} is pure rotation about the pin, so

$$a_{21} = \frac{a}{b} a_{11} = \frac{a}{b} \frac{1}{k_1}$$

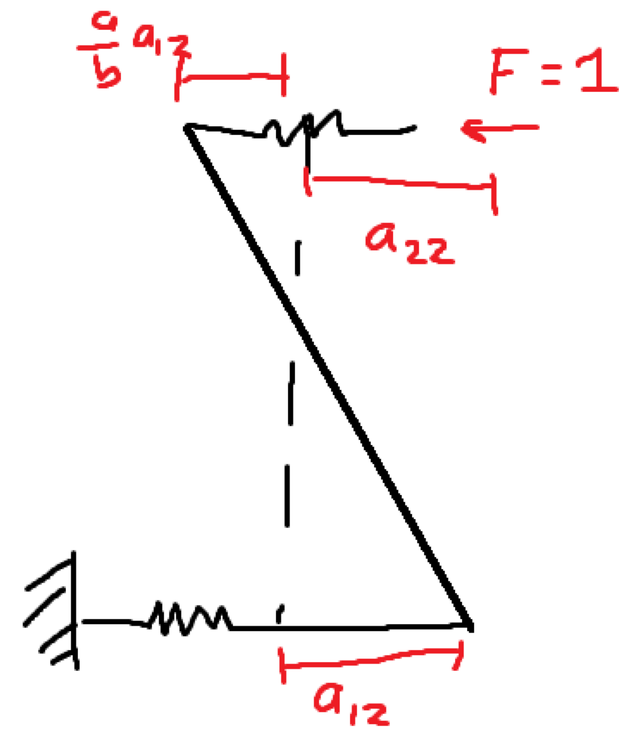

Figure 3: Resulting Displacements from Unit Force at Point 2.

Next, with a unit force at point 2, the resulting displacements are shown in Figure 3. By Maxwell's Reciprocity Theorem, the displacement of a_{12} is then

$$a_{12} = a_{21} = \frac{a}{b}\frac{1}{k_1}$$

and the displacement of a_{22} is

$$a_{22} = \frac{a}{b}a_{12} + \frac{1}{k_2} = \frac{a}{b}\frac{a}{b}\frac{1}{k_1} + \frac{1}{k_2}$$

Then, the flexibility influence is

$$\boxed{\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \frac{1}{k_1} & \frac{a}{b}\frac{1}{k_1} \\ \frac{a}{b}\frac{1}{k_1} & \frac{a^2}{b^2}\frac{1}{k_1} + \frac{1}{k_2} \end{bmatrix}}$$

(b)

The stiffness influence coefficients, starting from the definition.

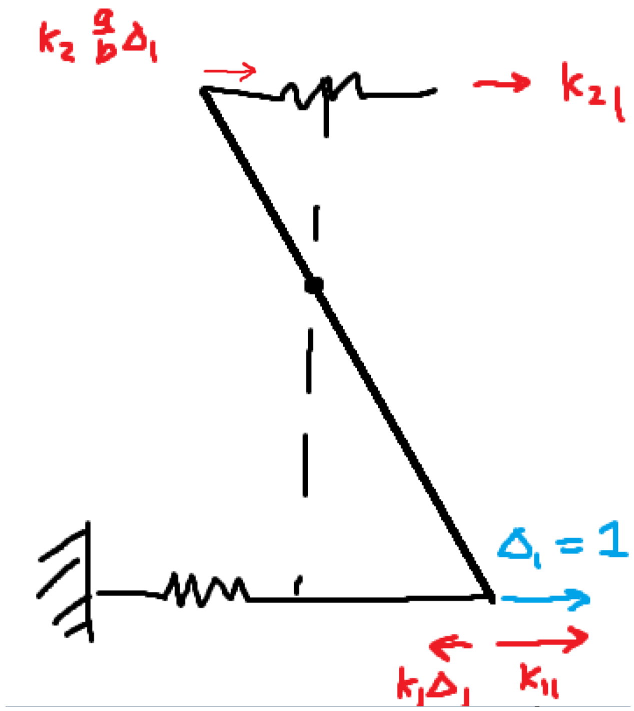

Figure 4: Resulting Forces from Unit Displacement at Point 1.

By definition, k_{ij} is the force at point i due to a unit displacement at point j, where all other coordinates are held fixed. Starting with a unit displacement at point 1, the resulting forces are shown in Figure 4.

Since all other coordinates are held fixed, the force at k_{11} can be determined by taking the moment about the pin, which gives,

$$\circlearrowleft \sum M_o := 0$$
$$\implies k_{11} = k_1 + \frac{a}{b} k_2$$

and the force at k_{21} is then,

$$k_{21} = -\frac{a}{b} k_1 = -\frac{a}{b} k_1$$

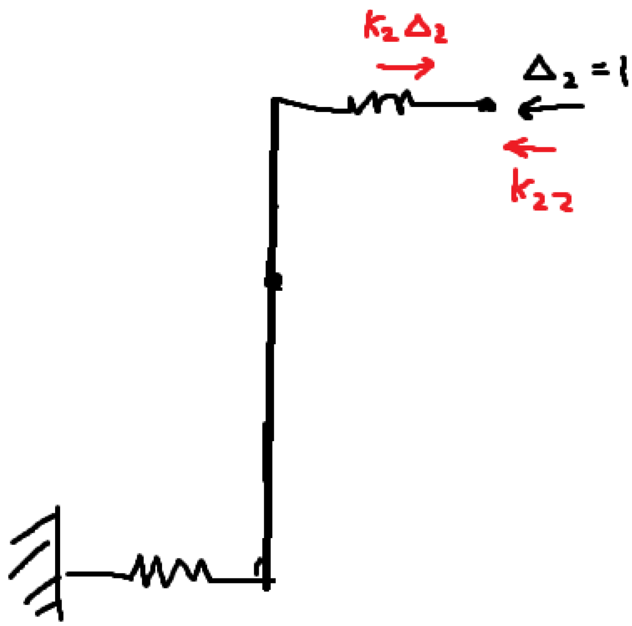

Figure 5: Resulting Forces from Unit Displacement at Point 2.

Next, with a unit displacement at point 2, the resulting forces are shown in Figure 5. By Maxwell's Reciprocity Theorem, the force at k_{12} is then,

$$k_{12} = k_{21} = -\frac{a}{b}k_1$$

and the force at k_{22} is then,

$$k_{22} = k_2$$

Then, the stiffness influence is

$$\boxed{\mathbf{K} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} = \begin{bmatrix} k_1 + \frac{a}{b}k_2 & -\frac{a}{b}k_1 \\ -\frac{a}{b}k_1 & k_2 \end{bmatrix}}$$

(c)

If $k_1 = k$, $k_2 = 2k$, $m_2 = m_1 = m$, and $b = 2a$, the equations of motion are:

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} \frac{3k}{2} & -k \\ -k & 2k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Determine the natural frequencies and mode shapes.

By Matlab,

```
syms k m
M = [m 0; 0 m];
K = [3*k/2 -k; -k 2*k];
[V, D] = eig(inv(M)*K);
p = diag(D);
simplify(p)
simplify(V)

>> ans =

(k*(17^(1/2) + 7))/(4*m)
-(k*(17^(1/2) - 7))/(4*m)


>> ans =

[1/4 - 17^(1/2)/4, 17^(1/2)/4 + 1/4]
[                1,                1]
```

Therefore,

$$\boxed{\begin{aligned} p_1 &= \frac{k}{4m}\left(7+\sqrt{17}\right) \\ p_2 &= \frac{k}{4m}\left(7-\sqrt{17}\right) \end{aligned}}$$

and

$$\boxed{\begin{aligned} \{\Phi\}^{①} &= \begin{bmatrix} \frac{1}{4}-\frac{\sqrt{17}}{4} \\ 1 \end{bmatrix} \\ \{\Phi\}^{②} &= \begin{bmatrix} \frac{1}{4}+\frac{\sqrt{17}}{4} \\ 1 \end{bmatrix} \end{aligned}}$$

Question 2

The installation of a magnetic resonance imaging (MRI) system requires the machine to be well isolated from vibration. The original isolation is shown where the support motion $y = Y_{sin}(\omega t)$ is due to the floor vibrating from the equipment in the room below. This equipment is assumed to run at 2400 rpm, and the natural frequency of the installation is estimated to be 15 Hz.

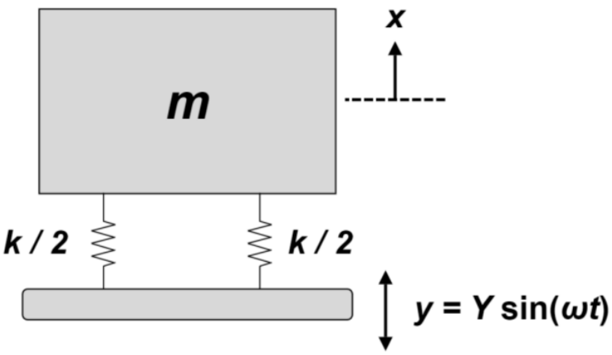

Figure 6: MRI system.

(a)

Estimate the amplitude of vibration of m compared to the amplitude of the floor motion.

This is a base excitation problem. The frequency is $\omega = \frac{2400}{60} = 40$ Hz. The natural frequency was given as $p = 15$ Hz. The amplitude compared to the floor motion is

$$\begin{aligned}\frac{\mathbb{X}}{a} &= \frac{\sqrt{1+\left(2\zeta\frac{\omega}{p}\right)^2}}{\sqrt{\left[1-\left(\frac{\omega}{p}\right)^2\right]^2+\left(2\zeta\frac{\omega}{p}\right)^2}} \\ &= \frac{\sqrt{1+\left(2\cdot 0\cdot\frac{40}{15}\right)^2}}{\sqrt{\left[1-\left(\frac{40}{15}\right)^2\right]^2+\left(2\cdot 0\cdot\frac{40}{15}\right)^2}} \\ &= \boxed{0.1636}\end{aligned}$$

(b)

As the calculation in (a) indicated that the motion of m was too large, the supplier suggests the installation of a damper to give a damping ratio of $\zeta = 0.2$. Will the damper reduce of increase the amplitude of motion? Calculate the difference the damper will make.

Since the machine is operating above resonance, the effects of damping should increase the amplitude of motion. To confirm this, we can apply the same equation as in part (a) with

$\zeta = 0.2$.

$$
\begin{aligned}
\frac{\mathbb{X}}{a} &= \frac{\sqrt{1+\left(2\zeta\frac{\omega}{p}\right)^2}}{\sqrt{\left[1-\left(\frac{\omega}{p}\right)^2\right]^2+\left(2\zeta\frac{\omega}{p}\right)^2}} \\
&= \frac{\sqrt{1+\left(2\cdot 0.2\cdot\frac{40}{15}\right)^2}}{\sqrt{\left[1-\left(\frac{40}{15}\right)^2\right]^2+\left(2\cdot 0.2\cdot\frac{40}{15}\right)^2}} \\
&= \boxed{0.2357}
\end{aligned}
$$

(c)

Another alternative is suggested by a trainee engineer who argued that the machine should be supported from above as well as from the vibration floor as illustrated (it is assumed that the upper support is fixed). Starting from a free body diagram, determine the equation of motion and an expression for the steady state amplitude of m. Compare the amplitude of motion of m for this case to the original design in (a) and the alternative design with the damper in (b).

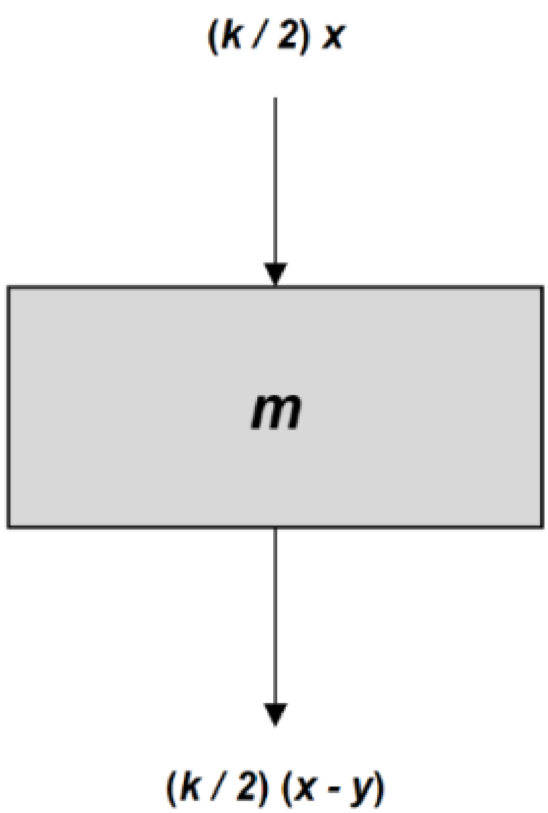

Figure 7: MRI system with upper support.

The equation of motion is then,

$$\begin{aligned}\uparrow \sum F_y &:= m\ddot{x} \\ &= -\frac{k}{2}x - \frac{k}{x}(x-y)\end{aligned}$$

Then the equation of motion is then,

$$\begin{aligned}m\ddot{x} + kx &= \frac{k}{2}y \\ m\ddot{x} + kx &= \underbrace{\frac{k}{2}Y}_{F_0}\sin(\omega t)\end{aligned}$$

Which has the amplitude,

$$\mathbb{X} = \frac{F_0/k}{\left|1 - \left(\frac{\omega}{p}\right)^2\right|} \tag{1}$$

$$\implies \frac{\mathbb{X}}{a} = \frac{\frac{1}{2}}{\left|1 - \left(\frac{40}{15}\right)^2\right|} \tag{2}$$

$$= \boxed{0.0818} \tag{3}$$

(d)

Another way to reduce the motion of m is to use a vibration absorber on the original design. i) If the absorber has a mass m_2 equal to 15% of the main mass m, calculate the amplitude of motion of the absorber mass at the operating speed (2400 rpm). The natural frequency of the large mass in isolation is known,

$$p_{11} = \sqrt{\frac{k}{m}} = 15 \text{ Hz}$$

To reduce vibrations, select the natural frequency of the absorber mass to be $p_{22} = \omega = 40$ Hz. The amplitude of motion of the absorber mass is then,

$$\begin{aligned}\frac{\mathbb{X}_2}{\mathbb{X}_0} &= -\left(\frac{p_{11}}{p_{22}}\right)^2 \frac{1}{\mu} \\ &= -\left(\frac{15}{40}\right)^2 \frac{1}{0.15} \\ &= \boxed{-0.9375}\end{aligned}$$

The natural frequencies can be determined by

$$\left(\frac{p_{22}}{p_{11}}\right)\left(\frac{p}{p_{22}}\right)^4 - [1+(1+\mu)\left(\frac{p_{22}}{p_{11}}\right)^2]\left(\frac{p}{p_{22}}\right)^2 + 1 = 0$$
$$\left(\frac{40}{15}\right)\left(\frac{40}{40}\right)^4 - [1+(1+0.15)\left(\frac{40}{15}\right)^2]\left(\frac{40}{40}\right)^2 + 1 = 0$$

The natural frequencies are then,

$$\boxed{\begin{aligned} p_1 &= 13.86 \text{ Hz} \\ p_2 &= 43.28 \text{ Hz} \end{aligned}}$$

Question 3

The system shown is assumed to have a rotating imbalance and is operated at 100 rpm and 200 rpm. At 100 rpm the steady state amplitude of vibration is 1 mm while at 200 rpm it is 5 mm.

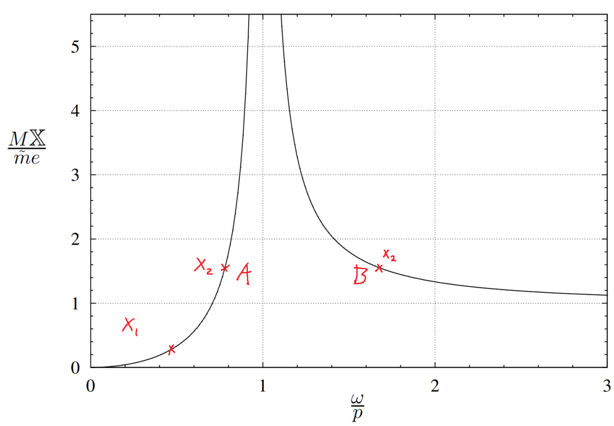

Figure 8: Frequency response of the system.

The system can be operating at either A or B for $\mathbb{X}_2 = 5$ mm. Assuming that the system is operating at A, the system is below resonance, which leads

$$\frac{M\mathbb{X}_2}{\tilde{m}e} = \frac{\left(\frac{\omega_2}{p}\right)^2}{1-\left(\frac{\omega_2}{p}\right)^2}$$

The system at $\mathbb{X}_1 = 1$ mm is operating below resonance, so,

$$\frac{M\mathbb{X}_1}{\tilde{m}e} = \frac{\left(\frac{\omega_1}{p}\right)^2}{1-\left(\frac{\omega_1}{p}\right)^2}$$

Solving for p,

$$p_A = 400 \text{ RPM}$$

If the system is operating at B, the system is above resonance, which leads

$$\frac{M\mathbb{X}_2}{\tilde{m}e} = \frac{1}{\left(\frac{\omega_2}{p}\right)^2 - 1}$$

Solving for p,

$$p_B = 163 \text{ RPM}$$

(a)

It is decided to operate the system shown at 200 rpm, however, the amplitude of vibration is too large, and a vibration absorber is to be added. Assuming that the natural frequency of the original system is at 150 rpm, what should the mass m_2 be as a fraction of m_1 to ensure that the lowest natural frequency of the combined system is 100 rpm?

Let the natural frequency of the mass added be $p_{22} = 200$, the frequency of the original system be $p_{11} = 150$, and the frequency of the combined system be $p_1 = 100$. Then this relation must be satisfied,

$$\left(\frac{p_{22}}{p_{11}}\right)\left(\frac{p}{p_{22}}\right)^4 - [1 + (1+\mu)\left(\frac{p_{22}}{p_{11}}\right)^2]\left(\frac{p}{p_{22}}\right)^2 + 1 = 0$$
$$\left(\frac{200}{150}\right)\left(\frac{100}{200}\right)^4 - [1 + (1+\mu)\left(\frac{200}{150}\right)^2]\left(\frac{100}{200}\right)^2 + 1 = 0$$

Solving for μ,

$$\mu = 0.9375$$

Question 4

The system shown is a two-dimensional approximation to an automobile.

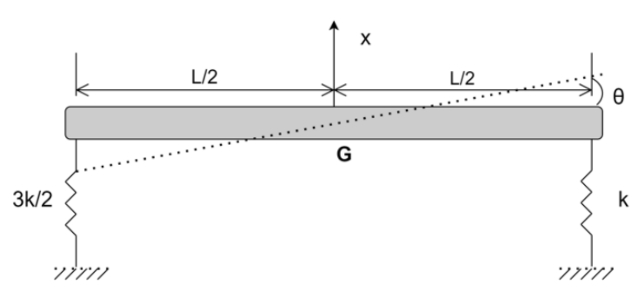

Figure 9: Two-dimensional automobile.

(a)

Using the coordinates shown determine the equations of motion.

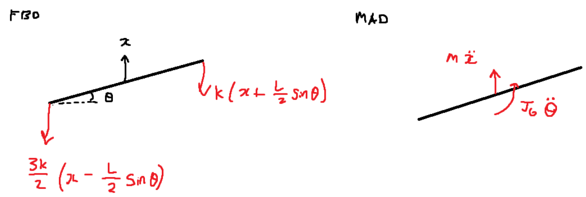

Figure 10: Freebody diagram and mass acceleration diagram.

The freebody diagram and mass acceleration diagram is shown in Figure 10. The equations of motion are then,

$$
\begin{aligned}
\uparrow \sum F_y := m\ddot{x} &= -\frac{3}{2}k(x - \frac{L}{2}\theta) - k(x + \frac{L}{2}\theta) \\
\implies m\ddot{x} &+ \frac{5}{2}kx - \frac{L}{4}k\theta = 0
\end{aligned}
$$

and,

$$
\begin{aligned}
\circlearrowleft \sum M_G := J_G\ddot{\theta} &= \frac{3}{2}k(x - \frac{L}{2}\theta)\frac{L}{2} - k(x + \frac{L}{2}\theta)\frac{L}{2} \\
\implies J_G\ddot{\theta} &+ \frac{5}{8}kL^2\theta - \frac{1}{4}kLx = 0
\end{aligned}
$$

The equations of motion are then,

$$
\boxed{
\begin{aligned}
m\ddot{x} + \frac{5}{2}kx - \frac{L}{4}k\theta &= 0 \\
J_G\ddot{\theta} + \frac{5}{8}kL^2\theta - \frac{1}{4}kLx &= 0
\end{aligned}
}
$$

(b)

If $J_G = \frac{mL^2}{6}$, the equations of motion become as what follows:

$$
\begin{bmatrix} m & 0 \\ 0 & \frac{mL^2}{6} \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} \frac{5}{2}k & -\frac{L}{4}k \\ -\frac{L}{4}k & \frac{5L^2}{8}k \end{bmatrix} \begin{Bmatrix} x \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}
$$

Determine the natural frequencies and mode shapes. With Matlab,

```
syms k m L
M = [m 0; 0 m*L^2/6];
K = [5/2*k -L/4*k; -L/4*k 5/8*L^2*k];
```

```
[V,D] = eig(inv(M)*K);
simplify(diag(D))
simplify(V)

>> ans =
(4*k)/m
(9*k)/(4*m)


>> ans =
[-L/6, L]
[   1, 1]
```

The natural frequencies and mode shapes are then,

$$\boxed{\begin{aligned} p_1 &= \frac{4k}{m} \\ p_2 &= \frac{9k}{4m} \\ \{\Phi^{①}\} &= \begin{bmatrix} -\frac{L}{6} \\ 1 \end{bmatrix} \\ \{\Phi^{②}\} &= \begin{bmatrix} L \\ 1 \end{bmatrix} \end{aligned}}$$

(c)

Sketch the mode shapes and clearly show the position on any nodes.

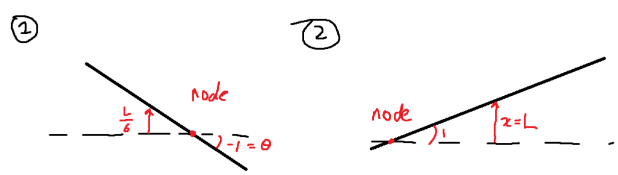

Figure 11: Mode shapes and nodes.

Question 5

The two degree of freedom system shown consists of a rigid bar AC of negligible mass, disc of mass m_1 which is pinned at C and rolls without slipping on the wall, and a mass m_2 suspended by a spring of stiffness k_2.

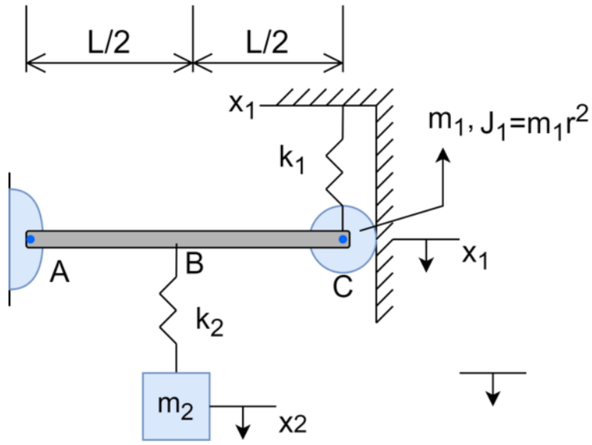

Figure 12: Two degree of freedom system

For the general case of m_1, m_2, k_1, k_2 determine the equations of motion. Be sure to include a detailed free-body diagram.

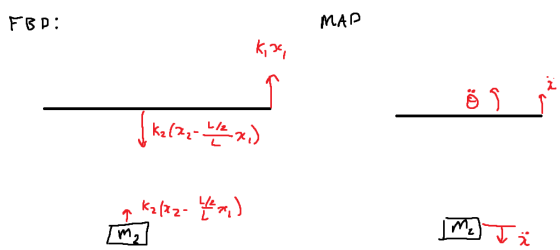

Figure 13: Free-body diagram

Summing the moment about the pivot point, A,

$$\circlearrowleft \sum M_A := J_A\alpha = (m_1r^2 + m_1L^2)\ddot{\theta}$$
$$= +k_2\left(x_2 - \frac{1}{2}x_1\right)\left(\frac{L}{2}\right) - k_1x_1L$$
$$= k_2x_2\frac{L}{2} - k_2\frac{L}{4}x_1 - k_1x_1L$$

using $\theta = x_1/r$,

$$(m_1r^2 + m_1L^2)\frac{\ddot{x}_1}{r} + k_1x_1L + k_2\frac{L}{4}x_1 - k_2x_2\frac{L}{2} = 0$$
$$\left[\frac{1}{r}\left(m_1r^2 + m_1L^2\right)\right]\ddot{x}_1 + \left[k_1L + k_2\frac{L}{4}\right]x_1 - k_2x_2\frac{L}{2} = 0$$

Summing the forces in the y direction of the mass, m_2,

$$
\begin{aligned}
\downarrow \sum F_y : &= m_2\ddot{x}_2 \\
&= -k_2(x_2 - \frac{L}{2}x_1)
\end{aligned}
$$

then,

$$
m_2\ddot{x}_2 + k_2x_2 - k_2\frac{L}{2}x_1 = 0
$$

The equations of motion are then,

$$
\begin{bmatrix} \left[\frac{1}{r}\left(m_1r^2 + m_1L^2\right)\right] & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1L + k_2\frac{L}{4} & -k_2\frac{L}{2} \\ \frac{-k_2L}{2} & k_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}
$$

(a)

If $k_1 = k_2 = k$ and $m_1 = m_2 = m$, determine the natural frequencies and corresponding mode shape. By Matlab,

```
syms k m L r
M = [1/r*(m*r^2 + m*L^2), 0; 0, m];
K = [k*L + k*L/4, -k*L/2; -k*L/2, k];
[V, D] = eig(inv(M)*K);
p = simplify(D)
simplify(V)
```

The natural frequencies are,

$$
\boxed{\begin{aligned} p_1 &= 0.412\frac{k}{m} \\ p_2 &= 1.213\frac{k}{m} \end{aligned}}
$$

And the corresponding mode shapes are,

$$
\boxed{\begin{aligned} \Phi^{1} &= \begin{Bmatrix} 0.851 \\ 1 \end{Bmatrix} \\ \Phi^{2} &= \begin{Bmatrix} -2.351 \\ 1 \end{Bmatrix} \end{aligned}}
$$