# 1   System Response for Unforced SDOF Systems

Table 1: Steady State Solutions for Unforced SDOF Systems

| System | Response |
|---|---|
| Undamped Spring Mass | $\dfrac{v_0}{p}\sin(pt) + x_0\cos(pt)$ |
| Damped Spring Mass | $e^{-\zeta pt}\left[\dfrac{v_0 + \zeta p x_0}{\sqrt{1-\zeta^2}\,p}\sin(\sqrt{1-\zeta^2}\,pt) + x_0\cos(\sqrt{1-\zeta^2}\,pt)\right]$ |

# 2   Steady State Solutions for Forced SDOF Systems

Table 2: Steady State Solutions for Forced SDOF Systems

| System | Steady State | DMF (or Amplitude Response) | Transmissibility |
|---|---|---|---|
| **Undamped Forced Spring Mass** | | | |
| Forced Spring Mass | $\left(\dfrac{F_0}{k}\right)\left(\dfrac{1}{1-\left(\frac{\omega}{p}\right)^2}\right)\sin(\omega t)$ | $\dfrac{\mathbb{X}}{\delta_{\mathrm{ST}}} = \dfrac{1}{\left|1-\left(\frac{\omega}{p}\right)^2\right|}$ | $\dfrac{F_{T_{\max}}}{F_0} = \dfrac{1}{\left|1-\left(\frac{\omega}{p}\right)^2\right|}$ |
| Rotating Imbalance | $\dfrac{F_0/k}{\left|1-\left(\frac{\omega}{p}\right)^2\right|}\sin(\omega t - \phi)$ | $\dfrac{M\mathbb{X}}{\tilde{m}e} = \dfrac{\left(\frac{\omega}{p}\right)^2}{\left|1-\left(\frac{\omega}{p}\right)^2\right|}$ | $\dfrac{F_{T_{\max}}}{\tilde{m}e\omega^2} = \dfrac{1}{\left|1-\left(\frac{\omega}{p}\right)^2\right|}$ |
| Base Excitation | $a\left(\dfrac{1}{1-\left(\frac{\omega}{p}\right)^2}\right)\sin(\omega t)$ | $\dfrac{\mathbb{X}}{a} = \dfrac{1}{\left|1-\left(\frac{\omega}{p}\right)^2\right|}$ | $\dfrac{F_{T_{\max}}}{ka} = \dfrac{\left(\frac{\omega}{p}\right)^2}{\left|1-\left(\frac{\omega}{p}\right)^2\right|}$ |
| **Damped Forced Spring Mass** | | | |
| Forced Spring Mass | $\dfrac{F_0/k}{\sqrt{\left[1-\left(\frac{\omega}{p}\right)^2\right]^2 + \left(2\zeta\frac{\omega}{p}\right)^2}}$ | $\dfrac{\mathbb{X}}{\delta_{\mathrm{ST}}} = \dfrac{1}{\sqrt{\left[1-\left(\frac{\omega}{p}\right)^2\right]^2 + \left(2\zeta\frac{\omega}{p}\right)^2}}$ | $\dfrac{F_{T_{\max}}}{F_0} = \dfrac{\sqrt{1+\left(2\zeta\frac{\omega}{p}\right)^2}}{\sqrt{\left[1-\left(\frac{\omega}{p}\right)^2\right]^2 + \left(2\zeta\frac{\omega}{p}\right)^2}}$ |
| Rotating Imbalance | $\dfrac{\tilde{m}e\omega^2/k}{\sqrt{\left[1-\left(\frac{\omega}{p}\right)^2\right]^2 + \left(2\zeta\frac{\omega}{p}\right)^2}}$ | $\dfrac{M\mathbb{X}}{\tilde{m}e} = \dfrac{\left(\frac{\omega}{p}\right)^2}{\sqrt{\left[1-\left(\frac{\omega}{p}\right)^2\right]^2 + \left(2\zeta\frac{\omega}{p}\right)^2}}$ | $\dfrac{F_{T_{\max}}}{\tilde{m}e\omega^2} = \dfrac{\sqrt{1+\left(2\zeta\frac{\omega}{p}\right)^2}}{\sqrt{\left[1-\left(\frac{\omega}{p}\right)^2\right]^2 + \left(2\zeta\frac{\omega}{p}\right)^2}}$ |
| Base Excitation | $\dfrac{a\sqrt{k^2+(c\omega)^2}/k}{\sqrt{\left[1-\left(\frac{\omega}{p}\right)^2\right]^2 + \left(2\zeta\frac{\omega}{p}\right)^2}}$ | $\dfrac{\mathbb{X}}{a} = \dfrac{\sqrt{1+\left(2\zeta\frac{\omega}{p}\right)^2}}{\sqrt{\left[1-\left(\frac{\omega}{p}\right)^2\right]^2 + \left(2\zeta\frac{\omega}{p}\right)^2}}$ | $\dfrac{F_{T_{\max}}}{ka} = \dfrac{\left(\frac{\omega}{p}\right)^2\sqrt{1+\left(2\zeta\frac{\omega}{p}\right)^2}}{\sqrt{\left[1-\left(\frac{\omega}{p}\right)^2\right]^2 + \left(2\zeta\frac{\omega}{p}\right)^2}}$ |
| | | $\dfrac{\mathbb{Z}}{a} = \dfrac{\left(\frac{\omega}{p}\right)^2}{\sqrt{\left[1-\left(\frac{\omega}{p}\right)^2\right]^2 + \left(2\zeta\frac{\omega}{p}\right)^2}}$ | |

General solution for forced damped SDOF system [1]

$$m_{\mathrm{eff}}\ddot{x} + c_{\mathrm{eff}}\dot{x} + k_{\mathrm{eff}}x = F_0\sin(\omega t - \alpha)$$

is given by

$$x(t) = \mathbb{X}\sin(\omega t - \alpha - \phi)$$

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[1] $F_0$ for spring, imbalance, and excitation is $F_0$, $me\omega^2$, and $a\sqrt{k^2+(c\omega)^2}$, respectively.

where

$$\mathbb{X} = \frac{F_0}{\sqrt{(k_{\text{eff}} - m_{\text{eff}}\omega^2)^2 + (c_{\text{eff}}\omega)^2}} = \frac{F_0/k}{\sqrt{\left[1 - \left(\frac{\omega}{p}\right)^2\right]^2 + \left(2\zeta\frac{\omega}{p}\right)^2}}$$

# 3  Steady State Solutions for Forced Non-Harmonic SDOF Systems

The general steady state solution for a periodic ($\tau = 2\pi/\omega$) forced damped SDOF system is given by:

$$x(t) = \frac{a_0}{2k} + \sum_{j=1}^{\infty} \frac{a_j/k}{\sqrt{\left[1 - \left(\frac{\omega}{p}\right)^2\right]^2 + \left(2\zeta\frac{\omega}{p}\right)^2}} \cos(j\omega t - \phi_j)$$

$$+ \sum_{j=1}^{\infty} \frac{b_j/k}{\sqrt{\left[1 - \left(\frac{\omega}{p}\right)^2\right]^2 + \left(2\zeta\frac{\omega}{p}\right)^2}} \sin(j\omega t - \phi_j)$$

where

$$\phi_j = \tan^{-1}\left[\frac{2\zeta\frac{\omega}{p}}{1 - \left(\frac{j\omega}{p}\right)^2}\right]$$

$$a_0 = \frac{2}{\tau}\int_0^\tau F(t)dt = 2F_{\text{avg}}$$

$$a_j = \frac{2}{\tau}\int_0^\tau F(t)\cos(j\omega t)dt, \quad j = 1, 2, 3, \ldots$$

$$b_j = \frac{2}{\tau}\int_0^\tau F(t)\sin(j\omega t)dt, \quad j = 1, 2, 3, \ldots$$

Todo: add common solutions such as step, ramp. Ask TA if you can infer some coefficients are zero based on some symmetry.

# 4  Transient Response of Spring-Mass Systems

Table 3: Particular Response of Spring-Mass Systems

| Input | $F_0(t)$ | Particular Response |
|---|---|---|
| Step | $F_0(t) = F_1$ | $x_p(t) = \dfrac{F_1}{k}$ |
| Ramp | $F_0(t) = \beta t$ | $x_p(t) = \dfrac{\beta}{k}t$ |
| Exponential | $F_0(t) = F_1 e^{-at}$ | $x_p(t) = \dfrac{F_1}{ma^2 + k}e^{-at}$ |

# 5  Multiple Degree of Freedom Systems

## 5.1  Unforced Undamped Systems

For an $N$ degree of freedom system,

- $N$ equations of motion are required to describe the system.
- $N$ natural frequencies and mode shapes are obtained from the eigenvalue problem.

The general form of the equation of motion for an $N$ degree of freedom system is given by

$$\mathbf{m_{eff}}\ddot{\mathbf{q}} + \mathbf{k_{eff}}\mathbf{q} = \mathbf{0}$$

where

- $\mathbf{m_{eff}}$ is a $N \times N$ mass matrix,
- $\mathbf{k_{eff}}$ is a $N \times N$ stiffness matrix,
- $\mathbf{q}$ is a $N \times 1$ vector of generalized coordinates,