Chapter 5: Non-Dimentionalization of the Flow

Physically and mathematically, the flow results (dynamics) should not change based on the size of a simulation setup.

$\hookrightarrow$ Most fluid dynamics analyses are completed in a dimensionless framework. This allows for scaling of the real problem.

$\hookrightarrow$ If the scaled conditions are maintained (i.e. Reynold's Number is the same), CFD simulations should be independent from geometrical or scalable physical parameters.

Note: The scalability of the flow condition holds only if the main flow behavior/dynamics is the same in terms of Re, ρ, μ, ...

Now, we return to our governing equations and discuss means to make them non-dimentionalized using normalization factors.

For example, velocity can be normalized using the freestream condition,

$$u_i = u_i^* u_\infty \implies u_i^* = \frac{u_i}{u_\infty}$$

where u_∞ is the freestream velocity.

Similarly,

$$t_0 = \frac{C}{u_\infty} \implies t^* = \frac{t}{t_0} = \frac{u_\infty t}{C}$$

For pressure,

$$P_{\text{dyn}} = \frac{1}{2}\rho u_\infty^2 \implies P^* = \frac{P}{P_{\text{dyn}}} = \frac{2P}{\rho u_\infty^2}$$

and it is given that

$$X_i^* = \frac{X_i}{C} \implies X_i = X_i^* C$$

$\hookrightarrow$ Now, let's begin with the continuity equation (incompressible)

$$\frac{\partial u_i}{\partial x_i} = 0$$

Substituting u_i and x_i with their non-dimensionalized counterparts,

$$\frac{\partial u_i^* u_\infty}{\partial x_i^* C} = 0$$

$$\frac{\cancel{u_\infty}}{\cancel{C}} \frac{\partial u_i^*}{\partial x_i^*} = 0$$

$$\frac{\partial u_i^*}{\partial x_i^*} = 0$$

$\hookrightarrow$ Now, let's look at the momentum equation (incompressible)

$$\frac{\partial u_i}{\partial t} + \frac{\partial (u_i u_j)}{\partial x_j} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j} + b_i$$

Then,

$$\begin{aligned}
\frac{\partial (u_i^* u_\infty)}{\partial (t^* t_0)} + \frac{\partial (u_i^* u_j^* u_\infty^2)}{\partial (x_j^* C)} &= -\frac{1}{\rho}\frac{\partial (P^* P_{\text{dyn}})}{\partial (x_i^* C)} + \nu \frac{\partial^2 (u_i^* u_\infty)}{\partial (x_j^* C)\partial (x_j^* C)} + g b_i^* \\
\frac{\partial (u_i^* u_\infty)}{\partial (t^* t_0)} + \frac{\partial (u_i^* u_j^* u_\infty^2)}{\partial (x_j^* C)} &= -\frac{1}{\rho}\frac{\partial (P^* P_{\text{dyn}})}{\partial (x_i^* C)} + \nu \frac{\partial^2 (u_i^* u_\infty)}{\partial (x_j^* C)\partial (x_j^* C)} + g b_i^* \\
\left(\frac{u_\infty}{t_0}\right)\frac{\partial u_i^*}{\partial t^*} + \left(\frac{u_\infty^2}{C}\right)\frac{\partial (u_i^* u_j^*)}{\partial x_j^*} &= -\frac{1}{\rho}\left(\frac{\frac{1}{2}\rho u_\infty^2}{C}\right)\frac{\partial P^*}{\partial x_i^*} + \left(\frac{\nu u_\infty}{C^2}\right)\frac{\partial^2 u_i^*}{\partial x_j^* \partial x_j^*} + g b_i^*
\end{aligned}$$

Now multiply by C/u_∞^2,

$$\frac{C}{u_\infty t_0}\frac{\partial u_i^*}{\partial t^*} + \frac{\partial (u_i^* u_j^*)}{\partial x_j^*} = -\frac{1}{2}\frac{\partial P^*}{\partial x_i} + \frac{\nu}{u_\infty C}\frac{\partial^2 u_i^*}{(\partial x_j^*)^2} + \frac{gC}{u_\infty^2} b_i^*$$

Therefore we obtain a number of important characteristic flow quantities:

1. Strouhal Number: This dimentionless number is wkd as a number to describe flow unsteadiness (periodicity). Hence, $f_i = 1/t_i$ in the frequency of unsteadiness, which relates to vortex formulation frequency, flapping wings, etc.

$$\text{St} = \frac{f_i C}{u_\infty} = \frac{C}{\underbrace{P_i}_{\text{Period}} u_\infty}$$

2. Reynold's Number: Describes the ratio of inertial to viscous effects.

$$\text{Re} = \frac{\rho u_\infty C}{\mu} = \frac{u_\infty C}{\nu}$$

3. Froude Number: Describes the ratio of inertial to external fields in the flow

$$\text{Fr} = \frac{u_\infty}{\sqrt{Cg}}$$

   This enables us to understand if the flow is driven by inertial forces or external effects (gravity)

From the energy equation, we will get the Peclet Number, which describes the ratio of convection to diffusion effects.

$$\text{Pe} = \frac{C u_\infty}{k/\rho C_p}$$