**MEC E 451 Winter 2024** **Assignment #3** **Due: March 1, 11:59 PM**

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**Instructions:**

- Submit your assignment as a single PDF file through eClass.
- Show all your steps and solution procedures including clear and well labelled FBD/MAD diagrams when needed.
- Make sure that your solution is well organised and that you are using appropriate headers for each question and sub-question.
- Scanned photos of your handwritten solution are acceptable as long as they are **legible**

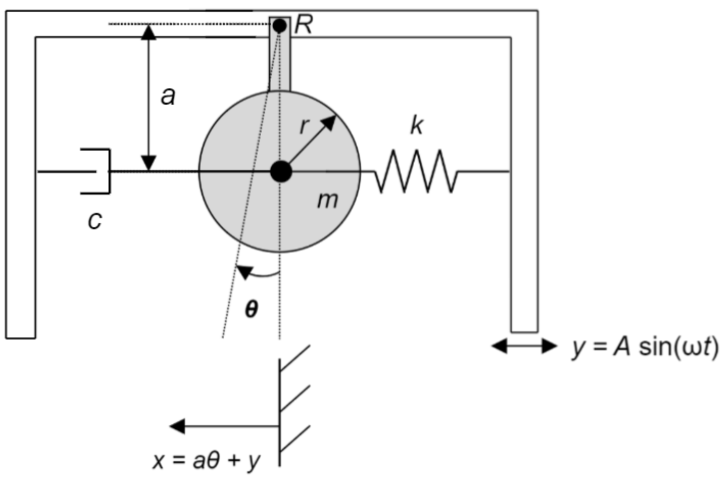
**Question 1 (10 points)**

The diagram models a gate valve which regulates the fluid flow in an engine. The gate valve is modelled as a pendulum (with a **massless** rod and uniform disk of mass $m$ and radius $r$) pinned to the engine at $R$, and is controlled by a spring/damper mechanism connecting the centre of the disk to the engine.

The engine's motion during operation is denoted as $y = A\sin(\omega t)$, and the distance from $R$ to the centre of the disk is $a$. Assuming small oscillations, the total horizontal displacement of the disk is denoted as $x$. The relative motion $z(t)$ between the gate and engine is given as:

$$z = x - y = a\theta$$

**Note**: the disk is fixed to the rod, so it does not spin about the point of connection with the rod, but the rod itself is pinned at $R$. For a uniform disk, $J_G = \frac{1}{2}mr^2$.



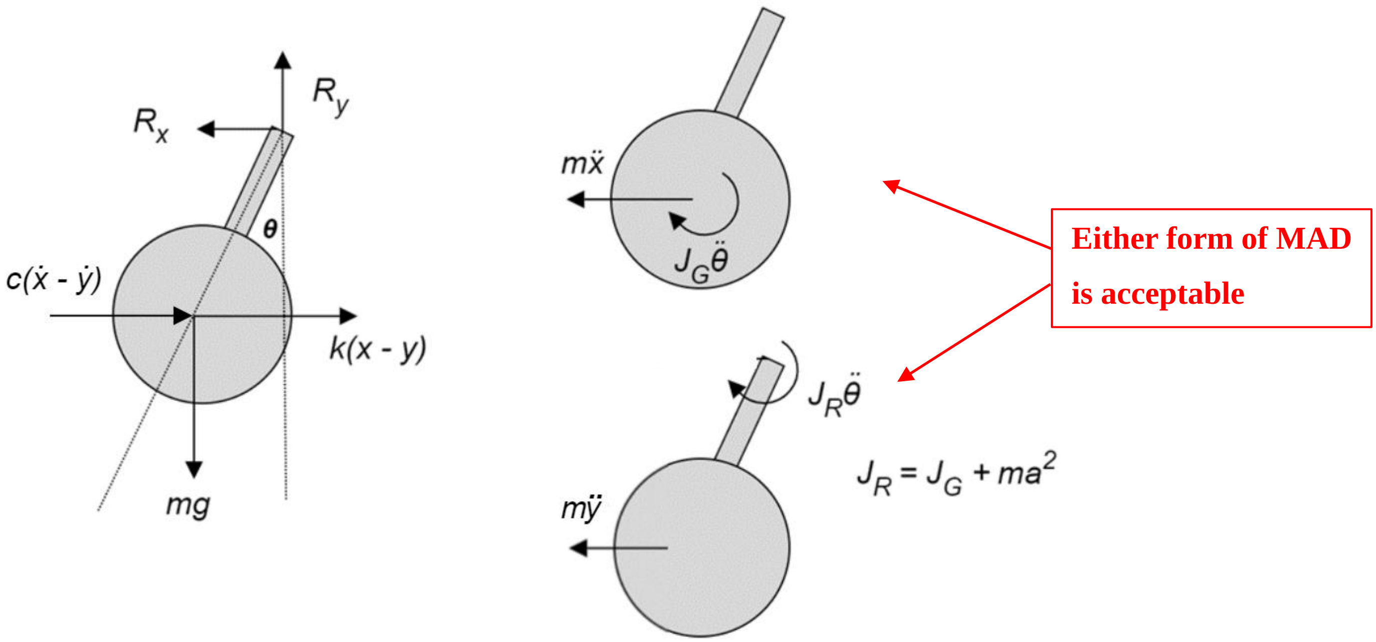

a) **(5 pts)** Determine the equation of motion for the relative motion of the gate with respect to the engine $z$, in terms of $m, k, a, g,$ and $r$. Assume small oscillations.

b) **(5 pts)** If $\omega = 2400$ rpm, $a = 70$ mm, $r = 35$ mm, $m = 2.25$ kg, $k = 8.75$ kN/m, and $c = 30$ Ns/m, determine the amplitude of the **relative motion** $\mathbb{Z}$ between the gate valve and engine in terms of $A$.

**QUESTION 1 SOLUTION**

**PART A) (5 pts)**

Since the gate valve is attached to the support at $R$, the net spring and damper forces are only due to the relative displacement/velocity between the engine and valve ($x - y = a\theta$ or $\dot{x} - \dot{y} = a\dot{\theta}$). However, the engine acceleration must be accounted for in the MAD.




Assuming small oscillations and using the MAD with $m\ddot{x}$, $J_G\ddot{\theta}$:

$$\sum M_R = a(m\ddot{x}) + J_G\ddot{\theta} = -a\theta(mg) - a\big(k(x - y)\big) - a\big(c(\dot{x} - \dot{y})\big)$$

Converting everything in terms of $z = a\theta$:

$$x - y = a\theta = z$$

$$am\big(a\ddot{\theta} + \ddot{y}\big) + \frac{1}{2}mr^2\ddot{\theta} = -(mg - ka)(a\theta) - ca\big(a\dot{\theta}\big)$$

$$\ddot{y} = -\omega^2 y$$

$$m\left(a^2 + \frac{r^2}{2}\right)\ddot{\theta} + ca^2\dot{\theta} + (mga + ka^2)\theta = ma\omega^2(A\sin(\omega t))$$

Writing in terms of $z = a\theta$:

$$m\left(1 + 0.5\left(\frac{r}{a}\right)^2\right)\ddot{z} + c\dot{z} + \left(\frac{mg}{a} + k\right)z = mA\omega^2 \sin(\omega t)$$

**PART B) (5 pts)**

The solution to the equation of motion is:

$$z(t) = \mathbb{Z}\sin(\omega t + \phi)$$

For damped relative motion base excitation, the equation of motion is in the form of a rotating imbalance with $F_0 = mA\omega^2$

$$\mathbb{Z} = \frac{\frac{mA\omega^2}{k_{eff}}}{\sqrt{\left(1 - \left(\frac{\omega}{p}\right)^2\right)^2 + \left(2\zeta\frac{\omega}{p}\right)^2}}$$

**Note, you CANNOT use**

$$\mathbb{Z} = \frac{A\left(\frac{\omega}{p}\right)^2}{\sqrt{\left(1 - \left(\frac{\omega}{p}\right)^2\right)^2 + \left(2\zeta\frac{\omega}{p}\right)^2}}$$

**Since $m$ in $mA\omega^2 \sin(\omega t)$ is not the same as $m_{eff}$ and therefore:**

$$\frac{m\omega^2}{k_{eff}} \neq \frac{\omega^2}{p^2}$$

$$\omega = 2400 \text{ rpm} = 251.327 \text{ rad/s}$$

$$m_{eff} = 2.25\left(1 + 0.5\left(\frac{35}{70}\right)^2\right) = 2.53125 \text{ kg}$$

$$k_{eff} = 8750 + \frac{2.25(9.81)}{0.070} = 9065.32 \text{ N/m}$$

$$p = \sqrt{\frac{9065.32 \text{ N/m}}{2.53125 \text{ kg}}} = 59.844 \text{ rad/s}$$

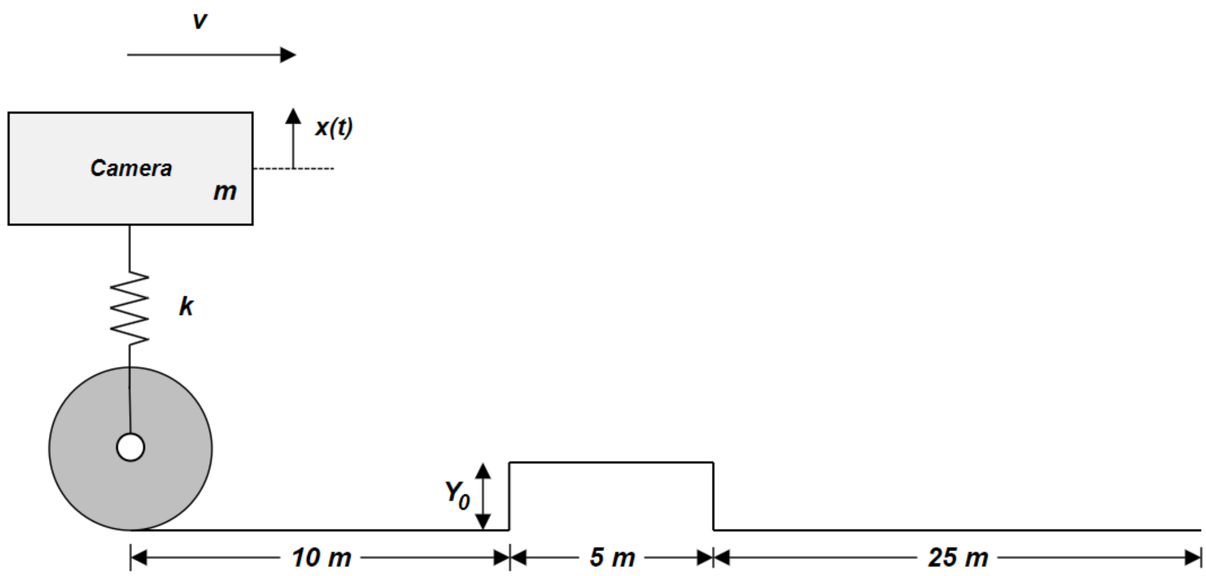
$$\zeta = \frac{c}{c_c} = \frac{c}{2\sqrt{m_{eff}k_{eff}}} = \frac{30}{2\sqrt{(2.53125)(9065.32)}} = 0.099$$

$$\mathbb{Z} = \frac{\frac{(2.25)(251.327)^2}{9065.32}}{\sqrt{\left(1 - \left(\frac{251.327}{59.844}\right)^2\right)^2 + \left(2(0.099)\left(\frac{59.844}{251.327}\right)\right)^2}}A = \mathbf{0.941}\boldsymbol{A}$$

**Question 2 (10 points)**

A film crew is shooting a movie and performs a tracking shot, which involves a moving camera that follows a subject. In this case the camera is situated on a wheeled cart, and the crew encounters a small bump on the road which causes the camera to oscillate vertically.

The total distance to be travelled by the camera is 40 m, with the length of each section shown below, and the combined mass of the camera and the cart is 210 kg. Assume the cart moves at a constant speed $v$ for the entire duration, and the height of the bump $Y_0$ is 5 cm.




a) **(5 pts)** Fortunately, the footage recorded while the camera moves on the bump will not be used in the final production, but do not want any oscillations to occur after the camera cart has cleared the bump. If the entire tracking shot is to be filmed over 10 seconds, determine the required wheel suspension stiffness $k$.

b) **(5 pts)** The production crew tries refilming the shot while doubling the camera speed ($2v$). In reference to the indicated coordinate $x$, determine the displacement ($x$) and velocity ($\dot{x}$) of the camera at the end of the shot (ie. at 40 m). Assume that the stiffness $k$ corresponds to the answer in part a).

**QUESTION 2 SOLUTION**

**PART A) (5 pts)**

The bump in the road can be modelled as a step input forcing function with magnitude $kY_0$. Prior to the bump, there are no oscillations, so the problem can be considered right when the step function is applied ($t = 0$ corresponds to when step function is applied)

The time at which the step function should be removed to have zero oscillations afterwards can be calculated (pg. 103 in course notes):

$$\hat{t} = \frac{2\pi}{p}$$

$$x(\hat{t}) = \frac{F_0}{k}\left[1 - \cos 2\pi\right] = 0,$$

$$\dot{x}(\hat{t}) = \frac{F_0\, p}{k}\sin 2\pi \qquad = 0.$$

As a result,

$$\hat{A} = 0, \qquad \hat{B} = 0,$$

so the response in this case is

$$x(t) = \begin{cases} \frac{F_0}{k}\left[1 - \cos pt\right], & t \le \frac{2\pi}{p}, \\ 0, & t > \frac{2\pi}{p}. \end{cases}$$

Using the given information:

$$v = \frac{d_{total}}{t_{total}} = \frac{40 \text{ m}}{10 \text{ s}} = 4 \text{ m/s}$$

$$\hat{t} = \frac{d_{bump}}{v} = \frac{5 \text{ m}}{4 \text{ m/s}} = 1.25 \text{ s}$$

$$\hat{t} = \frac{2\pi}{p} = 2\pi\sqrt{\frac{m}{k}}, \qquad k = m\left(\frac{2\pi}{\hat{t}}\right)^2 = (210 \text{ kg})\left(\frac{2\pi}{1.25 \text{ s}}\right)^2$$

$$k = 5305.9 \text{ N/m} = 5.31 \text{ kN/m}$$

**PART B) (5 pts)**

With a speed of $2v$ and the same mass, stiffness, the time at which the step function is removed is halved. This corresponds to ii) on page 103 of your course notes:

ii) $\hat{t} = \dfrac{\pi}{p}$

$$x(\hat{t}) = \frac{F_0}{k}\Big[1 - \cos\pi\Big] = 2\frac{F_0}{k},$$
$$\dot{x}(\hat{t}) = \frac{F_0\,p}{k}\sin\pi \qquad = 0.$$

As a result,

$$\hat{A} = 0, \qquad \hat{B} = \frac{2F_0}{k},$$

so the response in this case is

$$x(t) = \begin{cases} \dfrac{F_0}{k}\Big[1 - \cos pt\Big], & t \le \dfrac{\pi}{p}, \\ -\dfrac{2F_0}{k}\cos pt, & t > \dfrac{\pi}{p}. \end{cases}$$

After exiting the bump, this can be considered as a SDOF free vibration problem, using $\tau = t - \hat{t}$ as a reference (such that $\tau = 0$ when the step input is removed), where $F_0 = kY_0$:

$$x(\tau) = 2Y_0\cos(p\tau), \qquad \dot{x}(\tau) = -2Y_0 p\sin(p\tau)$$

**Note:** there is no negative sign in front of $x(\tau)$ because we are using $\tau$ as a reference instead of $t$:

$$\cos(p\tau) = \cos(pt - p\hat{t}) = \cos(pt - \pi) = -\cos(pt)$$

$$p = \sqrt{\frac{k}{m}} = \sqrt{\frac{5305.9}{210}} = 5.03\ \text{rad/s}$$

The camera velocity is two times $v$, corresponding to 8 m/s.

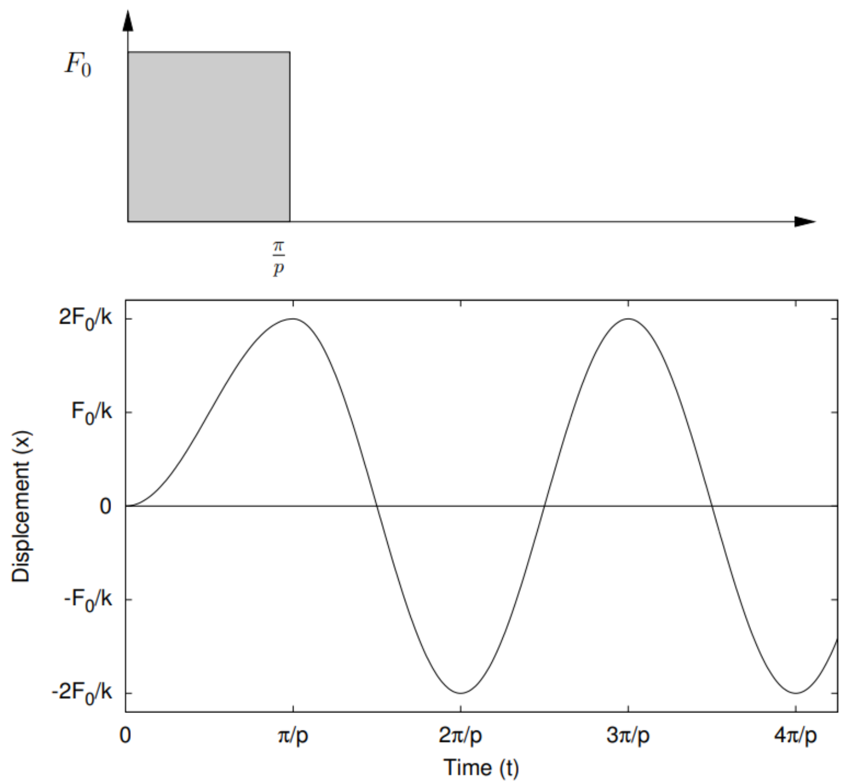
Therefore, the camera reaches the end of the shot at $\tau = \frac{25\ \text{m}}{8\ \text{m/s}} = 3.125$ s:

$$x(3.125) = 0.1\cos[(5.03\ \text{rad/s})(3.125\ \text{s})] = -0.1\ \text{m}$$

$$\dot{x}(3.125) = -2(0.05)(5.03)\sin[(5.03\ \text{rad/s})(3.125\ \text{s})] \cong 0\ \text{m/s}$$

$$x = -0.1\ \text{m}, \qquad \dot{x} = 0\ \text{m/s}$$

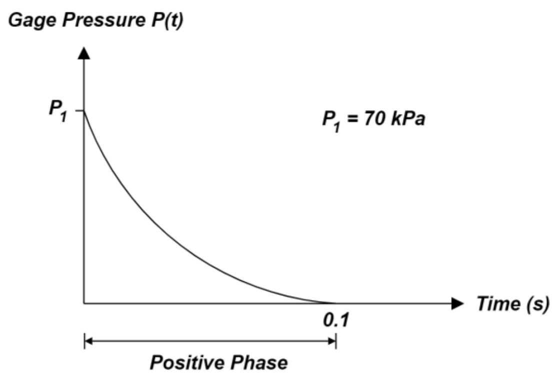
Another way to think about it is that the step function is removed at $t = \pi/p$. However, the distance from the end of the bump to the end of the shot (ie. at 40 m) is exactly five times the length of the bump. Since the camera is moving at a constant velocity, the end of the shot will occur at exactly $t = \frac{6\pi}{p}$, where $t$ is measured the same as the picture below (from pg. 104 in course notes). That point corresponds to a peak (maximum) of $\frac{2F_0}{k} = 2Y_0 = 0.1$ m. For undamped SDOF systems, maximum displacement implies zero velocity.

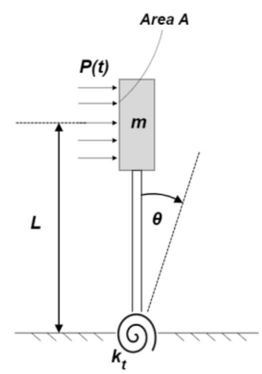


(b) $\tau = \frac{\pi}{p}$

## Question 3 (10 points)

Explosions send blast waves from the centre, which consist of an initial positive pressure wave front followed by a phase of negative pressure which pulls objects back towards the detonation centre. For this question, consider the positive pressure phase of a blast wave as shown below. Typically, the blast wave last 100 ms, and reaches a max pressure (gage) of around 70 kPa.




A nearby road sign (shown above on the right), is approximated by a **massless** rod supporting a uniform thin disk (viewed from the side) of mass $m$, radius $r$, and frontal area $A$. The centre of the disk is at a height $L = 2r$ from the ground, and the ground connection is modelled as a torsional spring $k_t$. Assume that the pressure-time relation during the positive phase is approximated as:

$$P(t) = \frac{P_1}{4}(e^{-at} + 3e^{-bt} - 6t), \qquad 0 \le t \le 0.1 \text{ s}$$

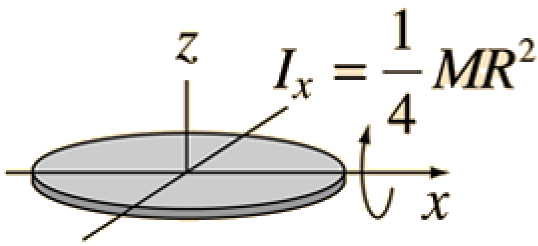
The moment of inertia of a thin disk about its central diameter is $J = \frac{1}{4}mr^2$ (see next page for clarification).

a. **(2 pts)** For the relation described above, determine the equation of motion for the sign during the positive phase in terms of $m, L, P(t), k_t, A$, and $g$. Use $\theta$ as your coordinate.

b. **(8 pts)** If the sign is initially at rest and $a = 10, b = 25$, $m = 1.5$ kg, $r = 0.25$ m, and $k_t = 1 \times 10^3$ Nm, calculate the response $\theta(t)$ of the sign during the positive phase of the blast wave. **You can neglect gravity in this case ($k_t$ dominates the effective stiffness).** <u>**Hint**</u>: To simplify calculations and algebra, work with dummy variables for the effective mass, stiffness, and coefficients of the particular solution rather than $m, L, P_1$, etc.

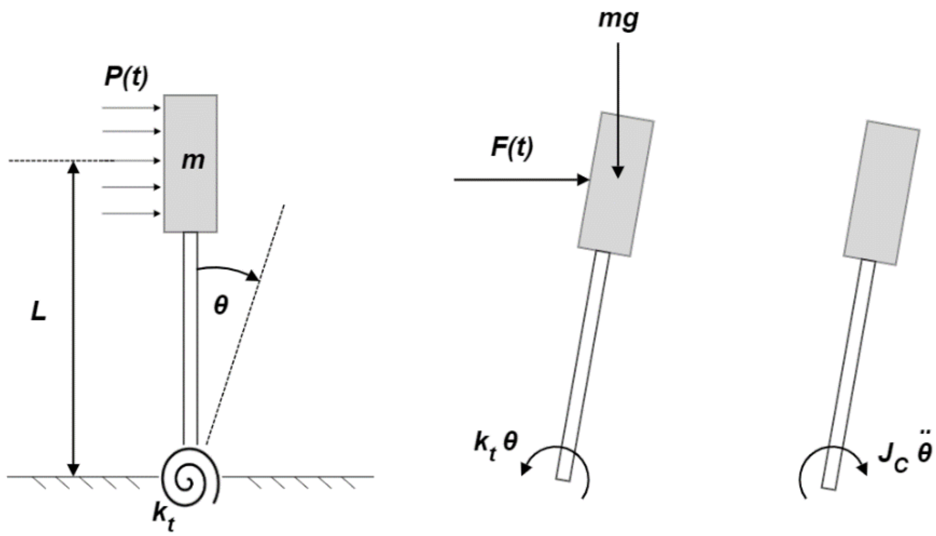
Moment of inertia about any diameter:

$$J = \frac{1}{4} m r^2$$



**QUESTION 3 SOLUTION**

**PART A) (2 pts)**




$F(t)$ is found as:

$$F(t) = A\,P(t)$$

Taking the moment about the ground:

$$\sum M_C = J_C\ddot{\theta} = -k_t\theta + AL\,P(t) + mgL\theta$$

$$J_C = \frac{1}{4}mr^2 + mL^2$$

Since $r = L/2$:

$$J_C = \frac{17}{16}mL^2$$

The equation of motion is therefore:

$$\frac{17}{16}mL^2\ddot{\theta} + (k_t - mgL)\theta = AL\,P(t)$$

**PART B) (8 pts)**

The solution comes from the homogeneous and particular solutions:

$$\theta(t) = \theta_H(t) + \theta_P(t)$$

$$\theta_H(t) = C_1 \sin(pt) + C_2 \cos(pt)$$

For an angular coordinate $\theta$, the forcing function is in the form of a moment:

$$M(t) = AL\, P(t) = \frac{P_1 AL}{4}(e^{-at} + 3e^{-bt} - 6t)$$

$M(t)$ is in the form $C_3 e^{-at} + C_4 e^{-bt} + C_5 t$. The particular solution can be obtained by superimposing the particular solution corresponding to each term in $M(t)$:

$$\theta_P(t) = \theta_{P1}(t) + \theta_{P2}(t) + \theta_{P3}(t)$$

Let $m_{eff} = \frac{17}{16} mL^2$ and $k_{eff} = k_t$ (neglecting gravity):

$$\theta_{P1}(t) = \frac{P_1 AL/4}{m_{eff}a^2 + k_{eff}} e^{-at}$$

$$\theta_{P2}(t) = 3\left(\frac{P_1 AL/4}{m_{eff}b^2 + k_{eff}}\right) e^{-bt}$$

$$\theta_{P3}(t) = -6\left(\frac{P_1 AL/4}{k_{eff}}\right) t$$

Let $C = \frac{P_1 AL/4}{m_{eff}a^2 + k_{eff}}$, $D = 3\left(\frac{P_1 AL/4}{m_{eff}b^2 + k_{eff}}\right)$, $E = -6\left(\frac{P_1 AL/4}{k_{eff}}\right)$

The full solution is then:

$$\theta(t) = C_1 \sin(pt) + C_2 \cos(pt) + Ce^{-at} + De^{-bt} + Et$$

$$\dot{\theta}(t) = C_1 p \cos(pt) - C_2 p \sin(pt) - aCe^{-at} - bDe^{-bt} + E$$

Applying initial conditions:

$$\theta(0) = C_2 + C + D = 0, \qquad C_2 = -(C + D)$$

$$\dot{\theta}(0) = C_1 p - aC - bD + E = 0, \qquad C_1 = \frac{aC + bD - E}{p}$$

Plugging in values:

$$L = 2r = 0.5 \text{ m}$$

$$A = \pi r^2 = 0.1963 \text{ m}^2$$

$$m_{eff} = \frac{17}{16} mL^2 = \frac{17}{16}(1.5 \text{ kg})(0.5 \text{ m})^2 = 0.3984 \text{ kg m}^2$$

$$k_{eff} = k_t = 1000 \text{ Nm}$$

$$p = \sqrt{k_{eff}/m_{eff}} = \sqrt{\frac{1000 \text{ Nm}}{0.3984 \text{ kgm}^2}} = 50.1 \text{ rad/s}$$

$$C = \frac{P_1 AL/4}{m_{eff}a^2 + k_{eff}} = \frac{(70000)(0.1963)(0.5)/4}{0.3984(10)^2 + 1000} = 1.652$$

$$D = 3\left(\frac{P_1 AL/4}{m_{eff}b^2 + k_{eff}}\right) = 3\left(\frac{(70000)(0.1963)(0.5)/4}{0.3984(25)^2 + 1000}\right) = 4.126$$

$$E = -6\left(\frac{P_1 AL/4}{k_{eff}}\right) = -6\left(\frac{(70000)(0.1963)(0.5)/4}{1000}\right) = -10.308$$

$$C_1 = \frac{aC + bD - E}{p} = 2.595$$

$$C_2 = -(C + D) = -5.779$$

The response $\theta(t)$ is given as:

$$\theta(t) = 2.60\sin(50.1t) - 5.78\cos(50.1t) + 1.65e^{-10t} + 4.13e^{-25t} - 10.31t$$

Q4. Known information:

$M = 100\text{kg}$

$K_{eff} = 700 \text{ kN/m}$

$F_0 = 350\text{ N} = me\omega^2$

$\omega = 3000 \text{ rpm}.$

$\xi = 0.2$

a) $x$?

$$\frac{MX}{\tilde{m}e} = \frac{\left(\frac{\omega}{p}\right)^2}{\left(\left(1-\left(\frac{\omega}{p}\right)^2\right)^2 + \left(2\xi\frac{\omega}{p}\right)^2\right)^{1/2}}$$

$$P = \sqrt{\frac{k}{m}} = \sqrt{\frac{700\times10^3}{100}} = 83.66 \text{ rad/sec}$$

$\omega = 3000\text{ rpm} = 50\text{ Hz} \rightarrow \omega^2 = 2500,\ \omega_r = 314.1592 \text{ rad/s}$

$F_0 = 350 \rightarrow \tilde{m}e\omega_r^2 = 350 \rightarrow \tilde{m}e = 0.0035$

$\rightarrow r = \frac{\omega}{p} = 3.75.$

$$\rightarrow X = \frac{0.0035}{100} \times \frac{(3.75)^2}{\left(\left(1-(3.75)^2\right)^2 + \left(2(0.2)(3.75)\right)^2\right)^{1/2}}$$

$\rightarrow X = 37.92 \text{ mm}.$

b) $$TR = \frac{\left(1 + \left(2\xi\frac{\omega}{p}\right)^2\right)^{1/2}}{\left(\left(1-\left(\frac{\omega}{p}\right)^2\right)^2 + \left(2\xi\omega/p\right)^2\right)^{1/2}}$$

$$\rightarrow TR = \frac{\left(1 + (2(0.2)(3.75))^2\right)^{1/2}}{\left((1-(3.75)^2)^2 + (2(0.2)(3.75))^2\right)^{1/2}}$$

$\longrightarrow TR = 0.1368.$

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Q5- Known Information:

The governing equations of the system will be as:

$$\frac{1}{2} m l^2 \ddot{\theta} = -T_f - mgl\sin\theta - kl^2 \sin\theta\cos\theta$$

$$\xrightarrow[\text{oscilations}]{\text{for small}} \frac{1}{2} m l^2 \ddot{\theta} + (mgl + kl^2)\theta = -T_f.$$

$$\longrightarrow x(t) = A\sin pt + B\cos pt + x_p(t).$$

then as the friction torque is acting as a step input:

$$x_p(t) = \frac{-T_f}{(mgl + kl^2)}.$$

Based on the initial condition:

$$\theta(0) = \pi/12, \quad \dot{\theta}(0) = 0$$

$$\rightarrow \dot{x}(t) = Ap\cos pt - Bp\sin pt$$

$$\rightarrow \dot{x}(0) = Ap\cos(0) - Bp\sin(0) = Ap = 0$$

$$\rightarrow A = 0 \rightarrow x(t) = B\cos p(t) + x_p(t)$$

$$\rightarrow x(0) = B\cos(0) - \frac{T_f}{mgl + kl^2} = \pi/12$$

$$\rightarrow B = \frac{\pi}{12} + \frac{T_f}{mgl + kl^2}$$

$$\rightarrow x(t) = \left(\frac{\pi}{12} + \frac{T_f}{mgl + kl^2}\right)\cos pt - \frac{T_f}{mgl + kl}.$$

$$\rightarrow x(t) = \frac{\pi}{12}\cos pt + \frac{T_f}{mgl + kl^2}(\cos pt - 1).$$

b) if there is no friction:

$$T_f = 0 \rightarrow x(t) = \frac{\pi}{12}\cos pt$$