2. SDOF Systems

2.1 Undamped SDOF System

2.1.1 Equation of Motion

$$m_{\text{eff}}\ddot{x} + k_{\text{eff}}x = 0$$
$$\ddot{x} + p^2 x = 0$$

where $p = \sqrt{\frac{k_{\text{eff}}}{m_{\text{eff}}}}$ is the natural frequency of the system. The general solution to this equation is

$$x(t) = A\sin(pt) + B\cos(pt)$$

If the system is subjected to initial conditions $x(0) = x_0$ and $\dot{x}(0) = v_0$, the solution becomes

$$x(t) = \left(\frac{v_0}{p}\right)\sin(pt) + x_0\cos(pt)$$

the single-term solution is

$$x(t) = \sqrt{x_0^2 + \left(\frac{v_0}{p}\right)^2}\sin(pt + \phi)$$
$$\phi = \arctan\left(\frac{x_0}{v_0/p}\right)$$

and period of oscillation is

$$\tau = \frac{2\pi}{p}$$

0.0.1 2.1.2 Energy Methods

When a spring is displaced from its equilibrium position by some x, the potential energy stored in the spring is

$$U = \frac{1}{2}kx^2$$

Similarly, the kinetic energy of the mass is

$$T = \frac{1}{2}m\dot{x}^2$$

0.1 2.2 Spring-Mass Vertical Systems

All equations hold from the previous section if you consider the spring from its equilibrium position. If the spring is considered at its unstretched length, then X_0, the static deflection, is added to the displacement x.

0.2 2.3 Equivalent Mass and Stiffness

0.2.1 2.3.1 Equivalent Mass

Effective mass can be found by finding the total kinetic energy of the system and equating it to the kinetic energy of an effective mass m_{eff}.

$$T = \frac{1}{2}m_{\text{eff}}\dot{q}^2 = \sum \frac{1}{2}m_i\dot{q}_i^2$$

where q is the generalized coordinate.

0.2.2 2.3.2 Equivalent Stiffness

Similarly, effective stiffness can be found by equating the total potential energy of the system to the potential energy of an effective spring k_{eff}.

$$U = \frac{1}{2}k_{\text{eff}}q^2 = \sum \frac{1}{2}k_i q_i^2$$

however, stiffness and flexibility approaches are preferred. For the stiffness approach, apply a unit displacement (Δ or $\theta = 1$) then find the force or moment. For the flexibility approach, apply a unit force or moment ($F = 1$ or $M = 1$) then find the displacement or angle. Assume the system is static.

For linear and angular displacements respectively,

$$F = k_{\text{eff}}\Delta$$
$$M = k_{\text{eff}}\theta$$

4. Forced Vibrations of SDOF Systems

4.1 Simple Spring-Mass Systems

Consider a simple spring mass system that is forced by $F_0 \sin(\omega t)$ in the direction of motion. The equation of motion is

$$m\ddot{x} + c\dot{x} + kx = F_0 \sin(\omega t)$$

The solution is

$$x(t) = x_H(t) + x_P(t)$$

Recall from Section 2.1.1 that the homogeneous solution is

$$x_H(t) = A\sin(pt) + B\cos(pt)$$

The particular solution can be found using the method of undetermined coefficients by assuming $x_P(t) = C\sin(\omega t) + D\cos(\omega t)$. The particular solution is

$$x_P(t) = \frac{F_0}{k - m\omega^2}\sin(\omega t)$$

So, the total response is

$$x(t) = A\sin(pt) + B\cos(pt) + \frac{F_0}{k - m\omega^2}\sin(\omega t)$$

Since most real systems have damping to some degree, we consider only the forced response, which comes from the particular solution. So let the steady state solution be

$$\begin{aligned} x(t) &= \frac{F_0}{k - m\omega^2}\sin(\omega t) \\ &= \frac{F_0}{k}\frac{1}{1 - \left(\frac{\omega}{p}\right)^2}\sin(\omega t) \\ &= X\sin(\omega t) \end{aligned}$$

Defining static deflection as

$$X_0 = \frac{F_0}{k}\frac{1}{1 - \left(\frac{\omega}{p}\right)^2}$$

then, the dynamic magnification factor is

$$\text{DMF} = \frac{X}{X_0} = \frac{1}{\sqrt{1 - \left(\frac{\omega}{p}\right)^2}}$$

4.1.1 Resonance Response

A different form for the particular solution is used when the system is at resonance. By assuming the form $x_P(t) = Ct\sin(\omega t) + Dt\cos(\omega t)$, the particular solution is

$$x_P(t) = -\frac{F_0 t}{2mp}\sin(\omega t)$$