Question 1

The metronome shown is used as a timing device for musicians. The essence of the device is an inverted 'T' mounted on springs. The natural frequency of the device is varied by moving the mass m to different heights L.

Assuming that the system undergoes only small oscillations about the axis through point O (the static equilibrium configuration is at $\theta = 0$):

(a) (5 pts) Determine the equation of motion of the system using θ as your coordinate. Neglect the mass of the 'T' bar. Be sure to include the force of gravity as there is no static deflection of the springs in the static equilibrium configuration.

(b) (5 pts) What is the limiting value of L so that the system is stable?

Solution

(a)

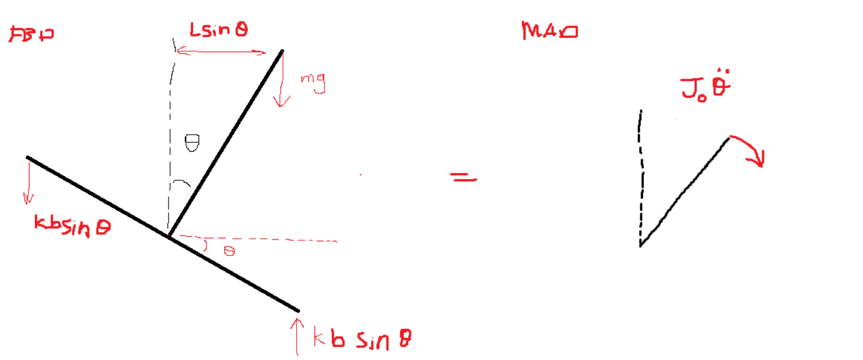

Figure 1: The metronome system

From the free body diagram and mass acceleration diagram, we can take the moment about the pivot point O to get the equation of motion:

$$\circlearrowright \sum M_O = J_O\ddot{\theta}$$
$$\implies -k(b\sin\theta)(b\cos\theta) - k(b\sin\theta)(b\cos\theta) + mgL\sin\theta = J_O\ddot{\theta}$$
$$\implies -2kb^2\sin\theta\cos\theta + mgL\sin\theta = mL^2\ddot{\theta}$$
$$\implies -\frac{2kb^2}{mL^2}\cos\theta\sin\theta + \frac{g}{L}\sin\theta = \ddot{\theta}$$

So the equation of motion is:

$$\boxed{\ddot{\theta} + \left(\frac{2kb^2}{mL^2}\cos\theta - \frac{g}{L}\right)\sin\theta = 0}$$

(b)

Applying linearization, we get:

$$\ddot{\theta} + \left(\frac{2kb^2}{mL^2} - \frac{g}{L}\right)\theta = 0$$

From differential equations, the solution is bounded if the roots of the characteristic equation have negative real parts. Therefore,

$$\frac{2kb^2}{mL^2} - \frac{g}{L} \geq 0$$

$$\boxed{L \leq \frac{2kb^2}{mg}}$$

Question 2

The system shown below consists of a pulley (radius r) pinned about point O, which is connected to a uniform rod (length L) that is pinned about point R. The system is supported by two springs k_1 and k_2, connected to points A and B, respectively. The cable connecting the pulley and rod can be considered approximately inextensible (ie. the pulley and rod are rigidly connected). Assume small oscillations.

(a) (5 pts) Determine the effective stiffness of the system with respect to θ_2 using the stiffness approach, by setting $\theta_2 = 1$.

(b) (5 pts) Assume that a downward force of 5 N is applied at the rod's centre of mass (point G) while the system is initially as shown above. Using your answer from part a), determine the resulting compression of spring k_2 if $r = 0.5$ m, $L = 3$ m, and $k_1 = k_2 = 100$ N/m.

Solution

(a)

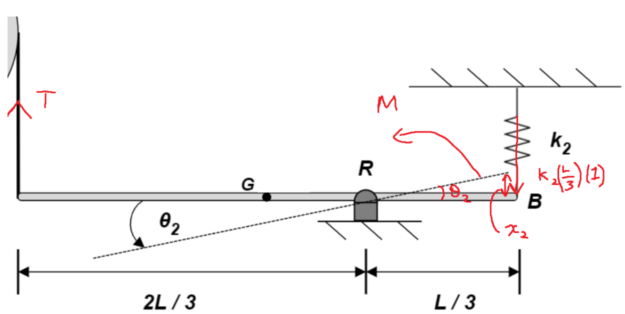

Figure 2: Free body diagram of the bar

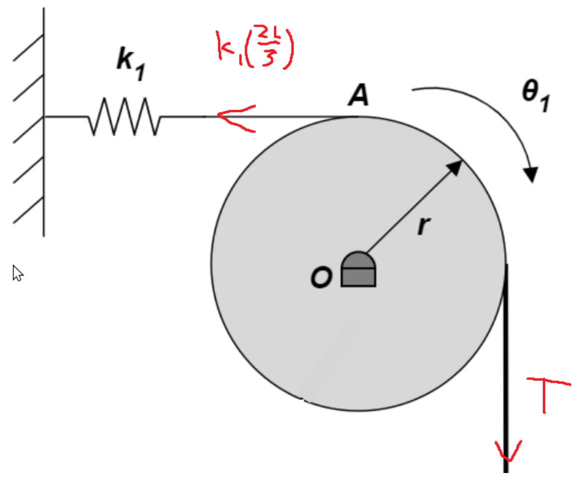

Figure 3: Free body diagram of the pulley

From the free body diagrams from the bar, we can take the sum of moment about the pivot point R

$$\circlearrowleft \sum M_R = 0$$
$$\implies -k_2\left(\frac{L}{3}\right)^2 - T\left(\frac{2L}{3}\right) + M = 0$$
$$\implies M = \frac{k_2 L^2}{9} + \frac{2L}{3}T$$

From the free body diagram of the pulley, we can take the sum of moments about the pivot point O

$$\circlearrowleft \sum M_O = 0$$
$$\implies k_1\left(\frac{2L}{3}\right)r - Tr = 0$$
$$\implies T = \frac{2k_1 L}{3}$$

Substituting T into the equation for M, we get:

$$M = \frac{k_2 L^2}{9} + \frac{2L}{3}\left(\frac{2k_1 L}{3}\right)$$
$$\implies M = \frac{4k_1 L^2}{9} + \frac{k_2 L^2}{9}$$

Since effective stiffness is defined as $M = k_{eff}\theta_2$, we get:

$$k_{eff} = \frac{M}{\theta_2} = \frac{M}{1}$$
$$\boxed{\implies k_{eff} = \frac{4k_1 L^2}{9} + \frac{k_2 L^2}{9}}$$

(b)

The 5N applied generates a moment

$$M = 5\left(\frac{2L}{3} - \frac{L}{2}\right) = 5\left(\frac{2(3)}{3} - \frac{3}{2}\right) = 2.5 \text{ Nm}$$

the effective stiffness is:

$$\begin{aligned} k_{eff} &= \frac{4k_1L^2}{9} + \frac{k_2L^2}{9} \\ &= \frac{4(100)(3)^2}{9} + \frac{(100)(3)^2}{9} \\ &= 400 + 100 \\ &= 500 \text{ Nm/rad} \end{aligned}$$

so the angular displacement is:

$$\begin{aligned} \theta_2 &= \frac{M}{k_{eff}} \\ &= \frac{2.5}{500} \\ &= 0.005 \text{ rad} \end{aligned}$$

using the small angle approximation, $x_2 = \frac{L}{3}\theta_2$,

$$x_2 = \frac{3}{3} \times 0.005$$

$$\boxed{\implies x_2 = 0.005 \text{ m}}$$

Question 3

A machine component is modeled as a system comprising a pendulum and a spring, illustrated in Fig 1. This setup combines gravitational and spring forces, with the pendulum contributing rotational dynamics influenced by gravity, and the spring providing a proportional restoring force. The geometric arrangement and interconnection of these components are depicted in Fig 1. (Assumption: the rotation is small enough so that the spring only deflects horizontally.)

(a) (5 pts) Derive the equation of motion for this system using the energy method.

(b) (3 pts) Linearize the equation of motion and derive a formula for the natural frequency of the system.

(c) (2 pts) Compare the natural frequency of this system with the case where there is no spring.

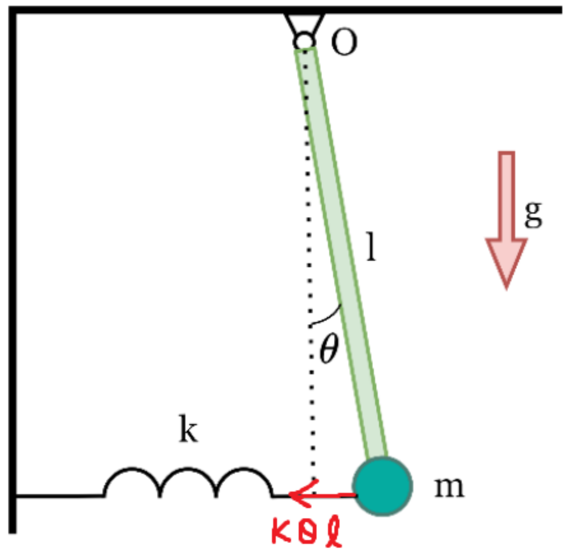

Figure 4: The pendulum system connected to a spring

Solution

(a)

First the total potential energy can be found using the pendulum mass and spring potential energy:

$$\begin{aligned} U_{\text{pendulum}} &= mgL(1-\cos\theta) \\ U_{\text{spring}} &= \frac{1}{2}k(L\sin\theta)^2 \\ \implies U_{\text{total}} &= mgL(1-\cos\theta) + \frac{1}{2}k(L\sin\theta)^2 \end{aligned}$$

Next, the total kinetic energy can be found using the pendulum mass

$$T_{\text{pendulum}} = \frac{1}{2}m\dot{x}^2 = \frac{1}{2}m(L\dot{\theta})^2$$

So total energy is:

$$\begin{aligned} E &= T + U \\ \implies E &= \frac{1}{2}m(L\dot{\theta})^2 + mgL(1-\cos\theta) + \frac{1}{2}k(L\sin\theta)^2 \end{aligned}$$

Since the system is conservative, the total energy is constant. So,

$$\begin{aligned} \frac{dE}{dt} &:= 0 \\ \frac{dE}{dt} &= mL^2\dot{\theta}\ddot{\theta} + mgL\sin\theta\dot{\theta} + kL^2\sin\theta\cos\theta\dot{\theta} \end{aligned}$$

$$\implies \ddot{\theta} + \frac{g}{L}\sin\theta + \frac{k}{m}\sin\theta\cos\theta = 0$$

$$\boxed{\ddot{\theta} + \left(\frac{g}{L} + \frac{k}{m}\cos\theta\right)\sin\theta = 0}$$

(b)

For small angles,

$$\cos\theta \approx 1$$
$$\sin\theta \approx \theta$$

So the equation of motion becomes:

$$\ddot{\theta} + \left(\frac{g}{L} + \frac{k}{m}\right)\theta = 0$$

The natural frequency is given by:

$$\boxed{p = \sqrt{\frac{g}{L} + \frac{k}{m}}}$$

(c)

The natural frequency of a normal pendulum is given by:

$$p_{\text{pendulum}} = \sqrt{\frac{g}{L}}$$
$$p_{\text{spring \& pendulum}} = \sqrt{\frac{g}{L} + \frac{k}{m}}$$

The spring adds a term k/m to the natural frequency of the system. Since $k/m > 0$,

$$\boxed{\implies p_{\text{spring \& pendulum}} > p_{\text{pendulum}}}$$

Question 4

The figure shows a traction elevator system used in high-rise residential buildings. These traction elevators consist of hoist cables connected to the top of the cab operated by a traction machine (electric motor) located in the penthouse. The system is modelled as a simple spring-mass system, where the spring represents the cable stiffness and the mass corresponds to the elevator cab and its occupants (counterweights are neglected). The elevator provides a rapid ascent/descent, while not causing excessive acceleration to the passengers or stress in the cable system. The situation under consideration is the stop after descent to the basement floor level for a 10 floor apartment building (assume 3.5 m per story). Assume that the traction motor stops instantly when reaching the basement floor (acts as a fixed support). The velocity of the cab before stopping is 1.5 m/s. The cables have an equivalent stiffness of a single cable with a radius of 2 cm and an elastic modulus of 100 GPa ($k = \frac{EA}{L}$).

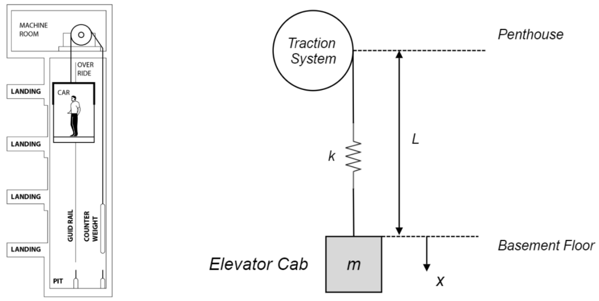

Figure 5: The traction elevator system

(a) Consider both cases with an unloaded cab (mass of 1.2 metric tonnes) and with a maximum capacity of 15 people with an average weight of 70 kg each. For each case, determine:

   (i) (1 pt) The overshoot of the cab past the basement floor level after stopping.

   (ii) (1 pt) The maximum acceleration felt by the occupants.

   (iii) (1 pt) The maximum stress in the cables.

(b) (3 pts) To reduce the maximum tension in the cables and acceleration of the cab, a coil spring (k = 600 kN/m) is inserted between the cable attachment and the cab. How does this change the maximum displacement, acceleration, and stress for both the loaded and unloaded cases?

(c) The results for the vibration analysis of the original elevator system (no coil spring) was done under the assumption of no damping. However, the system components have an inherent damping. a test was done on an UNLOADED cab and it was found that the cab's oscillation amplitude decreased by 50% in two cycles.

   (i) (2 pts) Determine the damping ratio for the LOADED case assuming viscous damping.

   (ii) (2 pts) For the LOADED case, estimate how much time is needed after reaching the ground floor so that the passengers feel virtually no vibration of the elevator cab. Assume that the vibrations essentially stop when the amplitude decreases to 8% of its maximum value. Hint: recall that the logarithmic decrement is measured between subsequent peaks, so you must account for the time from $t = 0$ to the first peak.

Solution

(a)

The system acts as a simple spring-mass system with initial conditions $x(0) = 0$ and $\dot{x}(0) =$ 1.5 m/s. Assume that the equilibrium position is at the basement floor level.

The equation of motion is:

$$\ddot{x} + \frac{k}{m}x = 0$$

with the general solution:

$$x(t) = A\cos(pt) + B\sin(pt)$$

with the solution to the initial conditions:

$$\begin{aligned} x(0) = 0 &\implies A = 0 \\ \dot{x}(0) = 1.5 &\implies B = 1.5/p \end{aligned}$$

So the solution is:

$$x(t) = \frac{1.5}{p}\sin(pt)$$

Next, the spring constant can be found by

$$k = \frac{EA}{L} = \frac{(100 \times 10^9)\pi(0.02)^2}{3.5 \times 10} = 3.59\text{Mn/m}$$

So the natural frequency for both cases is:

$$\begin{aligned} p_{\text{unloaded}} &= \sqrt{\frac{3.59 \times 10^6}{1200}} = 54.7\text{rad/s} \\ p_{\text{loaded}} &= \sqrt{\frac{3.59 \times 10^6}{1200 + 15 \times 70}} = 39.9\text{rad/s} \end{aligned}$$

The overshoot of the cab is simply the amplitude of the solution. So,

$$\boxed{\begin{aligned} x_{\text{unloaded,overshoot}} &= \frac{1.5}{54.7} = 0.0274\text{m} \\ x_{\text{loaded,overshoot}} &= \frac{1.5}{39.9} = 0.0376\text{m} \end{aligned}}$$

To find the maximum acceleration, we can take the second derivative of the solution

$$\ddot{x} = -1.5p\sin(pt)$$

The maximum acceleration is the amplitude of the second derivative. So,

$$\boxed{\begin{aligned} \ddot{x}_{\text{unloaded,max}} &= 1.5 \times 54.7 = 82.0\text{m/s}^2 \\ \ddot{x}_{\text{loaded,max}} &= 1.5 \times 39.9 = 59.9\text{m/s}^2 \end{aligned}}$$

The maximum stress in the cables is at the maximum displacement. So,

$$\sigma_{\max} = \frac{kx_{\max} + mg}{A}$$

$$\boxed{\begin{aligned} \implies \sigma_{\text{unloaded,max}} &= \frac{3.59 \times 10^6 \times 0.0274 + 1200 \times 9.81}{\pi(0.02)^2} = 87.6\text{MPa} \\ \implies \sigma_{\text{loaded,max}} &= \frac{3.59 \times 10^6 \times 0.0376 + (1200 + 15 \times 70) \times 9.81}{\pi(0.02)^2} = 125\text{MPa} \end{aligned}}$$

(b)

An additional spring is added in series with the cable. For a simple oscillator, as seen in the Effective Stiffness Examples.pdf, the effective stiffness is given by:

$$k_{\text{eff}} = \frac{k_1 k_2}{k_1 + k_2}$$

So the effective stiffness is:

$$k_{\text{eff}} = \frac{3.59 \times 10^6 \times 600 \times 10^3}{3.59 \times 10^6 + 600 \times 10^3} = 0.514\text{Mn/m}$$

So the new natural frequency for both cases is:

$$p_{\text{unloaded}} = \sqrt{\frac{0.514 \times 10^6}{1200}} = 20.70\text{rad/s}$$
$$p_{\text{loaded}} = \sqrt{\frac{0.514 \times 10^6}{1200 + 15 \times 70}} = 15.1\text{rad/s}$$

The overshoot of the cab is

$$\boxed{\begin{aligned} x_{\text{unloaded,overshoot}} &= \frac{1.5}{20.70} = 0.0725\text{m} \\ x_{\text{loaded,overshoot}} &= \frac{1.5}{15.1} = 0.0993\text{m} \end{aligned}}$$

The maximum acceleration is

$$\boxed{\begin{aligned} \ddot{x}_{\text{unloaded,max}} &= 1.5 \times 20.70 = 31.1\text{m/s}^2 \\ \ddot{x}_{\text{loaded,max}} &= 1.5 \times 15.1 = 22.7\text{m/s}^2 \end{aligned}}$$

The maximum stress in the cables is at the maximum displacement. So,

$$\sigma_{\max} = \frac{kx_{\max} + mg}{A}$$

$$\boxed{\begin{aligned} \implies \sigma_{\text{unloaded,max}} &= \frac{0.514 \times 10^6 \times 0.0725 + 1200 \times 9.81}{\pi(0.02)^2} = 39.0\text{MPa} \\ \implies \sigma_{\text{loaded,max}} &= \frac{0.514 \times 10^6 \times 0.0993 + (1200 + 15 \times 70) \times 9.81}{\pi(0.02)^2} = 58.2\text{MPa} \end{aligned}}$$

(c)

Using the unloaded system, the damping ratio can be found through the logarithmic decrement (eq. 3.19).

$$\delta = \frac{1}{q} \ln\left(\frac{x_p}{x_{p+q}}\right)$$
$$\zeta = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}}$$

since $x_0/x_{0+2} = 1/0.5 = 2$. So,

$$\delta = \frac{1}{2}\ln(2) = 0.3466$$
$$\zeta = \frac{0.3466}{\sqrt{4\pi^2 + 0.3466^2}} = 0.055079$$

From the definition of the damping ratio,

$$\zeta = \frac{c}{2mp}$$

So the damping coefficient is:

$$c = 2mp\zeta = 2 \times 1200 \times 54.7 \times 0.055079 = 7230.77\text{N s/m}$$

Calculating the damping ratio for the loaded case,

$$\boxed{\begin{aligned} \zeta &= \frac{c}{2mp} \\ &= \frac{7230.77}{2 \times (1200 + 15 \times 70) \times 39.9} \\ &= 0.0403 \end{aligned}}$$

(d)

For the time to reach 8% of the maximum amplitude, (eq. 3.20) can be used to find the time from the zeroth peak to the m-th peak.

$$\frac{\ln(x_p/x_{p+q})\sqrt{1-\zeta^2}}{2\pi\zeta} = q$$

then let $n = 0$, $x_0/x_{0+q} = 1/0.08 = 12.5$. So,

$$\begin{aligned} q &= \frac{\ln(12.5)\sqrt{1 - 0.0403^2}}{2\pi \times 0.0403} \\ &= 10 \end{aligned}$$

since the response is sin, the time from the start to the first peak is

$$t_{\text{zero to peak}} = \frac{\tau}{4}$$

So the total time is

$$\begin{aligned} t_{\text{total}} &= q\tau + \frac{\tau}{4} \\ &= \tau\left(10 + \frac{1}{4}\right) \\ &= \frac{2\pi}{\sqrt{1 - 0.0403^2} \times 39.9}\left(10 + \frac{1}{4}\right) \\ &= \boxed{1.615\text{s}} \end{aligned}$$

Question 5

The system shown, represents a floor of mass M supported by springs of stiffness k. A mass m is dropped on the floor from a height of h.

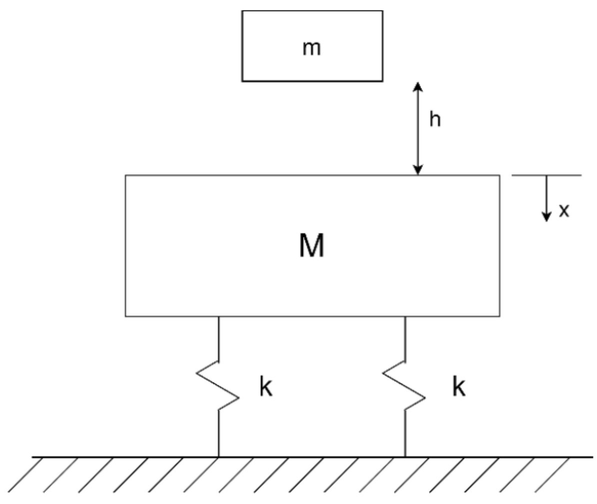

Figure 6: The floor and mass system

(a) (5 pts) Determine the motion of the floor using x, the displacement from the static equilibrium configuration before impact.

(b) (5 pts) Determine the maximum displacement for the case when $h = 0$. NOTE: Do not neglect the weight of the floor and the weight of mass m

Solution

(a)

First define

- State 1 is the initial state of m at rest at height h.
- State 2 is the state of m just before impact at height $h = 0$.
- State 3 is the state of m just after impact with M.

and assume

- The collision is perfectly inelastic.
- m is dropped with no initial velocity.
- M is initially at rest.
- The springs are initially at their equilibrium length.

We begin the analysis by considering Work-Energy Theorem on mass m initially and just before impact:

$$
\begin{aligned}
T_1 + U_{1\to 2} &= T_2 \\
\implies 0 + mgh &= \frac{1}{2}mv_2^2 \\
\implies v_2 &= \sqrt{2gh}
\end{aligned}
$$

Next, we employ Conservation of Momentum on the system of m and M before and after impact.

$$
\begin{aligned}
mv_2 &= (m+M)v_3 \\
\implies v_3 &= \frac{m}{m+M}v_2 \\
\implies v_3 &= \frac{m}{m+M}\sqrt{2gh}
\end{aligned}
$$

This gives us the initial velocity of the oscillator with $m_{\text{eff}} = m+M$. Next, we must determine the effect of m on the equilibrium position. The equilibrium position of the oscillator can be

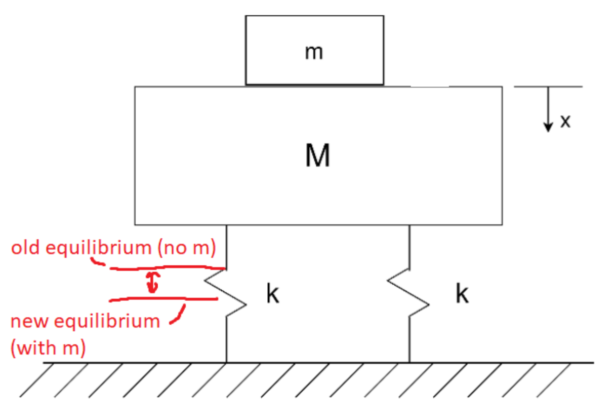

Figure 7: The equilibrium position of the oscillator

found using Hooke's Law. Since the springs are in parallel, the effective stiffness is the sum of the individual stiffnesses.

$$
\begin{aligned}
F &= k_{\text{eff}}\delta_{st} \\
\implies \delta_{st} &= \frac{mg}{k_{\text{eff}}} \\
&= \frac{mg}{2k}
\end{aligned}
$$

Since x is defined as positive sense downwards, the initial conditions are:

$$
\begin{aligned}
x(0) &= -\frac{mg}{2k} \\
\dot{x}(0) &= \frac{m}{m+M}\sqrt{2gh}
\end{aligned}
$$

Note that the negative sign on $x(0)$ is because the new equilibrium position is below the original equilibrium position. Now we define the natural frequency of this system:

$$\boxed{p = \sqrt{\frac{k_{\text{eff}}}{m_{\text{eff}}}} = \sqrt{\frac{2k}{m+M}}}$$

From (eq. 2.7),

$$x(t) = x_0 \cos(pt) + \frac{\dot{x}_0}{p} \sin(pt)$$

$$\boxed{x(t) = -\frac{mg}{2k} \cos(pt) + \frac{m\sqrt{2gh}}{p(m+M)} \sin(pt)}$$

(b)

Using the single-term form of the response,

$$x(t) = X \sin(pt + \phi)$$

$$X = \sqrt{x_0^2 + \left(\frac{\dot{x}_0}{p}\right)^2}$$

$$\phi = \tan^{-1}\left(\frac{x_0}{\dot{x}_0/p}\right)$$

So the maximum displacement is the amplitude, X, which is

$$\boxed{x_{\text{max}} = \sqrt{\left(-\frac{mg}{2k}\right)^2 + \left(\frac{m\sqrt{2gh}}{p(m+M)}\right)^2}}$$