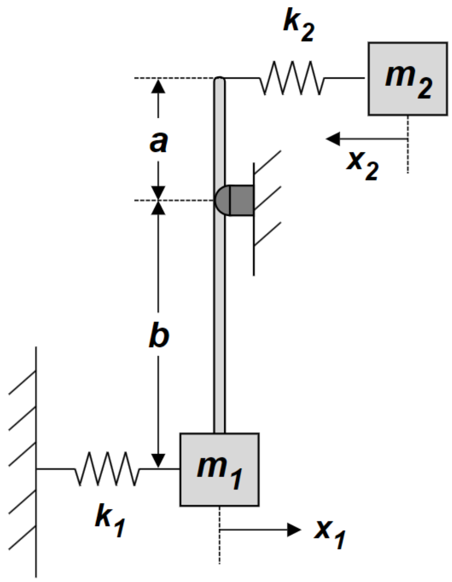
Question 1 (10 points)



For the two degree of freedom system shown, determine:

a) **(3 pts)** The flexibility influence coefficients, starting from the definition.

b) **(3 pts)** The stiffness influence coefficients, starting from the definition.

If $k_1 = k,\ \ k_2 = 2k,\ \ m_2 = m_1 = m,$ and $b = 2a,$ the equations of motion are:

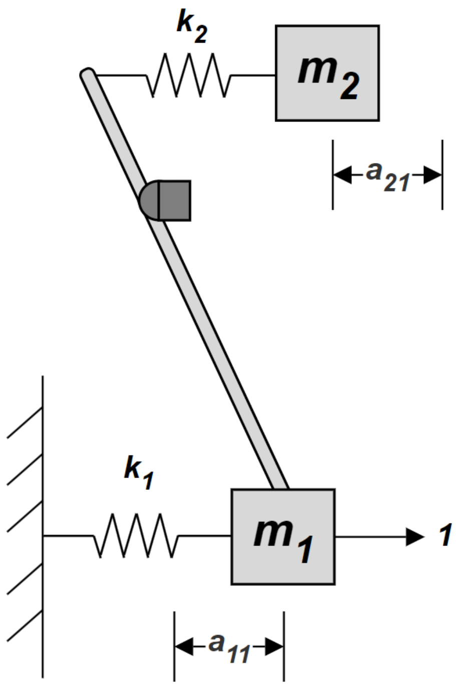
$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 3k/2 & -k \\ -k & 2k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \{0\}$$

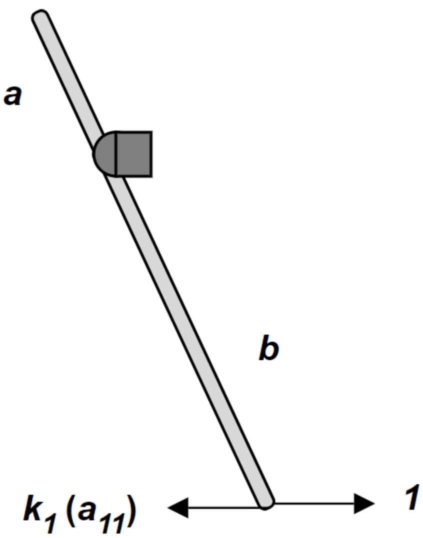
c) **(4 pts)** Determine the natural frequencies and mode shapes.

QUESTION 1 SOLUTION

Part a) (3 pts)

Flexibility Influence Coefficients: a_{ij}: the displacement of coordinate i when a unit load is applied at j, when **ALL COORDINATES ARE FREE TO MOVE**





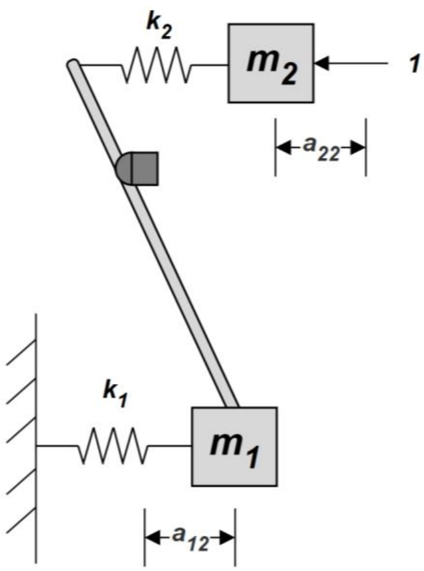
Because all coordinates are free to move, as the bar rotates and elongates spring k_2, there is nothing holding m_2 in place, which displaces by a_{21} so that the net change in length of spring k_2 is zero. Therefore spring k_2 does not apply any force to the bar.

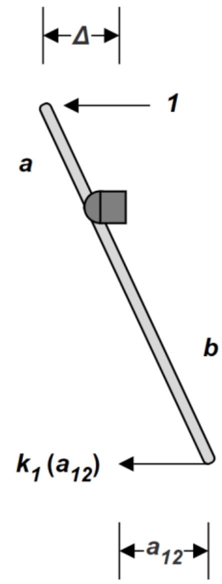
Similar triangles:

$$\frac{a_{21}}{a} = \frac{a_{11}}{b}, \qquad a_{21} = \frac{a}{b} a_{11}$$

$$\circlearrowleft +\sum M_{pin} = 1(b) - k_1(a_{11})b = 0$$

$$a_{11} = \frac{1}{k_1}, \qquad a_{21} = \frac{a}{b}\left(\frac{1}{k_1}\right)$$





The total displacement of mass m_2 is due to the displacement Δ due to the rotation of the rod **and also the displacement due to the compression of spring k_2 from the applied unit load.** This is because at the other end, the spring k_1 is attached to a fixed wall. Therefore, spring k_1 cannot extend to its resting length and will resist the rotation of the rod, so that spring k_2 also cannot be at its resting length and will be compressed because of the applied load.

$$a_{22} = \Delta + \delta_{spring}$$

Since the system is in static equilibrium, we can take an FBD of mass m_2 which shows that the force in the spring is equal to 1 (balances the applied unit load): $F_{spring} = 1 = k_2\delta_{spring}$

$$a_{22} = \Delta + \frac{1}{k_2}$$

$$\circlearrowleft +\sum M_{pin} = 1(a) - k_1(a_{12})b = 0$$

$$a_{12} = \frac{a}{b}\left(\frac{1}{k_1}\right)$$

Similar triangles:

$$\frac{\Delta}{a} = \frac{a_{12}}{b}, \qquad \Delta = \left(\frac{a}{b}\right)a_{12} = \left(\frac{a}{b}\right)^2\left(\frac{1}{k_1}\right)$$

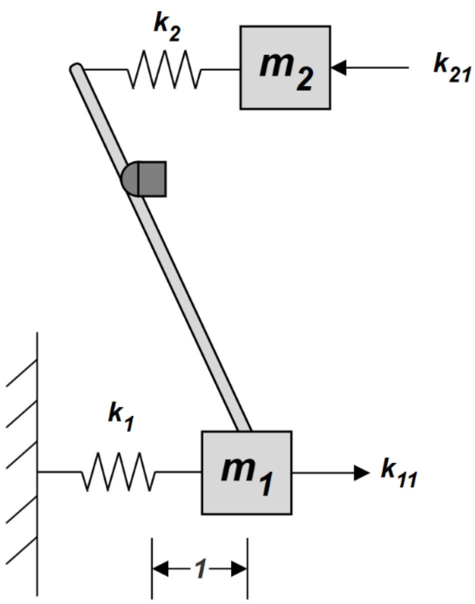
$$a_{22} = \left(\frac{a}{b}\right)^2\left(\frac{1}{k_1}\right) + \frac{1}{k_2}$$

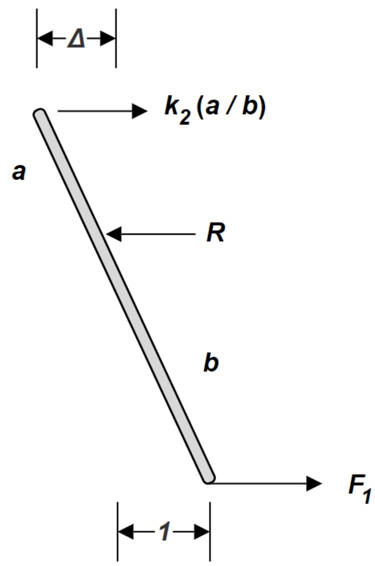
The flexibility matrix is therefore:

$$[a] = \begin{bmatrix} \dfrac{1}{k_1} & \dfrac{a}{b}\left(\dfrac{1}{k_1}\right) \\ \dfrac{a}{b}\left(\dfrac{1}{k_1}\right) & \left[\left(\dfrac{a}{b}\right)^2\left(\dfrac{1}{k_1}\right) + \dfrac{1}{k_2}\right] \end{bmatrix}$$

Part b) (3 pts)

Stiffness Influence Coefficients: k_{ij}: the force required at coordinate i to maintain a unit displacement of coordinate j, when **ALL OTHER COORDINATES ARE FIXED.**





Coordinate x_2 is fixed: therefore, the displacement Δ is entirely from the elongation of spring k_2.

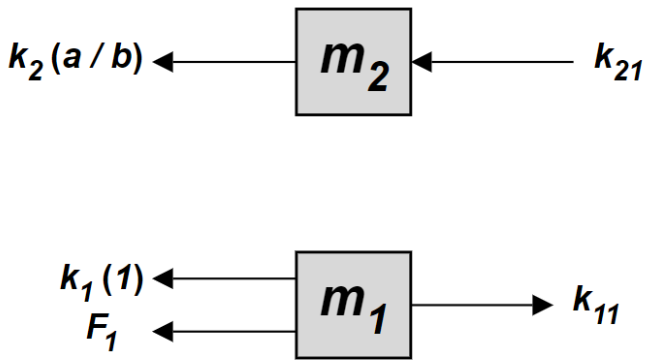
$$F_{spring2} = k_2\Delta$$

Similar triangles: $\Delta = \frac{a}{b}(1)$, so that $F_{spring2} = k_2\left(\frac{a}{b}\right)$

$$\circlearrowleft +\sum M_{pin} = k_2\left(\frac{a}{b}\right)a - F_1 b = 0$$

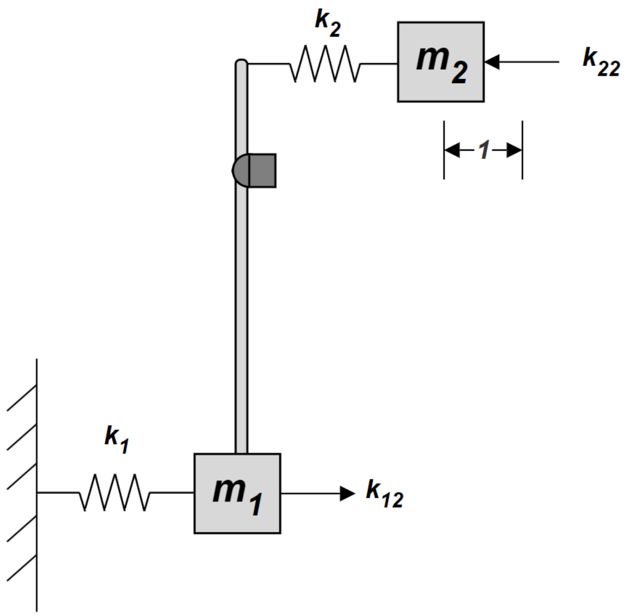
$$F_1 = k_2\left(\frac{a}{b}\right)^2$$

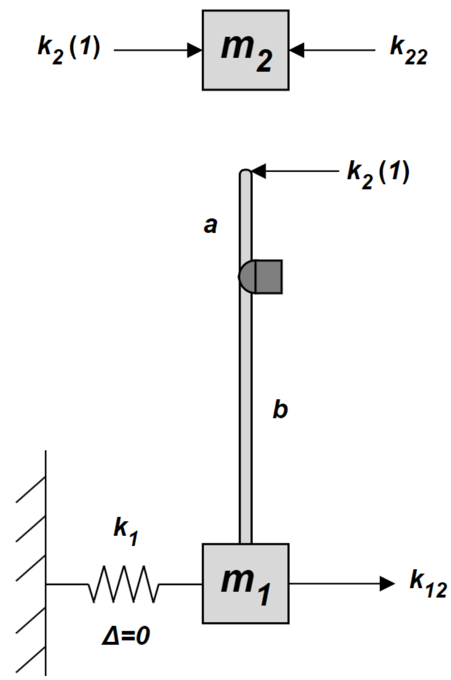
Remember that the forces (stiffness influence coefficients) are applied at each coordinate, which corresponds to the **masses**, not the rod.



$$k_{11} = F_1 + k_1, \qquad k_{21} = -k_2\left(\frac{a}{b}\right)$$

$$k_{11} = k_1 + \left(\frac{a}{b}\right)^2 k_2, \qquad k_{21} = -\left(\frac{a}{b}\right)k_2$$





Because **all other coordinates are fixed except for coordinate j (in this case $j = 2$),** x_1 is fixed at zero. Therefore, the rod does not rotate, and spring k_1 does not apply a force.

$$k_{12} = -\left(\frac{a}{b}\right)k_2, \qquad k_{22} = k_2$$

The stiffness matrix is therefore:

$$[k] = \begin{bmatrix} k_1 + \left(\frac{a}{b}\right)^2 k_2 & -\left(\frac{a}{b}\right)k_2 \\ -\left(\frac{a}{b}\right)k_2 & k_2 \end{bmatrix}$$

Part c) (4 pts)

In a mode:

$$\{x\} = \{A\}\sin(pt+\phi)$$

$$\begin{bmatrix} \frac{3}{2}k - mp^2 & -k \\ -k & 2k - mp^2 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Setting the determinant to zero:

$$\left(\frac{3}{2k} - mp^2\right)(2k - mp^2) + k^2 = 0$$

$$p^4 - \frac{7}{2}\left(\frac{k}{m}\right)p^2 + \frac{2k^2}{m^2} = 0$$

$$p_{1,2}^2 = \left(\frac{7}{4} \pm \sqrt{\frac{49}{16} - \frac{32}{16}}\right)\left(\frac{k}{m}\right) = \left(\frac{7}{4} \pm \frac{\sqrt{17}}{4}\right)\left(\frac{k}{m}\right)$$

$$p_1^2 = 0.7192\left(\frac{k}{m}\right), \qquad p_2^2 = 2.7808\left(\frac{k}{m}\right)$$

Can use any of the two equations from the matrix EOM (highlighted in purple) to determine mode shapes. Using the second equation:

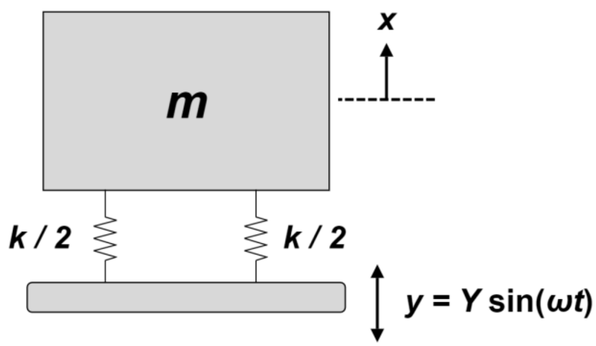
$$\frac{A_2}{A_1} = \frac{1}{2 - \left(\frac{m}{k}\right)p^2}$$

$$\left(\frac{A_2}{A_1}\right)^{(1)} = 0.7808$$

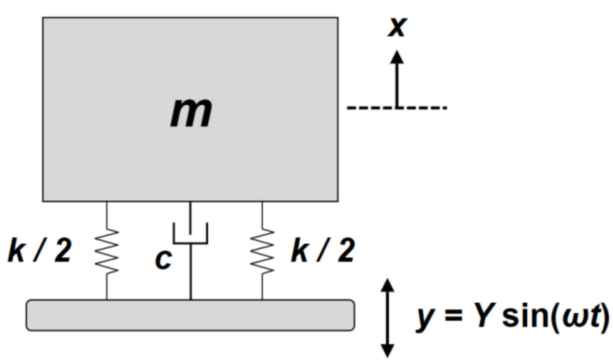
$$\left(\frac{A_2}{A_1}\right)^{(2)} = -1.2808$$

Question 2 (10 points)

The installation of a magnetic resonance imaging (MRI) system requires the machine to be well isolated from vibration. The original isolation is shown where the support motion $y = Ysin(\omega t)$ is due to the floor vibrating from the equipment in the room below. This equipment is assumed to run at 2400 rpm, and the natural frequency of the installation is estimated to be 15 Hz.



a) (1 pt) Estimate the amplitude of vibration of m compared to the amplitude of the floor motion.



b) (2 pts) As the calculation in (a) indicated that the motion of m was too large, the supplier suggests the installation of a damper to give a damping ratio of $\zeta = 0.2$. Will the damper reduce of increase the amplitude of motion? Calculate the difference the damper will make.

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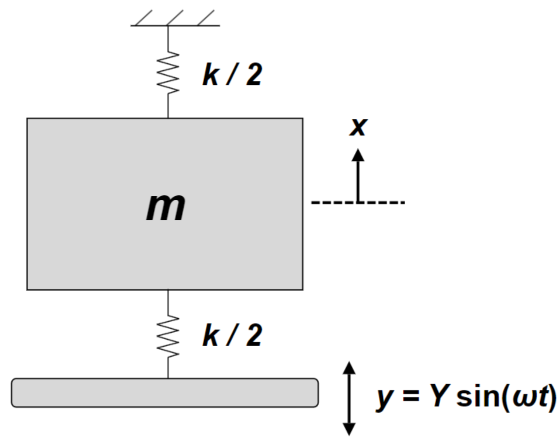
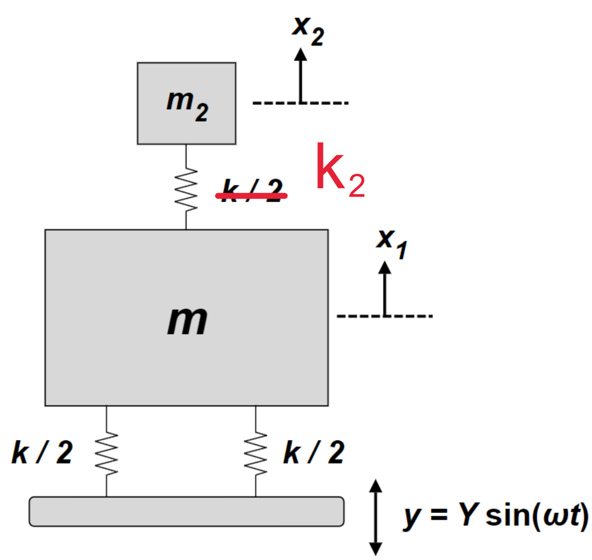


c) **(3 pts)** Another alternative is suggested by a trainee engineer who argued that the machine should be supported from above as well as from the vibration floor as illustrated (it is assumed that the upper support is fixed). Starting from a free body diagram, determine the equation of motion and an expression for the steady state amplitude of m. Compare the amplitude of motion of m for this case to the original design in (a) and the alternative design with the damper in (b).



d) Another way to reduce the motion of m is to use a vibration absorber on the original design.

i. **(2 pts)** If the absorber has a mass m_2 equal to 15% of the main mass m, calculate the amplitude of motion of the absorber mass at the operating speed (2400 rpm).

ii. **(2 pts)** What are the two natural frequencies of the combined system?

QUESTION 2 SOLUTION

Part a) (1 pt)

$$p = 15 \text{ Hz}, \qquad \omega = 40 \text{ Hz}$$

Undamped base excitation:

$$\frac{X}{Y} = \frac{1}{\left|1 - \left(\frac{\omega}{p}\right)^2\right|} = \frac{1}{\left(\frac{40}{15}\right)^2 - 1}$$

$$\frac{X}{Y} = 0.1636$$

Part b) (2 pts)

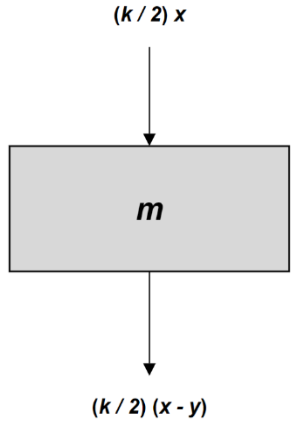
$$\zeta = 0.2, \qquad p = 15 \text{ Hz}, \qquad \omega = 40 \text{ Hz}$$

Damped base excitation:

$$\frac{X}{Y} = \frac{\sqrt{1 + \left(2\zeta\frac{\omega}{p}\right)^2}}{\sqrt{\left(1 - \left(\frac{\omega}{p}\right)^2\right)^2 + \left(2\zeta\frac{\omega}{p}\right)^2}} = \frac{\sqrt{1 + \left(2(0.2)\left(\frac{40}{15}\right)\right)^2}}{\sqrt{\left(1 - \left(\frac{40}{15}\right)^2\right)^2 + \left(2(0.2)\left(\frac{40}{15}\right)\right)^2}}$$

$$\frac{X}{Y} = 0.2357$$

Part c) (3 pts)



$$\uparrow +\sum F_x = m\ddot{x} = -\left(\frac{k}{2}\right)x - \left(\frac{k}{2}\right)(x-y)$$

$$m\ddot{x} + kx = \left(\frac{k}{2}\right)y$$

$$m\ddot{x} + kx = \frac{kY}{2}\sin(\omega t)$$

Undamped base excitation:

$$\frac{X}{Y} = \frac{1}{2}\left(\frac{1}{1-\left(\frac{\omega}{p}\right)^2}\right) = \frac{1}{2}(0.1636)$$

$$\frac{X}{Y} = 0.0818$$

Part d) (4 pts)

At $\omega = p_{22}$ (2400 rpm = 40 Hz):

$$\frac{X_2}{X_0} = -\left(\frac{p_{11}}{p_{22}}\right)^2\left(\frac{1}{\mu}\right) = \left(\frac{15}{40}\right)^2\left(\frac{1}{0.15}\right)$$

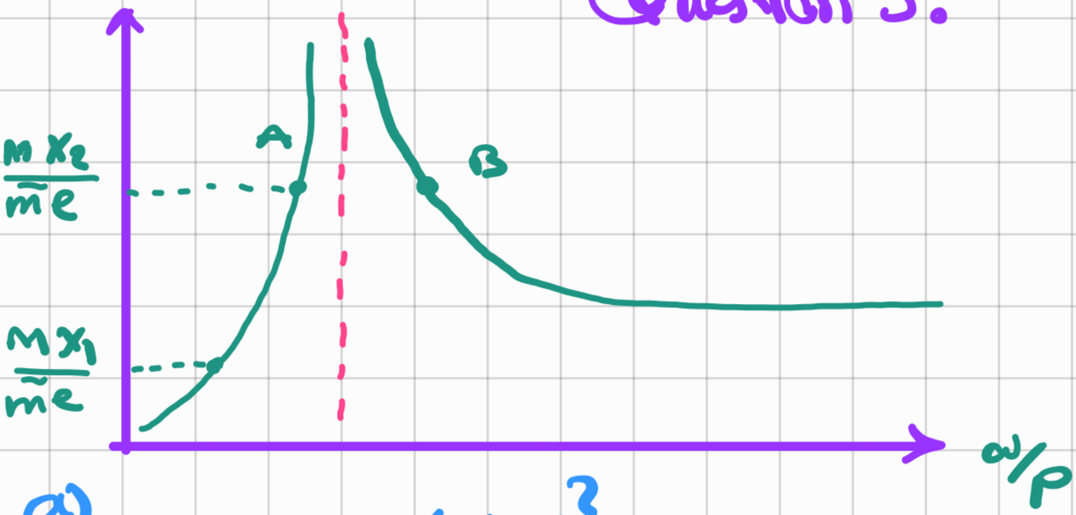
$$\frac{X_2}{X_0} = -0.9375$$

$$\left(\frac{p_{22}}{p_{11}}\right)^2\left(\frac{p}{p_{22}}\right)^4 - \left[1+(1+\mu)\left(\frac{p_{22}}{p_{11}}\right)^2\right]\left(\frac{p}{p_{22}}\right)^2 + 1 = 0$$

$$\left(\frac{40}{15}\right)^2\left(\frac{p}{40}\right)^4 - \left[1+(1+0.15)\left(\frac{40}{15}\right)^2\right]\left(\frac{p}{40}\right)^2 + 1 = 0$$

$p_1 = 832\text{ rpm} = 13.86\text{ Hz} = 87.08\text{ rad/s}$, $\quad p_2 = 2597\text{ rpm} = 43.28\text{ Hz} = 271.94\text{ rad/s}$

Question 3.



a)

$$\frac{m X_1}{\tilde{m} e} = \frac{(\omega_1/p)^2}{1-(\frac{\omega_1}{p})^2} , \quad X_1 = 1 \text{ mm} , \quad X_2 = 5 \text{ mm}$$

$$\frac{m X_2}{\tilde{m} e}^{(A)} = \frac{(\omega_2/p)^2}{1-(\frac{\omega_2}{p})^2} , \quad \frac{m X_2}{\tilde{m} e}^{(A)} = \frac{(\omega_2/p)^2}{(\frac{\omega_2}{p})^2-1}$$

For A:

$$\frac{m X_1}{\tilde{m} e} = \frac{(\frac{100}{p})^2}{1-(\frac{100}{p})^2} , \quad \frac{m X_2}{\tilde{m} e} = \frac{(\frac{200}{p})^2}{1-(\frac{200}{p})^2}$$

$$\frac{X_1}{X_2} = \frac{1}{5} = (\frac{100}{200})^2 \frac{(p^2 - 200^2)}{(p^2 - 100^2)}$$

$$\therefore 4p^2 - 4(100)^2 = 5p^2 - 5(200)^2 \longrightarrow p = 400 \text{ rpm.}$$

For (B):

$$\frac{m X_2}{\tilde{m} e} = \frac{(\frac{200}{p})^2}{(\frac{200}{p})^2-1}$$

$$\frac{X_1}{X_2} = \frac{1}{5} = (\frac{100}{200})^2 \frac{200^2 - p^2}{p^2 - 100^2} \longrightarrow 4p^2 - 4(100)^2$$

$$= 5(200)^2 - 5p^2 \longrightarrow p = \sqrt{\frac{24}{9}} \times 100 = 163 \text{ rpm.}$$

b) $$\left(\frac{P_{22}}{P_{11}}\right)^2\left(\frac{P}{P_{22}}\right)^4-\left[1+(1+\mu)\left(\frac{P_{22}}{P_{11}}\right)^2\right]\left(\frac{P}{P_{22}}\right)^2+1=0 \qquad (**)$$

$P_{22} = 200$

$P_{11} = 100$, $P_{11} = 150$

$$\rightarrow \left(\frac{200}{150}\right)^2\left(\frac{100}{200}\right)^4-\left[1+(1+\mu)\left(\frac{200}{150}\right)^2\right]\left(\frac{100}{200}\right)^2+1=0$$

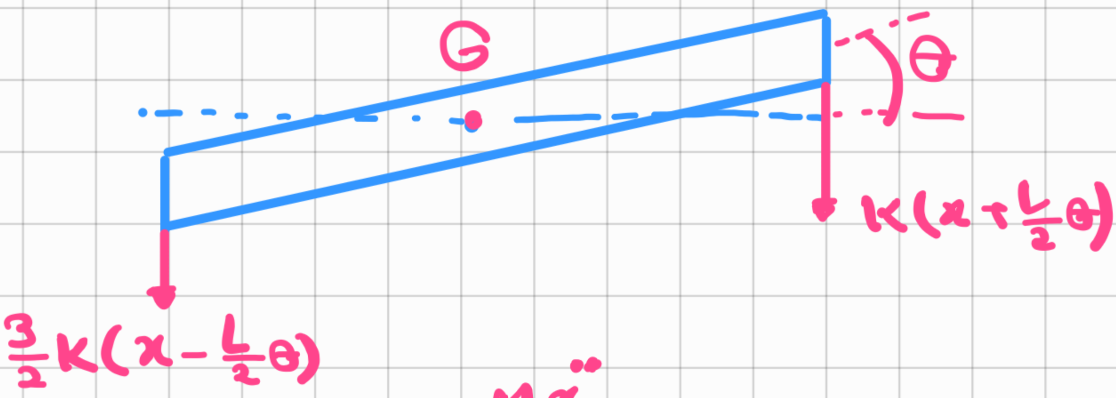
$\longrightarrow \mu = 15/16$ ∴

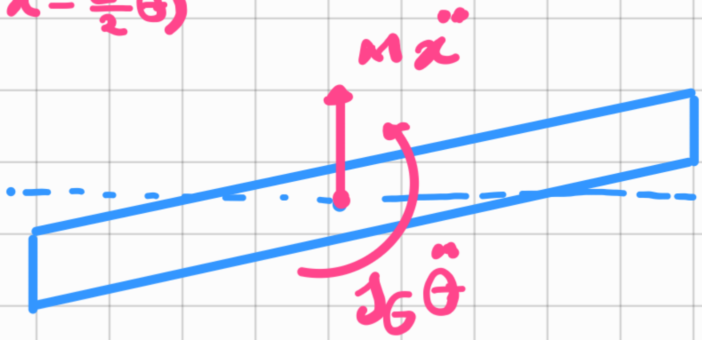
$$(*) \quad \mu = \left(\frac{P_{11}}{P_{22}}\right)\left\{\left[\left(\frac{P_{22}}{P_{11}}\right)^2\left(\frac{P}{P_{22}}\right)^4+1\right]\left(\frac{P_{22}}{P}\right)^2-1\right\}-1$$

↳ derived from (*) equation.

Question 4.

a)





$+\uparrow \sum F_x = m\ddot{x} \Rightarrow m\ddot{x} = -\frac{3}{2}k(x-\frac{l}{2}\theta) - k(x+\frac{l}{2}\theta)$

$\longrightarrow m\ddot{x} + \frac{5}{2}kx - \frac{k}{4}l\theta = 0 \quad \therefore$

$\circlearrowleft + \sum M_G = J_G\ddot{\theta} \rightarrow \frac{mL^2}{6}\ddot{\theta} = \frac{3}{2}k(x-\frac{l}{2}\theta)\frac{l}{2}$

$- k(x+\frac{l}{2}\theta)\, l/2$

$\longrightarrow \frac{ml^2}{6}\ddot{\theta} + \frac{5}{8}kl^2\theta - \frac{1}{4}klx = 0 \quad \therefore$

$$\longrightarrow \begin{bmatrix} m & 0 \\ 0 & \frac{mL^2}{6} \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} \frac{5}{2}k & -kl/4 \\ -\frac{kL}{4} & \frac{5}{8}kL^2 \end{bmatrix} \begin{Bmatrix} x \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (*)$$

b) $(\frac{5}{2}k - mp^2)X - \frac{1}{4}kL\Theta = 0$

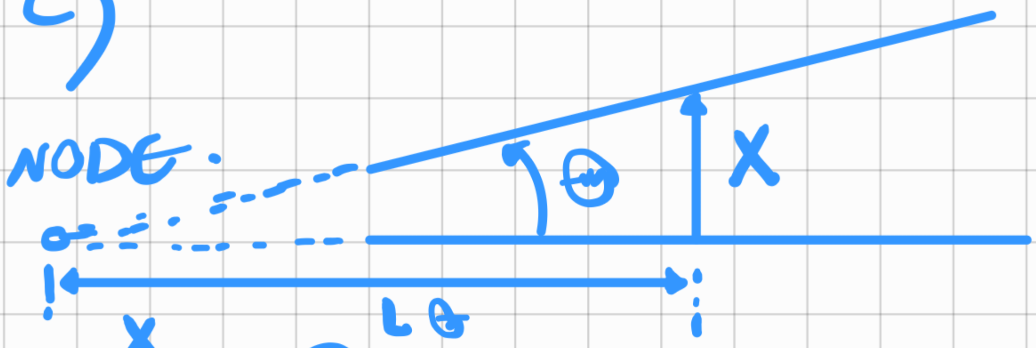
$$\longrightarrow \frac{\Theta}{X} = \frac{5k - mp^2}{\frac{1}{4}kL} = \frac{10 - 4\frac{m}{k}p^2}{L}$$

$$\longrightarrow \left(\frac{\Theta}{X}\right)^{(1)} = \frac{10 - 4(9/4)}{L} = \frac{1}{L}$$

$$\left(\frac{\Theta}{X}\right)^{(2)} = \frac{10 - 4(4)}{L} = \frac{-6}{L}$$

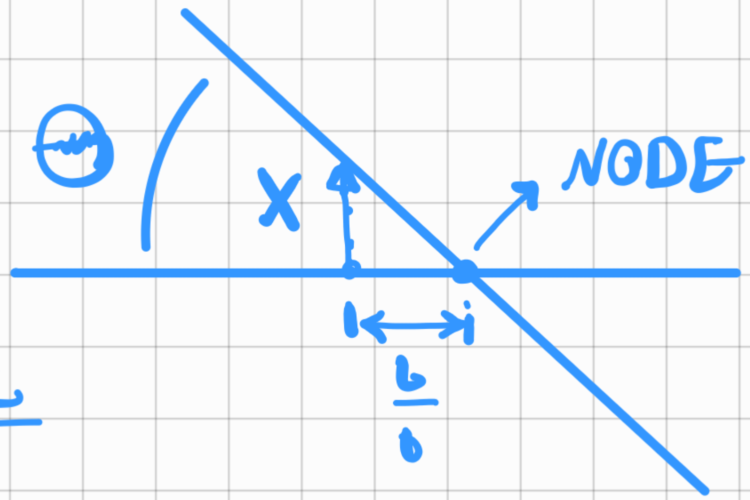
Continued...

c)



mode 1.

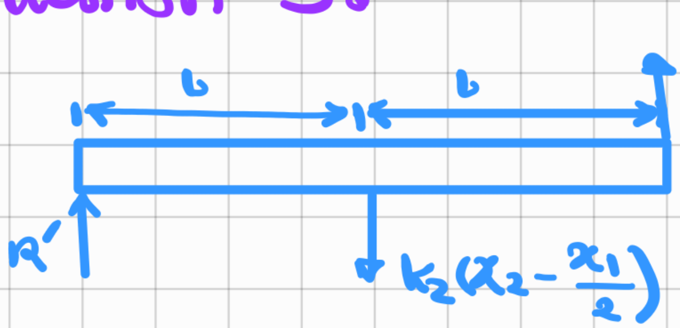
$$\frac{x}{L} = \theta$$



$$\frac{x}{\theta} = -\frac{L}{6}$$

Question 5.

a)



$$+\circlearrowleft \ \Sigma M_{R'} = R(2L) - K_2\left(x_2 - \frac{x_1}{2}\right) L = 0$$

$$\longrightarrow \ \therefore R = \frac{K_2}{2}\left(x_2 - \frac{x_1}{2}\right)$$

$$+\circlearrowleft \ \Sigma M_G = J_G \alpha = J_G \frac{\ddot{x}_1}{r} = -G r$$

where $G = -J_G \dfrac{\ddot{x}_1}{r^2}$

$$+\downarrow \sum F_{x_1} = m_1 \ddot{x}_1 = -kx_1 + \frac{k_2}{2}\left(x_2 - \frac{x_1}{2}\right) - J_G \frac{\ddot{x}_1}{r^2}$$

$$\rightarrow \therefore \left(m_1 + \frac{J_G}{r^2}\right)\ddot{x} = -kx_1 - k_2 x_1/4 + k_2 x_2/2$$

$$+\downarrow \sum F_{x_2} = m_2 \ddot{x}_2 = -k_2\left(x_2 - \frac{x_1}{2}\right)$$

$$\rightarrow \begin{bmatrix} \left(m_1 + \frac{J_G}{r^2}\right) & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} \left(k_1 + \frac{k_2}{4}\right) & -\frac{k_2}{2} \\ -\frac{k_2}{2} & k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

$$= \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\rightarrow \begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} \frac{5k}{4} & -\frac{k}{2} \\ -\frac{k}{2} & k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}.$$

$$\left(\frac{5k}{4} - 2mp^2\right) A_1 - \frac{k}{2} A_2 = 0$$

$$-\frac{k}{2} A_1 + (k - mp^2) A_2 = 0$$

$$\rightarrow \left(\frac{5k}{4} - 2mp^2\right)(k - mp^2) - \frac{k^2}{4} = 0$$

$$\Rightarrow 2m^2 p^4 - \frac{13}{4} mk p^2 + k^2 = 0$$

$$\rightarrow p^2 = \frac{13}{16} \pm \sqrt{\frac{169}{256} - \frac{128}{256}} \rightarrow p = \frac{13 \pm \sqrt{41}}{16} \frac{k}{m}$$

$$= 0.412 \frac{k}{m}, \quad 1.213\, k/m.$$

$$\frac{A_2}{A_1} = \frac{k/2}{k - mp^2} = \frac{1}{2 - \frac{2m}{k} p^2}$$

$$\longrightarrow \left(\frac{A_2}{A_1}\right)^{(1)} = 0.851$$

$$\left(\frac{A_2}{A_1}\right)^{(2)} = -2.351$$