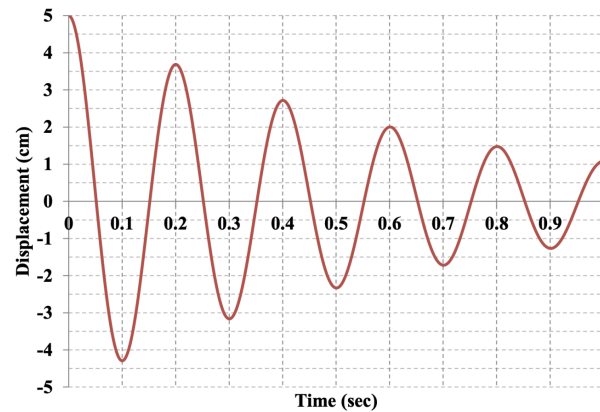


Question 1

The free vibration of a viscously damped SDOF system due to a non-zero initial displacement (zero initial velocity) is given in the graph shown below. Determine the following questions using this graph. Clearly indicate the values you obtain from the graph.



- (5 pts) Write a differential equation that governs the equation of motion of this system.
- (5 pts) If the same system was subjected only to a non-zero initial velocity of 100cm/sec (zero initial displacement), what would be the displacement response at $t = 0.25$ sec.

Solution

(a)

Notice that the graph exhibits the behaviour of a damped cosine wave. Using $x_0 = 5$ and $x_3 = 2$,

$$\begin{aligned}\delta &= \frac{1}{n} \ln \left(\frac{x_0}{x_3} \right) \\ &= \frac{1}{3} \ln \left(\frac{5}{2} \right) \\ &= 0.30543\end{aligned}$$

From this, we can find the damping ratio ζ ,

$$\begin{aligned}\zeta &= \frac{\delta}{\sqrt{4\pi^2 + \delta^2}} \\ &= \frac{0.30543}{\sqrt{4\pi^2 + 0.30543^2}} \\ &= 0.048553\end{aligned}$$

The period is $\tau = 0.1$ sec, so the natural frequency is

$$\begin{aligned} p &= \frac{2\pi}{\sqrt{1 - \zeta^2\tau}} \\ &= \frac{2\pi}{\sqrt{1 - 0.048553^2 \cdot 0.1}} \\ &= 62.91 \text{ rad/sec} \end{aligned}$$

Recall the equation of motion for a damped SDOF system,

$$m\ddot{x} + c\dot{x} + kx = 0$$

dividing by m and letting $p^2 = \frac{k}{m}$ and $2\zeta p = \frac{c}{m}$, we get

$$\ddot{x} + 2\zeta p\dot{x} + p^2x = 0$$

substituting in the values we found, we get

$$\ddot{x} + 2(0.048553)(62.91)\dot{x} + (62.91)^2x = 0$$

$$\boxed{\ddot{x} + 6.11\dot{x} + 3957.7x = 0}$$

(b)

The general solution to the damped SDOF system is given by

$$x(t) = e^{-\zeta pt} \left(A \sin \left(\sqrt{1 - \zeta^2} pt \right) + B \cos \left(\sqrt{1 - \zeta^2} pt \right) \right)$$

where A and B are constants. We can find these constants using the initial conditions. The initial displacement can be found from the graph to be 5 cm.

$$x(0) = 5 = A$$

Now, using initial velocity of 100 cm/sec, using (3.13) from the course notes,

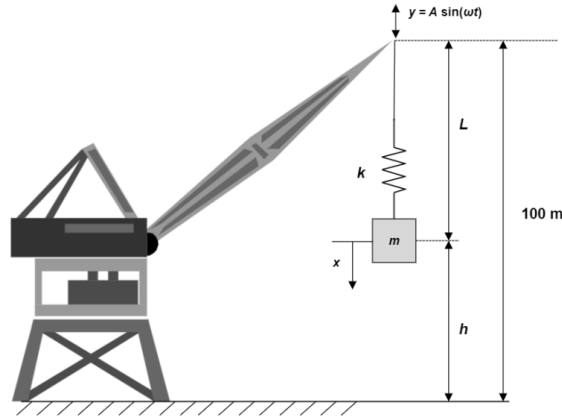
$$x(t) = e^{-\zeta pt} \left(\frac{v_0 + \zeta p x_0}{\sqrt{1 - \zeta^2}} \sin \left(\sqrt{1 - \zeta^2} pt \right) + x_0 \cos \left(\sqrt{1 - \zeta^2} pt \right) \right)$$

so,

$$\begin{aligned} x(0.25) &= e^{-0.048553 \cdot 62.91 \cdot 0.25} \left(\frac{100 + 0.048553 \cdot 62.91 \cdot 5}{\sqrt{1 - 0.048553^2}} \sin \left(\sqrt{1 - 0.048553^2} 62.91 \cdot 0.25 \right) \right. \\ &\quad \left. + 5 \cos \left(\sqrt{1 - 0.048553^2} 62.91 \cdot 0.25 \right) \right) \\ &= \boxed{-2.383 \text{ cm}} \end{aligned}$$

Question 2

A crane 100 m tall is loading a container full of feathers and delicate glass figurines weighing 5 metric tons onto a cargo ship. While the container is in the air, there is an emergency shutdown of the crane and the case is left hanging for a short time. During this time, winds cause the arm of the crane to vibrate vertically at a frequency of 4 Hz with an amplitude of 5 cm. The supporting cable has an effective stiffness of $k = \frac{100}{L}$ MN/m, where L is the length of exposed cable in metres (neglect changes in L due to vibration).



- (2 pts) Determine the cable length L at which the transmissibility would be exactly 1.
- (5 pts) Assuming steady state vibration, determine all possible heights h at which the crate would hit the ground.
- (3 pts) After the emergency is dealt with, operations resume as the wind continues to excite the crane arm. The crane needs to lift shipments 80 m high to place it onto the cargo ships. What is the smallest mass of cargo that can be lifted in these conditions without shaking with an amplitude greater than 4 cm? Assume $\omega > p$.

Solution

(a)

First, we must choose the proper model for the system. A simple undamped spring-mass with base excitation model is appropriate. Assume the weight of the cable is negligible.

The transmissibility is then

$$\text{TR} = \frac{\left(\frac{\omega}{p}\right)^2}{1 - \left(\frac{\omega}{p}\right)^2}$$

converting $f = 4$ Hz to $\omega = 2\pi f = 8\pi$ rad/sec, we can solve for p when $\text{TR} = 1$

$$\begin{aligned} 1 &= \frac{\left(\frac{8\pi}{p}\right)^2}{1 - \left(\frac{8\pi}{p}\right)^2} \\ 1 - \left(\frac{8\pi}{p}\right)^2 &= \left(\frac{8\pi}{p}\right)^2 \\ 1 &= 2\left(\frac{8\pi}{p}\right)^2 \\ \implies p &= 8\sqrt{2}\pi \end{aligned}$$

Since p is defined as

$$\begin{aligned} p &= \sqrt{\frac{k}{m}} \\ \implies k &= p^2 m = (8\sqrt{2}\pi)^2 (5 \times 10^3) = 0.64\pi^2 \times 10^6 \text{ N/m} = 0.64\pi^2 \text{ MN/m} \end{aligned}$$

and since $k = \frac{100}{L}$, we can solve for L ,

$$\boxed{L = \frac{100}{k} = \frac{100}{0.64\pi^2} = 15.8 \text{ m}}$$

(b)

The steady state vibration of the system is given by

$$x(t) = A \left(\frac{1}{1 - \left(\frac{\omega}{p}\right)^2} \right) \sin(\omega t)$$

where x is the distance from the equilibrium position, A is the amplitude, ω is the angular frequency of the excitation. Then, the amplitude is