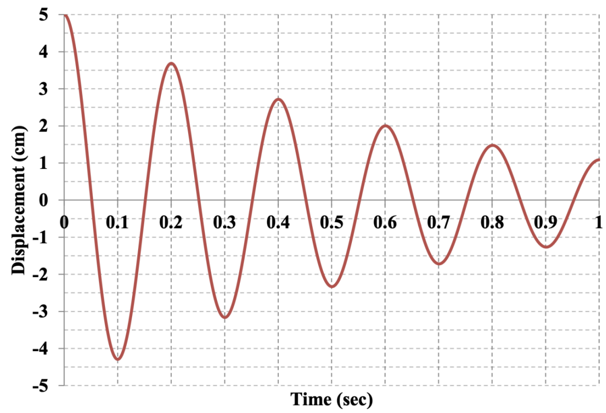
Question 1

The free vibration of a viscously damped SDOF system due to a non-zero initial displacement (zero initial velocity) is given in the graph shown below. Determine the following questions using this graph. Clearly indicate the values you obtain from the graph.



(a) (5 pts) Write a differential equation that governs the equation of motion of this system.

(b) (5 pts) If the same system was subjected only to a non-zero initial velocity of 100cm/sec (zero initial displacement), what would be the displacement response at t = 0.25 sec.

Solution

(a)

Notice that the graph exhibits the behaviour of a damped cosine wave. Using $x_0 = 5$ and $x_3 = 2$,

$$\begin{aligned}\delta &= \frac{1}{n}\ln\left(\frac{x_0}{x_3}\right)\\ &= \frac{1}{3}\ln\left(\frac{5}{2}\right)\\ &= 0.30543\end{aligned}$$

From this, we can find the damping ratio ζ,

$$\begin{aligned}\zeta &= \frac{\delta}{\sqrt{4\pi^2+\delta^2}}\\ &= \frac{0.30543}{\sqrt{4\pi^2+0.30543^2}}\\ &= 0.048553\end{aligned}$$

The period is $\tau = 0.1$ sec, so the natural frequency is

$$\begin{aligned} p &= \frac{2\pi}{\sqrt{1-\zeta^2}\tau} \\ &= \frac{2\pi}{\sqrt{1-0.048553^2}\cdot 0.1} \\ &= 62.91 \text{ rad/sec} \end{aligned}$$

Recall the equation of motion for a damped SDOF system,

$$m\ddot{x} + c\dot{x} + kx = 0$$

dividing by m and letting $p^2 = \frac{k}{m}$ and $2\zeta p = \frac{c}{m}$, we get

$$\ddot{x} + 2\zeta p\dot{x} + p^2 x = 0$$

substituting in the values we found, we get

$$\ddot{x} + 2(0.048553)(62.91)\dot{x} + (62.91)^2 x = 0$$

$$\boxed{\ddot{x} + 6.11\dot{x} + 3957.7x = 0}$$

(b)

The general solution to the damped SDOF system is given by

$$x(t) = e^{-\zeta pt}\left(A\sin\left(\sqrt{1-\zeta^2}pt\right) + B\cos\left(\sqrt{1-\zeta^2}pt\right)\right)$$

where A and B are constants. We can find these constants using the initial conditions. The initial displacement can be found from the graph to be 5 cm.

$$x(0) = 5 = A$$

Now, using initial velocity of 100 cm/sec, using (3.13) from the course notes,

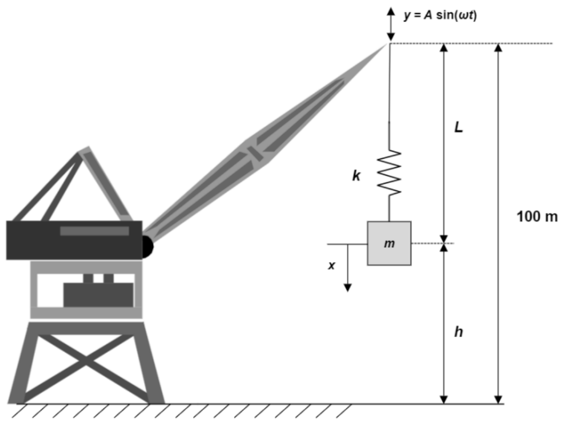
$$x(t) = e^{-\zeta pt}\left(\frac{v_0 + \zeta p x_0}{\sqrt{1-\zeta^2}}\sin\left(\sqrt{1-\zeta^2}pt\right) + x_0\cos\left(\sqrt{1-\zeta^2}pt\right)\right)$$

so,

$$\begin{aligned} x(0.25) = e^{-0.048553\cdot 62.91\cdot 0.25}&\left(\frac{100 + 0.048553\cdot 62.91\cdot 5}{\sqrt{1-0.048553^2}}\sin\left(\sqrt{1-0.048553^2}62.91\cdot 0.25\right)\right. \\ &\left. +5\cos\left(\sqrt{1-0.048553^2}62.91\cdot 0.25\right)\right) \\ = \boxed{-2.383 \text{ cm}} \end{aligned}$$

Question 2

A crane 100 m tall is loading a container full of feathers and delicate glass figurines weighing 5 metric tons onto a cargo ship. While the container is in the air, there is an emergency shutdown of the crane and the case is left hanging for a short time. During this time, winds cause the arm of the crane to vibrate vertically at a frequency of 4 Hz with an amplitude of 5 cm. The supporting cable has an effective stiffness of $k = \frac{100}{L}$ MN/m, where L is the length of exposed cable in metres (neglect changes in L due to vibration).



(a) (2 pts) Determine the cable length L at which the transmissibility would be exactly 1.

(b) (5 pts) Assuming steady state vibration, determine all possible heights h at which the crate would hit the ground.

(c) (3 pts) After the emergency is dealt with, operations resume as the wind continues to excite the crane arm. The crane needs to lift shipments 80 m high to place it onto the cargo ships. What is the smallest mass of cargo that can be lifted in these conditions without shaking with an amplitude greater than 4 cm? Assume $\omega > p$.

Solution

(a)

First, we must choose the proper model for the system. A simple undamped spring-mass with base excitation model is appropriate. Assume the weight of the cable is negligible.

The transmissibility is then

$$\mathrm{TR} = \frac{\left(\frac{\omega}{p}\right)^2}{1 - \left(\frac{\omega}{p}\right)^2}$$

converting $f = 4$ Hz to $\omega = 2\pi f = 8\pi$ rad/sec, we can solve for p when TR $= 1$

$$\begin{aligned} 1 &= \frac{\left(\frac{8\pi}{p}\right)^2}{1-\left(\frac{8\pi}{p}\right)^2} \\ 1-\left(\frac{8\pi}{p}\right)^2 &= \left(\frac{8\pi}{p}\right)^2 \\ 1 &= 2\left(\frac{8\pi}{p}\right)^2 \\ \implies p &= 8\sqrt{2}\pi \end{aligned}$$

Since p is defined as

$$\begin{aligned} p &= \sqrt{\frac{k}{m}} \\ \implies k &= p^2 m = (8\sqrt{2}\pi)^2(5\times 10^3) = 0.64\pi^2 \times 10^6 \text{ N/m} = 0.64\pi^2 \text{ MN/m} \end{aligned}$$

and since $k = \frac{100}{L}$, we can solve for L,

$$\boxed{L = \frac{100}{k} = \frac{100}{0.64\pi^2} = 15.8 \text{ m}}$$

(b)

The steady state vibration of the system is given by

$$x(t) = A\left(\frac{1}{1-\left(\frac{\omega}{p}\right)^2}\right)\sin(\omega t)$$

where x is the distance from the equilibrium position, A is the amplitude, ω is the angular frequency of the excitation. Then, the amplitude is

$$\begin{aligned} X &= A\left(\frac{1}{1-\left(\frac{\omega}{p}\right)^2}\right) \\ &= 0.05\left(\frac{1}{1-\left(\frac{8\pi}{8\sqrt{2}\pi}\right)^2}\right) \\ &= 0.1 \text{ m} \end{aligned}$$