Question 1

Street performers often use a "gut bucket", or washtub bass, to produce frequencies near the resonant frequency of the human chest cavity (10 Hz). This is a crude instrument made from a wire stretched between points B and C. The wire is plucked to create transverse vibrations, and the tension in the wire is created by applying a rigid force P to rigid member AD. The instrument uses a steel wire, with a diameter of 2 mm and density of 7800 kg/m^3.

(a)

(5 pts) Determine the required distance h so that the string has a fundamental frequency of 15 Hz when a force of 15 N is applied to member AD. We want the fundamental frequency to be $p = 2\pi f = 94.25$ rad/s. The fundamental frequency of a string is given by

$$\begin{aligned} p &= \frac{c\pi}{L} \\ &= \frac{\sqrt{\frac{T}{\rho}}\pi}{L} \end{aligned} \tag{1}$$

The linear density, ρ, of the wire is given by

$$\begin{aligned} \rho_{\text{lin}} &= \frac{\pi}{4} d^2 \rho_{\text{steel}} \\ &= \frac{\pi}{4}(0.002)^2(7800) \\ &= 0.0245 \text{ kg/m} \end{aligned}$$

Next, the free body diagram of the gut bucket is shown in Figure 1. Since the bucket is in

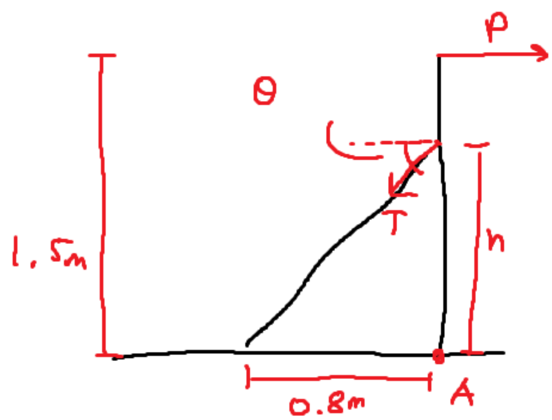

Figure 1: Free body diagram of the gut bucket

equilibrium, take the moment about the pin A,

$$\begin{aligned} \circlearrowleft \sum M_A &= 0 \\ &= Th\cos\theta - P(1.5) \end{aligned}$$

where $\theta = \tan^{-1}\left(\frac{h}{0.8}\right)$. The tension in the wire is given by

$$T = \frac{P(1.5)}{h\cos\left(\tan^{-1}\left(\frac{h}{0.8}\right)\right)}$$

Using the property that

$$\cos\left(\tan^{-1}(x)\right) = \frac{1}{\sqrt{1+x^2}}$$

the tension in the wire is given by

$$\begin{aligned} T &= \frac{P(1.5)\sqrt{1+\left(\frac{h}{0.8}\right)^2}}{h} \\ &= \frac{15(1.5)\sqrt{1+\left(\frac{h}{0.8}\right)^2}}{h} \\ &= \frac{22.5\sqrt{1+\left(\frac{h}{0.8}\right)^2}}{h} \end{aligned}$$

substituting the given values into Eq. (1),

$$\begin{aligned} p &= \frac{\sqrt{\frac{T}{0.0245}}\pi}{\sqrt{(0.8)^2+h^2}} \\ 94.25 &= \frac{\sqrt{\frac{22.5\sqrt{1+\left(\frac{h}{0.8}\right)^2}}{7800h}}\pi}{\sqrt{(0.8)^2+h^2}} \end{aligned}$$

Using MATLAB,

```
syms h
eqn = sqrt(22.5*sqrt((1 + (h/0.8)^2))/(0.0245*h))*pi/sqrt((0.8)^2 + h^2) == 2*pi*15;
sol = vpasolve(eqn, h)

>> sol = 0.99751616990113187949217696874991
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Therefore,

$$\boxed{h = 0.998 \text{ m}}$$

(b)

For a distance h of 1.2 m, what range of force (P_1 to P_2) is necessary to produce notes that cover an entire octave from 20 to 40 Hz? The fundamental frequency of the string is given by Eq. (1),

$$\begin{aligned} p &= \frac{\sqrt{\frac{T}{\rho}}\pi}{L} \\ \implies T &= \frac{p^2\rho L^2}{\pi^2} \end{aligned}$$

Calculating L and θ,

$$\begin{aligned} L &= \sqrt{(0.8)^2 + (1.2)^2} \\ &= 1.442 \text{ m} \\ \theta &= \tan^{-1}\left(\frac{1.2}{0.8}\right) \\ &= 56.301^\circ \end{aligned}$$

Recall from the free body diagram that

$$\begin{aligned} P &= \frac{Th\cos\theta}{1.5} \\ &= \frac{p^2\rho L^2 h\cos\theta}{1.5\pi^2} \end{aligned}$$

Then for P_1 and P_2,

$$\boxed{\begin{aligned} P_1 &= \frac{p_1^2\rho L^2 h\cos\theta}{1.5\pi^2} \\ &= \frac{(2\pi 20)^2(0.0245)(1.442)(1.2)\cos(56.301)}{1.5\pi^2} \\ &= 25.09 \text{ N} \\ P_2 &= \frac{p_2^2\rho L^2 h\cos\theta}{1.5\pi^2} \\ &= \frac{(2\pi 40)^2(0.0245)(1.442)(1.2)\cos(56.301)}{1.5\pi^2} \\ &= 100.36 \text{ N} \end{aligned}}$$

Question 2

Briefly describe the role of each component in determining the note that a string produces from a vibrational standpoint:

(a) The fretboard

The fretboard provides a surface for the guitarist to press down the strings against, effectively shortening the length of the vibrating portion of the string. By pressing the string against different frets, the guitarist changes the length of the vibrating portion, thereby altering the frequency of the produced note.

(b) The tuning pegs

The tuning pegs are used to adjust the tension of the strings. Tightening or loosening the strings with the tuning pegs changes their tension, which in turn alters their fundamental frequency when plucked.

(c) The bridge

The bridge of the guitar anchors the strings at the other end of the instrument. It also transmits the vibrations of the strings to the guitar body, contributing to the resonance and overall sound of the instrument.

(d) The frets

The frets are unevenly spaced along the guitar neck to account for the logarithmic nature of musical intervals. The spacing between frets decreases as you move up the neck because the frequency ratio between notes is not linear. The spacing is designed according to the principles of equal temperament to ensure that each fret corresponds to the correct pitch on the musical scale.

(e) Bending a note

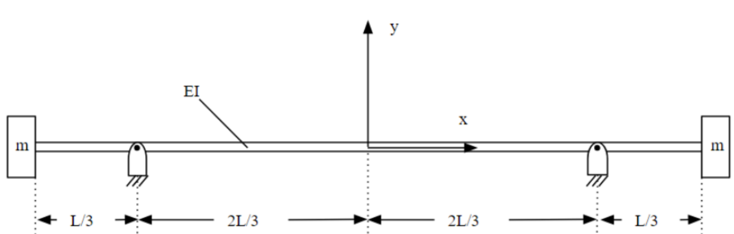
Bending a note involves applying pressure to the string with one or more fingers and then moving the string across the fretboard horizontally, effectively increasing the tension and stretching the string. This action increases the pitch of the note being played. From a vibrational standpoint, bending a note alters the tension and length of the vibrating portion of the string, which in turn changes the fundamental frequency and pitch of the produced sound. Additionally, bending can introduce subtle variations in the harmonic content of the sound, adding expressiveness and character to the note.

Question 3

(a)

A beam of negligible mass has two masses of M attached as shown. Estimate the lowest natural frequency using Rayleigh's Quotient. Assume

$$\begin{aligned}\mathbb{Y}(x) &= A(x - \frac{2L}{3})(x + \frac{2L}{3}) \\ &= A(x^2 - \frac{4L^2}{9})\end{aligned}$$



Recall the Rayleigh's Quotient for continuous systems:

$$p^2 = \frac{\displaystyle\int_{-L}^{L} EI \left(\frac{d^2\mathbb{Y}}{dx^2}\right)^2 dx + \sum_{i=1}^{s} k_i \left(\mathbb{Y}(x_i)\right)^2}{\displaystyle\int_{-L}^{L} \rho A \left(\mathbb{Y}\right)^2 dx + \sum_{j=1}^{n} m_j \left(\mathbb{Y}(x_j)\right)^2}$$

Let us evaluate convenient quantities for the given beam:

$$\begin{aligned}
\frac{d^2\mathbb{Y}}{dx^2} &= 2A \\
\mathbb{Y}(-L) &= A(L^2 - \frac{4L^2}{9}) = \frac{5AL^2}{9} \\
\mathbb{Y}(L) &= A(L^2 - \frac{4L^2}{9}) = \frac{5AL^2}{9}
\end{aligned}$$

Then the terms of the Rayleigh's Quotient become:

$$\begin{aligned}
\int_{-L}^{L} EI \left(\frac{d^2\mathbb{Y}}{dx^2}\right)^2 dx &= 4EIA^2 \int_{-L}^{L} dx \\
&= 8EIA^2L \\
\int_{-L}^{L} \overset{\text{negligible}}{\cancel{\rho}} A \left(\mathbb{Y}\right)^2 dx &= 0 \\
m_1 \left(\mathbb{Y}(-L)\right)^2 &= m\frac{25A^2L^4}{81} \\
m_2 \left(\mathbb{Y}(L)\right)^2 &= m\frac{25A^2L^4}{81}
\end{aligned}$$

Therefore, the Rayleigh's Quotient for the given beam is:

$$\begin{aligned}
p^2 &= \frac{8EIA^2L}{m\frac{50A^2L^4}{81}} \\
&= \frac{324EI}{25mL^3}
\end{aligned}$$

And the lowest natural frequency is:

$$\boxed{p = \sqrt{\frac{324}{25}} \cdot \sqrt{\frac{EI}{mL^3}}}$$

(b)

To stiffen the system a section is added to increase the natural frequency by 25%. Using the same trial function as above, estimate the parameter α to determine the length of the stiffened section.

The denominator of the Rayleigh's Quotient is the same as before. We only need to evaluate the numerator for the stiffened system. Using symmetry,

$$\begin{aligned}
2\left[\int_0^{\alpha L} 3EI\left(\frac{d^2\mathbb{Y}}{dx^2}\right)^2 dx + \int_{\alpha L}^{L} EI\left(\frac{d^2\mathbb{Y}}{dx^2}\right)^2 dx\right] &= 2EI\left[3\int_0^{\alpha L} 4A^2 dx + \int_{\alpha L}^{L} 4A^2 dx\right] \\
&= 2EI\left[12A^2\alpha L + 4A^2(1-\alpha)L\right] \\
&= 8EIA^2L(3\alpha + 1 - \alpha) \\
&= 8EIA^2L(2\alpha + 1)
\end{aligned}$$

The Rayleigh's Quotient for the stiffened system is then:

$$\begin{aligned}
p_2^2 &= \frac{8EIA^2L(2\alpha+1)}{m\frac{50A^2L^4}{81}} \\
&= \frac{324EI(2\alpha+1)}{25mL^3}
\end{aligned}$$

Since we want $p_2 = 1.25p_1$, we have:

$$\begin{aligned}
1.25^2 p_1^2 &= \frac{324EI(2\alpha+1)}{25mL^3} \\
\frac{25}{16}\cdot\frac{324EI}{25mL^3} &= \frac{324EI(2\alpha+1)}{25mL^3} \\
\frac{25}{16} &= 2\alpha + 1 \\
&\boxed{\alpha = \frac{9}{32}}
\end{aligned}$$

Question 4

A hollow shaft of length L contains an auger of negligible mass. As the speed of the auger is close to the natural frequency of the shaft it is necessary to change the natural frequency of the shaft by changing its outside diameter.

(a)

Determine an expression for the ratio of the natural frequencies before and after the outer radius B is altered. Assume the density of the material is γ. Note that the second moment of area is:

The mode shape of the shaft is given by Eq. (10.32),

$$\mathbb{Y}(x) = C_1\sin\beta x + C_2\cos\beta x + C_3\sinh\beta x + C_4\cosh\beta x$$

where the constants C_1, C_2, C_3, and C_4 are determined by the boundary conditions. Since the shaft is pinned-pinned,

$$\begin{aligned}\mathbb{Y}(0) &= 0\\ \mathbb{Y}(L) &= 0\\ \frac{d^2\mathbb{Y}}{dx^2}(0) &= 0\\ \frac{d^2\mathbb{Y}}{dx^2}(L) &= 0\end{aligned}$$

Solving for the constants, first at $x = 0$,

$$\begin{aligned}\mathbb{Y}(0) &= C_2 + C_4 = 0\\ \frac{d^2\mathbb{Y}}{dx^2}(0) &= -\beta^2 C_2 + \beta^2 C_4 = 0\\ \implies C_2 &= C_4 = 0\end{aligned}$$

Simplifying the mode shape,

$$\mathbb{Y}(x) = C_1 \sin\beta x + C_3 \sinh\beta x$$

Applying the boundary conditions at $x = L$,

$$\mathbb{Y}(L) = C_1 \sin\beta L + C_3 \sinh\beta L = 0 \tag{2}$$

$$\frac{d^2\mathbb{Y}}{dx^2}(L) = -\beta^2 C_1 \sin\beta L + \beta^2 C_3 \sinh\beta L = 0 \tag{3}$$

$$= -C_1 \sin\beta L + C_3 \sinh\beta L = 0 \tag{4}$$

Performing (2) + (4),

$$C_3 \sinh\beta L = 0$$

Since the assumption that $\beta \neq 0$ was made to arrive at (4), $C_3 = 0$. Using this in (2),

$$C_1 \sin\beta L = 0$$

Letting $C_1 = 0$ would result in a trivial solution. Therefore, $\sin\beta L = 0$, which implies $\beta L = n\pi$ for $n \in \mathbb{N}$. The fundamental value ($n = 1$) for β is then

$$\beta = \frac{1\pi}{L}$$

The natural frequency of the shaft is given by Eq. (10.30),

$$\begin{aligned}p &= (\beta L)^2 \sqrt{\frac{EI}{\gamma A L^4}}\\ &= (n\pi)^2 \sqrt{\frac{EI}{\gamma A L^4}}\end{aligned}$$

Then for the fundamental frequency,

$$\boxed{p_1 = \pi^2 \sqrt{\frac{EI}{\gamma A L^4}}}$$

Then for p_1/p_2, where the outer radius is altered,

$$\boxed{\begin{aligned} \frac{p_1}{p_2} &= \sqrt{\frac{I_1}{I_2}} \\ &= \sqrt{\frac{B_1^4 - A^4}{B_2^4 - A^4}} \end{aligned}}$$

(b)

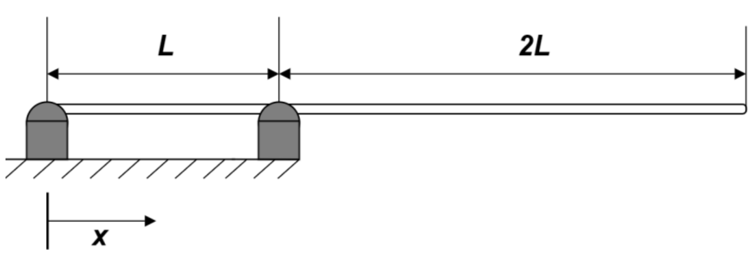
If $A = 0.1$ m and B is initially 0.112 m, what must the outer radius be in order that the frequency to be increased by 10%? For a simply supported beam, $p = \pi^2 \sqrt{\frac{EI}{mL^4}}$

Given that the frequency must be increased by 10%,

$$\begin{aligned} \frac{p_2}{p_1} &= 1.1 \\ &= \sqrt{\frac{B_2^4 - A^4}{B_1^4 - A^4}} \\ \implies 1.1^2 &= \frac{B_2^4 - 0.1^4}{0.112^4 - 0.1^4} \\ \implies &\boxed{B = 0.114 \text{ m}} \end{aligned}$$

Question 5

A diving board is modeled as a beam of length $3L$ and is simply supported at $x = 0$ and $x = L$. The diving board has a uniform density and cross-sectional area A.



(a)

Estimate the lowest natural frequency of the beam in terms of E, I, ρ, A, L, using Rayleigh's Quotient. Assume a mode shape of $\mathbb{Y}(x) = Lx - x^2$.

Recall the Rayleigh's Quotient for continuous systems:

$$p^2 = \frac{\displaystyle\int_0^{3L} EI\left(\frac{d^2\mathbb{Y}}{dx^2}\right)^2 dx + \sum_{i=1}^{s} k_i\left(\mathbb{Y}(x_i)\right)^2}{\displaystyle\int_0^{3L} \rho A\left(\mathbb{Y}\right)^2 dx + \sum_{j=1}^{n} m_j\left(\mathbb{Y}(x_j)\right)^2}$$

Let us evaluate convenient quantities for the given beam:

$$\begin{aligned}
\frac{d^2\mathbb{Y}}{dx^2} &= -2 \\
\left(\frac{d^2\mathbb{Y}}{dx^2}\right)^2 &= 4 \\
\mathbb{Y}^2 &= (Lx - x^2)^2 = L^2x^2 - 2Lx^3 + x^4
\end{aligned}$$

Then the terms of the Rayleigh's Quotient become:

$$\begin{aligned}
\int_0^{3L} EI\left(\frac{d^2\mathbb{Y}}{dx^2}\right)^2 dx &= 4EI\int_0^{3L} dx \\
&= 12EIL \\
\int_0^{3L} \rho A\left(\mathbb{Y}\right)^2 dx &= \rho A\int_0^{3L} L^2x^2 - 2Lx^3 + x^4 dx \\
&= \rho A\left[\frac{L^2x^3}{3} - \frac{2Lx^4}{4} + \frac{x^5}{5}\right]_0^{3L} \\
&= \rho A\left[\frac{L^2(3L)^3}{3} - \frac{2L(3L)^4}{4} + \frac{(3L)^5}{5}\right] \\
&= \frac{171\rho AL^5}{10}
\end{aligned}$$

Therefore, the Rayleigh's Quotient for the given beam is:

$$\begin{aligned}
p^2 &= \frac{12EIL}{\frac{171\rho AL^5}{10}} \\
&= \frac{120EI}{171\rho AL^4} \\
&= \frac{40EI}{57\rho AL^4}
\end{aligned}$$

And the lowest natural frequency is:

$$\boxed{p = \sqrt{\frac{40}{57}} \cdot \sqrt{\frac{EI}{\rho A L^4}}}$$

(b)

Divers have complained that the board is too bouncy. A proposed solution is to install a mass positioned as shown below to reduce the natural frequency to 90% of its original value. If the mass of the diving board is 60 kg, determine the mass M that must be installed.

The addition of the mass adds a term to the denominator of the Rayleigh's Quotient. The new term is

$$\begin{aligned} M\left(\mathbb{Y}(2L)\right)^2 &= M\left(L(2L) - (2L)^2\right)^2 \\ &= 4ML^4 \end{aligned}$$

The Rayleigh's Quotient for the system with the added mass is then

$$p_2^2 = \frac{12EIL}{\frac{171\rho AL^5}{10} + 4ML^4}$$

Since we want $p_2 = 0.9p_1$, we have

```
syms M L A rho EI
p_1_sq = 40 * EI / (57 * rho * A * L^4);
p_2_sq = 12 * EI * L / ((171 * rho * A * L^5) / 10 + 4 * M * L^4);

simplify(p_2_sq / p_1_sq)
eqn = p_2_sq / p_1_sq == 0.9^2;
eqn = subs(eqn, A*L*rho, 20);
M = solve(eqn, M)

>> (171*A*L*rho)/(40*M + 171*A*L*rho)

>> M = 20.055555555555555555555555555556
```

So,

$$\begin{aligned} \frac{p_2^2}{p_1^2} &= 0.9^2 \\ &= \frac{171AL\rho}{40M + 171AL\rho} \end{aligned}$$

Since the mass of the diving board is 60 kg,

$$\begin{aligned} & m = 3AL\rho = 60 \\ \implies & AL\rho = 20 \end{aligned}$$

Then,

$$
\begin{aligned}
0.9^2 &= \frac{171 \cdot 20}{40M + 171 \cdot 20} \\
0.81 &= \frac{3420}{40M + 3420} \\
\implies & \boxed{M = 20.056 \text{ kg}}
\end{aligned}
$$