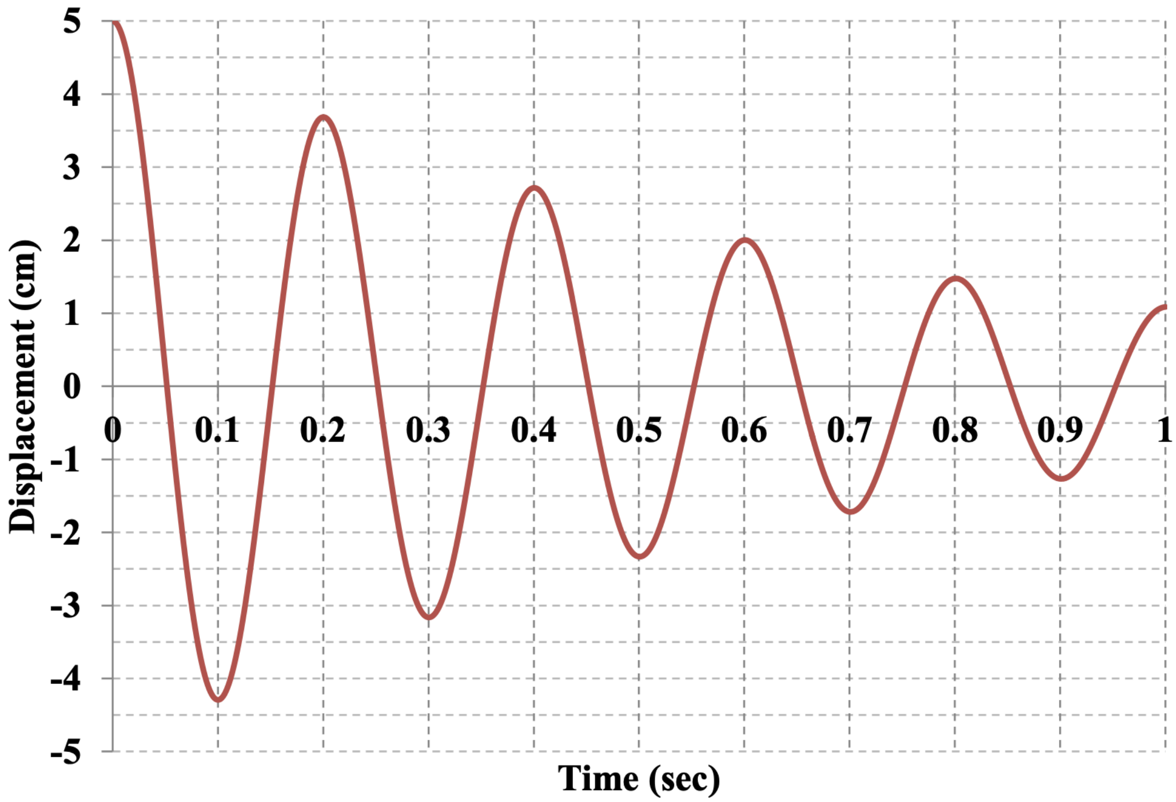
Question 1 (10 pts)

The free vibration of a viscously damped SDOF system due to a non-zero initial displacement (zero initial velocity) is given in the graph shown below. Determine the following questions using this graph. Clearly indicate the values you obtain from the graph.



a) **(5 pts)** Write a differential equation that governs the equation of motion of this system.

b) **(5 pts)** If the same system was subjected only to a non-zero initial velocity of 100cm/sec (zero initial displacement), what would be the displacement response at t = 0.25 sec.

Question 1 Solution

a) $m\ddot{u} + c\dot{u} + ku = 0$

$$\ddot{u} + \frac{c}{m}\dot{u} + \frac{k}{m}u = 0$$

$$\frac{c}{C_{cr}} = \xi = \frac{c}{2mW_n} \Rightarrow \frac{c}{m} = 2\xi W_n$$

$$\Rightarrow \ddot{u} + 2\xi W_n \dot{u} + W_n^2 u = 0$$

From the graph, $T_D = 0.2\,sec \Rightarrow W_D = \frac{2\pi}{T_D} = 10\pi = 31.42$ rad/sec

$$\xi = \frac{1}{2\pi n} \ln\left(\frac{u_1}{u_{1+n}}\right) \Rightarrow \xi = \frac{1}{2\pi \times 3} \ln\left(\frac{5}{2}\right) \Rightarrow \xi = 0.0485$$

$$\Rightarrow W_n = \frac{W_D}{\sqrt{1-\xi^2}} = 31.42 / \sqrt{1-(0.0485)^2} = 31.45 \text{ rad/sec}$$

$\Rightarrow$ the governing differential equation

$$\ddot{u} + 2(0.0485)(31.45)\dot{u} + (31.45)^2 u = 0$$

$$\Rightarrow \ddot{u} + 3.05u + 989.1u = 0 \quad \text{with} \quad u(0) = 5 \text{ cm/sec}, \quad \dot{u}(0) = 0$$

b) $$u(t) = e^{-\xi W_n t}\left[u(0)\cos W_D t + \frac{\dot{u}(0) + \xi W_n u(0)}{W_D} \sin W_D t\right]$$

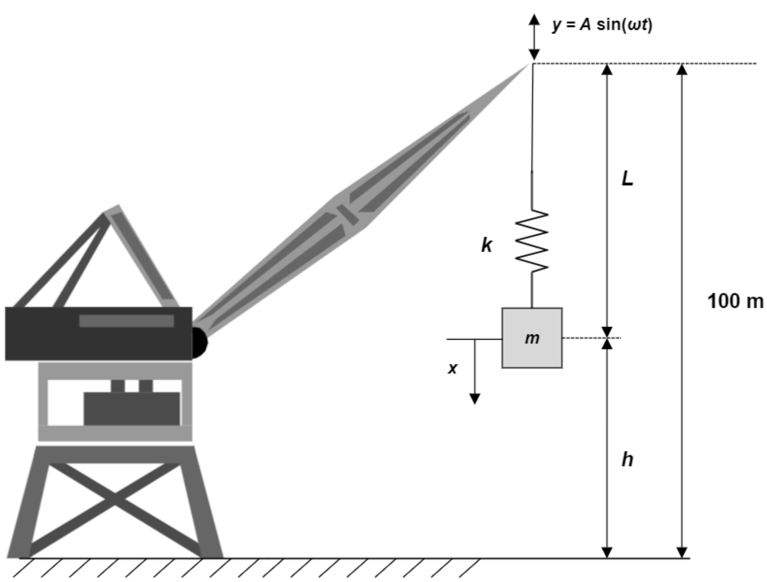
$$= e^{-0.0485 \times 31.45}\left[0 + \frac{100}{31.42} \sin 31.42t\right]$$

$$= e^{-1.52t}\left[3.18 \sin(31.42t)\right]$$

$$\Rightarrow u(0.25) = \ldots\ldots = 2.17\,cm$$

Question 2 (10 pts)

A crane 100 m tall is loading a container full of feathers and delicate glass figurines weighing 5 metric tons onto a cargo ship. While the container is in the air, there is an emergency shutdown of the crane and the case is left hanging for a short time. During this time, winds cause the arm of the crane to vibrate **vertically** at a frequency of 4 Hz with an amplitude of 5 cm. The supporting cable has an effective stiffness of $k = \frac{100}{L}$ MN/m, where L is the length of exposed cable in metres (neglect changes in L due to vibration).



a) **(2 pts)** Determine the cable length L at which the transmissibility would be exactly 1.

b) **(5 pts)** Assuming **steady state vibration,** determine the height h at which the crate would hit the ground (neglect the size of the crate and assume this corresponds to the length of the cable). **Note**: this corresponds to a different value of L than part a).

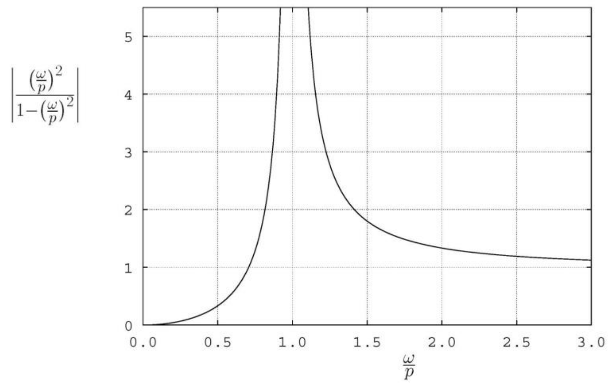
c) **(3 pts)** After the emergency is dealt with, operations resume as the wind continues to excite the crane arm. The crane needs to lift shipments 80 m high to place it onto the cargo ships. What is the largest mass of cargo that can be lifted in these conditions without shaking with an amplitude greater than 4 cm? Assume $\omega > p$.

QUESTION 2 SOLUTION

PART A) (2 pts)

For undamped base excitation:

$$TR = \frac{\left(\frac{\omega}{p}\right)^2}{\left|1 - \left(\frac{\omega}{p}\right)^2\right|} = 1$$



$TR = 1$ in the region where $\frac{\omega}{p} < 1$, so therefore assume $\omega < p$ and the absolute value bracket can be removed without having to multiply by -1:

$$\frac{\left(\frac{\omega}{p}\right)^2}{1 - \left(\frac{\omega}{p}\right)^2} = 1, \qquad \left(\frac{\omega}{p}\right)^2 = \frac{1}{2}$$

Using the given information:

$$\omega = 4\ Hz = 8\pi\ rad/s$$

$$p = \sqrt{\frac{k}{m}} = \sqrt{\frac{(100 \times 10^6)}{5000L}}$$

$$\left(\frac{\omega}{p}\right)^2 = \frac{(8\pi)^2(5000L)}{100 \times 10^6} = \frac{1}{2}$$

$$L = 15.83\ m$$

PART B) (5 pts)

<u>The crate will impact the ground when its amplitude is equal to the height h.</u>

For undamped base excitation:

$$X = \left| \frac{A}{1 - \left(\frac{\omega}{p}\right)^2} \right| = h$$

Where A is the amplitude of the crane arm (ie. the "base") of 5 cm.

The natural frequency p can be written in terms of L using the stiffness of the cable:

$$p^2 = \frac{k}{m} = \frac{100 \times 10^6}{5000L} = \frac{20000}{L}$$

A relation between L and h can be found using the height of the crane:

$$L = 100 - h$$

$$\left(\frac{\omega}{p}\right)^2 = \frac{(8\pi)^2 L}{20000} = 0.0032\pi^2(100 - h) = 0.32\pi^2 - 0.0032\pi^2 h$$

$$h = \left| \frac{A}{1 - 0.32\pi^2 + 0.0032\pi^2 h} \right| = \left| \frac{0.05}{0.0032\pi^2 h - 2.158} \right|$$

The absolute value must be considered, since ω/p can either be greater than or less than 1 depending on the length of the cable (which also determines its height).

$$h = \begin{cases} \dfrac{0.05}{0.0032\pi^2 h - 2.158} & \dfrac{\omega}{p} < 1 \\ \dfrac{0.05}{-0.0032\pi^2 h + 2.158} & \dfrac{\omega}{p} > 1 \end{cases}$$

As a result, there are multiple potential values of h.

CASE 1: $\underline{\omega/p < 1}$

$$h = \frac{0.05}{0.0032\pi^2 h - 2.158}$$

This gives a quadratic equation for h:

$$0.0032\pi^2 h^2 - 2.158h - 0.05 = 0$$

$$h = \frac{2.158 \pm \sqrt{2.158^2 - 4(0.0032\pi^2)(-0.05)}}{2(0.0032\pi^2)}$$

$$h = -0.0232 \text{ m}, \qquad 68.36 \text{ m}$$

$h \geq 0$, so the only valid solution for $\omega/\mathrm{p} < 1$ is:

$$h_1 = 68.36 \text{ m}$$

CASE 2: $\underline{\omega/p > 1}$

$$h = \frac{0.05}{2.158 - 0.0032\pi^2 h}$$

Similarly:

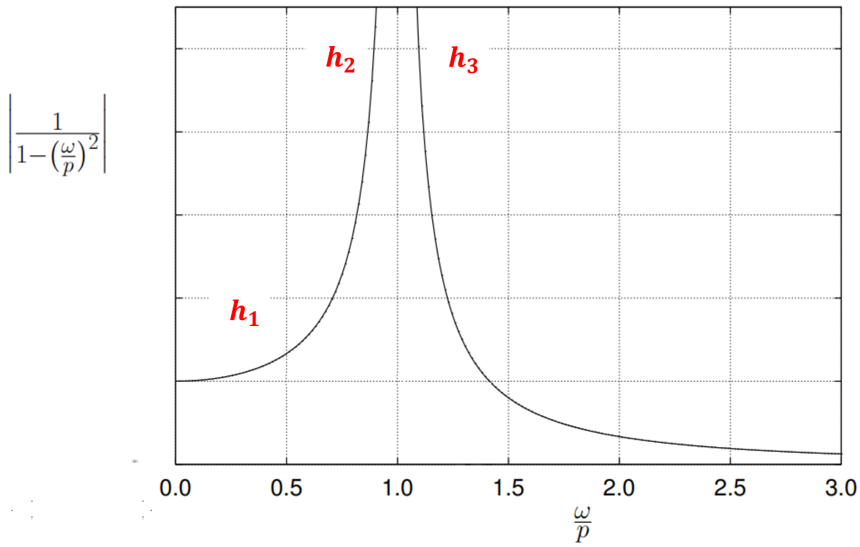
$$0.0032\pi^2 h^2 - 2.158h + 0.05 = 0$$

$$h = \frac{2.158 \pm \sqrt{2.158^2 - 4(0.0032\pi^2)(0.05)}}{2(0.0032\pi^2)}$$

$$h = 0.0231 \text{ m}, \qquad 68.31 \text{ m}$$

Therefore, the crate would hit the ground at a height of 2.31 cm, 68.31 m, and 68.36 m.

Side Note: h_1 (2.31 cm) corresponds to when the crate is very close to the ground to begin with. h_2 and h_3 (~68 m) correspond to the situation where the response of the crate is near resonance, resulting in a very large amplitude X:



PART C) (3 pts)

The dynamic magnification factor can be calculated using the information given about the amplitude of the crane arm A and the amplitude of the crate X:

$$\mathrm{DMF} = \frac{X}{A} = \frac{4\ cm}{5\ cm} = 0.8 = \left|\frac{1}{1-\left(\frac{\omega}{p}\right)^2}\right|$$

$$L = 100\ \mathrm{m} - 80\ \mathrm{m} = 20\ \mathrm{m}, \qquad k = 5 \times 10^6\ \mathrm{N/m}$$

$\omega > p$, so the absolute value brackets must be replaced with a factor of -1:

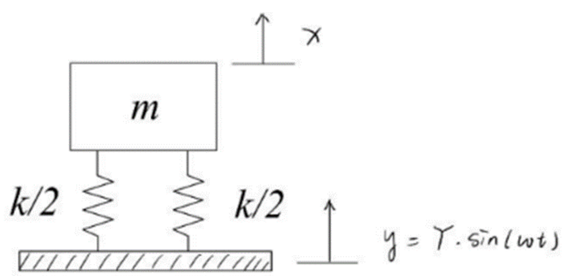
$$\mathrm{DMF} = \left|\frac{1}{1-\left(\frac{\omega}{p}\right)^2}\right| = \frac{1}{\frac{m\omega^2}{k} - 1}$$

$$m = \frac{\left(1 + \frac{1}{\mathrm{DMF}}\right)k}{\omega^2} = \frac{(1+1.25)(5 \times 10^6)}{(8\pi)^2}$$

$$m = 17810\ kg = 17.8\ \mathrm{tonnes}$$

Question 3 (10 pts)

National Institute of Nanotechnology (NINT) is a first-generation Nano-research facility and the first of its kind in Canada. The facility is a six-storey building located on the University of Alberta Campus. Along with research offices, wet laboratories and clean nano-fab space, the facility features several ultra sensitive electron microscopes. In order for these microscopes to operate in the nanoscale, they must be provided with an extremely stable environment that is free from movement and vibration. One model of isolation setup for the electron microscopes is shown as (1) where an excitation displacement is applied to the base in response of floor vibration. The floor vibration is assumed to have a frequency of 50 Hz. The natural frequency of these whole setup is measured as 10 Hz.



(1)

a) **(4 pts)** Estimate the amplitude of vibration of electron microscope by comparing it to the amplitude of the floor for the setup as (1).

b) **(5 pts)** The quality of the image from electron microscope is not satisfying since the amplitude is still too large. A dampler was added between the electron microscope and floor. What damping ratio should be chosen to make the amplitude of the vibration of electron microscope reduced to 10% of (1)?

c) **(1 pts)** If the damping ratio of the damper needs to be adjusted to increase the amplitude of vibration of electron microscope, should the laboratory staff adjust the damping factor up or down?

Question 3 Solution:

a)

Identify the forced vibration as the based excitation.

The equation of motion is the same, the particular solution is:

2 pts for correct equation

1 pts deduction if student uses a instead of Y

$$x_P(t) = Y\left(\frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right)\sin(\omega t)$$

Where the amplitude of microscope is $X = Y\left(\frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right)$.

1 pts for identify the correct amplitude of microscope

Y is the amplitude of floor.

ω is the frequency of floor, and ω_n is the natural frequency of the setup (a).

Find $\left|\frac{X}{Y}\right|$ in this case:

$$\left|\frac{X}{Y}\right| = \left|\frac{Y\left(\frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right)}{Y}\right| = \frac{1}{\left|1-\left(\frac{\omega}{\omega_n}\right)^2\right|}$$

1 pts for identify the correct relationship between X and Y.

Since the floor vibration frequency is 50 Hz = 314.16 rad/s, the setup (a) natural frequency is 10 Hz = 62.83 rad/s.

We have

$$\left|\frac{X}{Y}\right| = \frac{1}{\left|1-\left(\frac{\omega}{\omega_n}\right)^2\right|} = \frac{1}{\left|1-\left(\frac{314.16}{62.83}\right)^2\right|} = 0.042$$

2 pts for correct answer

1 pts deduction if student forget to use absolute value.

b) from (a), we got $X = Y \left| \frac{1}{1-(\frac{w}{w_n})^2} \right| = 0.042Y$

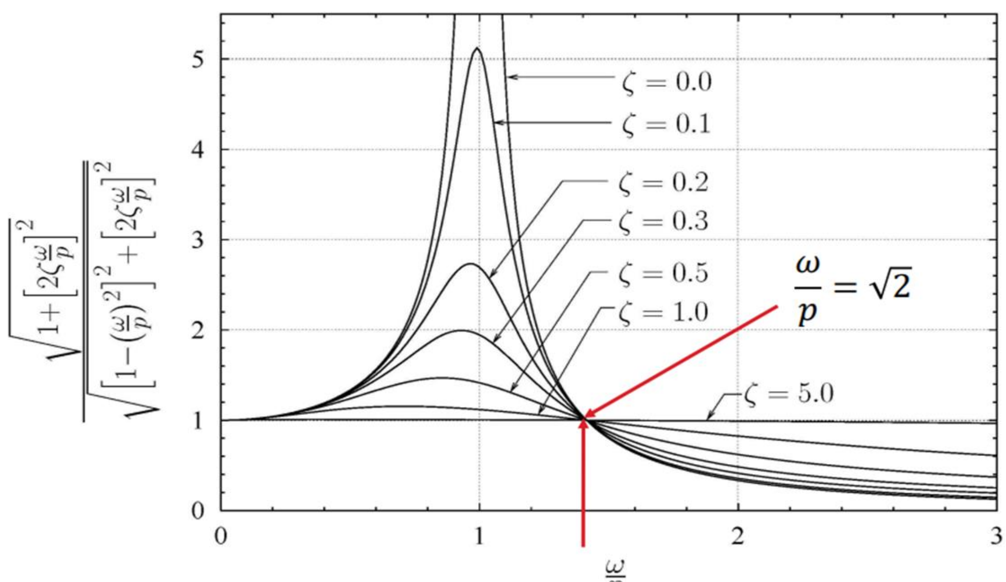
$$TR = 2 \times 0.042 = 0.084 = \frac{\sqrt{1+[2\xi(w/w_n)]^2}}{\sqrt{[1-(w/w_n)^2]^2+[2\xi(w/w_n)]^2}}$$

$$\Rightarrow 0.084 = \frac{\sqrt{1+[2\xi \times 5]^2}}{\sqrt{[1-25]^2+[2\times\xi\times 5]^2}}$$

$$\Rightarrow 0.084^2 = \frac{1+100\xi^2}{576+100\xi^2}$$

$$\Rightarrow \xi = 0.176, \quad 17.6\%$$

c)

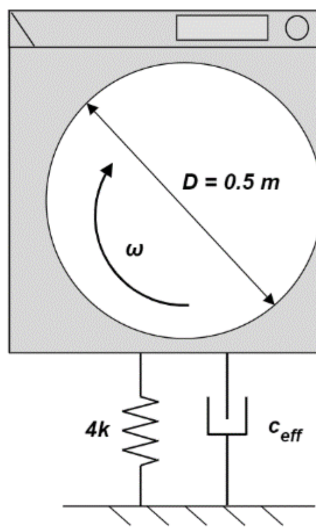


Since the frequency ratio (ω/p) is 5, it is the right part of the graph. Laboratory staff can increase the TR by increasing the damping ratio, thereby increasing the amplitude of the vibration of electron microscope.

Question 4 (10 pts)

A washing machine produces disruptive noise during its spin cycle due to an uneven distribution of clothes around its circumference. To reduce the noise, an engineering consulting firm has proposed that the machine be mounted on spring isolators at each corner (four in total).

The machine has a capacity of 10 kg of laundry, an **unloaded** mass of 450 kg, a drum diameter of 0.50 m, and spin cycle speed of 955 rpm. To address the noise levels, the transmitted force should be reduced by 75%. Assume vertical vibrations only and that c_{eff} is inherent to the system (damping from the spring isolators are negligible).



a) **(5 pts)** Determine the isolator spring constant needed to resolve the noise issue. Assume that the system is 25% damped **at full capacity** ($\zeta = 0.25$)

b) After installing an appropriate set of spring isolators, it was found that a 0.3 kg wet hoodie was spinning out of balance.

   i. **(2.5 pts)** Using the stiffness from part a), find the response amplitude due to this imbalance during the spin cycle for the washing machine when it is fully loaded.

   ii. **(2.5 pts)** After **removing the hoodie and resuming the same wash**, a wet towel with a mass of 0.5 kg began to spin out of balance. Determine the spring constants needed to maintain the same amplitude as part i).

QUESTION 4 SOLUTION

PART A (5 pts)

When the washing machine is fully loaded:

$$M = (450 + 10)\text{ kg} = 460\text{ kg}$$

$$\zeta = 0.25$$

The transmissibility is given by:

$$TR = \frac{\sqrt{1 + \left(2\zeta\frac{\omega}{p}\right)^2}}{\sqrt{\left(1 - \left(\frac{\omega}{p}\right)^2\right)^2 + \left(2\zeta\frac{\omega}{p}\right)^2}}$$

Let $\frac{\omega}{p} = r$.

$$TR^2[(1 - r^2)^2 + (2\zeta r)^2] = 1 + (2\zeta r)^2$$

If the transmitted force should be reduced by 75%, then the spring isolators should result in a transmissibility of 0.25. Plugging in $\text{TR} = 0.25, \zeta = 0.25$:

$$0.25^2(r^4 - 2r^2 + 1 + 0.25r^2) = 1 + 0.25r^2$$

$$0.0625r^4 - 0.359375r^2 - 0.9375 = 0$$

$$r^2 = -1.94845, \qquad 7.69845$$

Choosing the positive solution,

$$r = \frac{\omega}{p} = 2.77461, \qquad p = \frac{\omega}{r}$$

$$\omega = 955\text{ rpm} = 100\text{ rad/s}, \qquad p = \frac{100\text{ rad/s}}{2.77461} = 36.0411\text{ rad/s}$$

$$p = \sqrt{\frac{4k}{M}}, \qquad k = \frac{(36.0411)^2(460)}{4}$$

$$k = 149{,}381\text{ N/m} = 149.4\text{ kN/m}$$

PART B) (5 pts)

Part i)

Using $F_0 = \tilde{m}e\omega^2$ and $k_{eff} = 4k$, the response amplitude is given by:

$$X = \frac{\frac{\tilde{m}e\omega^2}{4k}}{\sqrt{\left[1-\left(\frac{\omega}{p}\right)^2\right]^2 + \left[2\zeta\frac{\omega}{p}\right]^2}}$$

Using the given information:

$$\tilde{m} = 0.3 \text{ kg}$$

$$e = 0.25 \text{ m}$$

$$\omega = 955 \text{ rpm} = 100 \text{ rad/s}$$

From part a):

$$p = 36.0411 \text{ rad/s}$$

$$r = \frac{\omega}{\omega_n} = 2.77461$$

$$X = \frac{\left(\frac{(0.3 \text{ kg})(0.25\text{m})(100 \text{ rad/s})^2}{4(149381 \text{ N/m})}\right)}{\sqrt{((1-(2.77461)^2)^2 + (2(0.25)(2.77461)^2}}$$

$$X = 0.000183 \text{ m} = 0.183 \text{ mm}$$

PART B) CONT'D

Part ii)

After removing the hoodie, the total mass of the washing machine decreases:

$$M' = 460 - 0.3 = 459.7 \text{ kg}$$

$$\widetilde{m}' = 0.50 \text{ kg}$$

$$\mathbb{X} = 0.000183 \text{ m}$$

$\zeta = 0.25$ is only valid for when the washing machine is fully loaded ($M = 460$ kg), but c_{eff} does not change.

Using information from part a):

$$c_{eff} = \zeta c_c = 2\zeta\sqrt{Mk_{eff}} = 0.25\left(2\sqrt{M(4k)}\right) = 0.25\left(2\sqrt{(460 \text{ kg})(4 \times 149381 \text{ N/m})}\right)$$

$$c_{eff} = 8289.5 \text{ Ns/m}$$

Then to find the spring constant needed to maintain the same amplitude:

$$\mathbb{X} = \frac{\widetilde{m}' e\omega^2}{\sqrt{(4k' - M'\omega^2)^2 + \left(c_{eff}\omega\right)^2}} = 0.000183 \text{ m}$$

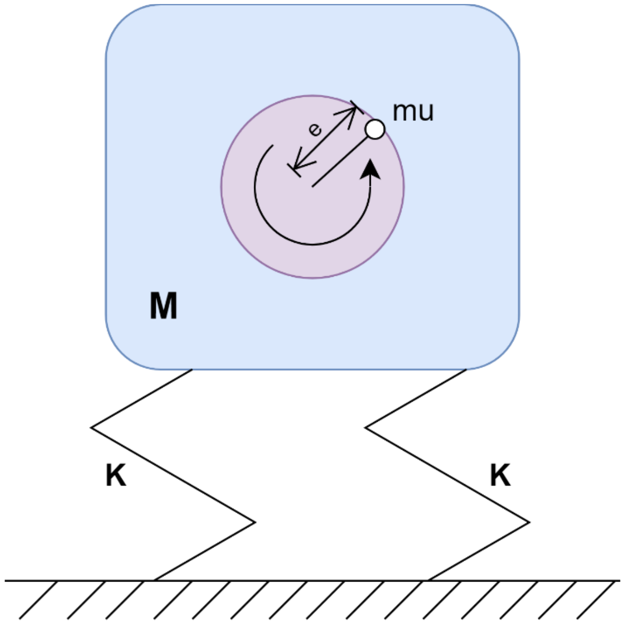
Plugging in values for $M', \widetilde{m}'$:

$$0.000183 = \frac{(0.50)(0.25)(100)^2}{\sqrt{(4k' - (459.7)(100)^2)^2 + [(8289.5)(100)]^2}}$$

$$k' = 2{,}839{,}713 \text{ N/m} = 2839.7 \text{ kN/m}$$

Question 5 (10 pts)

A motor with the mass 40 kg is supported with 4 springs, each of stiffness 250 N/m as it is illustrated in following figure. The rotor is unbalanced such that the unbalanced effect is equivalent mass of 5 kg located 50 mm from the axis of rotation. (Note: There is no damping in the system.)



a) **(5 pts)** Find the amplitude of vibration and the force transmitted to the foundation when the speed of motor is 1000 rpm.

b) **(5 pts.)** Compare the amplitude of vibration and the transmitted force calculated in part (a) with the case that the motor is running at speed of 60 rpm.

Question 5 Solution:

Known information: $M = 40\ kg, \widetilde{m} = 5\ kg, e = 0.05\ m, k_{eff} = 1000.$

a) The motor is running at speed of 1000 rpm:

$$\omega = 1000\ rpm = 104.7\frac{rad}{s}, p = \sqrt{\frac{K_{eff}}{M}} = \sqrt{\frac{1000}{40}} = 5\frac{rad}{s}, \frac{\omega}{p} = 20.94$$

Therefore, for calculation of the amplitude of vibration:

$$\frac{MX}{\widetilde{m}e} = \frac{\left(\frac{\omega}{p}\right)^2}{\left|1-\left(\frac{\omega}{p}\right)^2\right|} \to \frac{MX}{\widetilde{m}e} = \frac{20.94^2}{|1-20.94^2|} = 1.0022 \to X = \frac{5*0.05}{40} * 1.0022 = 0.00626\ m = 6.26\ mm$$

The transmitted force will be calculated as:

$$\frac{F_T}{\widetilde{m}e\omega^2} = \frac{1}{\left|1-\left(\frac{\omega}{p}\right)^2\right|} \to F_T = \frac{\widetilde{m}e\omega^2}{\left|1-\left(\frac{\omega}{p}\right)^2\right|} = \frac{5 * 0.05 * 104.7^2}{|1-(20.94)^2|} = 6.2643\ N$$

b) The motor is running at speed of 60 rpm:

$$\omega = 60\ rpm = 6.28\frac{rad}{s}, p = \sqrt{\frac{K_{eff}}{M}} = \sqrt{\frac{1000}{40}} = 5\frac{rad}{s}, \frac{\omega}{p} = 1.26$$

The same as previous section the amplitude of vibration can be calculated as:

$$\frac{MX}{\widetilde{m}e} = \frac{\left(\frac{\omega}{p}\right)^2}{\left|1-\left(\frac{\omega}{p}\right)^2\right|} \to \frac{MX}{\widetilde{m}e} = \frac{1.26^2}{|1-1.26^2|} = 2.7018 \to X = \frac{5*0.05}{40} * 2.7018$$
$$= 0.0168\ m = 16.8\ mm$$

Moreover, the transmitted force for the operating speed of 60 rpm is as follows:

$$\frac{F_T}{\widetilde{m}e\omega^2} = \frac{1}{\left|1-\left(\frac{\omega}{p}\right)^2\right|} \to F_T = \frac{\widetilde{m}e\omega^2}{\left|1-\left(\frac{\omega}{p}\right)^2\right|} = \frac{5 * 0.05 * 6.28^2}{|1-(1.26)^2|} = 16.77\ N$$