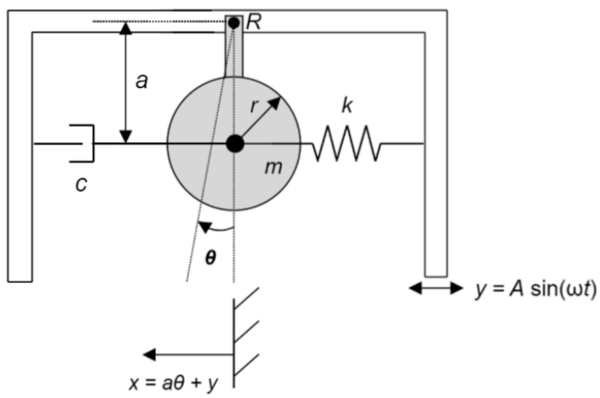
Question 1

The diagram models a gate valve which regulates the fluid flow in an engine. The gate valve is modelled as a pendulum (with a massless rod and uniform disk of mass m and radius r) pinned to the engine at R, and is controlled by a spring/damper mechanism connecting the centre of the disk to the engine.

The engine's motion during operation is denoted as $y = A\sin(\omega t)$, and the distance from R to the centre of the disk is a. Assuming small oscillations, the total horizontal displacement of the disk is denoted as x. The relative motion $z(t)$ between the gate and engine is given as:

$$z = x - y = a\theta$$

Note: the disk is fixed to the rod, so it does not spin about the point of connection with the rod, but the rod itself is pinned at R. For a uniform disk, $J = \frac{1}{2}mr^2$.



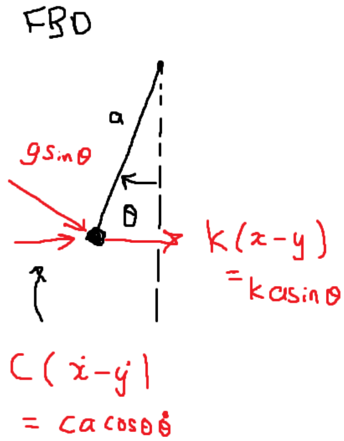
(a) *Determine the equation of motion for the relative motion of the gate with respect to the engine z, in terms of m, k, a, g, and r. Assume small oscillations.*

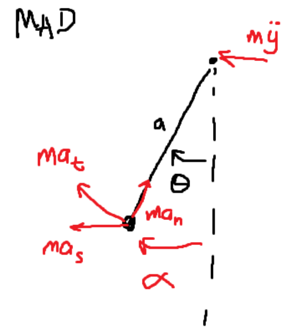
(b) *If $\omega = 2400$ rpm, $a = 70$ mm, $r = 35$ mm, $m = 2.25$ kg, $k = 8.75$ kN/m, and $c = 30$ Ns/m, determine the amplitude of the relative motion Z between the gate valve and engine in terms of A.*

Solution

(a)

Here is the FBD and MAD for the problem:





The equation of motion for the relative motion of the gate can be found by first taking the moment about the pin, R. Since the disk is undergoing rotation and translation, we consider general plane motion. First simplify the kinetic moment about the pin using the MAD diagram,

$$\begin{aligned}
\circlearrowright \sum M_R &= (\mathcal{M}_k)_R \\
&= J_G\ddot{\theta} + (a)m(a_G)_t + (a\cos\theta)m(a_G)_s \\
&= J_G\ddot{\theta} + ma(a\ddot{\theta}) + m(a\cos\theta)(\ddot{y}) \\
&= \underbrace{\left(J_G + ma^2\right)}_{J_R}\ddot{\theta} + ma\ddot{y}\cos\theta \qquad (1)
\end{aligned}$$

Next, summing the moments in the FBD,

$$\circlearrowright \sum M_R = -ka^2\sin\theta - ca^2\cos\theta\dot{\theta} - mga\sin\theta \qquad (2)$$

Equating (1) and (2),

$$J_R\ddot{\theta} + ma\ddot{y}\cos\theta = -ka^2\sin\theta - ca^2\cos\theta\dot{\theta} - mga\sin\theta$$

Assuming small oscillations, we can use the small angle approximation, $\sin\theta \approx \theta$ and $\cos\theta \approx 1$ to simplify the equation.

$$J_R\ddot{\theta} + ma\ddot{y} = -ka^2\theta - ca^2\dot{\theta} - mga\theta$$

Rearranging,

$$\left[J_R\right]\ddot{\theta} + \left[ca^2\right]\dot{\theta} + \left[ka^2 + mga\right]\theta = -ma\ddot{y}$$

Since $z = a\theta$, multiplying both sides by a,

$$\begin{aligned}
\left[J_R\right]a\ddot{\theta} + \left[ca^2\right]a\dot{\theta} + \left[ka^2 + mga\right]a\theta &= -ma^2\ddot{y} \\
\left[J_R\right]az + \left[ca^2\right]\dot{z} + \left[ka^2 + mga\right]z &= -ma^2\ddot{y}
\end{aligned}$$

Finally,

$$\boxed{\underbrace{\left[\frac{1}{2}mr^2 + ma^2\right]}_{m_{\text{eff}}}\ddot{z} + \underbrace{\left[ca^2\right]}_{c_{\text{eff}}}\dot{z} + \underbrace{\left[ka^2 + mga\right]}_{k_{\text{eff}}} z = \underbrace{ma^2 A\omega^2}_{F_o}\sin(\omega t)}$$

(b)

This is the standard form of a forced, damped harmonic oscillator. From Eq. (5.9) in the textbook, the amplitude is

$$\mathbb{Z} = \frac{F_o}{\sqrt{\left(k_{\text{eff}} - m_{\text{eff}}\omega^2\right)^2 + \left(c_{\text{eff}}\omega\right)^2}}$$

Determining the effective mass, damping, and stiffness,

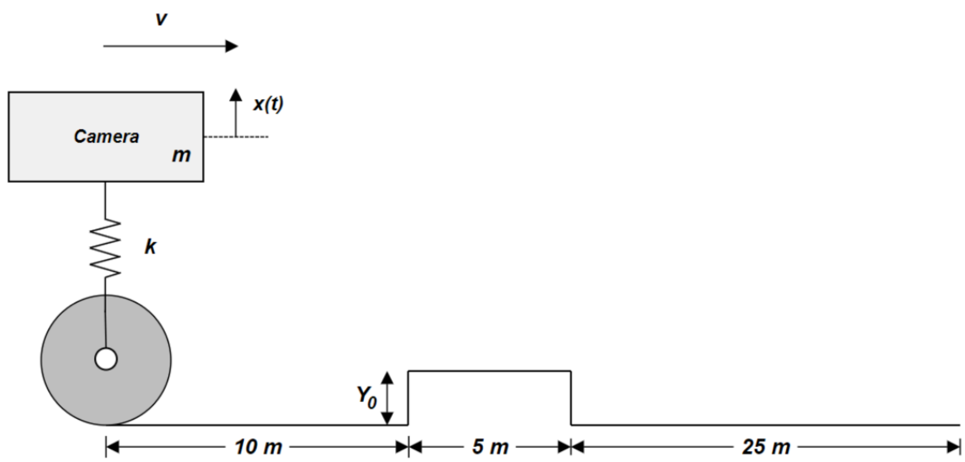
$$\begin{aligned}
m_{\text{eff}} &= \frac{1}{2}mr^2 + ma^2 = \frac{1}{2}(2.25)(0.035)^2 + (2.25)(0.070)^2 = 0.0124 \\
c_{\text{eff}} &= ca^2 = (30)(0.070)^2 = 0.147 \\
k_{\text{eff}} &= ka^2 + mga = (8.75 \times 10^3)(0.070)^2 + (2.25)(9.81)(0.070) = 44.420 \\
F_o &= ma^2 A\omega^2 = (2.25)(0.070)^2(2400 \times 2\pi/60)^2 A = 696.3993A
\end{aligned}$$

Substituting into the amplitude equation,

$$\begin{aligned}
\mathbb{Z} &= \frac{696.3993}{\sqrt{\left(44.420 - 0.0124(2400 \times 2\pi/60)^2\right)^2 + \left(0.147(2400 \times 2\pi/60)\right)^2}} A \\
&= \boxed{0.9414A}
\end{aligned}$$

Question 2

A film crew is shooting a movie and performs a tracking shot, which involves a moving camera that follows a subject. In this case the camera is situated on a wheeled cart, and the crew encounters a small bump on the road which causes the camera to oscillate vertically. The total distance to be travelled by the camera is 40 m, with the length of each section shown below, and the combined mass of the camera and the cart is 210 kg. Assume the cart moves at a constant speed v for the entire duration, and the height of the bump Y is 5 cm.



(a) *Fortunately the footage recorded while the camera moves on the bump will not be used in the final production, but the crew does not want any oscillations to occur after the camera cart has cleared the bump. If the entire tracking shot is to be filmed over 10 seconds, determine the required wheel suspension stiffness k.*

(b) *The production crew tries refilming the shot with a camera speed of $v/2$. In reference to the indicated coordinate x, determine the displacement (x) and velocity $\dot{x}$ of the camera at the end of the shot (i.e. at 40 m). Assume that the stiffness k corresponds to the answer in part (a).*

Solution

(a)

Assume that the cart's vertical displacement follows the ground exactly. Then at 10m, the forces on the camera are:

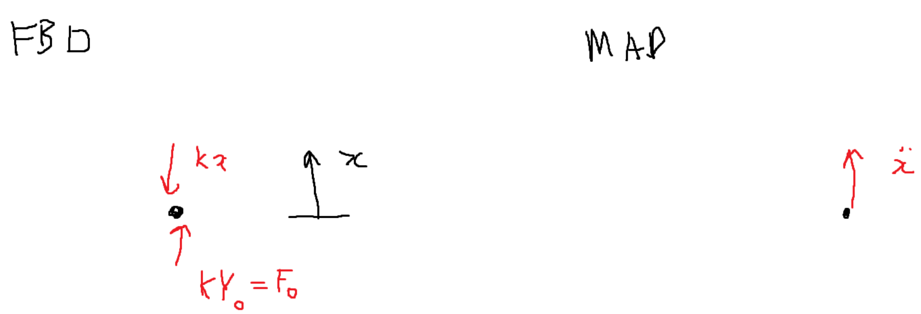


Figure 1: Free body diagram and mass acceleration diagram.

Define t as the time when the cart is at 10m. The initial conditions the time when the cart is at 10m, are:

$$x(0) = 0, \quad \dot{x}(0) = 0$$

From Eq. (7.3), the solution is given by:

$$\begin{aligned} x(t) &= \frac{F_0}{k}\left(1 - \cos(pt)\right) \\ &= Y_0\left(1 - \cos(pt)\right) \end{aligned}$$

From the textbook, we want to unload the cart at $t_2 = \frac{2\pi}{p}$ since this will be the time when the camera has zero displacement and zero velocity. We can solve for k using the following equation:

$$\begin{aligned} t_2 &= \frac{2\pi}{p} \\ t_2^2 &= \frac{4\pi^2}{k/m} \\ \implies k &= \frac{4\pi^2 m}{t_2^2} \end{aligned}$$

The time it takes for the cart to travel to the end of the bump is

$$t_2 = \frac{5}{v}$$

Then k is

$$\begin{aligned} k &= \frac{4\pi^2 m}{\left(\frac{5}{v}\right)^2} \\ &= \boxed{\frac{4\pi^2 m v^2}{25}} \end{aligned}$$

(b)

In part a), we found stiffness such that

$$\tau = \frac{2\pi}{p} = t_2 = \frac{5}{v}$$

Halving the speed of the cart will double the time it takes to travel the bump. The new time is then,

$$t_{2b} = \frac{5}{v/2} = \frac{10}{v} = \frac{4\pi}{p} = 2\tau$$

This means it will take 2 periods of oscillation for the cart to travel the bump. But at every integer multiple of τ, the displacement and velocity of the cart will be zero. Therefore, the displacement and velocity of the cart at the end of the shot will be zero.

$$\boxed{\begin{aligned} x(t_{40m}) &= 0 \\ \dot{x}(t_{40m}) &= 0 \end{aligned}}$$

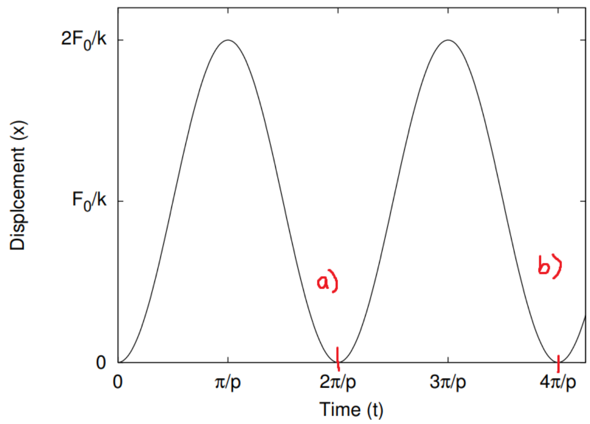
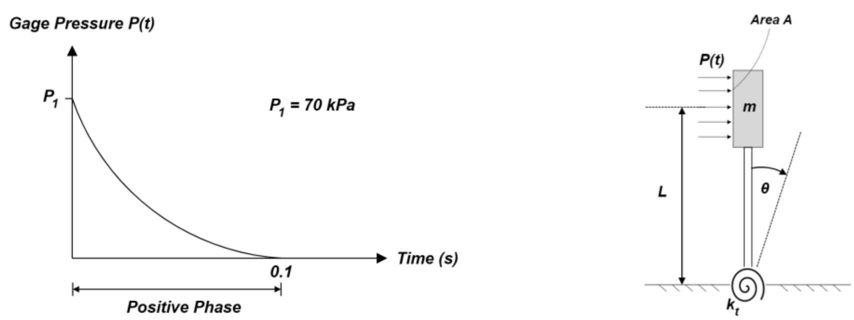


Figure 2: Displacement plot. Halving speed of the cart places it at $t = 4\pi/p$.

Question 3

Explosions send blast waves from the centre, which consist of an initial positive pressure wave front followed by a phase of negative pressure which pulls objects back towards the detonation centre. For this question, consider the positive pressure phase of a blast wave as shown below. Typically, the blast wave last 100 ms, and reaches a max pressure (gage) of around 70 kPa.



A nearby road sign (shown above on the right), is approximated by a massless rod supporting a uniform thin disk (viewed from the side) of mass m, radius r, and frontal area A. The centre of the disk is at a height $L = 2r$ from the ground, and the ground connection is modelled as a torsional spring k. Assume that the pressure-time relation during the positive phase is approximated as:

$$P(t) = \frac{P_1}{4}\left(e^{-at} + 3e^{-bt} - 6t\right), \quad 0 \leq t \leq 0.1 \text{ s}$$

The moment of inertia of a thin disk about its central diameter is $J = \frac{1}{4}mr^2$.

(a) *For the relation described above, determine the equation of motion for the sign during the positive phase in terms of m, L, $P(t)$, k, and A. Use as your coordinate t, $Ag\theta$.*

(b) *If the sign is initially at rest and $a = 10$, $b = 25$, $m = 1.5$ kg, $r = 0.25$ m, and $k = 1 \times 10^3$ Nm, calculate the response of the sign during the positive phase of the blast wave. You can neglect gravity in this case (k dominates the effective stiffness).*

Solution

(a)

The freebody diagram is as follows,

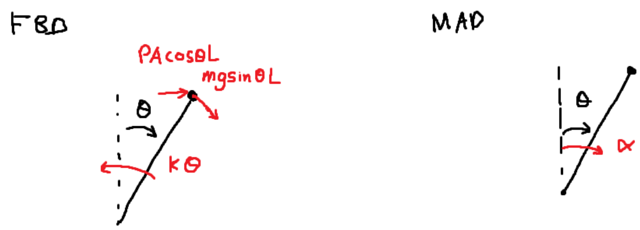


Figure 3: Freebody Diagram and Mass Acceleration Diagram

The moment of inertia is given about the central diameter. Since the sign is experiencing pure rotation about the pin, it is convenient to move the moment of inertia to the IC,

$$
\begin{aligned}
J_{\mathrm{IC}} &= J_G + mL^2 \\
&= \frac{1}{4}mr^2 + mL^2 \\
&= \frac{1}{4}m\left(\frac{L}{2}\right)^2 + mL^2 \\
&= \frac{1}{16}mL^2 + mL^2 \\
&= \frac{17}{16}mL^2
\end{aligned}
$$

The area of the sign is

$$
\begin{aligned}
A &= \pi r^2 \\
&= \pi\left(\frac{L}{2}\right)^2 \\
&= \frac{\pi L^2}{4}
\end{aligned}
$$

The equation of motion is given by,

$$
\begin{aligned}
\circlearrowright \sum M_{\mathrm{IC}} &= J_{\mathrm{IC}}\ddot{\theta} \\
&= -k_t\theta + PAL\cos\theta + mgL\sin\theta
\end{aligned}
$$

Rearranging,

$$
\boxed{\frac{17}{16}mL^2\ddot{\theta} + k_t\theta - mgL\sin\theta = \frac{\pi PL^3}{4}\cos\theta}
$$

Assuming small angle approximation, $\sin\theta \approx \theta$ and $\cos\theta \approx 1$,

$$\boxed{\underbrace{\left[\frac{17}{16}mL^2\right]}_{m_{\text{eff}}}\ddot{\theta} + \underbrace{\left[k_t - mgL\right]}_{k_{\text{eff}}}\theta = \underbrace{\frac{\pi P(t)L^3}{4}}_{F(t)}}$$

(b)

Let us rewrite the forcing term, $F(t)$, in a more convenient form,

$$\begin{aligned}
F(t) &= \frac{\pi L^3}{4}\cdot\frac{P_1}{4}\left(e^{-at} + 3e^{-bt} - 6t\right) \\
&= \frac{\pi P_1 L^3}{16}\left(e^{-at} + 3e^{-bt} - 6t\right) \\
&= \underbrace{\frac{\pi P_1 L^3}{16}}_{F_1}e^{-at} + \underbrace{\frac{3\pi P_1 L^3}{16}}_{F_2}e^{-bt} + \underbrace{\frac{-3\pi P_1 L^3}{8}}_{\beta}t
\end{aligned}$$

Neglecting gravity, the differential is now

$$\underbrace{\left[\frac{17}{16}mL^2\right]}_{m_{\text{eff}}}\ddot{\theta} + \underbrace{\left[k_t\right]}_{k_{\text{eff}}}\theta = \underbrace{\frac{\pi P_1 L^3}{16}}_{F_1}e^{-at} + \underbrace{\frac{3\pi P_1 L^3}{16}}_{F_2}e^{-bt} + \underbrace{\frac{-3\pi P_1 L^3}{8}}_{\beta}t$$

Substituting numbers,

$$\begin{aligned}
m_{\text{eff}} &= \frac{17}{16}\times 1.5\times(2\times 0.25)^2 = 0.3984375 \\
k_{\text{eff}} &= 1\times 10^3 \\
p &= \sqrt{\frac{k_{\text{eff}}}{m_{\text{eff}}}} = \sqrt{\frac{1000}{0.3984375}} = 50.1 \\
F_1 &= \frac{\pi\times 70\times 10^3\times(2\times 0.25)^3}{16} = 1718.0585 \\
F_2 &= \frac{3\pi\times 70\times 10^3\times(2\times 0.25)^3}{16} = 5154.1754 \\
\beta &= \frac{-3\pi\times 70\times 10^3\times(2\times 0.25)^3}{8} = -10308.3509
\end{aligned}$$

The homogeneous solution is given by,

$$\begin{aligned}
\theta_H(t) &= C_1\sin pt + C_2\cos pt \\
&= C_1\sin(50.1t) + C_2\cos(50.1t)
\end{aligned}$$

The particular solution due to F_1e^{-at} is given by Eq. (7.7) from the textbook,

$$\begin{aligned}
\theta_{p1}(t) &= \frac{F_1}{m_{\text{eff}}a^2 + k_{\text{eff}}}e^{-at} \\
&= \frac{1718.0585}{0.3984375\times 100 + 1000}e^{-10t} \\
&= 1.652e^{-10t}
\end{aligned}$$

The particular solution due to $F_2 e^{-bt}$ is given by Eq. (7.7) from the textbook,

$$\begin{aligned}\theta_{p2}(t) &= \frac{F_2}{m_{\text{eff}} b^2 + k_{\text{eff}}} e^{-bt} \\ &= \frac{5154.1754}{0.3984375 \times 625 + 1000} e^{-25t} \\ &= 4.1266 e^{-25t}\end{aligned}$$

The particular solution due to βt is given by Eq. (7.5) from the textbook,

$$\begin{aligned}\theta_{p3}(t) &= \frac{\beta}{k} t \\ &= \frac{-10308.3509}{1000} t \\ &= -10.3084t\end{aligned}$$

The total response is the sum of the homogeneous and particular solutions,

$$\begin{aligned}\theta(t) &= \theta_H(t) + \theta_{p1}(t) + \theta_{p2}(t) + \theta_{p3}(t) \\ &= C_1 \sin(50.1t) + C_2 \cos(50.1)t + 1.652e^{-10t} + 4.1266e^{-25t} - 10.3084t\end{aligned}$$

Applying the initial conditions,

$$\begin{aligned}\theta(0) &= 0 \\ \dot{\theta}(0) &= 0\end{aligned}$$

First, applying the initial condition $\theta(0) = 0$,

$$\begin{aligned}\theta_H(0) &= C_2 \\ \theta_{p1}(0) &= 1.652 \\ \theta_{p2}(0) &= 4.1266 \\ \theta_{p3}(0) &= 0\end{aligned}$$

Substituting the values into the total response,

$$\begin{aligned}\theta(0) = C_2 + 1.652 + 4.1266 &= 0 \\ C_2 &= -5.7786\end{aligned}$$

Next, applying the initial condition $\dot{\theta}(0) = 0$,

$$\begin{aligned}\dot{\theta}_H(0) &= 50.1C_1 \\ \dot{\theta}_{p1}(0) &= -10 \times 1.652 = -16.52 \\ \dot{\theta}_{p2}(0) &= -25 \times 4.1266 = -103.165 \\ \dot{\theta}_{p3}(0) &= -10.3084\end{aligned}$$

Substituting the values into the total response,

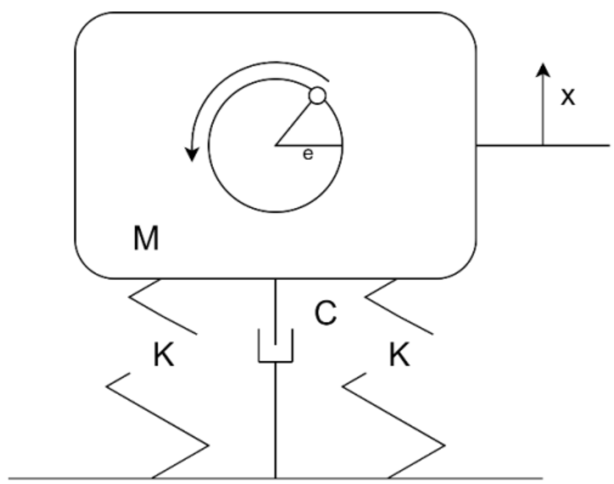
$$\begin{aligned}\dot{\theta}(0) &= 50.1C_1 - 16.52 - 103.165 - 10.3084 = 0\\ C_1 &= 2.59\end{aligned}$$

The response is then,

$$\boxed{\theta(t) = 2.59\sin(50.1t) - 5.7786\cos(50.1t) + 1.652e^{-10t} + 4.1266e^{-25t} - 10.3084t}$$

Question 4

A machine of 100 kg mass is supported on springs of total stiffness 700 kN/m and has an unbalanced rotating element, which results in a disturbing force of 350 N, at the speed of 3000 rpm. Assuming a damping ratio of $\zeta = 0.2$:



(a) *The amplitude of motion of the machine due to the unbalance.*

(b) *The transmissibility and the transmitted force to the base.*

Solution

(a)

This is a forced damped system with rotating imbalance. The amplitude of vibration is given from Eq. (5.18),

$$\mathbb{X} = \frac{F_o/k}{\sqrt{\left[1 - \left(\frac{\omega}{p}\right)^2\right]^2 + \left[2\zeta\frac{\omega}{p}\right]^2}}$$

All parameters are known except p. The natural frequency is

$$p = \sqrt{\frac{k}{m}} = \sqrt{\frac{700 \times 10^3}{100}} \times \frac{60}{2\pi} = 798.95 \text{ RPM}$$

then,

$$\begin{aligned}\mathbb{X} &= \frac{350/(700 \times 10^3)}{\sqrt{\left[1 - \left(\frac{3000}{798.95}\right)^2\right]^2 + \left[2 \times 0.2 \times \frac{3000}{798.95}\right]^2}} \\ &= \boxed{3.792 \times 10^{-5} \text{ m}}\end{aligned}$$

(b)

Transmissibility is given in Eq. (5.19),

$$TR = \frac{F_{T_{\text{Max}}}}{F_o} = \frac{\sqrt{1 + (2\zeta\frac{\omega}{p})^2}}{\sqrt{\left[1 - \left(\frac{\omega}{p}\right)^2\right]^2 + \left[2\zeta\frac{\omega}{p}\right]^2}}$$

Substituting the known values,

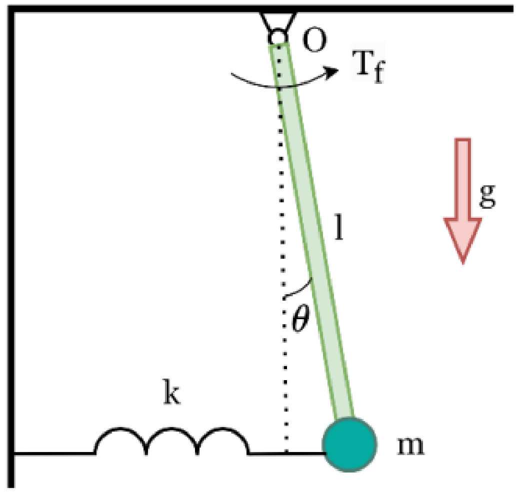
$$\begin{aligned}TR &= \frac{\sqrt{1 + (2 \times 0.2 \times \frac{3000}{798.95})^2}}{\sqrt{\left[1 - \left(\frac{3000}{798.95}\right)^2\right]^2 + \left[2 \times 0.2 \times \frac{3000}{798.95}\right]^2}} \\ &= \boxed{0.1369}\end{aligned}$$

Then, the transmitted force to the base is,

$$F_{T_{\text{Max}}} = TR \times F_o = 0.1369 \times 350 = \boxed{47.92 \text{ N}}$$

Question 5

There is a pendulum with mass of M connected to a wall with a spring of stiffness k. Also, there is a constant Coulomb friction torque of T acting on the pendulum. (Note: Assume that the oscillations are small).



(a) *Find the time response of the system and maximum angular acceleration of the pendulum. (Initial condition: $\theta = \frac{\pi}{12}$ and $\dot{\theta} = 0$)*

(b) *Compare the response of the system with the case where there is no friction.*

Solution

(a)

The freebody diagram is as follows,

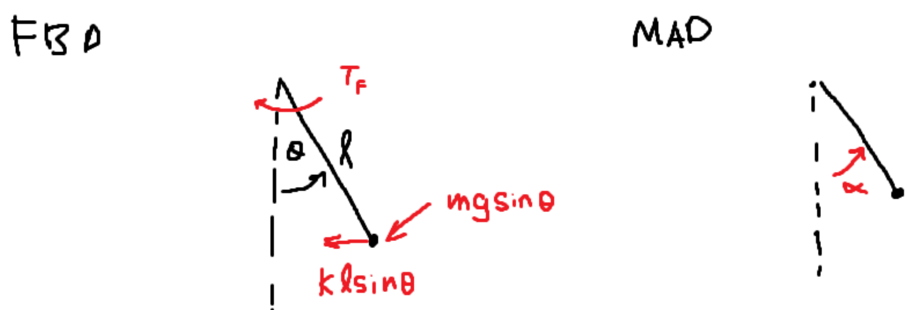


Figure 4: Freebody Diagram and Mass Acceleration Diagram

The equation of motion is given by,

$$\begin{aligned} \circlearrowleft \sum M_O &= I_O \ddot{\theta} \\ &= -k \sin\theta \cos\theta - mgl \sin\theta - T_f \end{aligned}$$

Assuming small angle approximation, $\sin\theta \approx \theta$ and $\cos\theta \approx 1$,

$$\underbrace{ml^2}_{m_{\text{eff}}} \ddot{\theta} + \underbrace{(k + mg)}_{k_{\text{eff}}} \theta = \underbrace{-T_f}_{F_0}$$

Assuming T_f is constant, from (7.2),

$$\begin{aligned}\theta &= A\sin pt + B\cos pt + \frac{F_0}{k_{\text{eff}}}\\ \dot{\theta} &= pA\cos pt - pB\sin pt\end{aligned}$$

Using the initial conditions,

$$\begin{aligned}\theta(0) &= \frac{\pi}{12}\\ \dot{\theta}(0) &= 0\end{aligned}$$

First, the initial velocity condition gives,

$$\begin{aligned}&\dot{\theta}(0) = pA\cos 0 - pB\sin 0 = 0\\ \implies &A = 0\end{aligned}$$

Now, the initial displacement condition gives,

$$\begin{aligned}&\theta(0) = B\cos 0 + \frac{F_0}{k_{\text{eff}}} = \frac{\pi}{12}\\ \implies &B = \frac{\pi}{12} - \frac{F_0}{k_{\text{eff}}}\end{aligned}$$

The solution is now,

$$\boxed{\theta(t) = \left(\frac{\pi}{12} - \frac{F_0}{k_{\text{eff}}}\right)\cos pt + \frac{F_0}{k_{\text{eff}}}}$$

The acceleration is given by,

$$\ddot{\theta} = -p^2\left(\frac{\pi}{12} - \frac{F_0}{k_{\text{eff}}}\right)\cos pt$$

where,

$$\boxed{\begin{aligned}m_{\text{eff}} &= ml^2\\ k_{\text{eff}} &= k + mg\\ F_0 &= -T_f\\ p &= \sqrt{\frac{k_{\text{eff}}}{m_{\text{eff}}}}\end{aligned}}$$

The maximum angular acceleration is given by the amplitude,

$$\boxed{\left|\ddot{\theta}\right|_{\max} = \left|p^2\left(\frac{\pi}{12} - \frac{F_0}{k_{\text{eff}}}\right)\right|}$$

(b)

The response of the system with no friction can be given by letting $T_f = -F_0 = 0$. The response is then,

$$\begin{aligned}\theta(t) &= \left(\frac{\pi}{12} - \frac{0}{k_{\text{eff}}}\right)\cos pt + \frac{0}{k_{\text{eff}}} \\ &= \boxed{\frac{\pi}{12}\cos pt}\end{aligned}$$