Conservation of Energy (Scalar)

Energy is a scalar quantity, and therefore it can be identified & formulated through the conservation of a scalar quantity:

Let's try ϕ = scalar,

$$\frac{\partial}{\partial t}\int_{\text{C.V.}} \rho\phi\, dV + \int_{\text{C.S.}} \rho\phi(\vec{u}\cdot\vec{n})\, dS = \oint_{\text{C.S.}} \Gamma\,\text{grad}(\phi)\cdot\vec{n}\, dS + \int_{\text{C.V.}} \underbrace{q_\phi}_{\text{source/sink}}\, dV$$

$$\implies \frac{\partial}{\partial t}(\rho\phi) + \frac{\partial}{\partial x_j}(\rho\phi u_j) = \frac{\partial}{\partial x_j}(\Gamma\frac{\partial\phi}{\partial x_j}) + q_\phi$$

Conservation of a scalar (Energy). In vector notation,

$$\partial_t(\rho\phi) + \text{div}(\rho\phi\vec{u}) = \text{div}(\Gamma\,\text{grad}(\phi)) + q_\phi$$

Where

- $\partial_t(\rho\phi)$ is the time rate of change of the scalar quantity ϕ (conservative term).
- $\text{div}(\rho\phi\vec{u})$ is the rate of change due to the flow due to $\vec{u}$ (advection term).
- $\text{div}(\Gamma\,\text{grad}(\phi))$ is the rate of change due to diffusion (Γ) (diffusion term).
- q_ϕ is the rate of production (source) or destruction (sink) of ϕ.

Chapter 4. Fundamental flows (Simplification)

Navier-Stokes equations are highly non-linear PDEs with no exact solutions. However, there are fundamental flow dynamics (simplified flows) based on assumptions and approximations that makes the mathematics easier to follow, solve, and interpret.

We are going to look at 4 simplified flow cases:

1. Incompressible flow (ρ = constant)
2. Invicid flow (Euler's flow) ($\mu \to 0$)
3. Creeping flow (Stokes flow) (Re $\ll 100$, inertial forces are negligible)
4. Potential flow (Re $\to 0$, Ma $\to 0$

Let's look at the conservation laws for each:

Incompressible flow

Incompressibility is defined as incapability of a fluid (i.e. liquid) to compress to a smaller size under internal/external loads. This, therefore, means that their **density** ρ does not change as long as we keep their mass the same.

Typically, liquids are incompressible, but air (gas) can become compressible at special conditions.

$$\underbrace{\text{Ma}}_{\vec{u}/\text{speed of sound}} > 0.3 \implies \text{compressible}$$

Continuity:

$$\overbrace{\cancel{\frac{\partial \rho}{\partial t}}}^{0(\text{Incomp})} + \boldsymbol{\nabla} \cdot (\rho \vec{u}) = 0$$
$$\implies \boldsymbol{\nabla} \cdot \vec{u} = 0$$

Momentum:

$$\frac{\partial}{\partial t}(\rho \vec{u}) + \boldsymbol{\nabla} \cdot (\rho \vec{u} \vec{u}) = -\boldsymbol{\nabla} P + \overbrace{\cancel{\frac{1}{3}\mu \nabla(\boldsymbol{\nabla} \cdot \vec{u})}}^{0,\ \text{continuity}} + \mu \nabla^2 \vec{u} + \rho \vec{b}$$

Expanding the $\boldsymbol{\nabla} \cdot (\rho \vec{u} \vec{u})$ term,

$$\begin{aligned} \boldsymbol{\nabla} \cdot (\rho \vec{u} \vec{u}) &= (\boldsymbol{\nabla} \cdot \rho \vec{u})\vec{u} + \rho \vec{u} \cdot \boldsymbol{\nabla} \vec{u} \\ &= \overbrace{\cancel{(\boldsymbol{\nabla} \cdot \rho \vec{u})}}^{0,\ \text{continuity}}\vec{u} + \rho \vec{u} \cdot \boldsymbol{\nabla} \vec{u} \\ &= \rho \vec{u} \cdot \boldsymbol{\nabla} \vec{u} \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{\partial}{\partial t}(\rho \vec{u}) + \rho \vec{u} \cdot \boldsymbol{\nabla} \vec{u} &= -\boldsymbol{\nabla} P + \mu \nabla^2 \vec{u} + \rho \vec{b} \\ \rho \frac{\partial \vec{u}}{\partial t} + \rho \vec{u} \cdot \boldsymbol{\nabla} \vec{u} &= -\boldsymbol{\nabla} P + \mu \nabla^2 \vec{u} + \rho \vec{b} \\ \frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \boldsymbol{\nabla} \vec{u} &= -\frac{\boldsymbol{\nabla} P}{\rho} + \underbrace{\frac{\mu}{\rho}}_{\nu} \nabla^2 \vec{u} + \vec{b} \end{aligned}$$

Invicid flow (Euler's flow)

Viscous forces can be important in flows close to a wall, where we have large velocity gradients (Also in wakes). As we should before, it is the combination of ν and $\vec{u}$ that forms the viscous effects in transport of fluids.

$\implies$ Vorticies $\to \nu$ may be important.

$\hookrightarrow$ if you are far from a surface or regions of large velocity gradients, the implication of viscocity becomes minimal.

↪ we quantify the effect of viscocity in the flow using Reynold's Number:

$$\mathrm{Re} = \frac{\rho u \overbrace{L}^{\text{characteristic length}}}{\mu} = \frac{\text{inertial forces}}{\text{viscous forces}}$$

if Re $\gg 1000 \implies \mu \to 0$ which means inertial forces dominate the flow (negligible viscous forces).

Continuity:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \vec{u}) = 0$$

No impact because no μ term.

Momentum:

$$\frac{\partial}{\partial t}(\rho \vec{u}) + \boldsymbol{\nabla} \cdot (\rho \vec{u}\vec{u}) = -\boldsymbol{\nabla} P + \cancel{\frac{1}{3}\mu \nabla(\boldsymbol{\nabla} \cdot \vec{u})} + \cancel{\mu \nabla^2 \vec{u}} + \rho \vec{b}$$

0, invicid

$$\boxed{\frac{\partial}{\partial t}(\rho \vec{u}) + \boldsymbol{\nabla} \cdot (\rho \vec{u}\vec{u}) = -\boldsymbol{\nabla} P + \rho \vec{b}}$$

Note: As we saw in the Navier-Stokes equation, the flow can only be dominated by the **Pressure** and **External forces**. This means that the invicid cond. cannot hold if we are dealing with areas of high straining (vorcity and wakes).

Note: Since we are assuring invicid condition, then the flow cannot slow down close to the stationary wall. $\implies$ Slip Boundary Condition.

Creeping flow (Stokes flow)

At high re, we just discussed that viscous effects are negligible. Contrarily, at low Re, (Re $\ll 100$) the effect of viscosity **dominates** the flow.

$\implies$ Inertial forces are negligible.

$$\mathrm{Re} \propto \frac{uD}{\nu} \begin{cases} \text{Either } u \to 0 \text{and/or} \\ D \to 0 \end{cases}$$

Continuity:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \vec{u}) = 0$$

No impact.

Momentum:

$$0 = -\frac{\boldsymbol{\nabla} P}{\rho} + \boldsymbol{\nabla} \cdot (\nu \boldsymbol{\nabla} \vec{u}) + \vec{b} + \vec{b}$$

This type of flow is mostly for porous media coating, or nano-fluidics.

Potential flow

One of the simpliest flows in fluid mechanics.

Based on two conditions:

1. Invicid flow ($\mu \to 0$)
2. Irrotational flow ($\vec{\omega} \to 0$)

provides approximation for initial flow conditions.

Chapter 5: Non-Dimentionalization of the Flow

Physically and mathematically, the flow results (dynamics) should not change based on the size of a simulation setup.

$\hookrightarrow$ Most fluid dynamics analyses are completed in a dimensionless framework. This allows for scaling of the real problem.

$\hookrightarrow$ If the scaled conditions are maintained (i.e. Reynold's Number is the same), CFD simulations should be independent from geometrical or scalable physical parameters.

Note: The scalability of the flow condition holds only if the main flow behavior/dynamics is the same in terms of Re, ρ, μ, ...

Now, we return to our governing equations and discuss means to make them non-dimentionalized using normalization factors.

For example, velocity can be normalized using the freestream condition,

$$u_i = u_i^* u_\infty \implies u_i^* = \frac{u_i}{u_\infty}$$

where u_∞ is the freestream velocity.

Similarly,

$$t_0 = \frac{C}{u_\infty} \implies t^* = \frac{t}{t_0} = \frac{u_\infty t}{C}$$

For pressure,

$$P_{\text{dyn}} = \frac{1}{2}\rho u_\infty^2 \implies P^* = \frac{P}{P_{\text{dyn}}} = \frac{2P}{\rho u_\infty^2}$$

and it is given that

$$X_i^* = \frac{X_i}{C} \implies X_i = X_i^* C$$

$\hookrightarrow$ Now, let's begin with the continuity equation (incompressible)

$$\frac{\partial u_i}{\partial x_i} = 0$$

Substituting u_i and x_i with their non-dimensionalized counterparts,

$$\begin{aligned}
\frac{\partial u_i^* u_\infty}{\partial x_i^* C} &= 0 \\
\cancel{\frac{u_\infty}{C}} \frac{\partial u_i^*}{\partial x_i^*} &= 0 \\
\frac{\partial u_i^*}{\partial x_i^*} &= 0
\end{aligned}$$

$\hookrightarrow$ Now, let's look at the momentum equation (incompressible)

$$\frac{\partial u_i}{\partial t} + \frac{\partial (u_i u_j)}{\partial x_j} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j} + b_i$$

Then,

$$\begin{aligned}
\frac{\partial (u_i^* u_\infty)}{\partial (t^* t_0)} + \frac{\partial (u_i^* u_j^* u_\infty^2)}{\partial (x_j^* C)} &= -\frac{1}{\rho}\frac{\partial (P^* P_{\text{dyn}})}{\partial (x_i^* C)} + \nu \frac{\partial^2 (u_i^* u_\infty)}{\partial (x_j^* C)\partial (x_j^* C)} + g b_i^* \\
\frac{\partial (u_i^* u_\infty)}{\partial (t^* t_0)} + \frac{\partial (u_i^* u_j^* u_\infty^2)}{\partial (x_j^* C)} &= -\frac{1}{\rho}\frac{\partial (P^* P_{\text{dyn}})}{\partial (x_i^* C)} + \nu \frac{\partial^2 (u_i^* u_\infty)}{\partial (x_j^* C)\partial (x_j^* C)} + g b_i^* \\
\left(\frac{u_\infty}{t_0}\right)\frac{\partial u_i^*}{\partial t^*} + \left(\frac{u_\infty^2}{C}\right)\frac{\partial (u_i^* u_j^*)}{\partial x_j^*} &= -\frac{1}{\rho}\left(\frac{\frac{1}{2}\rho u_\infty^2}{C}\right)\frac{\partial P^*}{\partial x_i^*} + \left(\frac{\nu u_\infty}{C^2}\right)\frac{\partial^2 u_i^*}{\partial x_j^* \partial x_j^*} + g b_i^*
\end{aligned}$$

Now multiply by C/u_∞^2,

$$\frac{C}{u_\infty t_0}\frac{\partial u_i^*}{\partial t^*} + \frac{\partial (u_i^* u_j^*)}{\partial x_j^*} = -\frac{1}{2}\frac{\partial P^*}{\partial x_i} + \frac{\nu}{u_\infty C}\frac{\partial^2 u_i^*}{(\partial x_j^*)^2} + \frac{gC}{u_\infty^2} b_i^*$$

Therefore we obtain a number of important characteristic flow quantities:

1. Strouhal Number: This dimentionless number is wkd as a number to describe flow unsteadiness (periodicity). Hence, $f_i = 1/t_i$ in the frequency of unsteadiness, which relates to vortex formulation frequency, flapping wings, etc.

$$\text{St} = \frac{f_i C}{u_\infty} = \frac{C}{\underbrace{P_i}_{\text{Period}} u_\infty}$$

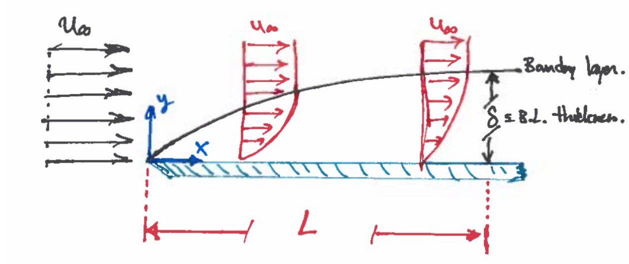

Figure 1: Boundary Layer Concept

2. Reynold's Number: Describes the ratio of inertial to viscous effects.

$$\mathrm{Re} = \frac{\rho u_\infty C}{\mu} = \frac{u_\infty C}{\nu}$$

3. Froude Number: Describes the ratio of inertial to external fields in the flow

$$\mathrm{Fr} = \frac{u_\infty}{\sqrt{Cg}}$$

   This enables us to understand if the flow is driven by inertial forces or external effects (gravity)

From the energy equation, we will get the Peclet Number, which describes the ratio of convection to diffusion effects.

$$\mathrm{Pe} = \frac{C u_\infty}{k/\rho C_p}$$

Boundary Layer Approximation

Boundary layers are one of the most common fluid flow phenomena observed in nature. In fact, the wind on an Earth's atmosphere is the result of a boundary layer developed by moving air next to the planet's surface. This is referred to as the atmospheric boundary. $\hookrightarrow$ Correct understanding of the flow dynamics inside the boundary layer is critical in technology development, control systems, and weather forecasting.

In classical fluid mechanics, we make unique assumptions, based in which we can apply certain approximating to our governing equations. For a laminar boundary layer, there are 3 assumption that drive our approximation process.

1. B.L are 2-D. $\implies$ at high Re, this can become problematic.
2. The thickness of the B.L. (δ) is small compared to the other characteristic length. $\implies$ Mostly true.
3. The flow velocity in the streamwise direction dominates. $\implies$ Provable.

Assuming that $\mathrm{Re}_l \gg 1$ and $\delta \ll \ell$, then we can rely on the following normalization factors (order of magnitude analysis). Let's start with the continuity equation (incompressible flow):

Table 1: Order of Magnitude Analysis

	Variable	**Order of Magnitude**	**Normalization Factor**
Streamline Velocity	u	u_∞	$u = u^* u_\infty$
Streamline Spatial Coordinate	x	ℓ	$x = x^* \ell$
Orthogonal Directional Coordinate	y	δ	$y = y * \delta$
Orthogonal Velocity	v	$\mathcal{V}$??	$v = v * \mathcal{V}$

$$\frac{\partial u_i}{\partial x_i} = 0$$
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$
$$\frac{\partial u^* u_\infty}{\partial x^* \ell} + \frac{\partial v^* \mathcal{V}}{\partial y^* \delta} = 0$$
$$\implies \left(\frac{u_\infty}{\ell}\right) \frac{\partial u^*}{\partial x^*} + \left(\frac{\mathcal{V}}{\delta}\right) \frac{\partial v^*}{\partial y^*} = 0$$
$$\implies \frac{\partial u^*}{\partial x^*} + \frac{\mathcal{V}\ell}{u_\infty \delta} \frac{\partial v^*}{\partial y^*} = 0$$

Based on the order of magnitude analysis,

$$\frac{\mathcal{V}\ell}{u_\infty \delta} = 1$$
$$\implies \mathcal{V} = \frac{u_\infty \delta}{\ell}$$

Therefore, if $\delta ll \ell$, then $\mathcal{V} \ll u_\infty$. This provies that streamline velocity dominates.

Now, let's move to the Navier-Stokes equation. First, the x-dir:

$$\cancelto{\frac{\partial u}{\partial t}}{S.S.} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

y-dir:

$$\cancelto{\frac{\partial v}{\partial t}}{S.S.} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

Let's look at the x-dir expression:

$$(u^* u_\infty)\frac{\partial(u^* u_\infty)}{\partial(x^*\ell)} + (v^*\mathcal{V})\frac{\partial(u^* u_\infty)}{\partial(y^*\delta)} = -\frac{1}{\rho}\frac{\partial P}{\partial(x^*\ell)} + \nu\left(\frac{\partial^2(u^* u_\infty)}{\partial(x^*\ell)^2} + \frac{\partial^2(u^* u_\infty)}{\partial(y^*\delta)^2}\right)$$
$$\implies \left(\frac{u_\infty^2}{\ell}\right)u^*\frac{\partial u^*}{\partial x^*} + \left(\frac{u_\infty\mathcal{V}}{\delta}\right)v^*\frac{\partial u^*}{\partial y^*} = -\frac{1}{\rho\ell}\frac{\partial P}{\partial x^*} + \nu\left(\left(\frac{u_\infty}{\ell^2}\right)\left(\frac{\partial^2 u^*}{\partial(x^*)^2} + \left(\frac{u_\infty}{\delta^2}\right)\frac{\partial^2 u^*}{\partial(y^*)^2}\right)\right)$$

Using $\mathcal{V} = \frac{u_\infty\delta}{\ell}$,

$$\left(\frac{u_\infty^2}{\ell}\right)u^*\frac{\partial u^*}{\partial x^*} + \left(\frac{u_\infty^2}{\ell}\right)v^*\frac{\partial u^*}{\partial y^*} = -\frac{1}{\rho\ell}\frac{\partial P}{\partial x^*} + \frac{\nu u_\infty}{\delta^2}\left(\cancelto{\left(\frac{\delta^2}{\ell^2}\right)}{0}\frac{\partial^2 u^*}{\partial(x^*)^2} + \frac{\partial^2 u^*}{\partial(y^*)^2}\right)$$

Note the term inside the ν term is zero since $\delta \ll \ell$. From order of magnitude analysis, we can say

Scaling of LHS Scaling of RHS

$$\frac{\delta^2}{\ell^2}\ \frac{\nu}{\ell u_\infty} = \frac{1}{\mathrm{Re}}$$

So, if $\delta^2 \ll \ell^2$, then $\frac{1}{\mathrm{Re}} \ll 1$. This implies the flow remains 2D for high Re.

At this point, we can look at the scaling for pressure,

$$\frac{u_\infty^2}{\ell}\ \text{(Pressure Scaling)}\left(\frac{1}{\rho}\frac{\partial P}{\partial x} = 0\left(\frac{\rho u_\infty^2}{\ell}\right)\right)$$

Let's apply the same process to the y-direction. The results are:

$$\left(\frac{u_\infty\mathcal{V}}{\ell}\right)u^*\frac{\partial v^*}{\partial x^*} + \left(\frac{\mathcal{V}^2}{\delta^2}\right)v^*\frac{\partial v^*}{\partial y^*} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \nu\left(\left(\frac{\mathcal{V}}{\ell^2}\right)\frac{\partial^2 v^*}{\partial(x^*)^2} + \left(\frac{\mathcal{V}}{\delta^2}\right)\frac{\partial^2 v^*}{\partial(y^*)^2}\right)$$

Since $\delta/\ell \ll 1$ and $u_\infty\mathcal{V}/\ell = 0$

$$\implies 0 = \frac{1}{\rho}\frac{\partial P}{\partial y^*}$$

We can see that

$$\text{Scaling of } \frac{\partial P}{\partial y}\ \frac{u_\infty^2\delta}{\ell^2}$$

We can again show that

$$\frac{\partial P}{\partial x} gg \frac{\partial P}{\partial y}$$

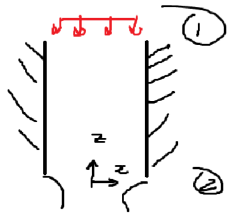

Figure 2: Viscous Liquid in a Narrow Channel

Example

Consider a viscous liquid in steady state that is in a downward motion in a narrow channel.

(a) Write the full Navier-Stokes equations in the z-direction with body forces using index notation.

(b) Simplify the obtained equation from (a) for a fully-developed, 2D, steady flow with cartesian viscosity across the channel width.

(a) We write the general form of the Navier-Stokes equation:

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) = -\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right) - \frac{\partial}{\partial x_i}\left(\frac{2}{3}\mu\frac{\partial u_k}{\partial x_k}\right) + \rho b_i$$

in the z-dir, $i = 3$:

$$\boxed{\partial_t(\rho w) + \partial_j(\rho w u_j) = -\partial_z P + \partial_j\left(\mu\left(\partial_j w + \partial_i u_j\right)\right) - \partial_z\left(\frac{2}{3}\mu\partial_k u_k\right) + \rho b_z}$$

(b) We need to write out our assumptions and their approximations Now, we can start

Assumption	Approximation
Viscous flow	No slip B.C. @ walls
Incompressibe flow	$\rho =$ constant
Steady State	$\partial_t = 0$
2D flow	$\partial_y = 0$
Fully developed	$\partial u_i/\partial x \gg \partial u_i/\partial z \implies \partial u_i/\partial z = 0$

applying the approximation to our governing equations. First, continuity,

$$\rho \cdot (\rho\vec{u}) = 0$$
$$\implies \partial_x u + \cancelto{\text{2D}}{\partial_y v} + \cancelto{\text{fully dev.}}{\partial_z w} = 0$$
$$\implies \partial_x u = 0$$
$$\implies u = c_1$$

Since we have no-slip at the walls,

$$u|_{x=\text{width}/2} = 0 \implies c_1 = 0$$

Now, let's move to our z-direction Navier-Stokes equation:

$$\begin{aligned}
\cancelto{\text{S.S.}}{\partial_t(\rho w)} + \partial_x(\rho u \cancelto{\text{cont.}}{w}) + \partial_y(\rho \cancelto{\text{2D}}{v w}) + \cancelto{\text{fully dev.}}{\partial_z}(\rho w^2) &= -\partial_z P + \partial_x\left(\mu\left(\cancelto{\text{2D}}{\partial_x w} + \cancelto{\text{fully dev.}}{\partial_z u}\right)\right) \\
&+ \cancelto{\text{2D}}{\partial_y}\left(\mu\left(\partial_y w + \partial_z v\right)\right) + \partial_z\left(\cancelto{\text{fully dev.}}{\mu(\partial_z w + \partial_z w)}\right) \\
&- \partial_z\left(\cancelto{\text{cont.}}{\frac{2}{3}\mu(\partial_x u + \partial_y v + \partial_z w)}\right) + \rho b_z
\end{aligned}$$

Therefore,

$$\begin{aligned}
0 &= -\partial_z P + \partial_x\left(\mu \partial_x w\right) + \rho b_z \\
\implies \partial_z P &= \mu \partial_{xx} w + \rho b_z
\end{aligned}$$

At location (2), we have that flow open to outside. This means that the flow cannot be fully developed.

$$P_2 = P_{\text{atm}}$$

Assumptions	Approximations
Incomp. flow	$\rho = \text{constant} \implies \nabla \cdot \vec{u} = 0$
2D flow	$\partial_y = 0$ and $v = 0$
Steady State	$\partial_t = 0$
Open to air	$P_2 = P_{\text{atm}}$
Gravity	$b_z = -g$

$$\begin{aligned}
\cancelto{\text{S.S.}}{\partial_t(\rho w)} + \partial_x(\rho u w) + \cancelto{\text{2D}}{\partial_y(\rho v w)} + \partial_z(\rho w^2) &= -\cancelto{\text{Atm.}}{\partial_z P} + \partial_x\left(\mu\left(\partial_x w + \partial_z u\right)\right) \\
&+ \partial_y\left(\cancelto{\text{cont.}}{\mu(\partial_y w + \partial_z v)}\right) + \partial_z\left(\mu\left(\partial_z w + \partial_z w\right)\right) \\
&- \partial_z\left(\cancelto{\text{cont.}}{\frac{2}{3}\mu(\partial_x u + \partial_y v + \partial_z w)}\right) + \rho b_z
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\partial_x(uw) + \partial_z(\rho w^2) &= \nu \partial_x\left(\partial_x w + \partial_z u\right) + \nu \partial_z\left(2\partial_z w\right) - g \\
\implies w\partial_x u + u\partial_x w + 2w\partial_z w &= \nu\left(\partial_{xx} w + \partial_{xz} u + 2\partial_{zz} w\right) - g
\end{aligned}$$

Consider a heat conduction at steady state with no source or sink. The boundary conditions are:

$$
\begin{aligned}
@x = 0, &\quad T = T_1 \\
@x = L, &\quad T = T_2
\end{aligned}
$$

Now, solve for temperature distribution across the solid in a 1D case.

$$\rho C_p \partial_t T + \rho C_p \partial_x^2 T = k \partial_x^2 T + Q$$

Now assumptions,

Assumptions	Approximations
Steady State	$\partial_t = 0$
No flow	$u = 0$
No source/sink	$Q = 0$

$$
\begin{aligned}
\implies 0 &= k \partial_x^2 T \\
T &= c_1 x + c_2
\end{aligned}
$$

Now, applying boundary conditions,

$$
\begin{aligned}
T|_{x=0} = T_1 &\implies c_2 = T_1 \\
T|_{x=L} = T_2 \implies c_1 L + T_1 = T_2 &\implies c_1 = \frac{T_2 - T_1}{L}
\end{aligned}
$$

Therefore,

$$T = \frac{T_2 - T_1}{L} x + T_1$$

Example 3

Consider the example above but with the following changes

- There is a constant heat source in the material
- The wall at $x = L$ is adiabatic ($\partial_x T = 0$)
- The B.C at $x = 0$ stays $T = T_1$

$$\cancelto{\text{S.S}}{\rho C_p \partial_t T} + \cancelto{\text{No flow}}{\rho C_p \partial_x u} \partial_x T = k \partial_x^2 T + Q$$

So,

$$
\begin{aligned}
k \partial_x^2 T &= -Q \\
\implies T &= \frac{-Q}{2k} x^2 + c_1 x + c_2
\end{aligned}
$$

Using the boundary conditions,

$$T = \frac{-Q}{2k}\left(\frac{x^2}{2} - Lx\right) + T_1$$