Chapter 5: Non-Dimentionalization of the Flow

Physically and mathematically, the flow results (dynamics) should not change based on the size of a simulation setup.

$\hookrightarrow$ Most fluid dynamics analyses are completed in a dimensionless framework. This allows for scaling of the real problem.

$\hookrightarrow$ If the scaled conditions are maintained (i.e. Reynold's Number is the same), CFD simulations should be independent from geometrical or scalable physical parameters.

Note: The scalability of the flow condition holds only if the main flow behavior/dynamics is the same in terms of Re, ρ, μ, ...

Now, we return to our governing equations and discuss means to make them non-dimentionalized using normalization factors.

For example, velocity can be normalized using the freestream condition,

$$u_i = u_i^* u_\infty \implies u_i^* = \frac{u_i}{u_\infty}$$

where u_∞ is the freestream velocity.

Similarly,

$$t_0 = \frac{C}{u_\infty} \implies t^* = \frac{t}{t_0} = \frac{u_\infty t}{C}$$

For pressure,

$$P_{\text{dyn}} = \frac{1}{2}\rho u_\infty^2 \implies P^* = \frac{P}{P_{\text{dyn}}} = \frac{2P}{\rho u_\infty^2}$$

and it is given that

$$X_i^* = \frac{X_i}{C} \implies X_i = X_i^* C$$

$\hookrightarrow$ Now, let's begin with the continuity equation (incompressible)

$$\frac{\partial u_i}{\partial x_i} = 0$$

Substituting u_i and x_i with their non-dimensionalized counterparts,

$$\begin{aligned}
\frac{\partial u_i^* u_\infty}{\partial x_i^* C} &= 0 \\
\frac{\cancel{u_\infty}}{\cancel{C}} \frac{\partial u_i^*}{\partial x_i^*} &= 0 \\
\frac{\partial u_i^*}{\partial x_i^*} &= 0
\end{aligned}$$

$\hookrightarrow$ Now, let's look at the momentum equation (incompressible)

$$\frac{\partial u_i}{\partial t} + \frac{\partial (u_i u_j)}{\partial x_j} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j} + b_i$$

Then,

$$\begin{aligned}
\frac{\partial (u_i^* u_\infty)}{\partial (t^* t_0)} + \frac{\partial (u_i^* u_j^* u_\infty^2)}{\partial (x_j^* C)} &= -\frac{1}{\rho}\frac{\partial (P^* P_{\text{dyn}})}{\partial (x_i^* C)} + \nu \frac{\partial^2 (u_i^* u_\infty)}{\partial (x_j^* C)\partial (x_j^* C)} + g b_i^* \\
\frac{\partial (u_i^* u_\infty)}{\partial (t^* t_0)} + \frac{\partial (u_i^* u_j^* u_\infty^2)}{\partial (x_j^* C)} &= -\frac{1}{\rho}\frac{\partial (P^* P_{\text{dyn}})}{\partial (x_i^* C)} + \nu \frac{\partial^2 (u_i^* u_\infty)}{\partial (x_j^* C)\partial (x_j^* C)} + g b_i^* \\
\left(\frac{u_\infty}{t_0}\right)\frac{\partial u_i^*}{\partial t^*} + \left(\frac{u_\infty^2}{C}\right)\frac{\partial (u_i^* u_j^*)}{\partial x_j^*} &= -\frac{1}{\rho}\left(\frac{\frac{1}{2}\rho u_\infty^2}{C}\right)\frac{\partial P^*}{\partial x_i^*} + \left(\frac{\nu u_\infty}{C^2}\right)\frac{\partial^2 u_i^*}{\partial x_j^* \partial x_j^*} + g b_i^*
\end{aligned}$$

Now multiply by C/u_∞^2,

$$\frac{C}{u_\infty t_0}\frac{\partial u_i^*}{\partial t^*} + \frac{\partial (u_i^* u_j^*)}{\partial x_j^*} = -\frac{1}{2}\frac{\partial P^*}{\partial x_i} + \frac{\nu}{u_\infty C}\frac{\partial^2 u_i^*}{(\partial x_j^*)^2} + \frac{gC}{u_\infty^2} b_i^*$$

Therefore we obtain a number of important characteristic flow quantities:

1. Strouhal Number: This dimentionless number is wkd as a number to describe flow unsteadiness (periodicity). Hence, $f_i = 1/t_i$ in the frequency of unsteadiness, which relates to vortex formulation frequency, flapping wings, etc.

$$\text{St} = \frac{f_i C}{u_\infty} = \frac{C}{\underbrace{P_i}_{\text{Period}} u_\infty}$$

2. Reynold's Number: Describes the ratio of inertial to viscous effects.

$$\text{Re} = \frac{\rho u_\infty C}{\mu} = \frac{u_\infty C}{\nu}$$

3. Froude Number: Describes the ratio of inertial to external fields in the flow

$$\text{Fr} = \frac{u_\infty}{\sqrt{Cg}}$$

   This enables us to understand if the flow is driven by inertial forces or external effects (gravity)

From the energy equation, we will get the Peclet Number, which describes the ratio of convection to diffusion effects.

$$\text{Pe} = \frac{C u_\infty}{k/\rho C_p}$$

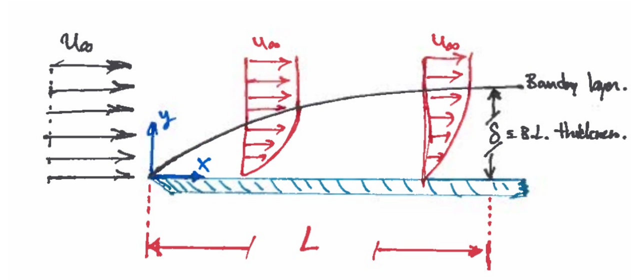

Figure 1: Boundary Layer Concept

Table 1: Order of Magnitude Analysis

	Variable	**Order of Magnitude**	**Normalization Factor**
Streamline Velocity	u	u_∞	$u = u^* u_\infty$
Streamline Spatial Coordinate	x	ℓ	$x = x^* \ell$
Orthogonal Directional Coordinate	y	δ	$y = y * \delta$
Orthogonal Velocity	v	$\mathcal{V}$??	$v = v * \mathcal{V}$

Boundary Layer Approximation

Boundary layers are one of the most common fluid flow phenomena observed in nature. In fact, the wind on an Earth's atmosphere is the result of a boundary layer developed by moving air next to the planet's surface. This is referred to as the atmospheric boundary. $\hookrightarrow$ Correct understanding of the flow dynamics inside the boundary layer is critical in technology development, control systems, and weather forecasting.

In classical fluid mechanics, we make unique assumptions, based in which we can apply certain approximating to our governing equations. For a laminar boundary layer, there are 3 assumption that drive our approximation process.

1. B.L are 2-D. $\implies$ at high Re, this can become problematic.
2. The thickness of the B.L. (δ) is small compared to the other characteristic length. $\implies$ Mostly true.
3. The flow velocity in the streamwise direction dominates. $\implies$ Provable.

Assuming that $\mathrm{Re}_l \gg 1$ and $\delta \ll \ell$, then we can rely on the following normalization factors (order of magnitude analysis). Let's start with the continuity equation (incompressible flow):

$$
\begin{gathered}
\frac{\partial u_i}{\partial x_i} = 0 \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \\
\frac{\partial u^* u_\infty}{\partial x^* \ell} + \frac{\partial v^* \mathcal{V}}{\partial y^* \delta} = 0 \\
\implies \left(\frac{u_\infty}{\ell}\right) \frac{\partial u^*}{\partial x^*} + \left(\frac{\mathcal{V}}{\delta}\right) \frac{\partial v^*}{\partial y^*} = 0 \\
\implies \frac{\partial u^*}{\partial x^*} + \frac{\mathcal{V}\ell}{u_\infty \delta} \frac{\partial v^*}{\partial y^*} = 0
\end{gathered}
$$

Based on the order of magnitude analysis,

$$
\begin{gathered}
\frac{\mathcal{V}\ell}{u_\infty \delta} = 1 \\
\implies \mathcal{V} = \frac{u_\infty \delta}{\ell}
\end{gathered}
$$

Therefore, if $\delta ll \ell$, then $\mathcal{V} \ll u_\infty$. This provies that streamline velocity dominates.

Now, let's move to the Navier-Stokes equation. First, the x-dir:

$$
\cancelto{\frac{\partial u}{\partial t}}{\text{S.S.}} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)
$$

y-dir:

$$
\cancelto{\frac{\partial v}{\partial t}}{\text{S.S.}} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)
$$

Let's look at the x-dir expression:

$$
\begin{aligned}
(u^* u_\infty)\frac{\partial (u^* u_\infty)}{\partial (x^* \ell)} + (v^* \mathcal{V})\frac{\partial (u^* u_\infty)}{\partial (y^* \delta)} &= -\frac{1}{\rho}\frac{\partial P}{\partial (x^* \ell)} + \nu\left(\frac{\partial^2 (u^* u_\infty)}{\partial (x^* \ell)^2} + \frac{\partial^2 (u^* u_\infty)}{\partial (y^* \delta)^2}\right) \\
\implies \left(\frac{u_\infty^2}{\ell}\right) u^* \frac{\partial u^*}{\partial x^*} + \left(\frac{u_\infty \mathcal{V}}{\delta}\right) v^* \frac{\partial u^*}{\partial y^*} &= -\frac{1}{\rho\ell}\frac{\partial P}{\partial x^*} + \nu\left(\left(\frac{u_\infty}{\ell^2}\right)\left(\frac{\partial^2 u^*}{\partial (x^*)^2} + \left(\frac{u_\infty}{\delta^2}\right)\frac{\partial^2 u^*}{\partial (y^*)^2}\right)\right)
\end{aligned}
$$

Using $\mathcal{V} = \frac{u_\infty \delta}{\ell}$,

$$
\left(\frac{u_\infty^2}{\ell}\right) u^* \frac{\partial u^*}{\partial x^*} + \left(\frac{u_\infty^2}{\ell}\right) v^* \frac{\partial u^*}{\partial y^*} = -\frac{1}{\rho\ell}\frac{\partial P}{\partial x^*} + \frac{\nu u_\infty}{\delta^2}\left(\cancelto{\left(\frac{\delta^2}{\ell^2}\right)}{0}\frac{\partial^2 u^*}{\partial (x^*)^2} + \frac{\partial^2 u^*}{\partial (y^*)^2}\right)
$$

Note the term inside the ν term is zero since $\delta \ll \ell$. From order of magnitude analysis, we can say

Scaling of LHS Scaling of RHS

$$\frac{\delta^2}{\ell^2} \; \frac{\nu}{\ell u_\infty} = \frac{1}{\mathrm{Re}}$$

So, if $\delta^2 \ll \ell^2$, then $\frac{1}{\mathrm{Re}} \ll 1$. This implies the flow remains 2D for high Re.

At this point, we can look at the scaling for pressure,

$$\frac{u_\infty^2}{\ell} \text{ (Pressure Scaling) } \left(\frac{1}{\rho}\frac{\partial P}{\partial x} = 0\left(\frac{\rho u_\infty^2}{\ell}\right)\right)$$

Let's apply the same process to the y-direction. The results are:

$$\left(\frac{u_\infty \mathcal{V}}{\ell}\right) u^* \frac{\partial v^*}{\partial x^*} + \left(\frac{\mathcal{V}^2}{\delta^2}\right) v^* \frac{\partial v^*}{\partial y^*} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \nu\left(\left(\frac{\mathcal{V}}{\ell^2}\right)\frac{\partial^2 v^*}{\partial (x^*)^2} + \left(\frac{\mathcal{V}}{\delta^2}\right)\frac{\partial^2 v^*}{\partial (y^*)^2}\right)$$

Since $\delta/\ell \ll 1$ and $u_\infty \mathcal{V}/\ell = 0$

$$\implies 0 = \frac{1}{\rho}\frac{\partial P}{\partial y^*}$$

We can see that

$$\text{Scaling of } \frac{\partial P}{\partial y} \; \frac{u_\infty^2 \delta}{\ell^2}$$

We can again show that

$$\frac{\partial P}{\partial x} gg \frac{\partial P}{\partial y}$$