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# 1 Tensor Notation

Table 1: Different Rank Tensors

| Rank | Name | Notation | Example |
|---|---|---|---|
| 0 | Scalar | $T$ | Temperature, Pressure, Volume |
| 1 | Vector | $T_i$ | Force, Velocity, Vorticity |
| 2 | Matrix | $T_{ij}$ | Stress, Strain, Rate of Deformation |
| 3 | Third Order Tensor | $T_{ijk}$ | ... |
| 4 | Fourth Order Tensor | $T_{ijkl}$ | ... |
| ... | ... | ... | ... |

## 1.1 Einstein Summation Convention

The Einstein Summation Convention is a shorthand notation for writing out long sums. It is used to simplify the notation of tensor operations. The convention is as follows:

- If an index appears more than once in a term, it is a dummy index. It is summed over.
- If an index appears once in a term, it is a free index. The number of free indices is the rank of the tensor.

### 1.1.1 Examples

#### 1.1.1.1 Example 1

Consider,

$$A_{ij} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

There are two free indices, $i$ and $j$. The rank of the tensor is 2.

#### 1.1.1.2 Example 2

Consider,

$$\partial_i u_{ij} = \begin{bmatrix} \partial_1 u_{11} + \partial_2 u_{12} + \partial_3 u_{13} \\ \partial_1 u_{21} + \partial_2 u_{22} + \partial_3 u_{23} \\ \partial_1 u_{31} + \partial_2 u_{32} + \partial_3 u_{33} \end{bmatrix}$$

There is one free index, $j$, and one dummy index, $i$. The rank of the tensor is 1.

## 1.2 Kronecker Delta

The Kronecker Delta is a mathematical operator that is used to represent the identity matrix. It is defined as:

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

### 1.2.1 Properties (in 3D)

- $\delta_{ij} = \delta_{ji}$
- $\delta_{ij} A_{jk} = A_{ik}$
- $\delta_{ij} \delta_{jk} = \delta_{ik}$
- $\delta_{ij} \delta_{ij} = \delta_{ii} = 3$
- $a_{ij} \delta_{ij} = \delta_{ij} a_{ij} = a_{ii}$

## 1.3 Levi-Civita Symbol

The Levi-Civita Symbol is a mathematical operator that is used to represent the permutation of indices. It is defined as:

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{if } ijk \text{ is an even permutation of 123} \\ -1 & \text{if } ijk \text{ is an odd permutation of 123} \\ 0 & \text{if any two indices are equal} \end{cases}$$

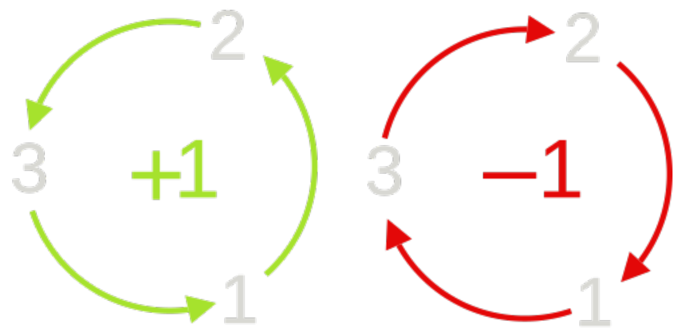


Figure 1: Levi-Civita Even and Odd Permutations

For example,

$$\varepsilon_{123} = 1$$
$$\varepsilon_{231} = 1$$
$$\varepsilon_{312} = 1$$
$$\varepsilon_{132} = -1$$
$$\varepsilon_{213} = -1$$
$$\varepsilon_{321} = -1$$
$$\varepsilon_{122} = 0$$
$$\varepsilon_{113} = 0$$
$$\varepsilon_{111} = 0$$

Since this is a third order tensor, there are 27 components. From these, 3 of these components are $+1$, 3 are $-1$, and 21 are 0.

#### 1.3.1 Properties (in 3D)

- $\varepsilon_{ijk} = \varepsilon_{jki} = \varepsilon_{kij}$
- $\varepsilon_{ijk}\varepsilon_{ijk} = 6$
- $\varepsilon_{ijk}\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$
- $\varepsilon_{ijk} = -\varepsilon_{ikj}$
- $a_{ij}\varepsilon_{ijk} = \varepsilon_{ijk}a_{ij}$

## 1.4 Vector and Tensor Operations

### 1.4.1 Multiplication of a Vector by a Scalar

$$\alpha \vec{A} = \vec{B}$$

$$\alpha A_i = B_i$$

### 1.4.2 Dot Product of Two Vectors

$$\vec{A} \cdot \vec{B} = C$$

$$A_i B_i = C$$

### 1.4.3 Cross Product of Two Vectors

$$\vec{A} \times \vec{B} = \vec{C}$$

$$\varepsilon_{ijk} A_j B_k = C_i$$

### 1.4.4 Dot Product of Two Tensors (Tensor Product)

$$A \otimes B = A_{ij} B_{jk} = C_{ik}$$

### 1.4.5 Double Dot Product of Two Tensors

$$A : B = A_{ij} B_{ij} = C$$

A property of the double dot product is that it is commutative,

$$A : B = B : A$$

### 1.4.6 Nabla Operator

The Nabla operator is a vector differential operator. It is defined as:

$$\nabla = \frac{\partial}{\partial x_i} = \partial_i$$

In vector notation,

$$\nabla = \begin{bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{bmatrix}$$

#### 1.4.6.1 Gradient of a Scalar

The gradient of a scalar is a vector. It is defined as:

$$\nabla T = \frac{\partial T}{\partial x_i} = \partial_i T$$

#### 1.4.6.2 Gradient of a Vector

The gradient of a vector is a tensor. It is defined as:

$$\begin{aligned}\nabla \vec{A} &= \frac{\partial A_j}{\partial x_i} = \partial_i A_j \\ &= \begin{bmatrix} \partial_1 A_1 & \partial_1 A_2 & \partial_1 A_3 \\ \partial_2 A_1 & \partial_2 A_2 & \partial_2 A_3 \\ \partial_3 A_1 & \partial_3 A_2 & \partial_3 A_3 \end{bmatrix}\end{aligned}$$

#### 1.4.6.3 Divergence of a Vector

The divergence of a vector is a scalar. It is defined as:

$$\begin{aligned}\nabla \cdot \vec{A} &= \frac{\partial A_i}{\partial x_i} = \partial_i A_i = \mathrm{div}(\vec{A}) \\ &= \partial_1 A_1 + \partial_2 A_2 + \partial_3 A_3 \\ &= C\end{aligned}$$

#### 1.4.6.4 Divergence of a Tensor

The divergence of a rank 2 tensor is a vector. It is defined as:

$$\begin{aligned}\nabla \cdot A &= \frac{\partial A_{ij}}{\partial x_i} = \partial_i A_{ij} \\ &= \begin{bmatrix} \partial_1 A_{11} + \partial_2 A_{21} + \partial_3 A_{31} \\ \partial_1 A_{12} + \partial_2 A_{22} + \partial_3 A_{32} \\ \partial_1 A_{13} + \partial_2 A_{23} + \partial_3 A_{33} \end{bmatrix}\end{aligned}$$

#### 1.4.6.5 Curl of a Vector

The curl of a vector is a vector. It is defined as:

$$\nabla \times \vec{A} = \varepsilon_{ijk}\partial_j A_k = \text{curl}(\vec{A})$$
$$= \begin{bmatrix} \partial_2 A_3 - \partial_3 A_2 \\ \partial_3 A_1 - \partial_1 A_3 \\ \partial_1 A_2 - \partial_2 A_1 \end{bmatrix}$$

#### 1.4.6.6 Laplace of a Scalar

The Laplace of a scalar is a scalar. It is defined as:

$$\nabla^2 \phi = \text{div}(\text{grad}(\phi)) = \nabla \cdot (\nabla \phi)$$
$$= \partial_i(\partial_i \phi) = \partial_i \partial_i \phi$$
$$= \partial_1\partial_1\phi + \partial_2\partial_2\phi + \partial_3\partial_3\phi$$

#### 1.4.6.7 Laplace of a Vector

The Laplace of a vector is a vector. It is defined as:

$$\nabla^2 \vec{A} = \text{div}(\text{grad}(\vec{A})) = \nabla \cdot (\nabla \vec{A})$$
$$= \partial_i(\partial_i A_j) = \partial_i \partial_i A_j$$
$$= \begin{bmatrix} \partial_1\partial_1 A_1 + \partial_2\partial_2 A_1 + \partial_3\partial_3 A_1 \\ \partial_1\partial_1 A_2 + \partial_2\partial_2 A_2 + \partial_3\partial_3 A_2 \\ \partial_1\partial_1 A_3 + \partial_2\partial_2 A_3 + \partial_3\partial_3 A_3 \end{bmatrix}$$

#### 1.4.6.8 Vector Outer Product (Dyadic Product)

The vector outer product is a rank 2 tensor. It is defined as:

$$\vec{A}\vec{B} = \vec{A}\vec{B}^T = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}$$

$$= \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix}$$

$$= A_i B_j = C_{ij}$$

### 1.5 Summary of Tensor Operations

Table 2: Summary of Tensor Operations

| Description | Vector Notation | Einstein Notation |
|---|---|---|
| Multiplication of a Vector by a Scalar | $\alpha\vec{A}$ | $\alpha A_i$ |
| Dot Product of Two Vectors | $\vec{A}\cdot\vec{B}$ | $A_iB_i$ |
| Cross Product of Two Vectors | $\vec{A}\times\vec{B}$ | $\varepsilon_{ijk}A_jB_k$ |
| Dot Product of Two Tensors (Tensor Product) | $A\otimes B$ | $A_{ij}B_{jk}=C_{ik}$ |
| Double Dot Product of Two Tensors | $A:B$ | $A_{ij}B_{ij}=C$ |
| Gradient of a Scalar | $\nabla T$ | $\partial_i T$ |
| Gradient of a Vector | $\nabla\vec{A}$ | $\partial_i A_j$ |
| Divergence of a Vector | $\nabla\cdot\vec{A}$ | $\partial_i A_i$ |
| Divergence of a Tensor | $\nabla\cdot A$ | $\partial_i A_{ij}$ |
| Curl of a Vector | $\nabla\times\vec{A}$ | $\varepsilon_{ijk}\partial_j A_k$ |
| Laplace of a Scalar | $\nabla^2\phi$ | $\partial_i\partial_i\phi$ |
| Laplace of a Vector | $\nabla^2\vec{A}$ | $\partial_i\partial_i A_j$ |
| Vector Outer Product (Dyadic Product) | $\vec{A}\vec{B}$ | $A_iB_j=C_{ij}$ |

# 2 Flow Descriptions

## 2.1 Continuity Equation

The continuity equation is a statement of the conservation of mass. The general form is given by:

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\vec{u})=0$$

$$\underbrace{\frac{\partial\rho}{\partial t}+\vec{u}\cdot\nabla\rho}_{\frac{D\rho}{Dt}}+\rho\nabla\cdot\vec{u}=0$$

where $D/Dt$ is the material derivative,

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{u} \cdot \nabla$$

## 2.2 Momentum Equation

The momentum equation is a statement of the conservation of momentum. There are many, many forms. The one most useful for this course is:

$$\boxed{\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j}(\rho v_j v_i) = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\right] - \frac{\partial}{\partial x_i}\left[\frac{2}{3}\mu\frac{\partial v_k}{\partial x_k}\right] + \rho b_i} \tag{1}$$

In cartesian coordinates, first in the $x$ direction,

$$\boxed{\begin{aligned}\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho u^2) + \frac{\partial}{\partial y}(\rho uv) + \frac{\partial}{\partial z}(\rho uw) = -\frac{\partial p}{\partial x} &+ \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial x}\right)\right] \\ &+ \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right] \\ &- \frac{\partial}{\partial x}\left[\frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right] + \rho b_x\end{aligned}} \tag{2}$$

Then in the $y$ direction,

$$\boxed{\begin{aligned}\frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}(\rho uv) + \frac{\partial}{\partial y}(\rho v^2) + \frac{\partial}{\partial z}(\rho vw) = -\frac{\partial p}{\partial y} &+ \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] \\ &+ \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial y}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right] \\ &- \frac{\partial}{\partial y}\left[\frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right] + \rho b_y\end{aligned}} \tag{3}$$

Finally in the $z$ direction,

$$\boxed{\begin{aligned}\frac{\partial}{\partial t}(\rho w) + \frac{\partial}{\partial x}(\rho uw) + \frac{\partial}{\partial y}(\rho vw) + \frac{\partial}{\partial z}(\rho w^2) = -\frac{\partial p}{\partial z} &+ \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right] \\ &+ \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w}{\partial z} + \frac{\partial w}{\partial z}\right)\right] \\ &- \frac{\partial}{\partial z}\left[\frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right] + \rho b_z\end{aligned}} \tag{4}$$

### 2.3 Energy Equation

The general internal energy equation is too annoying to write out and isn't particularly useful. The one most useful for this course is the incompressible Newtonian fluid energy equation:

$$\boxed{\begin{aligned}\rho C_p\left(\frac{\partial T}{\partial t}+\vec{u}\cdot\nabla T\right) &= k\nabla^2 T+\mu\left[2\left(\frac{\partial u}{\partial x}\right)^2+2\left(\frac{\partial v}{\partial y}\right)^2+2\left(\frac{\partial w}{\partial z}\right)^2\right.\\ &\left.\quad+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)^2\right]\end{aligned}} \tag{5}$$

### 2.4 Common Simplifications

Here are some common simplifications that are often made to the momentum and energy equations:

Table 3: Common Simplifications

| Description | Consequece |
|---|---|
| Incompressible Flow | $\rho = \text{constant}$ |
| Newtonian Fluid | $\mu = \text{constant}$ |
| Steady Flow | $\frac{\partial}{\partial t} = 0$ |
| 2D Flow | $\frac{\partial \vec{u}}{\partial z} = 0,\ w = 0$ |
| Fully Developed Flow | $\frac{\partial \vec{u}}{\partial x} = 0$ |
| Gravity | $\vec{b} = -g\hat{k}$ |
| Constant Pressure | $p = \text{constant}$ |

## 3 Discretization Methods

### 3.1 Finite Volume Method

Basically, take a differential equation and integrate it over a control volume. This gives us a discrete form of the equation.

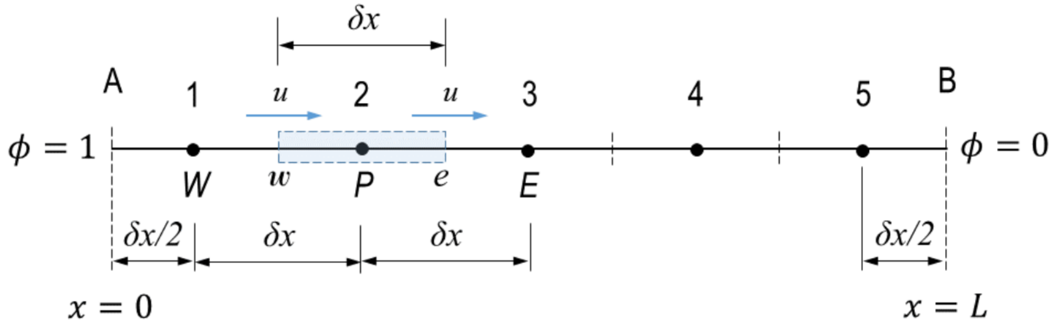


Figure 2: Grid generation for finite volume method

Consider the diffusion-convection equation with no source term:

$$\frac{d}{dx}(\rho\vec{u}\phi) = \frac{d}{dx}(\Gamma\frac{d\phi}{dx})$$

Integrating over the control volume:

$$\int_{\mathrm{CV}} \frac{d}{dx}(\rho\vec{u}\phi)dV = \int_{\mathrm{CV}} \frac{d}{dx}(\Gamma\frac{d\phi}{dx})dV$$

$$\int_{\Delta x} \frac{d}{dx}(\rho\vec{u}\phi)Sdx = \int_{\Delta x} \frac{d}{dx}(\Gamma\frac{d\phi}{dx})Sdx$$

$$[\rho\vec{u}\phi]_w^e = \left[\Gamma\frac{d\phi}{dx}\right]_w^e$$

$$[\rho\vec{u}\phi]_e - [\rho\vec{u}\phi]_w = \left[\Gamma\frac{d\phi}{dx}\right]_e - \left[\Gamma\frac{d\phi}{dx}\right]_w$$

### 3.1.1 Central Difference Scheme

The derivative terms can be approximated using the central difference scheme.

$$\boxed{\begin{aligned} \frac{d\phi}{dx}_e &= \frac{\phi_E - \phi_P}{\Delta x_{PE}} \quad (6) \\ \frac{d\phi}{dx}_w &= \frac{\phi_P - \phi_W}{\Delta x_{WP}} \quad (7) \end{aligned}}$$

For $\phi$ at the non-node points, the central difference scheme calculates the values of $\phi$ at $e$ and $w$ using linear interpolation:

$$\boxed{\begin{aligned} \phi_e &= \frac{\phi_P + \phi_E}{2} \quad (8) \\ \phi_w &= \frac{\phi_P + \phi_W}{2} \quad (9) \end{aligned}}$$

And at Node 1 ($\phi_W = \phi_A$),

$$\boxed{\begin{aligned} \frac{d\phi}{dx}_w &= \frac{\phi_P - \phi_A}{\Delta x_{AP}} \quad (10) \\ \phi_w &= \phi_A \quad (11) \end{aligned}}$$

And at Node N ($\phi_E = \phi_B$),

$$\boxed{\begin{aligned} \frac{d\phi}{dx}_e &= \frac{\phi_B - \phi_P}{\Delta x_{PB}} \quad (12) \\ \phi_e &= \phi_B \quad (13) \end{aligned}}$$

Substituting these values into the finite volume equation:

$$\rho\vec{u}\frac{\phi_E - \phi_P}{\Delta x_{PE}} - \rho\vec{u}\frac{\phi_P - \phi_W}{\Delta x_{WP}} = \Gamma\frac{\phi_E - \phi_P}{\Delta x_{PE}} - \Gamma\frac{\phi_P - \phi_W}{\Delta x_{WP}}$$

And then you can write it in the form

$$a_P\phi_P = a_W\phi_W + a_E\phi_E + q$$

#### 3.1.2 Upwind Scheme

For flow in positive direction (W to E), the upwind scheme calculates the values of $\phi_e$ and $\phi_w$ using:

$$\boxed{\begin{aligned} \phi_e &= \phi_P \quad (14) \\ \phi_w &= \phi_W \quad (15) \end{aligned}}$$

And in the negative direction (E to W), the upwind scheme calculates the values of $\phi_e$ and $\phi_w$ using:

$$\boxed{\begin{aligned} \phi_e &= \phi_E \quad (16) \\ \phi_w &= \phi_P \quad (17) \end{aligned}}$$

The central difference scheme is used to evaluate the derivative terms.