Question 1

Consider two arbitrary vectors $\vec{a}$ and $\vec{b}$. Use index notation to show that the following relationship is true:

$$(\vec{a} \times \vec{b}) \cdot (\vec{a} \times \vec{b}) = (\vec{a} \cdot \vec{a})(\vec{b} \cdot \vec{b}) - (\vec{a} \cdot \vec{b})^2$$

Solution

First, recall the definition of the cross product in index notation,

$$(\vec{a} \times \vec{b}) = \epsilon_{ijk} a_j b_k = C_i$$

the definition of the dot product,

$$(\vec{a} \cdot \vec{b}) = a_i b_i$$

an identity for Levi-Civita symbol,

$$\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$$

and the Kronecker delta,

$$\delta_{ij} A_{mj} = A_{mi}$$

Then,

$$\begin{aligned}
(\vec{a} \times \vec{b}) \cdot (\vec{a} \times \vec{b}) &= C_i C_i \\
&= (\epsilon_{ijk} a_j b_k)(\epsilon_{ilm} a_l b_m) \\
&= \epsilon_{ijk}\epsilon_{ilm} a_j b_k a_l b_m \\
&= (\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}) a_j b_k a_l b_m \\
&= (\delta_{jl}\delta_{km} a_j b_k a_l b_m) - (\delta_{jm}\delta_{kl} a_j b_k a_l b_m) \\
&= (a_j b_k a_j b_k) - (a_j b_k a_k b_j)
\end{aligned}$$

because we are dealing with vectors and scalar products, we can utilize commutativity

$$\begin{aligned}
&= (a_j a_j b_k b_k) - (a_j a_k b_k b_j) \\
&= (a_j a_j)(b_k b_k) - (a_j a_k)(b_k b_j) \\
&= (\vec{a} \cdot \vec{a})(\vec{b} \cdot \vec{b}) - (\vec{a} \cdot \vec{b})^2
\end{aligned}$$

which matches the RHS. □

Question 2

Using index notation, prove the following expression is true:

$$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = \vec{c}[\vec{d} \cdot (\vec{a} \times \vec{b})] - \vec{d}[\vec{c} \cdot (\vec{a} \times \vec{b})]$$

Use index notation and replace the indices only at the end.

Solution

Similar to Question 1, expand LHS,

$$
\begin{aligned}
\underbrace{(\vec{a} \times \vec{b})}_{P_j} \times \underbrace{(\vec{c} \times \vec{d})}_{Q_k} &= \epsilon_{ijk} P_j Q_k \\
&= \epsilon_{ijk} P_j (\epsilon_{klm} c_l d_m) \\
&= \epsilon_{ijk} \epsilon_{klm} P_j c_l d_m \\
&= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) P_j c_l d_m \\
&= (\delta_{il}\delta_{jm} P_j c_l d_m) - (\delta_{im}\delta_{jl} P_j c_l d_m) \\
&= (P_j c_i d_j) - (P_j c_j d_i) \\
&= c_i [d_j P_j] - d_i [c_j P_j] \\
&= \vec{c}[\vec{d} \cdot (\vec{a} \times \vec{b})] - \vec{d}[\vec{c} \cdot (\vec{a} \times \vec{b})] \quad \square
\end{aligned}
$$