Question 1

Integrate eq. (2) over the internal control volumes (2 - 3) using control volume 2 as a representative volume. Assume constant cross section value $S = 1$.

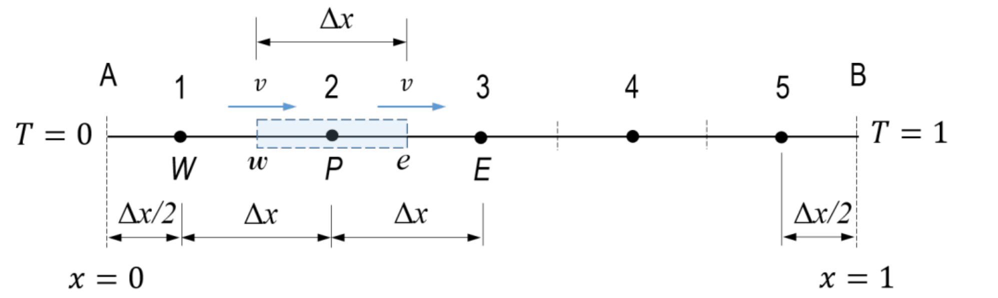


Figure 1: Simulation Domain Mesh

Eq. 2 is given by:

$$\frac{d\hat{T}}{dx} = \frac{d}{dx}\left(\frac{1}{\mathrm{Pe}}\frac{d\hat{T}}{dx}\right)$$

Then integrating over the control volume,

$$\int_{\Delta x} \frac{d\hat{T}}{dx} S dx = \int_{\Delta x} \frac{d}{dx}\left(\frac{1}{\mathrm{Pe}}\frac{d\hat{T}}{dx}\right) S dx$$

$$\hat{T}\Big|_w^e = \left(\frac{1}{\mathrm{Pe}}\frac{d\hat{T}}{dx}\right)\Big|_w^e$$

Then,

$$\boxed{\underbrace{\hat{T}_e - \hat{T}_w}_{\text{Convective}} = \underbrace{\frac{1}{\mathrm{Pe}}\left[\frac{d\hat{T}}{dx}\right]_e - \frac{1}{\mathrm{Pe}}\left[\frac{d\hat{T}}{dx}\right]_w}_{\text{Diffusive}}}$$

Question 2

Approximate the convective term of the obtained equation using the upwind scheme. For upwind scheme with east as positive flow direction,

$$\hat{T}_w = \hat{T}_W$$
$$\hat{T}_e = \hat{T}_P$$

Then, the convective term becomes,

$$\begin{aligned}\text{Convective} &= \hat{T}_e - \hat{T}_w \\ &= \boxed{\hat{T}_P - \hat{T}_W}\end{aligned}$$

Question 3

Approximate the diffusive term of the equation using the central difference scheme.

For central difference scheme,

$$\left[\frac{d\hat{T}}{dx}\right]_e = \frac{\hat{T}_E - \hat{T}_P}{\Delta x}$$

$$\left[\frac{d\hat{T}}{dx}\right]_w = \frac{\hat{T}_P - \hat{T}_W}{\Delta x}$$

Then, the diffusive term becomes,

$$\begin{aligned}\text{Diffusive} &= \frac{1}{\text{Pe}}\left[\frac{d\hat{T}}{dx}\right]_e - \frac{1}{\text{Pe}}\left[\frac{d\hat{T}}{dx}\right]_w \\ &= \frac{1}{\text{Pe}}\left(\frac{\hat{T}_E - \hat{T}_P}{\Delta x}\right) - \frac{1}{\text{Pe}}\left(\frac{\hat{T}_P - \hat{T}_W}{\Delta x}\right) \\ &= \boxed{\frac{1}{\text{Pe}}\left(\frac{\hat{T}_E - 2\hat{T}_P + \hat{T}_W}{\Delta x}\right)}\end{aligned}$$

Question 4

Substitute approximations obtained in Questions 2 and 3 back into equation derived in Question 1. Rearrange the obtained equation into a form that can be used to solve for $\hat{T}_P$.

$$a_P T_P = a_W T_W + a_E T_E$$

and identify coefficients a_P, a_W, a_E, q_T^u, and q_T^P.

Substituting back into the equation derived in Question 1,

$$\hat{T}_P - \hat{T}_W = \frac{1}{\text{Pe}}\left(\frac{\hat{T}_E - 2\hat{T}_P + \hat{T}_W}{\Delta x}\right)$$

Rearranging,

$$\boxed{\underbrace{\left[\frac{2}{\text{Pe}\Delta x} + 1\right]}_{a_P}\hat{T}_P = \underbrace{\left[\frac{1}{\text{Pe}\Delta x} + 1\right]}_{a_W}\hat{T}_W + \underbrace{\left[\frac{1}{\text{Pe}\Delta x}\right]}_{a_E}\hat{T}_E + \underbrace{[0]q_T^u}_{q_T^u} + \underbrace{[0]q_T^P}_{q_T^P}}$$

Question 5

Integrate eq. (2) over boundary control volume 1 and approximate the convective term using upwind scheme. Use central difference for approximation of the diffusive term. Assume constant cross section value $S = 1$. Integrating Eq. 2, but over control volume 1,

$$\int_{\Delta x} \frac{d\hat{T}}{dx} S dx = \int_{\Delta x} \frac{d}{dx}\left(\frac{1}{\text{Pe}}\frac{d\hat{T}}{dx}\right) S dx$$
$$\hat{T}\Big|_A^e = \left(\frac{1}{\text{Pe}}\frac{d\hat{T}}{dx}\right)\Big|_A^e$$

Then,

$$\boxed{\underbrace{\hat{T}_e - \hat{T}_A}_{\text{Convective}} = \underbrace{\frac{1}{\text{Pe}}\left[\frac{d\hat{T}}{dx}\right]_e - \frac{1}{\text{Pe}}\left[\frac{d\hat{T}}{dx}\right]_A}_{\text{Diffusive}}}$$

For the convective term, upwind scheme is used,

$$\hat{T}_A = \hat{T}_A$$
$$\hat{T}_e = \hat{T}_P$$

Then,

$$\begin{aligned}\text{Convective} &= \hat{T}_e - \hat{T}_A \\ &= \hat{T}_P - \hat{T}_A\end{aligned}$$

For the diffusive term, a central difference scheme is used,

$$\left[\frac{d\hat{T}}{dx}\right]_e = \frac{\hat{T}_E - \hat{T}_P}{\Delta x}$$
$$\left[\frac{d\hat{T}}{dx}\right]_A = \frac{\hat{T}_P - \hat{T}_A}{\Delta x/2}$$

Substituting back into the equation derived,

$$\boxed{\hat{T}_E - \hat{T}_P = \frac{1}{\text{Pe}}\left(\frac{\hat{T}_E + 2\hat{T}_A - 3\hat{T}_P}{\Delta x}\right)}$$

Question 6

Rearrange the obtained equation into a general form

$$a_P T_P = a_W T_W + a_E T_E + q_T^u$$

and identify coefficients a_P, a_W, a_E, q_T^u, and q_P^T.

From Q5, the equation obtained is,

$$\hat{T}_E - \hat{T}_P = \frac{1}{\text{Pe}}\left(\frac{\hat{T}_E + 2\hat{T}_A - 3\hat{T}_P}{\Delta x}\right)$$

Expanding the equation,

$$\underbrace{\left[\frac{3}{\text{Pe}\Delta x} - 1\right]}_{a_P}\hat{T}_P = \underbrace{[0]}_{a_W}\hat{T}_W + \underbrace{\left[\frac{1}{\text{Pe}\Delta x} - 1\right]}_{a_E}\hat{T}_E + \underbrace{\frac{2}{\text{Pe}\Delta x}\hat{T}_A}_{q_T^u} + \underbrace{0}_{q_P^T}$$

Question 7

Integrate eq. (2) over boundary control volume 5 and approximate the convective term using upwind scheme. Use central difference scheme for the approximation of the diffusive term. Assume constant cross section value $S = 1$.

Integrating Eq. 2, but over control volume 5,

$$\int_{\Delta x}\frac{d\hat{T}}{dx}Sdx = \int_{\Delta x}\frac{d}{dx}\left(\frac{1}{\text{Pe}}\frac{d\hat{T}}{dx}\right)Sdx$$
$$\hat{T}\bigg|_w^B = \left(\frac{1}{\text{Pe}}\frac{d\hat{T}}{dx}\right)\bigg|_w^B$$

Then,

$$\boxed{\underbrace{\hat{T}_w - \hat{T}_B}_{\text{Convective}} = \underbrace{\frac{1}{\text{Pe}}\left[\frac{d\hat{T}}{dx}\right]_w - \frac{1}{\text{Pe}}\left[\frac{d\hat{T}}{dx}\right]_B}_{\text{Diffusive}}}$$

For the convective term, upwind scheme is used,

$$\hat{T}_B = \hat{T}_P$$
$$\hat{T}_w = \hat{T}_W$$

Then,

$$\begin{aligned}\text{Convective} &= \hat{T}_w - \hat{T}_B \\ &= \hat{T}_W - \hat{T}_P\end{aligned}$$

For the diffusive term, a central difference scheme is used,

$$\left[\frac{d\hat{T}}{dx}\right]_w = \frac{\hat{T}_P - \hat{T}_W}{\Delta x}$$
$$\left[\frac{d\hat{T}}{dx}\right]_B = \frac{\hat{T}_B - \hat{T}_P}{\Delta x/2}$$

Substituting back into the equation derived,

$$\boxed{\hat{T}_W - \hat{T}_P = \frac{1}{\mathrm{Pe}}\left(\frac{3\hat{T}_P - \hat{T}_W - 2\hat{T}_B}{\Delta x}\right)}$$

Question 8

Rearrange the obtained equation into a general form

$$a_P T_P = a_W T_W + a_E T_E + q_T^u$$

and identify coefficients a_P, a_W, a_E, q_T^u, and q_P^T.

From Q7, the equation obtained is,

$$\hat{T}_W - \hat{T}_P = \frac{1}{\mathrm{Pe}}\left(\frac{3\hat{T}_P - \hat{T}_W - 2\hat{T}_B}{\Delta x}\right)$$

Expanding the equation,

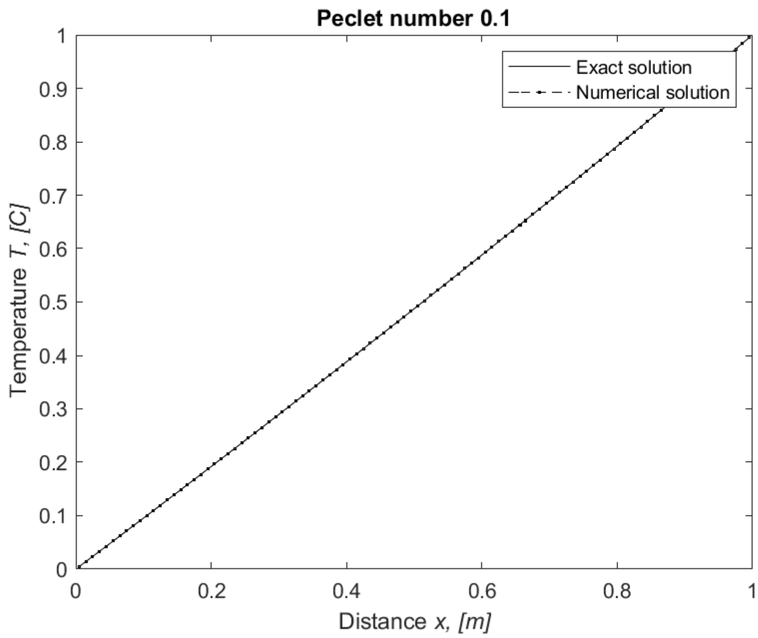
$$\underbrace{\left[\frac{-3}{\mathrm{Pe}\Delta x} - 1\right]}_{a_P}\hat{T}_P = \underbrace{\left[\frac{-1}{\mathrm{Pe}\Delta x} - 1\right]}_{a_W}\hat{T}_W + \underbrace{[0]}_{a_E}\hat{T}_E + \underbrace{\frac{-2}{\mathrm{Pe}\Delta x}\hat{T}_B}_{q_T^u} + \underbrace{0}_{q_P^T}$$

Question 9

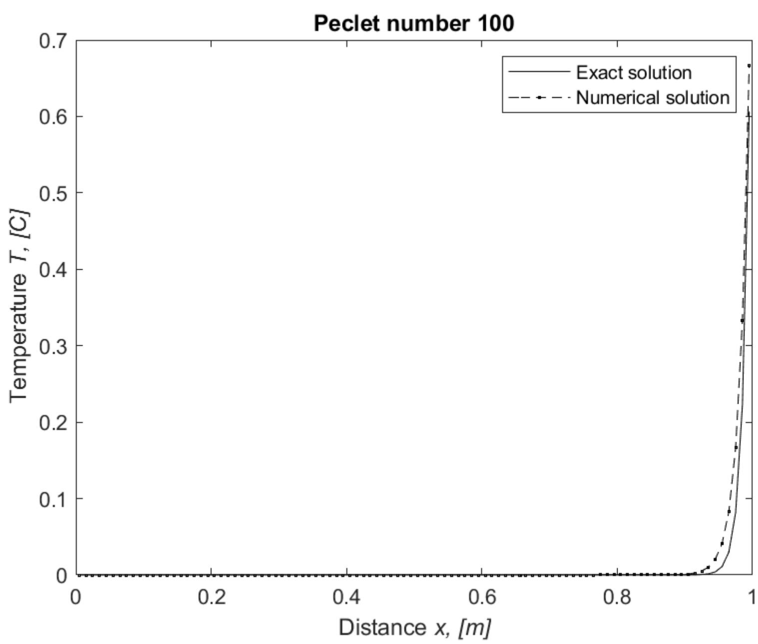
Modify a MATLAB script provided on eClass with the derived coefficients. Solve the problem with the modified code for two different Peclet numbers equal to 0.1 and 100 using N = 100. Compare the numerical results to the analytical solution, which is

$$T(x) = \frac{\exp(\mathrm{Pe}\cdot x) - 1}{\exp(\mathrm{Pe}) - 1}$$

Plot numerical and analytic distribution on two different plots corresponding to different Peclet numbers. Explain the results. Calculate a maximum absolute error for each Peclet number.

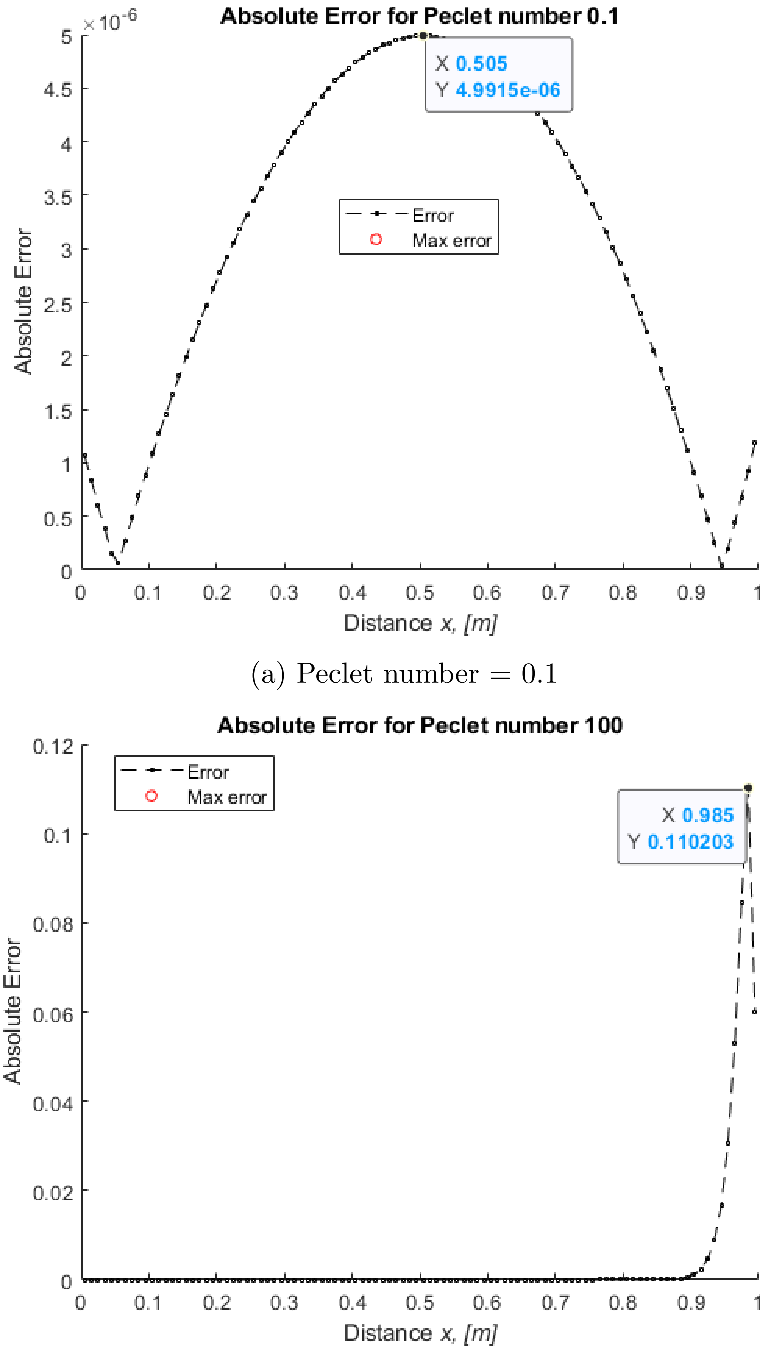


(a) Peclet number = 0.1



(b) Peclet number = 100

Figure 2: Analytical and numerical solutions for Peclet number = 0.1 and 100



(a) Peclet number = 0.1

(b) Peclet number = 100

Figure 3: Absolute error for Peclet number = 0.1 and 100

Peclet number is the ratio between the rate of convection and the rate of diffusion. When the Peclet number is small, the rate of diffusion is much larger than the rate of convection. This means that the temperature distribution is dominated by the diffusion term. As a result, the temperature distribution is smooth and the temperature gradient is small.

When the Peclet number is large, the rate of convection is much larger than the rate of diffusion. This means that the temperature distribution is dominated by the convection term. As a result, the temperature distribution is steep and the temperature gradient is large.

The numerical solution is in good agreement with the analytical solution for both Peclet numbers. The maximum absolute error for Peclet number = 0.1 is 5E-6 and the maximum absolute error for Peclet number = 100 is 0.11.

A finer mesh is required to capture the steep temperature distribution for Peclet number = 100.

```
clear
clc
%% Define simulation parameters
N = 100;            % 100 CVs
L = 1.0;            % [m] Length of the rod
dx = L/N;           % [m] Distance between the nodal points [Don't
   change]
Pe_1 = 0.1;         % Peclet number 1
Pe_2 = 100;         % Peclet number 2
T_A = 0;            % [C] Temperature at the left boundary
T_B = 1;            % [C] Temperature at the right boundary

%% Define variable size [Don't change]
aW_1 = zeros(N,1);
aE_1 = zeros(N,1);
aP_1 = zeros(N,1);
qP_1 = zeros(N,1);
qu_1 = zeros(N,1);
AA_1 = zeros(N,1);
CD_1 = zeros(N,1);

aW_2 = zeros(N,1);
aE_2 = zeros(N,1);
aP_2 = zeros(N,1);
qP_2 = zeros(N,1);
qu_2 = zeros(N,1);
AA_2 = zeros(N,1);
CD_2 = zeros(N,1);

%% Calculate coefficients of the discretized equations for internal
   nodes ------------
T_1 = convection_diffusion_1D(N,Pe_1,dx, T_A, T_B);
T_2 = convection_diffusion_1D(N,Pe_2,dx, T_A, T_B);

%% Analytical solution ------
```

```
x = dx/2:dx:(L-dx/2);
T_hat_anal_1 = (exp(Pe_1*x) - 1)/(exp(Pe_1) - 1);
T_hat_anal_2 = (exp(Pe_2*x) - 1)/(exp(Pe_2) - 1);

%% Error calculation
error_1 = abs(T_hat_anal_1 - T_1);
error_2 = abs(T_hat_anal_2 - T_2);

[T_1_max_error, T_1_max_error_index] = max(error_1);
[T_2_max_error, T_2_max_error_index] = max(error_2);

%% Plot the results
% Peclet number 1
figure(1)
y=dx/2:dx:(L-dx/2);
plot(x,T_hat_anal_1, 'k-', y, T_1, 'ks--', 'MarkerSize',2)
xlabel('Distance \itx, [m]')
ylabel('Temperature \itT, [C]')
title('Peclet number 0.1')
legend('Exact solution','Numerical solution')
saveas(gcf,'peclet_0.1.png')

% Peclet number 2
figure(2)
plot(x,T_hat_anal_2, 'k-', y, T_2, 'ks--', 'MarkerSize',2)
xlabel('Distance \itx, [m]')
ylabel('Temperature \itT, [C]')
title('Peclet number 100')
legend('Exact solution','Numerical solution')
saveas(gcf,'peclet_100.png')

% Error for Peclet number 1
figure(3)
hold on
plot(x, error_1, 'ks--', 'MarkerSize',2)
plot(x(T_1_max_error_index), T_1_max_error, 'ro', 'MarkerSize',5)
xlabel('Distance \itx, [m]')
ylabel('Absolute Error')
title('Absolute Error for Peclet number 0.1')
legend('Error','Max error', 'Location', 'best')
saveas(gcf,'error_0.1.png')

% Error for Peclet number 2
figure(4)
hold on
plot(x, error_2, 'ks--', 'MarkerSize',2)
plot(x(T_2_max_error_index), T_2_max_error, 'ro', 'MarkerSize',5)
```

```
xlabel('Distance \itx, [m]')
ylabel('Absolute Error')
title('Absolute Error for Peclet number 100')
legend('Error','Max error', 'Location', 'best')
saveas(gcf,'error_100.png')


%% Function to solve the 1D convection-diffusion equation
function [T] = convection_diffusion_1D(N,Pe,dx, T_A, T_B)
    %% Calculate coefficients of the discretized equations for
   internal nodes ------------
    for i=2:(N-1)
        aE(i) = 1/(Pe*dx);
        aW(i) = 1/(Pe*dx) + 1;
        aP(i) = 2/(Pe*dx) + 1;
    end
    %% Calculate coefficients for the boundary volumes
   -------------------------------
    % West (A point)
    aE(1) = 1/(Pe*dx) - 1;
    aW(1) = 0;
    qP(1) = 0;
    qu(1) = 2/(Pe*dx) * T_A;
    aP(1) = 3/(Pe*dx) - 1;
    % East (B point)
    aE(N) = 0;
    aW(N) = -1/(Pe*dx) - 1;
    qP(N) = 0;
    qu(N) = -2/(Pe*dx) * T_B;
    aP(N) = -3/(Pe*dx) - 1;

    %% Solution of the system of algebraic equations
    %  TDMA procedure
    AA(1)=aE(1)/aP(1);
    CD(1)=qu(1)/aP(1);
    for i=2:N
        AA(i)=aE(i)/(aP(i)-aW(i)*AA(i-1));
        CD(i)=(aW(i)*CD(i-1)+qu(i))/(aP(i)-aW(i)*AA(i-1));
    end
    %
    T(N)=CD(N);
    for i=N-1:-1:1
        T(i)=AA(i)*T(i+1)+CD(i);
    end
end
```