

## Question 1

Consider two arbitrary vectors  $\vec{a}$  and  $\vec{b}$ . Use index notation to show that the following relationship is true:

$$(\vec{a} \times \vec{b}) \cdot (\vec{a} \times \vec{b}) = (\vec{a} \cdot \vec{a})(\vec{b} \cdot \vec{b}) - (\vec{a} \cdot \vec{b})^2$$

## Solution

First, recall the definition of the cross product in index notation,

$$(\vec{a} \times \vec{b}) = \epsilon_{ijk} a_j b_k = C_i$$

the definition of the dot product,

$$(\vec{a} \cdot \vec{b}) = a_i b_i$$

an identity for Levi-Civita symbol,

$$\epsilon_{ijk} \epsilon_{ilm} = \delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl}$$

and the Kronecker delta,

$$\delta_{ij} A_{mj} = A_{mi}$$

Then,

$$\begin{aligned} (\vec{a} \times \vec{b}) \cdot (\vec{a} \times \vec{b}) &= C_i C_i \\ &= (\epsilon_{ijk} a_j b_k)(\epsilon_{ilm} a_l b_m) \\ &= \epsilon_{ijk} \epsilon_{ilm} a_j b_k a_l b_m \\ &= (\delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl}) a_j b_k a_l b_m \\ &= (\delta_{jl} \delta_{km} a_j b_k a_l b_m) - (\delta_{jm} \delta_{kl} a_j b_k a_l b_m) \\ &= (a_j b_k a_j b_k) - (a_j b_k a_k b_j) \end{aligned}$$

because we are dealing with vectors and scalar products, we can utilize commutativity

$$\begin{aligned} &= (a_j a_j b_k b_k) - (a_j a_k b_k b_j) \\ &= (a_j a_j)(b_k b_k) - (a_j a_k)(b_k b_j) \\ &= (\vec{a} \cdot \vec{a})(\vec{b} \cdot \vec{b}) - (\vec{a} \cdot \vec{b})^2 \end{aligned}$$

which matches the RHS.  $\square$

## Question 2

Using index notation, prove the following expression is true:

$$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = \vec{c}[\vec{d} \cdot (\vec{a} \times \vec{b})] - \vec{d}[\vec{c} \cdot (\vec{a} \times \vec{b})]$$

Use index notation and replace the indices only at the end.

## Solution

Similar to Question 1, expand LHS,

$$\begin{aligned}
 \underbrace{(\vec{a} \times \vec{b})}_{P_j} \times \underbrace{(\vec{c} \times \vec{d})}_{Q_k} &= \epsilon_{ijk} P_j Q_k \\
 &= \epsilon_{ijk} P_j (\epsilon_{klm} c_l d_m) \\
 &= \epsilon_{ijk} \epsilon_{klm} P_j c_l d_m \\
 &= (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) P_j c_l d_m \\
 &= (\delta_{il} \delta_{jm} P_j c_l d_m) - (\delta_{im} \delta_{jl} P_j c_l d_m) \\
 &= (P_j c_i d_j) - (P_j c_j d_i) \\
 &= c_i [d_j P_j] - d_i [c_j P_j] \\
 &= \vec{c} [\vec{d} \cdot (\vec{a} \times \vec{b})] - \vec{d} [\vec{c} \cdot (\vec{a} \times \vec{b})] \quad \square
 \end{aligned}$$

done

## Question 3

Consider the general form of momentum balance

$$\rho \frac{dv_i}{dt} = \frac{\partial T_{ji}}{\partial x_j} + \rho b_i \quad (1)$$

where  $d/dt$  is the total derivative;  $v_i$  is the velocity;  $\rho$  denotes density;  $T_{ij}$  is the stress tensor and  $b_i$  is a body force. This equation says that inertial and body forces are balanced by the gradients of the stress tensor.

The general form of the stress tensor reads

$$T_{ij} = -p\delta_{ij} + \mu \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) - \frac{2}{3}\mu \frac{\partial v_k}{\partial x_k} \delta_{ij} + \kappa \left( \frac{\partial v_k}{\partial x_k} \delta_{ij} \right) \quad (2)$$

where  $p$  is the pressure,  $\mu$  is the shear (dynamic) viscosity and  $\kappa$  is the dialation viscosity. For each question assume the viscosities  $\mu$  and  $\kappa$  are constant.

- Using equations (1) and (2) derive the general form of Navier-Stokes equations for compressible viscous fluid. Note: only index notation can be used. The final form of equations should be presented in operator (vector) form.
- Assuming  $\kappa = 0$ , write the full set of Navier-Stokes equations in Cartesian coordinates assuming two-dimensional flow (two equations corresponding to  $x$  and  $y$  directions).

## Solution

(a)

First, deal with the stress tensor.

$$T_{ji} = -p\delta_{ji} + \mu \left( \frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j} \right) - \frac{2}{3}\mu \frac{\partial v_k}{\partial x_k} \delta_{ji} + \kappa \left( \frac{\partial v_k}{\partial x_k} \delta_{ji} \right)$$

Differentiate with respect to  $x_j$ ,

$$\frac{\partial T_{ji}}{\partial x_j} = -\frac{\partial p}{\partial x_j} \delta_{ji} + \mu \left( \frac{\partial^2 v_j}{\partial x_j \partial x_i} + \frac{\partial^2 v_i}{\partial x_j^2} - \frac{2}{3} \frac{\partial^2 v_k}{\partial x_j \partial x_k} \delta_{ji} \right) + \kappa \left( \frac{\partial^2 v_k}{\partial x_j \partial x_k} \delta_{ji} \right)$$

Substituting into equation 1,

$$\begin{aligned} \rho \frac{\partial v_i}{\partial t} + \rho v_j \frac{\partial v_i}{\partial x_j} &= -\frac{\partial p}{\partial x_j} \delta_{ji} + \mu \left( \frac{\partial^2 v_j}{\partial x_j \partial x_i} + \frac{\partial^2 v_i}{\partial x_j^2} - \frac{2}{3} \frac{\partial^2 v_k}{\partial x_j \partial x_k} \delta_{ji} \right) + \kappa \left( \frac{\partial^2 v_k}{\partial x_j \partial x_k} \delta_{ji} \right) + \rho b_i \\ &= -\frac{\partial p}{\partial x_i} + \mu \left( \frac{\partial^2 v_j}{\partial x_j \partial x_i} + \frac{\partial^2 v_i}{\partial x_j^2} - \frac{2}{3} \frac{\partial^2 v_k}{\partial x_i \partial x_k} \right) + \kappa \left( \frac{\partial^2 v_k}{\partial x_i \partial x_k} \right) + \rho b_i \end{aligned}$$

Converting the RHS to vector form,

$$\begin{aligned} \text{RHS} &= -\nabla p + \mu \left[ \nabla(\nabla \cdot \vec{v}) + \nabla^2 \vec{v} - \frac{2}{3} \nabla(\nabla \cdot \vec{v}) \right] + \kappa \nabla(\nabla \cdot \vec{v}) + \rho \vec{b} \\ &= -\nabla p + \mu \nabla^2 \vec{v} + \left[ \frac{\mu}{3} + \kappa \right] \nabla(\nabla \cdot \vec{v}) + \rho \vec{b} \end{aligned}$$

Converting the LHS to vector form,

$$\text{LHS} = \rho \left( \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right)$$

Combining the LHS and RHS,

$$\boxed{\rho \left( \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right) = -\nabla p + \mu \nabla^2 \vec{v} + \left[ \frac{\mu}{3} + \kappa \right] \nabla(\nabla \cdot \vec{v}) + \rho \vec{b}}$$

(b)

Assuming  $\kappa = 0$ , the equation reduces to

$$\rho \left( \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right) = -\nabla p + \mu \nabla^2 \vec{v} + \frac{\mu}{3} \nabla(\nabla \cdot \vec{v}) + \rho \vec{b}$$

In Cartesian coordinates (x, y), assuming  $\vec{v} = (u, v)$  and  $\vec{b} = (b_x, b_y)$ . In the x-direction,

$$\boxed{\rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = -\frac{\partial p}{\partial x} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + \frac{\mu}{3} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} \right) + \rho b_x}$$

In the y-direction,

$$\boxed{\rho \left( \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) = -\frac{\partial p}{\partial y} + \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) + \frac{\mu}{3} \left( \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial y^2} \right) + \rho b_y}$$

## Question 4

Consider the heat transfer in incompressible flow of Newtonian fluid. The following form of energy conservation equation holds for this case:

$$\rho C_p \left( \frac{\partial T}{\partial t} + \vec{v} \cdot \nabla T \right) = k \nabla^2 T + \frac{\mu}{2} \Phi^2$$

where  $T$  is the temperature,  $t$  is time,  $\vec{v}$  is the velocity,  $\rho$  is density,  $C_p$  is the heat capacity,  $k$  is the thermal conductivity,  $\mu$  is the shear (dynamic) viscosity, and  $\Phi_{ij}$  is the rate of shear tensor defined as

$$\Phi_{ij} = \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}$$

- What is the physical interpretation of the following terms:  $\rho C_p(\vec{v} \cdot \nabla T)$ ,  $k \nabla^2 T$  and  $\frac{\mu}{2} \Phi^2$ .
- Write equation (3) in Cartesian coordinates assuming three-dimensional flow. Explain all the steps how the term  $\Phi_{ij}^2$  unfolds.

## Solution

(a)

- $\rho C_p(\vec{v} \cdot \nabla T)$  is the rate of change of energy due to convection.
- $k \nabla^2 T$  is the rate of change of energy due to conduction.
- $\frac{\mu}{2} \Phi^2$  is the rate of change of energy due to viscous dissipation (friction).

(b)

First deal with  $\Phi_{ij}^2$ ,

$$\Phi_{ij}^2 = \sum_{i,j}^3 \Phi_{ij} \Phi_{ij}$$

Then,

$$\begin{aligned} \Phi_{ij}^2 &= \Phi_{11}\Phi_{11} + \Phi_{21}\Phi_{21} + \Phi_{31}\Phi_{31} \\ &\quad + \Phi_{12}\Phi_{12} + \Phi_{22}\Phi_{22} + \Phi_{32}\Phi_{32} \\ &\quad + \Phi_{13}\Phi_{13} + \Phi_{23}\Phi_{23} + \Phi_{33}\Phi_{33} \end{aligned}$$

In Cartesian coordinates, assuming  $v_i = (u, v, w)$ ,  $x_i = (x, y, z)$ ,

$$\begin{aligned}
 \Phi_{ij}^2 &= \left( \frac{\partial u}{\partial x} + \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)^2 + \left( \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right)^2 \\
 &\quad + \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial y} + \frac{\partial v}{\partial y} \right)^2 + \left( \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right)^2 \\
 &\quad + \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right)^2 + \left( \frac{\partial w}{\partial z} + \frac{\partial w}{\partial z} \right)^2 \\
 &= 4 \left( \frac{\partial u}{\partial x} \right)^2 + 4 \left( \frac{\partial v}{\partial x} \right)^2 + 4 \left( \frac{\partial w}{\partial x} \right)^2 \\
 &\quad + 2 \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 + 2 \left( \frac{\partial v}{\partial y} + \frac{\partial v}{\partial y} \right)^2 + 2 \left( \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right)^2
 \end{aligned}$$

Now the rest of the terms,

$$\begin{aligned}
 \rho C_p \left[ \frac{\partial T}{\partial t} + \left( u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z} \right) \right] &= k \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) \\
 &\quad + \frac{\mu}{2} \left[ 4 \left( \frac{\partial u}{\partial x} \right)^2 + 4 \left( \frac{\partial v}{\partial x} \right)^2 + 4 \left( \frac{\partial w}{\partial x} \right)^2 \right. \\
 &\quad + 2 \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 + 2 \left( \frac{\partial v}{\partial y} + \frac{\partial v}{\partial y} \right)^2 \\
 &\quad \left. + 2 \left( \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right)^2 \right]
 \end{aligned}$$