Question 1

Write down the full set of governing equations (continuity, momentum and energy).

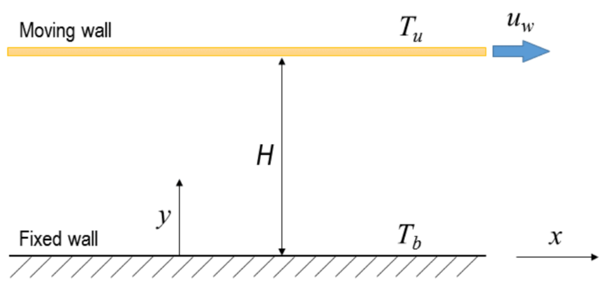

Figure 1: Fluid flow betweeen parallel plates. The bottom plate is fixed, the upper plate is moving horizontally with velocity u_w. THe fluid is incompressible and Newtonian.

Solution

We first begin with continuity,

$$\boxed{\frac{\partial \rho}{\partial t} + \vec{u} \cdot \nabla \rho + \rho(\nabla \cdot \vec{u}) = 0}$$

then, the momentum equation,

$$\boxed{\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) = -\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right] - \frac{\partial}{\partial x_i}\left(\frac{2}{3}\mu\frac{\partial u_k}{\partial x_k}\right) + \rho b_i}$$

lastly, the energy equation,

$$\boxed{\rho C_p\left(\frac{T}{\partial t} + \vec{v} \cdot \nabla T\right) = k\nabla^2 T + \frac{\mu}{2}\Phi^2}$$

where $\Phi_{i,j} = \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$.

Question 2

Write down the momentum conservation equations in Cartesian coordinates. Simplify them using conditions outlined in bullet points.

Solution

Table 1: Assumptions and simplifying consequences

Assumption	**Consequence**
Steady State (S.S.)	$\frac{\partial}{\partial t} = 0$
Incompressible	$\rho = \text{constant}$
Newtonian	$\mu = \text{constant}$
2D Flow	$u_z = 0$, $\partial_z \vec{u} = 0$
Parallel	$u_y = 0$, $\partial_y \vec{u} = 0$
Fully developed in x (F.D.)	$v_x = v_x(y)$,
Constant Pressure in x (C.P.)	$\frac{\partial P}{\partial x} = 0$
Gravity in z	$\vec{b} = -g\hat{k}$

We begin with the continuity equation,

$$\cancelto{\text{S.S.}}{\frac{\partial \rho}{\partial t}} + \vec{u} \cdot \cancelto{\text{Incmp.}}{\nabla \rho} + \rho(\nabla \cdot \vec{u}) = 0$$

this simplifies to,

$$\frac{\partial}{\partial x} u_x + \cancelto{\text{Par.}}{\frac{\partial}{\partial y} u_y} + \cancelto{\text{2D}}{\frac{\partial}{\partial z} u_z} = 0$$

$$\implies \boxed{\frac{\partial u_x}{\partial x} = 0}$$

Next, in the x-direction with the momentum equation,

$$\cancelto{\text{S.S.}}{\frac{\partial}{\partial t}(\rho u_x)} + \frac{\partial}{\partial x_j}(\rho u_x u_j) = -\cancelto{\text{C.P.}}{\frac{\partial P}{\partial x}} + \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial u_x}{\partial x_j} + \frac{\partial u_j}{\partial x_x}\right)\right] - \frac{\partial}{\partial x}\left(\frac{2}{3}\mu\frac{\partial u_k}{\partial x_k}\right) + \rho b_x$$

expanding the index notation and simplifying,

$$\rho\left(\cancelto{\text{Cont.}}{\frac{\partial}{\partial x}(u_x^2)} + \cancelto{\text{Par.}}{\frac{\partial}{\partial y}(u_x u_y)} + \cancelto{\text{2D}}{\frac{\partial}{\partial z}(u_x u_z)}\right) = \frac{\partial}{\partial x}\left[\mu\cancelto{\text{Cont.}}{\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_x}{\partial x}\right)}\right]$$

$$+ \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u_x}{\partial y} + \cancelto{\text{Par.}}{\frac{\partial u_y}{\partial x}}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\cancelto{\text{2D}}{\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}}\right)\right]$$

$$- \frac{\partial}{\partial x}\left[\frac{2}{3}\mu\cancelto{\text{Cont.}}{\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right)}\right] + \rho\cancelto{\text{Grav.}}{b_x}$$

which results in

$$\boxed{\frac{\partial^2 u_x}{\partial y^2} = 0}$$

Next, in the y-direction with the momentum equation,

$$\cancelto{\text{S.S.}}{\frac{\partial}{\partial t}(\rho u_y)} + \frac{\partial}{\partial x_j}(\rho u_y u_j) = -\frac{\partial P}{\partial y} + \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial u_y}{\partial x_j} + \frac{\partial u_j}{\partial x_y}\right)\right] - \frac{\partial}{\partial y}\left(\frac{2}{3}\mu\frac{\partial u_k}{\partial x_k}\right) + \rho b_y$$

expanding the index notation and simplifying,

$$\begin{aligned}\rho\left(\cancelto{\text{Par.}}{\frac{\partial}{\partial x}}(u_x u_y) + \cancelto{\text{Par.}}{\frac{\partial}{\partial y}}(u_y^2) + \cancelto{\text{Par.}}{\frac{\partial}{\partial z}}(u_y u_z)\right) &= -\frac{\partial P}{\partial y} + \frac{\partial}{\partial x}\left[\mu\left(\cancelto{\text{Par.}}{\frac{\partial u_y}{\partial x}} + \cancelto{\text{Cont.}}{\frac{\partial u_x}{\partial y}}\right)\right] \\ &\quad + \frac{\partial}{\partial y}\left[\cancelto{\text{Par.}}{\mu\left(\frac{\partial u_y}{\partial y} + \frac{\partial u_y}{\partial y}\right)}\right] + \frac{\partial}{\partial z}\left[\mu\cancelto{\text{2D}}{\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right)}\right] \\ &\quad - \frac{\partial}{\partial y}\left[\frac{2}{3}\mu\cancelto{\text{Cont.}}{\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right)}\right] + \rho \cancelto{\text{Grav.}}{b_y}\end{aligned}$$

resulting in

$$\boxed{\frac{\partial P}{\partial y} = 0}$$

Finally, in the z-direction with the momentum equation,

$$\cancelto{\text{S.S.}}{\frac{\partial}{\partial t}(\rho u_z)} + \frac{\partial}{\partial x_j}(\rho u_z u_j) = -\frac{\partial P}{\partial z} + \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial u_z}{\partial x_j} + \frac{\partial u_j}{\partial x_z}\right)\right] - \frac{\partial}{\partial z}\left(\frac{2}{3}\mu\frac{\partial u_k}{\partial x_k}\right) + \rho b_z$$

expanding the index notation and simplifying,

$$\begin{aligned}\rho\left(\cancelto{\text{2D}}{\frac{\partial}{\partial x}}(u_x u_z) + \cancelto{\text{2D}}{\frac{\partial}{\partial y}}(u_y u_z) + \cancelto{\text{2D}}{\frac{\partial}{\partial z}}(u_z^2)\right) &= -\frac{\partial P}{\partial z} + \frac{\partial}{\partial x}\left[\mu\left(\cancelto{\text{2D}}{\frac{\partial u_z}{\partial x}} + \cancelto{\text{2D}}{\frac{\partial u_x}{\partial z}}\right)\right] \\ &\quad + \frac{\partial}{\partial y}\left[\mu\left(\cancelto{\text{2D}}{\frac{\partial u_z}{\partial y}} + \cancelto{\text{2D}}{\frac{\partial u_y}{\partial z}}\right)\right] + \frac{\partial}{\partial z}\left[\mu\cancelto{\text{2D}}{\left(\frac{\partial u_z}{\partial z} + \frac{\partial u_z}{\partial z}\right)}\right] \\ &\quad - \frac{\partial}{\partial z}\left[\frac{2}{3}\mu\cancelto{\text{Cont.}}{\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right)}\right] + \rho b_z\end{aligned}$$

resulting in

$$\boxed{\frac{\partial P}{\partial z} = -\rho g}$$

Question 3

Solve the simplified momentum equation.

Solution

Using no-slip boundary conditions, the boundary conditions are

$$u_x(0) = 0$$
$$u_x(h) = u_w$$

The simplified momentum equation in the x-direction is given by

$$\frac{\partial^2 u_x}{\partial y^2} = 0$$

which has the general solution

$$u_x(y) = Ay + B$$

where A and B are constants. Applying the boundary conditions, we find

$$u_x(0) = B = 0$$
$$u_x(h) = Ah = u_w$$

which gives

$$u_x(y) = \frac{u_w}{h} y$$

This linear profile is also known as Couette flow. Next, the simplified momentum equation in the y-direction is given by

$$\frac{\partial P}{\partial y} = 0$$

which has the general solution

$$P = C(z)$$

where $C(z)$ is a function of z. Lastly, the simplified momentum equation in the z-direction is given by

$$\frac{\partial P}{\partial z} = -\rho g$$

which has the general solution

$$P(z) = -\rho g z + D$$

where D is a constant.

So in summary,

$$\boxed{\begin{aligned} \vec{u} &= \left[\frac{u_w}{h} y\right] \hat{i} \\ P &= -\rho g z + D \end{aligned}}$$

Question 4

Simplify the energy equation.

Solution

On top of the assumptions from Table 1, we need additional assumptions to address temperature.

Table 2: Assumptions and simplifying consequences for energy equation

Assumption	**Consequence**
2D Flow	$\partial_z T = 0$
Fully developed in x (F.D.)	$T = T(y)$, $\partial_x T = 0$

The energy equation in the absence of a heat source is given by

$$\rho C_p \left(\frac{\partial T}{\partial t} + \vec{v} \cdot \nabla T \right) = k \nabla^2 T + \frac{\mu}{2} \Phi^2$$

From Assignment 1, the full Cartesian expansion is

$$\rho C_p \left[\underbrace{\cancel{\frac{\partial T}{\partial t}}}_{\text{S.S}} + \left(\underbrace{\cancel{u_x \frac{\partial T}{\partial x}}}_{\text{F.D.}} + \underbrace{\cancel{u_y \frac{\partial T}{\partial y}}}_{\text{Par.}} + \underbrace{\cancel{u_z \frac{\partial T}{\partial z}}}_{\text{2D}} \right) \right] = k \left(\underbrace{\cancel{\frac{\partial^2 T}{\partial x^2}}}_{\text{F.D.}} + \frac{\partial^2 T}{\partial y^2} + \underbrace{\cancel{\frac{\partial^2 T}{\partial z^2}}}_{\text{2D}} \right)$$
$$+ \frac{\mu}{2} \left[4 \underbrace{\cancel{\left(\frac{\partial u_x}{\partial x} \right)^2}}_{\text{Cont.}} + 4 \underbrace{\cancel{\left(\frac{\partial u_y}{\partial x} \right)^2}}_{\text{Par.}} + 4 \underbrace{\cancel{\left(\frac{\partial u_z}{\partial x} \right)^2}}_{\text{2D}} \right.$$
$$+ 2 \left(\frac{\partial u_x}{\partial y} + \underbrace{\cancel{\frac{\partial u_y}{\partial x}}}_{\text{Par.}} \right)^2 + 2 \underbrace{\cancel{\left(\frac{\partial u_y}{\partial y} + \frac{\partial u_y}{\partial y} \right)^2}}_{\text{Par.}}$$
$$\left. + 2 \underbrace{\cancel{\left(\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z} \right)^2}}_{\text{2D}} \right]$$

which simplifies to

$$0 = k \frac{\partial^2 T}{\partial y^2} + \mu \left(\frac{\partial u_x}{\partial y} \right)^2$$

Using the results from Q3,

$$\boxed{\frac{\partial^2 T}{\partial y^2} = -\frac{\mu}{k} \left(\frac{u_w}{H} \right)^2}$$

Question 5

Solve the simplified energy equation.

Solution

The simplified energy equation is given by

$$\frac{\partial^2 T}{\partial y^2} = -\frac{\mu}{k}\left(\frac{u_w}{H}\right)^2$$

Integrating twice with respect to y,

$$\begin{aligned}\frac{\partial T}{\partial y} &= -\frac{\mu}{k}\left(\frac{u_w}{H}\right)^2 y + C_1 \\ T &= -\frac{\mu}{2k}\left(\frac{u_w}{H}\right)^2 y^2 + C_1 y + C_2\end{aligned}$$

Applying the boundary conditions $T(0) = T_b$ and $T(H) = T_u$,

$$\begin{aligned}&T_b = C_2 \\ &T_u = -\frac{\mu}{2k}\left(\frac{u_w}{H}\right)^2 H^2 + C_1 H + C_2 \implies C_1 = \frac{T_u - T_b + \frac{\mu}{2k}\left(\frac{u_w}{H}\right)^2 H^2}{H}\end{aligned}$$

Thus, the temperature profile is given by

$$\boxed{T(y) = -\frac{\mu}{2k}\left(\frac{u_w}{H}\right)^2 y^2 + \left(T_u - T_b + \frac{\mu}{2k}\left(\frac{u_w}{H}\right)^2 H^2\right)\frac{y}{H} + T_b}$$