Conservation of Energy (Scalar)

Energy is a scalar quantity, and therefore it can be identified & formulated through the conservation of a scalar quantity:

Let's try $\phi =$ scalar,

$$\frac{\partial}{\partial t}\int_{\text{C.V.}} \rho\phi \, dV + \int_{\text{C.S.}} \rho\phi(\vec{u}\cdot\vec{n})\, dS = \oint_{\text{C.S.}} \Gamma \,\text{grad}(\phi)\cdot\vec{n}\, dS + \int_{\text{C.V.}} \underbrace{q_\phi}_{\text{source/sink}} \, dV$$

$$\implies \frac{\partial}{\partial t}(\rho\phi) + \frac{\partial}{\partial x_j}(\rho\phi u_j) = \frac{\partial}{\partial x_j}(\Gamma\frac{\partial\phi}{\partial x_j}) + q_\phi$$

Conservation of a scalar (Energy). In vector notation,

$$\partial_t(\rho\phi) + \text{div}(\rho\phi\vec{u}) = \text{div}(\Gamma \,\text{grad}(\phi)) + q_\phi$$

Where

- $\partial_t(\rho\phi)$ is the time rate of change of the scalar quantity ϕ (conservative term).
- $\text{div}(\rho\phi\vec{u})$ is the rate of change due to the flow due to $\vec{u}$ (advection term).
- $\text{div}(\Gamma \,\text{grad}(\phi))$ is the rate of change due to diffusion (Γ) (diffusion term).
- q_ϕ is the rate of production (source) or destruction (sink) of ϕ.

Chapter 4. Fundamental flows (Simplification)

Navier-Stokes equations are highly non-linear PDEs with no exact solutions. However, there are fundamental flow dynamics (simplified flows) based on assumptions and approximations that makes the mathematics easier to follow, solve, and interpret.

We are going to look at 4 simplified flow cases:

1. Incompressible flow ($\rho =$ constant)
2. Invicid flow (Euler's flow) ($\mu \to 0$)
3. Creeping flow (Stokes flow) (Re $\ll 100$, inertial forces are negligible)
4. Potential flow (Re $\to 0$, Ma $\to 0$

Let's look at the conservation laws for each:

Incompressible flow

Incompressibility is defined as incapability of a fluid (i.e. liquid) to compress to a smaller size under internal/external loads. This, therefore, means that their **density** ρ does not change as long as we keep their mass the same.

Typically, liquids are incompressible, but air (gas) can become compressible at special conditions.

$$\underbrace{\text{Ma}}_{\vec{u}/\text{speed of sound}} > 0.3 \implies \text{compressible}$$

Continuity:

$$\begin{aligned}\overbrace{\cancel{\frac{\partial \rho}{\partial t}}}^{0(\text{Incomp})} + \boldsymbol{\nabla}\cdot(\rho\vec{u}) &= 0\\ \implies \boldsymbol{\nabla}\cdot\vec{u} &= 0\end{aligned}$$

Momentum:

$$\frac{\partial}{\partial t}(\rho\vec{u}) + \boldsymbol{\nabla}\cdot(\rho\vec{u}\vec{u}) = -\boldsymbol{\nabla}P + \overbrace{\cancel{\frac{1}{3}\mu\nabla(\boldsymbol{\nabla}\cdot\vec{u})}}^{0,\ \text{continuity}} + \mu\nabla^2\vec{u} + \rho\vec{b}$$

Expanding the $\boldsymbol{\nabla}\cdot(\rho\vec{u}\vec{u})$ term,

$$\begin{aligned}\boldsymbol{\nabla}\cdot(\rho\vec{u}\vec{u}) &= (\boldsymbol{\nabla}\cdot\rho\vec{u})\vec{u} + \rho\vec{u}\cdot\boldsymbol{\nabla}\vec{u}\\ &= \overbrace{\cancel{(\boldsymbol{\nabla}\cdot\rho\vec{u})\vec{u}}}^{0,\ \text{continuity}} + \rho\vec{u}\cdot\boldsymbol{\nabla}\vec{u}\\ &= \rho\vec{u}\cdot\boldsymbol{\nabla}\vec{u}\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{\partial}{\partial t}(\rho\vec{u}) + \rho\vec{u}\cdot\boldsymbol{\nabla}\vec{u} &= -\boldsymbol{\nabla}P + \mu\nabla^2\vec{u} + \rho\vec{b}\\ \rho\frac{\partial\vec{u}}{\partial t} + \rho\vec{u}\cdot\boldsymbol{\nabla}\vec{u} &= -\boldsymbol{\nabla}P + \mu\nabla^2\vec{u} + \rho\vec{b}\\ \frac{\partial\vec{u}}{\partial t} + \vec{u}\cdot\boldsymbol{\nabla}\vec{u} &= -\frac{\boldsymbol{\nabla}P}{\rho} + \underbrace{\frac{\mu}{\rho}}_{\nu}\nabla^2\vec{u} + \vec{b}\end{aligned}$$

Invicid flow (Euler's flow)

Viscous forces can be important in flows close to a wall, where we have large velocity gradients (Also in wakes). As we should before, it is the combination of ν and $\vec{u}$ that forms the viscous effects in transport of fluids.

$\implies$ Vorticies $\to$ ν may be important.

$\hookrightarrow$ if you are far from a surface or regions of large velocity gradients, the implication of viscocity becomes minimal.

$\hookrightarrow$ we quantify the effect of viscocity in the flow using Reynold's Number:

$$\mathrm{Re} = \frac{\rho u \overbrace{L}^{\text{characteristic length}}}{\mu} = \frac{\text{inertial forces}}{\text{viscous forces}}$$

if Re $\gg 1000 \implies \mu \to 0$ which means inertial forces dominate the flow (negligible viscous forces).

Continuity:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \vec{u}) = 0$$

No impact because no μ term.

Momentum:

$$\frac{\partial}{\partial t}(\rho \vec{u}) + \boldsymbol{\nabla} \cdot (\rho \vec{u}\vec{u}) = -\boldsymbol{\nabla} P + \cancelto{0,\ \text{invicid}}{\frac{1}{3}\mu \nabla(\boldsymbol{\nabla} \cdot \vec{u})} + \cancel{\mu \nabla^2 \vec{u}} + \rho \vec{b}$$

$$\boxed{\frac{\partial}{\partial t}(\rho \vec{u}) + \boldsymbol{\nabla} \cdot (\rho \vec{u}\vec{u}) = -\boldsymbol{\nabla} P + \rho \vec{b}}$$

Note: As we saw in the Navier-Stokes equation, the flow can only be dominated by the **Pressure** and **External forces**. This means that the invicid cond. cannot hold if we are dealing with areas of high straining (vorcity and wakes).

Note: Since we are assuring invicid condition, then the flow cannot slow down close to the stationary wall. $\implies$ Slip Boundary Condition.

Creeping flow (Stokes flow)

At high re, we just discussed that viscous effects are negligible. Contrarily, at low Re, (Re $\ll 100$) the effect of viscosity **dominates** the flow.

$\implies$ Inertial forces are negligible.

$$\mathrm{Re} \propto \frac{uD}{\nu} \begin{cases} \text{Either } u \to 0 \text{and/or} \\ D \to 0 \end{cases}$$

Continuity:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \vec{u}) = 0$$

No impact.

Momentum:

$$0 = -\frac{\boldsymbol{\nabla} P}{\rho} + \boldsymbol{\nabla} \cdot (\nu \boldsymbol{\nabla} \vec{u}) + \vec{b} + \vec{b}$$

This type of flow is mostly for porous media coating, or nano-fluidics.

Potential flow

One of the simpliest flows in fluid mechanics.

Based on two conditions:

1. Invicid flow ($\mu \to 0$)
2. Irrotational flow ($\vec{\omega} \to 0$)

provides approximation for initial flow conditions.