Question 1

Consider two arbitrary vectors $\vec{a}$ and $\vec{b}$. Use index notation to show that the following relationship is true:

$$(\vec{a} \times \vec{b}) \cdot (\vec{a} \times \vec{b}) = (\vec{a} \cdot \vec{a})(\vec{b} \cdot \vec{b}) - (\vec{a} \cdot \vec{b})^2$$

Solution

First, recall the definition of the cross product in index notation,

$$(\vec{a} \times \vec{b}) = \epsilon_{ijk} a_j b_k = C_i$$

the definition of the dot product,

$$(\vec{a} \cdot \vec{b}) = a_i b_i$$

an identity for Levi-Civita symbol,

$$\epsilon_{ijk} \epsilon_{ilm} = \delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl}$$

and the Kronecker delta,

$$\delta_{ij} A_{mj} = A_{mi}$$

Then,

$$\begin{aligned}
(\vec{a} \times \vec{b}) \cdot (\vec{a} \times \vec{b}) &= C_i C_i \\
&= (\epsilon_{ijk} a_j b_k)(\epsilon_{ilm} a_l b_m) \\
&= \epsilon_{ijk} \epsilon_{ilm} a_j b_k a_l b_m \\
&= (\delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl}) a_j b_k a_l b_m \\
&= (\delta_{jl} \delta_{km} a_j b_k a_l b_m) - (\delta_{jm} \delta_{kl} a_j b_k a_l b_m) \\
&= (a_j b_k a_j b_k) - (a_j b_k a_k b_j)
\end{aligned}$$

because we are dealing with vectors and scalar products, we can utilize commutativity

$$\begin{aligned}
&= (a_j a_j b_k b_k) - (a_j a_k b_k b_j) \\
&= (a_j a_j)(b_k b_k) - (a_j a_k)(b_k b_j) \\
&= (\vec{a} \cdot \vec{a})(\vec{b} \cdot \vec{b}) - (\vec{a} \cdot \vec{b})^2
\end{aligned}$$

which matches the RHS. □

Question 2

Using index notation, prove the following expression is true:

$$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = \vec{c}[\vec{d} \cdot (\vec{a} \times \vec{b})] - \vec{d}[\vec{c} \cdot (\vec{a} \times \vec{b})]$$

Use index notation and replace the indices only at the end.

Solution

Similar to Question 1, expand LHS,

$$
\begin{aligned}
\underbrace{(\vec{a}\times\vec{b})}_{P_j}\times\underbrace{(\vec{c}\times\vec{d})}_{Q_k} &= \epsilon_{ijk}P_jQ_k \\
&= \epsilon_{ijk}P_j(\epsilon_{klm}c_ld_m) \\
&= \epsilon_{ijk}\epsilon_{klm}P_jc_ld_m \\
&= (\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl})P_jc_ld_m \\
&= (\delta_{il}\delta_{jm}P_jc_ld_m)-(\delta_{im}\delta_{jl}P_jc_ld_m) \\
&= (P_jc_id_j)-(P_jc_jd_i) \\
&= c_i[d_jP_j]-d_i[c_jP_j] \\
&= \vec{c}[\vec{d}\cdot(\vec{a}\times\vec{b})]-\vec{d}[\vec{c}\cdot(\vec{a}\times\vec{b})] \quad \square
\end{aligned}
$$

done

Question 3

Consider the general form of momentum balance

$$\rho\frac{dv_i}{dt}=\frac{\partial T_{ji}}{\partial x_j}+\rho b_i \tag{1}$$

vhere d/dt is the total derivative; v_i is the velocity; ρ denotes density; T_{ij} is the stress tensor and b_i is a body force. This equation says that inertial and body forces are balanced by the gradients of the stress tensor.

The general form of the stress tensor reads

$$T_{ij}=-p\delta_{ij}+\mu\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right)-\frac{2}{3}\mu\frac{\partial v_k}{\partial x_k}\delta_{ij}+\kappa\left(\frac{\partial v_k}{\partial x_k}\delta_{ij}\right) \tag{2}$$

vhere p is the pressure, μ is the shear (dynamic) viscosity and κ is the dialation viscocity. For each question assume the viscosities μ and κ are constant.

(a) Using equations (1) and (2) derive the general form of Navier-Stokes equations for compressible viscous fluid. Note: only index notation can be used. The final form of equations should be presented in operator (vector) form.

(b) Assuming $\kappa = 0$, vrite the full set of Navier-Stokes equations in Cartesian coordinates assuming tvo-dimensional flov (tvo equations corresponding to x and y directions).

Solution

(a)

First, deal vith the stress tensor.

$$T_{ji} = -p\delta_{ji} + \mu\left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j}\right) - \frac{2}{3}\mu\frac{\partial v_k}{\partial x_k}\delta_{ji} + \kappa\left(\frac{\partial v_k}{\partial x_k}\delta_{ji}\right)$$

Differentiate vith respect to x_j,

$$\frac{\partial T_{ji}}{\partial x_j} = -\frac{\partial p}{\partial x_j}\delta_{ji} + \mu\left(\frac{\partial^2 v_j}{\partial x_j \partial x_i} + \frac{\partial^2 v_i}{\partial x_j^2} - \frac{2}{3}\frac{\partial^2 v_k}{\partial x_j \partial x_k}\delta_{ji}\right) + \kappa\left(\frac{\partial^2 v_k}{\partial x_j \partial x_k}\delta_{ji}\right)$$

Substituting into equation 1,

$$\begin{aligned}\rho\frac{\partial v_i}{\partial t} + \rho v_j\frac{\partial v_i}{\partial x_j} &= -\frac{\partial p}{\partial x_j}\delta_{ji} + \mu\left(\frac{\partial^2 v_j}{\partial x_j \partial x_i} + \frac{\partial^2 v_i}{\partial x_j^2} - \frac{2}{3}\frac{\partial^2 v_k}{\partial x_j \partial x_k}\delta_{ji}\right) + \kappa\left(\frac{\partial^2 v_k}{\partial x_j \partial x_k}\delta_{ji}\right) + \rho b_i \\ &= -\frac{\partial p}{\partial x_i} + \mu\left(\frac{\partial^2 v_j}{\partial x_j \partial x_i} + \frac{\partial^2 v_i}{\partial x_j^2} - \frac{2}{3}\frac{\partial^2 v_k}{\partial x_i \partial x_k}\right) + \kappa\left(\frac{\partial^2 v_k}{\partial x_i \partial x_k}\right) + \rho b_i\end{aligned}$$

Converting the RHS to vector form,

$$\begin{aligned}\text{RHS} &= -\nabla p + \mu\left[\nabla(\nabla\cdot\vec{v}) + \nabla^2\vec{v} - \frac{2}{3}\nabla(\nabla\cdot\vec{v})\right] + \kappa\nabla(\nabla\cdot\vec{v}) + \rho\vec{b} \\ &= -\nabla p + \mu\nabla^2\vec{v} + \left[\frac{\mu}{3} + \kappa\right]\nabla(\nabla\cdot\vec{v}) + \rho\vec{b}\end{aligned}$$

Converting the LHS to vector form,

$$\text{LHS} = \rho\left(\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v}\right)$$

Combining the LHS and RHS,

$$\boxed{\rho\left(\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v}\right) = -\nabla p + \mu\nabla^2\vec{v} + \left[\frac{\mu}{3} + \kappa\right]\nabla(\nabla\cdot\vec{v}) + \rho\vec{b}}$$

(b)

Assuming $\kappa = 0$, the equation reduces to

$$\rho\left(\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v}\right) = -\nabla p + \mu\nabla^2\vec{v} + \frac{\mu}{3}\nabla(\nabla\cdot\vec{v}) + \rho\vec{b}$$

In Cartesian coordinates (x, y), assuming $\vec{v} = (u, v)$ and $\vec{b} = (b_x, b_y)$. In the x-direction,

$$\boxed{\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \frac{\mu}{3}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y}\right) + \rho b_x}$$

In the y-direction,

$$\boxed{\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \frac{\mu}{3}\left(\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial y^2}\right) + \rho b_y}$$

Question 4

Consider the heat transfer in incompressible flow of Newtonian fluid. The following form of energy conservation equation holds for this case:

$$\rho C_p \left(\frac{\partial T}{\partial t} + \vec{v} \cdot \nabla T \right) = k \nabla^2 T + \frac{\mu}{2} \Phi^2$$

where T is the temperature, t is time, $\vec{v}$ is the velocity, ρ is density, C_p is the heat capacity, k is the thermal conductivity, μ is the shear (dynamic) viscosity, and Φ_{ij} is the rate of shear tensor defined as

$$\Phi_{ij} = \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}$$

(a) What is the physical interpretation of the following terms: $\rho C_p(\vec{v} \cdot \nabla T)$, $k\nabla^2 T$ and $\frac{\mu}{2}\Phi^2$.

(b) Write equation (3) in Cartesian coordinates assuming three-dimensional flow. Explain all the steps how the term Φ_{ij}^2 unfolds.

Solution

(a)

1. $\rho C_p(\vec{v} \cdot \nabla T)$ is the rate of change of energy due to convection.
2. $k\nabla^2 T$ is the rate of change of energy due to conduction.
3. $\frac{\mu}{2}\Phi^2$ is the rate of change of energy due to viscous dissipation (friction).

(b)

First deal with Φ_{ij}^2,

$$\Phi_{ij}^2 = \sum_{i,j}^{3} \Phi_{ij}\Phi_{ij}$$

Then,

$$\begin{aligned} \Phi_{ij}^2 = {} & \Phi_{11}\Phi_{11} + \Phi_{21}\Phi_{21} + \Phi_{31}\Phi_{31} \\ & + \Phi_{12}\Phi_{12} + \Phi_{22}\Phi_{22} + \Phi_{32}\Phi_{32} \\ & + \Phi_{13}\Phi_{13} + \Phi_{23}\Phi_{23} + \Phi_{33}\Phi_{33} \end{aligned}$$

In Cartesian coordinates, assuming $v_i = (u, v, w)$, $x_i = (x, y, z)$,

$$\begin{aligned}
\Phi_{ij}^2 = &\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)^2 \\
&+ \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)^2 \\
&+ \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z} + \frac{\partial w}{\partial z}\right)^2 \\
= &\,4\left(\frac{\partial u}{\partial x}\right)^2 + 4\left(\frac{\partial v}{\partial x}\right)^2 + 4\left(\frac{\partial w}{\partial x}\right)^2 \\
&+ 2\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial y}\right)^2 + 2\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)^2
\end{aligned}$$

Now the rest of the terms,

$$\boxed{\begin{aligned}
\rho C_p \left[\frac{\partial T}{\partial t} + \left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}\right)\right] = &\, k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) \\
&+ \frac{\mu}{2}\left[4\left(\frac{\partial u}{\partial x}\right)^2 + 4\left(\frac{\partial v}{\partial x}\right)^2 + 4\left(\frac{\partial w}{\partial x}\right)^2\right. \\
&\quad + 2\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial y}\right)^2 \\
&\quad \left. +2\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)^2\right]
\end{aligned}}$$