Question 1

Write down the full set of governing equations (continuity, momentum and energy).

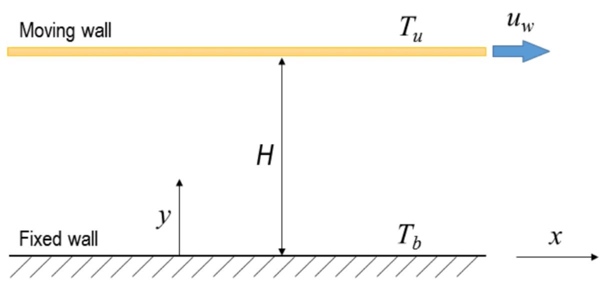

Figure 1: Fluid flow betweeen parallel plates. The bottom plate is fixed, the upper plate is moving horizontally with velocity u_w. THe fluid is incompressible and Newtonian.

Solution

We first begin with continuity,

$$\boxed{\frac{\partial \rho}{\partial t} + \vec{u} \cdot \nabla \rho + \rho(\nabla \cdot \vec{u}) = 0}$$

then, the momentum equation,

$$\boxed{\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) = -\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right] - \frac{\partial}{\partial x_i}\left(\frac{2}{3}\mu\frac{\partial u_k}{\partial x_k}\right) + \rho b_i}$$

lastly, the energy equation,

$$\boxed{\rho C_p\left(\frac{\partial T}{\partial t} + \vec{v} \cdot \nabla T\right) = k\nabla^2 T + \frac{\mu}{2}\Phi^2}$$

where $\Phi_{i,j} = \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$.

Question 2

Write down the momentum conservation equations in Cartesian coordinates. Simplify them using conditions outlined in bullet points.

Solution

There is no pressure gradient pushing the fluid in the x-direction. The flow is established by viscous stresses caused by the movement of the upper plate.

Table 1: Assumptions and simplifying consequences

Assumption	**Consequence**
Steady State (S.S.)	$\frac{\partial}{\partial t} = 0$
Incompressible	$\rho = \text{constant}$
Newtonian	$\mu = \text{constant}$
2D Flow	$u_z = 0,\ \partial_z \vec{u} = 0$
Parallel	$u_y = 0,\ \partial_y \vec{u} = 0$
Fully developed in x (F.D.)	$v_x = v_x(y),$
Constant Pressure in x (C.P.)	$\frac{\partial P}{\partial x} = 0$
Gravity in z	$\vec{b} = -g\hat{k}$

We begin with the continuity equation,

$$\cancelto{\text{S.S.}}{\frac{\partial \rho}{\partial t}} + \vec{u}\cdot\cancelto{\text{Incmp.}}{\nabla\rho} + \rho(\nabla\cdot\vec{u}) = 0$$

this simplifies to,

$$\frac{\partial}{\partial x}u_x + \cancelto{\text{Par.}}{\frac{\partial}{\partial y}u_y} + \cancelto{\text{2D}}{\frac{\partial}{\partial z}u_z} = 0$$

$$\implies \boxed{\frac{\partial u_x}{\partial x} = 0}$$

Next, in the x-direction with the momentum equation,

$$\cancelto{\text{S.S.}}{\frac{\partial}{\partial t}(\rho u_x)} + \frac{\partial}{\partial x_j}(\rho u_x u_j) = -\cancelto{\text{C.P.}}{\frac{\partial P}{\partial x}} + \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial u_x}{\partial x_j} + \frac{\partial u_j}{\partial x_x}\right)\right] - \frac{\partial}{\partial x}\left(\frac{2}{3}\mu\frac{\partial u_k}{\partial x_k}\right) + \rho b_x$$

expanding the index notation and simplifying,

$$\rho\left(\cancelto{\text{Cont.}}{\frac{\partial}{\partial x}(u_x^2)} + \cancelto{\text{Par.}}{\frac{\partial}{\partial y}(u_x u_y)} + \cancelto{\text{2D}}{\frac{\partial}{\partial z}(u_x u_z)}\right) = \frac{\partial}{\partial x}\left[\cancelto{\text{Cont.}}{\mu\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_x}{\partial x}\right)}\right]$$

$$+ \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u_x}{\partial y} + \cancelto{\text{Par.}}{\frac{\partial u_y}{\partial x}}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\cancelto{\text{2D}}{\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}}\right)\right]$$

$$- \frac{\partial}{\partial x}\left[\cancelto{\text{Cont.}}{\frac{2}{3}\mu\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right)}\right] + \rho\cancelto{\text{Grav.}}{b_x}$$

which results in

$$\boxed{\frac{\partial^2 u_x}{\partial y^2} = 0}$$

Next, in the y-direction with the momentum equation,

$$\cancelto{\text{S.S.}}{\frac{\partial}{\partial t}(\rho u_y)} + \frac{\partial}{\partial x_j}(\rho u_y u_j) = -\frac{\partial P}{\partial y} + \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial u_y}{\partial x_j} + \frac{\partial u_j}{\partial x_y}\right)\right] - \frac{\partial}{\partial y}\left(\frac{2}{3}\mu\frac{\partial u_k}{\partial x_k}\right) + \rho b_y$$

expanding the index notation and simplifying,

$$\begin{aligned}\rho\left(\cancelto{\text{Par.}}{\frac{\partial}{\partial x}}(u_x u_y) + \cancelto{\text{Par.}}{\frac{\partial}{\partial y}}(u_y^2) + \cancelto{\text{Par.}}{\frac{\partial}{\partial z}}(u_y u_z)\right) &= -\frac{\partial P}{\partial y} + \frac{\partial}{\partial x}\left[\mu\left(\cancelto{\text{Par.}}{\frac{\partial u_y}{\partial x}} + \cancelto{\text{Cont.}}{\frac{\partial u_x}{\partial y}}\right)\right] \\ &\quad + \frac{\partial}{\partial y}\left[\cancelto{\text{Par.}}{\mu\left(\frac{\partial u_y}{\partial y} + \frac{\partial u_y}{\partial y}\right)}\right] + \frac{\partial}{\partial z}\left[\mu\cancelto{\text{2D}}{\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right)}\right] \\ &\quad - \frac{\partial}{\partial y}\left[\frac{2}{3}\mu\cancelto{\text{Cont.}}{\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right)}\right] + \rho\cancelto{\text{Grav.}}{b_y}\end{aligned}$$

resulting in

$$\boxed{\frac{\partial P}{\partial y} = 0}$$

Finally, in the z-direction with the momentum equation,

$$\cancelto{\text{S.S.}}{\frac{\partial}{\partial t}(\rho u_z)} + \frac{\partial}{\partial x_j}(\rho u_z u_j) = -\frac{\partial P}{\partial z} + \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial u_z}{\partial x_j} + \frac{\partial u_j}{\partial x_z}\right)\right] - \frac{\partial}{\partial z}\left(\frac{2}{3}\mu\frac{\partial u_k}{\partial x_k}\right) + \rho b_z$$

expanding the index notation and simplifying,

$$\begin{aligned}\rho\left(\cancelto{\text{2D}}{\frac{\partial}{\partial x}}(u_x u_z) + \cancelto{\text{2D}}{\frac{\partial}{\partial y}}(u_y u_z) + \cancelto{\text{2D}}{\frac{\partial}{\partial z}}(u_z^2)\right) &= -\frac{\partial P}{\partial z} + \frac{\partial}{\partial x}\left[\mu\left(\cancelto{\text{2D}}{\frac{\partial u_z}{\partial x}} + \cancelto{\text{2D}}{\frac{\partial u_x}{\partial z}}\right)\right] \\ &\quad + \frac{\partial}{\partial y}\left[\mu\left(\cancelto{\text{2D}}{\frac{\partial u_z}{\partial y}} + \cancelto{\text{2D}}{\frac{\partial u_y}{\partial z}}\right)\right] + \frac{\partial}{\partial z}\left[\mu\cancelto{\text{2D}}{\left(\frac{\partial u_z}{\partial z} + \frac{\partial u_z}{\partial z}\right)}\right] \\ &\quad - \frac{\partial}{\partial z}\left[\frac{2}{3}\mu\cancelto{\text{Cont.}}{\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right)}\right] + \rho b_z\end{aligned}$$

resulting in

$$\boxed{\frac{\partial P}{\partial z} = -\rho g}$$

Question 3

Solve the simplified momentum equation.

Solution

Using no-slip boundary conditions, the boundary conditions are

$$u_x(0) = 0$$
$$u_x(H) = u_w$$

The simplified momentum equation in the x-direction is given by

$$\frac{\partial^2 u_x}{\partial y^2} = 0$$

which has the general solution

$$u_x(y) = Ay + B$$

where A and B are constants. Applying the boundary conditions, we find

$$u_x(0) = B = 0$$
$$u_x(H) = AH = u_w$$

which gives

$$u_x(y) = \frac{u_w}{H} y$$

This linear profile is also known as Couette flow. Next, the simplified momentum equation in the y-direction is given by

$$\frac{\partial P}{\partial y} = 0$$

which has the general solution

$$P = C(z)$$

where $C(z)$ is a function of z. Lastly, the simplified momentum equation in the z-direction is given by

$$\frac{\partial P}{\partial z} = -\rho g$$

which has the general solution

$$P(z) = -\rho g z + D$$

where D is a constant.

So in summary,

$$\boxed{\begin{aligned} \vec{u} &= \left[\frac{u_w}{H} y\right] \hat{i} \\ P &= -\rho g z + D \end{aligned}}$$

Question 4

Simplify the energy equation.

Solution

On top of the assumptions from Table 1, we need additional assumptions to address temperature.

Table 2: Assumptions and simplifying consequences for energy equation

Assumption	**Consequence**
2D Flow	$\partial_z T = 0$
Fully developed in x (F.D.)	$T = T(y)$, $\partial_x T = 0$

The energy equation in the absence of a heat source is given by

$$\rho C_p \left(\frac{\partial T}{\partial t} + \vec{v} \cdot \nabla T \right) = k \nabla^2 T + \frac{\mu}{2} \Phi^2$$

From Assignment 1, the full Cartesian expansion is

$$\begin{aligned}
\rho C_p \left[\cancelto{\text{S.S}}{\frac{\partial T}{\partial t}} + \left(\cancelto{\text{F.D.}}{u_x \frac{\partial T}{\partial x}} + \cancelto{\text{Par.}}{u_y \frac{\partial T}{\partial y}} + \cancelto{\text{2D}}{u_z \frac{\partial T}{\partial z}} \right) \right] &= k \left(\cancelto{\text{F.D.}}{\frac{\partial^2 T}{\partial x^2}} + \frac{\partial^2 T}{\partial y^2} + \cancelto{\text{2D}}{\frac{\partial^2 T}{\partial z^2}} \right) \\
&\quad + \frac{\mu}{2} \left[4 \cancelto{\text{Cont.}}{\left(\frac{\partial u_x}{\partial x} \right)^2} + 4 \cancelto{\text{Par.}}{\left(\frac{\partial u_y}{\partial x} \right)^2} + 4 \cancelto{\text{2D}}{\left(\frac{\partial u_z}{\partial x} \right)^2} \right. \\
&\quad + 2 \left(\frac{\partial u_x}{\partial y} + \cancelto{\text{Par.}}{\frac{\partial u_y}{\partial x}} \right)^2 + 2 \cancelto{\text{Par.}}{\left(\frac{\partial u_y}{\partial y} + \frac{\partial u_y}{\partial y} \right)^2} \\
&\quad \left. + 2 \cancelto{\text{2D}}{\left(\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z} \right)^2} \right]
\end{aligned}$$

which simplifies to

$$0 = k \frac{\partial^2 T}{\partial y^2} + \mu \left(\frac{\partial u_x}{\partial y} \right)^2$$

Using the results from Q3,

$$\boxed{\frac{\partial^2 T}{\partial y^2} = -\frac{\mu}{k} \left(\frac{u_w}{H} \right)^2}$$

Question 5

Solve the simplified energy equation.

Solution

The simplified energy equation is given by

$$\frac{\partial^2 T}{\partial y^2} = -\frac{\mu}{k}\left(\frac{u_w}{H}\right)^2$$

Integrating twice with respect to y,

$$\begin{aligned}\frac{\partial T}{\partial y} &= -\frac{\mu}{k}\left(\frac{u_w}{H}\right)^2 y + C_1 \\ T &= -\frac{\mu}{2k}\left(\frac{u_w}{H}\right)^2 y^2 + C_1 y + C_2\end{aligned}$$

Applying the boundary conditions $T(0) = T_b$ and $T(H) = T_u$,

$$\begin{aligned}T_b &= C_2 \\ T_u &= -\frac{\mu}{2k}\left(\frac{u_w}{H}\right)^2 H^2 + C_1 H + C_2 \implies C_1 = \frac{T_u - T_b + \frac{\mu}{2k}\left(\frac{u_w}{H}\right)^2 H^2}{H}\end{aligned}$$

Thus, the temperature profile is given by

$$\boxed{T(y) = -\frac{\mu}{2k}\left(\frac{u_w}{H}\right)^2 y^2 + \left(\frac{T_u - T_b + \frac{\mu}{2k}\left(\frac{u_w}{H}\right)^2 H^2}{H}\right) y + T_b}$$

Question 6

Complete the numerical simulations using STAR CCM+ in the Seminar using the provided procedures. Include

(1) *Contour of temperature*

(2) *Contour of velocity*

(3) *Vertical profiles of temperature and velocity*

Furthermore, the results files of your simulations must be included in your Assignment submission.

Solution

The simulation was run as outlined in the seminar document.

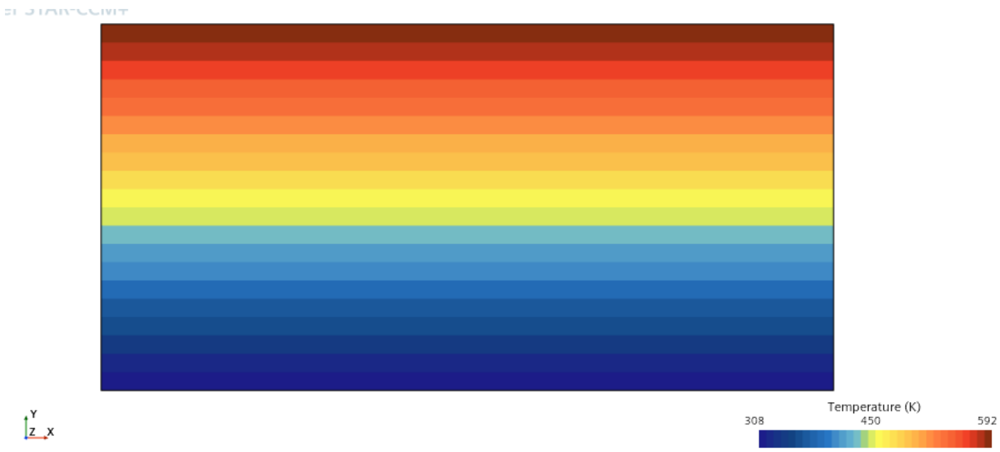

Figure 2: Temperature Contour

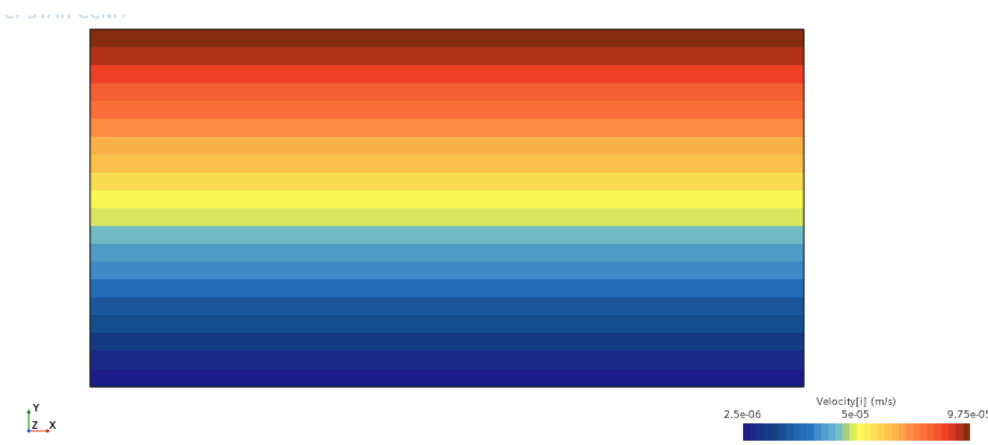

Figure 3: Velocity Contour

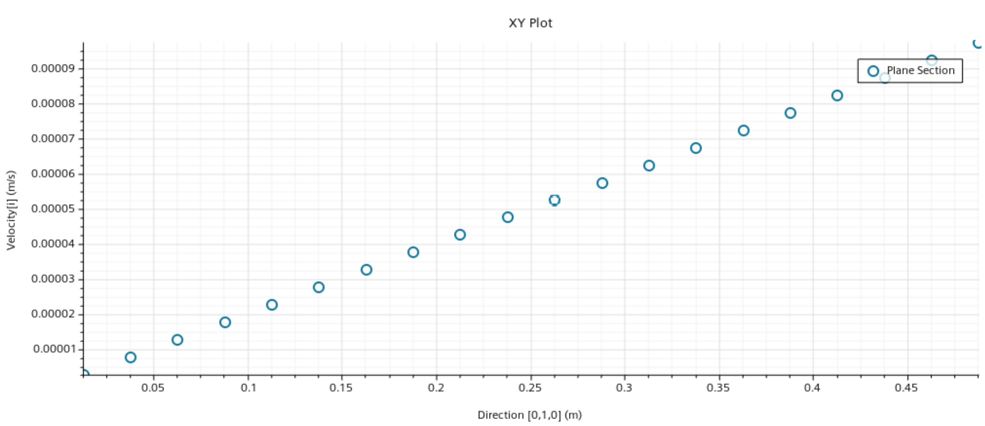

Figure 4: Velocity Vertical Plot

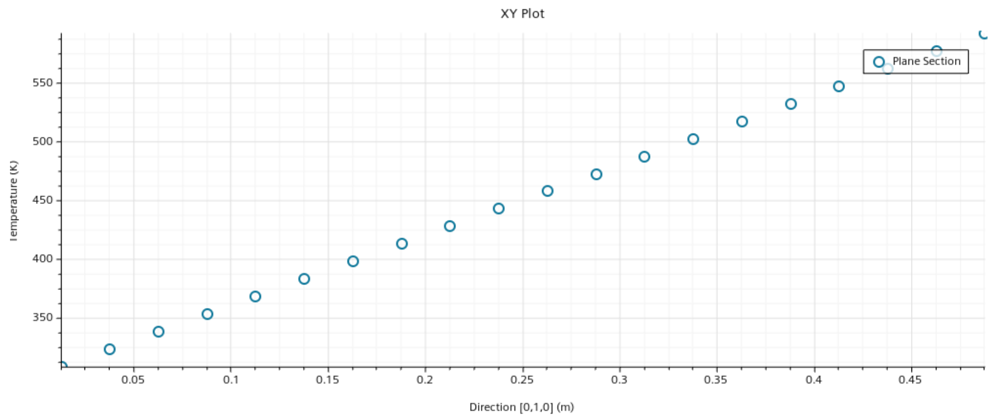

Figure 5: Temperature Vertical Plot

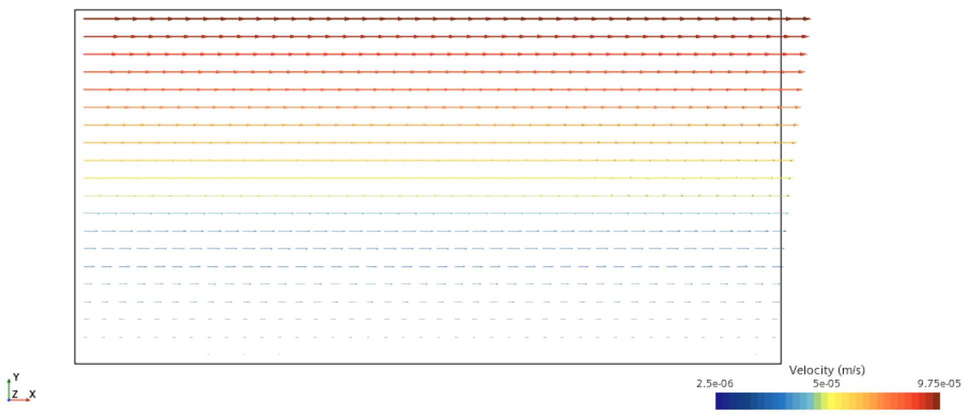

Figure 6: Velocity Vector Profile

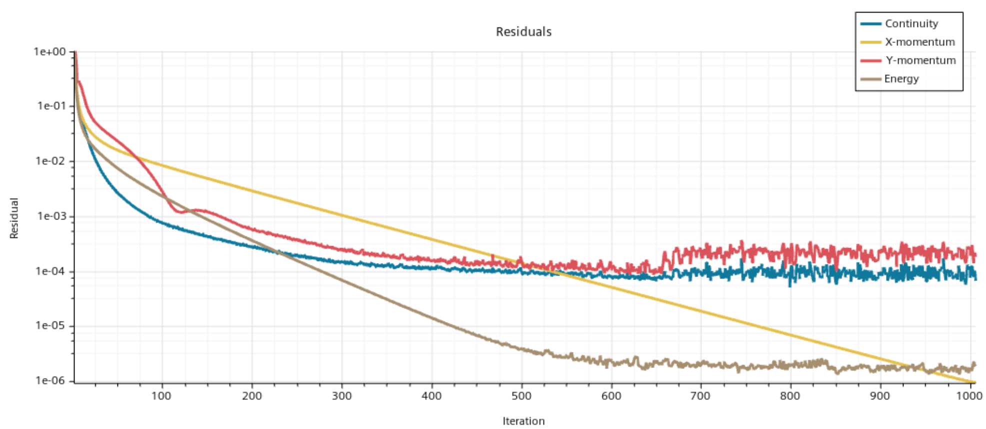

Figure 7: Residuals

Numerical vs Analytical Results

The values from the problem statement were given in Table 3. The analytical solution for

Table 3: Parameter values

Variable	Value
u_w	0.0001 m/s
H	0.5 m
μ	0.001 Pa· s
k	0.6 W/m· K
T_b	300 K
T_u	600 K

velocity was found to be

$$\begin{aligned}u_x(y) &= \frac{u_w}{H}y \\ &= \frac{0.0001}{0.5}y \\ &\boxed{= 0.0002y}\end{aligned}$$

The analytical solution for temperature was found to be

$$T(y) = ay^2 + by + c$$

where

$$\begin{aligned}a &= -\frac{\mu}{2k}\left(\frac{u_w}{H}\right)^2 \\ &= -\frac{0.001}{2\cdot 0.6}\left(\frac{0.0001}{0.5}\right)^2 \\ &= -3.33333\times 10^{-11} \\ b &= \frac{T_u - T_b + \frac{\mu}{2k}\left(\frac{u_w}{H}\right)^2 H^2}{H} \\ &= \frac{600-300+\frac{0.001}{2\cdot 0.6}\left(\frac{0.0001}{0.5}\right)^2 0.5^2}{0.5} \\ &= 600 \\ c &= T_b \\ &= 300\end{aligned}$$

so,

$$\boxed{T(y) = -3.33 \times 10^{-11} y^2 + 600y + 300}$$

Numerical vs analytical results are compared in Tables 4 and 5. Overall, the assumptions made were valid as the numerical and analytical results were in very close agreement. A linear velocity profile was observed, as expected from the Couette flow. The temperature profile had a linear profile but this is due to the small value from the viscous dissipation term. The residuals did not converge to zero, but were sufficiently small to be considered converged.

Table 4: Comparison of numerical and analytical for velocity

y (m)	u_N (m/s)	u_A (m/s)	Abs Difference (m/s)
0.4875	9.75E-05	9.75E-05	0.00E+00
0.4625	9.25E-05	9.25E-05	0.00E+00
0.4375	8.75E-05	8.75E-05	0.00E+00
0.4125	8.25E-05	8.25E-05	0.00E+00
0.3875	7.75E-05	7.75E-05	0.00E+00
0.3625	7.25E-05	7.25E-05	0.00E+00
0.3375	6.75E-05	6.75E-05	0.00E+00
0.3125	6.25E-05	6.25E-05	0.00E+00
0.2875	5.75E-05	5.75E-05	0.00E+00
0.2625	5.25E-05	5.25E-05	0.00E+00
0.2375	4.75E-05	4.75E-05	0.00E+00
0.2125	4.25E-05	4.25E-05	0.00E+00
0.1875	3.75E-05	3.75E-05	0.00E+00
0.1625	3.25E-05	3.25E-05	0.00E+00
0.1375	2.75E-05	2.75E-05	0.00E+00
0.1125	2.25E-05	2.25E-05	0.00E+00
0.0875	1.75E-05	1.75E-05	0.00E+00
0.0625	1.25E-05	1.25E-05	0.00E+00
0.0375	7.50E-06	7.50E-06	0.00E+00
0.0125	2.50E-06	2.50E-06	0.00E+00

Table 5: Comparison of numerical and analytical for temperature

y (m)	T_N (K)	T_A (K)	Abs Difference (K)
0.4875	592.5	592.5	0.00E+00
0.4625	577.5	577.5	6.10E-05
0.4375	562.5	562.5	6.10E-05
0.4125	547.5	547.5	0.00E+00
0.3875	532.5	532.5	6.10E-05
0.3625	517.5	517.5	6.10E-05
0.3375	502.5	502.5	6.10E-05
0.3125	487.5	487.5	6.10E-05
0.2875	472.5	472.5	6.10E-05
0.2625	457.5	457.5	3.05E-05
0.2375	442.5	442.5	3.05E-05
0.2125	427.5	427.5	3.05E-05
0.1875	412.5	412.5	3.05E-05
0.1625	397.5	397.5	3.05E-05
0.1375	382.5	382.5	3.05E-05
0.1125	367.5	367.5	0.00E+00
0.0875	352.5	352.5	0.00E+00
0.0625	337.5	337.5	0.00E+00
0.0375	322.5	322.5	0.00E+00
0.0125	307.5	307.5	0.00E+00