

Question 1

Consider two arbitrary vectors \vec{a} and \vec{b} . Use index notation to show that the following relationship is true:

$$(\vec{a} \times \vec{b}) \cdot (\vec{a} \times \vec{b}) = (\vec{a} \cdot \vec{a})(\vec{b} \cdot \vec{b}) - (\vec{a} \cdot \vec{b})^2$$

Solution

First, recall the definition of the cross product in index notation,

$$(\vec{a} \times \vec{b}) = \epsilon_{ijk} a_j b_k = C_i$$

the definition of the dot product,

$$(\vec{a} \cdot \vec{b}) = a_i b_i$$

an identity for Levi-Civita symbol,

$$\epsilon_{ijk} \epsilon_{ilm} = \delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl}$$

and the Kronecker delta,

$$\delta_{ij} A_{mj} = A_{mi}$$

Then,

$$\begin{aligned} (\vec{a} \times \vec{b}) \cdot (\vec{a} \times \vec{b}) &= C_i C_i \\ &= (\epsilon_{ijk} a_j b_k)(\epsilon_{ilm} a_l b_m) \\ &= \epsilon_{ijk} \epsilon_{ilm} a_j b_k a_l b_m \\ &= (\delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl}) a_j b_k a_l b_m \\ &= (\delta_{jl} \delta_{km} a_j b_k a_l b_m) - (\delta_{jm} \delta_{kl} a_j b_k a_l b_m) \\ &= (a_j b_k a_j b_k) - (a_j b_k a_k b_j) \end{aligned}$$

because we are dealing with vectors and scalar products, we can utilize commutativity

$$\begin{aligned} &= (a_j a_j b_k b_k) - (a_j a_k b_k b_j) \\ &= (a_j a_j)(b_k b_k) - (a_j a_k)(b_k b_j) \\ &= (\vec{a} \cdot \vec{a})(\vec{b} \cdot \vec{b}) - (\vec{a} \cdot \vec{b})^2 \end{aligned}$$

which matches the RHS. \square

Question 2

Using index notation, prove the following expression is true:

$$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = \vec{c}[\vec{d} \cdot (\vec{a} \times \vec{b})] - \vec{d}[\vec{c} \cdot (\vec{a} \times \vec{b})]$$

Use index notation and replace the indices only at the end.

Solution

Similar to Question 1, expand LHS,

$$\begin{aligned}
 \underbrace{(\vec{a} \times \vec{b})}_{P_j} \times \underbrace{(\vec{c} \times \vec{d})}_{Q_k} &= \epsilon_{ijk} P_j Q_k \\
 &= \epsilon_{ijk} P_j (\epsilon_{klm} c_l d_m) \\
 &= \epsilon_{ijk} \epsilon_{klm} P_j c_l d_m \\
 &= (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) P_j c_l d_m \\
 &= (\delta_{il} \delta_{jm} P_j c_l d_m) - (\delta_{im} \delta_{jl} P_j c_l d_m) \\
 &= (P_j c_i d_j) - (P_j c_j d_i) \\
 &= c_i [d_j P_j] - d_i [c_j P_j] \\
 &= \vec{c} [\vec{d} \cdot (\vec{a} \times \vec{b})] - \vec{d} [\vec{c} \cdot (\vec{a} \times \vec{b})] \quad \square
 \end{aligned}$$

done

Question 3

Consider the general form of momentum balance

$$\rho \frac{dv_i}{dt} = \frac{\partial T_{ji}}{\partial x_j} + \rho b_i \quad (1)$$

where d/dt is the total derivative; v_i is the velocity; ρ denotes density; T_{ij} is the stress tensor and b_i is a body force. This equation says that inertial and body forces are balanced by the gradients of the stress tensor.

The general form of the stress tensor reads

$$T_{ij} = -p\delta_{ij} + \mu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) - \frac{2}{3}\mu \frac{\partial v_k}{\partial x_k} \delta_{ij} + \kappa \left(\frac{\partial v_k}{\partial x_k} \delta_{ij} \right) \quad (2)$$

where p is the pressure, μ is the shear (dynamic) viscosity and κ is the dialation viscosity. For each question assume the viscosities μ and κ are constant.

- Using equations (1) and (2) derive the general form of Navier-Stokes equations for compressible viscous fluid. Note: only index notation can be used. The final form of equations should be presented in operator (vector) form.
- Assuming $\kappa = 0$, write the full set of Navier-Stokes equations in Cartesian coordinates assuming two-dimensional flow (two equations corresponding to x and y directions).

Solution

(a)

First, deal with the stress tensor.

$$T_{ji} = -p\delta_{ji} + \mu \left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j} \right) - \frac{2}{3}\mu \frac{\partial v_k}{\partial x_k} \delta_{ji} + \kappa \left(\frac{\partial v_k}{\partial x_k} \delta_{ji} \right)$$

Differentiate with respect to x_j ,

$$\frac{\partial T_{ji}}{\partial x_j} = -\frac{\partial p}{\partial x_j} \delta_{ji} + \mu \left(\frac{\partial^2 v_j}{\partial x_j \partial x_i} + \frac{\partial^2 v_i}{\partial x_j^2} - \frac{2}{3} \frac{\partial^2 v_k}{\partial x_j \partial x_k} \delta_{ji} \right) + \kappa \left(\frac{\partial^2 v_k}{\partial x_j \partial x_k} \delta_{ji} \right)$$

Substituting into equation 1,

$$\rho \frac{dv_i}{dt} = -\frac{\partial p}{\partial x_j} \delta_{ji} + \mu \left(\frac{\partial^2 v_j}{\partial x_j \partial x_i} + \frac{\partial^2 v_i}{\partial x_j^2} - \frac{2}{3} \frac{\partial^2 v_k}{\partial x_j \partial x_k} \delta_{ji} \right) + \kappa \left(\frac{\partial^2 v_k}{\partial x_j \partial x_k} \delta_{ji} \right) + \rho b_i$$