Question 1

Write down the full set of governing equations (continuity, momentum and energy).

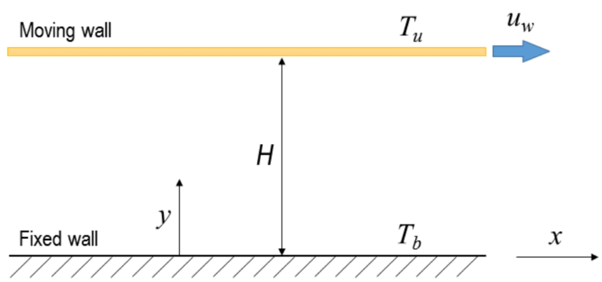

Figure 1: Fluid flow betweeen parallel plates. The bottom plate is fixed, the upper plate is moving horizontally with velocity u_w. THe fluid is incompressible and Newtonian.

Solution

We first begin with continuity,

$$\boxed{\frac{\partial \rho}{\partial t} + \vec{u} \cdot \nabla \rho + \rho(\nabla \cdot \vec{u}) = 0}$$

then, the momentum equation,

$$\boxed{\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) = -\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right] - \frac{\partial}{\partial x_i}\left(\frac{2}{3}\mu\frac{\partial u_k}{\partial x_k}\right) + \rho b_i}$$

lastly, the energy equation,

$$\boxed{\rho C_p\left(\frac{T}{\partial t} + \vec{v} \cdot \nabla T\right) = k\nabla^2 T + \frac{\mu}{2}\Phi^2}$$

where $\Phi_{i,j} = \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$.

Question 2

Write down the momentum conservation equations in Cartesian coordinates. Simplify them using conditions outlined in bullet points.

Table 1: Assumptions and simplifying consequences

Assumption	**Consequence**
Steady State (S.S.)	$\frac{\partial}{\partial t} = 0$
Incompressible	$\rho = \text{constant}$
Newtonian	$\mu = \text{constant}$
2D Flow	$u_z = \partial_z = 0$
Parallel	$u_y = \partial_y = 0$
Fully developed (F.D.)	$v_x = v_x(y)$, $T = T(y)$, $\partial_x = 0$

We begin with the continuity equation,

$$\cancelto{\text{S.S.}}{\frac{\partial \rho}{\partial t}} + \vec{u}\cdot\cancelto{\text{Incmp.}}{\nabla\rho} + \rho(\nabla\cdot\vec{u}) = 0$$

this simplifies to,

$$\frac{\partial}{\partial x}u_x + \frac{\partial}{\partial y}u_y + \frac{\partial}{\partial z}u_z = 0$$

We begin in the x-direction with the momentum equation,

$$\cancelto{\text{S.S.}}{\frac{\partial}{\partial t}(\rho u_x)} + \frac{\partial}{\partial x_j}(\rho u_x u_j) = -\cancelto{\text{F.D.}}{\frac{\partial P}{\partial x}} + \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial u_x}{\partial x_j} + \frac{\partial u_j}{\partial x_x}\right)\right] - \frac{\partial}{\partial x}\left(\frac{2}{3}\mu\frac{\partial u_k}{\partial x_k}\right) + \rho b_x$$

which simplifies to,

$$\rho\left(\cancelto{\text{F.D.}}{\frac{\partial}{\partial x}}(u_x^2) + \cancelto{\text{Par.}}{\frac{\partial}{\partial y}}(u_x u_y) + \cancelto{\text{2D}}{\frac{\partial}{\partial z}}(u_x u_z)\right) = \mu\left(\cancelto{\text{Par.}}{\frac{\partial^2 u_x}{\partial y^2}} + \cancelto{\text{2D}}{\frac{\partial^2 u_x}{\partial z^2}}\right) + \rho b_x$$

$$\boxed{0 = b_x}$$

Next, we move to the y-direction,

$$\cancelto{\text{S.S.}}{\frac{\partial}{\partial t}(\rho u_y)} + \frac{\partial}{\partial x_j}(\rho u_y u_j) = -\cancelto{\text{F.D.}}{\frac{\partial P}{\partial y}} + \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial u_y}{\partial x_j} + \frac{\partial u_j}{\partial x_y}\right)\right] - \frac{\partial}{\partial y}\left(\frac{2}{3}\mu\frac{\partial u_k}{\partial x_k}\right) + \rho b_y$$

which simplifies to,

$$\rho\left(\cancelto{\text{Par.}}{\frac{\partial}{\partial x}}(u_x u_y) + \cancelto{\text{F.D.}}{\frac{\partial}{\partial y}}(u_y^2) + \cancelto{\text{2D}}{\frac{\partial}{\partial z}}(u_y u_z)\right) = \mu\left(\cancelto{\text{Par.}}{\frac{\partial^2 u_y}{\partial x^2}} + \cancelto{\text{2D}}{\frac{\partial^2 u_y}{\partial z^2}}\right) + \rho b_y$$

$$\boxed{0 = b_y}$$

Question 3

Consider the general form of momentum balance

$$\rho \frac{dv_i}{dt} = \frac{\partial T_{ji}}{\partial x_j} + \rho b_i \tag{1}$$

vhere d/dt is the total derivative; v_i is the velocity; ρ denotes density; T_{ij} is the stress tensor and b_i is a body force. This equation says that inertial and body forces are balanced by the gradients of the stress tensor.

The general form of the stress tensor reads

$$T_{ij} = -p\delta_{ij} + \mu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) - \frac{2}{3}\mu \frac{\partial v_k}{\partial x_k}\delta_{ij} + \kappa \left(\frac{\partial v_k}{\partial x_k}\delta_{ij} \right) \tag{2}$$

vhere p is the pressure, μ is the shear (dynamic) viscosity and κ is the dialation viscocity. For each question assume the viscosities μ and κ are constant.

(a) Using equations (1) and (2) derive the general form of Navier-Stokes equations for compressible viscous fluid. Note: only index notation can be used. The final form of equations should be presented in operator (vector) form.

(b) Assuming $\kappa = 0$, vrite the full set of Navier-Stokes equations in Cartesian coordinates assuming tvo-dimensional flov (tvo equations corresponding to x and y directions).

Solution

(a)

First, deal vith the stress tensor.

$$T_{ji} = -p\delta_{ji} + \mu \left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j} \right) - \frac{2}{3}\mu \frac{\partial v_k}{\partial x_k}\delta_{ji} + \kappa \left(\frac{\partial v_k}{\partial x_k}\delta_{ji} \right)$$

Differentiate vith respect to x_j,

$$\frac{\partial T_{ji}}{\partial x_j} = -\frac{\partial p}{\partial x_j}\delta_{ji} + \mu \left(\frac{\partial^2 v_j}{\partial x_j \partial x_i} + \frac{\partial^2 v_i}{\partial x_j^2} - \frac{2}{3}\frac{\partial^2 v_k}{\partial x_j \partial x_k}\delta_{ji} \right) + \kappa \left(\frac{\partial^2 v_k}{\partial x_j \partial x_k}\delta_{ji} \right)$$

Substituting into equation 1,

$$\begin{aligned} \rho \frac{\partial v_i}{\partial t} + \rho v_j \frac{\partial v_i}{\partial x_j} &= -\frac{\partial p}{\partial x_j}\delta_{ji} + \mu \left(\frac{\partial^2 v_j}{\partial x_j \partial x_i} + \frac{\partial^2 v_i}{\partial x_j^2} - \frac{2}{3}\frac{\partial^2 v_k}{\partial x_j \partial x_k}\delta_{ji} \right) + \kappa \left(\frac{\partial^2 v_k}{\partial x_j \partial x_k}\delta_{ji} \right) + \rho b_i \\ &= -\frac{\partial p}{\partial x_i} + \mu \left(\frac{\partial^2 v_j}{\partial x_j \partial x_i} + \frac{\partial^2 v_i}{\partial x_j^2} - \frac{2}{3}\frac{\partial^2 v_k}{\partial x_i \partial x_k} \right) + \kappa \left(\frac{\partial^2 v_k}{\partial x_i \partial x_k} \right) + \rho b_i \end{aligned}$$

Converting the RHS to vector form,

$$\begin{aligned}\text{RHS} &= -\nabla p + \mu\left[\nabla(\nabla\cdot\vec{v}) + \nabla^2\vec{v} - \frac{2}{3}\nabla(\nabla\cdot\vec{v})\right] + \kappa\nabla(\nabla\cdot\vec{v}) + \rho\vec{b} \\ &= -\nabla p + \mu\nabla^2\vec{v} + \left[\frac{\mu}{3} + \kappa\right]\nabla(\nabla\cdot\vec{v}) + \rho\vec{b}\end{aligned}$$

Converting the LHS to vector form,

$$\text{LHS} = \rho\left(\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v}\right)$$

Combining the LHS and RHS,

$$\boxed{\rho\left(\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v}\right) = -\nabla p + \mu\nabla^2\vec{v} + \left[\frac{\mu}{3} + \kappa\right]\nabla(\nabla\cdot\vec{v}) + \rho\vec{b}}$$

(b)

Assuming $\kappa = 0$, the equation reduces to

$$\rho\left(\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v}\right) = -\nabla p + \mu\nabla^2\vec{v} + \frac{\mu}{3}\nabla(\nabla\cdot\vec{v}) + \rho\vec{b}$$

In Cartesian coordinates (x, y), assuming $\vec{v} = (u, v)$ and $\vec{b} = (b_x, b_y)$. In the x-direction,

$$\boxed{\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \frac{\mu}{3}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x\partial y}\right) + \rho b_x}$$

In the y-direction,

$$\boxed{\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \frac{\mu}{3}\left(\frac{\partial^2 u}{\partial x\partial y} + \frac{\partial^2 v}{\partial y^2}\right) + \rho b_y}$$

Question 4

Consider the heat transfer in incompressible flow of Newtonian fluid. The following form of energy conservation equation holds for this case:

$$\rho C_p\left(\frac{\partial T}{\partial t} + \vec{v}\cdot\nabla T\right) = k\nabla^2 T + \frac{\mu}{2}\Phi^2$$

where T is the temperature, t is time, $\vec{v}$ is the velocity, ρ is density, C_p is the heat capacity, k is the thermal conductivity, μ is the shear (dynamic) viscosity, and Φ_{ij} is the rate of shear tensor defined as

$$\Phi_{ij} = \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}$$

(a) What is the physical interpretation of the following terms: $\rho C_p(\vec{v}\cdot\nabla T)$, $k\nabla^2 T$ and $\frac{\mu}{2}\Phi^2$.

(b) Write equation (3) in Cartesian coordinates assuming three-dimensional flow. Explain all the steps how the term Φ_{ij}^2 unfolds.

Solution

(a)

1. $\rho C_p(\vec{v} \cdot \nabla T)$ is the rate of change of energy due to convection.
2. $k\nabla^2 T$ is the rate of change of energy due to conduction.
3. $\frac{\mu}{2}\Phi^2$ is the rate of change of energy due to viscous dissipation (friction).

(b)

First deal with Φ_{ij}^2,

$$\Phi_{ij}^2 = \sum_{i,j}^{3} \Phi_{ij}\Phi_{ij}$$

Then,

$$\begin{aligned}\Phi_{ij}^2 = \Phi_{11}\Phi_{11} + \Phi_{21}\Phi_{21} + \Phi_{31}\Phi_{31} \\ + \Phi_{12}\Phi_{12} + \Phi_{22}\Phi_{22} + \Phi_{32}\Phi_{32} \\ + \Phi_{13}\Phi_{13} + \Phi_{23}\Phi_{23} + \Phi_{33}\Phi_{33}\end{aligned}$$

In Cartesian coordinates, assuming $v_i = (u, v, w)$, $x_i = (x, y, z)$,

$$\begin{aligned}\Phi_{ij}^2 &= \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)^2 \\ &\quad + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)^2 \\ &\quad + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z} + \frac{\partial w}{\partial z}\right)^2 \\ &= 4\left(\frac{\partial u}{\partial x}\right)^2 + 4\left(\frac{\partial v}{\partial x}\right)^2 + 4\left(\frac{\partial w}{\partial x}\right)^2 \\ &\quad + 2\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial y}\right)^2 + 2\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)^2\end{aligned}$$

Now the rest of the terms,

$$\boxed{\begin{aligned}\rho C_p\left[\frac{\partial T}{\partial t}+\left(u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z}\right)\right] &= k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)\\ &+\frac{\mu}{2}\left[4\left(\frac{\partial u}{\partial x}\right)^2+4\left(\frac{\partial v}{\partial x}\right)^2+4\left(\frac{\partial w}{\partial x}\right)^2\right.\\ &\quad+2\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2+2\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial y}\right)^2\\ &\quad\left.+2\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)^2\right]\end{aligned}}$$