Background on Valuations

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Definition 1 (Discrete Valuation Ring (D.V.R.)).
A discrete valuation ring is a principal ideal domain O with a unique maximal ideal $p \neq 0$.

Definition 2 (Uniformizing Parameter).
Let $\mathfrak{p}$ be the unique maximal ideal of D.V.R. O. Since O is a PID, there exists a prime π in O such that $\mathfrak{p} = (\pi)$. Such a π is called a uniformizing parameter.

Definition 3 (Discrete Valuation).
A discrete valuation is a function $v_{\mathfrak{p}} : K^* \to \mathbb{Z}$ where K is the field of fractions of O. The valuation $v_{\mathfrak{p}}$ can be extended to K by setting $v_{\mathfrak{p}}(0) = \infty$. Further $v_{\mathfrak{p}}$ satisfies the following

I. $v_{\mathfrak{p}}(ab) = v_{\mathfrak{p}}(a) + v_{\mathfrak{p}}(b)$.

II. $v_{\mathfrak{p}}(a+b) \geq min\{v_{\mathfrak{p}}(a), v_{\mathfrak{p}}(b)\}$.

Proposition 1 (Nonarchimedean).
A valuaion $v(x)$ is nonarchimedean if and only if it satisfies $v(x+y) \leq max\{v(x), v(y)\}$. All discrete valuations are nonarchimedean.

Proposition 2.
Let O be a noetherian integral domain. O is a Dedekind domain if and only if, for all prime ideals $\mathfrak{p} \neq 0$, the localizations $O_{(\mathfrak{p})}$ are discrete valuation rings.

Definition 4 (Prolongation of a valuation).
If $A \subset B$ are rings with B integral over A, $\mathfrak{p}$ a prime in A, and $\mathfrak{q}$ prime in B dividing $\mathfrak{p}$ with ramification index $e_{\mathfrak{q}}$ then $v_{\mathfrak{q}}(x) = e_{\mathfrak{q}} v_{\mathfrak{p}}(x)$ is a prolongation of $v_{\mathfrak{p}}$ with index $e_{\mathfrak{q}}$.