COLORADO STATE UNIVERSITY MATHEMATICS

Higher Ramification Groups

Dean Bisogno

April 19, 2016

1 ABSTRACT

2 DEFINITIONS AND NOTATION

2.1 DISCRETE VALUATIONS

We borrow our definitions in this section from Dummit and Foote, 2004.

Definition 1 (Discrete Valuations)**.**
A discrete valuation on a field K is a surjective function $v_K : K^* \to \mathbb{Z}$ such that

I. $v_K(xy) = v(x) + v(y)$ *for all* $x, y \in K^X$.

II. $v_K(x + y) \geq min\{v_K(x), v_K(y)\}$ *for all* $x, y \in K^X$ *such that* $x + y \neq 0$.

Further, we call the subring $\{x \in K | v_K(x) \geq 0\} \cup \{0\}$ *the valuation ring of* K.

Definition 2 (Discrete Valuation Ring (D.V.R.))**.**
If R is an integral domain and R is the valuation ring of the field of fractions on R, then we call R a discrete valuation ring. The ring R also has a local ring with unique maximal ideal $M = \{r \in R | v_K(x) > 0\}$.

Note that in Math 566 we proved (exercise 26 section 7.1) that for any unit $u \in K^*$ that $v_K(u) = 0$. Thus it follows that $R \setminus M$ is the set of invertible elements of R.

Definition 3 (Uniformizing Parameter)**.**
If R is a D.V.R. of K via v_K, then an element $t \in R$ such that $v_K(t) = 1$ is a uniformizing parameter for R. Theorem 7, section 16.2 in Dummit and Foote, 2004 shows that t is unique (up to multiplication by a unit) and generates the unique maximal ideal of R.

2.2 Field Completion

With a valuation v_K on a field K then for any real number $a \in (0,1)$ we can induce an absolute value on K by

$$||x|| = \begin{cases} a^{v_K(x)} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

which satisfies the usual conditions of a metric as outlined by Serre, 1995. A topology is then induced on K via the absolute value metric, and we can denote the completion of K with respect to the valuation v_K by K_v. Note also that the metrics induced by different choices of a are topologically equivalent, so the completion is dependent only on v_K. Further, v_K extends in the completion of K to a discrete valuation (which we will continue to call v_K) on K_v (Serre, 1995).

2.3 Higher Ramification Groups

Consider now a finite Galois extension $L|K$ with Galois group G, and associated discrete valuations v_K and v_L and valuation rings O_L, and O_K.

Definition 4 (Higher Ramification Groups)**.**
For every real number $i > -1$ we define the i^{th} ramification group of $L|K$ by

$$G_i = G_i(L|K) = \{\sigma \in G | v_L(\sigma(a) - a) \geq i + 1 \ \forall a \in O_L\}.$$

Clearly this induces the following filtration on G:

$$\{G_i(L|K)\}_{i \geq -1} = G_{-}1 \supseteq G_0 \supseteq G_1 \supseteq \dots$$

Proposition 1.
For any $a \in O_L$ and $\sigma \in G$, $v_L(\sigma(a) - a) \geq i + 1$ if and only if $\sigma(a) \cong a \mod P_L^{i+1}$.

Proposition 2.
The first two ramification groups are $G_0 = G$ and $G_1 = I$.

3 The Upper Numbering and Ramification Jump

Definition 5 (Upper Numbering the Higher Ramification groups)**.**
Consider the function

$$t = \varphi(s) = \int_0^s \frac{dx}{[G_0 : G_x]}$$

called the Herbrand function which has inverse map ψ. Then for any real $s \geq -1$ let $G_s = G_{\lceil s \rceil}$ and renumber the ramification groups by $G^t(L|K) = G_s(L|K)$ where $s = \psi(t)$.

Definition 6 (Ramification Jump)**.**
If $G^t(L|K) \neq G^{t+\epsilon}(L|K)$ for any $\epsilon \geq 0$ then we call t a ramification jump.

Theorem 1 (Hasse-Arf)**.**
For a finite abelian extension $L|K$, the jumps of the filtration $\{G^i(L|K)\}_{i\geq -1}$ are rational integers.

3.1 Application to Wild Riemann-Hurwitz

We can rephrase Riemann-Hurwitz such that if field $\mathbb{F}$ has characteristic p and $p|e$ where e is the ramification index of a point under a degree d covering map $\phi : X \to Y$ of curves over F Then

$$2g_Y - 2 = d(2g_X - 2) + \sum_{x \in X} \sum_{i=0}^{\infty} |G_i(x)| - 1$$

References

Cassels, J. & Frohlich, A. (1967). *Algebraic number theory*. Academic Press Inc. (London) LTD.

Chonoles, Z. (2010). An introduction to higher ramification groups. Retrieved from http://math.uchicago.edu/~chonoles/expository-notes/ramificationgroups.pdf

Clark, P. (2009). Lecture notes on local fields. Retrieved from http://math.uga.edu/~pete/local.pdf

Dummit, D. & Foote, R. (2004). *Abstract algebra*. John Wiley & Sons, Inc.

Neukirch, J. (1999). *Algebraic number theory*. Springer-Verlag Berlin Heidelberg.

Serre, J.-P. (1995). *Local fields*. Springer-Verlag New York.

Stitchenoth, H. (1993). *Algebraic function fields and codes*. Springer-Verlag Berlin Heidelberg.