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Higher Ramification Groups

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1 Abstract

2 Definitions and Notation

2.1 Discrete Valuations

We borrow our definitions in this section from Dummit and Foote, 2004.

Definition 1 (Discrete Valuations)**.**
A discrete valuation on a field K is a surjective function $\nu_K : K^* \to \mathbb{Z}$ such that

I. $\nu_K(xy) = \nu(x) + \nu(y)$ *for all* $x, y \in K^X$.

II. $\nu_K(x+y) \geq min\{\nu_K(x), \nu_K(y)\}$ *for all* $x, y \in K^X$ *such that* $x + y \neq 0$.

Further, we call the subring $\{x \in K | \nu_K(x) \geq 0\} \cup \{0\}$ *the valuation ring of* K.

Definition 2 (Discrete Valuation Ring (D.V.R.))**.**
If R is an integral domain and R is the valuation ring of the field of fractions on R, then we call R a discrete valuation ring. The ring R also has a local ring with unique maximal ideal $M = \{r \in R | \nu_K(x) > 0\}$.

Note that in Math 566 we proved (exercise 26 section 7.1) that for any unit $u \in K^*$ that $\nu_K(u) = 0$. Thus it follows that $R \setminus M$ is the set of invertible elements of R.

Definition 3 (Uniformizing Parameter)**.**
If R is a D.V.R. of K via ν_K, then an element $t \in R$ such that $\nu_K(t) = 1$ us a uniformizing parameter for R. Theorem 7, section 16.2 in Dummit and Foote, 2004 shows that t is unique and generates the unique maximal ideal of R.

2.2 Field Completion

With a valuation v_K on a field K then for any real number $a \in (0,1)$ we can induce an absolute value on K by

$$||x|| = \begin{cases} a^{v_K(x)} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

which satisfies the usual conditions of a metric as outlined by Serre, 1995. A topology is then induced on K via the absolute value metric, and we can denote the completion of K with respect to the valuation v_K by K_v. Note also that the metrics induced by different choices of a are topologically equivalent, so the completion is dependent only on v_K. Further, v_K extends in the completion of K to a discrete valuation (which we will continue to call v_K) on K_v (Serre, 1995).

2.3 Higher Ramification Groups

Consider now a finite Galois extension $L|K$ with Galois group G, and associated discrete valuations v_K and v_L and valuation rings O_L, and O_K.

Definition 4 (Higher Ramification Groups)**.**
For every real number $i \geq -1$ we define the i^{th} ramification group of $L|K$ by

$$G_i = G_i(L|K) = \{\sigma \in G | v_L(\sigma(a) - a) \geq i + 1 \ \forall a \in O_L\}.$$

Proposition 1.
For any $a \in O_L$ and $\sigma \in G$, $v_L(\sigma(a) - a) \geq i + 1$ if and only if $\sigma(a) \cong a mod P_L^{k+1}$.

Proposition 2.
The first two ramification groups are $G_{-1} = G$ and $G_0 = I$.

3 The Upper Numbering and Ramification Jump

Definition 5 (Upper Numbering the Higher Ramification groups)**.**
Consider the function

$$t = \varphi(s) = \int_0^s \frac{dt}{[G_0 : G_t]}$$

Which has inverse map ψ. Then $G^t(L|K) = G_s(L|K)$ where $s = \psi(t)$.

Definition 6 (Ramification Jump)**.**
If for any $\epsilon > 0$ $G^t(L|K) \neq G^{t+\epsilon}(L|K)$ then we call t a ramification jump.

Theorem 1 (Hasse-Arf)**.**
For a finite abelian extension $L|K$, the jumps of the filtration $\{G^i(L|K)\}_{i \geq -1}$ are rational integers.

References

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