1
The Laplace Transform

1.1 Introduction

⇒ Axioms are proposed and theorems are proved by invoking these axioms logically.

1.2 The Laplace Transform

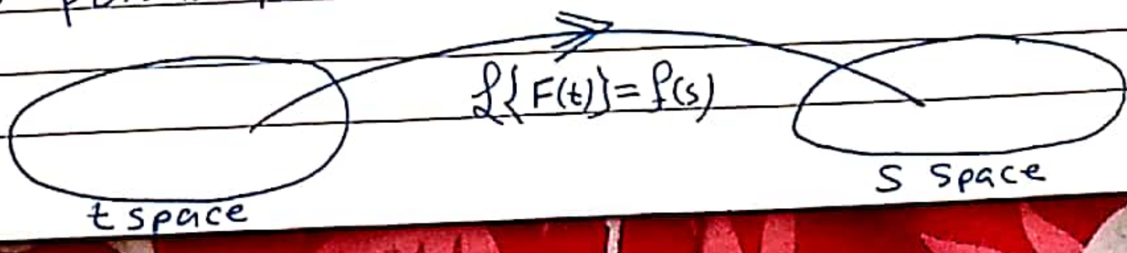
Given a Suitable function $F(t)$ the Laplace Transform, written $f(s)$ is defined by:-

$$f(s) = \int_0^{\infty} F(t) e^{-st} \, dt$$

Improper integral ⇒ Integral that has either or both limits ∞ or integrand that approaches infinity at one or more points in the range of integration

⇒ The notation $\mathcal{L}\{F(t)\}$ is used to denote the Laplace Transform of the function $F(t)$.

⇒ Another way of looking at the Laplace Transform is as a mapping from points in the t domain to points in the s domain.



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⇒ Another aspect of Laplace Transforms that needs mentioning at this stage is that the variable s often has to take Complex values.

Definition 1.1: If an interval $[0, t_0]$ say can be partitioned into a finite number of subintervals $[0, t_1], [t_1, t_2], [t_2, t_3] \cdots [t_n, t_0]$ with $0, t_1, t_2 \cdots t_n$ an increasing sequence of times and such that a given function $f(t)$ is continuous in each of these subintervals but not necessarily at the end points themselves, then $f(t)$ is piecewise continuous in the interval $[0, t_0]$.

⇒ Is it possible to have two different functions to have the same Laplace Transform?

⇓

⇒ As long as we restrict ourselves to piecewise continuous functions this ceases to be a problem.

⇒ If $F_1(t) = F(t)$ except at finite number of points where they differ by finite value then $\mathcal{L}\{F_1(t)\} = \mathcal{L}\{F(t)\}$.

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★ Conditions for the existence of the Laplace Transform

★ Riemann Integral

Let $F(x)$ be a function which is defined and is bounded in the interval $a < x < b$ and suppose the m and M are respectively the lower and upper bound of $F(x)$ in this interval. Take a set of points

$$a = x_0, x_1, x_2 \cdots x_{r-1}, x_r, \cdots x_n = b$$

and write $\delta_r = x_r - x_{r-1}$. Let M_r, m_r be the bounds of $F(x)$ in the subinterval (x_{r-1}, x_r) and form the sum

$$S = \sum_{r=1}^{n} M_r \delta_r \quad \text{——— (1)}$$

$$s = \sum_{r=1}^{n} m_r \delta_r \quad \text{——— (2)}$$

⇒ These are called respectively the upper and lower riemann sums corresponding to the mode of subdivision.

$\mathcal{L}\{F(t)\}$ -

⇒ It is certainly clear that $S > s$.

⇒ Let M be the minimum of all possible M_r and m be the maximum of all possible m_r.

⇒ lower bound of the set S is therefore $M(b-a)$ and upper bound for the set s is $m(b-a)$.

⇒ These bounds are of course rough. There are exact bounds for S and s, call them J and I respectively.

⇒ If $I = J$, $F(x)$ is said to be Riemann integrable in (a,b) and the value of the integral is I or J and is denoted by

$$I = J = \int_a^b F(x)\, dx$$

⇒ {Functions that have a finite number of finite discontinuities are included}.

⇒ Excluded functions are those that have singularities such as $\ln(x)$ or $1/x$ or function with infinite discontinuities or function with finite discontinuity at infinite points.

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1.3) Elementary Properties

Theorem 1.2 (Linearity): If $F_1(t)$ and $F_2(t)$ are two functions whose Laplace Transform exist, then

$$\mathcal{L}\{aF_1(t) + bF_2(t)\} = a\mathcal{L}\{F_1(t)\} + b\mathcal{L}\{F_2(t)\}$$

where a and b are arbitrary constant.

Proof: $\mathcal{L}\{aF_1(t) + bF_2(t)\}$

$$\Rightarrow \int_0^\infty (aF_1(t) + bF_2(t)) e^{-st} dt$$

$$\Rightarrow \int_0^\infty aF_1(t) e^{-st} dt + \int_0^\infty bF_2(t) e^{-st} dt$$

$$\Rightarrow a\int_0^\infty F_1(t) e^{-st} dt + b\int_0^\infty F_2(t) e^{-st} dt$$

$$\Rightarrow a\mathcal{L}\{F_1(t)\} + b\mathcal{L}\{F_2(t)\}$$

Theorem 1.3 (First Shift Theorem): If it is possible to choose constants M and α such that $|F(t)| \leq Me^{\alpha t}$, that is F(t) is of exponential order, then

$$\mathcal{L}\{e^{-bt}F(t)\} = f(s+b)$$

Provided $b < \alpha$. (In practice if F(t) is of exponential order then the constant α can be chosen such that this inequality hold).

Proof: $$\mathcal{L}\{e^{-bt}F(t)\} = \lim_{T\to\infty}\int_0^T e^{-st}e^{-bt}F(t)\,dt$$

$$= \int_0^\infty e^{-(s+b)t}F(t)\,dt$$

$$= f(s+b)$$

$$\left\{ \mathcal{L}\{F(t)\} = f(s) \right\}$$

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Example 1.4 : Find the laplace transform of the function $F(t) = t$.

$$\mathcal{L}\{t\} = \int_0^\infty t e^{-st} dt$$

$$= t \int_0^\infty e^{-st} dt - \int_0^\infty \left(\int_0^\infty e^{-st} dt \right) dt$$

$$= \left[\frac{t}{-s}\right]_0^\infty \left[e^{-st}\right]_0^\infty - \frac{1}{s^2}\left[e^{-st}\right]_0^\infty$$

$$= \left[\frac{-t}{s}\right]_0^\infty (0-1) - \frac{1}{s^2}(0-1)$$

$$= \left[\frac{t}{s}\right]_0^\infty + \frac{1}{s^2} = \cancel{\frac{1+ts}{s^2}} \quad \frac{1}{s^2}$$

Corollary: $\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$

Proof: $\mathcal{L}(t^n) = \int_0^\infty t^n e^{-st} dt = \left[\frac{-t^n}{s} e^{-st}\right]_0^\infty + \int \frac{n t^{n-1}}{s} e^{-st} dt$

$$= \frac{n}{s} \mathcal{L}(t^{n-1})$$

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$$\Rightarrow \mathcal{L}(t^n) = \frac{n}{s}\{\mathcal{L}(t^{n-1})\}$$

$$= \frac{n(n-1)}{s^2}\{\mathcal{L}(t^{n-2})\}$$

$$\boxed{\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}} \quad \{\text{by Induction}\}$$

* Example 1.5: Find the laplace transformation of $\mathcal{L}(te^{at})$ and deduce the value of $\mathcal{L}(t^n e^{at})$, where a is a real constant and n is a positive integer.

$$\Rightarrow \mathcal{L}\{t^n e^{at}\} = f(s-a)$$

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

$$\Rightarrow \mathcal{L}\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}}$$

★ Laplace Transform of Trignometric functions

⇒ We shall calculate $\mathcal{L}\{\sin(t)\}$ and $\mathcal{L}\{\cos(t)\}$.

⇒ Laplace transform of other common trignometric functions (tan, cot, csc & sec) do not exist as they have singularity for finite time t.

In order to find the Laplace Transform of $\sin(t)$ and $\cos(t)$ it is best to determine $\mathcal{L}(e^{it})$

$$\mathcal{L}(e^{it}) = \int_0^\infty e^{-st} e^{it} dt$$

$$= \int_0^\infty e^{(i-s)t} dt$$

$$= \left[\frac{e^{(i-s)t}}{i-s}\right]_0^\infty = \frac{1}{s-i}$$

$$\boxed{\mathcal{L}(e^{it}) = \frac{s}{s^2+1} + i\frac{1}{s^2+1}}$$

Now, $\mathcal{L}(e^{it}) = \mathcal{L}\{\cos(t)\} + i\mathcal{L}(\sin(t))$

$$\Rightarrow \boxed{\mathcal{L}\{\cos t\} = \frac{s}{s^2+1}}$$

$$\Rightarrow \boxed{\mathcal{L}\{\sin t\} = \frac{1}{s^2+1}}$$

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* Example 1.6: Find the Laplace transform of the function represented by $F(t)$ where,

$$F(t) = \begin{cases} t & 0 \leq t \leq t_0 \\ 2t_0 - t & t_0 \leq t \leq 2t_0 \\ 0 & t > 2t_0 \end{cases}$$

$$\Rightarrow \mathcal{L}\{F(t)\} = \int_0^{\infty} e^{-st} F(t)\, dt$$

$$\Rightarrow \int_0^{t_0} t e^{-st} dt + \int_{t_0}^{2t_0} (2t_0 - t) e^{-st} dt$$

$$\Rightarrow \left[\frac{-t}{s} e^{-st}\right]_0^{t_0} + \int_0^{t_0} + \frac{e^{-st}}{s} dt$$

$$+ \left[\frac{-(2t_0 - t)}{s} e^{-st}\right]_{t_0}^{2t_0} - \int_{t_0}^{2t_0} + \frac{e^{-st}}{s} dt$$

$$\Rightarrow \left(-\frac{t_0}{s} e^{-st_0} - 0\right) + \left[\frac{-e^{-st}}{s^2}\right]_0^{t_0}$$

$$+ \left\{0 - \left(-\frac{t_0}{s} e^{-st_0}\right)\right\} - \left[\frac{-e^{-st}}{s^2}\right]_{t_0}^{2t}$$

$$\Rightarrow \frac{1}{s^2}\left[1 - e^{-st_0}\right]^2$$

* **Example 1.7:** Determine the Laplace Transform of the Step function $F(t)$ defined by

$$F(t) = \begin{cases} 0 & 0 \leq t < t_0 \\ a & t \geq t_0 \end{cases}$$

$$\Rightarrow \mathcal{L}\{F(t)\} = \int_0^{\infty} F(t) e^{-st} dt$$

$$= \int_0^{t_0} 0 e^{-st} + \int_{t_0}^{\infty} a e^{-st}$$

$$= \left[\frac{-a}{s} e^{-st}\right]_{t_0}^{\infty} = \frac{a e^{-st_0}}{s}$$

Theorem 1.8: If $\mathcal{L}\{F(t)\} = f(s)$ then $\mathcal{L}\{tF(t)\}$

$= -\frac{d}{ds} f(s)$ and in general $\boxed{\mathcal{L}\{t^n F(t)\} = (-1)^n \frac{d^n}{ds^n} f(s)}$

Proof: Let $\mathcal{L}\{F(t)\} = \int_0^{\infty} e^{-st} F(t)\, dt = f(s)$

$$\frac{d f(s)}{ds} = \frac{d}{ds} \int_0^{\infty} e^{-st} F(t)\, dt = \int_0^{\infty} -t e^{-st} F(t)\, dt$$

$$= -\int_0^{\infty} e^{-st} t F(t)\, dt = -\mathcal{L}\{tF(t)\}$$

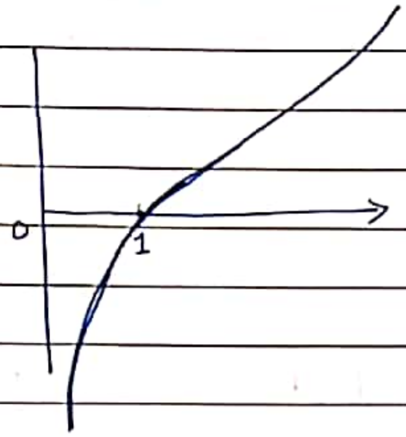
* Example 1.9: Determine the Laplace Transformation of the function $t \sin(t)$.

$$\Rightarrow \mathcal{L}\{t \sin(t)\} = -\frac{d}{ds}\mathcal{L}\{\sin t\} = -\frac{d}{ds}\left(\frac{1}{s^2+1}\right)$$

$$= +\frac{(2s)\times 1}{(s^2+1)^2} = +\frac{2s}{(s^2+1)^2}$$

Exercises

1. ⓐ



$\Rightarrow$ I think it is not possible as between 0 & 1 $|F(x)| \leq Me^{\alpha x}$ is violated.

$$\int_0^\infty \ln x \, e^{-sx} \, dx$$

ⓑ e^{3t}

$$\int_0^\infty e^{3t} e^{-st} \, dt = \int_0^\infty e^{(3-s)t} \, dt = \left[\frac{1}{3-s} e^{(3-s)t} \right]_0^\infty$$

$$= \frac{-1}{3-s} = \frac{1}{s-3}$$

ⓓ $e^{\frac{1}{t}}$

$$\int_0^\infty e^{\frac{1}{t}} e^{-st} \, dt = \int_0^\infty e^{\left(\frac{1}{t} - st\right)} \, dt$$

$\cancel{y = \frac{1}{t} - st \quad dy = \left(\frac{-1}{t^2} - s\right) dt}$

$y = \frac{1}{t}$

$dy = \frac{-1}{t^2} dt = -y^2 dt$

$$\Rightarrow \int_0^\infty e^{y-st} \frac{-dy}{y^2} = \int_0^\infty \frac{-1}{y^2} e^{y-st} dy$$

e) $\int_0^\infty \frac{e^{-st}}{t} dt$

2> ⓐ $\mathcal{L}\{e^{Kt}\} = \mathcal{L}\{1 \times e^{Kt}\} = f(s-K)$

$$f(s) = \int_0^\infty e^{-st} dt = \left[\frac{-1}{s} e^{-st}\right]_0^\infty = 0 - \frac{-1}{s} = \frac{1}{s}$$

$$\Rightarrow \mathcal{L}\{e^{Kt}\} = \frac{1}{s-K}$$

ⓑ $\mathcal{L}\{t^2\} = \frac{2!}{s^{2+1}} = \frac{2}{s^3}$

3> ⓐ $\mathcal{L}\{t^2 e^{-3t}\} = f(s+3)$

$$f(s) = \mathcal{L}\{t^2\} = \frac{2!}{s^{2+1}} = \frac{2}{s^3}$$

$$\Rightarrow \mathcal{L}\{t^2 e^{-3t}\} = \frac{2}{(s+3)^3}$$

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(b) $\mathcal{L}\{4t + 6e^{4t}\}$

$\Rightarrow \mathcal{L}\{4t\} + \mathcal{L}\{6e^{4t}\}$

$\Rightarrow 4\mathcal{L}\{t\} + 6\mathcal{L}\{e^{4t}\}$

$\Rightarrow \frac{4}{s^2} \cancel{+ 6 \times f(s-4)}$ $\left\{ f(s) = \frac{1}{s} \right\}$

$\Rightarrow \frac{4}{s^2} + \frac{6}{s-4}$ //

(c) $\mathcal{L}\{e^{-4t} \sin 5t\}$

$\Rightarrow f(s+4) = \frac{5}{(s+4)^2+25} = \frac{5}{s^2+16+8s+25}$

$f(s) = \mathcal{L}\{\sin 5t\} = \cancel{1}\ \frac{5}{s^2+25}$

$\cancel{f(s) = \int_0^\infty \sin(5t) e^{-st} dt = \frac{1}{y}\int_0^\infty \sin y\, e^{\frac{-sy}{s}} dy}$

$\cancel{y = st \quad dy = s\,dt}$

$\cancel{\int_0^\infty F(kt) e^{-st} dt}$ $\left\{ \mathcal{L}\{\sin(kt)\} = \frac{k}{s^2+k^2} \right\}$ //

$\Rightarrow \frac{5}{s^2+8s+41}$ //

4) $\int_0^\infty F(t)e^{-st}dt$

$\Rightarrow \int_0^1 te^{-st}dt + \int_1^2 (2-t)e^{-st}dt$

6) (a) $\sin(\omega t+\phi) = \sin(y+\phi)$

$\mathcal{L}(F(t)e^{kt}) = f(s+k)$

$\mathcal{L}\{\sin(\omega t+\phi)\} = \int_0^\infty \sin(\omega t+\phi)e^{-kt}dt$

$\mathcal{L}\{F(t+\phi)\} = \int_0^\infty F(t+\phi)e^{-st}dt$

$= \int_\phi^\infty \frac{1}{e^{-s\phi}} F(t+\phi)\, e^{-s(t+\phi)}\, d(t+\phi)$

$= \frac{1}{e^{-s\phi}} \int_\phi^\infty F(t)e^{-st}dt$

$= \frac{1}{e^{-s\phi}}\left[\mathcal{L}\{F(t)\} - \int_0^\phi F(t)e^{-st}dt\right]$