Least mean squares (LMS) Estimation

★ **LMS estimation in the absence of observations**

⇒ Unknown Θ ; prior $P_\Theta(\theta)$ or $f_\Theta(\theta)$

⇒ Criterion:

⇒ Mean Square Error (MSE): $E[(\Theta-\hat{\theta})^2]$

⇒ Minimize mean square error

$$E[\Theta^2] - 2E[\Theta]\hat{\theta} + \hat{\theta}^2$$

$$\frac{d}{d\hat{\theta}}(\cdot) = -2E[\Theta] + 2\hat{\theta} = 0$$

$$\hat{\theta} = E[\Theta]$$

{Estimator which minimize MSE and LMS estimator}

⇒ $MSE = E[(\Theta - E[\Theta])^2] = Var(\Theta)$

★ **LMS estimation of Θ based on X**

⇒ Unknown Θ ; prior $P_\Theta(\theta)$ or $f_\Theta(\theta)$

⇒ Interested in a point estimate $\hat{\theta}$.

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⇒ Observation X; model $p_{X|\Theta}(x|\theta)$ or $f_{X|\Theta}(x|\theta)$

⇒ We want to minimize conditional mean squared error:

$$E\left[(\Theta-\theta)^2 \mid X=x\right]$$

$$\Rightarrow \hat{\theta} = E[\Theta \mid X=x]$$

$$\Rightarrow E\left[(\Theta - E[\Theta \mid X=x])^2 \mid X=x\right]$$

$$\leqslant E\left[(\Theta - g(x))^2 \mid X=x\right] \;\forall x$$

$$\Rightarrow E\left[(\Theta - E[\Theta \mid X])^2 \mid X\right] \leqslant E\left[(\Theta - g(X))^2 \mid X\right]$$

$$\Rightarrow E\left[(\Theta - E[\Theta \mid X])^2\right] \leqslant E\left[(\Theta - g(X))^2\right]$$

{using Law of iterated expectation}

⇒ $\hat{\Theta}_{LMS} = E[\Theta \mid X]$ minimizes $E[(\Theta - g(X))^2]$, over all estimators $\hat{\Theta} = g(X)$

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★ LMS performance evaluation

⇒ LMS estimate: $\hat{\theta} = E[\Theta \mid X = x]$

Estimator: $\hat{\Theta} = E[\Theta \mid X]$

⇒ Expected performance, once we have a measurement:

$$MSE = E\left[\left(\Theta - E[\Theta \mid X = x]\right)^2 \mid X = x\right] = Var(\Theta \mid X = x)$$

⇒ Expected performance of the design:

$$MSE = E\left[\left(\Theta - E[\Theta \mid X]\right)^2\right] = E\left[Var(\Theta \mid X)\right]$$

⇒ LMS relevant to estimation (not hypothesis testing)

⇒ Same as MAP if the posterior is unimodal and symmetric around the mean.

★ LMS estimation with multiple observations or unknowns

⇒ Unknown Θ ; prior $p_{\Theta}(\theta)$

→ Interested in a point estimate $\hat{\theta}$

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$\Rightarrow$ Observation $X = (X_1, \cdots X_n)$

$\Rightarrow$ Model $P_{X|\Theta}(x|\theta)$

$\Rightarrow$ LMS estimate: $E[\Theta | X = x]$

$\Rightarrow$ If Θ is a vector, apply to each component separately

$$\Theta = (\theta_1, \cdots \theta_m)$$

$$\hat{\Theta}_j = E[\Theta_j | X = x]$$

★ **Properties of the estimation error in LMS estimation**

$\Rightarrow$ Estimator: $\hat{\Theta} = E[\Theta | X]$

$\Rightarrow$ Error: $\tilde{\Theta} = \hat{\Theta} - \Theta$

$\Rightarrow$ $E[\tilde{\Theta} | X = x] = 0$ —— (1)

$\Rightarrow$ $\mathrm{cov}(\tilde{\Theta}, \hat{\Theta}) = 0$ —— (2)

$\Rightarrow$ $\mathrm{Var}(\Theta) = \mathrm{Var}(\hat{\Theta}) + \mathrm{Var}(\tilde{\Theta})$ —— (3)