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Linear Least Mean Square Estimation
(LLMS)

★ LLMS formulation

Unknown: Θ

Observation: X

⇒ Minimize $E[(\hat{\Theta}-\Theta)^2]$ {objective}

⇒ Estimator $\hat{\Theta} = g(X) \rightarrow \hat{\Theta}_{LMS} = E[\Theta|X]$

⇒ Consider estimator of Θ of the form

$$\hat{\Theta} = aX + b$$

⇒ Minimize $E[(\Theta - aX - b)^2]$ wrt a, b

★ Solution to the LLMS problem

⇒ Minimize $E[(\Theta - aX - b)^2]$ wrt a, b {objective}

⇒ Suppose a is already known:

↳ So value of b that minimizes above quantity is

$$b = E[\Theta - aX] = E[\Theta] - aE[X]$$

$$\boxed{b = E[\Theta] - aE[X]}$$

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$$\Rightarrow E\left[\left(\theta - aX - E(\theta - aX)\right)^2\right]$$

$$= Var(\theta - aX)$$

$$= Var(\theta) + a^2 Var(X) - 2a\,Cov(\theta, X)$$

$$\Rightarrow \frac{d}{da}(\cdot) = 0$$

$$\Rightarrow 2a\,Var(X) - 2\,Cov(\theta, X) = 0$$

$$a = \frac{Cov(\theta, X)}{Var(X)}$$

$$\Rightarrow \boxed{\hat{\theta}_L = E[\theta] + \left(\frac{Cov(\theta, X)}{Var(X)}\right)(X - E[X])}$$

$$\rho \frac{\sigma_\theta}{\sigma_X}$$

Correlation

$$\Rightarrow \boxed{E[(\hat{\theta} - \theta)^2] = (1 - \rho^2)\,Var(\theta)}$$

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⋆ LLMS with multiple observations

⇒ Unknown Θ; observations $X = (X_1, \cdots X_n)$

⇒ Consider an estimators of the form:

$$\hat{\Theta} = a_1 X_1 + \cdots + a_n X_n + b$$

⇒ Find best choice of $a_1, \cdots a_n, b$

minimize $E\left[(a_1 X_1 + \cdots + a_n X_n + b - \Theta)^2\right]$

⇒ If $E[\Theta | X]$ is linear in X then

$$\hat{\Theta}_{LMS} = \hat{\Theta}_{LLMS}$$

⇒ Only means, Variance, covariances matter.

⇒ If multiple unknown Θ_i, apply to each one separately.