Positive Definit Matrix

$\boxed{A = A^T}$

⇒ Symmetric matrix with all the eigen values are positive

→ All the pivots are positive.

→ All Sub-determinants are positive.

① Tests for Positive definite Matrix

- → All Eigen Values are positive
- → All the pivots are positive
- → All Sub-determinants are positive
- → $x^T A x > 0$ {except at zero Vector}

  x → Any Vector

Positive Semidefinit

- → Matrix is Singular
- → All Eigen Values are ≥ 0
- → " Pivots are ≥ 0
- → " Sub-determinant ≥ 0

$x^T A x$ for 2D Case

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} ax_1 + bx_2 \\ bx_1 + cx_2 \end{bmatrix}$$

$$\Rightarrow ax_1^2 + 2bx_1x_2 + cx_2^2$$ (Quadratic form)

Graph of $f(x,y) = \bar{x}^T A \bar{x}$

$$= ax^2 + 2bxy + cy^2$$

Example Let $A = \begin{bmatrix} 2 & 6 \\ 6 & 7 \end{bmatrix}$

So $f(x,y) = 2x^2 + 12xy + 7y^2$ → Not Positive definite

→ It has a saddle point.

→ { Point on the surface with a relitive minima along one axial direction and relative maxima along the crossing axis. }

Condition for minima

→ first derivative Zero $\dfrac{du}{dx} = 0$

→ second derivative Positive. $\dfrac{d^2u}{dx^2} > 0$

$f(x_1, x_2 \cdots x_n)$ → first derivative is zero for minima

→ Matrix of 2^{nd} derivative is Pos definit for minima

$$\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} \rightarrow \text{Second derivative Matrix}$$

$$\boxed{f_{xx} f_{yy} > 2 f_{xy}^2}$$

A^TA is Positive definite

Let A be m by n matrix. $\{m > n\}$

A^TA
- → Square
- → Symmetric

$x^T A^T A x \Rightarrow (Ax)^T (Ax) \Rightarrow$ length Square

$\rightarrow \|Ax\|^2 \geq 0$

$\Rightarrow \|Ax\|^2 > 0$ is Rank of $A = n$
