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Derived Distribution

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★ Derived Distribution (The discrete case)

$Y = g(X)$ $\left\{ X \text{ and } Y \text{ are two discrete random variable} \right\}$

$$P_Y(y) = P(g(x) = y)$$

$$= \sum_{x: g(x) = y} P_X(x)$$

Linear function

If $Y = aX + b$

$$\boxed{P_Y(y) = P_X\left(\frac{y-b}{a}\right)}$$

Linear function of a continuous r.V

$Y = aX + b$ $\forall\, a > 0$

$$F_Y(y) = P(Y \leq y) = P(aX + b \leq y)$$

$$= P\left(X \leq \frac{y-b}{a}\right) = F_X\left(\frac{y-b}{a}\right)$$

$$\boxed{f_Y(y) = f_X\left(\frac{y-b}{a}\right) \cdot \frac{1}{|a|}}$$

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★ Linear function of a normal r.v. is normal

$X \sim N(\mu, \sigma^2)$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}$$

Let $Y = aX + b$ $a \neq 0$

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

$$= \frac{1}{|a|} \frac{1}{\sqrt{2\pi}\sigma} e^{-\left(\frac{y-b}{a}-\mu\right)^2/2\sigma^2}$$

$$= \frac{1}{\sqrt{2\pi}\sigma|a|} e^{-(y-b-a\mu)^2/2\sigma^2a^2}$$

$Y \sim N(a\mu+b, a^2\sigma^2)$

★ A general function $g(X)$ of a continuous r.v.

⇒ Two step procedure:

① Find the CDF of Y: $F_Y(y) = P(Y \leq y)$

② Differentiate: $f_Y(y) = \frac{dF_Y(y)}{dy}$

⇒ A general formula for the PDF of $Y = g(x)$ when g is monotonic.

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dg^{-1}}{dy}(y) \right|$$

★ <u>A function of multiple r.v's $Z = g(X, Y)$</u>

① Find the CDF of Z: $F_Z(z) = P(Z \leq z)$

$$= P(g(X,Y) \leq z)$$

② Differentiate : $f_Z(z) = \dfrac{d F_Z(z)}{dz}$

★ <u>Simulation: Generate sample of X with given CDF $F_X(\cdot)$</u>

⇒ Modern computer generally have a random number generator which generates samples from a Uniform distribution.

$$\boxed{\text{RNG}} \xrightarrow[\text{Uniform } [0,1]]{u} \boxed{g} \longrightarrow g(u)$$

⇒ Find g so that $g(u) \sim F_X$

$$\boxed{g = F_X^{-1}}$$