Discrete Random Variable

★ **Variance** (A measure of the spread of PMF)

⇒ Random variable X, with $\mu = E[X]$

⇒ Definition of Variance : $Var(X) = E[(X-\mu)^2]$

$$Var(X) = E[g(X)] \text{ where } g(X) = (X-\mu)^2$$

$$= \sum_{x} g(x) P_X(x)$$

$$= \sum_{x} (x-\mu)^2 P_X(x)$$

⇒ Standard deviation ⇒ $\sigma_X = \sqrt{Var(X)}$

★ **Properties of Variance**

(1) $Var(aX+b) = a^2 Var(X)$

Let $Y = aX + b$

$$E[Y] = E[aX+b] = aE[X] + b$$
$$= a\mu + b$$

$$Var(Y) = E\left[(Y - E(Y))^2\right]$$

$$= E\left[(aX + \cancel{b} - (a\mu + \cancel{b}))^2\right]$$

$$= E\left[a^2(X-\mu)^2\right]$$

$$= a^2 E\left[(X-\mu)^2\right]$$

$$= a^2 Var(X)$$

(2) $Var(X) = E[X^2] - (E[X])^2$

$$Var(X) = E\left[(x-\mu)^2\right]$$

$$\Rightarrow E\left[x^2 + \mu^2 - 2x\mu\right]$$

$$\Rightarrow E[x^2] + \mu^2 - 2\mu E[X]$$

$$\Rightarrow E[X^2] + (E[X])^2 - 2(E[X])^2$$

$$\Rightarrow E[X^2] - (E[X])^2$$

★ Variance of the Bernoulli

$E(X) = p$

- $Var(X) = \sum_{x} (x - E[X])^2 P_X(x)$

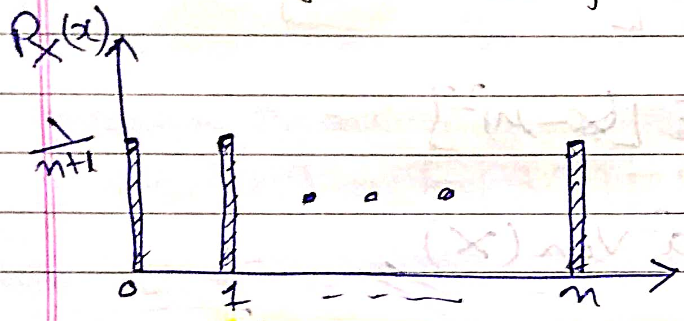
$$= (1-p)^2 p + (0-p)^2 (1-p) = p(1-p)$$

• $Var(X) = E[X^2] - (E[X])^2$

$= p - p^2$

$= p(1-p)$

★ Variance of the uniform distribution



$Var(X) = E[X^2] - (E[X])^2$

$\Rightarrow \frac{1}{n+1}(0^2 + 1^2 + \cdots n^2) - \left(\frac{n}{2}\right)^2$

$\Rightarrow \frac{1}{12} n(n+2)$

⇒ For general case where it starts from a and ends at b.

$n = b - a$

$Var(X) = \frac{1}{2}(b-a)(b-a+2)$

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★ Conditional PMF and expectation, given an event

$$P_{X|A}(x) = P(X = x|A) \quad \{\text{Conditional PMF}\}$$

{PMF given A has occured}

- $$\sum_{x} P_{X|A}(x) = 1$$

- $$E[X|A] = \sum_{x} x\, P_{X|A}(x) \quad \{\text{Conditional Expectation}\}$$

- $$E[g(X)|A] = \sum_{x} g(x) P_{X|A}(x)$$

★ Total Expectation theorem

$\Rightarrow$ $P(B) = P(A_1)P(B|A_1) + \cdots + P(A_n)P(B|A_n)$

{Total probability theorem}

where, B is some event of interest & $A_1, A_2 \cdots A_n$ is the partition of SS

$\Rightarrow$ Let X is a random variable in SS and $B = \{X = x\}$

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$\Rightarrow P_X(x) = P(A_1) P_{X|A_1}(x) + \cdots + P(A_n) P_{X|A_n}(x)$

$$\Rightarrow \sum_x x P_X(x) = P(A_1) \sum_x x P_{X|A_1}(x) + \cdots + P(A_n) \sum_x x P_{X|A_n}(x)$$

$$\Rightarrow \boxed{E[X] = P(A_1) E[X|A_1] + \cdots + P(A_n) E[X|A_n]}$$

{Total Expectation Theorem}

★ <u>Mean of geometric random variable</u>

$$P_X(K) = (1-p)^{K-1} p$$

$\Rightarrow E[X-1] = E[X] - 1$ {Linearity of $E[X]$}

$$\Rightarrow E[X] = 1 + E[X-1]$$

$$p \cdot E[X-1 \mid x=1] = p * 0$$

$+$

$$(1-p) E[X-1 \mid X > 1] = (1-p) E[X]$$

$$\Rightarrow E[X] = 1 + (1-p)E[X]$$

$$\Rightarrow E[X] = 1/p$$

★ Multiple random variable & join PMFs

⇒ Let X and Y be two random variable with PMF P_X and P_Y respectively.

⇒ Joint PMF:

$$\boxed{P_{X,Y}(x,y) = P(X=x \;\&\; Y=y)}$$

$$\sum_x \sum_y P_{X,Y}(x,y) = 1$$

⇒ Individual PMF are called Marginal PMF:

$$P_X(x) = \sum_y P_{X,Y}(x,y)$$

$$P_Y(y) = \sum_x P_{X,Y}(x,y)$$

★ Functions of multiple random variable

$$Z = g(X,Y)$$

So $P_Z(z) = P(g(x,y) = z) = \sum_{(x,y):\, g(x,y)=z} P_{X,Y}(x,y)$

$$\boxed{E[g(X,Y)] = \sum_{x}\sum_{y} g(x,y)\, P_{X,Y}(x,y)}$$

{Expected value rule}

★ <u>Linearity of Expectation</u>

$$\boxed{E[X+Y] = E[X] + E[Y]}$$

Let $g(x,y) = x+y$

$$E[X,Y] = \sum_{x}\sum_{y} (x+y)\, P_{X,Y}(x,y)$$

$$= \sum_{x}\left(\sum_{y} x\, P_{X,Y}(x,y)\right) + \sum_{x}\sum_{y} y\, P_{XY}(x,y)$$

$$\Rightarrow \sum_{x} x \sum_{y} P_{XY}(x,y) + \sum_{y} y \sum_{x} P_{X,Y}(x,y)$$

$$\Rightarrow \sum_{x} x\, P_X(x) + \sum_{y} y\, P_Y(y)$$

$$\Rightarrow E[X] + E[Y]$$

★ The mean of the binomial random variable

X = Number of successes in n independent trials.

Probability of success = P

$$E[X] = \sum_{K=0}^{n} K \binom{n}{K} P^K (1-P)^{n-K}$$

{Quite difficult to solve}

{Same}

$$\Rightarrow \text{Let } X_i = \begin{cases} 1 & \text{if } i^{th} \text{ trial is a success} \\ 0 & \text{otherwise} \end{cases}$$

$$X = X_1 + X_2 + \dots + X_n$$

$$E[X] = E[X_1] + E[X_2] + \dots + E[X_n]$$

$$= P + P + \dots + P$$

$$= nP$$

$$Var[X] = Var[X_1] + \dots V[X_n]$$

$$= nP(1-P)$$

{As $X_1, X_2 \dots X_n$ are independent random variable}