# Contents

# Laplace Transformation

Book: An Introduction to Laplace Transforms and Fourier Series.

P.P.G. Dyke

1. The Laplace Transform

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*3. Convolution and the Solution of Ordinary Differential equations.

CHAPTER 1

# The Laplace Transform

# 1
# The Laplace Transform

## 1.1 Introduction

⇒ Axions are proposed and theorems are proved by invoking these axioms logically.

## 1.2 The Laplace Transform

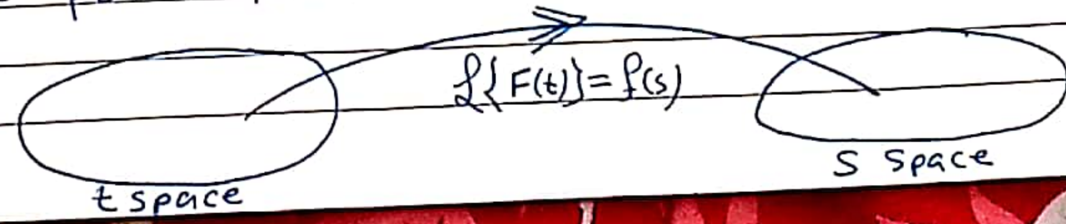
Given a Suitable function $F(t)$ the Laplace Transform, written $f(s)$ is defined by:-

$$f(s) = \int_0^\infty F(t)e^{-st}\,dt$$

{Improper integral ⇒ Integral that has either or both limits ∞ or integrand that approaches infinity at one or more points in the range of integration}

⇒ The notation $\mathcal{L}\{F(t)\}$ is used to denote the Laplace Transform of the function $F(t)$.

⇒ Another way of looking at the Laplace Transform is as a mapping from points in the t domain to points in the s domain.



⇒ Another aspect of Laplace Transforms that needs mentioning at this stage is that the variable s often has to take Complex values.

# Definition 1.1: If an interval $[0\ t_0]$ say can be partitioned into a finite number of subintervals $[0\ t_1], [t_1\ t_2]\ [t_2\ t_3] \cdots [t_n\ t_0]$ with $0, t_1, t_2 \cdots t_n$ an increasing sequence of times and such that a given function $f(t)$ is continuous in each of these subintervals but not necessarily at the end points themselves, then $f(t)$ is piecewise continuous in the interval $[0\ t_0]$.

⇒ Is it possible to have two different functions to have the same Laplace Transform?

⇓

⇒ As long as we restrick ourselves to piecewise continuous functions this ceases to be a problem

⇒ If $F_1(t) = F(t)$ except at finite number of points where they differ by finite value then $\mathcal{L}\{F_1(t)\} = \mathcal{L}\{F(t)\}$.

★ Conditions for the existence of the Laplace Transform

★ Riemann Integral

Let $F(x)$ be a function which is defined and is bounded in the interval $a < x < b$ and suppose the $m$ and $M$ are respectively the lower and upper bound of $F(x)$ in this interval. Take a set of points

$$a = x_0, x_1, x_2 \cdots x_{r-1}, x_r, \cdots x_n = b$$

and write $\delta_r = x_r - x_{r-1}$. Let $M_r, m_r$ be the bounds of $F(x)$ in the subinterval $(x_{r-1}, x_r)$ and form the sum

$$S = \sum_{r=1}^{n} M_r \delta_r \quad \text{———} \quad (1)$$

$$s = \sum_{r=1}^{n} m_r \delta_r \quad \text{———} \quad (2)$$

⇒ These are called respectively the upper and lower riemann sums corresponding to the mode of subdivision.

$\mathcal{L}\{F(t)\}$ -

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⇒ It is certainly clear that $S > s$.

⇒ Let $M$ be the minimum of all possible $M_r$ and $m$ be the maximum of all possible $m_r$.

⇒ Lower bound of the Set $S$ is therefore $M(b-a)$ and upper bound for the set $s$ is $m(b-a)$.

⇒ These bounds are of course rough. There are exact bounds for $S$ and $s$, call them $J$ and $I$ respectively.

⇒ If $I = J$, $F(x)$ is said to be Riemann integrable in $(a, b)$ and the value of the integral is $I$ or $J$ and is denoted by

$$I = J = \int_a^b F(x)\,dx$$

⇒ {Functions that have a finite number of finite discontinuities are included}

⇒ Excluded functions are those that have singularities such as $\ln(x)$ or $1/f(x)$ or function with infinite discontinuity or function with finite discontinuity at infinite points.

## 1.3 Elementary Properties

# Theorem 1.2 (Linearity): If $F_1(t)$ and $F_2(t)$ are two functions whose Laplace Transform exist, then

$$\mathcal{L}\{aF_1(t) + bF_2(t)\} = a\mathcal{L}\{F_1(t)\} + b\mathcal{L}\{F_2(t)\}$$

where $a$ and $b$ are arbitrary constant.

Proof: $\mathcal{L}\{aF_1(t) + bF_2(t)\}$

$$\Rightarrow \int_0^\infty (aF_1(t) + bF_2(t))e^{-st}\,dt$$

$$\Rightarrow \int_0^\infty aF_1(t)e^{-st}\,dt + \int_0^\infty bF_2(t)e^{-st}\,dt$$

$$\Rightarrow a\int_0^\infty F_1(t)e^{-st}\,dt + b\int_0^\infty F_2(t)e^{-st}\,dt$$

$$\Rightarrow a\mathcal{L}\{F_1(t)\} + b\mathcal{L}\{F_2(t)\}$$

# Theorem 1.3 (First Shift Theorem): If it is possible to choose constants M and $\alpha$ such that $|F(t)| \leq Me^{\alpha t}$, that is F(t) is of exponential order, then

$$\mathcal{L}\{e^{-bt}F(t)\} = f(s+b)$$

Provided $b < \alpha$. (In practice if F(t) is of exponential order then the constant $\alpha$ can be chosen such that this inequality hold).

Proof: $$\mathcal{L}\{e^{-bt}F(t)\} = \lim_{T\to\infty}\int_0^T e^{-st}e^{-bt}F(t)\,dt$$

$$= \int_0^\infty e^{-(s+b)t}F(t)\,dt$$

$$= f(s+b)$$

$$\left\{ \mathcal{L}\{F(t)\} = f(s) \right\}$$

**Example 1.4**: Find the laplace transform of the function $F(t) = t$.

$$\mathcal{L}\{t\} = \int_0^\infty t e^{-st}\,dt$$

$$= t\int_0^\infty e^{-st}\,dt - \int_0^\infty\left(\int_0^\infty e^{-st}\,dt\right)dt$$

$$= \left[\frac{t}{-s}\right]_0^\infty \left[e^{-st}\right]_0^\infty - \frac{1}{s^2}\left[e^{-st}\right]_0^\infty$$

$$= \left[\frac{-t}{s}\right]_0^\infty (0-1) - \frac{1}{s^2}(0-1)$$

$$= \left[\frac{t}{s}\right]_0^\infty + \frac{1}{s^2} = \cancel{\frac{1+ts}{s^2}}\ \frac{1}{s^2}$$

# Corollary: $\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$

**Proof:** $\mathcal{L}(t^n) = \int_0^\infty t^n e^{-st}\,dt = \left[\frac{-t^n}{s} e^{-st}\right]_0^\infty + \int \frac{n t^{n-1}}{s} e^{-st}\,dt$

$$= \frac{n}{s}\mathcal{L}(t^{n-1})$$

$$\Rightarrow \mathcal{L}(t^n) = \frac{n}{s}\{\mathcal{L}(t^{n-1})\}$$

$$= \frac{n(n-1)}{s^2}\{\mathcal{L}(t^{n-2})\}$$

$$\boxed{\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}} \quad \{\text{by Induction}\}$$

* Example 1.5: Find the laplace transformation of $\mathcal{L}(te^{at})$ and deduce the value of $\mathcal{L}(t^n e^{at})$, where a is a real constant and n is a positive integer.

$$\Rightarrow \mathcal{L}\{t^n e^{at}\} = f(s-a)$$

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

$$\Rightarrow \mathcal{L}\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}}$$

★ Laplace Transform of Trignometric functions

⇒ We shall calculate $\mathcal{L}\{\sin(t)\}$ and $\mathcal{L}\{\cos(t)\}$.

⇒ Laplace transform of other common trignometric functions (tan, cot, csc & sec) do not exist as they have singularity for finite time t.

In order to find the Laplace Transform of $\sin(t)$ and $\cos(t)$ it is best to detetermine $\mathcal{L}(e^{it})$

$$\mathcal{L}(e^{it}) = \int_0^\infty e^{-st} e^{it} dt$$

$$= \int_0^\infty e^{(i-s)t} dt$$

$$= \left[\frac{e^{(i-s)t}}{i-s}\right]_0^\infty = \frac{1}{s-i}$$

$$\boxed{\mathcal{L}(e^{it}) = \frac{s}{s^2+1} + i\frac{1}{s^2+1}}$$

Now, $\mathcal{L}(e^{it}) = \mathcal{L}\{\cos(t)\} + i\mathcal{L}(\sin(t))$

$$\Rightarrow \boxed{\mathcal{L}\{\cos t\} = \frac{s}{s^2+1}}$$

$$\Rightarrow \boxed{\mathcal{L}\{\sin t\} = \frac{1}{s^2+1}}$$

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* Example 1.6: Find the Laplace transform of the function represented by F(t) where,

$$F(t) = \begin{cases} t & 0 \leq t \leq t_0 \\ 2t_0 - t & t_0 \leq t \leq 2t_0 \\ 0 & t > 2t_0 \end{cases}$$

$$\Rightarrow \mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt$$

$$\Rightarrow \int_0^{t_0} t e^{-st} dt + \int_{t_0}^{2t_0} (2t_0 - t) e^{-st} dt$$

$$\Rightarrow \left[\frac{-t}{s} e^{-st}\right]_0^{t_0} + \int_0^{t_0} + \frac{e^{-st}}{s} dt$$

$$+ \left[\frac{-(2t_0 - t)}{s} e^{-st}\right]_{t_0}^{2t_0} - \int_{t_0}^{2t_0} + \frac{e^{-st}}{s} dt$$

$$\Rightarrow \left(-\frac{t_0}{s} e^{-st_0} - 0\right) + \left[\frac{-e^{-st}}{s^2}\right]_0^{t_0}$$

$$+ \left\{0 - \left(-\frac{t_0}{s} e^{-st_0}\right)\right\} - \left[\frac{-e^{-st}}{s^2}\right]_{t_0}^{2t}$$

$$\Rightarrow \frac{1}{s^2}\left[1 - e^{-st_0}\right]^2$$

* Example 1.7: Determine the Laplace Transform of the Step function $F(t)$ defined by

$$F(t) = \begin{cases} 0 & 0 \leq t < t_0 \\ a & t \geq t_0 \end{cases}$$

$$\Rightarrow \mathcal{L}\{F(t)\} = \int_0^{\infty} F(t) e^{-st} dt$$

$$= \int_0^{t_0} 0 e^{-st} + \int_{t_0}^{\infty} a e^{-st}$$

$$= \left[-\frac{a}{s} e^{-st}\right]_{t_0}^{\infty} = \frac{a e^{-st_0}}{s}$$

# Theorem 1.8: If $\mathcal{L}\{F(t)\} = f(s)$ then $\mathcal{L}\{t F(t)\}$ $= -\frac{d}{ds} f(s)$ and in general $\boxed{\mathcal{L}\{t^n F(t)\} = (-1)^n \frac{d^n}{ds^n} f(s)}$

Proof: Let $\mathcal{L}\{F(t)\} = \int_0^{\infty} e^{-st} F(t)\, dt = f(s)$

$$\frac{d f(s)}{ds} = \frac{d}{ds} \int_0^{\infty} e^{-st} F(t)\, dt = \int_0^{\infty} -t e^{-st} F(t)\, dt$$

$$= -\int_0^{\infty} e^{-st}\, t F(t)\, dt = -\mathcal{L}\{t F(t)\}$$

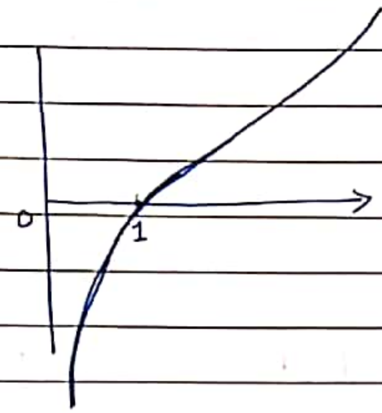
* Example 1.9: Determine the Laplace Transformation of the function $t\sin(t)$.

$$\Rightarrow \mathcal{L}\{t\sin(t)\} = -\frac{d}{ds}\mathcal{L}\{\sin t\} = -\frac{d}{ds}\left(\frac{1}{s^2+1}\right)$$

$$= +\frac{(2s)\times 1}{(s^2+1)^2} = +\frac{2s}{(s^2+1)^2}$$

# Exercises

1. (a)




$\Rightarrow$ I think it is not possible as between 0 & 1 $|F(x)| \leq Me^{\alpha x}$ is violated.

$$\int_0^\infty \ln x \, e^{-sx} \, dx$$

(b) $e^{3t}$

$$\int_0^\infty e^{3t} e^{-st} \, dt = \int_0^\infty e^{(3-s)t} \, dt = \left[ \frac{1}{3-s} e^{(3-s)t} \right]_0^\infty$$

$$= \frac{-1}{3-s} = \frac{1}{s-3}$$

(d) $e^{\frac{1}{t}}$

$$\int_0^\infty e^{\frac{1}{t}} e^{-st} \, dt = \int_0^\infty e^{\left(\frac{1}{t} - st\right)} dt$$

$\cancel{y = \frac{1}{t} - st \quad dy = \left(\frac{-1}{t^2} - s\right) dt}$

$y = \frac{1}{t}$

$dy = \frac{-1}{t^2} dt = -y^2 dt$

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$$\Rightarrow \int_0^\infty e^{y-st} \frac{-dy}{y^2} = \int_0^\infty \frac{-1}{y^2} e^{y-st} dy$$

e) $\int_0^\infty \frac{e^{-st}}{t} dt$

2) (a) $\mathcal{L}\{e^{Kt}\} = \mathcal{L}\{1 \times e^{Kt}\} = f(s-K)$

$$f(s) = \int_0^\infty e^{-st} dt = \left[\frac{-1}{s} e^{-st}\right]_0^\infty = 0 - \frac{-1}{s} = \frac{1}{s}$$

$$\Rightarrow \mathcal{L}\{e^{Kt}\} = \frac{1}{s-K}$$

(b) $\mathcal{L}\{t^2\} = \frac{2!}{s^{2+1}} = \frac{2}{s^3}$

3) (a) $\mathcal{L}\{t^2 e^{-3t}\} = f(s+3)$

$$f(s) = \mathcal{L}\{t^2\} = \frac{2!}{s^{2+1}} = \frac{2}{s^3}$$

$$\Rightarrow \mathcal{L}\{t^2 e^{-3t}\} = \frac{2}{(s+3)^3}$$

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(b) $\mathcal{L}\{4t + 6e^{4t}\}$

$\Rightarrow \mathcal{L}\{4t\} + \mathcal{L}\{6e^{4t}\}$

$\Rightarrow 4\mathcal{L}\{t\} + 6\mathcal{L}\{e^{4t}\}$

$\Rightarrow \frac{4}{s^2} + 6 \times f(s-4)$ $\left\{ f(s) = \frac{1}{s} \right\}$

$\Rightarrow \frac{4}{s^2} + \frac{6}{s-4}$

(c) $\mathcal{L}\{e^{-4t} \sin 5t\}$

$\Rightarrow f(s+4) = \frac{5}{(s+4)^2+25} = \frac{5}{s^2+16+8s+25}$

$f(s) = \mathcal{L}\{\sin 5t\} = \cancel{\phantom{xx}} \frac{5}{s^2+25}$

$\cancel{f(s) = \int_0^\infty \sin(5t)\, e^{-st}\, dt = \frac{1}{y}\int_0^\infty \sin y\, e^{\frac{-sy}{s}}\, dy}$

$\cancel{y = 5t \quad dy = 5dt}$

$\cancel{\int_0^\infty F(kt)\, e^{-st}\, dt}$ $\left\{ \mathcal{L}\{\sin(kt)\} = \frac{k}{s^2+k^2} \right\}$

$\Rightarrow \frac{5}{s^2+8s+41}$

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4) $\int_0^\infty F(t)e^{-st}dt$

$\Rightarrow \int_0^1 te^{-st}dt + \int_1^2 (2-t)e^{-st}dt$

b) (a) $\sin(\omega t+\varphi) = \sin(y+\varphi)$

$\mathcal{L}(F(t)e^{kt}) = f(s+k)$

$\mathcal{L}\{\sin(\omega t+\varphi)\} = \int_0^\infty \sin(\omega t+\varphi)e^{-kt}dt$

$\mathcal{L}\{F(t+\varphi)\} = \int_0^\infty F(t+\varphi)e^{-st}dt$

$= \int_\varphi^\infty \frac{1}{e^{-s\varphi}} F(t+\varphi)e^{-s(t+\varphi)}d(t+\varphi)$

$= \frac{1}{e^{-s\varphi}}\int_\varphi^\infty F(t)e^{-st}dt$

$= \frac{1}{e^{-s\varphi}}\left[\mathcal{L}\{F(t)\} - \int_0^\varphi F(t)e^{-st}dt\right]$

CHAPTER 2

# Furture Properties of the Laplace Transforms

# 2

# Further Properties of the Laplace Transform

## 2.1) Real Functions

# Definition 2.1: Heaviside's Unit step function, or simply the unit step function is defined as:

$$H(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

## 2.2) Derivative property of the Laplace Transform

# Theorem 2.2: Suppose a differentiable function $F(t)$ has a Laplace Transform $f(s)$, we can find the Laplace Transform.

$$\mathcal{L}\{F'(t)\} = \int_0^\infty e^{-st} F'(t)\,dt = -F(0) + sf(s)$$

Proof:

$$\int_0^\infty e^{-st} F'(t)\,dt = \left[F(t) e^{-st}\right]_0^\infty + \int_0^\infty se^{-st} F(t)\,dt$$

$$= -F(0) + s\,f(s)$$

where $F(0)$ is the value of $F(t)$ at $t = 0$

# Theorem 2.3: If $F(t)$ is a twice differentiable function of $t$ then

$$\mathcal{L}\{F''(t)\} = s^2 f(s) - sF(0) - F'(0)$$

The general result proved by induction is :-

$$\boxed{\mathcal{L}\{F^{n}(t)\} = s^{n}f(s) - s^{n-1}F(0) - s^{n-2}F'(0) - \cdots - F^{n-1}(0)}$$

Let $g(t) = \int_0^t F(u)\,du$

$$g'(t) = F(t)$$

$$\mathcal{L}\{g'(t)\} = s\mathcal{L}\{g(t)\} - g(0)$$

$$= s\mathcal{L}\{g(t)\}$$

$$\Rightarrow \boxed{\mathcal{L}\{g(t)\} = \frac{f(s)}{s}} \quad \left\{f(s) = \mathcal{L}\{F(t)\}\right\}$$

2.3) Heaviside's Unit Step Function

$$\mathcal{L}\{H(t-t_0)\} = \int_0^\infty H(t-t_0)\,e^{-st}\,dt$$

Now, Since $H(t-t_0) = 0 \quad \forall\ t < t_0$

$$\mathcal{L}\{H(t-t_0)\} = \int_{t_0}^\infty e^{-st}\,dt = \frac{e^{-st_0}}{s}$$

# Theorem 2.5 (Second Shift Theorem): If $F(t)$ is a function of exponential order in $t$ then,

$$\mathcal{L}\{H(t-t_0)F(t-t_0)\} = e^{-st_0} f(s)$$

Where $f(s)$ is the Laplace Transform of $F(t)$.

**Proof:** $\mathcal{L}\{H(t-t_0)F(t-t_0)\} = \int_0^\infty H(t-t_0)F(t-t_0)e^{-st}\,dt$

$$= \int_{t_0}^\infty F(t-t_0)e^{-st}\,dt$$

$$= e^{-st_0}\int_0^\infty F(u)e^{-su}\,du \quad (u = t-t_0)$$

$$= e^{-st_0} f(s)$$

* Example 2.6: $F(t) = \begin{cases} \sin(t) & t \geq 3 \\ 0 & t < 3 \end{cases}$

$$\Rightarrow F(t) = H(t-3)\sin t$$
$$= H(t-3)\sin(t-3+3)$$
$$= H(t-3)\{\sin(t-3)\cos 3 + \cos(t-3)\sin 3\}$$

$$\mathcal{L}\{F(t)\} = \cos 3\,\mathcal{L}\{H(t-3)\sin(t-3)\}$$
$$+ \sin 3\,\mathcal{L}\{H(t-3)\cos(t-3)\}$$

$$\mathcal{L}\{H(t)\} = e^{-3s}\cos 3 \frac{1}{s^2+1} + e^{-3s}\sin 3 \frac{s}{s^2+1}$$

$$\Rightarrow \mathcal{L}\{H(t)\} = \frac{(\cos 3 + s\sin 3)e^{-3s}}{s^2+1}$$

## 2.4) Inverse of Laplace Transform

# Defination 2.7: If $F(t)$ has the Laplace Transform $f(s)$, that is

$$\mathcal{L}\{F(t)\} = f(s)$$

then the inverse Laplace Transform is defined by,

$$\mathcal{L}^{-1}\{f(s)\} = F(t)$$

and is unique apart from null functions.

# Theorem 2.8: The inverse Laplace Transform is linear.

$$\mathcal{L}^{-1}\{a f_1(s) + b f_2(s)\} = a\mathcal{L}^{-1}\{f_1(s)\} + b\mathcal{L}^{-1}\{f_2(s)\}$$

Proof: $\mathcal{L}\{aF_1(t) + bF_2(t)\} = a\mathcal{L}\{F_1(t)\} + b\mathcal{L}\{F_2(t)\}$

$$= a f_1(s) + b f_2(s)$$

$$\mathcal{L}^{-1}\{a f_1(s) + b f_2(s)\} = aF_1(t) + bF_2(t)$$

$$\Rightarrow \mathcal{L}^{-1}\{a f_1(s) + b f_2(s)\} = a\,\mathcal{L}^{-1}\{f_1(s)\} + b\,\mathcal{L}^{-1}\{f_2(s)\}$$

* <u>Example 2.9</u> $\mathcal{L}^{-1}\left\{\frac{a}{s^2-a^2}\right\}$

$$\frac{a}{s^2-a^2} = \frac{a}{(s+a)(s-a)} = \frac{A}{s+a} + \frac{B}{s-a}$$

$$a = A(s-a) + B(s+a)$$
$$= (A+B)s + \cancel{(A-B)} + (B-A)a = a$$

$$A+B=0$$
$$B-A=1$$
$$B = 1/2 \quad A = -\frac{1}{2}$$

$$\Rightarrow \frac{a}{s^2-a^2} = \frac{1}{2(s-a)} - \frac{1}{2(s+a)} = \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right]$$

$$\frac{1}{2}\left[\mathcal{L}^{-1}\left(\frac{1}{s-a}\right) - \mathcal{L}^{-1}\left(\frac{1}{s+a}\right)\right] = \frac{1}{2}\left[e^{at}\mathcal{L}^{-1}\left(\frac{1}{s}\right) - e^{-at}\mathcal{L}^{-1}\left(\frac{1}{s}\right)\right]$$

$$\Rightarrow \frac{1}{2}\left[e^{at} - e^{-at}\right] = \sinh(at)$$

* Example 2.10: $\mathcal{L}^{-1}\left\{\frac{s^2}{(s+3)^3}\right\}$

$$\frac{s^2}{(s+3)^3} = \frac{A}{s+3} + \frac{B}{(s+3)^2} + \frac{C}{(s+3)^3}$$

$$= \frac{A(s+3)^2 + B(s+3) + C}{(s+3)^3}$$

$$= \frac{A(s^2+9+6s) + B(s+3) + C}{(s+3)^3}$$

$$\frac{s^2}{(s+3)^3} = \frac{As^2 + (B+6A)s + C+3B+9A}{(s+3)^3}$$

$A = 1$

$6A + B = 0 \Rightarrow B = -6$

$C + 3B + 9A = 0 \Rightarrow C + (-18) + (9\times1) = 0$

$C - 9 = 0$

$C = 9$

$$\frac{s^2}{(s+3)^3} = \frac{1}{s+3} + \frac{-6}{(s+3)^2} + \frac{9}{(s+3)^3}$$

$$\mathcal{L}^{-1}\left\{\frac{s^2}{(s+3)^3}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} - \mathcal{L}^{-1}\left\{\frac{6}{(s+3)^2}\right\} + \mathcal{L}^{-1}\left\{\frac{9}{(s+3)^3}\right\}$$

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$$\Rightarrow e^{-3t}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - e^{-3t}\mathcal{L}^{-1}\left\{\frac{6}{s^2}\times\frac{1!}{1!}\right\} + e^{-3t}\mathcal{L}^{-1}\left\{\frac{9\times 2!}{s^3\times 2!}\right\}$$

$$\Rightarrow e^{-3t}\times 1 - e^{-3t}\times 6\times t + e^{-3t}\times\frac{9}{2}\times t^2$$

$$\Rightarrow e^{-3t}\left\{1-6t+\frac{9}{2}t^2\right\}$$

* Example 2.11 : Determine the following inverse Laplace Transforms.

a) $\mathcal{L}^{-1}\left(\frac{(s+3)}{s(s+2)(s-1)}\right)$

$$\Rightarrow \frac{s+3}{s(s-1)(s+2)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+2}$$

$$\Rightarrow \frac{A(s-1)(s+2) + Bs(s+2) + Cs(s-1)}{s(s-1)(s+2)}$$

$$\Rightarrow \frac{A(s^2+s-2) + B(s^2+2s) + C(s^2-s)}{s(s-1)(s+2)}$$

$$\Rightarrow \frac{(A+B+C)s^2 + (A+2B-C)s + (-2A)}{\cancel{s(s-1)(s+2)}}$$

$-2A = 3 \Rightarrow A = -3/2$

$A+B+C=0$ $\quad$ $A+2B-C=1$

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$\Rightarrow A - 2(A+C) - C = 1$

$\Rightarrow A - 2A - 2C - C = 1$

$-A - 3C = 1$

$\cancel{C = -A/3 = -1/2}$

$C = \frac{-A-1}{3} = \frac{\frac{3}{2}-1}{3} = \frac{1}{6}$

$B = -(A+C) = \cancel{-\left(+\frac{1}{6}+\frac{1}{2}\right) = -\left(\frac{-2}{2}\right) = 1}$

$= -\left(\frac{-3}{2}+\frac{1}{6}\right) = -\left(\frac{-9+1}{6}\right) = +\frac{8}{6} = \frac{4}{3}$

$$\frac{S+3}{S(S-1)(S+2)} = \frac{-3/2}{S} + \frac{4/3}{S-1} + \frac{1/6}{S+2}$$

$$\mathcal{L}^{-1}\left\{\frac{S+3}{S(S-1)(S+2)}\right\} = -\frac{3}{2}\mathcal{L}^{-1}\left\{\frac{1}{S}\right\} + \frac{4}{3}\mathcal{L}^{-1}\left\{\frac{1}{S-1}\right\} + \frac{1}{6}\mathcal{L}^{-1}\left\{\frac{1}{S+2}\right\}$$

$$= -\frac{3}{2} + \frac{4e^{t}}{3} + \frac{1}{6} \times e^{-2t}$$

(b) $\mathcal{L}^{-1}\left\{\frac{e^{-7s}}{(s+3)^3}\right\}$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+3)^3}\right\} = \frac{1}{2}t^2 e^{-3t}$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-7s}}{(s+3)^3}\right\} = \begin{cases} \frac{1}{2}(t-7)^2 e^{-3(t-7)} & \forall\ t > 7 \\ 0 & \forall\ 0 \le t \le 7 \end{cases}$$

## 2.5) Limiting Theorems

# Theorem 2.12 (Initial Value): If the indicated limits exist then,

$$\lim_{t \to 0} F(t) = \lim_{s \to \infty} s f(s)$$

(The left hand side is F(0) of course, or F(0+) if $\lim_{t \to 0} F(t)$ is not unique.)

Proof: $\mathcal{L}\{F'(t)\} = s f(s) - F(0)$

# Theorem 2.13 (Final Value): If the limits indicated exist, then

$$\lim_{t \to \infty} F(t) = \lim_{s \to 0} s f(s)$$

## 2.6) The Impulse function

⇒ It is sometime also called Dirac's δ function after the pioneering theoratical physicist P.A.M Dirac (1902 – 1984)

# Defination 2.15: The Dirac-δ function δ(t) is defined as having the following properties.

i) $\delta(t) = 0 \ \forall t, \ t \neq 0$

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(ii) $$\int_{-\infty}^{+\infty} h(t)\,\delta(t)\,dt = h(0)$$

for any function $h(t)$ continuous in $(-\infty, \infty)$

⇒ Dirac-δ function can be thought of as the limiting case of a top hat function of unit area as it becomes infinitesimally thin but infinitely tall.

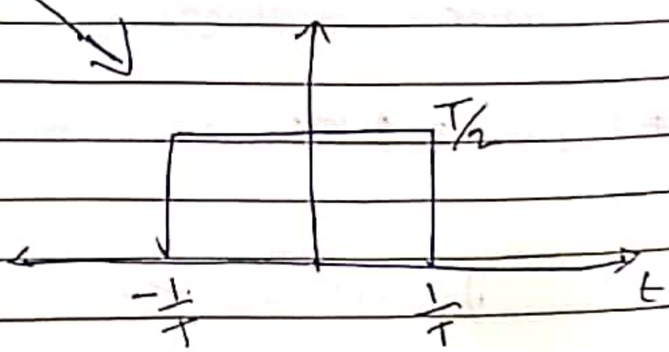
$$\delta(t) = \lim_{T\to\infty} T_p(t)$$

Top hat function

$$T_p = \begin{cases} 0 & t \leq -1/T \\ \frac{T}{2} & -1/T < t < 1/T \\ 0 & t \geq 1/T \end{cases}$$

$$\int_{-\infty}^{\infty} h(t) \lim_{T\to\infty} T_p(t)\,dt = \lim_{T\to\infty} \int_{-\infty}^{\infty} h(t)\,T_p(t)\,dt$$

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1$$



$$\boxed{\mathcal{L}\{\delta(t)\} = 1}$$

⇒ The integral in the definition can be written as :-

$$\int_{-\infty}^{\infty} h(t) \lim_{T_p \to \infty} T_p(t)\, dt = \lim_{T \to \infty} \int_{-\infty}^{\infty} h(t)\, T_p(t)\, dt$$

⇒ The area under curve $h(t)T_p(t)$ approaches $h(0)$ as $T \to \infty$.

⇒ For sufficiently large value of $T$, the interval $[-1/T, 1/T]$ will be small enough for the value of $h(t)$ not to differ very much from the value at origin.

$$\Rightarrow h(t) = h(0) + \varepsilon(t)$$

$\varepsilon(t)$ ↳ tends to zero as $T \to \infty$

⇒ The Integral thus can be seen to tend to $h(0)$ as $T \to \infty$ and the property is established.

⇒ In definition of $\delta(t)$ strictly, the first condition is redundant only second condition is necessary, but it is convenient to retain it.

⇒ $\delta(t)$ is not a true function because it has not been defined for $t=0$. $\delta(0)$ has no value.

# Laplace Transformation of $\delta(t)$

$$\int_{-\infty}^{\infty} \delta(t) e^{-st} dt = \int_{0^-}^{\infty} \delta(t) e^{-st} dt = 1$$

$$\boxed{\mathcal{L}\{\delta(t)\} = 1}$$

⇒ Function $\delta(t-t_0)$ represents an impulse that is centered on the time $t=t_0$.

⇒ It can be considered to be the limit of the function $K(t)$ where $K(t)$ is the displaced top hat function defined by:

$$K(t) = \begin{cases} 0 & t < \frac{t_0 - 1}{2T} \\ T/2 & \frac{t_0 - 1}{2T} < t_0 < \frac{t_0+1}{2T} \\ 0 & t > \frac{t_0+1}{2T} \end{cases}$$

as $T \to \infty$

$$\text{So} \int_{-\infty}^{\infty} h(t)\, \delta(t-t_0)\, dt = h(t_0)$$

$$\boxed{\mathcal{L}\{\delta(t-t_0)\} = e^{-st_0}}$$

# Filtering Property

$$\int_{-\infty}^{\infty} h(t)\, \delta(t-t_0)\, dt = h(t_0)$$

Let $h(t) = e^{-st} f(t)$ and $t_0 = a \geq 0$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-st} f(t)\, \delta(t-a)\, dt = e^{-st} f(a)$$

$$\Rightarrow \int_{0}^{\infty} e^{-st} f(t)\, \delta(t-a)\, dt = e^{-st} f(a)$$

$$\Rightarrow \boxed{\mathcal{L}\{f(t)\, \delta(t-a)\} = e^{-as} f(a)}$$

⇒ One property that is particularly useful in the Context of Laplace Transforms is the value of the integral.

$$\int_{-\infty}^{t} \delta(u-u_0)\,dt = \begin{cases} 0 & t < u_0 \\ 1 & t > u_0 \end{cases}$$

$$\Rightarrow \int_{-\infty}^{t} \delta(u-u_0)\,dt = H(t-u_0)$$

Heaviside's Unit Step Function

2

⇒ On differentiating the result.

$$\boxed{\delta(u-u_0) = H'(u-u_0)}$$

"The impulse function is the derivative of Heaviside Unit Step Function"

⇒ It is possible to define a whole string of derivatives $\delta'(t)$, $\delta''(t)$ etc...

In general,

$$\int_{-\infty}^{\infty} h(t)\, \delta^{n}(t)\, dt = (-1)^{n} h^{n}(0)$$

It is easy to deduce that,

$$\boxed{\mathcal{L}\{\delta^{n}(t)\} = \int_{0^-}^{\infty} e^{-st} \delta^{n}(t)\, dt = s^{n}}$$

## 2.7) Periodic Functions

# Definition 2.18: If $F(t)$ is a function that obeys the rule

$$F(t) = F(t+\tau)$$

$\forall$ some real $\tau$ for all values of $t$ then $F(t)$ is called a periodic function with period $\tau$.

# Theorem 2.19: Let $F(t)$ have period $T > 0$ so that $F(t) = F(t+T)$. Then

$$\mathcal{L}\{F(t)\} = \frac{\int_0^T e^{-st} F(t)\, dt}{1 - e^{-sT}}$$

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classmate

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<u>Proof:</u> $\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt$

$$= \int_0^T e^{-st} F(t)\, dt + \int_T^{2T} e^{-st} F(t)\, dt$$

$$+ \int_{2T}^{3T} e^{-st} F(t)\, dt + \int_{(n-1)T}^{nT} e^{-st} F(t)\, dt + \cdots$$

Consider the Integral

$$\int_{(n-1)T}^{nT} e^{-st} F(t)\, dt$$

Let $u = t - (n-1)T \Rightarrow du = dt$

@ $t \to (n-1)T \quad u \to 0$

$t \to nT \quad u \to T$

$$\Rightarrow \int_0^T e^{-s(u+(n-1)T)} F(u+(n-1)T)\, du$$

$$\Rightarrow e^{-s(n-1)T} \int_0^T e^{-su} F(u)\, du \quad \forall n = 1, 2, 3 \ldots$$

So $\int_0^\infty e^{-st}F(t)\,dt = \left(1+e^{-sT}+e^{-2sT}+\cdots\right)\int_0^T e^{-st}F(t)\,dt$

$$\Rightarrow \boxed{\mathcal{L}\{F(t)\} = \frac{\int_0^T e^{-st}F(t)\,dt}{1-e^{-sT}}}$$

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CHAPTER 3

# Convolution and the Solution of Ordinary Differential Equation

# 3

# Convolution and the Solution of Ordinary Differential Equation

## 3.1) Introduction

"Mastery of the Convolution theorem greatly extends the power of Laplace Transforms to Solve ODEs"

## 3.2) Convolution

# **Defination 3.1:** The Convolution of two given functions $f(t)$ and $g(t)$ is written $f*g$ and is defined by the integral

$$f*g = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

**Note:** The only Condition that is necessary to impose on the function $f$ & $g$ is that their behaviour be such that the integral on the right exists.

↳ Piecewise Continuity of both in the interval $[0, t]$ is certainly Sufficient.

# Definition 3.2: If an interval $[0, t_0]$ say can be partitioned into a finite number of subintervals $[0, t_1]$, $[t_1, t_2]$, $[t_2, t_3]$ ---- $[t_n, t_0]$ with $0, t_1, t_2$ ---- $t_n$ to an increasing sequence of time and such that a given function $f(t)$ is continuous in each of these subintervals but not necessarily at the end points themselves, then $f(t)$ is piecewise continuous in the interval $[0, t_0]$.

# Theorem 3.3 (Symmetry): $f * g = g * f$

# Theorem 3.4 (Convolution): If $f(t)$ and $g(t)$ are two functions of exponential order, and writing $\mathcal{L}\{f\} = \bar{f}(s)$ and $\mathcal{L}\{g\} = \bar{g}(s)$ as the two Laplace Transforms then $\mathcal{L}^{-1}\{\bar{f}\bar{g}\} = f * g$ where $*$ is the convolution operator.

Proof

$$\mathcal{L}\{f(t) * g(t)\} = \int_0^\infty e^{-st} \int_0^t f(\tau) g(t-\tau)\, d\tau\, dt$$

$$\Rightarrow \int_0^\infty \int_0^t e^{-st} f(\tau) g(t-\tau)\, d\tau\, dt$$

Changing order of integration

$$\Rightarrow \boxed{\int_0^\infty \int_\tau^\infty e^{-st} f(\tau) g(t-\tau)\, dt\, d\tau}$$

??

$$\Rightarrow \int_0^\infty f(\tau)\left\{\int_\tau^\infty e^{-st} g(t-\tau)\,dt\right\} d\tau$$

let $u = t-\tau$

$$\text{So} \int_\tau^\infty e^{-st} g(t-\tau)\,dt = \int_0^\infty e^{-s(u+\tau)} g(u)\,du$$

$$= e^{-s\tau}\int_0^\infty e^{-su} g(u)\,du$$

$$= e^{-s\tau}\,\bar{g}(s)$$

$$\text{So } \mathcal{L}\{f*g\} = \int_0^\infty f(\tau)\, e^{-s\tau}\,\bar{g}(s)\,d\tau$$

$$= \bar{g}(s)\int_0^\infty f(\tau)\, e^{-s\tau}\,d\tau$$

$$\mathcal{L}\{f*g\} = \bar{g}(s)\,\bar{f}(s).$$

$$\Rightarrow \boxed{f(t)*g(t) = \mathcal{L}^{-1}\{\bar{f}\,\bar{g}\}}$$

Borel's Theorem

# Example 3.5: find value of $\cos(t) * \sin(t)$.

$$\cos(t) * \sin(t) = \int_0^t \cos(\tau)\sin(t-\tau)\,d\tau$$

$$\Rightarrow \frac{1}{2}\int_0^t \sin(t) + \sin(t-2\tau)\,d\tau$$

$$\Rightarrow \frac{1}{2}\left\{ \Big[\tau \sin(t)\Big]_0^t + \frac{1}{2}\Big[\cos(t-2\tau)\Big]_0^t \right\}$$

$$= \frac{1}{2}\big(t\sin(t)\big) + \frac{1}{4}\big(\cancel{\cos(t) - \cos(t)}\big)$$

$$= \frac{1}{2} t \sin t$$

# Example 3.7: Find the following Inverse Laplace Transforms:

a) $\mathcal{L}^{-1}\left\{\dfrac{s}{(s^2+1)^2}\right\}$

$\Rightarrow$

$$\mathcal{L}\{\cos(t)\}\,\mathcal{L}\{\sin(t)\} = \frac{s}{(s^2+1)^2}$$

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \cos(t) * \sin(t) = \frac{1}{2} t \sin t$$