2

Further Properties of the Laplace Transform

2.1) Real Functions

Definition 2.1: Heaviside's Unit step function, or Simply the unit step function is defined as:

$$H(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geqslant 0 \end{cases}$$

2.2) Derivative property of the Laplace Transform

Theorem 2.2: Suppose a differentiable function $F(t)$ has a Laplace Transform $f(s)$, we can find the Laplace Transform:

$$\mathcal{L}\{F'(t)\} = \int_0^\infty e^{-st} F'(t)\,dt = -F(0) + sf(s)$$

Proof: $$\int_0^\infty e^{-st} F'(t)\,dt = \left[F(t)\,e^{-st}\right]_0^\infty + \int_0^\infty s e^{-st} F(t)\,dt$$

$$= -F(0) + s f(s)$$

where $F(0)$ is the value of $F(t)$ at $t=0$

Theorem 2.3: If $F(t)$ is a twice differentiable function of t then

$$\mathcal{L}\{F''(t)\} = s^2 f(s) - sF(0) - F'(0)$$

The general result proved by induction is :-

$$\boxed{\mathcal{L}\{F^{n}(t)\} = s^{n}f(s) - s^{n-1}F(0) - s^{n-2}F'(0) - \cdots - F^{n-1}(0)}$$

Let $g(t) = \int_0^t F(u)\,du$

$$g'(t) = F(t)$$

$$\mathcal{L}\{g'(t)\} = s\,\mathcal{L}\{g(t)\} - g(0)$$

$$= s\,\mathcal{L}\{g(t)\}$$

$$\Rightarrow \boxed{\mathcal{L}\{g(t)\} = \frac{f(s)}{s}} \qquad \left\{ f(s) = \mathcal{L}\{F(t)\} \right\}$$

2.3) Heaviside's Unit step Function

$$\mathcal{L}\{H(t-t_0)\} = \int_0^{\infty} H(t-t_0)\,e^{-st}\,dt$$

Now, Since $H(t-t_0) = 0 \;\forall\; t < t_0$

$$\mathcal{L}\{H(t-t_0)\} = \int_{t_0}^{\infty} e^{-st}\,dt = \frac{e^{-st_0}}{s}$$

Theorem 2.5 (Second Shift Theorem): If $F(t)$ is a function of exponential order in t then,

$$\mathcal{L}\{H(t-t_0)F(t-t_0)\} = e^{-st_0} f(s)$$

where $f(s)$ is the Laplace Transform of $F(t)$.

Proof: $$\mathcal{L}\{H(t-t_0)F(t-t_0)\} = \int_0^\infty H(t-t_0)F(t-t_0)e^{-st}\,dt$$

$$= \int_{t_0}^\infty F(t-t_0)e^{-st}\,dt$$

$$= e^{-st_0}\int_0^\infty F(u)e^{-su}\,du \quad (u = t-t_0)$$

$$= e^{-st_0} f(s)$$

* Example 2.6: $F(t) = \begin{cases} \sin(t) & t \geq 3 \\ 0 & t < 3 \end{cases}$

$$\Rightarrow F(t) = H(t-3)\sin t$$
$$= H(t-3)\sin(t-3+3)$$
$$= H(t-3)\{\sin(t-3)\cos 3 + \cos(t-3)\sin 3\}$$

$$\mathcal{L}\{F(t)\} = \cos 3\,\mathcal{L}\{H(t-3)\sin(t-3)\}$$
$$+ \sin 3\,\mathcal{L}\{H(t-3)\cos(t-3)\}$$

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$$\mathcal{L}\{H(t)\} = e^{-3s}\cos 3 \frac{1}{s^2+1} + e^{-3s}\sin 3 \frac{s}{s^2+1}$$

$$\Rightarrow \mathcal{L}\{H(t)\} = \frac{(\cos 3 + s\sin 3)e^{-3s}}{s^2+1}$$

2.4) Inverse of Laplace Transform

Defination 2.7: If $F(t)$ has the Laplace Transform $f(s)$, that is

$$\mathcal{L}\{F(t)\} = f(s)$$

then the inverse Laplace Transform is defined by,

$$\mathcal{L}^{-1}\{f(s)\} = F(t)$$

and is unique apart from null functions.

Theorem 2.8: The inverse Laplace Transform is linear.

$$\mathcal{L}^{-1}\{a f_1(s) + b f_2(s)\} = a\mathcal{L}^{-1}\{f_1(s)\} + b\mathcal{L}^{-1}\{f_2(s)\}$$

Proof: $\mathcal{L}\{aF_1(t) + bF_2(t)\} = a\mathcal{L}\{F_1(t)\} + b\mathcal{L}\{F_2(t)\}$

$$= a f_1(s) + b f_2(s)$$

$$\mathcal{L}^{-1}\{a f_1(s) + b f_2(s)\} = aF_1(t) + bF_2(t)$$

$$\Rightarrow \mathcal{L}^{-1}\{a f_1(s) + b f_2(s)\} = a\,\mathcal{L}^{-1}\{f_1(s)\} + b\,\mathcal{L}^{-1}\{f_2(s)\}$$

* Example 2.9 $\mathcal{L}^{-1}\left\{\frac{a}{s^2-a^2}\right\}$

$$\frac{a}{s^2-a^2} = \frac{a}{(s+a)(s-a)} = \frac{A}{s+a} + \frac{B}{s-a}$$

$$a = A(s-a) + B(s+a)$$

$$= (A+B)s + \cancel{(A-B)} + (B-A)a = a$$

$$A+B=0$$

$$B-A=1$$

$$B = 1/2 \quad A = -\frac{1}{2}$$

$$\Rightarrow \frac{a}{s^2-a^2} = \frac{1}{2(s-a)} - \frac{1}{2(s+a)} = \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right]$$

$$\frac{1}{2}\left[\mathcal{L}^{-1}\left(\frac{1}{s-a}\right) - \mathcal{L}^{-1}\left(\frac{1}{s+a}\right)\right] = \frac{1}{2}\left[e^{at}\mathcal{L}^{-1}\left(\frac{1}{s}\right) - e^{-at}\mathcal{L}^{-1}\left(\frac{1}{s}\right)\right]$$

$$\Rightarrow \frac{1}{2}\left[e^{at} - e^{-at}\right] = \sinh(at)$$

* **Example 2.10:** $\mathcal{L}^{-1}\left\{\frac{s^2}{(s+3)^3}\right\}$

$$\frac{s^2}{(s+3)^3} = \frac{A}{s+3} + \frac{B}{(s+3)^2} + \frac{C}{(s+3)^3}$$

$$= \frac{A(s+3)^2 + B(s+3) + C}{(s+3)^3}$$

$$= \frac{A(s^2+9+6s) + B(s+3) + C}{(s+3)^3}$$

$$\frac{s^2}{(s+3)^3} = \frac{As^2 + (B+6A)s + C + 3B + 9A}{(s+3)^3}$$

$A = 1$

$6A + B = 0 \Rightarrow B = -6$

$C + 3B + 9A = 0 \Rightarrow C + (-18) + (9\times 1) = 0$

$C - 9 = 0$

$C = 9$

$$\frac{s^2}{(s+3)^3} = \frac{1}{s+3} + \frac{-6}{(s+3)^2} + \frac{9}{(s+3)^3}$$

$$\mathcal{L}^{-1}\left\{\frac{s^2}{(s+3)^3}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} - \mathcal{L}^{-1}\left\{\frac{6}{(s+3)^2}\right\} + \mathcal{L}^{-1}\left\{\frac{9}{(s+3)^3}\right\}$$

$$\Rightarrow e^{-3t} \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - e^{-3t} \mathcal{L}^{-1}\left\{\frac{6}{s^2} \times \frac{1!}{1!}\right\} + e^{-3t} \mathcal{L}^{-1}\left\{\frac{9 \times 2!}{s^3 \times 2!}\right\}$$

$$\Rightarrow e^{-3t} \times 1 - e^{-3t} \times 6 \times t + e^{-3t} \times \frac{9}{2} \times t^2$$

$$\Rightarrow e^{-3t}\left\{1 - 6t + \frac{9}{2}t^2\right\}$$

* Example 2.11 : Determine the following inverse Laplace Transforms.

a) $\mathcal{L}^{-1}\left(\frac{(s+3)}{s(s+2)(s-1)}\right)$

$$\Rightarrow \frac{s+3}{s(s-1)(s+2)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+2}$$

$$\Rightarrow \frac{A(s-1)(s+2) + Bs(s+2) + Cs(s-1)}{s(s-1)(s+2)}$$

$$\Rightarrow \frac{A(s^2+s-2) + B(s^2+2s) + C(s^2-s)}{s(s-1)(s+2)}$$

$$\Rightarrow \frac{(A+B+C)s^2 + (A+2B-C)s + (-2A)}{s(s-1)(s+2)}$$

$-2A = 3 \Rightarrow A = -3/2$

$A+B+C=0$ $\qquad$ $A+2B-C=1$

$\Rightarrow A - 2(A+C) - C = 1$

$\Rightarrow A - 2A - 2C - C = 1$

$-A - 3C = 1$ $\qquad C = \frac{-A-1}{3} = \frac{\frac{3}{2}-1}{3} = \frac{1}{6}$

$\cancel{C = \frac{-A}{3} = \frac{1}{2}}$

$B = -(A+C) = \cancel{-\left(\frac{+1}{6}+\frac{1}{2}\right) = -\left(\frac{-2}{2}\right) = 1}$

$= -\left(\frac{-3}{2}+\frac{1}{6}\right) = -\left(\frac{-9+1}{6}\right) = +\frac{8}{6} = \frac{4}{3}$

$$\frac{s+3}{s(s-1)(s+2)} = \frac{-3/2}{s} + \frac{4/3}{s-1} + \frac{1/6}{s+2}$$

$$\mathcal{L}^{-1}\left\{\frac{s+3}{s(s-1)(s+2)}\right\} = -\frac{3}{2}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \frac{4}{3}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \frac{1}{6}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}$$

$$= -\frac{3}{2} + \frac{4e^{t}}{3} + \frac{1}{6} \times e^{-2t}$$

(b) $\mathcal{L}^{-1}\left\{\frac{e^{-7s}}{(s+3)^3}\right\}$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+3)^3}\right\} = \frac{1}{2}t^2 e^{-3t}$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-7s}}{(s+3)^3}\right\} = \begin{cases} \frac{1}{2}(t-7)^2 e^{-3(t-7)} & \forall\ t>7 \\ 0 & \forall\ 0 \le t \le 7 \end{cases}$$

2.5) Limiting Theorems

Theorem 2.12 (Initial Value): If the indicated limits exist then,

$$\lim_{t \to 0} F(t) = \lim_{s \to \infty} s f(s)$$

(The left hand side is $F(0)$ of course, or $F(0+)$ if $\lim_{t \to 0} F(t)$ is not unique.)

Proof: $\mathcal{L}\{F'(t)\} = s f(s) - F(0)$

Theorem 2.13 (Final Value): If the limits indicated exist, then

$$\lim_{t \to \infty} F(t) = \lim_{s \to 0} s f(s)$$

2.6) The Impulse function

$\Rightarrow$ It is sometime also called Dirac's δ function after the pioneering theoretical physicist P.A.M Dirac (1902 – 1984)

Defination 2.15: The Dirac-δ function $\delta(t)$ is defined as having the following properties.

i) $\delta(t) = 0 \;\forall t,\; t \neq 0$

(ii) $\int_{-\infty}^{+\infty} h(t)\,\delta(t)\,dt = h(0)$

for any function $h(t)$ continuous in $(-\infty, \infty)$

⇒ Dirac-δ function can be thought of as the limiting case of a <u>top hat function</u> of unit area as it becomes infinitesimally thin but infinitely tall.

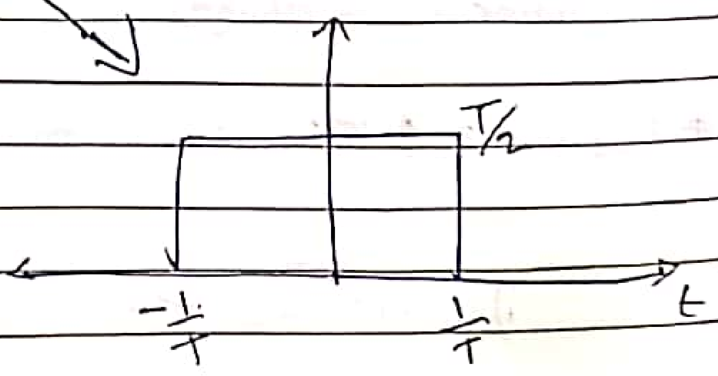
$$\delta(t) = \lim_{T\to\infty} T_p(t)$$

Top hat function

$$T_p = \begin{cases} 0 & t \le -1/T \\ \frac{T}{2} & -1/T < t < 1/T \\ 0 & t \ge 1/T \end{cases}$$

$$\int_{-\infty}^{\infty} h(t) \lim_{T\to\infty} T_p(t)\,dt = \lim_{T\to\infty} \int_{-\infty}^{\infty} h(t)\,T_p(t)\,dt$$

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1$$



$$\boxed{\mathcal{L}\{\delta(t)\} = 1.}$$

⇒ The integral in the definition can be written as :-

$$\int_{-\infty}^{\infty} h(t) \lim_{T_p \to \infty} T_p(t)\,dt = \lim_{T \to \infty} \int_{-\infty}^{\infty} h(t)\,T_p(t)\,dt$$

⇒ The area under curve $h(t)T_p(t)$ approaches $h(0)$ as $T \to \infty$.

⇒ For sufficiently large value of T, the interval $[-1/T, 1/T]$ will be small enough for the value of $h(t)$ not to differ very much from the value at origin.

$$\Rightarrow h(t) = h(0) + \underbrace{\varepsilon(t)}_{\text{tends to zero as } T \to \infty}$$

⇒ The Integral thus can be seen to tend to $h(0)$ as $T \to \infty$ and the property is established.

⇒ In definition of $\delta(t)$ strictly, the first condition is redundant only second condition is necessary but it is convenient to retain it.

⇒ $\delta(t)$ is not a true function because it has not been defined for $t=0$. $\delta(0)$ has no value.

Laplace Transformation of $\delta(t)$:

$$\int_{-\infty}^{\infty} \delta(t) e^{-st} dt = \int_{0^-}^{\infty} \delta(t) e^{-st} dt = 1$$

$$\boxed{\mathcal{L}\{\delta(t)\} = 1}$$

⇒ Function $\delta(t-t_0)$ represents an impulse that is centered on the time $t = t_0$.

⇒ It can be considered to be the limit of the function $K(t)$ where $K(t)$ is the displaced top hat function defined by:

$$K(t) = \begin{cases} 0 & t < \frac{t_0 - 1}{2T} \\ T/2 & \frac{t_0 - 1}{2T} < t_0 < \frac{t_0 + 1}{2T} \\ 0 & t > \frac{t_0 + 1}{2T} \end{cases}$$

as $T \to \infty$

So $$\int_{-\infty}^{\infty} h(t)\,\delta(t-t_0)\,dt = h(t_0)$$

$$\boxed{\mathcal{L}\{\delta(t-t_0)\} = e^{-st_0}}$$

Filtering Property

$$\int_{-\infty}^{\infty} h(t)\,\delta(t-t_0)\,dt = h(t_0)$$

Let $h(t) = e^{-st} f(t)$ and $t_0 = a \geq 0$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-st} f(t)\,\delta(t-a)\,dt = e^{-st} f(a)$$

$$\Rightarrow \int_{0}^{\infty} e^{-st} f(t)\,\delta(t-a)\,dt = e^{-st} f(a)$$

$$\Rightarrow \boxed{\mathcal{L}\{f(t)\,\delta(t-a)\} = e^{-as} f(a)}$$

$\Rightarrow$ One property that is particularly useful in the context of Laplace Transforms is the value of the integral:

$$\int_{-\infty}^{t} \delta(u-u_0)\,dt = \begin{cases} 0 & t < u_0 \\ 1 & t > u_0 \end{cases}$$

$$\Rightarrow \int_{-\infty}^{t} \delta(u-u_0)\,dt = H(t-u_0)$$

$\rightarrow$ Heaviside's Unit Step Function

$\Rightarrow$ On differentiating the result:

$$\boxed{\delta(u-u_0) = H'(u-u_0)}$$

$\rightarrow$ "The impulse function is the derivative of Heaviside Unit Step Function"

$\Rightarrow$ It is possible to define a whole string of derivatives $\delta'(t)$, $\delta''(t)$ etc...

In general,

$$\int_{-\infty}^{\infty} h(t)\, \delta^{n}(t)\, dt = (-1)^{n} h^{n}(0)$$

It is easy to deduce that,

$$\boxed{\mathcal{L}\{\delta^{n}(t)\} = \int_{0^-}^{\infty} e^{-st} \delta^{n}(t)\, dt = s^{n}}$$

2.7) Periodic Functions

Definition 2.18: If $F(t)$ is a function that obeys the rule

$$F(t) = F(t+\tau)$$

$\forall$ some real τ for all values of t then $F(t)$ is called a periodic function with period τ.

Theorem 2.19: Let $F(t)$ have period $T>0$ so that $F(t) = F(t+T)$. Then

$$\mathcal{L}\{F(t)\} = \frac{\int_{0}^{T} e^{-st} F(t)\, dt}{1-e^{-sT}}$$

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Proof: $\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)\,dt$

$$= \int_0^T e^{-st} F(t)\,dt + \int_T^{2T} e^{-st} F(t)\,dt$$

$$+ \int_{2T}^{3T} e^{-st} F(t)\,dt + \int_{(n-1)T}^{nT} e^{-st} F(t)\,dt + \cdots$$

Consider the Integral

$$\int_{(n-1)T}^{nT} e^{-st} F(t)\,dt$$

let $u = t-(n-1)T \Rightarrow du = dt$

@ $t \to (n-1)T \quad u \to 0$

$t \to nT \quad u \to T$

$$\Rightarrow \int_0^T e^{-s(u+(n-1)T)} F(u+(n-1)T)\,du$$

$$\Rightarrow e^{-s(n-1)T} \int_0^T e^{-su} F(u)\,du \quad \forall\, n = 1,2,3\cdots$$

So $\int_0^\infty e^{-st}F(t)\,dt = \left(1 + e^{-sT} + e^{-2sT} + \cdots\right)\int_0^T e^{-st}F(t)\,dt$

$$\Rightarrow \boxed{\mathcal{L}\{F(t)\} = \frac{\int_0^T e^{-st}F(t)\,dt}{1 - e^{-sT}}}$$