Lecture - 9

Autonomous Linear dynamical System

⇒ Continuous-time autonomous LDS has form

$$\dot{x} = Ax \qquad \{\text{No input}\}$$

→ $x(t) \in \mathbb{R}^n$ is called the state

→ n is the state dimension.

→ A is the dynamics matrix

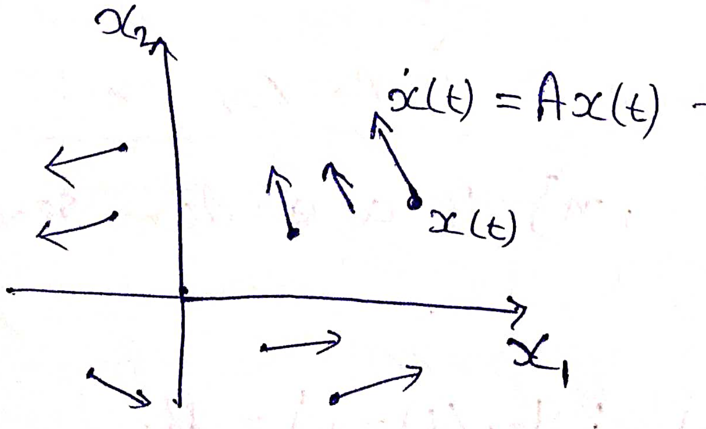
(System is time-invariant if A doesn't depend on t)

If x is scalar

$$\frac{dx}{dt} = Ax \Rightarrow \int_{x(0)}^{x(t)} \frac{dx}{x} = \int_0^t A\,dt$$

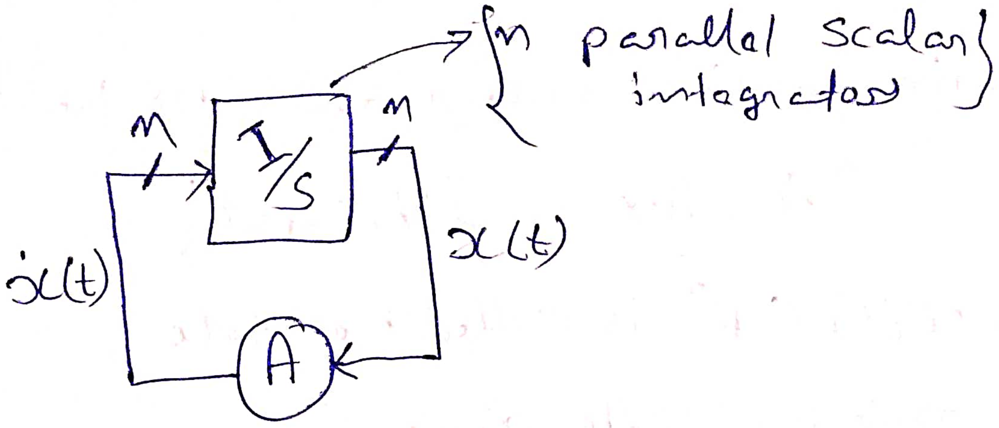
$$\Rightarrow \ln(x(t)) - \ln(x(0)) = At$$

$$\ln\left(\frac{x(t)}{x(0)}\right) = At \Rightarrow \boxed{x(t) = x(0)e^{At}}$$



Vector field

★ Block diagram representation of $\dot{x} = Ax$



⇒ If Sum of Columns of A is zero:

Example $A = \begin{bmatrix} -K_1 & 0 & 0 \\ K_1 & -K_2 & 0 \\ 0 & K_2 & 0 \end{bmatrix}$

$$\frac{d}{dt}\left(1^T x(t)\right) = 1^T \dot{x}(t)$$
$$= (1^T A) x(t)$$
$$= 0$$

$$\Rightarrow \quad 1^T x(t) = 1^T x(0)$$

★ Finite-state discrete-time Markov chain

⇒ $z(t) \in \{1, \ldots, n\}$ is a random sequence with

$$\text{Prob}\left(z(t+1) = i \mid z(t) = j\right) = P_{ij}$$

where $P \in R^{n \times n}$ is the matrix of transition probabilities.

⇒ Can represent probability distribution of $z(t)$ as n-vectors

$$p(t) = \begin{bmatrix} \text{Prob}(z(t)=1) \\ \vdots \\ \text{Prob}(z(t)=n) \end{bmatrix}$$

$$\boxed{p(t+1) = P\,p(t)}$$

→ A discrete time linear dynamical system.

★ <u>Numerical Integration of Continuous System</u>

⇒ Compute approximate solution of $\dot{x} = Ax$, $x(0) = x_0$

⇒ Suppose h is small time step.

(x doesn't change much in h)

$$x(t+h) \approx x(t) + h\dot{x}(t)$$

$$x(t+h) \approx (I + hA)x(t)$$

$$\boxed{x(kh) \approx (I + hA)^k x(0)}$$

→ Called forward Euler method.

→ Never used in practice.

* Higher order linear dynamical system

$$x^{(K)} = A_{K-1}\, x^{(K-1)} + \dots + A_1 x^{(1)} + A_0 x$$

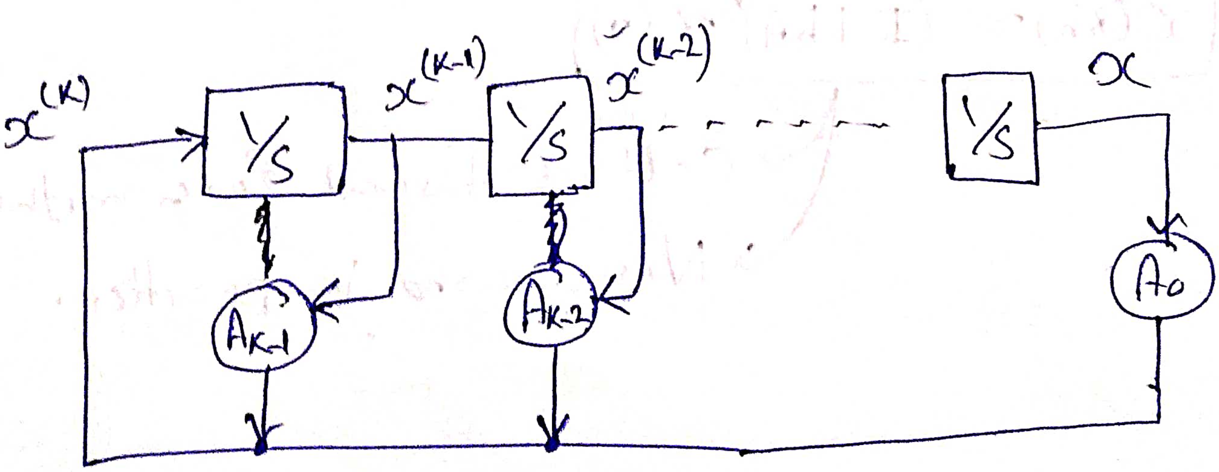
$$x(t) \in R^n$$

where $x^{(m)}$ denotes m^{th} derivative

$\Rightarrow$ define a new variable $z = \begin{bmatrix} x \\ x^{(1)} \\ \vdots \\ x^{(K-1)} \end{bmatrix} \in R^{nK}$

$$\dot{z} = \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & I \\ A_0 & A_1 & \cdots & \cdots & A_{K-1} \end{bmatrix} z \quad \Rightarrow \boxed{\dot{z} = A z}$$

(First order LSD with bigger state)



* Mechanical System

⇒ Mechanical System with K degree of freedom undergoing small motions:

$$M\ddot{q} + D\dot{q} + Kq = 0$$

⇒ $q(t) \in R^K$ is the vector of generalized displacements.

⇒ M is the mass matrix

⇒ K is the stiffness matrix

⇒ D is the damping matrix

⇒ With state $x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}$ we have

$$\dot{x} = \begin{bmatrix} \dot{q} \\ \ddot{q} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{bmatrix} x$$

* Linearization near equilibrium point

⇒ nonlinear, time-invariant differential equation (DE)

$$\dot{x} = f(x) \quad \text{where } f: R^n \to R^n$$

⇒ Suppose x_e is an equilibrium point (i.e. $f(x_e) = 0$)

⇒ now suppose $x(t)$ is near x_e, so

$$\dot{x}(t) = f(x) = \underbrace{f(x_e)}_{0} + Df(x_e)(x(t) - x_e)$$

$- \dot{x}_e$

Let $\delta x(t) = x(t) - x_e$

So $\boxed{\dot{\delta x}(t) = Df(x_e)\, \delta x(t)}$

★ Linearization along trajectory

⇒ Suppose $x_{traj} : R_+ \to R^n$

Satisfies $\dot{x}_{traj}(t) = f(x_{traj}(t), t)$

⇒ Suppose $x(t)$ is another trajectory

i.e. $\dot{x}(t) = f(x(t), t)$, and is near $x_{traj}(t)$

$$\frac{d}{dt}(x - x_{traj}) = f(x,t) - f(x_{traj}, t)$$

$$\approx D_x f(x_{traj}, t)(x - x_{traj})$$

⇒ (time-varying) LDS

$$\boxed{\dot{\delta x} = D_x f(x_{traj}, t)\, \delta x}$$

→ Called linearized or variational system along trajectory x_{traj}.