Introduction: Vector & Tensor

⇒ If physical events and entities are to be quantified, then a reference frame & a Coordinate System within that frame must be introduced. → {Temporary structure outside}

⇒ One the other hand, as a frame & Coordinate are mere Scaffolding, it should be possible to express the laws of physics in frame & coordinate free form. → (ie. Invariant form)

⇒ We shall study how, within a fixed frame, the mathematical representation of a physical object or law changes when one coordinate system is replaced by another.

★ Three dimensional Euclidean Space

⇒ Three dimensional Euclidean Space E_3 may be characterized by a set of axioms that expresses relationship among primitive undefined quantities called point, line etc.

⇒ These relationships, so closely corresponds to the results of ordinary measurements of distance in the physical world that, until the appearance of general relativity, it was thought that Euclidean geometry was the Kinematic model of the universe.

* Directed line Segment

→ ~~Directed~~
An arrow is an ordered Pair of points (A, B).
A → Tail, B → Head

→ It is Customary to represent it as $\overline{AB}$.

⇒ Two arrows are said to be equivalent if one Can be brought into Coincidence with the other by parallel translation.

⇒ The Set of all arrows equivalent to a given arrow is called the (geometric) Vector of that arrow.

→ Length of the Vector $\overline{V}$ is defined as to be the length of any one of the arrow. $\{|\overline{V}|\}$

⇒ We may choose, arbitrarily, a point O in E_3 and Call it the origin.

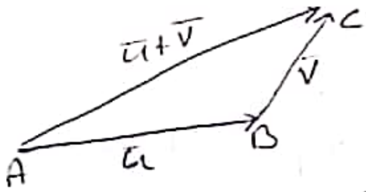
→ The vector X from O to a point P is called the position of P.

* Addition of two Vector

⇒ Addition of two vectors $\overline{U}$ & $\overline{V}$ may be defined in two equivalent ways:-

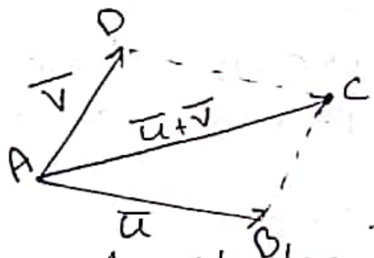
A) Head to Tail Rule

Take any arrow representing u, say $\overline{AB}$. For this choice there is a unique arrow $\overline{BC}$ representing $\overline{v}$. ~~$\overline{v+u}$~~ $\overline{u}+\overline{v}$ is defined ~~to~~ be the vector of the arrow $\overline{AC}$.



B) The Parallelogram Rule

Let $\overline{u}$ & $\overline{v}$ be represented by any two arrows having coincident tails, say $\overline{AB}$ & $\overline{AD}$. Then $\overline{u}+\overline{v}$ is the vector of the arrow $\overline{AC}$, where C is the vertex opposite A of the parallelogram having $\overline{AB}$ & $\overline{AD}$ as co-terminal edges.



★ Multiplication of a Vector $\overline{V}$ by Scalar α

If $\overline{AB}$ is an arrow representing $\overline{V}$, then $\alpha\overline{V}$ is the vector of the arrow $\alpha\overline{AB}$.

Linear Vector Space → The set of all geometric vectors, together with the operations of addition and multiplication by a scalar, form a <u>linear vector space</u>.

★ **Things that Vector may Represent**

→ Displacement
→ Force acting on a particle.
→ Finite rotation of a rigid body about an axis. (have direction & magnitude)

Different types of objects are represented by Vectors that belong to Different Vector Space

Vector addition may or may not Reflect an attribute of the objects represented.

⇒ For displacements, forces, or Velocities, there are obvious physical analogues of Vector addition.

⇒ For successive finite rotations of a rigid body about a fixed point, there is not.

★ **Cartesian Coordinate**

↳ { Descarts Characterize 3D Euclidean Space in algebraic terms as follows }

⇒ Through the Origin draw three mutually Perpendicular but otherwise arbitrarily chosen lines.

⇒ Call one of these the x-axis and on it place a point $I \neq O$. The ray from O containing I is called the positive x-axis.

⇒ $\overline{OI}$ is called the unit arrow along the x-axis and we denote its vector by $\overline{e_x}$.

⇒ Choose one of the remaining lines through O, call it the y-axis, and place on it a point J such that the length of $\overline{OJ}$ is equal to that of $\overline{OI}$.

↳ $\overline{OJ}$ is called the y-unit arrow and we denote its vector by $\overline{e}_y$.

⇒ The remaining line through O is called the z-axis and by arbitrarily adopting the ~~remaining line~~ "right hand thumb rule", we may place a unique point K on the z-axis.

↳ Such that the length of $\overline{OK}$ is equal to that of $\overline{OI}$. $\overline{OK}$ is the z-unit arrow and $\overline{e}_z$ denotes its vector.

⇒ Any point P may be represented by an ordered triple of real number (x, y, z), called the Cartesian coordinates of P.

⇒ When a vector V is represented by the arrow whose tail is the origin O, then the coordinates of the head of this arrow, say $(V_x\ V_y\ V_z)$ are called the Cartesian components of $\overline{V}$.

$$\left\{ \overline{V} \sim (V_x\ V_y\ V_z) \right\}$$

So $\overline{e}_x \sim (1, 0, 0)$ $\quad \overline{e}_y \sim (0, 1, 0)$ $\quad \overline{e}_z \sim (0, 0, 1)$

⇒ With a way of assigning Cartesian ~~coordinate~~ components $(V_x\ V_y\ V_z)$ to a vector V, we may easily deduce the following relations:-

1. $|\bar{V}| = \sqrt{V_x^2 + V_y^2 + V_z^2}$ {by Pythagorean theorem}

2. If α is a real number, then
$$\alpha\bar{V} \sim (\alpha V_x \;\; \alpha V_y \;\; \alpha V_z)$$

3. If $\bar{\omega} \sim (\omega_x \;\; \omega_y \;\; \omega_z)$ then
$$\bar{V} \pm \bar{\omega} = (V_x \pm \omega_x,\ V_y \pm \omega_y,\ V_z \pm \omega_z)$$

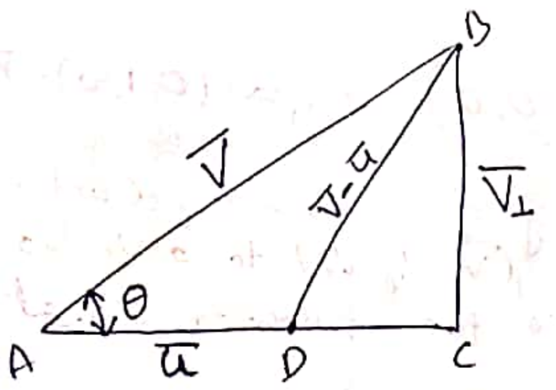
4. $\bar{V} = \bar{\omega}$
$\rightarrow V_x = \omega_x$ & $V_y = \omega_y$ & $V_z = \omega_z$

★ <u>The Dot product</u>

↳ The dot product of two vectors $\bar{u}$ & $\bar{V}$ denoted by $\bar{u} \cdot \bar{V}$ arises in many different physical & geometric contexts.

⇒ It can be defined by the formula
$$\bar{u} \cdot \bar{V} = |\bar{u}|\,|\bar{V}| \cos\theta$$
where θ is angle between $\bar{u}$ & $\bar{V}$.

⇒ We dont know how to compute θ So we need to come up with a definition of $\bar{u} \cdot \bar{V}$ that is independent of θ.



The double-headed arrow in θ indicates that θ is always nonnegative regardless of relative orientation of $\bar{u}$ & $\bar{V}$.

$\Rightarrow \bar{V}$ may be express as sum of vectors

- $\rightarrow |V| \cos\theta$ parallel to $\bar{u}$
- $\rightarrow \bar{V}_\perp$ $\perp$ to u

$$|\bar{V}|^2 = |\bar{V}|^2 \cos^2\theta + |\bar{V}_\perp|^2 \quad \{\text{from } \triangle ABC\}$$

$$|\bar{V}-\bar{u}|^2 = (|\bar{V}| \cos\theta - |\bar{u}|)^2 + |\bar{V}_\perp|^2 \quad \{\text{from } \triangle BOC\}$$

$$= -2|\bar{u}||\bar{V}|\cos\theta + |\bar{V}|^2\cos^2\theta + |\bar{u}|^2 + |\bar{V}_\perp|^2$$

$$= -2|\bar{u}||\bar{V}|\cos\theta + |\bar{u}|^2 + |\bar{V}|^2$$

$$\Rightarrow \boxed{\bar{u}\cdot\bar{V} = |\bar{u}||\bar{V}|\cos\theta = \frac{1}{2}\left(|\bar{u}|^2 + |\bar{V}|^2 - |V-u|^2\right)}$$

$\Rightarrow$ Two vectors whose dot product is zero are said to be orthogonal ($\perp$).

$\Rightarrow$ To evaluate $\bar{u}\cdot\bar{V}$ on a computer we need a component representation.

If $\bar{u} \sim (u_x\ u_y\ u_z)$ $\quad \bar{V} \sim (V_x\ V_y\ V_z)$

$$\boxed{\bar{u}\cdot\bar{V} = u_x V_x + u_y V_y + u_z V_z}$$

$\Rightarrow$ Dot product is a geometric invariant

{does not change by changing ~~coordinate~~ Cartesian axis}

★ Cartesian Base Vector

⇒ Given the Cartesian Component $(V_x\ V_y\ V_z)$ of any Vector $\bar{V}$ allow us to set :-

$$(V_x\ V_y\ V_z) = (V_x\ 0\ 0) + (0\ V_y\ 0) + (0\ 0\ V_z)$$
$$= V_x(1\,0\,0) + V_y(0\,1\,0) + V_z(0\,0\,1)$$

$$\Rightarrow \bar{V} = V_x\bar{e}_x + V_y\bar{e}_y + V_z\bar{e}_z$$

⇒ The Set $\{\bar{e}_x\ \bar{e}_y\ \bar{e}_z\}$ is called the Standard Cartesian basis, and its elements the Cartesian base Vectors.

★ The interpretation of Vector Addition

⇒ Suppose for example, that $\bar{V}$ represents the rotation of a rigid body about a fixed point.

↳ Let the direction & magnitude of $\bar{V}$ represent, respectively the axis & angle of rotation, using the right hand thumb rule to insure a Unique Correspondence.

⇒ If this rotation is followed by another, represented by a vector $\bar{U}$, then it is a fact from Kinematics that these two successive rotation are equivalent to a single rotation that we may represent by a Vector $\bar{W}$.

⇒ However, in general $\overline{V}+\overline{U} \neq \overline{W}$ ie Successive finite rotations do not add like Vectors.

<u>Note</u>

Rotations may also be represented by matrixes. If the two Successive rotations are represented by matrices V and U, then they are equivalent to single rotation representation by the matrix $W = UV$. $\overline{U}, \overline{V}$ or $\overline{W}$ may be identified, respectively with the single real eigenvector of U, V or W.

"~~Vector addition does~~

★ <u>The Cross Product</u>

⇒ The Cross product arises in mechanics when we want to Compute the torque of a force about a point.

↳ In electromagnetics when we want to compute the force on a charge moving in a magnetic field.

⇒ The Cross product of two Vectors $\overline{U}$ & $\overline{V}$ is denoted by $\overline{U}\times\overline{V}$ and defined to be the right oriented area of a parallelogram having $\overline{U}$ and $\overline{V}$ as Co-terminal edges.

$$|\overline{U}\times\overline{V}| = |\overline{U}||\overline{V}| \sin\theta$$

Where the direction of $\overline{U}\times\overline{V}$ is that of the thumb on the right hand when the fingers are curled from $\overline{U}$ to $\overline{V}$.

$$\boxed{\overline{U}\times\overline{V} = -\overline{V}\times\overline{U}} \quad \{\text{from definition}\}$$

Scanned by CamScanner

* Alternative Interpretation of the Dot and Cross Product. Tensors

⇒ In physics we learn that a constant force $\overline{F}$ acting through a displacement $\overline{D}$ does $\overline{F} \cdot \overline{D}$ unit of work.

⇒ Force $\overline{F}$ may be thought of mathematically as a representation of a linear function that sends any vector $\overline{D}$ into a real number.

⇒ Analogous but more elaborate considerations hold for the cross product.

⇒ Suppose force $\overline{F}$ acts at a point p with position $\overline{X}$. Representing the torque about O as $\overline{X} \times \overline{F}$ leads to the notion of linear operators that sends vectors into vectors.

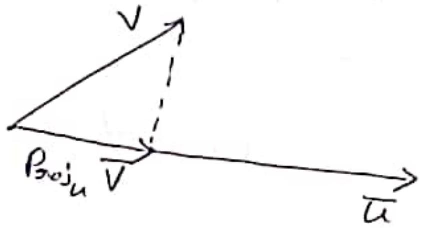
↳ Such operators are called **2nd order tensor.**

⇒ The name tensor comes from elasticity theory where in a loaded elastic body the stress tensor acting on a unit vector normal to a plane through a point delivers the tension acting across the plane at the point.

⇒ Other important 2nd order tensors include the inertia tensor in rigid body dynamics, the strain tensor in elasticity and the momentum-flux tensor in fluid dynamics.

Projection tensor {2nd order tensor}

$$\mathrm{Proj}_u \bar{V} = (\bar{V}\cdot\hat{u})\hat{u}$$



⇒ Left side may be interpreted as the action of the operator Proj_u on vector V.

⇒ <u>To qualify for the title of tensor, Proj_u must be linear:</u>

$$\mathrm{Proj}_u(\beta\bar{V} + \gamma\bar{\omega}) = \beta\,\mathrm{Proj}_u\bar{V} + \gamma\,\mathrm{Proj}_u\bar{\omega}$$

⇒ The direct product $\bar{u}\bar{v}$ of two vectors $\bar{u}$ and $\bar{v}$ is a tensor that sends any vector $\bar{\omega}$ into a new vector according to the rule:

$$\bar{u}\bar{v}(\bar{\omega}) = \bar{u}(\bar{v}\cdot\bar{\omega})$$

⇒ Thus in particular,

$$\mathrm{Proj}_u = \hat{u}\hat{u}$$

⇒ Tensors such as Proj_u, that can be represented as direct products, are called <u>dyads</u>.

↳ We shall see that any 2nd order tensor can be represented as a linear combination of dyads.

<u>Note</u>

Many authors denotes the direct product $\bar{u}\bar{v}$ by $\bar{u}\otimes\bar{v}$.

★ Definations

Two 2nd order tensor S and T are said to be equal if their action on all vectors V is the same.

$$S = T \iff Sv = Tv \quad \forall v$$

⇒ The zero and identity (unit) tensor are denoted and defined, respectively by

→ $0V = 0 \quad \forall V$ {zero tensor}

→ $1V = V$ {Identity tensor}

⇒ The <u>transpose of 2nd order tensor T</u> is defined as the unique 2nd order tensor T^T such that

$$u \cdot Tv = v \cdot T^T u \quad \forall u, v$$

⇒ A 2nd order tensor T is said to be

(a) Symmetric if $T = T^T$

(b) Skew if $T = -T^T$.

(c) Singular if there exist a $V \neq 0$ such that $Tv = 0$.

⇒ An arbitrary tensor T may always be decomposed into the sum of a Symmetric & Skew tensor as follows:

$$T = \underbrace{\frac{1}{2}(T + T^T)}_{\text{Symmetric}} + \underbrace{\frac{1}{2}(T - T^T)}_{\text{Skew}}$$

★ The Cartesian Components of a Second Order Tensor

⇒ Cartesian Component of a second order tensor T fall out almost automatically when we apply T to any Vector V expressed in terms of its Cartesian Components.

$$TV = T(V_x e_x + V_y e_y + V_z e_z)$$
$$= V_x Te_x + V_y Te_y + V_z Te_z$$

But Te_x Te_y & Te_z are Vectors and therefore may be expressed in terms of their Cartesian Components, which we label as follows:-

$$Te_x = T_{xx} e_x + T_{yx} e_y + T_{zx} e_z$$
$$Te_y = T_{xy} e_x + T_{yy} e_y + T_{zy} e_z$$
$$Te_z = T_{xz} e_x + T_{yz} e_y + T_{zz} e_z$$

⇒ The 9 coefficients T_{xx} T_{xy} .... T_{zz} are the Cartesian Components of T.

⇒ We indicate this by writing $\bar{T} \sim T$ where T^T is the matrix of coefficients appearing above.

$$\bar{T} \sim T = \begin{bmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{bmatrix}$$

$$T_{xx} = e_x \cdot Te_x \;,\; T_{xy} = \cancel{e_x} \cdot Te_y \quad etc\ldots$$

★ The Cartesian Basis for Second Order Tensor

Let us consider tensor in 2-dimension

$$Tv = V_x (T_{xx} e_x + T_{yx} e_y) + V_y (T_{xy} e_x + T_{yy} e_y)$$

But $V_x e_x = (\bar{V} \cdot e_x) e_x = e_x e_x (V)$

So $Tv = (T_{xx} e_x e_x + T_{xy} e_x e_y + T_{yx} e_y e_x$
$+ T_{yy} e_y e_y) V$

$$\Rightarrow \boxed{T = T_{xx} e_x e_x + T_{xy} e_x e_y + T_{yx} e_y e_x + T_{yy} e_y e_y}$$

⇒ Thus in 3D any 2nd order tensor can be represented as a unique linear combination of 9 dyads.