Lecture 13

Linear dynamical system with input & Output

★ **Inputs and Outputs**

⇒ Continuous-time LDS has form

$$\boxed{\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}}$$

→ drift term

→ input term

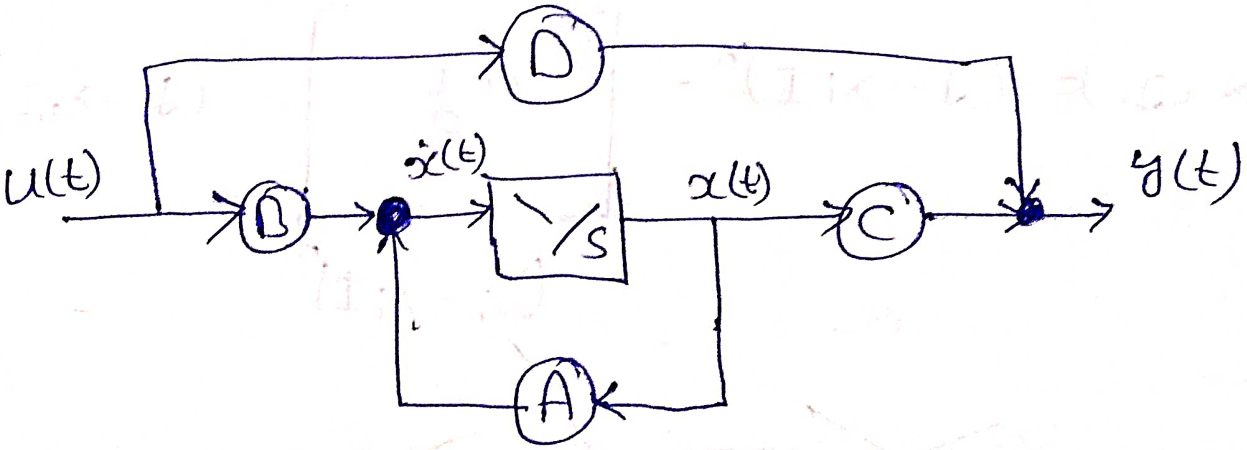
★ **Interpretations**

$\dot{x} = Ax + b_1 u_1 + \cdots + b_m u_m$, where $B = [b_1, \cdots b_m]$

⇒ State derivative is sum of autonomous term Ax and one term per input $(b_i u_i)$

⇒ Each input u_i gives another degree of freedom for $\dot{x}$ (assuming columns of B independent)

★ **Block diagram**



* Transfer matrix

⇒ Let take Laplace transform of $\dot{x} = Ax + Bu$

$$sX(s) - x(0) = AX(s) + BU(s)$$

$$\Rightarrow X(s) = (sI - A)^{-1}x(0) + (sI - A)^{-1}BU(s)$$

So,

$$x(t) = \underbrace{e^{tA}x(0)}_{\text{unforced or autonomous response}} + \int_0^t e^{(t-\tau)A}Bu(\tau)\,d\tau$$

⇒ $e^{tA}B$ ⇒ Called input to state impulse matrix

$(sI - A)^{-1}B$ ⇒ Called input to state transfer matrix.

$$\Rightarrow Y(s) = C(sI - A)^{-1}x(0) + \left(C(sI - A)^{-1}B + D\right)U(s)$$

$$\Rightarrow y(t) = \underbrace{Ce^{tA}x(0)}_{\text{Due to initial condition}} + \int_0^t Ce^{(t-\tau)A}Bu(\tau)\,d\tau + Du(t)$$

⇒ $H(s) = C(sI - A)^{-1}B + D$ is called transfer matrix

⇒ $h(t) = Ce^{tA}B + D\delta(t)$ is called impulse matrix or impulse response. (δ is Dirac delta function)

⇒ With zero initial condition we have:

$$Y(s) = H(s)\,U(s) \qquad y = h * u$$

where * is convolution.

→ H_{ij} is transfer function from input u_j to output y_i.

★ <u>Impulse matrix</u>

⇒ Impulse matrix $h(t) = Ce^{tA}B + D\delta(t)$

⇒ With $x(0) = 0$ $\quad y = h * u$

$$y_i(t) = \sum_{j=1}^{m} \int_0^t h_{ij}(t-\tau)\, u_j(\tau)\, d\tau$$

★ <u>Step matrix</u>

⇒ The step matrix or <u>step response matrix</u> is given by

$$s(t) = \int_0^t h(\tau)\, d\tau$$

⇒ For invertible A, we have

$$s(t) = CA^{-1}(e^{tA} - I)B + D$$

★ DC on static gain matrix

⇒ transfer matrix at $s=0$ is

$$H(0) = -CA^{-1}B + D \in \mathbb{R}^{m \times p}$$

⇒ DC transfer matrix describes system under static conditions i.e. x, u, y constant:

$$0 = \dot{x} = Ax + Bu$$
$$y = Cx + Du$$

eliminate x to get $y = H(0)u$

{ If A is not invertable, then it means there are inputs for which you cannot solve this equation }

⇒ If system is stable

$$H(0) = \int_0^\infty h(t)\,dt = \lim_{t \to \infty} s(t)$$

⇒ if $u(t) \to u_\infty \in \mathbb{R}^m$ then $y(t) \to y_\infty \in \mathbb{R}^p$

where $\boxed{y_\infty = H(0)u_\infty}$

★ Discritization with piecewise constant Input.

linear System

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

⇒ Suppose $u_d : Z_+ \to R^m$ is a sequence and

$$\underbrace{u(t) = u_d(K)}_{\text{also called Zero order hold}} \quad \forall\; Kh < t < (K+1)h$$

$K = 0, 1 \cdots$

$$\underbrace{x_d(K) = x(Kh) \quad y_d(K) = y(Kh)}_{\text{Sampling}}$$

⇒ $h > 0$ is called the saple interval ($\forall\, x$ & y) or update interval (for u)

$$x_d(K+1) = x((K+1)h)$$

$$= e^{hA}x(Kh) + \int_0^h e^{\tau A} B u((K+1)h - \tau)\, d\tau$$

$$= e^{hA}x_d(K) + \left(\int_0^h e^{\tau A}\, d\tau\right) B u_d(K)$$

⇒ x_d, u_d and y_d satisfy discrite-time LDS eq:

$$\boxed{\begin{aligned} x_d(K+1) &= A_d x_d(K) + B_d u_d(K) \\ y_d(K) &= C_d x_d(K) + D_d u_d(K) \end{aligned}}$$

{This is also called discritized system}

Where,

$$A_d = e^{hA} \qquad B_d = \left(\int_0^h e^{\tau A} d\tau \right) B$$

$$C_d = C \qquad D_d = D$$

$\Rightarrow$ If A is invertable, we can express integral as,

$$\int_0^h e^{\tau A} d\tau = A^{-1} (e^{hA} - I)$$

Stability: If eigenvalues of A are $\lambda_1, \ldots \lambda_n$ then eigenvalues of A_d are $e^{h\lambda_1}, \ldots, e^{h\lambda_2}$

$\Rightarrow$ discritization preserves stability properties since

$$R\lambda_i < 0 \Leftrightarrow |e^{h\lambda_i}| < 1$$

$$\forall h > 0$$

★ Causality

$\Rightarrow$ Current state $(x(t))$ and output $(y(t))$ depends on past input $(u(\tau))$ $\forall \tau \leq t$.

★ Idea of State

- future output depends only on current state and future input.
- State summarizes effect of past input.

★ Change of Coordinates

⇒ Start with LDS $\dot{x} = Ax + Bu$, $y = Cx + D$

⇒ Change coordinates in R^n to $\tilde{x}$ with $x = T\tilde{x}$

then,

$$\dot{\tilde{x}} = T^{-1}\dot{x} = T^{-1}(Ax + Bu) = T^{-1}AT\tilde{x} + T^{-1}Bu$$

⇒ hence LDS can be expressed as

$$\dot{x} = \tilde{A}\tilde{x} + \tilde{B}u$$
$$y = \tilde{C}\tilde{x} + \tilde{D}u$$

⇒ where,

$$\tilde{A} = T^{-1}AT \;\Big|\; \tilde{B} = T^{-1}B \;\Big|\; \tilde{C} = CT \;\Big|\; \tilde{D} = D$$

⇒ TF is same (since u, y aren't affected)

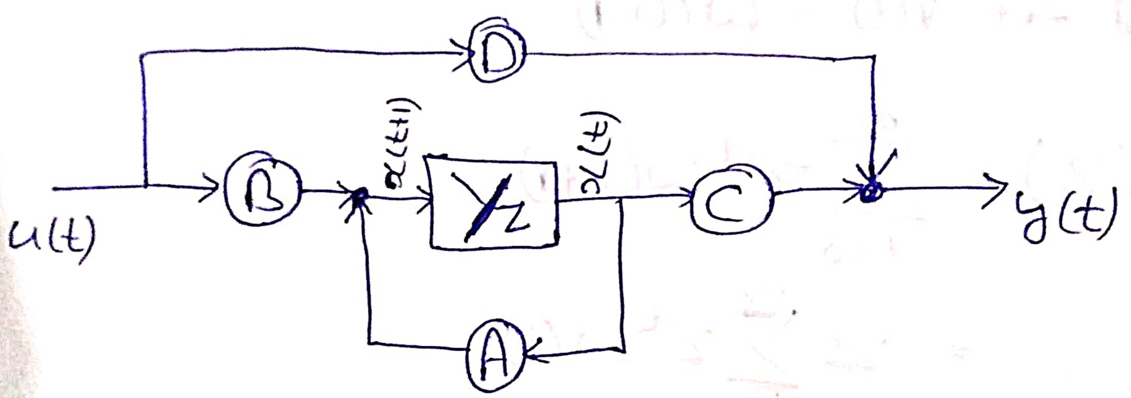
$$\tilde{C}(sI - \tilde{A})^{-1}\tilde{B} + \tilde{D} = C(sI - A)^{-1}B + D$$

★ Standard forms of LDS

⇒ Can change coordinates to put A in various forms (diagonal, real modal, Jordan etc...)

★ Discrete-time system

$$x(t+1) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$



⇒ z^{-1} block is unit delayer.

$$\boxed{x(t) = A^t x(0) + \sum_{\tau=0}^{t-1} A^{(t-1-\tau)} Bu(\tau)}$$

hence,

$$y(t) = CA^t x(0) + h * u$$

where * is discrete time convolution.

$$h(t) = \begin{cases} D & t=0 \\ CA^{t-1}B & t>0 \end{cases}$$

is the impulse response

★ Z transform

⇒ Suppose $w \in R^{p \times q}$ is a sequence (discrete-time signal)

$$W(z) = \sum_{t=0}^{\infty} z^{-t} w(t)$$

⇒ time-advance signal $w(t+1)$ · z-transform:

Let $v(t) = w(t+1)$

$$V(z) = \sum_{t=0}^{\infty} z^{-t} w(t+1)$$

$$= z \sum_{t=1}^{\infty} z^{-t} w(t)$$

$$\boxed{V(z) = zW(z) - zw(0)}$$

★ Discrete-time transfer function

⇒ take Z-transform of system equations:

$$\Rightarrow x(t+1) = Ax(t) + Bu(t)$$

$$\Rightarrow zX(z) - zx(0) = AX(z) + BU(z),$$

$$\Rightarrow \boxed{X(z) = (zI-A)^{-1} zx(0) + (zI-A)^{-1} BU(z)}$$

$$\Rightarrow \boxed{Y(z) = H(z)U(z) + C(zI-A)^{-1} zx(0)}$$

where, $H(z) = C(zI-A)^{-1}B + D$

$(zI-A)^{-1} = z^{-1}I + z^{-2}A + z^{-3}A^2 + \cdots$