Discrete Random Variable III

* Conditional PMF

$$P_{X|A}(x|A) = P(X=x|A)$$

Let $A = \{Y=y\}$

$$P_{X|Y}(x|y) = P(X=x|Y=y)$$

$$= \frac{P(X=x, Y=y)}{P(Y=y)}$$

$$\boxed{P_{X|Y}(x|y) = \frac{P_{X,Y}(x,y)}{P_Y(y)}}$$

* Conditional Expectation

$$E[X] = \sum_x x P_X(x) \qquad E[X|A] = \sum_x x P_{X|A}(x)$$

Let $A = \{Y=y\}$

$$\boxed{E[X|Y=y] = \sum_x x P_{X|Y}(x|y)}$$

$$P_X(x) = P(A_1) P_{X|A_1}(x) + \cdots + P(A_n) P_{X|A_n}(x)$$

$Y = \{y_1, \cdots y_n\}$ Let $A_i = \{y = y_i\}$

$$\boxed{P_X(x) = \sum_y P_Y(y)\, P_{X|Y}(x|y)}$$

$$E[X] = P(A_1)\, E[X|A_1] + \dots + P(A_n)\, E[X|A_n]$$

$$\boxed{E[X] = \sum_y P_Y(y)\, E[X|Y=y]}$$

★ **Independence**

⇒ Two events are independent ⇒ $P(A \cap B) = P(A) \cdot P(B)$

⇒ A R.V and an event

$$P(X=x \text{ and } A) = P(X=x) \cdot P(A) \quad \forall x$$

$$\downarrow$$

$$P_{X|A}(x) = P_X(x) \quad \forall x$$

⇒ Two Random Variable

$$P(X=x \text{ and } Y=y) = P(X=x) \cdot P(Y=y) \quad \forall x,y$$

$$\downarrow$$

$$\boxed{P_{XY}(x,y) = P_X(x)\, P_Y(y)} \quad \forall x,y$$

$$\downarrow$$

$$P_{X|Y}(x|y) = P_X(x) \quad \forall x,y$$

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★ Independence and Expectations

⇒ If X, Y are independent

$$\boxed{E[XY] = E[X]\,E[Y]}$$

$E[g(X,Y)]$ where $g(x,y) = xy$

$$\Rightarrow \sum_x \sum_y xy\, P_{XY}(x,y)$$

$$\Rightarrow \sum_x \sum_y xy\, P_X(x)\, P_Y(y)$$

$$\Rightarrow \sum_x x P_X(x) \sum_y y P_Y(y)$$

$$\Rightarrow E[X]\,E[Y]$$

//

⇒ If X, Y are independent then $g(X)$ and $h(Y)$ are also independent.

★ Independence and Variance

⇒ If X, Y are independent

$$Var(X+Y) = Var(X) + Var(Y)$$

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★ 7_ Suppliments

★ The inclusion-exclusion formula

⇒ Objective: $P(A_1 \cup A_2 \cup A_3)$

⇒ Let X_i be an indicator random variable of event A_i:

$$\Rightarrow \quad X_i = \begin{cases} 1 & \text{if } A_i \text{ occurs} \\ 0 & \text{otherwise} \end{cases}$$

Event	Indicator variable
A_i	X_i
A_i^c	$1-X_i$
$A_i \cap A_j$	$X_i X_j$
$A_i^c \cap A_j^c$	$(1-X_i)(1-X_j)$
$A_i \cup A_j = (A_i^c \cap A_j^c)^c$ {de morgans Law}	$1-(1-X_i)(1-X_j)$

$P(A_i) = E[X_i]$ {Already Proved}

$$P(A_1 \cup A_2 \cup A_3) = E[1-(1-X_1)(1-X_2)(1-X_3)]$$

$$= E[X_1 + X_2 + X_3 - X_1X_2 - X_2X_3 - X_1X_3 + X_1X_2X_3]$$

$$\Rightarrow E[X_1] + E[X_2] + E[X_3] - E[X_1X_2] - E[X_2X_3]$$
$$- E[X_1X_3] + E[X_1X_2X_3]$$

$$\Rightarrow P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_2 \cap A_3) - P(A_3 \cap A_1)$$
$$+ P(A_1 \cap A_2 \cap A_3)$$

★ <u>The variance of the geometric PMF</u>

$$\mathrm{Var}(X) = E[X^2] - (\underbrace{E[X]}_{1/p})^2$$

$$E[X^2] = \underbrace{P(X=1)}_{p}\,\underbrace{E[X^2 \mid X=1]}_{1}$$
$$+$$
$$\underbrace{P(X>1)}_{(1-p)}\,E[X^2 \mid X>1]$$

$$E[X^2 \mid X>1] \rightarrow E[(X+1)^2] \rightarrow E[X^2] + 2\underbrace{E[X]}_{1/p} + 1$$

$$\Rightarrow E[X^2] = p + (1-p)\left(E[X^2] + \frac{2}{p} + 1\right)$$

$$\Rightarrow E[X^2] = \frac{2}{p^2} - \frac{1}{p}$$

$$\Rightarrow Var[X] = \frac{2}{p^2} - \frac{1}{p} - \frac{1}{p^2} = \frac{1-p}{p^2}$$

$$\boxed{Var(X) = \frac{1-p}{p^2}}$$