Lecture 12

Jordan canonical form

⇒ Jordan Canonical form is essentially the closest you can get to diagonalization when you cant diagonalize the matrix

⇒ Any matrix $A \in \mathbb{R}^{n \times n}$ can be put in jordan canonical form by a similarity transform.

$$T^{-1}AT = J = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_q \end{bmatrix}$$

$$\boxed{n = \sum_{i=1}^{q} n_i}$$

Where,

$$J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix} \in \mathbb{C}^{n_i \times n_i}$$

(Jordan block of size n_i with eigenvalue λ_i)

⇒ J is upper bidiagonal.

⇒ J diagonal is the special case of n Jordan block of size $n_i = 1$.

⇒ Jordan form is unique (up to permutations of the block)

⇒ Can have multiple blocks with same eigenvalue.

Note: JCF is a conceptual tool, never used in numerical computations.

⇒ Characteristic polynomial under similarity transform doesn't change.

$$\chi(s) = \det(sI - A) = (s-\lambda_1)^{n_1} + \cdots + (s-\lambda_q)^{n_q}$$

⇒ $\dim N(\lambda I - A)$ is the number of Jordan blocks with eigen value λ.

★ <u>Generalized eigenvectors</u>

⇒ $T^{-1}AT = J = \text{diag}(J_1, \cdots J_q)$

$\left\{ \text{Columns of } T \text{ are called generalized eigen vectors} \right\}$

⇒ Express T as

$$T = [T_1\, T_2, \cdots T_q]$$

⇒ where $T_i \in \mathbb{C}^{n \times n_i}$ are the columns of T associated with i^{th} Jordan block J_i.

⇒ $AT_i = T_i J_i$

⇒ Let $T_i = [v_{i1}\; v_{i2} \cdots v_{in_i}]$

⇒ then we have

$$A v_{i1} = \lambda_i v_{i1}$$

$\forall\, j = 2, \dots, n_i$

$$A v_{ij} = v_{ij-1} + \lambda_i v_{ij}$$

⇒ The vectors $v_{i1}, \dots v_{in_i}$ are sometimes called generalized eigen vectors.

★ Jordan form LDS

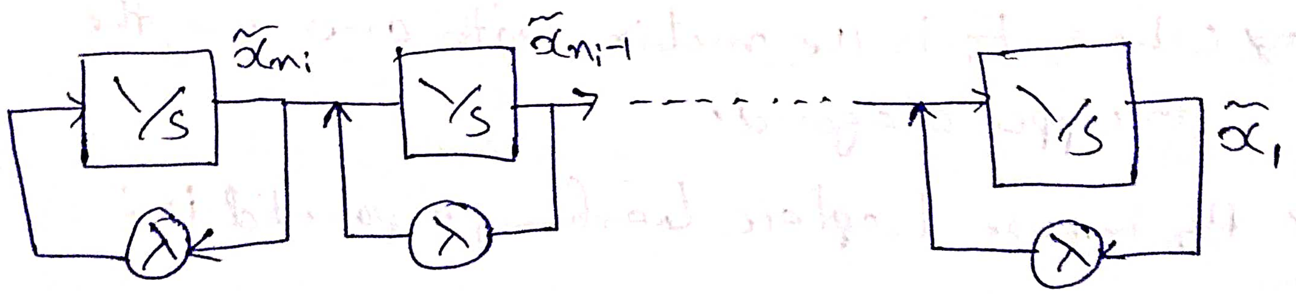
⇒ Let $\dot{x} = Ax$

⇒ by change of coordinate $x = T\tilde{x}$

$$\boxed{\dot{\tilde{x}} = J\tilde{x}}$$

⇒ System is decomposed into Independent "Jordan block Systems"

$$\dot{\tilde{x}}_i = J_i \tilde{x}_i$$



★ Resolvent, exponential of Jordan block

⇒ Resolvent of $K\times K$ Jordan block with eigenvalue λ:

$$(sI - J_\lambda)^{-1} = \begin{bmatrix} s-\lambda & -1 & & \\ & s-\lambda & & \\ & & \ddots & -1 \\ & & & s-\lambda \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} (s-\lambda)^{-1} & (s-\lambda)^{-2} & \cdots & (s-\lambda)^{-K} \\ & (s-\lambda)^{-1} & & (s-\lambda)^{-K+1} \\ & & \ddots & \vdots \\ & & & (s-\lambda)^{-1} \end{bmatrix}$$

$$= (s-\lambda)^{-1} I + (s-\lambda)^{-2} F_1 + \cdots + (s-\lambda)^{-K} F_{K-1}$$

⇒ where, F_i is the matrix with ones on the ith upper diagonal.

⇒ By Inverse Laplace transform, exponential is:

$$e^{tJ_\lambda} = e^{t\lambda}\left(I + tF_1 + \cdots + \frac{t^{K-1}}{(K-1)!} F_{K-1}\right)$$

★ Generalized modes

⇒ Consider $\dot{x} = Ax$, with

$$x(0) = a_1 v_{i1} + \cdots + a_{n_i} v_{in_i} = T_i a$$

⇒ then, $x(t) = T e^{Jt} \tilde{x}(0) = T_i e^{J_i t} a$

⇒ trajectory stays in span of generalized eigenvectors.

⇒ Coefficients have form $p(t) e^{\lambda t}$, where p is polynomial.

⇒ Such solutions are called generalized modes of the system.

⇒ with general $x(0)$ we can write

$$x(t) = e^{tA} x(0) = T e^{tJ} T^{-1} x(0) = \sum_{i=1}^{q} T_i e^{tJ_i} \left(s_i^T x(0) \right)$$

where,

$$T^{-1} = \begin{bmatrix} s_1^T \\ \vdots \\ s_q^T \end{bmatrix}$$

⇒ hence all solutions of $\dot{x} = Ax$ are linear combinations of (generalized) modes.

* Cayley-Hamilton Theorem

"For any $A \in \mathbb{R}^{n \times n}$ we have $\chi(A) = 0$, where $\chi(s) = \det(sI - A)$"

Corollary: for every $p \in \mathbb{Z}_+$ we have

$$A^p \in \text{span}\{I, A, A^2, \dots, A^{n-1}\}$$

$\Rightarrow$ Let $f(u) = \alpha_0 + \alpha_1 u + \alpha_2 u^2 + \dots$

$$f(A) = \alpha_0 I + \alpha_1 A + \alpha_2 A^2 + \dots \quad \forall A \in \mathbb{R}^{n \times n}$$

{overloading function f for matrix}

$\Rightarrow$ Using the above corollary, any $f(A)$ can be expressed as

$$\boxed{f(A) = \gamma_0 I + \gamma_1 A + \dots + \gamma_{n-1} A^{n-1}}$$

$$\Rightarrow \chi(A) = A^n + a_{n-1} A^{n-1} + \dots + a_0 I = 0$$

$$I = A\left(\left(\frac{-a_1}{a_0}\right) I + \left(\frac{-a_2}{a_0} A\right) + \dots + \left(\frac{-1}{a_0} A^{n-1}\right)\right)$$

$$A^{-1} = -\frac{a_1}{a_0} I - \frac{a_2}{a_0} A \dots - \frac{1}{a_0} A^{n-1}$$

$\Rightarrow$ A is invertible $\Leftrightarrow a_0 \neq 0$

★ <u>Proof of Cayley-Hamilton Theorem</u>

$\Rightarrow$ first assume A is diagonalizable $T^{-1}AT = \Lambda$

$$\chi(s) = (s-\lambda_1)(s-\lambda_2)\cdots(s-\lambda_n)$$

$$\Rightarrow \chi(A) = \chi(T\Lambda T^{-1}) = T\,\chi(\Lambda)\,T^{-1}$$

$\Rightarrow$ it suffices to show $\chi(\Lambda) = 0$

$$\chi(\Lambda) = (\Lambda - \lambda_1 I)\cdots(\Lambda - \lambda_n I)$$

$$= \mathrm{diag}(0, \lambda_2-\lambda_1, \cdots \lambda_n - \lambda_1)\cdots\mathrm{diag}(\lambda_1 - \lambda_n, \cdots, 0)$$

$$= 0$$

$\Rightarrow$ Now let's do general case $T^{-1}AT = J$

$$\chi(s) = (s-\lambda_1)^{n_1}\cdots(s-\lambda_q)^{n_q}$$

Suffices to $\chi(J_i) = 0$

$$\chi(J_i) = (J-\lambda_i I)^{n}\cdots\underbrace{\begin{bmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & \ddots \\ & & & \ddots \end{bmatrix}^{n_i}}_{(J_i - \lambda_i I)^{n}}\cdots(J_i - \lambda_q I)^{n_q} = 0$$