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General Bases and Tensor Notation

⇒ While the laws of mechanics can be written in coordinate-free form, they can be solved, in most cases, only if expressed in component form.

⇒ An aim of tensor analysis is to embrace arbitrary coordinate systems and their associated bases, yet to produce formulas for computing invariants, such as the dot product, that are as simple as the Cartesian forms.

★ General Bases

Let $\{g_1, g_2, g_3\}$ denotes any fixed set of non-coplanar vectors. then any vector v may be represented uniquely as:

$$v = v^1 g_1 + v_2 g_2 + v^3 g_3 = \sum_1^3 v^i g_i$$

⇒ The set $\{g_1, g_2, g_3\}$ is called basis and its elements base vectors.

↳ Base vectors need not be unit length nor mutually ⊥

★ Jacobian of a Basis is nonzero

⇒ If $G = [g_1\ g_2 \cdots]$ denotes the $n \times n$ matrix whose columns are the Cartesian components of $g_1, g_2 \cdots$, then $\{g_1, g_2 \cdots\}$ is a basis if & only if $\det G \neq 0$.

⇒ Using almost standard terminology, we shall call G the Jacobian matrix of $\{g_1, g_2 \dots\}$ and $J = \det G$ the Jacobian of $\{g_1, g_2 \dots\}$.

★ The Summation Convention

{The Summation Convention, invented by Einstein, gives tensor analysis much of its appeal.}

$$V = v^1 g_1 + v^2 g_2 + v^3 g_3 = \sum_{i=1}^{3} v^i g_i$$

⇒ Without any loss of information we may drop the summation symbol & write simply,

$$V = v^i g_i$$

Note

The Summation Convention applies only when one dummy index is "on the roof" and the other is "in the cellar".

eg :: $v^i V_i = v^1 V_1 + v^2 V_2 + v^3 V_3$

but $v^i v^i = v^1 v^1$ or $v^2 v^2$ or $v^3 v^3$

{Cartesian tensor notation, is the one exception to this}

⋆ Computing the Dot Product in a General Basis

Suppose we wish to compute the dot product of a vector $u = u^i g_i$ with a vector $v = v^i g_i$.

$$u \cdot v = u^i v^j (g_i \cdot g_j)$$

⇒ This extended expression of $u \cdot v$ is a nine-term mess. We can clean it up by introducing a set of reciprocal base vectors.

⋆ Reciprocal Base Vector

Let $u = u^1 g_1 + u^2 g_2$ {In given basis $\{g_1 g_2\}$}

$$\cancel{v = v^1 g_1 + v^2 g_2}$$

$$\cancel{u \cdot v = (u^1 g_1 + u^2 g_2)}$$

Let $v = v_1 g^1 + v_2 g^2$ {In new basis $\{g^1 g^2\}$}

So, $u \cdot v = (u^1 g_1 + u^2 g_2) \cdot (v_1 g^1 + v_2 g^2)$

$$= u^1 v_1 g_1 g^1 + u^1 v_2 g_1 g^2 + u^2 v_1 g_2 g^1 + u^2 v_2 g_2 g^2$$

⇒ The idea here is not to choose g^1 & g^2 so that the above expression reduces to

$$u \cdot v = u^1 v_1 + u^2 v_2$$

⇒ Let $\{g_1 g_2 \dots\} = \{g_j\}$ be a basis. Then $\det G \neq 0$

⇒ This implies that G^{-1} exist.

⇒ The element in the i^{th} row of G^{-1} may be regarded as the Cartesian components of a vector g^i.

$$G^{-1} = \begin{bmatrix} g^1 \\ g^2 \\ \vdots \end{bmatrix} = [g^1\, g^2 \dots]^T = [g^i]$$

⇒ Consistent with this notation we may say $G \equiv [g_j]$ when we wish to regard G as a collection of collum vectors.

⇒ $G^{-1}G = I$ is equivalent to the statement

$$g^i g_j = \delta^i_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

⇒ The symbol δ_j^i is called Kronecker delta.

⇒ The set $\{g^1\, g^2 \dots\} \equiv \{g^i\}$ is called a reciprocal basis and its elements reciprocal base vectors.

★ <u>The roof (Contravariant) and Cellar (Covariant) Components of a Vector</u>

If $\{g_i\}$ is a basis then not only may we express any vector V as $V^i g_i$, we may also represent V as a linear combination of the reciprocal base vectors, thus

$$V = V_i g^i$$

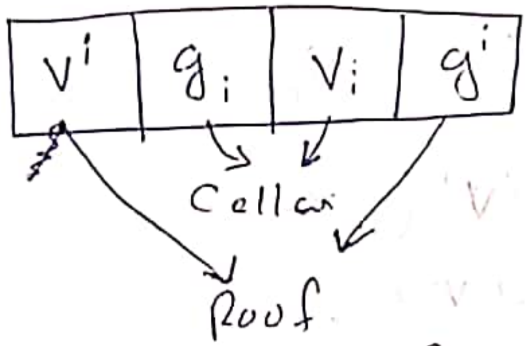
⇒ Backing with the tradition, we shall call the coefficients V^i the roof components of V and the g_i the cellar base vectors.

⇒ Like V_i should be called the Cellar Component and g^i the roof base vectors

⇒ The Conventional names of V^i and V_i are:

- V^i ⇒ Contravariant Component of V
- V_i ⇒ Covariant Component of V

⇒ A^i_j is sometimes used to denote the element of a matrix A that sits in the i^{th} row & j^{th} Column.



★ <u>Simplification of the Component form of the Dot product in a general basis</u>

Let us set $U = U^i g_i$ & $V = V_j g^j$

Then $U \cdot V = U^i V_j g_i \cdot g^j = U^i V_j \delta_i^j$

$= U^1 V_1 + U^2 V_2 + U^3 V_3$

★ Computing the Cross Product in a General Basis

Some ~~#~~ time it is convenient to denote the roof and cellar components of a vector V by $(V)^i$ and $(V)_i$ respectively.

$$u \times v = (u \times v)_K \, g^K$$

⇒ To compute the cellar component of $u \times v$ we set $u = u^i g_i$ & $v = v^j g_j$

So $$(u \times v)_K = (u \times v) \cdot g_K$$

$$= u^i v^j \, (g_i \times g_j) \cdot g_K$$

$$= u^i v^j \, \epsilon_{ijK}$$

⇒ The $3 \times 3 \times 3 = 27$ symbols ϵ_{ijK} are called cellar components of the permutation tensor P.

$$\epsilon_{ijK} = \begin{cases} +J & \text{if } (i,j,K) \text{ is an even permutation of } (1,2,3) \\ -J & \text{if } (ijK) \text{ is an odd permutation of } (1,2,3) \\ 0 & \text{if two or more indices are equal.} \end{cases}$$

Thus, $$u \times v = \epsilon_{ijK} u^i v^j g^K$$

⇒ To compute the roof components of $u \times v$, we mimic the above procedure, but use roof instead of cellar base vector.

$$(u \times v)^K = \epsilon^{ijk} u_i v_j$$

⇒ ϵ^{ijk} are the 27 roof components of the permutation tensor defined as follows:-

$$\epsilon^{ijk} = (g^i \times g^j) \cdot g^K = \begin{cases} +J^{-1} & \text{if } (ijK) \text{ is an even permutation of } (1,2,3) \\ -J^{-1} & \text{if } (ijK) \text{ is an odd permutation of } (1,2,3) \\ 0 & \text{if two or more indices are equal.} \end{cases}$$

$$\boxed{\epsilon_{ijK} = J^2 \epsilon^{ijk}}$$

$$\epsilon^{ijK} \epsilon_{pqr} = \begin{vmatrix} \delta^i_p & \delta^i_q & \delta^i_r \\ \delta^j_p & \delta^j_q & \delta^j_r \\ \delta^K_p & \delta^K_q & \delta^K_r \end{vmatrix}$$

If we set $r = K$ then,

$$\epsilon^{ijK} \epsilon_{pqK} = \delta^i_p \delta^j_q - \delta^i_q \delta^j_p$$

★ <u>A second order tensor has four sets of components in general.</u>

Consider a second order tensor T and in general basis $\{g_j\}$, the action of T on each of the basis vector is known, said

$$T g_j = T_j$$

Now each vector T_j may be expressed as a linear combination of the given basis vectors or their reciprocals.

⇒ Choosing the latter, we may write

$$T_j = T_{ij}\, g^i$$

⇒ The 9 components T_{ij} are called the Cellar components of T.

$$\boxed{T_{ij} = g_i \cdot T g_j}$$

If v is a arbitrary vector

$$\begin{aligned} Tv &= T(v^j g_j) \\ &= v^j T g_j \\ &= v^j T_{ij}\, g^i \\ &= T_{ij}\, v^j g^i \\ &= T_{ij}\,(v \cdot g^j)\, g^i \\ Tv &= T_{ij}\, g^i g^j (v) \end{aligned}$$

So $\boxed{T = T_{ij}\, g^i g^j}$

We see that $\{g^i g^j\}$ is the basis for the set of all ~~tensors~~ 2nd order tensors.

⇒ Repeating the above line of reasoning, but with the roles of the Cellar and roof base vectors reversed we have :-

$$T g^j = T^j = T^{ij} g_i \Rightarrow T = T^{ij} g_i g_j$$

⇒ The g coefficient of T^{ij} are called the soof components of T.

↳ T^{ij} are the components of T in the basis $\{g_i g_j\}$.

$$T_{ij} = g^i \cdot T g^j$$

⇒ There are two additional sets of components that can be defined, namely

$$T_{\cdot j}^{i} = g^i \cdot T g_j$$

$$T_j^{\cdot i} = g_j \cdot T g^i$$

⇒ These are called mixed components of T.

⇒ The dots are used as distinguishing marks because in general $T_{\cdot j}^{i} \neq T_j^{\cdot i}$

⇒ It is easy to show that T has the following representations in terms of its mixed components

$$T = T_{\cdot j}^{i} g_i g^j = T_j^{\cdot i} g^j g_i .$$

$\left\{ \begin{array}{l} T_{\cdot j}^{i} \text{ are components of } T \text{ in the basis } \{g_i g^j\} \\ \& \ T_j^{\cdot i} \text{ are components of } T \text{ in the basis } \{g^j g_i\} \end{array} \right\}$

⇒ If T is symmetric $\{T = T^T\}$

⇓

$$T_{ij} = T_{ji}$$

$T_{\cdot j}^{i} = T_j^{\cdot i}$ {does not imply $[T_{\cdot j}^{i}]$ & $[T_j^{\cdot i}]$ are symmetric}

* Change of Basis

⇒ Within a given frame, Vectors and tensors are blissfully unaware of the bases we choose to represent them.

↳ That is they are geometric invariants.

⇒ Under a change of basis it is their components that change, not they themselves.

⇒ Let us start by assuming that each element of the new basis is a known linear combination of the elements of old.

$$\tilde{g}_1 = A_1^1 g_1 + A_1^2 g_2 + A_1^3 g_3$$

$$\tilde{g}_2 = A_2^1 g_1 + \cdots \qquad \text{—— (1)}$$

$$\tilde{g}_3 = A_3^1 g_1 + \cdots$$

We may summarize (1) in either matrix or index form as :-

$$\tilde{G} = GA \quad \text{or} \quad \tilde{g}_j = A_j^i g_i$$

$$G = \tilde{G} A^{-1} \quad \text{or} \quad g_j = (A^{-1})_j^i \tilde{g}_i$$

$$\tilde{G}^{-1} = A^{-1} G^{-1} \quad \text{or} \quad \tilde{g}^i = (A^{-1})_j^i g^j$$

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$\Rightarrow$ The relations between the new and old components of any vector V follow immediately because:

$$\tilde{V}_j = \tilde{g}_j \cdot V = A_j^{\,i}\, g_i \cdot V \;\; \& \;\; \tilde{V}^i = \tilde{g}^i \cdot V = (A^{-1})^i_{\,j}\, g^j \cdot V$$

So $\tilde{V}_j = A_j^{\,i}\, V_i \;,\; \tilde{V}^i = (A^{-1})^i_{\,j}\, V^j$

Likewise for any 2nd order tensor T

$$\tilde{T}_{ij} = A_i^{\,k} A_j^{\,p}\, T_{kp} \qquad \tilde{T}^{\,i}_{\cdot j} = (A^{-1})^i_{\,k}\, A_j^{\,p}\, T^{k}_{\cdot p}$$

$$\tilde{T}_j^{\,\cdot i} = A_j^{\,k} (A^{-1})^i_{\,p}\, T_k^{\,\cdot p} \qquad \tilde{T}^{ij} = (A^{-1})^i_{\,k} (A^{-1})^j_{\,p}\, T^{kp}$$