# Lecture 10

## Solution via Laplace transform and matrix exponential

### ★ Laplace transform of matrix valued function

Suppose $z: R_+ \to R^{p \times q}$

Laplace transform: $Z = \mathcal{L}(z)$ where $Z: D \subseteq \mathbb{C} \to \mathbb{C}^{p \times q}$ is defined by

$$\boxed{Z(s) = \int_0^\infty e^{-st} z(t)\, dt}$$

{Integral of matrix is done term by term}

→ D is the domain or region of convergence of Z.

→ D includes atleast $\{s \mid \Re s > a\}$

where $|z_{ij}(t)| \leq a e^{at}$

$\forall\ t \geq 0 \quad i = 1 \dots p$

$j = 1 \dots q$

### ★ Derivative Property

$$\mathcal{L}(\dot{z}) = s Z(s) - z(0)$$

$$\boxed{\mathcal{L}(\dot{z}) = s \mathcal{L}(z) - z(0)}$$

## ★ Laplace transform solution of $\dot{x} = Ax$

⇒ Consider continuous-time time-invariant (TILSD)

$$\dot{x} = Ax$$

$$\forall\ t \geq 0 \quad \text{where } x(t) \in R^n$$

⇒ taking Laplace transform:

$$sX(s) - x(0) = AX(s)$$

$$X(s) = (sI - A)^{-1} x(0)$$

$$\Rightarrow \quad x(t) = \mathcal{L}^{-1}\left((sI-A)^{-1}\right) x(0)$$

## ★ Resolvent and State transition matrix

→ $(sI-A)^{-1}$ is called the resolvent of A.

⇒ resolvent defined for $s \in C$ except eigenvalues of A.

⇒ $\Phi(t) = \mathcal{L}^{-1}\left((sI-A)^{-1}\right)$ is called the <u>state-transition matrix</u>, it maps the initial state to the state at time t.

$$\boxed{x(t) = \Phi(t)\, x(t)}$$

## ★ Characteristic Polynomial

$\mathcal{X}(s) = \det(sI - A)$ is called the characteristic polynomial of A.

- Polynomial of degree $n$, with leading coefficient one (i.e. $s^n$)
- roots of $\mathcal{X}$ are the eigenvalues of A

## ★ Eigenvalues of A and poles of resolvent

⇒ i,j entry of resolvent $(sI-A)^{-1}$ can be expressed via Cramer's rule as

$$(-1)^{i+j}\frac{\det \Delta_{ij}}{\det(sI-A)}$$

where $\Delta_{ij}$ is $sI-A$ with $j^{th}$ row & $i^{th}$ column deleted

## ★ Matrix exponential

$(I-C)^{-1} = I + C + C^2 + \cdots$ (if series converges)

So, $(sI-A)^{-1} = \frac{1}{s}(I - A/s)^{-1} = \frac{I}{s} + \frac{A}{s^2} + \frac{A^2}{s^3} + \cdots$

{valid for $|s|$ large enough}.

$$\Phi(t) = \mathcal{L}^{-1}\left((sI-A)^{-1}\right) = I + tA + \frac{(tA)^2}{2!} + \cdots$$

$$e^{ta} = 1 + ta + \frac{(ta)^2}{2!} + \cdots$$

{looks very similar to previous eq}

⇒ Let define matrix exponential as

$$e^{M} = I + M + \frac{M^2}{2!} + \cdots$$

$$\forall M \in R^{n \times n}$$

⇒ With this definition state-transition matrix is

$$\phi(t) = \mathcal{L}^{-1}\left((sI - A)^{-1}\right) = e^{tA}$$

- $e^{A+B} \neq e^{A} e^{B}$
- $e^{A+B} = e^{A} e^{B}$ if $AB = BA$

Example

$$\forall s, t \in R$$

$$e^{(tA + sA)} = e^{tA} e^{sA}$$

★ Time transfer property

for $\dot{x} = Ax$ we know

$$x(t) = \phi(t)\, x(0) = e^{tA} x(0)$$

⇒ The matrix $e^{tA}$ propogates initial condition into state at time t.

$$x(\tau+t) = e^{tA}x(\tau) \quad \forall \tau \in R$$

(The matrix $e^{tA}$ propagates state t seconds forward in time)

★ **Sampling a Continuous-time System**

⇒ Suppose $\dot{x} = Ax$

⇒ Sample $x$ at times $t_1 \leq t_2 < \cdots$ define $z(K) = x(t_K)$

⇒ then, $z(K+1) = e^{(t_{K+1}-t_K)A} z(K)$

⇒ for uniform sampling $t_{K+1} - t_K = h$ so,

$$\boxed{z(K+1) = e^{hA} z(K)} \quad \{\text{descrite-time LDS}\}$$

★ **Piecewise constant system**

⇒ Consider time-varying LDS $\dot{x} = A(t)x$ with

$$A(t) = \begin{cases} A_0 & 0 \leq t < t_1 \\ A_1 & t_1 \leq t < t_2 \\ \vdots & \end{cases}$$

where $0 < t_1 < t_2 \cdots$

$\forall t \in [t_i, t_{i+1}]$

$$x(t) = \underbrace{e^{(t-t_i)A_i} \cdots e^{(t_3-t_2)A_2} e^{(t_2-t_1)A_1} e^{t_1 A_0}}_{\phi(t,0)} x(0)$$

## * Qualitative behavior of $x(t)$

$\Rightarrow$ Suppose $\dot{x} = Ax$, $x(t) \in R^n$

then, $x(t) = e^{tA} x(0)$

$$X(s) = (sI - A)^{-1} x(0)$$

$\Rightarrow$ ith component of $X_i(s)$ has form

$$X_i(s) = \frac{a_i(s)}{\mathcal{X}(s)}$$

Polynomial of deg < n

Characteristic polynomial (degree n)

$\Rightarrow$ Thus the poles of $X_i(s)$ are all eigenvalues of A

# First assume eigenvalues are distinct, so $X_i(s)$ cannot have repeated poles.

then $x_i(t)$ has form

$$x_i(t) = \sum_{j=1}^{n} \beta_{ij} e^{\lambda_j t}$$

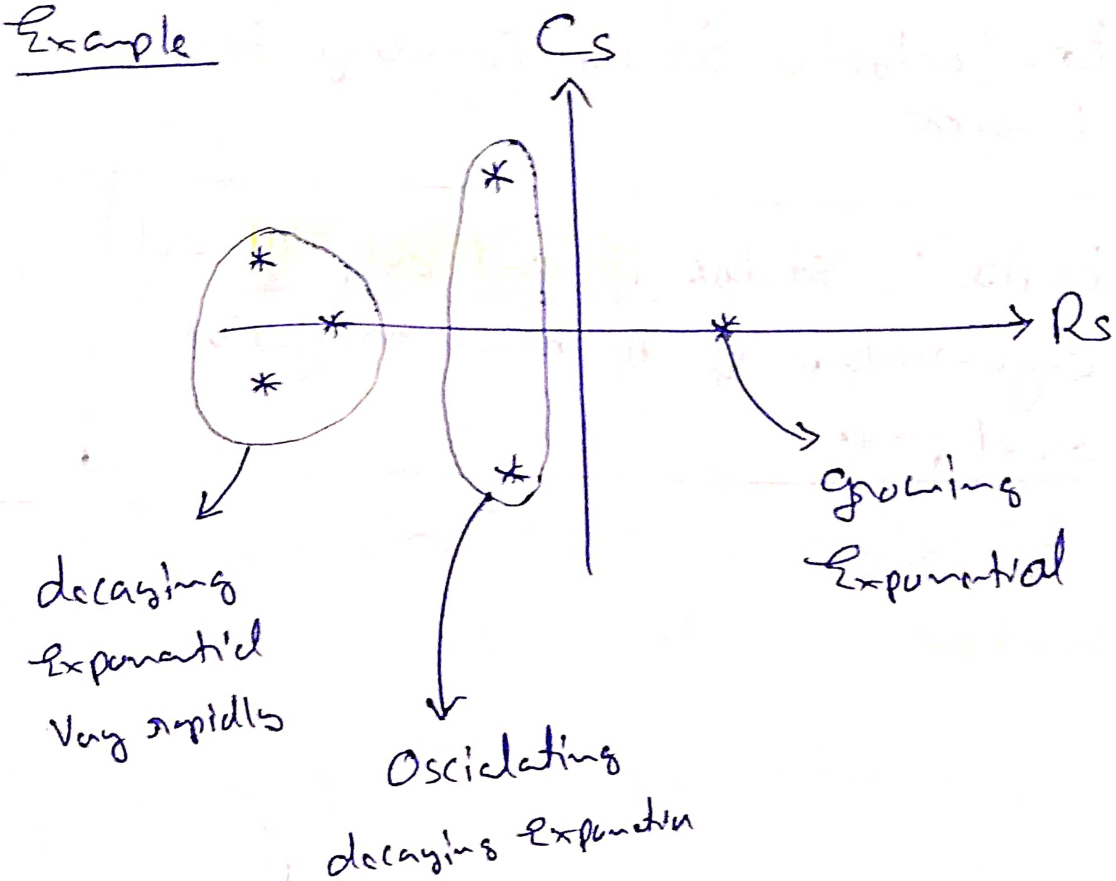
depends of $x(0)$ (linearly)

$\Rightarrow$ Eigenvalues determine (possible) qualitative behavior of $x_i$

① real eigenvalue $\lambda$ corresponds to an exponentially decaying or growing term $e^{\lambda t}$ in solution.

② Complex eigenvalue $\lambda = \sigma + j\omega$ corresponds to decaying or growing sinusoidal term $e^{\sigma t}\cos(\omega t + \varphi)$ in solution.

<u>Example</u>




⇒ Now suppose A has repeated eigenvalues, so $X_i$ can has repeated poles.

⇒ Express eigenvalues as $\lambda_1, \ldots, \lambda_r$ (distinct) with multiplicities $n_1, \ldots, n_r$ respectively $(n_1 + \cdots + n_r = n)$

⇒ then $x_i(t)$ has form

$$x_i(t) = \sum_{j=1}^{r} P_{ij}(t)\, e^{\lambda_j t}$$

↳ Polynomial of deg $< n_j$

## ★ Stability

We say system $\dot{x} = Ax$ is stable if

$e^{tA} \to 0$ as $t \to \infty$

Equivalently

→ All trajectories $\dot{x} = Ax$ converge to 0 as $t \to \infty$

FACT ⇒ $\dot{x} = Ax$ is stable if and only if all eigenvalues of $A$ have negative real parts.