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Discrete Random Variables I

Random Variable

→ It is a numerical quantity whose value is determined by the outcome of a probabilistic experiment.

⇒ Discrete random variable takes value in finite or countable set.
↓
infinit

⇒ A random variable associates a value to every possible outcome.

⇒ **Mathematically:** A function from the sample space Ω to the real number.

⇒ It can take discrete or continuous value.

Notation: random variable X
numerical value x

⇒ We can have several random variable defined on the same sample space.

★ Probability mass function (PMF) of a discrete r.v X

⇒ It is the "probability law" or "probability distribution" of X.

$P(X=x)$ ⇒ Probability of the event that random variable X is equal to x.
or
$P_X(x)$

Properties

① $P_X(x) \geq 0$

② $\sum_x P_X(x) = 1$

★ The Simplest random variable: Bernoulli with parameter $p \in [0,1]$

$$X = \begin{cases} 1 & \text{w.p } p \\ 0 & \text{w.p } 1-p \end{cases}$$

⇒ Models a trial that results in success/failure.

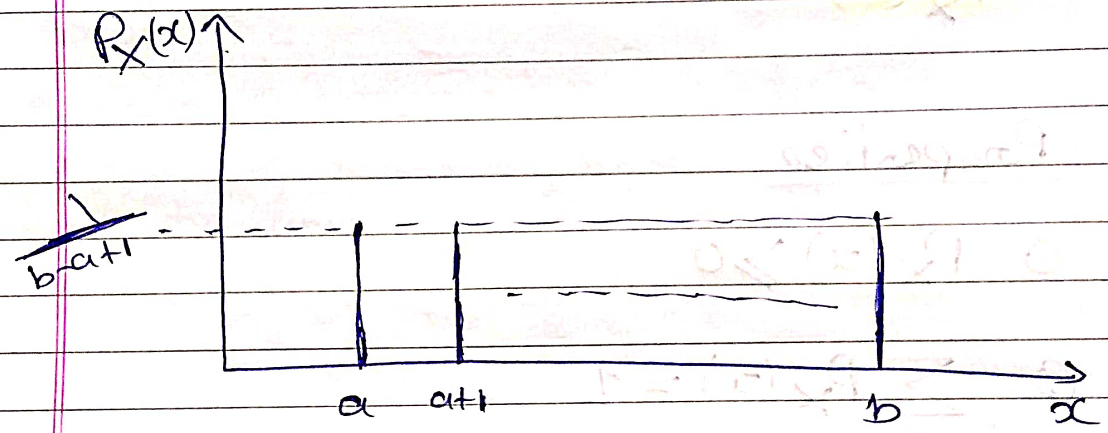
↳ Indicator r.v of an event A:

↳ $I_A = 1$ iff A occurs

↳ $I_A = 0$ if A^c occurs

★ Discrete uniform random variable

- Experiment: Pick one of $a, a+1, \dots b$ at random; all equally likely.
- Sample space: $\{a, a+1, \dots b\}$
- Random Variable X: $X(\omega) = \omega$



★ Binomial random variable

- Experiment: n independent tosses of a coin with $P(\text{Heads}) = p$.
- Sample Space: Set of sequences of H and T of length n
- Random variable X: number of Heads observed
- Model of: number of successes in a given number of independent trials.

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$$\boxed{P_X(k) = \binom{n}{k} p^k (1-p)^{n-k}} \quad \forall\, k = 0, 1 \ldots n$$

★ Geometric random variable

- Experiment: Infinitely many independent tosses of a coin: $P(\text{Heads}) = p$
- Sample space: Set of infinite sequence of H and T
- Random variable: X; number of tosses until the first Heads
- Model of: waiting times; number of trials until a success.

$$\boxed{P_X(k) = (1-p)^{k-1} p} \quad \forall\, k = 1, 2 \ldots$$

★ Expectation / mean of a random variable

$$\boxed{E[X] = \sum_x x P_X(x)}$$

→ Interpretation: Average in large number of independent repetitions of the experiment.

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★ Expectation of a Bernoulli r.v

$$X = \begin{cases} 1, & \text{w.p } p \\ 0, & \text{w.p } 1-p \end{cases}$$

$$E[X] = 1 * p + 0 * (1-p) = p$$

★ Expectation of a Uniform r.v

Uniform on $0, 1, \cdots n$

$$E[X] = 0 * \frac{1}{n+1} + 1 * \frac{1}{n+1} + \cdots n * \frac{1}{n+1}$$

$$= \frac{1}{n+1} (0 + 1 + \cdots n)$$

$$= \frac{1}{n+1} * \frac{n(n+1)}{2} = n/2$$

★ Elementary properties of expectation

① If $X \geq 0$, then $E[X] \geq 0$

② If $a \leq X \leq b$, then $a \leq E[X] \leq b$

③ If c is a constant, $E[c] = c$

★ The Expected value rule for calculating $(E[g(X)])$

⇒ Let X be a r.v and let $Y = g(X)$

$$E[Y] = \sum_{y} y P_Y(y)$$

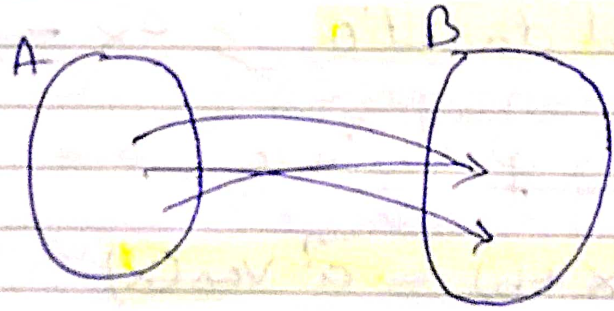
$$\boxed{E[Y] = \sum_{x} g(x) P_X(x)} \; *$$

★ Linearity of expectation

$$E[aX+b] = aE[X] + b$$

S-suppliment. 1

★ Function



$f: A \to B$

{ Every element of A should be mapped to exactly one one element of B }