(13)

Conditional Expectation and Variance

★ Conditional Expectation as a random variable

$$g(y) = E[X|Y=y] = \sum_x x P_{X|Y}(x|y)$$

⇒ $g(Y)$ is the random variable that takes the value $E[X|Y=y]$ if Y happens to take the value y.

⇒ Definition: $E[X|Y] = g(Y)$

- It is a function of Y.
- It is a random variable.

★ The mean of $E[X|Y]$

{Law of iterated expectations}

$$E[E[X|Y]] = \sum_y g(Y) P_Y(y)$$

$$= \sum_y E[X|Y=y] P_Y(y) \quad \{\text{Total Expectation theorem}\}$$

$$= \sum[X]$$

★ The conditional variance as a random variable

$$\text{Var}(X|Y=y) = E\left[\left(X - E[X|Y=y]\right)^2 \,|\, Y=y\right]$$

⇒ $\text{Var}(X|Y)$ is the r.v that takes the value $\text{Var}(X|Y=y)$, when $Y=y$.

⇒ **Law of total variance:**

$$\boxed{\text{Var}(X) = E[\text{Var}(X|Y)] + \text{Var}(E(X|Y))}$$