Contents

CHAPTER 1

Lecture 1

Lecture-1

\# n equation, n unknown ← Fundamental problem of Linear Algebra

- → Row picture
- → Column picture
- → Matrix form

Matrix ⇒ It is just a rectangular array of numbers

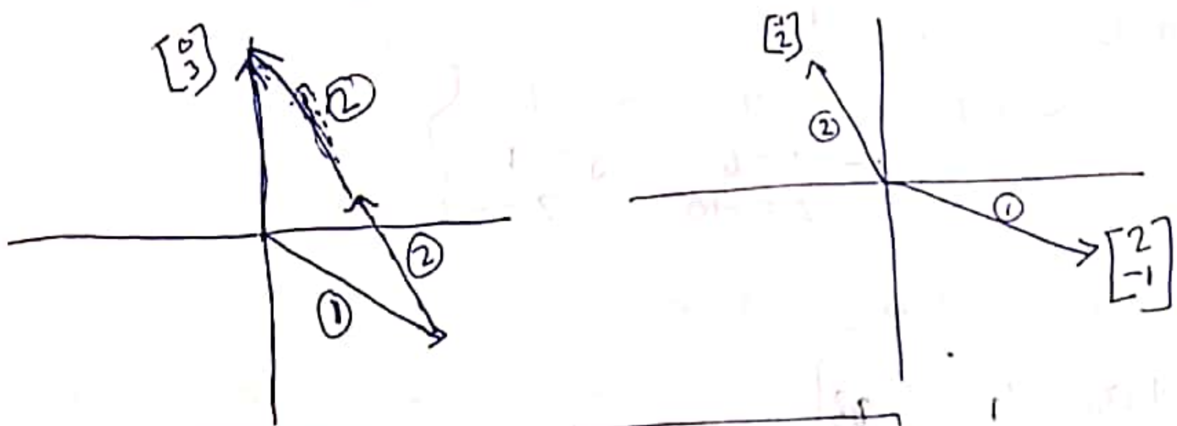
Example

$$2x - y = 0$$
$$-x + 2y = 3$$

$$\longrightarrow \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$\downarrow$$

$$x \begin{bmatrix} 2 \\ -1 \end{bmatrix} + y \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$ ← Linear Combination of Column



$\boxed{\text{Singular}}$ $\boxed{\text{Non-Invertable}}$

$Ax = b$ ← Matrix form

CHAPTER 2

Lecture 2

Lecture-2

1 { → Elimination
→ Back-Substitution

2 { → Elimination matrix
→ Matrix multiplication

Elimination

$$x + 2y + z = 2$$
$$3x + 8y + z = 12$$
$$4y + z = 2$$

$$\underbrace{\begin{bmatrix}1 & 2 & 1\\3 & 8 & 1\\0 & 4 & 1\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}x\\y\\z\end{bmatrix}}_{x} = \underbrace{\begin{bmatrix}2\\12\\2\end{bmatrix}}_{b} \qquad Ax = b$$

Pivot

$$\underbrace{\begin{matrix}\boxed{1} & 2 & 1\\ 3 & 8 & 1\\ 0 & 4 & 1\end{matrix}}_{A}\ \underbrace{\begin{matrix}2\\12\\2\end{matrix}}_{b} \Longrightarrow \begin{matrix}\boxed{1} & 2 & 1 & 2\\ 0 & \boxed{2} & -2 & 6\\ 0 & 4 & 1 & 2\end{matrix} \Longrightarrow \underbrace{\begin{matrix}\boxed{1} & 2 & 1\\ 0 & \boxed{2} & -2\\ 0 & 0 & 5\end{matrix}}_{u}\ \underbrace{\begin{matrix}2\\6\\-10\end{matrix}}_{c}$$

back Substitution

$$\left.\begin{array}{ll} x + 2y + z = 2 & x = 2\\ 2y - 2z = 6 & y = 1\\ 5z = -10 & z = -2 \end{array}\right\}$$

Elimination matrix

$$\begin{array}{r c} \text{Row 1} \rightarrow & \\ \text{Row 2} \rightarrow & \\ \text{Row 3} \rightarrow & \end{array}\begin{bmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}$$

Column 1   Column 2   Column 3

a_{ij} — i: Row no, j: Column no

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix}$$

$$= b_{11} \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} + b_{21} \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} + b_{31} \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} \quad \left\{ \text{Column method} \right\}$$

$$= \begin{bmatrix} (a_{11}\ a_{12}\ a_{13}) \cdot (b_{11}\ b_{21}\ b_{31}) \\ (a_{21}\ a_{22}\ a_{23}) \cdot (b_{11}\ b_{21}\ b_{31}) \\ (a_{31}\ a_{32}\ a_{33}) \cdot (b_{11}\ b_{21}\ b_{31}) \end{bmatrix} \quad \left\{ \text{Row method} \right\}$$

Dot Product

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} =$$

$$\begin{bmatrix} (a_{11}\ a_{12}\ a_{13}) \cdot (b_{11}\ b_{21}\ b_{31}) & (a_{11}\ a_{12}\ a_{13}) \cdot (b_{12}\ b_{22}\ b_{32}) & (a_{11}\ a_{12}\ a_{13}) \cdot (b_{13}\ b_{23}\ b_{33}) \\ (a_{21}\ a_{22}\ a_{23}) \cdot (b_{11}\ b_{21}\ b_{31}) & (a_{21}\ a_{22}\ a_{23}) \cdot (b_{12}\ b_{22}\ b_{32}) & (a_{21}\ a_{22}\ a_{23}) \cdot (b_{13}\ b_{23}\ b_{33}) \\ (a_{31}\ a_{32}\ a_{33}) \cdot (b_{11}\ b_{21}\ b_{31}) & (a_{31}\ a_{32}\ a_{33}) \cdot (b_{11}\ b_{21}\ b_{31}) & (a_{31}\ a_{32}\ a_{33}) \cdot (b_{11}\ b_{21}\ b_{31}) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ \boxed{0} & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix}$$

→ Row Operation

$$= \begin{bmatrix} a_{11}(b_{11} \ b_{12} \ b_{13}) + a_{12}(b_{21} \ b_{22} \ b_{23}) + a_{13}(b_{31} \ b_{32} \ b_{33}) \\ a_{21}(b_{11} \ b_{12} \ b_{13}) + a_{22}(b_{21} \ b_{22} \ b_{23}) + a_{23}(b_{31} \ b_{32} \ b_{33}) \\ a_{31}(b_{11} \ b_{12} \ b_{13}) + a_{32}(b_{21} \ b_{22} \ b_{23}) + a_{33}(b_{31} \ b_{32} \ b_{33}) \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}}_{E_{32}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_{21}} \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix}$$

(Elimination matrix)

$$E_{32}(E_{21}A) = U$$

$$\Rightarrow (E_{32}E_{21})A = U$$

{Associative law}

~~{Commutative Law}~~

$$= \left[b_{11}\begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} + b_{21}\begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix} + b_{31}\begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix}, \; b_{12}\begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} + b_{22}\begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix} + b_{32}\begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix}, \; b_{13}\begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} \; b_{23}\begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix} \; b_{33}\begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix} \right]$$

Column Operation

If A^{-1} is inverse of matrix A then

$$A^{-1}A = I$$

CHAPTER 3

Lecture 3

Lecture 3

→ Matrix Multiplication (4 ways)

→ Inverse of A AB & A^T

→ Gauss-Jordan / find A^{-1}

→ First way : {Normal Element by Element}

→ Second way {Combination of Rows}

→ Third way {Combination of Columns}

→ Forth way

$$AB = \sum (\text{col's of } A) \times (\text{rows of } B)$$

Example

$$\begin{bmatrix} 2 & 7 \\ 3 & 8 \\ 4 & 9 \end{bmatrix} \begin{bmatrix} 1 & 6 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 6 \end{bmatrix} + \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \begin{bmatrix} 0 & 0 \end{bmatrix}$$

Block Multiplication

$$\left[\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right] \left[\begin{array}{c|c} B_1 & B_2 \\ \hline B_3 & B_4 \end{array}\right] = \left[\begin{array}{c|c} A_1B_1 + A_2B_3 & \\ \hline & \end{array}\right]$$

Matrices

Inverses {Square Matrix}

Consider a Square matrix A, If you can find some Vectore x such that

$$Ax = 0 \quad \{x \text{ is not zero Vector}\}$$

then A is non Invertible

Proof

Consider A is Invertible & we find a non zero x.

then, $Ax = 0$

$\Rightarrow A^{-1}Ax = A^{-1}0$

$\Rightarrow x = 0$

which is not true hence A is no Invertible

$$A^{-1}A = I = AA^{-1}$$

True but hard to proof

Gauss Jordan

Example

$$\underbrace{\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}}_{A} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{array}\right] \leftarrow A^{-1}$$

CHAPTER 4

Lecture 4

Lecture - 4

→ Inverse of AB A^T

→ Product of Elimination matrix

$A = LU$ (no row exchange)

→ Upper triangle

→ Lower triangle

✓ $(AB)^{-1} = B^{-1}A^{-1}$

$(AB)^T = B^T A^T$

$AA^{-1} = I$

$(AA^{-1})^T = I^T \Rightarrow (A^{-1})^T (A)^T = I$

✓ $(A^T)^{-1} = (A^{-1})^T$

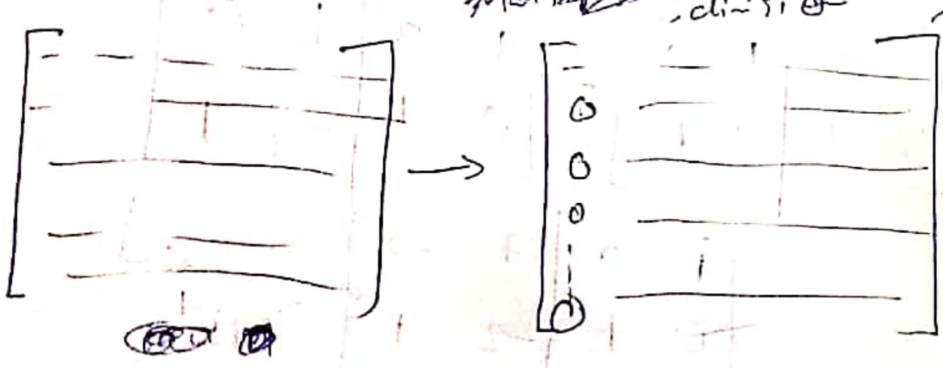
$E_{32}E_{31}E_{21}A = U$ (No row exchange)

$$A = \underbrace{E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}U}_{LU}$$

How many operations on $n \times n$ matrix A?

→ (Addition, Substraction ~~Multiplication~~ Multiplication division)

→ about n^2



Count = $n^2 + (n-1)^2 + \dots + 2^2 + 1^2$ {Roughly}

$\approx \frac{n^3}{3}$

Permutation matrix (3×3)

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \quad \begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix} \quad \begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$$

$$\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix} \quad \begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix} \quad \begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}$$

$$\boxed{P^{-1} = P^T}$$

about n^2

CHAPTER 5

Lecture 5

Lecture-5

R^TR is always Symmetric.

$$(R^TR)^T = R^T(R^T)^T = R^TR$$

Vector Space

→ Collection of Vector or Space of Vector but not jus any bunch of Vector.

Rule for being Vector Space → ⇒ We must be able to do any Linear Combination of Vectors in Vector Space without going out of it.

Example R^2 ⇒ All 2-Dim real Vectors

R^3 ⇒ All 3-Dim real Vectors

R^n ⇒ All n-dimentional real Vector

Subspace ⇒ A subspace is a Subset of Vector Space that Satifies the rules of Vector Space.

(ie $w+v$ in Subspace
cw in Subspace)

All Subspace of R^2

⇒ all of R^2

⇒ Any line passing through $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ L

⇒ Zero Vector only Z

If

⇒

⇒

#

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CHAPTER 6

Lecture 6

Lecture-6

→ Vector Space and Subspace
→ Column Space of A : Solving $Ax = b$
→ Nullspace of A:

If S and T are two Subspaces -

⇒ $S \cup T$ is not a Subspace
⇒ $S \cap T$ is a Subspace.

→ Let us consider U & V vectors in the intersection of S & T.
⇒ Linear Combination of U & V is in S because U & V is in S.
⇒ Linear Combination of " " T " " " T.
⇒ So Linear Combination of U & V is in the intersection of S & T
⇒ So $S \cap T$ is a Subspace

Column Space of Matrix A.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}_{4\times 3} \longrightarrow R^4 \text{ Subspace}$$

→ C(A)

→ all linear Combination of the Column.

Dose $Ax = b$ have Solution for every b?

4 equation, 3 Unknown

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

{Not independent (i.e. sum of first two)} → So for every b there is no x

I Can Solve $Ax = b$ exactly when b is in Column Space of A.

Null Space N(A) of A

→ all Solutions $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ to $Ax = 0$.

always R^3

⇒ Solutions of $Ax = 0$ gives a Subspace

$$\left\{ \begin{array}{l} Av = 0 \ \& \ Aw = 0 \\ \text{then } A(v + w) = 0 \end{array} \right\}$$

CHAPTER 7

Lecture 7

Lecture-7

→ Computing the null Space ($Ax=0$)
→ Pivot Variables - free Variables
→ Special Solution - $\text{rref}(A) = R$

Consider

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}$$

we are intrested in finding null Space of A.

$\Rightarrow Ax = 0$

{we need to find all x for which above equation is true}

$$A \rightarrow \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

Pivot Columns

Pivots

$$U \leftarrow \begin{bmatrix} \boxed{1} & 2 & 2 & 2 \\ 0 & 0 & \boxed{2} & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

free Columns

{Rank of A = no of Pivots = 2}

$Ux = 0$

$$x_1 + 2x_2 + 2x_3 + 2x_4 = 0$$
$$2x_3 + 4x_4 = 0$$

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

{echelon form}

{We can take any value of x_2 & x_4 as them combine with free columns}

Let us take $x_2 = 1$ & $x_4 = 0$

So $x_3 = 0$

$x_1 = -2$

$$X = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Let us take $x_2 = 0$ & $x_4 = 1$

So $x_3 = -2$

$x_1 = 2$

$$X = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

So, Nul

N(A)

R = r

A → [1 2 3

{Matlab to ge ⇒ r

So, Null Space A is

$$N(A) = c\begin{bmatrix}-2\\1\\0\\0\end{bmatrix} + d\begin{bmatrix}2\\0\\-2\\1\end{bmatrix}$$

R = reduced row echelon form

- → Zero above & below pivots
- → & Pivots = 1

$$A \rightarrow \begin{bmatrix}1 & 2 & 2 & 2\\2 & 4 & 6 & 8\\3 & 6 & 8 & 10\end{bmatrix} \rightarrow \begin{bmatrix}1 & 2 & 2 & 2\\0 & 0 & 2 & 4\\0 & 0 & 0 & 0\end{bmatrix}$$

Echelon form

$$\begin{bmatrix}\boxed{1} & 2 & 0 & -2\\0 & 0 & \boxed{1} & 2\\0 & 0 & 0 & 0\end{bmatrix}$$

Reduced row echelon form

{Matlab command to get R ⇒ rref(A)}

$Rx = 0$

$$\begin{bmatrix}I & F\\0 & 0\end{bmatrix}\begin{bmatrix}x_{pivot}\\x_{free}\end{bmatrix} = 0 \Rightarrow x_{pivot} I + x_{free} F = 0$$

$$x_{pivot} = \cancel{-FI} - x_{free} F$$

$$\begin{bmatrix}-F\\I\end{bmatrix}$$

CHAPTER 8

Lecture 8

Lecture-8

→ Complete Solution of $Ax=b$

→ Rank r

Solving $Ax=b$

⇒ Solvability Condition on b

→ $Ax=b$ solvable when b is in $C(A)$

{Column Space of A}

Complete Solution of $Ax=b$

① $X_{particular}$

→ Set all free Variables to Zero.

→ Solve $Ax=b$ for the pivot Variables.

↳ (X_p)

Add

② $X_{nullspace}$ (X_n)

All vector from null space

Complete Solution: $\boxed{X_c = X_p + X_n}$

Proof

$$A X_p = b$$

$$A X_n = 0$$

$$A(X_p + X_n) = b$$

$r < m \quad r < n$

$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$

(0 or ∞ Solution to $Ax=b$)

$m \times n$ matrix of rank r

{Number of pivots} ← {Rank of a matrix A is the dimension of the vector space generated by its column.}

{Tells you everything about number of solution}

$\{r \leq m, \; r \leq n\}$

Full Column rank $r = n$

→ No free variable

→ {Unique Solution if it exist}

(zero or one solution)

Full Row rank $r = m$

→ Can solve $Ax = b$ for every b

→ Left with $n - r$ free variables

$\boxed{r = m = n}$ {Full rank}

→ Matrix is invertible

→ $R = I$

→ Solution always exist

$r = m = n$ $R = I$ 1 solution of $Ax = b$	$r = n < m$ $R = \begin{bmatrix} I \\ 0 \end{bmatrix}$ (0 or 1 solution to $Ax = b$)	$r = m < n$ $R = [I \; F]$ ∞ solution to $Ax = b$

CHAPTER 9

Lecture 9

Lecture - 9

- → Linear independence
- → Spanning a Space
- → BASIS and Dimension.

Independence: Vectors $x_1, x_2, x_3 \cdots x_n$ are independent if no combination gives zero vector. {except zero combination}

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n \neq 0$$

Zero vector is dependent to every vector.

$V_1, V_2 \cdots V_n$ are columns of A

- → They are indipendent if nullspace of A is {Zero Vector} | rank = n
- → They are dependent if $Ac = 0$ ∀ some non zero c. | rank < n

Vector $V_1, \cdots V_l$ Span a Space means: The space consists of all combination of tose vectors.

Basis

(1)

(2)

Ev

So

D

#Basis for Space is a Sequence of Vectors $v_1, v_2 \dots v_d$ with two properties:

(1) They are Independent

(2) They Span the space.

#Every basis for the Space has the Same number of vectors.

$\uparrow$

Dimention of that Space

#Dimention of null Space $= n - \text{Rank}(A)$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad \|$

no of free Variable

CHAPTER 10

Lecture 10

Lecture - 10

4 Subspace

- → Column Space $\boxed{C(A)}$ → in R^m
- → Null Space $\boxed{N(A)}$ → in R^n
- → Row Space $\boxed{C(A^T)}$ → in R^n
  - → All combination of rows
  - → All Combination of Column of A^T
- → Null Space of A^T $\boxed{N(A^T)}$ → in R^m

Left null Space of A

dimention of Row Space = dimention of Column space = Rank of A

dimention of Null Space of A = n - Rank

dimention of Null Space of A^T = m - Rank

New Vector Space ! (M) → Matrix Space

↳ All 3×3 matrices !! {matrix follow all the rules of vector}

Subspace: → All upper △ matrix
→ All Symmetric matrices.
→ diagonal matrix

CHAPTER 11

Lecture 11

Lecture-11

→ Bases of new Vector Space
→ Rank one matrices
→ Small World graphs

Basis for M = all 3×3 matrix

$$\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\0&1&0\end{bmatrix}, \cdots, \begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}$$

→ dimention of Space 9

Basis for M - all 3×3 Symmetric Matrix.

$$\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix} \begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix} \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}$$

$$\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix} \begin{bmatrix}0&0&1\\0&0&0\\1&0&0\end{bmatrix} \begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}$$

→ dimention of Space 6

Sum of Element of S & U.

$S+U$ = all 3×3 matrix

→ dimention of Space 9

$$\dim(S+U) = \dim(S) + \dim(U) - \dim(S \cap U)$$

\# $\frac{d^2y}{dx^2} + y = 0$

$y = \underbrace{\cos x \ , \ \sin x}_{\text{Basis}}$

<u>Complete Solution</u>: $y = C_1 \cos x + C_2 \sin x$ {Null Space}

dimension = 2 = Order of differential Equation.

<u>Rank 1 matrix</u>

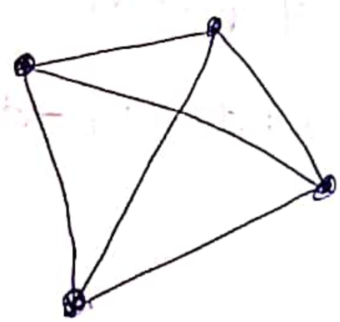
<u>Example</u> $\begin{bmatrix} 1 & 4 & 5 \\ 2 & 8 & 10 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 4 & 5 \end{bmatrix}$

Any Rank 1 matrix $A = uv^T$

<u>Column Vectors</u>

\# Any matrix of Rank r can be broken into r matrix of rank 1.

\# <u>Graph</u> ⇒ It a bunch of nodes and edges Connecting the nodes



→ 4 node
→ 6 edges

(2) Minimum Separation of two nodes.

CHAPTER 12

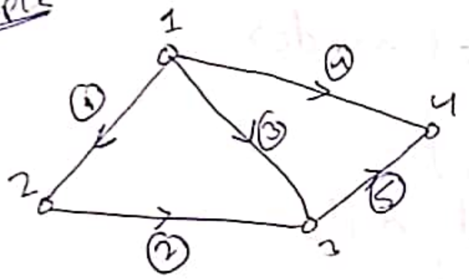
Lecture 12

Lecture - 12

→ Graphs & Networks
→ Incidence Matrices
→ Kirchhoff's Laws

Graph : Bunch of nodes and Edges connecting those nodes.

Example



$n = 4$ nodes

$m = 5$ edges

Incidence Matrix

$$A = \begin{array}{c} \text{node} \\ \\ \\ \\ \\ \\ \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} \text{edge} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}$$

Loop ⇒ Edges ①, ② & ③ forms a loop

→ Loops correspond to linearly dependent rows.

$$Ax = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_2 - x_1 \\ x_3 - x_2 \\ x_3 - x_1 \\ x_4 - x_1 \\ x_4 - x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

\# $X = x_1, x_2, x_3, x_4$
as potential at nodes

\# $(x_2 - x_1)\ (x_3 - x_2)$ ———
as potential difference
across edges

$$X = C \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad \dim N(A) = 1$$

Rank = 3

$y_1, y_2, \ldots, y_5$ be the Currents on the edges.

$A^T y = 0$ {Kirchhof's Current Law}

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Basis for null space of A^T

$$\begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

Graph with no loop $\Rightarrow$ <u>Tree</u>

$$\dim N(A^T) = m - r$$

$\dim N(A^T)$ → (no of loops)

m → (number of Edges)

r → (number of nodes -1)

$$\boxed{\begin{pmatrix}\text{no of}\\ \text{nodes}\end{pmatrix} - \begin{pmatrix}\text{no of}\\ \text{Edges}\end{pmatrix} + \begin{pmatrix}\text{no of}\\ \text{Loops}\end{pmatrix} = 1}$$

→ Euler's formula

$Ax = e$ ← Potential differ.
← Voltage Source Comes here

$y = Ce$

$A^T y = 0$ ← Current Source Can be put here

$$\boxed{A^T C A x = f}$$

$A^T C A$ → It is always Symmetric.

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CHAPTER 13

Lecture 13

Lecture 13

→ Review of previous lectures.

CHAPTER 14

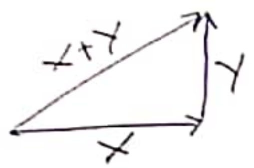
Lecture 14

Lecture-14

→ Orthogonal Vectors & Subspace
→ nullSpace ⊥ row Space
→ $N(A^TA) = N(A)$

Orthogonal Vectors

Condition for orthogonality



$$\boxed{x^Ty = 0}$$

$$|x|^2 + |y|^2 = |x+y|^2$$

$$\cancel{x^Tx} \quad \cancel{y^Ty} \quad (x+y)^T(x+y)$$

$$\to \cancel{x^Tx} + \cancel{y^Ty} + \cancel{y^Tx}$$
$$2x^Ty + \cancel{y^Tx}$$

$$x^Ty = 0$$

Subspace S is orthogonal to Subspace T.
Means: Every Vector in S is orthogonal to every Vector in T.

\# Row space is orthogonal to null space.
\# Column space is orthogonal to null space of A^T.

⇒ Nullspace and row space are orthogonal Complements in R^n.

↳ Nullspace contains all vectors ⊥ row space.

$Ax = b$ when there is no Solution.

A^TA

$n\times m$ $m\times n$

$n\times n$ → Square

→ Symmetric

$\boxed{Ax = b}$

↓

$\boxed{A^T A \hat{x} = A^T b}$

⇒ A^TA is invertible if (exactly) A has indipendent Columns.

$\text{Rank}(A^TA) = \text{Rank}(A)$

CHAPTER 15

Lecture 15

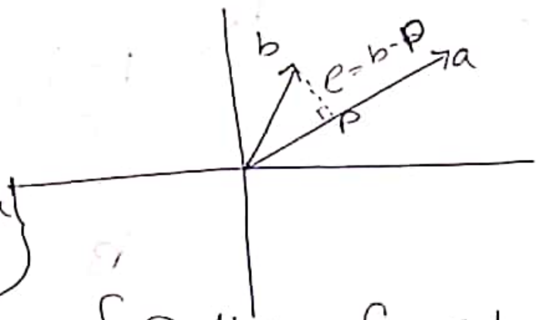
Lecture - 15

→ Projection

→ Least Square projection Matrix

$\boxed{P = xa}$

{P is along a with some magnitude}



$a^T(b - xa) = 0$ {Condition for e to be ⊥ to a}

$x a^T a = a^T b$

$\boxed{x = \frac{a^T b}{a^T a}} \rightarrow \boxed{P = a\frac{a^T b}{a^T a}}$

Projection Matrix $= Pb = \frac{aa^T}{a^Ta} b$

→ Symmetric

→ $P^2 = P$

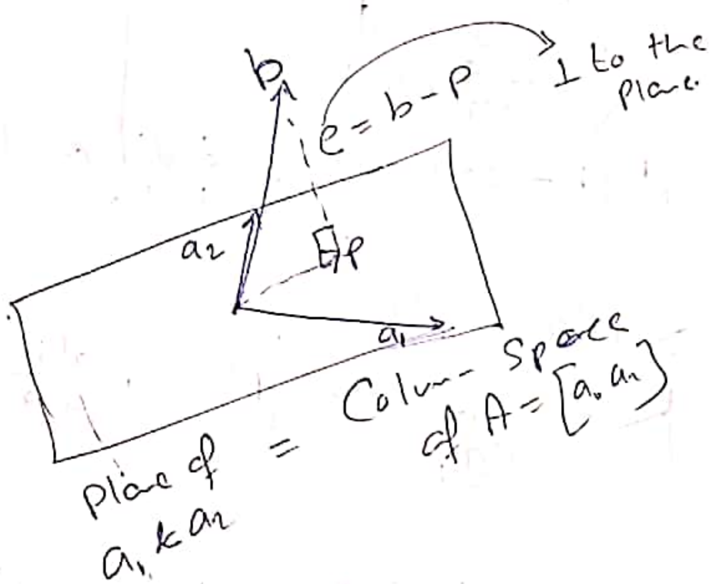
Column Space of P is a line thru a

Rank = 1

Why Projection

Because $Ax = b$ may have no solution

Solve $A\hat{x} = P$ → Projection of b onto Column Space



$$P = \hat{x}_1 a_1 + \hat{x}_2 a_2 = A\hat{x}$$

$P = A\hat{x}$ Find $\hat{x}$

Key: $b - A\hat{x}$ is $\perp$ to Column Space of A.

$$a_1^T(b - A\hat{x}) = 0$$

$$a_2^T(b - A\hat{x}) = 0$$

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$$\begin{bmatrix} a_1^T \\ a_2^T \end{bmatrix}(b - A\hat{x}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$A^T(b - A\hat{x}) = 0$$

$e \rightarrow$ It is in null space of A^T

$$A^TA\hat{x} = A^Tb$$

$$\hat{x} = (A^TA)^{-1}A^Tb$$

$$P = A\hat{x} = A(A^TA)^{-1}A^T b$$

$\rightarrow$ Projection Matrix

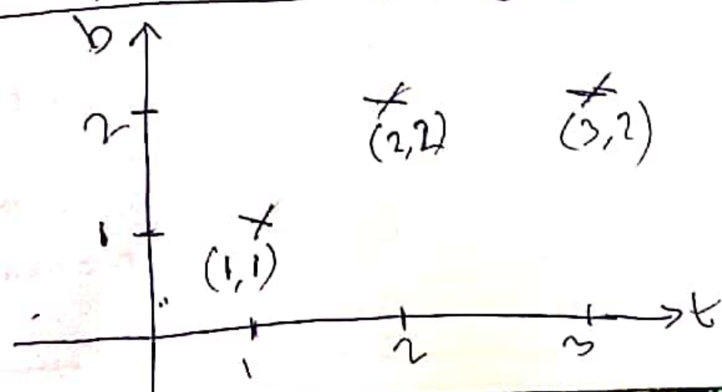
$$(A^TA)^{-1} \neq A^{-1}A^{T^{-1}}$$

$\rightarrow$ {because A is not invertible. Not a Square Matrix.}

$\rightarrow P^T = P$

$\rightarrow P^2 = P$

Least Square fitting by a line



$b = C + Dt$ {Equation of line with C & D unknown}

$$\boxed{\begin{aligned} C + D &= 1 \\ C + 2D &= 2 \\ C + 3D &= 2 \end{aligned}} \longrightarrow \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} C \\ D \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}}_{b}$$

{Conditions the line passing through all the point will satisfy}

$A\hat{x} = p$

best C & D

{Projection of b on Column Space of A}

CHAPTER 16

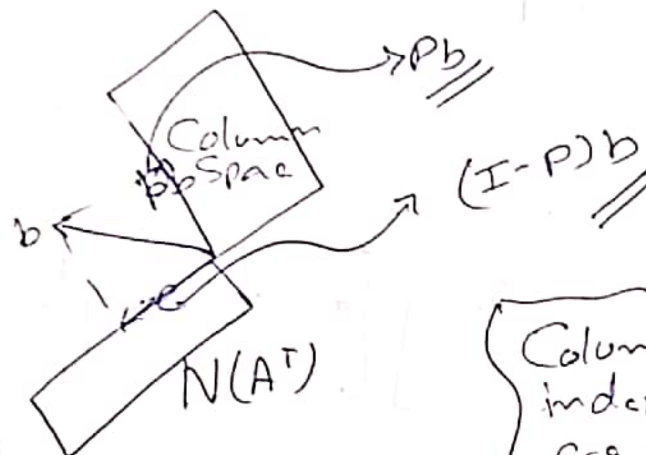
Lecture 16

Lecture-16

→ Projections
→ Least Square & best Straight line.

$$P = A(A^TA)^{-1}A^T$$

Projection matrix



Columns are definitely independent if they are ⊥ Unit Vectors → Orthonormal Vectors

$$A(A^TA)^{-1}A^Tb = A\hat{x}$$

$$\boxed{A^TA\hat{x} = A^Tb}$$

If A has independent Columns then A^TA is invertible.

⇒ Suppose $A^TAx = 0$, x must be 0. {Matrix is invertable if its null space is only zero}

$$x^TA^TAx = 0$$

$$\Rightarrow (Ax)^T(Ax) = 0 \rightarrow Ax = 0 \rightarrow x = 0$$ {as A has independent columns}

CHAPTER 17

Lecture 17

Lecture - 17

→ Orthogonal basis $q_1 \cdots q_n$
→ Orthogonal matrix Q
→ Gram - Schmidt $A \to Q$

Orthonormal Vector

$$q_i^T q_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

$$Q = \begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix}$$

$$Q^T Q = \begin{bmatrix} q_1^T \\ \vdots \\ q_n^T \end{bmatrix} \begin{bmatrix} q_1 \cdots q_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

I

Orthonormal matrix

If Q is square then $Q^T Q = I$
$\implies Q^T = Q^{-1}$

Q has Orthonormal Columns

Project onto its Column Space

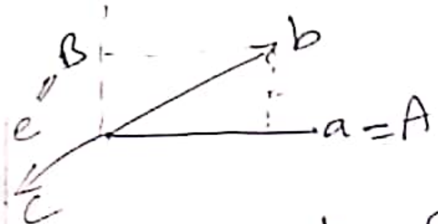
$$P = Q \underbrace{(Q^T Q)^{-1}}_{I} Q^T = \underbrace{Q Q^T}_{I \text{ if } Q \text{ is square}}$$

Gram-Schmidt

Independent Vectors $a, b, c \rightarrow A, B, C$ Orthogonal $\rightarrow$ Orthonormal

$$q_1 = \frac{A}{\|A\|} \quad q_2 = \frac{B}{\|B\|}$$

$$q_3 = \frac{C}{\|C\|}$$



$$B = b - Pb = b - A\frac{A^T b}{A^T A}$$

$$A^T B = A^T\left(b - A\frac{A^T b}{A^T A}\right) = A^T b - A^T A\,\frac{A^T b}{A^T A} = 0$$

$$C = c - A\frac{A^T}{A^T A}c - B\frac{B^T}{B^T B}c$$

$$\left\{ A \longrightarrow Q \right\}$$

A: Normal Independent Column Matrix

Q: Orthonormal Matrix

$$A = QR$$

R: Upper △

CHAPTER 18

Lecture 18

Lecture - 18

(Determinants) → {defined for square matrix}

det A / |A|

Invertible ——— $|A| \neq 0$

Non-Invertible ——— $|A| = 0$

① $\det I = 1$

② Exchange rows : reverse sign of determine

③ → ⓐ If we multiply t with one of the row than determinant becom t times.

$$\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

→ ⓑ $$\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$$

Linear for each row.

④ ~~row~~ Two equal result in Zero determinant.

⑤ Substract $l \times$ row I from row K.

↳ determinant Dosen't change.

⑥ Row

⑦ d

⑧ |A

⑨ de

⑩ d

⑥ Row of Zeros $\longrightarrow \det A = 0$

⑦ $$\begin{vmatrix} d_1 & \times & \times & \times \\ 0 & d_2 & \times & \times \\ 0 & 0 & d_3 \cdots & \times \\ 0 & 0 & 0 & d_n \\ 0 & 0 & 0 & \end{vmatrix} = d_1 d_2 d_3 \cdots d_n$$

⑧ $|A| = 0$ when A is Singular.

⑨ $\det AB = (\det A)(\det B)$

$$\left\| \det A^{-1} = \frac{1}{\det A} \right\|$$

$$\det A^n = (\det A)^n$$

~~det mA~~

$$\det mA = \det A \times m^n$$

$\rightarrow$ {If n = dimension of A}

⑩ $\det A^T = \det A$

CHAPTER 19

Lecture 19

Lecture 19

→ Formula for det A ($n!$ terms).
→ Cofactor formula
→ Tridiagonal matrices

$$\det A = \sum_{n! \text{ terms}} \pm a_{1\alpha} a_{2\beta} a_{3\gamma} \cdots a_{n\omega}$$

$(\alpha, \beta, \gamma \cdots \omega)$ = Permutation of $(1, 2 \ldots n)$

⇒ Cofactor is a way to connect $n \times n$ determinant to $(n-1)\times(n-1)$ determinant

$$\text{Cofactor of } a_{ij} = \pm \det \begin{pmatrix} n-1 \text{ matrix} \\ \text{with row } i \\ \text{and col } j \text{ erased} \end{pmatrix}$$

$$= C_{ij}$$

$+$ if $i+j$ = even
$-$ if $i+j$ = odd

$$\boxed{\det A = a_{11} C_{11} + a_{12} C_{12} + \cdots + a_{1n} C_{1n}}$$

{along row 1}

CHAPTER 20

Lecture 20

Lecture-20

→ Formula for A^{-1}
→ Cramers Rule for $x = A^{-1}b$
→ $|\det A| =$ Volume of box

$$\boxed{A^{-1} = \frac{1}{\det A} C^T}$$

...n)

$$Ax = b$$

$$x = A^{-1}b = \frac{1}{\det A} C^T b$$

Cramers Rule

$$x_1 = \frac{\det B_1}{\det A} \qquad B_1 = \begin{bmatrix} b & \begin{matrix} n-1 \\ \text{columns} \\ A \end{matrix} \end{bmatrix}$$

{A with Column 1 replaced with b}

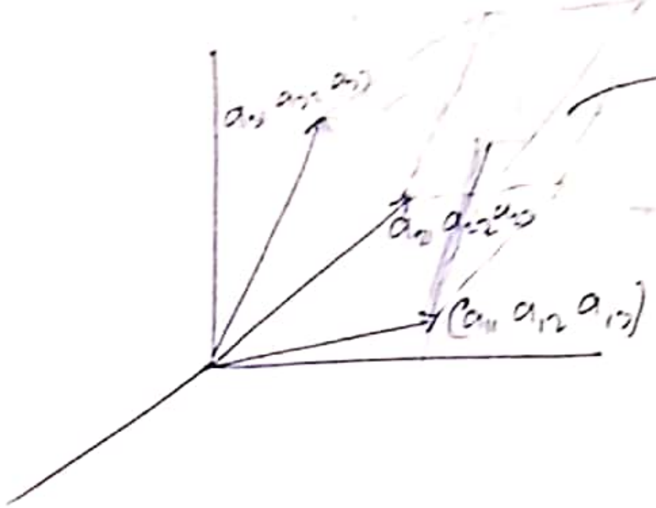
$$x_2 = \frac{\det B_2}{\det A}$$

$$\vdots$$

$$x_j = \frac{\det B_j}{\det A}$$

{A with Column j replaced with b}

det A = Volume of Box



$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

CHAPTER 21

Lecture 21

Lecture-21

→ Eigen Values - Eigen Vectors

→ $\det[A - \lambda I] = 0$

→ Trace $= \lambda_1 + \lambda_2 + \cdots + \lambda_n$

Ax parallel to x → Eigen Vector

Matrix, Vector

$$Ax = \lambda x$$

Eigen Value

Eigen Vectors

Eigen Values

If A is Singular, $\lambda = 0$ is eigen value.

\# What are the x's and λ's for Projection matrix.

⇒ Any x's in the plane will be a eigen Vectors

⇒ λ's $= 1$

⇒ $x \perp$ to the plane will be a eigen Vector

⇒ $\lambda = 0$

\# Example

$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\lambda = 1$

$x = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $\quad \lambda = -1$

$\Rightarrow N \times N$ will have N eigen value

$\Rightarrow$ Sum of λ's $= a_{11} + a_{nn} + \cdots a_{nn}$

Fact

definition

Trace $\Rightarrow$ Sum of the element on the main diagonal

$$tr(A) = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + \cdots a_{nn}$$

How to Solve $Ax = \lambda x$

$$\Rightarrow (A - \lambda I)x = 0$$

It has to be Singular for some non zero x.

$$\Rightarrow \boxed{|A - \lambda I| = 0}$$

Eigen Value Equation

Let Q be matrix which rotates every vector by $90°$

$$Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \det(Q - \lambda I)$$

$$\lambda^2 + 1 = 0$$

$$\lambda = \pm i$$

In Case of upper Δ matrix eigen Value are in the diagonal.

CHAPTER 22

Lecture 22

Lecture-22

→ Diagonalizing a matrix $\boxed{S^{-1}AS=\Lambda}$

→ Powers of A / equation $u_{k+1} = Au_k$.

$(A-\lambda I) \longrightarrow$ Singular

It is the eigen Value of A.

$Ax = \lambda x$

$S^{-1}AS = \Lambda$

Columns of 'S' are eigen Vectors of A.

→ Suppose we have n linearly independent eigen of A.
→ Put them in Columns of S.

$$AS = A\begin{bmatrix} \vdots & | & & | \\ x_1 & x_2 & \cdots & x_n \\ | & | & & | \end{bmatrix} = \lambda_1 x_1 + \lambda_2 x_2 - - - \lambda_n x_n$$

$$= \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 & & \\ 0 & \lambda_2 & & \\ 0 & & \ddots & \\ 0 & & & \\ 0 & & & \lambda_n \end{bmatrix}$$

$$AS = S\Lambda$$

diagonal eigenvalue Matrix.

$$\Rightarrow \boxed{S^{-1}AS = \Lambda}$$

If $Ax = \lambda x$

$$A^2 x = \lambda A x = \lambda^2 x$$

Theorem

$A^K \to 0$ as $K \to \infty$

if at $|\lambda_i| < 1$

$\Rightarrow$ A is sure to have n indipendent eigen vectors (and be diagonalizable)

if all the λ's are different.

$\Rightarrow$ Start with a given Vector U_0

Equation $U_{K+1} = AU_K$ {first order difference equation}

$U_1 = AU_0$

$U_2 = A^2 U_0$

$$\boxed{U_K = A^K U_0}$$

To really Solve:

$$U_0 = C_1 X_1 + C_2 X_2 + \cdots + C_n X_n$$

$$A^{100} U_0 = C_1 \lambda_1^{100} X_1 + C_2 \lambda_2^{100} X_2 + \cdots + C_n \lambda_n^{100} X_n$$

$$= \Lambda^{100} S c$$

$$\boxed{U_{100} = \Lambda^{100} S c}$$

Fib

Fibonacci example: 0, 1, 1, 2, 3, 5, 8, 13 ...

$$F_{k+2} = F_{k+1} + F_k \quad \text{—— (1)}$$

Let $U_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$

$$F_{k+1} = F_{k+1} \quad \text{—— (2)}$$

using (1) & (2) as system of equation

$$U_{k+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$$

$$\Rightarrow U_{k+1} = AU_k$$

$$\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$

$$U_0 = \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$c_1 x_1 + c_2 x_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 - \lambda - 1 = 0$$

$$\lambda_i = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

$$x_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} \longleftarrow \lambda_1 = \frac{1}{2}(1+\sqrt{5}) = 1.618$$

$$x_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix} \longleftarrow \lambda_2 = \frac{1}{2}(1-\sqrt{5}) \approx -0.618$$

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CHAPTER 23

Lecture 23

Lecture-23

→ Differential equation $\frac{du}{dt} = Au$

→ Exponential e^{At} of a matrix.

Example

$$\frac{du_1}{dt} = -u_1 + 2u_2$$

$$\frac{du_2}{dt} = u_1 - 2u_2$$

$u(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ {Initial Condition}

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad A = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}$$

$$\begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$\rightarrow \lambda = 0, -3$

$x_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Solution:

$$u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2$$

$$c_1 = \frac{1}{3} \quad c_2 = \frac{1}{3}$$

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{1}{3} e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$u(\infty) = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

\# for stability $Re\,\lambda < 0$

\# Steady State $\lambda = 0$

\# Blow up if $Re\,\lambda > 0$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$u(0) = Sc$$

tial
ndition

$\frac{du}{dt} = Au$

Let $u = Sv$ {uncoupling}

$\Rightarrow S\frac{dv}{dt} = ASv$

$\Rightarrow \frac{dv}{dt} = S^{-1}ASv = \Lambda v$

$\frac{dv_1}{dt} = \lambda_1 v_1$

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$\cancel{v_1 = c_1 e^{\lambda}} \Rightarrow v(t) = e^{\Lambda t} v(0)$

$u(t) = Se^{\Lambda t}S^{-1} u(0) = e^{At} u(0)$

$$\boxed{e^{At} = Se^{\Lambda t}S^{-1}}$$

$$e^{At} = I + At + \frac{(At)^2}{2} + \frac{(At)^3}{6} + \cdots + \frac{(At)^n}{n!} + \cdots$$

$$(I - At)^{-1} = I + At + (At)^2 + \cdots + (At)^n + \cdots$$

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & \ddots \; e^{\lambda_n t} \end{bmatrix}$$

$y'' + by' + ky = 0 \qquad y' = y'$

$u = \begin{bmatrix} y' \\ y \end{bmatrix}$

A

$u' = \begin{bmatrix} -b & -k \\ 1 & 0 \end{bmatrix} u$

$\boxed{u' = Au}$

CHAPTER 24

Lecture 24

(Linear Algebra)

Lecture 24a

→ Markov matrices
(Steady State)

→ Fourier Series projection.

A is markov matrice if

① All entity ≥ 0

② All Columns add to 1.

eg: $A = \begin{bmatrix} 0.1 & 0.01 & 0.3 \\ 0.2 & 0.99 & 0.3 \\ 0.7 & 0.0 & 0.4 \end{bmatrix}$

→ $\lambda = 1$ is an eigen Value

→ All other $|\lambda_i| < 1$

⇒ Eigen Value of A are Same as Eigen Value of A^T.

⇒ Projection with Orthonormal basis

$V = x_1 q_1 + x_2 q_2 + \dots x_n q_n$

{$x_1 \dots x_n$ are Scalars} {$q_1 \dots q_n$ are othonormal basis}

$V = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ ⇒ $QX = V$

$X = Q^{-1}V$

$\boxed{X = Q^T V}$

$x_1 = q_1^T V$

$$f(x) = a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots$$

{fourier Series}

$\Rightarrow$ 1, $\cos x$, $\sin x$, $\cos 2x$ --- are function basis.

for Vectors

$$v^T w = v_1 w_1 + \cdots + v_n w_n$$

for functions

$$f^T g = \int_0^{2\pi} f(x)\, g(x)\, dx$$

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos x \, dx$$

Lecture (24 b)

(Review)

1. $Q = [q_1, q_2 \cdots q_n]$

   Projection – Least Square

   (Gram Schmit)

2. det A

   → Properties

   → Big formula

   → $A^{-1} = \text{Cofactor}/|A|$

3. Eigen Values $Ax = \lambda x$

   $\det(A - \lambda I) = 0$

   Diagonalize $S^{-1}AS = \Lambda$

   Powers A^k

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CHAPTER 25

Lecture 25

Lecture-25

→ Symmetric matrices
Eigenvalues / Eigenvectors

→ Start: Positive Definite Matrices

Symmetric Matrices ⇒ $A = A^T$

→ Eigen Values → Real

→ Eigen Vectors → Orthogonal (Can be chosen)

$A = S\Lambda S^{-1}$ {In general}

$A = Q\Lambda Q^{-1} = Q\Lambda Q^T$ {Symmetric Matrix}

⇒ Every Symmetric matrix is a combination of Projection matrix.

Signs of pivots (for Symmetric matrix) are same as sign of eigen value.

→ Product of pivot = Product of eigen value {for Symmetric matrix}

Positive definite Symmetric matrix

→ all eigen Values are real & positive.

→ all pivots are positive.

→ Determinant are positive.

All Sub ———

CHAPTER 26

Lecture 26

Lecture - 26

Complex Vector Matrices

inner product
Discrete Fourier
Fast Fourier Transform = FFT

Fourier matrix F_n

$$Z^H \xrightarrow{\text{Hermitian}} \bar{Z}^T$$

Inner Product of Y & X $\Rightarrow Y^H X$ {∀ Complex numbers}

$\Rightarrow$ Symmetric {for Complex} $\Rightarrow Y^H = Y$

Called Hermitian matrix.

eg $\Rightarrow \begin{bmatrix} 2 & 3+i \\ 3-i & 5 \end{bmatrix}$

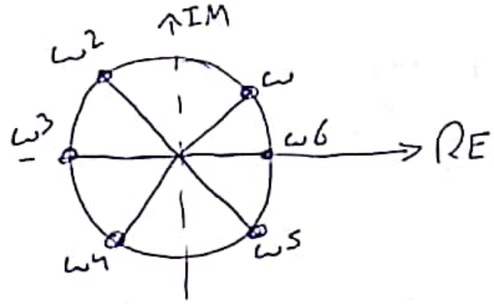
Fourier Matrix → Columns are orthogonal & normal

$$F_n = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & w & w^2 & \cdots & w^{n-1} \\ 1 & w^2 & w^4 & \cdots & w^{2(n-1)} \\ 1 & w^3 & w^6 & \cdots & w^{3(n-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & w^{n-1} & w^{2(n-1)} & \cdots & w^{(n-1)^2} \end{bmatrix}$$

w^{ij} {i, j begins from 0 to n−1}

$w^n = 1 \qquad w = e^{\frac{i2\pi}{n}}$

for $n=0$ $\quad W = e^{i\frac{2\pi}{6}}$



$F_n^H F_n = I$

→ Inverse of F_n

$$\boxed{\omega_{(n)}^2 = \omega_{(n/2)}}$$

Permutation matrix

$$[F_{64}] = \begin{bmatrix} I & D \\ I & -D \end{bmatrix} \begin{bmatrix} F_{32} & 0 \\ 0 & F_{32} \end{bmatrix} \begin{bmatrix} \ddots & \\ & \ddots \end{bmatrix} \quad \{FFT\}$$

64^2
Calculation

$2(32)^2 + \text{fix} \rightarrow 32$
Calculation

$$\begin{bmatrix} 1 & \omega & \cdots & \omega^{31} \end{bmatrix}$$

CHAPTER 27

Lecture 27

Lecture - 27

(26:37)

→ Positive definit Matrix (Tests)
→ Tests for Minimum ($x^TAx > 0$)
→ Ellipsoids in R^n

$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$

① $\lambda_1 > 0$ & $\lambda_2 > 0$

② $a > 0$ & $ac - b^2 > 0$

③ Pivots $a > 0$ $\frac{ac-b^2}{a} > 0$

④ $x^TAx > 0$

Min of $f(x_1, x_2 \cdots x_n)$ ~ (Matrix of 2nd derivative is positive definit)

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CHAPTER 28

Lecture 28

:37

Lecture -29

→ A^TA is positive definit!

→ Similar matrices

$$x^T(A^TA)x = (Ax)^T(Ax) = |Ax|^2 > 0$$

Hence positive definit

Similar Matrix

A & B are $n \times n$ matrix

A & B are Similar if for some M

$$B = M^{-1}AM$$

⇒ Similar matrix have same eigen Value.

$$Ax = \lambda x \qquad \{B = M^{-1}AM\}$$

$$AMM^{-1}x = \lambda x$$

$$\Rightarrow (M^{-1}AM)M^{-1}x = \lambda M^{-1}x$$

$$\Rightarrow B(M^{-1}x) = \lambda(M^{-1}x)$$

So λ is the eigen Value of B.

but eigen Vector is not same

(Eigen Vector of B) = M^{-1} (Eigen Vector of A)

BAD Case $\lambda_1 = \lambda_2 = 4$

Small family $M^{-1}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} M = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$

Big family $\begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix} \rightarrow$ Jordan form

Jordan Block

$$J_i = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 & 0 \\ 0 & 0 & \lambda_3 & 1 & \\ & & & \ddots & \end{bmatrix}$$

Every Sq matrix A is similar to a Jordan matrix J

$$J = \begin{bmatrix} \boxed{J_1} & & \\ & \boxed{J_2} & \\ & & \boxed{J_3} \end{bmatrix}$$

block = # Eigen Vector

CHAPTER 29

Lecture 29

Lecture-29

→ Singular Value Decomposition = SVD

→ $A = U \cdot \Sigma V^T$ // Σ diagonal, U, V orthogonal

⇒ $AV = U\Sigma$

$A = U\Sigma V^{-1} = U\Sigma V^T$ {V is orthonormal}

$A^TA = (V\Sigma^T \cancel{U^T})(\cancel{U}\Sigma V^T)$

$= V \begin{bmatrix} \sigma_1^2 & & \\ & \sigma_2^2 & \\ & & \ddots \end{bmatrix} V^T$

$V \longrightarrow$ Eigen Vector of A^TA

$U \rightarrow$ Eigen Vector of AA^T

Example

$$\begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{32} & 0 \\ 0 & \sqrt{18} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

CHAPTER 30

Lecture 30

Lecture 30

Linear Transformations

Without Coordinates : no matrix
With Coordinates : MATRIX

Example 1: Projection   Example 2: Rotation

$T: R^2 \longrightarrow R^2$

$V \longrightarrow T(V)$
{Matrix}   {Matrix}

Linear Transformation

$\rightarrow T(V+W) = T(V) + T(W)$
$\rightarrow T(cV) = cT(V)$
$\Downarrow$
$$\boxed{T(cV+dW) = cT(V) + dT(W)}$$

Example 3: Matrix A
$T(V) = AV$

$T: R^3 \longrightarrow R^2$

Information needed to know $T(V)$ for all inputs.
$T(V_1), T(V_2) --- T(V_n)$ for any basis $V_1 --- V_n$

CHAPTER 31

Appendix 1

6 {from Introduction to linear Algebra → Gilbert Strang}

Eigenvalues & Eigenvectors

1. Introduction to Eigen Value

→ Eigenvalues have their greatest importance in dynamic problems.

Eigen Vectors

→ Almost all vectors changes direction, when they are multiplied by A. Certain exceptional Vector x are in the Same direction as Ax. Those are the "eigen vectors"

$$\boxed{Ax = \lambda x}$$

λ: Eigen Value of A

x: Eigen Vector

All Vectors are eigenvectors of I. {As $Ix = x$}
→ & All eigen value of I is $\lambda = 1$.

⇒ Most of n by 2 matrices have two eigen vector directions & two eigen value.

$Ax - \lambda x = 0$

$(A - \lambda I)x = 0$ — ①

for a non zero eigen Vector x eq ① is Possible Only if $A - \lambda I$ is Singular.

⇒ $\boxed{|A - \lambda I| = 0}$ → from here we can find λ's

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When A is Squared, the eigenvectors stay the Same. The eigenvalues are Squared.

$$Ax = \lambda x$$

$$\Rightarrow AAx = A(\lambda x)$$

$$\Rightarrow A^2x = \lambda(Ax)$$

$$\Rightarrow \boxed{A^2x = \lambda^2 x} \quad \{As, Ax = \lambda x\}$$

Eigen Value Eigen Vector

Every Vector is linear Combination of the eigen Vectors.

The product of the n eigen values equals the determinant.

$$\lambda_1 + \lambda_2 + \cdots + \lambda_n = |A|$$

The Sum of the n eigen Values equals the Sum of the n diagonal entries. (Trace)

$$\lambda_1 + \lambda_2 + \cdots \lambda_n = \text{trace} = a_{11} + a_{22} + \cdots + a_{nn}$$

2. Diagonalizing a Matrix

⇒ The matrix A turns into a diagonal matrix Λ when we use the eigenvectors properly.

↳ This is matrix form of our Key idea.

the

<u>Diagonalization</u> $\Rightarrow$ Suppose the n by n matrix A has n linearly independent eigenvectors $x_1 \cdots x_n$.

$\rightarrow$ Put them into the columns of an eigenvector matrix S.

$\rightarrow$ Then $S^{-1}AS$ is the eigen value matrix. (Λ)

$S \Rightarrow$ Eigenvector matrix

$\Lambda \Rightarrow$ Eigen value matrix

$$S = \begin{bmatrix} x_1 & x_2 & x_3 & \cdots & x_n \end{bmatrix} \qquad \Lambda = S^{-1}AS = \begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \lambda_3 & & \\ & & & \ddots & \\ & & & & \lambda_n \end{bmatrix}$$

Eigen vectors

<u>Capital lambda</u>

Matrix A is diagonalized.

<u>Proof</u>

$$AS = A\begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} = \begin{bmatrix} Ax_1 & Ax_2 & \cdots & Ax_n \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \cdots & \lambda_n x_n \end{bmatrix}$$

$\Rightarrow$ The trick is to split the matrix AS in S times Λ.

$$\Rightarrow \begin{bmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \cdots & \lambda_n x_n \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \lambda_3 & \\ & & & \ddots \lambda_n \end{bmatrix}$$

$$\Rightarrow AS = S\Lambda$$

$$\Rightarrow \boxed{\Lambda = S^{-1}AS} \text{ or } \boxed{A = S\Lambda S^{-1}}$$

⇒ The matrix S has an inverse, because its Columns (the eigen vectors of A) were assumed to be linearly independent.

↳ Without n independent eigenvectors we cannot diagonalize.

$$\boxed{A^n = S\Lambda^n S^{-1}}$$

$$A = S\Lambda S^{-1}$$

$$A^2 = (S\Lambda S^{-1})(S\Lambda S^{-1}) = S\Lambda(S^{-1}S)\Lambda S$$

$$A^2 = S\Lambda^2 S \quad \text{Similarly}$$

\# Suppose the eigenvalues $\lambda_1 \cdots \lambda_n$ are all different. Then it is automatic that the eigenvectors $x_1 \cdots x_n$ are independent. <u>Any matrix that has no repeated eigenvalues Can be diagonalized</u>

\# The eigenvectors in S Come in the same order as the eigen values in Λ.

\# Some matrix have too few eigenvectors. (repeated eigenvalues). Those matrix cannot be diagonalized.

★ Solving difference Equation

{Fibonacci number F_{100}}

$$\boxed{F_{k+2} = F_{k+1} + F_k}$$

⇒ The Key is to begin with matrix equation $u_{k+1} = A u_k$.

Let $u_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} \Rightarrow u_{k+1} = \begin{bmatrix} F_{k+2} \\ F_{k+1} \end{bmatrix}$

$$\begin{bmatrix} F_{k+2} \\ F_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$$

$$u_{k+1} = A u_k \quad \left\{ A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \right\}$$

$$u_1 = A u_0$$

$$u_2 = A^2 u_0$$

$$u_{100} = A^{100} u_0 = \begin{bmatrix} F_{101} \\ F_{100} \end{bmatrix}$$

$u_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ {Starting of Fibonacci number}

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \rightarrow \left\{ \begin{aligned} \lambda_1 &= \frac{1+\sqrt{5}}{2} = 1.618 \\ \lambda_2 &= \frac{1-\sqrt{5}}{2} = -0.618 \end{aligned} \right\}$$

$$x_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}$$

$$x_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$$

$$u_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\lambda_1 - \lambda_2} \left(\begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} - \begin{bmatrix} \lambda_2 \\ 2 \end{bmatrix} \right)$$

$$u_0 = \left(\frac{x_1 - x_2}{\lambda_1 - \lambda_2} \right)$$

$$U_{100} = A^{100}\left(\frac{x_1 - x_2}{\lambda_1 - \lambda_2}\right)$$

$$\Rightarrow \frac{A^{99}}{\lambda_1 - \lambda_2}\left(\lambda_1 x_1 - \lambda_2 x_2\right)$$

$$\Rightarrow \frac{A^{98}}{\lambda_1 - \lambda_2}\left(\lambda_1^2 x_1 - \lambda_2^2 x_2\right)$$

$$\Rightarrow \frac{\lambda_1^{100} x_1 - \lambda_2^{100} x_2}{\lambda_1 - \lambda_2}$$

$$F_{100} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{100} - \left(\frac{1-\sqrt{5}}{2}\right)^{100}\right] \simeq 3.54 \times 10^{20}$$

So K^{th} term of Fibonacci number $= \dfrac{\lambda_1^K - \lambda_2^K}{\lambda_1 - \lambda_2}$

3. Application to Differential Equations

↳ The whole point of this section is to convert constant-coefficient differential equation into linear algebra.

\# n equations $\frac{du}{dt} = Au$ starting from the vector $u(0)$ at $t=0$.

$\Rightarrow$ $e^{\lambda_1 t}x_1, e^{\lambda_2 t}x_2 \dots e^{\lambda_n t}x_n$ are n solutions.

$\Rightarrow$ So general solution will be linear combination of these solutions.

$\mathcal{S}$

$\Rightarrow$ T

\# T

$$\frac{du}{dt} = Au$$
$$\Downarrow$$
$$u = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2 + \cdots + c_n e^{\lambda_n t} x_n$$

Second Order equation

⇒ The most important equation in mechanics:-

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0 \quad \{\text{Let } m=1\}$$

Le $u = \begin{bmatrix} y \\ \dot{y} \end{bmatrix} \longrightarrow \dot{u} = \begin{bmatrix} \dot{y} \\ \ddot{y} \end{bmatrix}$

$$\begin{bmatrix} \dot{y} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k & -b \end{bmatrix} \begin{bmatrix} y \\ \dot{y} \end{bmatrix}$$

$\dot{u} = Au$

→ $\lambda_1\ \lambda_2$ be is eigen Value

→ $x_1 = \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix}$ $x_2 = \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}$

$$\boxed{u(t) = c_1 e^{\lambda_1 t}\begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} + c_2 e^{\lambda_2 t}\begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}}$$

The Exponential Matrix

↳ We want to write the Solution $u(t)$ in a new form $e^{At}u(0)$.

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots$$
$$e^{At} = I + (At) + \frac{1}{2}(At)^2 + \frac{1}{6}(At)^3 + \cdots$$

Putting $A = S\Lambda S^{-1}$ we get

$$\boxed{e^{At} = S e^{\Lambda t} S^{-1}}$$

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{bmatrix}$$

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CHAPTER 32

Appendix 2

$\boxed{A = A^T}$

Positive Definit Matrix

$\Rightarrow$ Symmetric matrix with all the eigen Values are positive.

→ All the pivots are positive.

→ All Sub-determinants are positive.

① Tests for Positive definite Matrix

→ All Eigen Values are positive

→ All the pivots are positive

→ All sub-determinants are positive

→ $x^T A x > 0$ {except at zero vector}

→ Any Vector

Positive Semidefinit

→ Matrix is Singular

→ All Eigen Values are ≥ 0

→ " Pivots are ≥ 0

→ " Sub-determinant ≥ 0

$X^T A X$ for 2D Case

$$[x_1 \; x_2]\begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [x_1 \; x_2]\begin{bmatrix} ax_1 + bx_2 \\ bx_1 + cx_2 \end{bmatrix}$$

$$\Rightarrow ax_1^2 + 2bx_1x_2 + cx_2^2$$ (Quadratic form)

Graph of $f(x,y) = \bar{x}^T A \bar{x}$

$$= ax^2 + 2bxy + cy^2$$

Example Let $A = \begin{bmatrix} 2 & 6 \\ 6 & 7 \end{bmatrix}$

So $f(x,y) = 2x^2 + 12xy + 7y^2$ → Not Positive definite

It has a Saddle point.

{ Point on the surface with a relative minima along one axial direction and relative maxima along the crossing axis. }

Condition for minima

→ first derivative Zero $\frac{du}{dx} = 0$

→ second derivative Positive. $\frac{d^2u}{dx^2} > 0$

$f(x_1, x_2 \cdots x_n)$ → first derivative is zero for minima

→ Matrix of 2nd derivative is Pos definit for minima

$$\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} \rightarrow \text{Second derivative Matrix}$$

$$\boxed{f_{xx} f_{yy} > 2 f_{xy}^2}$$

A^TA is Positive definite

Let A be m by n matrix. $\{m > n\}$

A^TA
- → Square
- → Symmetric

$x^T A^T A x \Rightarrow (Ax)^T (Ax) \Rightarrow$ length Square
→ $\|Ax\|^2 \geq 0$

$\Rightarrow \|Ax\|^2 > 0$ is Rank of $A = n$

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