Lecture 11

Eigenvectors and diagonalization

★ Eigenvectors and eigenvalues

⇒ $\lambda \in C$ is a eigenvalue of $A \in C^{n\times n}$ if

$$\chi(\lambda) = \det(\lambda I - A) = 0$$

Equivalent

- There exists nonzero $v \in C^n$ st $(\lambda I - A)v = 0$

$$\boxed{Av = \lambda v}$$

any such v is called an eigenvector of A
(associated with eigenvalue λ)

- There exists nonzero $\omega \in C^n$ st $\omega^T(\lambda I - A) = 0$

$$\omega^T A = \lambda \omega^T$$

any such ω is called a left eigenvector of A.

⇒ If v is an eigenvector of A with eigenvalue λ, then so is αv, for any $\alpha \in C$, $\alpha \neq 0$.

⇒ Even when A is real, eigenvalue λ and eigenvector v can be complex.

- Conjugate symmetry

$\Rightarrow$ If A is real and $v \in \mathbb{C}^n$ is a eigenvector associated with $\lambda \in \mathbb{C}$, then $\overline{v}$ is an eigenvector associated with $\overline{\lambda}$

$$\overline{Av} = \overline{\lambda v} \Rightarrow A\overline{v} = \overline{\lambda}\,\overline{v}$$

★ Dynamic Interpretation

$\Rightarrow$ Suppose $Av = \lambda v$ $\quad V = 0$

if $\dot{x} = Ax$ and $x(0) = v$

then $x(t) = e^{\lambda t} v$

$$x(t) = e^{tA}v = \left(I + tA + \frac{(tA)^2}{2!} + \cdots\right)v$$

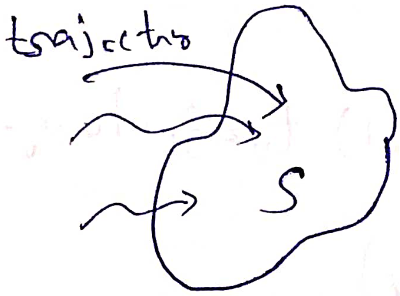
$$= v + \lambda t v + \frac{(\lambda t)^2}{2!}v + \cdots$$

$$= \underline{\underline{e^{\lambda t} v}}$$

$\Rightarrow$ Solution $x(t) = e^{\lambda t}v$ is called mode of system $\dot{x} = Ax$ (associated with eigenvalue λ)

★ Invariant sets

{A set ~~S ⊆ R^n~~ $S \subseteq \mathbb{R}^n$ is invariant under $\dot{x} = Ax$ if whenever $x(t) \in S$, then $x(\tau) \in S \ \forall \tau > t$.}



⇒ Invariant set can have only one element

$$S = \{x_0\}$$

if $Ax_0 = 0$

→ {x_0 is in null space of A}

→ {These are equilibrium points}

Example

⇒ line $\{tv \mid t \in \mathbb{R}\}$ is invariant

{given v is eigenvector}

★ Complex eigenvectors

⇒ Suppose $Av = \lambda v$, $v \neq 0$, λ is complex

⇒ for $a \in \mathbb{C}$ (complex) trajectory $a e^{\lambda t} v$ satisfies $\dot{x} = Ax$.

⇒ hence so does (real) trajectory.

$$x(t) = \mathcal{R}(a e^{\lambda t} v)$$

$$= e^{\sigma t} \begin{bmatrix} v_{re} & v_{im} \end{bmatrix} \begin{bmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{bmatrix} \begin{bmatrix} \alpha \\ -\beta \end{bmatrix}$$

Where,

$$v = v_{re} + j v_{im}, \quad \lambda = \sigma + j\omega, \quad a = \alpha + j\beta$$

⇒ trajectory stays in invariant plane span$\{v_{re}, v_{im}\}$

★ Dynamic interpretation of Left eigenvectors

⇒ Suppose, $\omega^T A = \lambda \omega^T$ $\omega \neq 0$

⇒ Let x be a solution of $\dot{x} = Ax$ $\{x$ is a trajectory$\}$

⇒ $$\frac{d}{dt}(\omega^T x) = \omega^T \dot{x} = \omega^T A x = \lambda(\omega^T x)$$

⇒ So $\omega^T x$ satisfies a scalar 1st order differential equation: $\dot{y} = \lambda y$

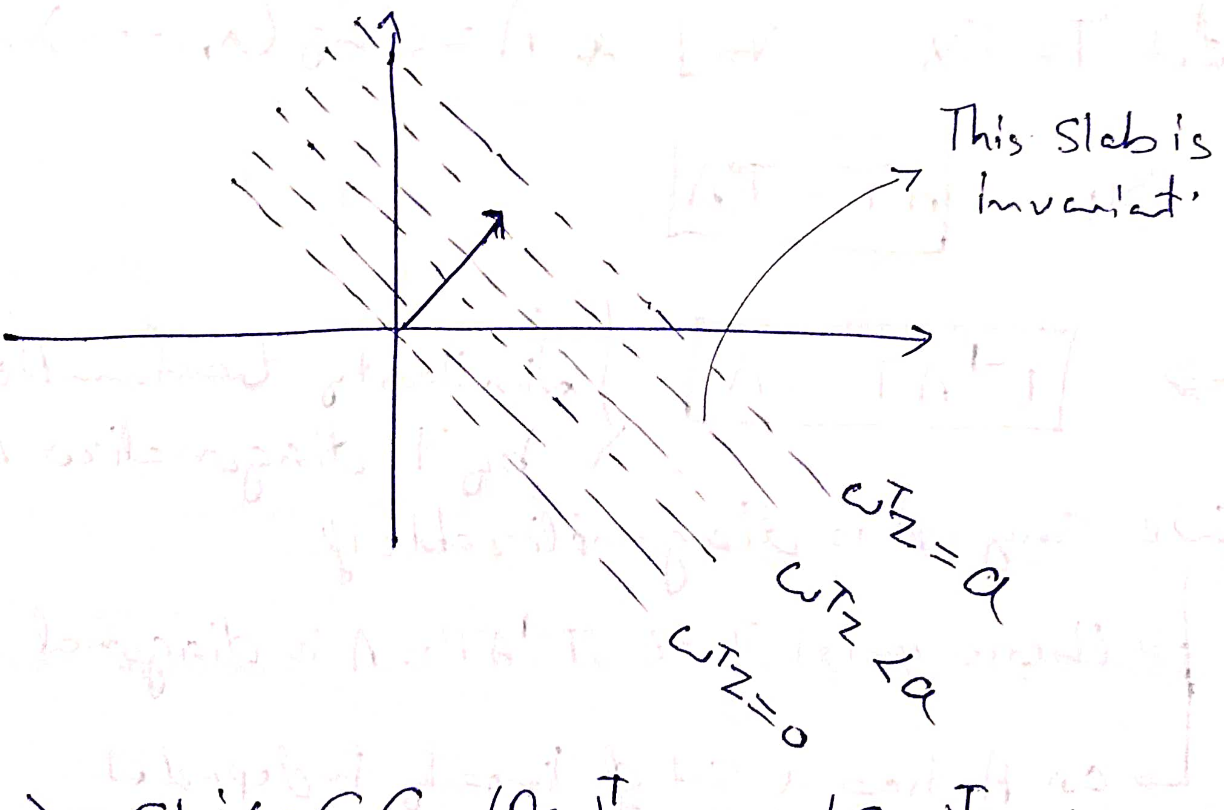
$y = e^{\lambda t} y(0)$

$\Rightarrow \boxed{\omega^T x = e^{\lambda t} \omega^T x(0)}$

$\Rightarrow$ even if trajectory x is complicated, $\omega^T x$ is simple.

$\Rightarrow$ if $\lambda \in \mathbb{R}$, $\lambda < 0$, half space $\{z \mid \omega^T z \leq a\}$ is invariant $(\forall a \geq 0)$

$\omega^T x \downarrow$



$\Rightarrow$ for $\lambda = \sigma + j\omega \in \mathbb{C}$, $(\Re\omega)^T x$ and $(\Im\omega)^T x$ both have form

$$e^{\sigma t}(\alpha \cos(\omega t) + \beta \sin(\omega t))$$

★ Diagonalization

⇒ Suppose $v_1 \cdots v_n$ is a linearly independent set of eigenvectors of $A \in \mathbb{R}^{n \times n}$

$$Av_i = \lambda_i v_i \quad \forall i = 1, \cdots n$$

$$A[v_1 \cdots v_n] = [v_1 \cdots v_n] \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

Let $T = [v_1 \cdots v_n]$ & $\Lambda = \text{diag}(\lambda_1, \cdots \lambda_n)$

So, $\boxed{AT = T\Lambda}$

$\Rightarrow \quad \boxed{T^{-1}AT = \Lambda}$ {Similarity transformation by T diagonalizes A}

⇒ We say A is diagonalizable if

- → there exist T s.t $T^{-1}AT = \Lambda$ is diagonal
- → or A has a set of linearly independent eigenvectors.

{If A is not diagonalizable, it is sometimes called defective}

⇒ Not all matrices are diagonalizable.

★ <u>Distinct eigenvalues</u>

<u>Fact</u>: If A has distinct eigenvalues i.e. $\lambda_i \neq \lambda_j$ for $i \neq j$, then A is diagonalizable.

{Converse is false}

★ <u>Diagonalization and left eigenvectors</u>

⇒ rewrite $T^{-1}AT = \Lambda$ as $T^{-1}A = \Lambda T^{-1}$

$$\begin{bmatrix} w_1^T \\ w_2^T \\ \vdots \\ w_n^T \end{bmatrix} A = \Lambda \begin{bmatrix} w_1^T \\ \vdots \\ w_n^T \end{bmatrix}$$

where $w_1^T, \dots w_n^T$ are rows of T^{-1}.

⇒ thus,

$$\boxed{w_i^T A = \lambda_i w_i^T}$$

⇒ The rows of T^{-1} are (linearly independent) left eigenvectors, normalized so that

$$\boxed{w_i^T v_j = \delta_{ij}} \xleftrightarrow{\text{Equivalent}} \boxed{T^{-1}T = I}$$

★ Model form

⇒ Suppose A is diagonalizable by T

⇒ Lets define new coordinates by

$$x = T\tilde{x}$$

⇒ $\dot{x} = Ax \Rightarrow T\dot{\tilde{x}} = AT\tilde{x}$

$$\Rightarrow \dot{\tilde{x}} = T^{-1}AT\tilde{x}$$

$$\Rightarrow \boxed{\dot{\tilde{x}} = \Lambda\tilde{x}}$$

⇒ In new coordinate system, system is diagonal (decoupled)

⇒ trajectories consists of n independent modes.

$$\boxed{\tilde{x}_i(t) = e^{\lambda_i t}\tilde{x}_i(0)}$$

★ Real model form

⇒ When eigenvalue (hence T) are complex, system can be put in real model form:

$$\boxed{S^{-1}AS = \operatorname{diag}\left(\Lambda_r, \begin{bmatrix}\sigma_{r+1} & \omega_{r+1}\\ -\omega_{r+1} & \sigma_{r+1}\end{bmatrix}, \cdots \begin{bmatrix}\sigma_n & \omega_n\\ -\omega_n & \sigma_n\end{bmatrix}\right)}$$

⇒ Where $\Lambda_r = \operatorname{diag}(\lambda_1 \cdots \lambda_r)$ are the real eigenvalues and $\lambda_i = \sigma_i + j\omega_i \ \forall\, i = r+1, \cdots n$ are complex eigenvalues.

$\Rightarrow$ diagonalization simplifies many matrix expressions

$$(sI-A)^{-1} = (sTT^{-1} - T\Lambda T^{-1})^{-1}$$
$$= (T(sI-\Lambda)T^{-1})^{-1}$$
$$= T(sI-\Lambda)^{-1}T^{-1}$$
$$= T\,\mathrm{diag}\left(\frac{1}{s-\lambda_1}, \cdots, \frac{1}{s-\lambda_n}\right)T^{-1}$$

$$A^K = (T\Lambda T^{-1})^K$$
$$= (T\Lambda T^{-1}) \cdots (TAT^{-1})$$
$$= T\Lambda^K T^{-1}$$
$$= T\,\mathrm{diag}(\lambda_1^k \cdots \lambda_n^k)T^{-1}$$

$$e^A = I + A + A^2/2! + \cdots$$
$$= I + T\Lambda T^{-1} + \frac{(TAT^{-1})^2}{2!}$$
$$= T(I + \Lambda + \Lambda^2/2! + \cdots)T^{-1}$$
$$= Te^{\Lambda}T^{-1}$$
$$= T\,\mathrm{diag}(e^{\lambda_1}, \cdots e^{\lambda_n})T^{-1}$$

$\Rightarrow$ For any analytic function $f: \mathbb{R} \to \mathbb{R}$ (given by power series) we can define $f(A)$ $\forall A \in \mathbb{R}^{n\times n}$ as

$$f(A) = \beta_0 I + \beta_1 A + \beta_2 A^2 + \beta_3 A^3 + \cdots$$

Where, $$f(a) = \beta_0 + \beta_1 a + \beta_2 a^2 + \cdots$$

So $$\boxed{f(A) = T \operatorname{diag}(f(\lambda_i)) T^{-1}}$$

★ <u>Solution via diagonalization</u>

$\Rightarrow$ Assume A is diagonalizable.

$\Rightarrow$ Consider LDS $\dot{x} = Ax$ with $T^{-1}AT = \Lambda$

$\Rightarrow$ then,

$$x(t) = e^{tA} x(0)$$

$$= T \cdot e^{\Lambda t} T^{-1} x(0)$$

$$= [v_1, \cdots v_n] \begin{bmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} w_1^T \\ \vdots \\ w_n^T \end{bmatrix} x(0)$$

$$\Rightarrow [v_1, \cdots v_n] \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} w_1^T x(0) \\ \vdots \\ w_n^T x(0) \end{bmatrix}$$

$$x(t) = [V_1, \dots V_n] \begin{bmatrix} e^{\lambda_1 t} \omega_1^T x(0) \\ \vdots \\ e^{\lambda_n t} \omega_n^T x(0) \end{bmatrix}$$

$$\boxed{x(t) = \sum_{i=1}^{n} e^{\lambda_i t} \left(\omega_i^T x(0)\right) V_i}$$

⇒ thus, any trajectory can be expressed as linear combination of modes.

<u>Interpretation</u>

- (left eigenvectors) decompose initial state $x(0)$ into modal components $\omega_i^T x(0)$
- $e^{\lambda_i t}$ term propagates ith mode forward t seconds.

★ <u>Application: for what $x(0)$ do we have $x(t) \to 0$ as $t \to \infty$?</u>

⇒ divide eigenvalues into those with negative real parts

$$\Re\lambda_1 < 0, \; \dots, \; \Re\lambda_s < 0$$

⇒ and the others,

$$\Re\lambda_{s+1} \geq 0, \; \dots, \; \Re\lambda_n \geq 0$$

⇒ from

$$x(t) = \sum_{i=1}^{n} e^{\lambda_i t} \left(\omega_i^T x(0)\right) V_i$$

⇒ Condition for $x(t) \to 0$ is:

$$\boxed{x(0) \in \text{Span}\{v_1, \cdots v_s\}}$$

as, $x(0) = \sum_{i=1}^{n} (w_i^T x(0))\, v_i$

or

$w_i^T x(0) = 0, \ \forall\ i = s+1, \cdots n$

★ Stability of discrete-time System

⇒ Suppose A diagonalizable,

⇒ Consider discrete-time ~~L~~DS,

$x(t+1) = Ax(t) \longrightarrow \boxed{x(K) = A^K x(0)}$

⇒ if $A = T\Lambda T^{-1} \Rightarrow A^K = T\Lambda^K T^{-1}$

⇒ then, $x(t) = A^t x(0) = T\Lambda^t T^{-1} x(0)$

$$= \sum_{i=1}^{n} \lambda_i^t (w_i^T x(0))\, v_i$$

⇒ for all $x(0)$ if and only if

$|\lambda_i| < 1, \ \forall\ i = 1, \cdots n$

the discrete-time System is stable

(Fact: This Statement is true even when A is not diagonalizable)