Lecture-23

→ Differential equation $\frac{du}{dt} = Au$

→ Exponential e^{At} of a matrix.

Example

$$\frac{du_1}{dt} = -u_1 + 2u_2 \qquad u(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ \{Initial Condition\}}$$

$$\frac{du_2}{dt} = u_1 - 2u_2$$

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad A = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \rightarrow \lambda = 0, -3$$

$$\begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \qquad x_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Solution:

$$u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2$$

$$c_1 = \frac{1}{3} \quad c_2 = \frac{1}{3}$$

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{1}{3} e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$u(\infty) = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

for stability $Re\,\lambda < 0$

Steady State $\lambda = 0$

Blow up if $Re\,\lambda > 0$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$u(0) = Sc$$

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ndition

$\frac{du}{dt} = Au$

Let $u = Sv$ {uncoupling}

$$\Rightarrow S\frac{dv}{dt} = ASv$$

$$\Rightarrow \frac{dv}{dt} = S^{-1}ASv = \Lambda v$$

$$\frac{dv_1}{dt} = \lambda_1 v_1$$

$$\vdots \qquad \vdots$$

$$\cancel{v_1 = c_1 e^{\lambda}} \Rightarrow v(t) = e^{\Lambda t} v(0)$$

$$u(t) = S e^{\Lambda t} S^{-1} u(0) = e^{At} u(0)$$

$$\boxed{e^{At} = S e^{\Lambda t} S^{-1}}$$

$$e^{At} = I + At + \frac{(At)^2}{2} + \frac{(At)^3}{6} + \cdots + \frac{(At)^n}{n!} + \cdots$$

$$(I - At)^{-1} = I + At + (At)^2 + \cdots + (At)^n + \cdots$$

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & \ddots \; e^{\lambda_n t} \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

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$$y'' + by' + ky = 0 \qquad y' = y'$$

$$u = \begin{bmatrix} y' \\ y \end{bmatrix}$$

$$u' = \overbrace{\begin{bmatrix} -b & -k \\ 1 & 0 \end{bmatrix}}^{A} u$$

$$\boxed{u' = Au}$$