Lecture - 15

Symmetric matrices, quadratic forms, matrix norm & SVD

★ Eigenvalues of Symmetric matrices

⇒ Suppose $A \in \mathbb{R}^{n \times n}$ is Symmetric $\boxed{\text{i.e. } A = A^T}$

<u>Fact</u>: The eigenvalues of A are real.

⇒ Let $Av = \lambda v$, $v \neq 0$ $v \in \mathbb{C}^n$

⇒ then

$$\bar{v}^T A v = \bar{v}^T (Av) = \bar{v}^T (\lambda v) = \lambda (\bar{v}^T v)$$

⇒ but also

$$\bar{v}^T A v = \bar{v}^T A^T v = (A\bar{v})^T v = (\overline{Av})^T v$$
$$= (\overline{\lambda v})^T v = \bar{\lambda} (\bar{v}^T v)$$

⇒ So we have $\lambda = \bar{\lambda}$ i.e. $\lambda \in \mathbb{R}$

★ Eigen vectors of Symmetric matrices

<u>Fact</u>: There is a set of orthonormal eigenvectors of A.

$A \in \mathbb{R}^{n \times n}$ $\quad Aq_i = \lambda q_i$ $\quad i \in \{1 \cdots n\}$

$$q_i^T q_j = \delta_{ij}$$

⇒ In matrix form, there is an orthogonal Q s.t.

$$Q^{-1}AQ = Q^TAQ = \Lambda$$

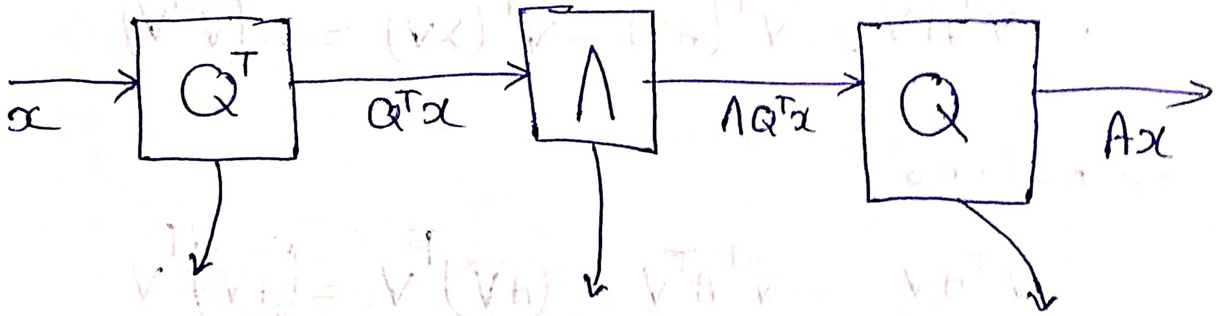
hence, $A = Q\Lambda Q^T = \sum_{i=1}^{n} \lambda_i (q_i q_i^T)$

Outer product

{Projection onto the line} ← ~~diad~~ (dyads)

★ Interpretations

$A = Q\Lambda Q^T$



{Resolve x into q_i coordinates} {Scale coordinates by λ_i} {reconstitute with basis q_i}

★ Proof (case of λ_i distinct)

Suppose $v_1, \dots v_n$ is a set of linearly independent eigenvectors of A.

$$Av_i = \lambda_i v_i \qquad \|v_i\| = 1$$

then,

$$v_i^T(Av_j) = \lambda_j v_i^T v_j = (Av_i)^T v_j = \lambda_i v_i^T v_j$$

So, $(\lambda_i - \lambda_j) V_i^T V_j = 0$

if $\lambda_i \neq \lambda_j \;\; \forall i \neq j$

$$\Rightarrow V_i^T V_j = 0$$

- In this case we say: independent Eigenvectors are orthogonal.
- In general case (λ_i not distinct) we must say: independent eigenvectors can be chosen to be orthogonal.

★ Quadratic form

$\Rightarrow$ A function $f: \mathbb{R}^n \to \mathbb{R}$ of the form

$$f(x) = x^T A x = \sum_{i,j=1}^{n} A_{ij} x_i x_j$$

is called a quadratic form.

$$(x^T A x)^T = x^T A^T x = x^T A x$$

$\Rightarrow$ In a quadratic form we may as well assume $A = A^T$ since,

$$x^T A x = x^T \underbrace{\left(\frac{A + A^T}{2}\right)}_{\text{Called the Symmetric Part of } A} x$$

<u>Uniqueness:</u> If $x^T A x = x^T B x \ \forall x \in R^n$ and $A = A^T$, $B = B^T$, then $A = B$.

- $\|Bx\|^2 = x^T B B^T x$
- $\sum_{i=1}^{n-1} (x_{i+1} - x_i)^2$
- $\|Fx\|^2 - \|Gx\|^2$

$= x^T (F^T F - G^T G) x$

Example of Quadratic form

$\Rightarrow$ Set defined by quadratic form:

- $\{x \mid f(x) = a\}$ is called quadratic surface
- $\{x \mid f(x) \leq a\}$ is called a quadratic region

★ Inequalities for Quadratic form

$\Rightarrow$ Suppose $A = A^T$, $A = Q \Lambda Q^T$ with eigenvalues sorted so $\lambda_1 \geq \cdots \geq \lambda_n$

$$\begin{aligned} x^T A x &= x^T Q \Lambda Q^T x \\ &= (Q^T x)^T \Lambda (Q^T x) \\ &= \sum \lambda_i (q_i^T x)^2 \\ &\leq \lambda_1 \sum_{i=1}^{n} (q_i^T x)^2 = \lambda_1 \|Q^T x\|^2 \\ &= \lambda_1 \|x\|^2 \end{aligned}$$

$$\boxed{x^T A x \leq \lambda_1 x^T x}$$

$\Rightarrow$ Similar argument shows

$x^T A x \geq \lambda_n \|x\|^2$, So we have

$$\boxed{\lambda_n x^T x \leq x^T A x \leq \lambda_1 x^T x}$$

$$\boxed{q_1^T A q_1 = \lambda_1 \|q_1\|^2} \qquad \boxed{q_n^T A q_n = \lambda_n \|q_n\|^2}$$

{Inequalities are tight}

★ <u>Positive semidefinite and positive definite matrices</u>

$\Rightarrow$ Suppose $A = A^T \in \mathbb{R}^{n \times n}$

$\Rightarrow$ we say A is positive semidefinite if $x^T A x \geq 0 \; \forall x$.

- $\rightarrow$ denoted $A \succeq 0$
- $\rightarrow$ $A \succeq 0$ if and only if $\lambda_{min}(A) \geq 0$.

{All eigenvalues are nonnegative}

$\Rightarrow$ We say A is positive definite if $x^T A x > 0 \; \forall x \neq 0$

- $\rightarrow$ denoted $A \succ 0$
- $\rightarrow$ $A \succ 0$ if and only if $\lambda_{min}(A) > 0$.

{All eigenvalues are positive}

★ Matrix Inequalities

⇒ We say A is negative semidefinite if $-A \geq 0$

⇒ We say A is negative definite if $-A > 0$

⇒ Otherwise we say A is Indefinite.

If $B = B^T \in \mathbb{R}^n$ we say $A \geq B$ if $A - B \geq 0$

$A < B$ if $B - A > 0$

etc...

⇒ Many properties that you'd guess holds actually do:

- If $A \geq B$ & $C \geq D$ then $A + C \geq B + D$
- If $B \leq 0$ then $A + B \leq A$
- If $A \geq 0$ & $\alpha \geq 0$ then $\alpha A \geq 0$
- If $A \geq 0$, then $A^2 > 0$
- If $A > 0$ then $A^{-1} > 0$

⇒ Matrix inequality is only a partial order, we have

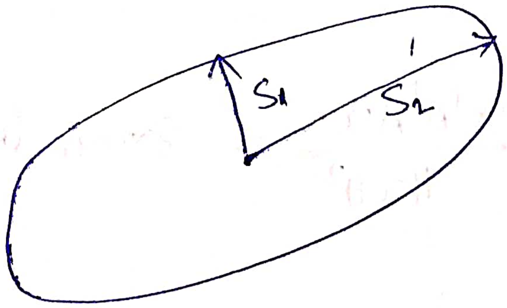
$$A \ngeq B \;,\; B \ngeq A$$

⇒ Such matrices are called incomparable.

★ Ellipsoids

$\Rightarrow$ If $A = A^T > 0$, the set

$$\mathcal{E} = \{x \mid x^T A x \leq 1\}$$



{ An Ellipsoid in $\mathbb{R}^n$ centered at 0 }

$\Rightarrow$ Semi axis are given by $s_i = \lambda_i^{-1/2} q_i$

$\Rightarrow$ In direction q_1, $x^T A x$ is large, hence ellipsoid is thin in direction q_1.

$\Rightarrow$ In direction q_n, $x^T A x$ is small, hence ellipsoid is fat in direction q_n

$\Rightarrow$ $\sqrt{\lambda_{max}/\lambda_{min}}$ gives maximum eccentricity.

★ Gain of a matrix in a direction

$\Rightarrow$ Suppose $A \in \mathbb{R}^{m \times n}$ (not necessarily square or symmetric)

$\Rightarrow$ For $x \in \mathbb{R}^n$ $\|Ax\|/\|x\|$ gives the amplification factor or gain of A in the direction x.

★ Matrix norm

⇒ The maximum gain

$$\max_{x \neq 0} \frac{\|Ax\|}{\|x\|}$$

is called matrix norm of A and is denoted $\|A\|$.

$$\max_{x \neq 0} \frac{\|Ax\|^2}{\|x\|^2} = \max_{x \neq 0} \frac{x^T A^T A x}{\|x\|^2} = \lambda_{max}(A^T A)$$

$$\Rightarrow \boxed{\|A\| = \sqrt{\lambda_{max}(A^T A)}}$$

⇒ Similarly the minimum gain is given by

$$\min_{x \neq 0} \|Ax\| / \|x\| = \sqrt{\lambda_{min}(A^T A)}$$

- max gain input direction is $x = q_1$, eigenvector of $A^T A$ associated with λ_{max}.
- min gain input direction is $x = q_n$, eigenvector of $A^T A$ associated with λ_{min}.

★ Properties of matrix norm

① Consistent with vector norm: matrix norm of $a \in R^{n \times 1}$ is $\sqrt{\lambda_{max}(a^T a)} = \sqrt{a^T a}$

② $\forall x \quad \|Ax\| \leq \|A\| \|x\|$

③ Scaling: $\|\alpha A\| = |\alpha| \, \|A\|$

④ triangle inequality: $\|A+B\| \leq \|A\| + \|B\|$

⑤ definiteness: $\|A\| = 0 \Leftrightarrow A = 0$

⑥ norm of product: $\|AB\| \leq \|A\| \, \|B\|$

★ <u>Singular value decomposition</u>

$$\boxed{A = U \Sigma V^T}$$

Where,

- $A \in R^{m \times n}$, $\text{Rank}(A) = r$
- $U \in R^{m \times r}$, $U^T U = I$
- $V \in R^{n \times r}$, $V^T V = I$
- $\Sigma = \text{diag}(\sigma_1, \cdots \sigma_r)$ where $\sigma_1 \geq \cdots \geq \sigma_r > 0$

$\Rightarrow U = [u_1, \cdots u_r]$, $V = [v_1, \cdots v_r]$

$$A = U \Sigma V^T = \sum_{i=1}^{r} \sigma_i u_i v_i^T \quad \{\text{dyadic expansion}\}$$

- σ_i are the (non zero) singular values of A.
- v_i are the <u>right or input singular vectors</u> of A.
- u_i are the <u>left or output singular vectors</u> of A.

$\Rightarrow A^T A = (U \Sigma V^T)^T (U \Sigma V) = V \Sigma^2 V^T$

hence:

- V_i are eigenvectors of A^TA
- $\sigma_i = \sqrt{\lambda_i(A^TA)} \quad \forall\, i \leq r$
- $\|A\| = \sigma_1$

$\Rightarrow$ Similarly,

$$AA^T = (U\Sigma V^T)(U\Sigma V^T)^T = U\Sigma^2 U^T$$

- U_i are eigenvectors of AA^T (corresponding to nonzero eigenvalues)

$\Rightarrow$ $U_1, \dots, U_r$ are orthonormal basis of range (A)

$\Rightarrow$ $V_1, \dots, V_r$ are orthonormal basis of $N(A)^\perp$