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A Brief on Tensor Analysis

By

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CHAPTER 1

Introduction: Vector and Tensor

Introduction: Vector & Tensor

⇒ If physical events and entities are to be quantified, then a reference frame & a Coordinate System within that frame must be introduced. → {Temporary Structure outside}

⇒ On the other hand, as a frame & Coordinate are mere Scaffolding, it should be possible to express the laws of physics in frame & coordinate free form. → (ie. Invariant form)

⇒ We shall study how, within a fixed frame, the mathematical representation of a physical object or law changes when one Coordinate system is replaced by another.

★ Three dimensional Euclidean Space

⇒ Three dimensional Euclidean Space E_3 may be Characterized by a set of axioms that expresses relationship among primitive Undefined quantities called point, line etc.

⇒ These relationships, so closely Corresponds to the results of ordinary measurements of distance in the physical world that, until the appearance of general relativity, it was thought that Euclidean geometry was the Kinematic model of the universe.

★ Directed line Segment

→ ~~Directed~~
An arrow is an ordered Pair of points (A, B).

A → Tail, B → Head

→ It is customary to represent it as $\overline{AB}$.

⇒ Two arrows are said to be equivalent if one can be brought into coincidence with the other by parallel translation.

⇒ The set of all arrows equivalent to a given arrow is called the (geometric) Vector of that arrow.

→ Length of the Vector $\overline{V}$ is defined as to be the length of any one of the arrow. $\{|\overline{V}|\}$

⇒ We may choose, arbitrarily, a point O in E_3 and call it the origin.

→ The vector X from O to a point P is called the position of P.

★ Addition of two Vector

⇒ Addition of two vectors $\overline{u}$ & $\overline{V}$ may be defined in two equivalent ways :-

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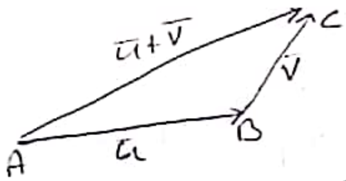
A) Head to Tail Rule

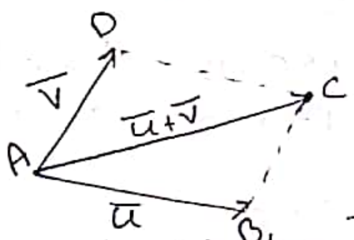
Take any arrow representing u, say $\overline{AB}$. For this choice there is a unique arrow $\overline{BC}$ representing $\overline{V}$.

~~$\overline{V}+\overline{U}$~~ $\overline{u}+\overline{v}$ is defined ~~to~~ be the vector of the arrow $\overline{AC}$.



B) The Parallelogram Rule

Let $\overline{u}$ & $\overline{V}$ be represented by any two arrows $\overline{AB}$ & $\overline{AD}$ having coincident tails, say. Then $\overline{u}+\overline{V}$ is the vector of the arrow $\overline{AC}$, where C is the vertex opposite A of the parallelogram having $\overline{AB}$ & $\overline{AD}$ as co-terminal edges.



★ Multiplication of a Vector $\overline{V}$ by Scalar α

If $\overline{AB}$ is an arrow representing $\overline{V}$, then $\alpha\overline{V}$ is the vector of the arrow $\alpha\overline{AB}$.

Linear Vector Space → The set of all geometric vectors, together with the operations of addition and multiplication by a scalar, form a <u>linear vector space</u>.

★ <u>Things that Vector may Represent</u>

→ Displacement
→ Force acting on a particle.
→ Finite rotation of a rigid body about an axis. (have direction & magnitude)

<u>Different types of objects are represented by Vectors that belong to Different Vector Space</u>

<u>Vector addition may or may not Reflect an attribute of the objects represented.</u>

⇒ For displacements, forces, or Velocities, there are obvious physical analogues of Vector addition.

⇒ For successive finite rotations of a rigid body about a fixed point, there is not.

★ <u>Cartesian Coordinate</u>

↳ { Descarts Characterize 3D Euclidean Space in algebraic terms as follow }

⇒ Through the Origin draw three mutually Perpendicular but otherwise arbitrarily chosen lines.

⇒ Call one of these the x-axis and on it place a point $I \neq O$. The ray from O containing I is called the positive x-axis.

⇒ $\overline{OI}$ is called the unit arrow along the x-axis and we denote its vector by $\overline{e_x}$.

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⇒ Choose one of the remaining lines through O, call it the y-axis, and place on it a point J such that the length of $\overline{OJ}$ is equal to that of $\overline{OI}$.

↳ $\overline{OJ}$ is called the y-unit arrow and we denote its vector by $\overline{e}_y$.

⇒ The remaining line through O is called the z-axis and by arbitrarily adopting the ~~remaining line~~ "right hand thumb rule", we may place a unique point K on the z-axis.

↳ Such that the length of $\overline{OK}$ is equal to that of $\overline{OI}$. $\overline{OK}$ is the z-unit arrow and $\overline{e}_z$ denotes its vector.

⇒ Any point P may be represented by an ordered triple of real number (x, y, z), called the Cartesian coordinates of P.

⇒ When a vector V is represented by the arrow whose tail is the origin O, then the coordinates of the head of this arrow, say $(V_x\ V_y\ V_z)$ are called the Cartesian components of $\overline{V}$.

$$\left\{ \overline{V} \sim (V_x\ V_y\ V_z) \right\}$$

So $\overline{e}_x \sim (1,0,0)$ $\overline{e}_y \sim (0,1,0)$ $\overline{e}_z \sim (0,0,1)$

⇒ With a way of assigning Cartesian ~~coordinate~~ components $(V_x\ V_y\ V_z)$ to a vector V, we may easily deduce the following relations:-

1. $|\overline{V}| = \sqrt{V_x^2 + V_y^2 + V_z^2}$ {by Pythagorean theorem}

2. If α is a real number, then

$$\alpha\overline{V} \sim (\alpha V_x \; \alpha V_y \; \alpha V_z)$$

3. If $\overline{\omega} \sim (\omega_x \; \omega_y \; \omega_z)$ then

$$\overline{V} \pm \overline{\omega} = (V_x \pm \omega_x, V_y \pm \omega_y, V_z \pm \omega_z)$$

4. $\overline{V} = \overline{\omega}$

$\rightarrow V_x = \omega_x$ & $V_y = \omega_y$ & $V_z = \omega_z$

★ The Dot product

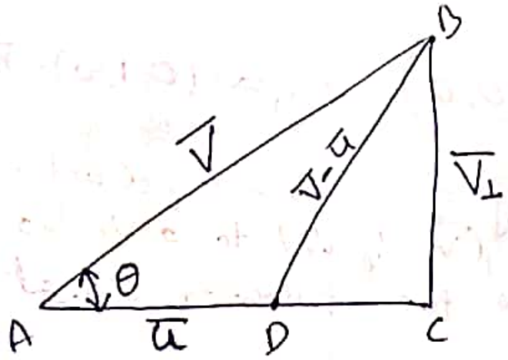
→ The dot product of two vectors $\overline{u}$ & $\overline{V}$ denoted by $\overline{u} \cdot \overline{V}$ arises in many different physical & geometric contexts.

⇒ It can be defined by the formula

$$\overline{u} \cdot \overline{V} = |\overline{u}| \, |\overline{V}| \cos\theta$$

where θ is angle between $\overline{u}$ & $\overline{V}$.

⇒ We dont know how to compute θ So we need to come up with a definition of $\overline{u} \cdot \overline{V}$ that is independent of θ.



The double-headed arrow in θ indicates that θ is always nonnegative regardless of relative orientation of $\overline{u}$ & $\overline{V}$.

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⇒ $\overline{V}$ may be express as sum of vectors

- → $|V|\cos\theta$ parallel to $\overline{u}$
- → $\overline{V}_\perp$ $\perp$ to u

$$|\overline{V}|^2 = |\overline{V}|^2\cos^2\theta + |\overline{V}_\perp|^2 \qquad \{\text{from } \triangle ABC\}$$

$$|\overline{v}-\overline{u}|^2 = (|\overline{V}|\cos\theta - |\overline{u}|)^2 + |\overline{V}_\perp|^2 \qquad \{\text{from } \triangle BOC\}$$

$$= -2|\overline{u}||\overline{v}|\cos\theta + |\overline{v}|^2\cos^2\theta + |\overline{u}|^2 + |\overline{V}_\perp|^2$$

$$= -2|\overline{u}||\overline{v}|\cos\theta + |\overline{u}|^2 + |\overline{v}|^2$$

$$\Rightarrow \boxed{\overline{u}\cdot\overline{v} = |\overline{u}||\overline{v}|\cos\theta = \frac{1}{2}\left(|\overline{u}|^2 + |\overline{v}|^2 - |v-u|^2\right)}$$

⇒ Two vectors whose dot product is zero are said to be orthogonal ($\perp$).

⇒ To evaluate $\overline{u}\cdot\overline{v}$ on a computer we need a component representation.

If $\overline{u} \sim (u_x\ u_y\ u_z)$ $\overline{v} \sim (v_x\ v_y\ v_z)$

$$\boxed{\overline{u}\cdot\overline{v} = u_x v_x + u_y v_y + u_z v_z}$$

⇒ Dot product is a geometric invariant

{does not change by changing ~~Coords~~ Cartesian axis}

★ Cartesian Base Vector

⇒ Given the Cartesian Component $(V_x\ V_y\ V_z)$ of any Vector $\bar{V}$ allow us to Set :-

$$(V_x\ V_y\ V_z) = (V_x\ 0\ 0) + (0\ V_y\ 0) + (0\ 0\ V_z)$$
$$= V_x(1\,0\,0) + V_y(0\,1\,0) + V_z(0\,0\,1)$$

$$\Rightarrow \bar{V} = V_x\bar{e}_x + V_y\bar{e}_y + V_z\bar{e}_z$$

⇒ The Set $\{\bar{e}_x\ \bar{e}_y\ \bar{e}_z\}$ is called the <u>Standard Cartesian basis</u>, and its elements the <u>Cartesian base vectors</u>.

★ The interpretation of Vector Addition

⇒ Suppose for example, that $\bar{V}$ represents the rotation of a rigid body about a fixed point.

↳ Let the direction & magnitude of $\bar{V}$ represent, respectively the axis & angle of rotation, using the right hand thumb rule to insure a Unique Correspondence.

⇒ If this rotation is followed by another, represented by a vector $\bar{u}$, then it is a fact from Kinematics that these two successive rotation are equivalent to a single rotation that we may represent by a Vector $\bar{\omega}$.

⇒ However, in general $\bar{V}+\bar{u} \neq \bar{w}$ ie Successive finite rotations do not add like Vectors.

<u>Note</u>

Rotations may also be represented by matrixes. If the two successive rotations are represented by matrices V and U, then they are equivalent to single rotation representation by the matrix $W = UV$. $\bar{u}, \bar{V}$ or $\bar{w}$ may be identified, respectively with the single real eigenvector of U, V or W.

~~"Vector addition does~~

★ <u>The Cross Product</u>

⇒ The Cross product arises in mechanics when we want to compute the torque of a force about a point.

↳ In electromagnetics when we want to compute the force on a charge moving in a magnetic field.

⇒ The Cross product of two Vectors $\bar{u}$ & $\bar{V}$ is denoted by $\bar{u}\times\bar{V}$ and defined to be the right oriented area of a parallelogram having $\bar{u}$ and $\bar{V}$ as co-terminal edges.

$$|\bar{u}\times\bar{V}| = |\bar{u}||\bar{V}| \sin\theta$$

Where the direction of $\bar{u}\times\bar{V}$ is that of the thumb on the right hand when the fingers are curled from $\bar{u}$ to $\bar{V}$.

$$\boxed{\bar{u}\times\bar{V} = -\bar{V}\times\bar{u}}$$ {from definition}

★ Alternative Interpretation of the Dot and Cross Product. Tensors

⇒ In physics we learn that a constant force $\overline{F}$ acting through a displacement $\overline{D}$ does $\overline{F} \cdot \overline{D}$ unit of work.

⇒ Force $\overline{F}$ may be thought of mathematically as a representation of a <u>linear function</u> that sends any vector $\overline{D}$ into a real number.

⇒ Analogous but more elaborate considerations hold for the cross product.

⇒ Suppose force $\overline{F}$ acts at a point p with position $\overline{X}$. Representing the torque about O as $\overline{X} \times \overline{F}$ leads to the notion of linear operators that sends vectors into vectors.

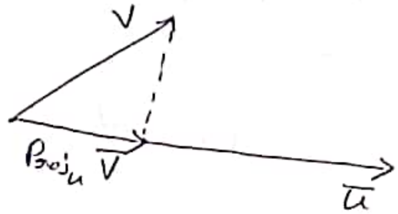
↳ Such operators are called **2nd order tensor.**

⇒ The name tensor comes from elasticity theory where in a loaded elastic body the <u>stress tensor</u> acting on a unit vector normal to a plane through a point delivers the tension acting across the plane at the point.

⇒ Other important 2nd order tensors include the <u>inertia tensor</u> in rigid body dynamics, the <u>strain tensor</u> in elasticity and the <u>momentum-flux tensor</u> in fluid dynamics.

Projection tensor {2nd order tensor}

$$\mathrm{Proj}_u \overline{V} = (\overline{V}\cdot\hat{u})\hat{u}$$



⇒ Left side may be interpreted as the action of the operator Proj_u on vector V.

⇒ To qualify for the title of tensor, Proj_u must be linear:

$$\mathrm{Proj}_u(\beta\overline{V}+\gamma\overline{W}) = \beta\,\mathrm{Proj}_u\overline{V} + \gamma\,\mathrm{Proj}_u\overline{W}$$

⇒ The direct product $\overline{u}\overline{v}$ of two vectors $\overline{u}$ and $\overline{v}$ is a tensor that sends any vector $\overline{w}$ into a new vector according to the rule.

$$\overline{u}\overline{v}(\overline{w}) = \overline{u}\,(\overline{v}\cdot\overline{w})$$

⇒ Thus in particular,

$$\mathrm{Proj}_u = \hat{u}\hat{u}$$

⇒ Tensors such as Proj_u, that can be represented as direct products, are called dyads.

↳ We shall see that any 2nd order tensor can be represented as a linear combination of dyads.

Note

Many authors denotes the direct product $\overline{u}\overline{v}$ by $\overline{u}\otimes\overline{v}$.

★ Definations

Two 2nd order tensor S and T are said to be equal if their action on all vectors v is the same.

$$S = T \iff Sv = Tv \quad \forall v$$

⇒ The zero and identity (unit) tensor are denoted and defined, respectively by

→ $0v = 0 \quad \forall v$ {Zero tensor}

→ $1v = v$ {Identity tensor}

⇒ The transpose of 2nd order tensor T is defined as the unique 2nd order tensor T^T such that

$$u \cdot Tv = v \cdot T^T u \quad \forall u, v$$

⇒ A 2nd order tensor T is said to be

(a) Symmetric if $T = T^T$

(b) Skew if $T = -T^T$

(c) Singular if there exist a $v \neq 0$ such that $Tv = 0$.

⇒ An arbitrary tensor T may always be decomposed into the sum of a symmetric & skew tensor as follows:

$$T = \underbrace{\frac{1}{2}(T + T^T)}_{\text{Symmetric}} + \underbrace{\frac{1}{2}(T - T^T)}_{\text{Skew}}$$

★ The Cartesian Components of a Second Order Tensor

⇒ Cartesian Component of a Second order tensor T fall out almost automatically when we apply T to any Vector v expressed in terms of its Cartesian Components.

$$Tv = T(v_x e_x + v_y e_y + v_z e_z)$$
$$= v_x T e_x + v_y T e_y + v_z T e_z$$

But Te_x, Te_y & Te_z are Vectors and therefore may be expressed in terms of their Cartesian Components, which we label as follows:-

$$Te_x = T_{xx} e_x + T_{yx} e_y + T_{zx} e_z$$
$$Te_y = T_{xy} e_x + T_{yy} e_y + T_{zy} e_z$$
$$Te_z = T_{xz} e_x + T_{yz} e_y + T_{zz} e_z$$

⇒ The 9 coefficients T_{xx}, T_{xy} ---- T_{zz} are the Cartesian Components of T.

⇒ We indicate this by writing $\bar{T} \sim T$ where T^T is the matrix of Coefficients appearing above.

$$\bar{\bar{T}} \sim T = \begin{bmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{bmatrix}$$

$$T_{xx} = e_x \cdot Te_x \;, \quad T_{xy} = e_x \cdot Te_y \quad \text{etc...}$$

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★ The Cartesian Basis for Second Order Tensor

Let us consider tensor in 2-dimension

$$Tv = V_x (T_{xx} e_x + T_{yx} e_y) + V_y (T_{xy} e_x + T_{yy} e_y)$$

But $V_x e_x = (\bar{V} \cdot e_x) e_x = e_x e_x (V)$

So $$Tv = (T_{xx} e_x e_x + T_{xy} e_x e_y + T_{yx} e_y e_x + T_{yy} e_y e_y) V$$

$$\Rightarrow \boxed{T = T_{xx} e_x e_x + T_{xy} e_x e_y + T_{yx} e_y e_x + T_{yy} e_y e_y}$$

⇒ Thus in 3D any 2nd order tensor can be represented as a unique linear combination of 9 dyads.

CHAPTER 2

General Bases and Tensor Notation

2

General Bases and Tensor Notation

⇒ While the laws of mechanics can be written in coordinate-free form, they can be solved, in most cases, only if expressed in component form.

⇒ An aim of tensor analysis is to embrace arbitrary coordinate systems and their associated bases, yet to produce formulas for computing invariants, such as the dot product, that are as simple as the Cartesian forms.

★ General Bases

Let $\{g_1, g_2, g_3\}$ denotes any fixed set of non-coplanar vectors. Then any vector v may be represented uniquely as:

$$v = v^1 g_1 + v_2 g_2 + v^3 g_3 = \sum_{1}^{3} v^i g_i$$

⇒ The set $\{g_1, g_2, g_3\}$ is called basis and its elements base vectors.

↳ {Base vectors need not be unit length nor mutually ⊥}

★ Jacobian of a Basis is nonzero

⇒ If $G = [g_1\ g_2 \cdots]$ denotes the $n \times n$ matrix whose columns are the Cartesian components of $g_1, g_2 \cdots$, then $\{g_1, g_2 \cdots\}$ is a basis if & only if $\det G \neq 0$.

⇒ Using almost standard terminology, we shall call G the Jacobian matrix of $\{g_1, g_2 \cdots\}$ and $J = \det G$ the Jacobian of $\{g_1, g_2 \cdots\}$.

★ The Summation Convention

{The Summation Convention, invented by Einstein, gives tensor analysis much of its appeal.}

$$V = v^1 g_1 + v^2 g_2 + v^3 g_3 = \sum_{i=1}^{3} v^i g_i$$

⇒ Without any loss of information we may drop the summation symbol & write simply,

$$V = v^i g_i$$

Note

The Summation Convention applies only when one dummy index is "on the roof" and the other is "in the cellar".

eg:: $v^i V_i = V^1 V_1 + V^2 V_2 + V^3 V_3$

but $v^i v^i = v^1 v^1$ or $v^2 v^2$ or $v^3 v^3$

{Cartesian tensor notation, is the one exception to this}

* Computing the Dot Product in a General Basis

Suppose we wish to compute the dot product of a vector $u = u^i g_i$ with a vector $v = v^i g_i$.

$$u \cdot v = u^i v^j (g_i \cdot g_j)$$

⇒ This extended expression of $u \cdot v$ is a nine-term mess. We can clean it up by introducing a set of reciprocal base vectors.

* Reciprocal Base Vector

Let $u = u^1 g_1 + u^2 g_2$ {a given basis $\{g_1, g_2\}$}

$$\cancel{v = v^1 g_1 \; v^2 g_2}$$

$$\cancel{u \cdot v = (u^1 g_1 + u^2 g_2)}$$

Let $v = v_1 g^1 + v_2 g^2$ {a new basis $\{g^1 \, g^2\}$}

So, $u \cdot v = (u^1 g_1 + u^2 g_2) \cdot (v_1 g^1 + v_2 g^2)$

$$= u^1 v_1 \, g_1 g^1 + u^1 v_2 \, g_1 g^2 + u^2 v_1 \, g_2 g^1 + u^2 v_2 \, g_2 g^2$$

⇒ The idea here is not to choose g^1 & g^2 so that the above expression reduces to

$$u \cdot v = u^1 v_1 + u^2 v_2$$

⇒ Let $\{g_1, g_2 \dots\} = \{g_j\}$ be a basis. Then $\det G \neq 0$

⇒ This implies that G^{-1} exist.

⇒ The element in the i^{th} row of G^{-1} may be regarded as the Cartesian Components of a vector g^i.

$$G^{-1} = \begin{bmatrix} g^1 \\ g^2 \\ \vdots \end{bmatrix} = [g^1\ g^2 \dots]^T = [g^i]$$

⇒ Consistent with this notation we may say $G \equiv [g_j]$ when we wish to regard G as a collection of collum vectors.

⇒ $G^{-1}G = I$ is equivalent to the statement

$$g^i g_j = \delta^i_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

⇒ The symbol δ_j^i is called Kronecker delta.

⇒ The set $\{g^1\ g^2 \dots\} \equiv \{g^i\}$ is called a reciprocal basis and its elements reciprocal base vectors.

★ <u>The roof (Contravariant) and Cellar (Covariant) Components of a Vector</u>

If $\{g_i\}$ is a basis then not only may we express any vector V as $V^i g_i$, we may also represent V as a linear combination of the reciprocal base vectors, thus

$$V = V_i g^i$$

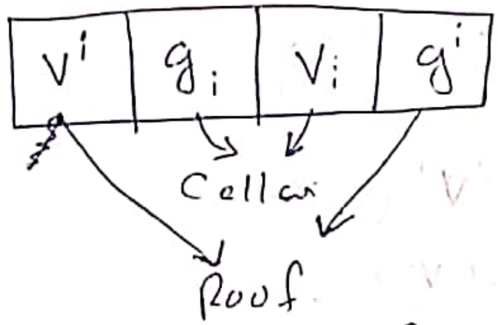
⇒ Backing with the tradition, we shall call the coefficients V^i the roof components of V and the g_i the cellar base vectors.

⇒ Like v_i should be called the cellar component and g^i the roof base vectors.

⇒ The conventional names of v^i and v_i are:

- v^i ⇒ Contravariant component of v
- v_i ⇒ Covariant component of v

⇒ A^i_j is sometimes used to denote the element of a matrix A that sits in the i^{th} row & j^{th} column.



★ Simplification of the component form of the dot product in a general basis

Let us set $u = u^i g_i$ & $v = v_j g^j$

Then $$u \cdot v = u^i v_j g_i g^j = u^i v_j \delta_i^j$$
$$= u^1 v_1 + u^2 v_2 + u^3 v_3$$

★ Computing the Cross product in a General Basis

Some time it is Convenient to denote the roof and Cellar Components of a Vector V by $(V)^i$ and $(V)_i$ respectively.

$$u \times v = (u \times v)_k g^k$$

⇒ To Compute the Cellar Component of $u \times v$ we Set $u = u^i g_i$ & $v = v^j g_j$

$$\text{So } (u \times v)_k = (u \times v) \cdot g_k$$

$$= u^i v^j (g_i \times g_j) \cdot g_k$$

$$= u^i v^j \epsilon_{ijk}$$

⇒ The $3 \times 3 \times 3 = 27$ Symbols ϵ_{ijk} are called Cellar Components of the permutation tensor P.

$$\epsilon_{ijk} = \begin{cases} +J & \text{if } (i,j,k) \text{ is an even permutation of } (1,2,3) \\ -J & \text{if } (ijk) \text{ is an odd permutation of } (1,2,3) \\ 0 & \text{if two or more indicies are equal.} \end{cases}$$

Thus, $u \times v = \epsilon_{ijk} u^i v^j g^k$

⇒ To compute the contravariant components of $u \times v$, we mimic the above procedure, but use contravariant instead of Cartesian base vectors.

$$(u \times v)^k = \epsilon^{ijk} u_i v_j$$

⇒ ϵ^{ijk} are the 27 contravariant components of the permutation tensor defined as follows:-

$$\epsilon^{ijk} = (g^i \times g^j) \cdot g^k = \begin{cases} +J^{-1} & \text{if } (ijk) \text{ is an even permutation of } (1,2,3) \\ -J^{-1} & \text{if } (ijk) \text{ is an odd permutation of } (1,2,3) \\ 0 & \text{if two or more indices are equal.} \end{cases}$$

$$\boxed{\epsilon_{ijk} = J^2 \epsilon^{ijk}}$$

$$\epsilon^{ijk}\epsilon_{pqr} = \begin{vmatrix} \delta^i_p & \delta^i_q & \delta^i_r \\ \delta^j_p & \delta^j_q & \delta^j_r \\ \delta^k_p & \delta^k_q & \delta^k_r \end{vmatrix}$$

If we set $r = k$ then,

$$\epsilon^{ijk}\epsilon_{pqk} = \delta^i_p \delta^j_q - \delta^i_q \delta^j_p$$

★ <u>A second order tensor has four sets of components in general.</u>

Consider a second order tensor T and in general basis $\{g_j\}$, the action of T on each of the basis vector is known, say

$$T g_j = T_j$$

Now each vector T_j may be expressed as a linear combination of the given basis vectors or their reciprocals.

⇒ Choosing the latter, we may write

$$T_j = T_{ij}\, g^i$$

⇒ The 9 components T_{ij} are called the Cellar components of T.

$$\boxed{T_{ij} = g_i \cdot T g_j}$$

If v is a arbitrary vector

$$Tv = T(v^j g_j)$$
$$= v^j T g_j$$
$$= v^j T_{ij}\, g^i$$
$$= T_{ij}\, v^j g^i$$
$$= T_{ij}\, (v \cdot g^j)\, g^i$$
$$Tv = T_{ij}\, g^i g^j (v)$$

So $\boxed{T = T_{ij}\, g^i g^j}$

We see that $\{g^i g^j\}$ is the basis for the set of all ~~tensor~~ 2nd order tensor.

⇒ Repeating the above line of reasoning, but with the roles of the Cellar and roof base vector reversed we have :-

$$Tg^j = T^j = T^{ij} g_i \Rightarrow T = T^{ij} g_i g_j$$

⇒ The 9 coefficient of T^{ij} are called the root components of T.

↳ T^{ij} are the components of T in the basis $\{g_i g_j\}$.

$$T_{ij} = g^i \cdot T g^j$$

⇒ There are two additional sets of components that can be defined, namely

$$T_{\cdot j}^{i} = g^i \cdot T g_j$$

$$T_j^{\cdot i} = g_j \cdot T g^i$$

⇒ These are called mixed components of T.

⇒ The dots are used as distinguishing marks because in general $T_{\cdot j}^{i} \neq T_j^{\cdot i}$

⇒ It is easy to show that T has the following representations in terms of its mixed components

$$T = T_{\cdot j}^{i} g_i g^j = T_j^{\cdot i} g^j g_i \cdot$$

{ $T_{\cdot j}^{i}$ are components of T in the basis $\{g_i g^j\}$

& $T_j^{\cdot i}$ are components of T in the basis $\{g^j g_i\}$. }

⇒ If T is symmetric $\{T = T^T\}$

⇓

$$T_{ij} = T_{ji}$$

$T_{\cdot j}^{i} = T_j^{\cdot i}$ {does not imply $[T_{\cdot j}^{i}]$ & $[T_j^{\cdot i}]$ are symmetric}

★ Change of Basis

⇒ Within a given frame, Vectors and tensors are blissfully unaware of the bases we choose to represent them.

↳ That is they are geometric invariants.

⇒ Under a change of basis it is their components that change, not they themselves.

⇒ Let us start by assuming that each element of the new basis is a known linear combination of the elements of old.

$$\tilde{g}_1 = A^1_1 g_1 + A^2_1 g_2 + A^3_1 g_3$$

$$\tilde{g}_2 = A^1_2 g_1 + \cdots \qquad \text{--- (1)}$$

$$\tilde{g}_3 = A^1_3 g_1 + \cdots$$

We may summarize (1) in either matrix or index form as :-

$$\tilde{G} = GA \quad \text{or} \quad \tilde{g}_j = A^i_j g_i$$

$$G = \tilde{G}A^{-1} \quad \text{or} \quad g_j = (A^{-1})^i_j \tilde{g}_i$$

$$\tilde{G}^{-1} = A^{-1}G^{-1} \quad \text{or} \quad \tilde{g}^i = (A^{-1})^i_j g^j$$

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⇒ The relations between the new and old Components of any Vector V follow immediately because.

$$\tilde{V}_j = \tilde{g}_j \cdot V = A_j^{\,i} g_i \cdot V \;, \quad \tilde{V}^i = \tilde{g}^i \cdot V = (A^{-1})^i_j g^j \cdot V$$

So $\tilde{V}_j = A_j^{\,i} V_i \;, \; \tilde{V}^i = (A^{-1})^i_j V^j$

Likewise for any 2nd order tensor T

$$\tilde{T}_{ij} = A_i^{\,K} A_j^{\,P} T_{KP} \quad \Big| \quad \tilde{T}_{\cdot j}^{\,i} = (A^{-1})^i_K A_j^{\,P} T_{\cdot P}^{\,K}$$

$$\tilde{T}_j^{\,\cdot i} = A_j^{\,K} (A^{-1})^i_P T_K^{\,\cdot P} \quad \Big| \quad \tilde{T}^{ij} = (A^{-1})^i_K (A^{-1})^j_P T^{KP}$$