Controllability and State transfer

★ State transfer

⇒ Consider $\dot{x} = Ax + Bu$ (or $x(t+1) = Ax(t) + Bu(t)$) over time interval $[t_i, t_f]$.

⇒ We say input $u : [t_i, t_f] \to R^m$ steers or transfers state from $x(t_i)$ to $x(t_f)$.

★ Reachability

⇒ Consider state transfer from $x(0) = 0$ to $x(t)$

⇒ We say $x(t)$ is reachable (in t seconds or epochs)

⇒ We define $R_t \in R^n$ as the set of points reachable in t seconds or epochs.

⇒ For CT system $\dot{x} = Ax + Bu$

$$R_t = \left\{ \int_0^t e^{(t-\tau)A} Bu(\tau)\, d\tau \;\middle|\; u : [0,t] \to R^m \right\}$$

⇒ and for DT system $x(t+1) = Ax(t) + Bu(t)$

$$R_t = \left\{ \sum_{\tau=0}^{t-1} A^{t-1-\tau} Bu(\tau) \;\middle|\; \begin{matrix} u(t) \in R^m \\ \forall t = 0, \dots t-1 \end{matrix} \right\}$$

⇒ $R_t \subseteq R_s$ if $t \leq s$

⇒ We define the reachable set R as the set of Points reachable for some t

$$R = \bigcup_{t \geq 0} R_t$$

★ Reachability for discrete-time LDS

⇒ For DT system $x(t+1) = Ax(t) + Bu(t)$, $x(t) \in R^n$

$$x(t) = C_t \begin{bmatrix} u(t-1) \\ \vdots \\ u(0) \end{bmatrix}$$

where $C_t = [B, AB, \cdots, A^{t-1}B]$

⇒ So reachable set at t is $R_t = \text{range}(C_t)$

⇒ by C-H theorem we can express each $A^K \ \forall \ K \geq n$ as linear combination of $A^0, \cdots A^{n-1}$

↳ hence for $t \geq n$ $\text{range}(C_t) = \text{range}(C_n)$

⇒ thus we have,

$$R_t = \begin{cases} \text{range}(C_t) & t < n \\ \text{range}(C) & t \geq n \end{cases}$$

where $C = C_n$ is called the Controllability matrix.

⇒ The system is Controllable if Controllability matrix is rank n.

★ Controllable System

⇒ A system is called reachable or controllable if all states are reachable.

↳ System is reachable if & only if Rank $(C) = n$

★ General state transfer

⇒ With $t_f > t_i$

$$x(t_f) = A^{(t_f - t_i)} x(t_i) + C_{t_f - t_i} \begin{bmatrix} u(t_f - 1) \\ \vdots \\ u(t_i) \end{bmatrix}$$

⇒ hence can transfer $x(t_i)$ to $x(t_f) = x_{des}$

$$x_{des} - A^{(t_f - t_i)} x(t_i) \in R_{t_f - t_i}$$

⇒ Important special case: driving state to zero.

{Sometime called regulating or Controlling state}

★ Least-norm input for reachability

⇒ Assume system is reachable, Rank $(C_t) = n$

⇒ to steer $x(0) = 0$ to $x(t) = x_{des}$, input $u(0), \dots u(t-1)$ must satisfy

$$x_{des} = C_t \begin{bmatrix} u(t-1) \\ \vdots \\ u(0) \end{bmatrix}$$

⇒ Among all u that steer $x(0)=0$ to $x(t)=x_{des}$, the one that minimizes

$$\sum_{\tau=0}^{t-1} \| u(\tau)^2 \|$$

is given by

$$\begin{bmatrix} u_m(t-1) \\ \vdots \\ u_m(0) \end{bmatrix} = C_t^T (C_t C_t^T)^{-1} x_{des}$$

★ <u>Continuous-time reachability</u>

⇒ Consider $\dot{x} = Ax + Bu$, with $x(t) \in \mathbb{R}^n$

⇒ Reachable set at time t is

$$R_t = \left\{ \int_0^t e^{(t-\tau)A} Bu(\tau) d\tau \;\middle|\; u : [0,t] \to \mathbb{R}^m \right\}$$

⇒ <u>Fact</u>: for $t > 0$, $R_t = R = \text{range}(C)$, where

$$C = [B \; AB \; \cdots \; A^{n-1}B]$$

is the controllability matrix of (A,B)

⇒ For continuous-time system, any reachable point can be reached as fast as you like.