Linear models with Normal Noise

★ **Recognizing normal PDFs**

$X \sim N(\mu, \sigma^2)$

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$\Rightarrow f_X(x) = c \cdot e^{-(\alpha x^2 + \beta x + \gamma)} \qquad \alpha > 0$$

$$\alpha x^2 + \beta x + \gamma = \alpha\left(x^2 + \frac{\beta}{\alpha}x + \frac{\gamma}{\alpha}\right)$$

$$= \alpha\left(\left(x + \frac{\beta}{2\alpha}\right)^2 - \frac{\beta^2}{4\alpha^2} + \frac{\gamma}{\alpha}\right)$$

$$f_X(x) = c \cdot e^{-\alpha\left(x + \frac{\beta}{2\alpha}\right)^2} e^{-\alpha\left(-\frac{\beta^2}{4\alpha^2} + \frac{\gamma}{\alpha}\right)}$$

$$\boxed{\mu = -\frac{\beta}{2\alpha}} \qquad \boxed{\sigma^2 = \frac{1}{2\alpha}} \qquad \left\{\text{Hence Normal}\right\}$$

★ **Estimating a normal random variable in the presence of additive noise**

$X = \Theta + W \qquad \Theta, W \sim N(0,1)$; independent

Date
Page

$$\boxed{f_{\Theta|X}(\theta|x) = \frac{f_{\Theta}(\theta)\, f_{X|\Theta}(x|\theta)}{f_X(x)}}$$ {Bayes rule}

$$f_{X|\Theta}(x|\theta) : \quad X = \theta + W \sim N(\theta, 1)$$

$$f_{\Theta|X}(\theta|x) = \frac{1}{f_X(x)} c \cdot e^{-\frac{1}{2}\theta^2} \cdot c e^{-\frac{1}{2}(x-\theta)^2}$$

$$= c(x)\, e^{-\text{quadratic}(\theta)}$$

{Normal PDF}

$$\hat{\theta}_{MAP} = \hat{\theta}_{LMS} = E[\Theta | X = x]$$

$$\downarrow$$

$$\min_{\theta} \left[\frac{1}{2}\theta^2 + \frac{1}{2}(x-\theta)^2\right]$$

$$\downarrow$$

$$\theta + (x - \theta) = 0$$

$$\Rightarrow \theta = x/2$$

$\Rightarrow$ These conclusion is valid for a general Θ and W.

$$\hat{\theta} = ax + b$$ {Modification}

Date

Page

★ The case of multiple observations

$X_1 = \Theta + W_1$ $\qquad \Theta \sim N(x_0, \sigma_0^2)$

$\vdots \qquad \vdots$ $\qquad W_i \sim N(0, \sigma_i^2)$

$X_n = \Theta + W_n$

$\Theta, W_1, \dots, W_n$ are independent.

$$f_{\Theta|X}(\theta|x) = \frac{f_\Theta(\theta) f_{X|\Theta}(x|\theta)}{f_X(x)}$$

Here X is a vector.

$f_{X_i|\Theta}(x_i|\theta)$: $X_i = \theta + W_i \sim N(\theta, \sigma_i^2)$

$\quad \llcorner \; c_i \cdot e^{-\frac{(x_i-\theta)^2}{2\sigma_i^2}}$

$$f_{X|\Theta}(x|\theta) = f_{X_1, X_2 \dots X_n|\Theta}(x|\theta)$$

This is a shorthand notation for the joint pdf of the random variables $X_1 \dots X_n$ conditional on the random variable Θ.

$$f_{X|\Theta}(x|\theta) = \prod_{i=1}^{n} f_{X_i|\Theta}(x_i|\theta)$$

{as $X_1 X_2 \cdots X_n$ are independent}

$$f_{\Theta|X}(\theta|x) = \frac{1}{f_X(x)} \cdot c_0 e^{-\frac{(\theta-x_0)^2}{2\sigma_0^2}} \prod_{i=1}^{n} c_i e^{-\frac{(x_i-\theta)^2}{2\sigma_i^2}}$$

{is Normal}.

$$f_{\Theta|X}(\theta|x) = c \cdot e^{-(\text{quad}(\theta))}$$

$$\text{quad}(\theta) = \frac{(\theta-x_0)^2}{2\sigma_0^2} + \frac{(\theta-x_1)^2}{2\sigma_1^2} + \cdots + \frac{(\theta-x_n)^2}{2\sigma_n^2}$$

$\Rightarrow$ To find Peak we need to find value of θ for which quad (θ) is minimum.

$$\frac{d}{d\theta}(\text{quad}(\theta)) = 0$$

$$\Rightarrow \sum_{i=0}^{n} \frac{\theta - x_i}{\sigma_i^2} = 0$$

$$\Rightarrow \theta \sum_{i=0}^{n} \frac{1}{\sigma_i^2} = \sum_{i=0}^{n} \frac{x_i}{\sigma_i^2}$$

Date
Page

$$\hat{\Theta}_{MAP} = \hat{\Theta}_{LMS} = E[\Theta | X = x] = \frac{\sum_{i=0}^{n} \frac{x_i}{\sigma_i^2}}{\sum_{i=0}^{n} \frac{1}{\sigma_i^2}}$$

$\Rightarrow$ Estimate is Linear

$$\hat{\Theta} = a_0 + a_1 x_1 + \cdots + a_n x_n$$

★ <u>The mean squared error</u>

$$\Rightarrow E[(\Theta - \hat{\Theta})^2 | X = x] = E[(\Theta - \hat{\theta})^2 | X = x]$$

$$\text{Var}(\Theta | X = x)$$

$$\Rightarrow \alpha = \frac{1}{2\sigma_0^2} + \frac{1}{2\sigma_1^2} + \cdots + \frac{1}{2\sigma_n^2}$$

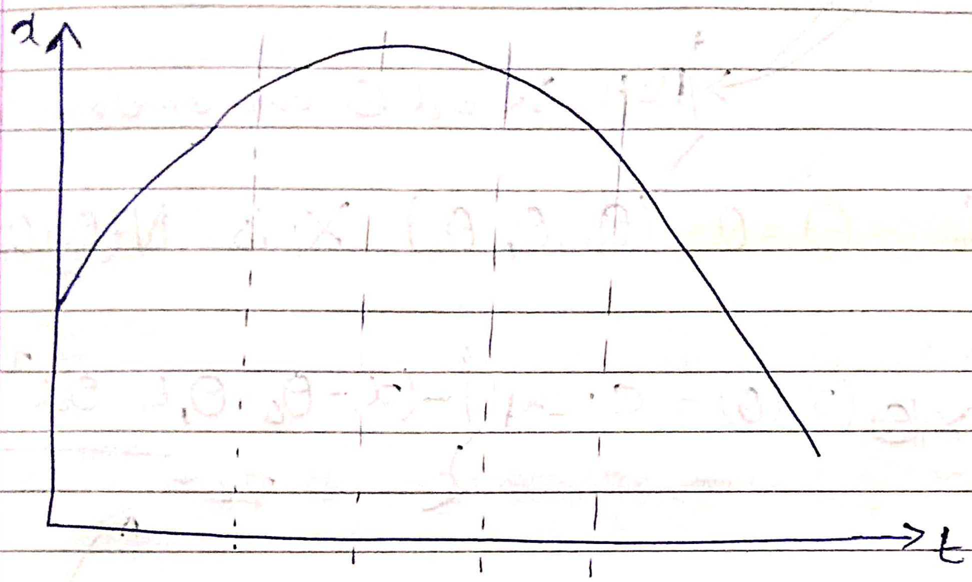
$$\text{Var}(\Theta | X = x) = \left(\frac{1}{\sum_{i=0}^{n} \frac{1}{\sigma_i^2}} \right)$$

$$\Rightarrow E[(\Theta - \hat{\Theta})^2] \quad \{\text{without observation}\}$$

$$\Rightarrow \int \underbrace{E[(\Theta - \hat{\Theta})^2 | X = x]} f_X(x)\, dx$$

$$\Rightarrow \left(\frac{1}{\left(\sum_{i=0}^{n} \frac{1}{\sigma_i^2} \right)} \right)$$

★ <u>The case of multiple parameters: trajectory Estimation</u>



$\Rightarrow$ $x(t) = \theta_0 + \theta_1 t + \theta_2 t^2$

Random variables $\Theta_0, \Theta_1, \Theta_2$
independent & prior f_{Θ_j}

$\Rightarrow$ Measurement at time $t_1 \cdots t_n$

$X_i = \Theta_0 + \Theta_1 t_i + \Theta_2 t_i^2 + W_i \qquad i = 1, \cdots n$

Noise model f_{W_i}

independent W_i; independent from Θ_j

$\Rightarrow$ assume $\Theta_j \sim N(0, \sigma_j^2)$

$W_i \sim N(0, \sigma^2)$

$$f_{\Theta|X}(\theta|x) = \frac{f_\Theta(\theta)\, f_{X|\Theta}(x|\theta)}{f_X(x)}$$

{Both X and Θ are random vectors}

$\Rightarrow$ Given $\Theta = \theta = (\theta_0, \theta_1, \theta_2)$, X_i is $N(\theta_0 + \theta_1 t_i + \theta_2 t_i^2, \sigma^2)$

$$f_{X_i|\Theta}(x_i|\theta) = c \cdot \exp\left\{ \frac{-(x_i - \theta_0 - \theta_1 t_i - \theta_2 t_i^2)^2}{2\sigma^2} \right\}$$

$$f_{\Theta|X}(\theta|x) = \frac{1}{f_X(x)} \prod_{j=0}^{2} f_{\Theta_j}(\theta_j) \prod_{i=1}^{n} f_{X_i|\Theta}(x_i|\theta)$$

$$\Rightarrow c(x) \exp\left\{ -\frac{1}{2}\left(\frac{\theta_0^2}{\sigma_0^2} + \frac{\theta_1^2}{\sigma_1^2} + \frac{\theta_2^2}{\sigma_2^2} \right) - \frac{1}{2\sigma^2} \sum (x_i - \theta_0 - \theta_1 t_i - \theta_2 t_i^2)^2 \right\}$$

MAP estimate

$\rightarrow$ Maximize over $(\theta_0\ \theta_1\ \theta_2)$

$\frac{\delta}{\delta \theta_j}(\text{quad}(\theta)) = 0$ 3 equations, 3 unknowns

linear

Date
Page

★ Linear normal model

⇒ Θ_j and X_i are linear functions of independent normal random variable.

⇒ Inferencing within this class of model goes under the name Linear regression.

⇒ $f_{\Theta|X}(\theta|x) = c(x)\exp\left(-\text{quad}(\theta_1 \cdots \theta_m)\right)$

⇒ Map estimate: maximize over $(\theta_1, \cdots \theta_m)$ (minimize quadratic function)

$$\hat{\Theta}_{MAP,j} = \text{Linear function of } X = (X_1, \cdots X_n)$$

⇒ $\hat{\Theta}_{MAP,j} = E[\Theta_j | X]$

⇒ marginal posterior PDF of Θ_j: $f_{\Theta_j|X}(\theta_j|x)$ is normal.

⇒ $E[(\hat{\Theta}_{i,MAP} - \hat{\Theta}_i)^2 | X = x]$: Same for all x.