Lecture-22

→ Diagonalizing a matrix $\boxed{S^{-1}AS=\Lambda}$
→ Powers of A /equation $u_{k+1}=Au_k$.

$(A-\lambda I) \rightarrow$ Singular

It is the eigen value of A.

$Ax=\lambda x$

$S^{-1}AS=\Lambda$

Columns of 'S' are eigen vectors of A.

→ Suppose we have n linearly independent eigen of A.
→ Put them in columns of S.

$$AS = A\begin{bmatrix} \vdots & \vdots & & \vdots \\ x_1 & x_2 & \cdots & x_n \\ \vdots & \vdots & & \vdots \end{bmatrix} = \lambda_1 x_1 + \lambda_2 x_2 \cdots \lambda_n x_n$$

$$= \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 & & \\ 0 & \lambda_2 & & \\ 0 & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

$$AS = S\Lambda$$

diagonal eigenvalue Matrix.

$$\Rightarrow \boxed{S^{-1}AS=\Lambda}$$

If $Ax = \lambda x$

$$A^2x = \lambda Ax = \lambda^2 x$$

Theorem

$A^K \to 0$ as $K \to \infty$

if at $|\lambda_i| < 1$

$\Rightarrow$ A is sure to have n indipendent eigen vectors (and be diagonalizable)

if all the λ's are different.

$\Rightarrow$ Start with a given Vector u_0

Equation $u_{K+1} = Au_K$ {first Order difference Equation}

$$u_1 = Au_0$$
$$u_2 = A^2 u_0$$
$$\boxed{u_K = A^K u_0}$$

To really Solve:

$$u_0 = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

$$A^{100} u_0 = c_1 \lambda_1^{100} x_1 + c_2 \lambda_2^{100} x_2 + \cdots + c_n \lambda_n^{100} x_n$$

$$= \Lambda^{100} Sc$$

$$\boxed{u_{100} = \Lambda^{100} Sc}$$

Fibonacci Example: 0, 1, 1, 2, 3, 5, 8, 13 ...

$$F_{k+2} = F_{k+1} + F_k \quad \text{--- (1)}$$

Let $U_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$

$$F_{k+1} = F_{k+1} \quad \text{--- (2)}$$

using (1) & (2) as system of equation

$$U_{k+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$$

$$\Rightarrow U_{k+1} = A U_k \qquad \cancel{\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}}$$

$$U_0 = \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 - \lambda - 1 = 0$$

$$\lambda_i = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

$$c_1 x_1 + c_2 x_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$x_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} \longleftarrow \lambda_1 = \frac{1}{2}(1+\sqrt{5}) = 1.618$$

$$x_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix} \longleftarrow \lambda_2 = \frac{1}{2}(1-\sqrt{5}) \approx -0.618$$