Math 53 {Multivariable Calculus}

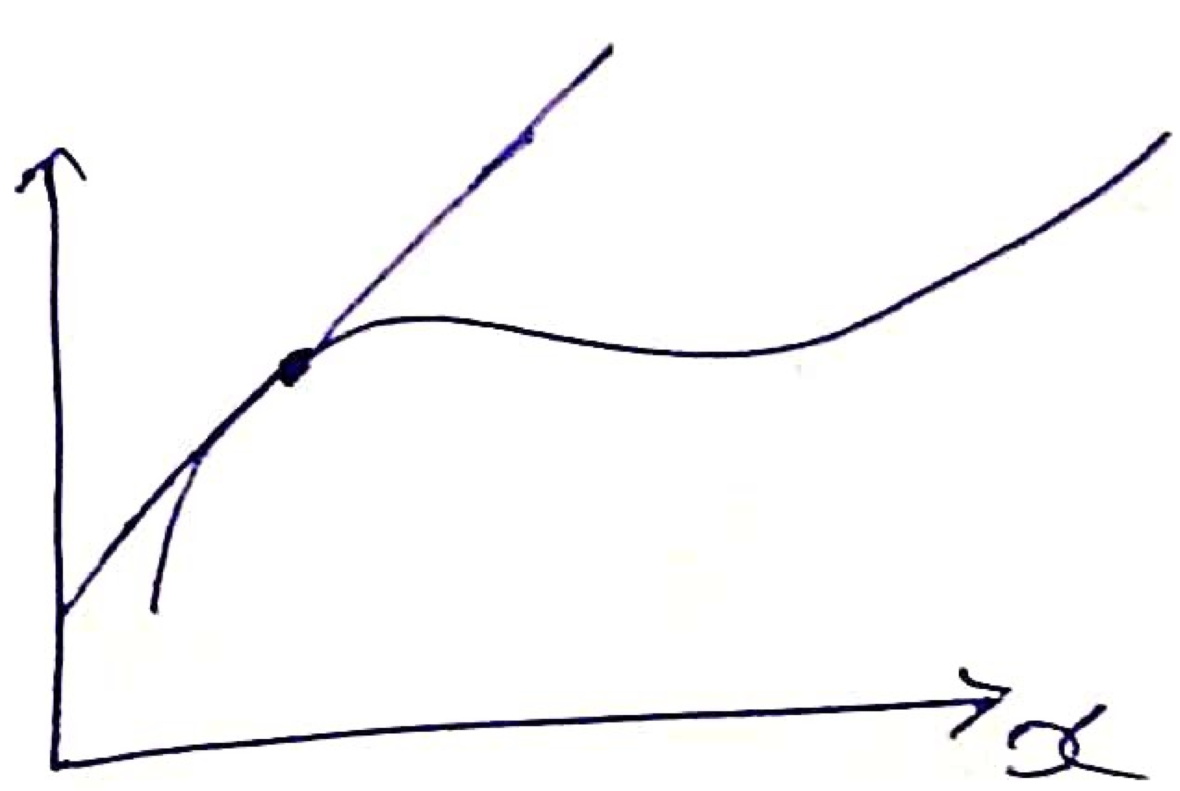
Part 1: Geometric Preliminaries

① Introduction to the Course, Parametrized Curves

★ Review & Introduction

Single variable Calculus

$y = f(x)$ function



$$\frac{dy}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

★(derivative) if this limit exists

→ Useful for optimization problem.
Minimum or maximum of f occurs
where $f' = 0$ (or on boundary of domain
or where f' is not defined)

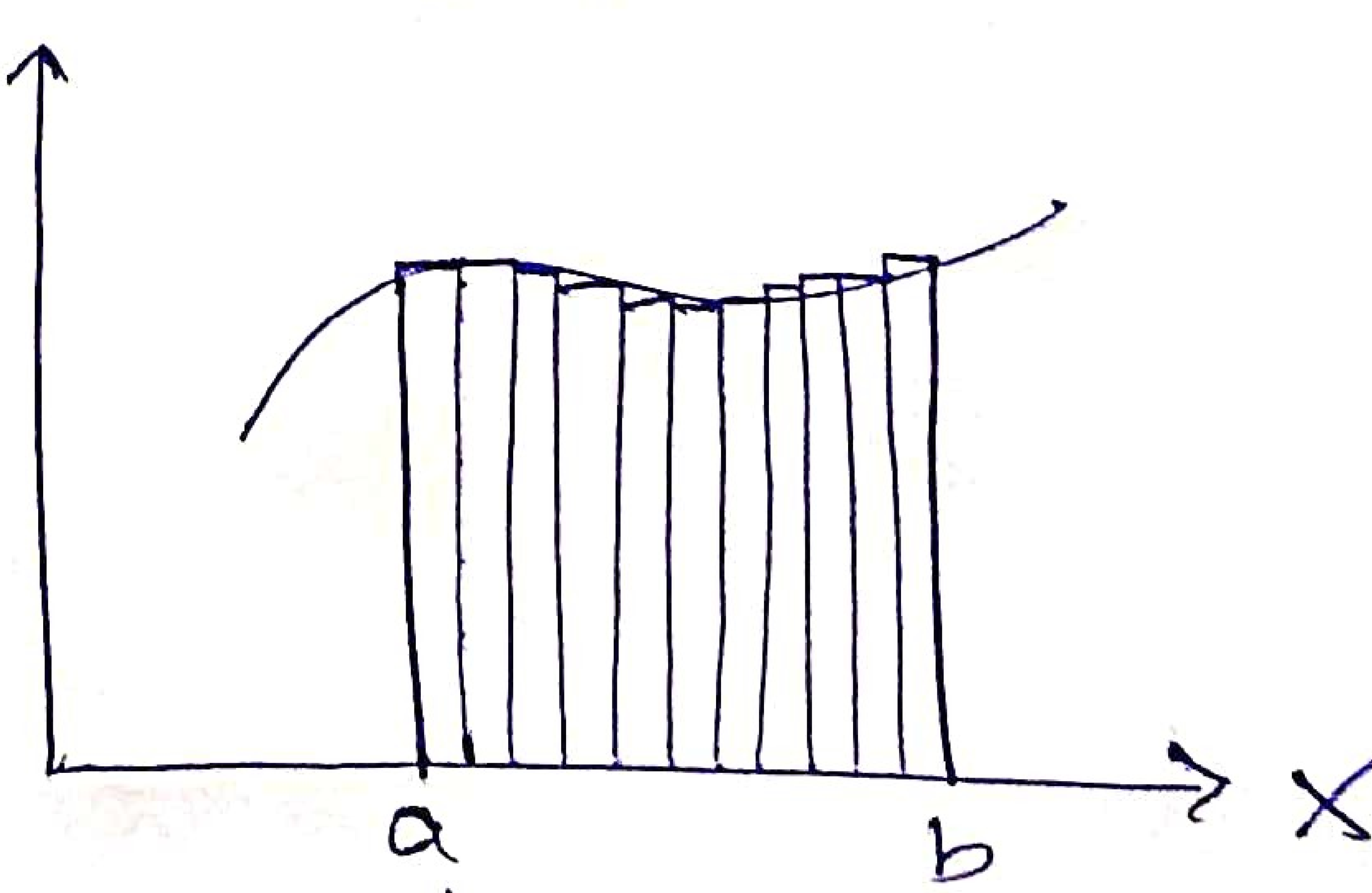
★ Integration

$$\int_a^b f(x)\,dx = \text{"area under the graph"}$$

$a = x_0 < x_1 < x_2 \cdots < x_n = b$

$\Delta x = x_i - x_{i-1} = \frac{b-a}{n}$

$x_i^* =$ a sample point with
$x_{i-1} \leqslant x_i^* \leqslant x_i$



{n subintervals}

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{f(x_i^*)}_{\{\text{height of the } i^{th} \text{ rectangle}\}} \underbrace{\Delta x}_{\{\text{width}\}}$$

★ Fundamental theorem of Calculus

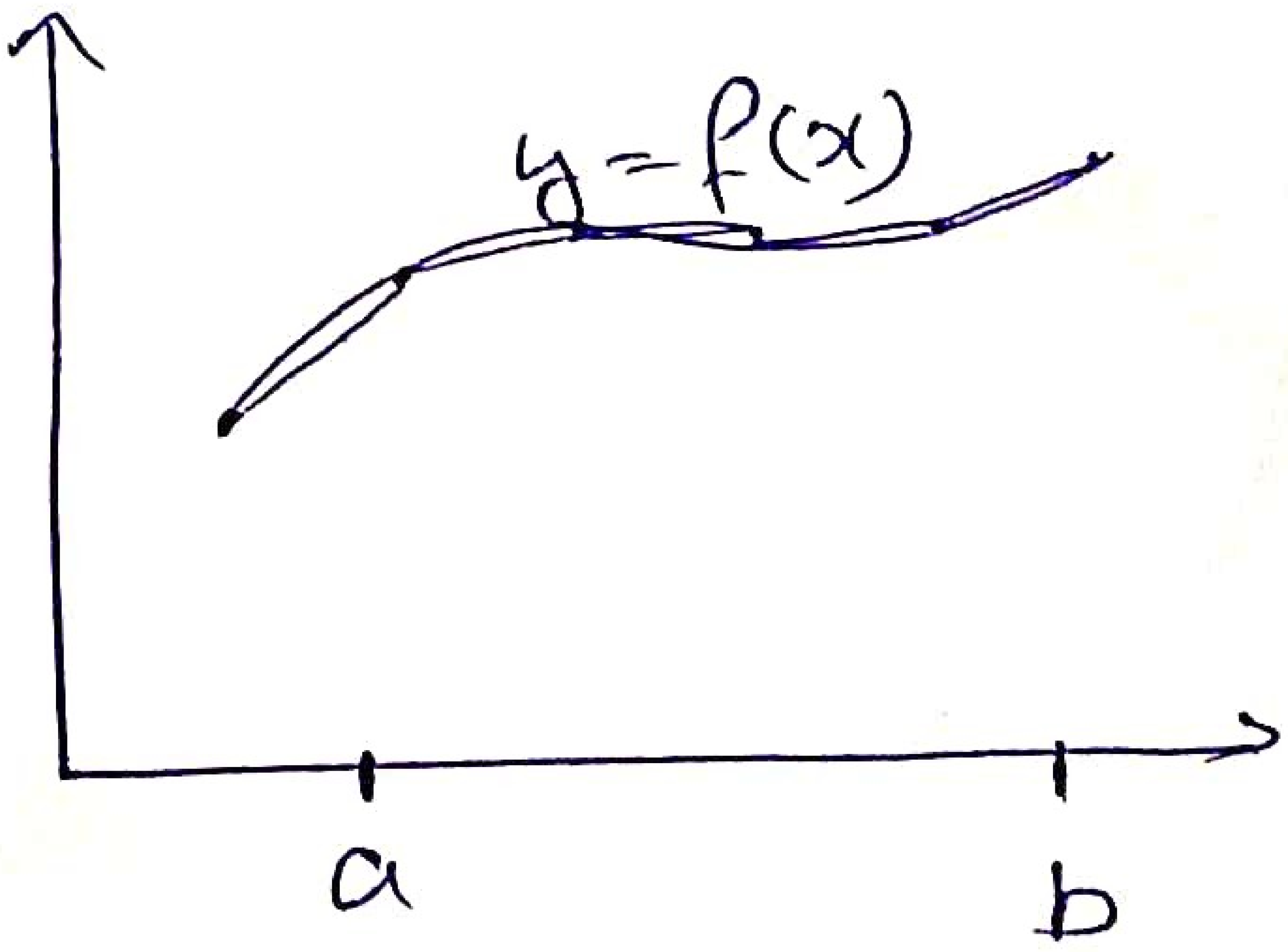
① $\int_a^b f'(x)\,dx = f(b) - f(a)$

② $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$

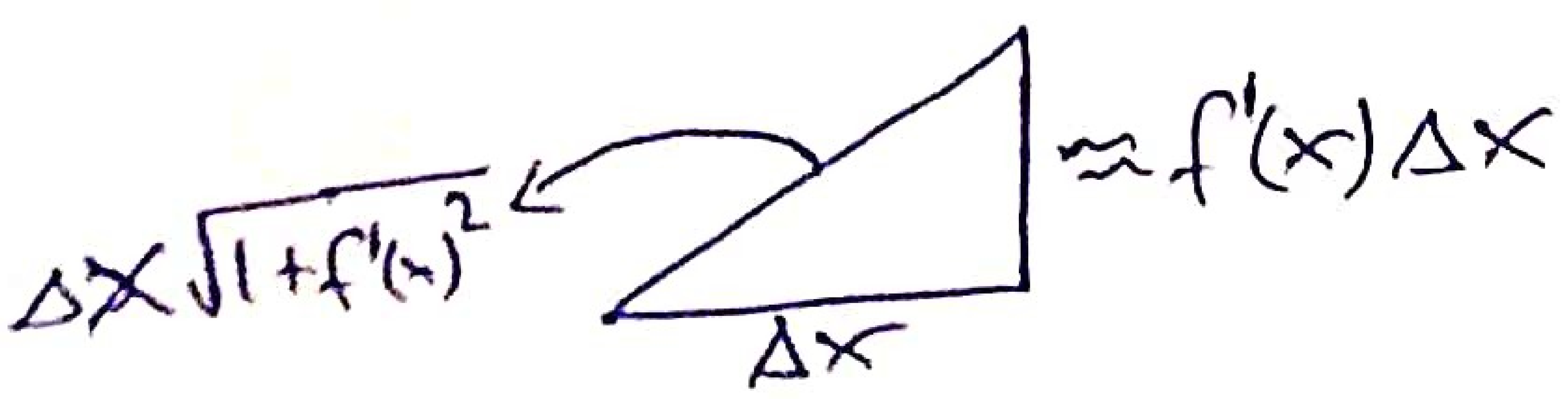
{First one can be deduced from the Second one}

★ Other useful formulas

⇒ Length of the graph $y=f(x)$ from $(a,f(a))$ to $(b,f(b))$



$$L = \int_a^b \sqrt{1+f'(x)^2}\,dx$$



⇒ Area of Surface of revolution around X-axis:

$$A = \int_a^b 2\pi f(x)\sqrt{1+f'(x)^2}\,dx$$

⇒ In this Course:

- More general curves & surfaces in 2d and 3d.
- Functions of 2 or 3 variables
- Partial derivatives
  - Minima & Maxima of functions of 2 or 3 variables.
- Integration in 2 or 3 dimension
- Fundamental Theorem of Line Integrals
- Green's Theorem
- Stokes Theorem
- Divergence Theorem

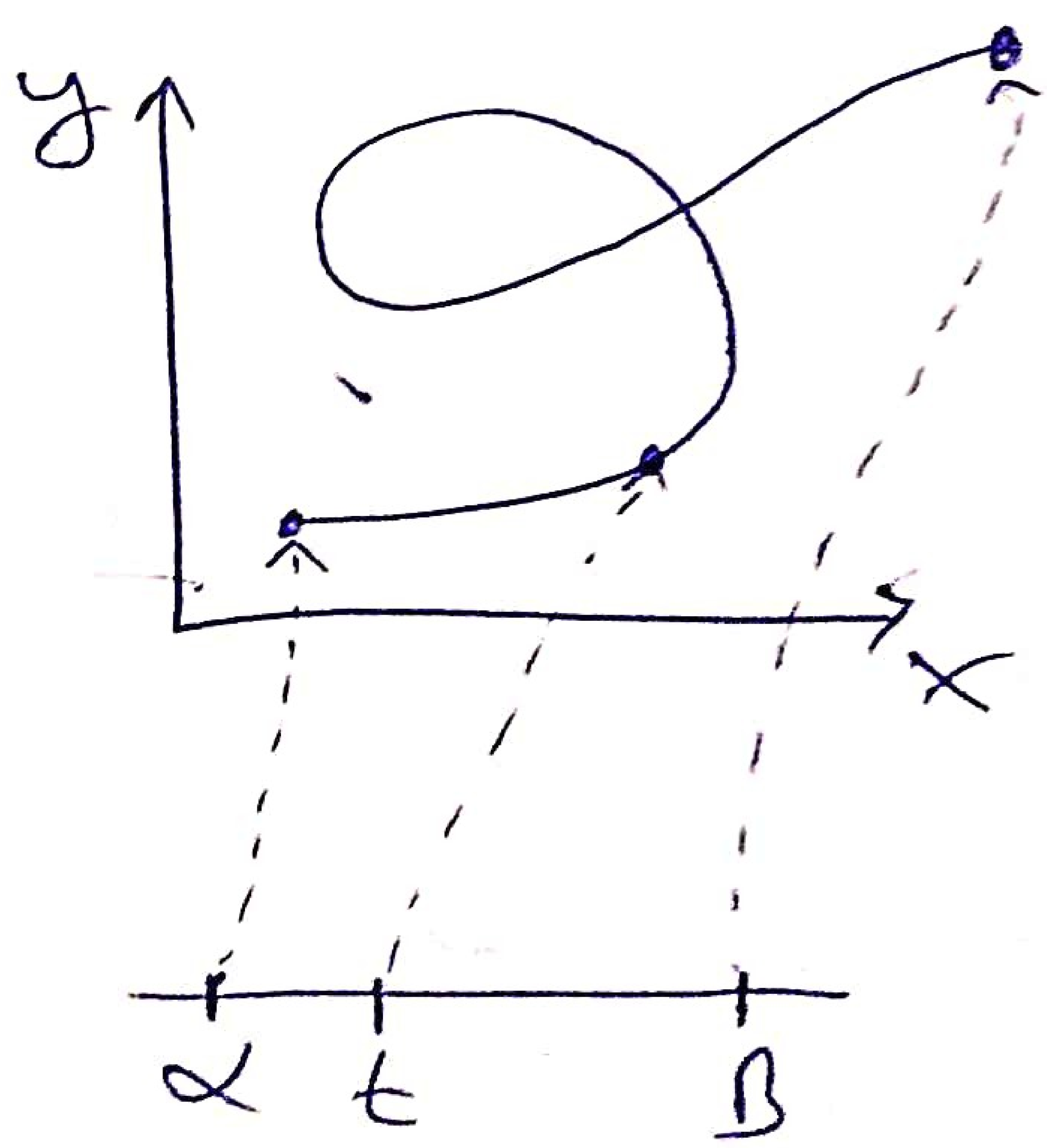
→ Generalization of Fundamental Theorem of Calculus

★ <u>Introduction to parametrized curves</u>

$x = f(t)$

$y = g(t)$

$\alpha \leq t \leq \beta$ → Parameter (You can think of it as time)

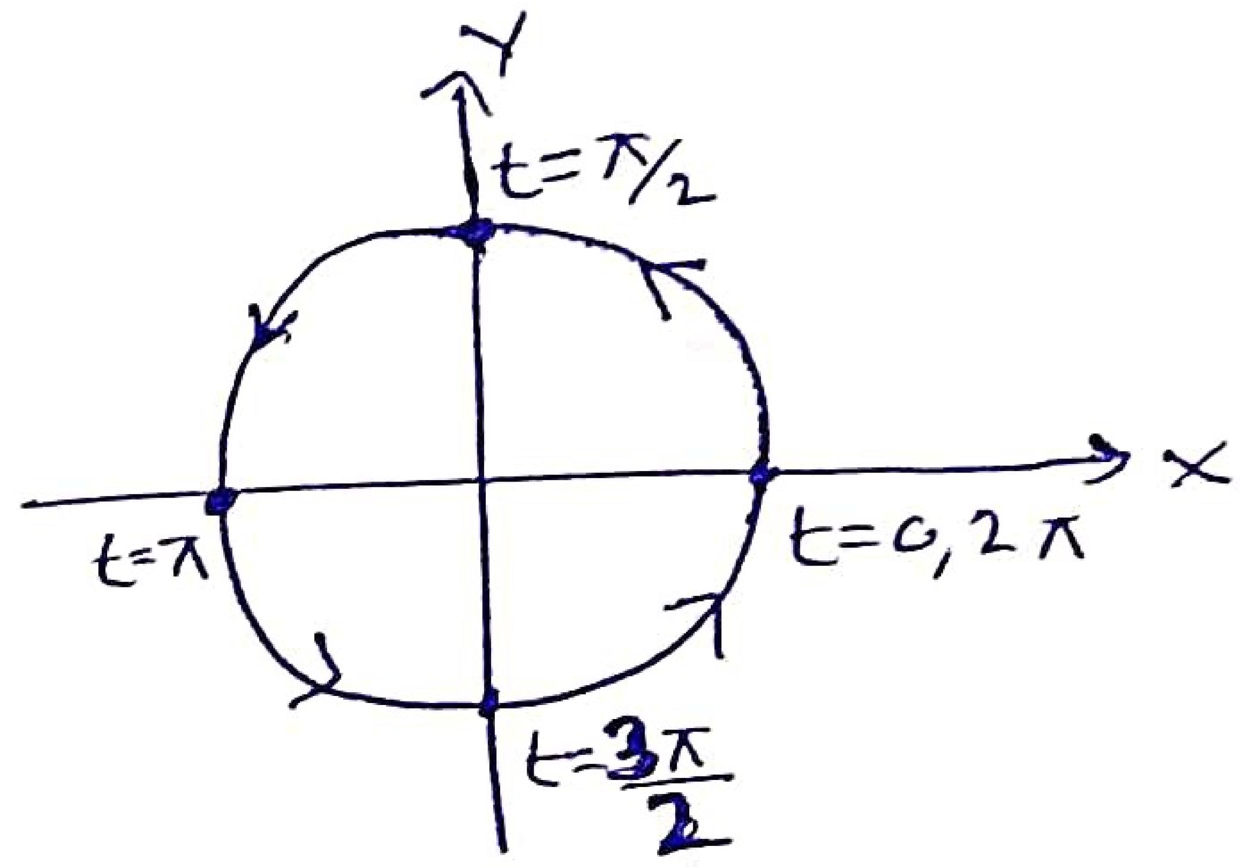


Example 1

$X = \cos t$

$Y = \sin t$

$0 \le t \le 2\pi$



{It is customary to draw arrows which indicate the direction in which t is increasing

$\downarrow$

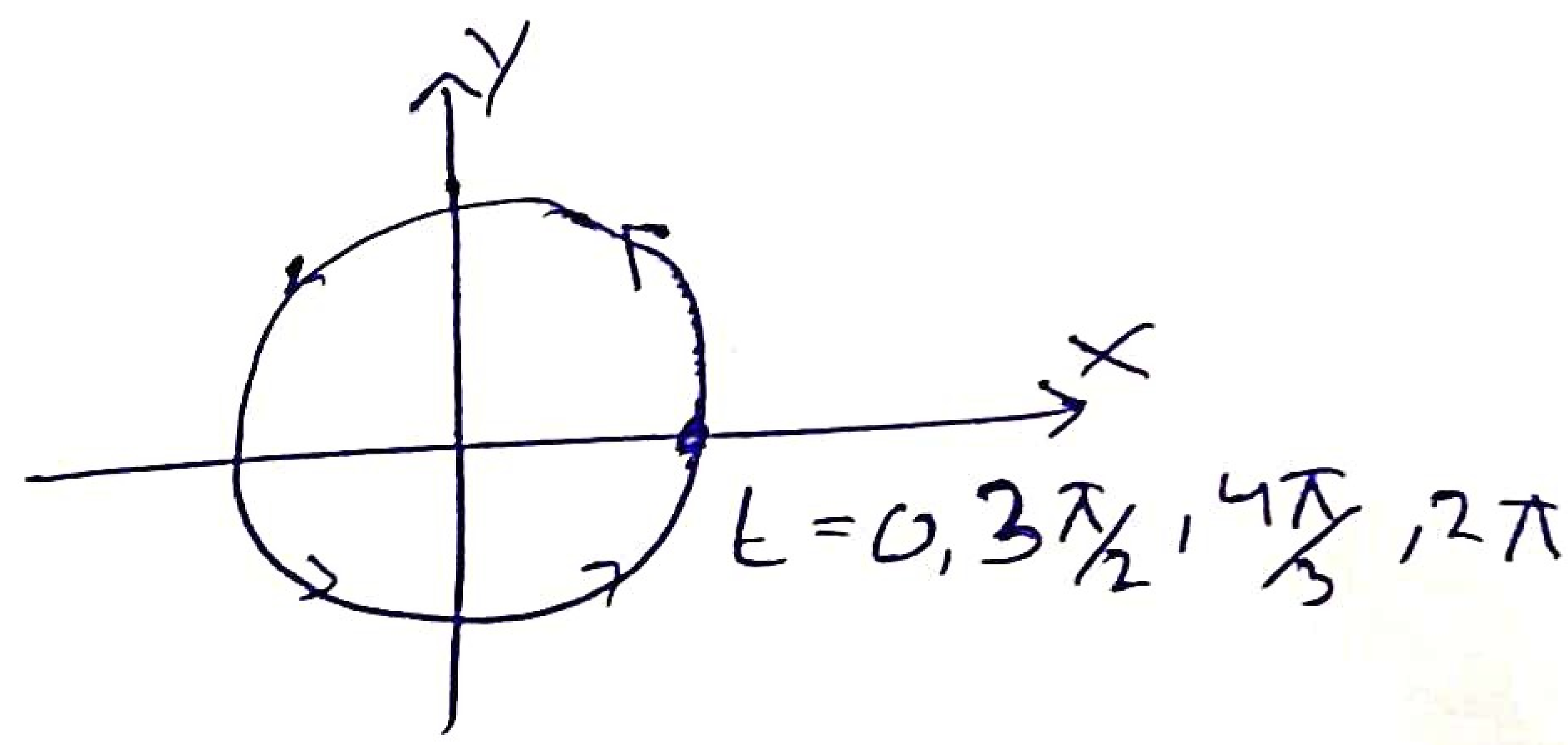
Unit circle, going around counterclockwise at unit speed.

Example 2

$X = \cos(3t)$

$Y = \sin(3t)$

$0 \le t \le 2\pi$



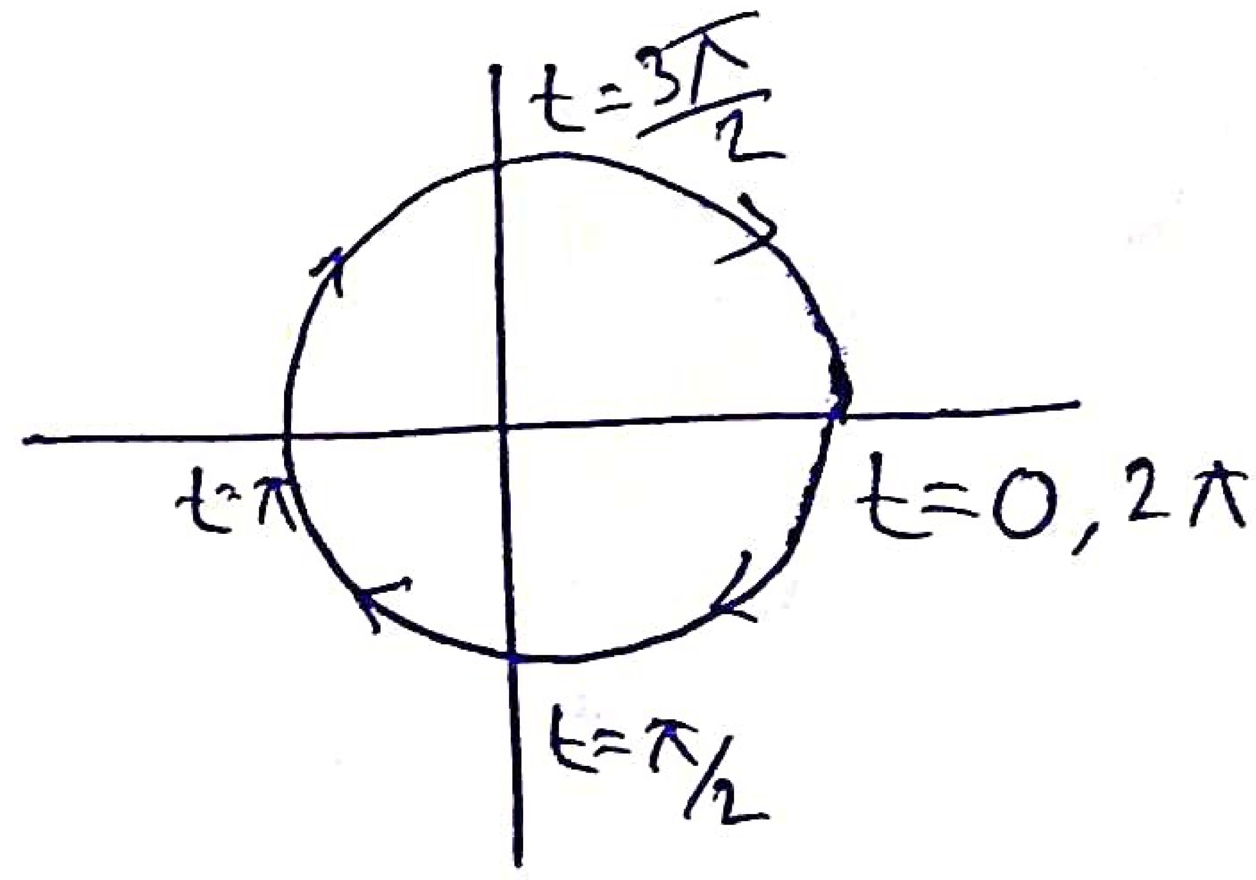
$\downarrow$

Unit circle going around counterclockwise three times.

Example 3

$X = \cos(-t) = \cos t$

$Y = \sin(-t) = -\sin t$



$\downarrow$

Unit circle going around clockwise

$\Rightarrow$ Parametrize Curve

$\hookrightarrow$ Curve + Parametrization
(time-table)

* Slope of a parametrized curve $x=f(t)$ $y=g(t)$ $\alpha \leq t \leq \beta$

$$\text{Slop} = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g'(t)}{f'(t)} \quad \text{if } f'(t) \neq 0$$

If $f'(t)=0$ & $g'(t)\neq 0$ $\text{Slop}=\infty$

__Justification:__ If $f'(t)\neq 0$, then locally the curve is a graph $y=h(x)$

$$y=h(x) \longrightarrow \frac{dy}{dx} = h'(x) \text{ or } \frac{dh}{dx}$$

$$g(t) = h(f(t))$$

$$g'(t) = \frac{dh}{dx} f'(t)$$

$\rightarrow \text{Slope} = \frac{dy}{dx}$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{g'(t)}{f'(t)}}$$