Continuous Random Variable III

★ Conditional PDFs, given another random variable

$$P_{X|Y}(x|y) = P(X=x \mid Y=y) = \frac{P_{XY}(x,y)}{P_Y(y)}$$

{This is problematic in continuous case}

$$\cancel{f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} \quad \text{if } f_Y(y) \neq 0}$$

⇒ Let us consider a conditional of $x < X < x+\delta$ given $y < Y < y+\epsilon$

$$P(x < X < x+\delta \mid y < Y < y+\epsilon)$$

$$= \frac{f_{XY}(x,y)\,\delta\epsilon}{f_Y(y)\epsilon} = \frac{f_{XY}(x,y)}{f_Y(y)}\,\delta$$

⇒ Let us define $\boxed{f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)}}$

$$\boxed{P(X \in A \mid Y=y) = \int_A f_{X|Y}(x|y)\,dx}$$

Date

Page

★ **Total probability and expectation theorem**

$$f_X(x) = \int_{-\infty}^{\infty} f_Y(y)\, f_{X|Y}(x|y)\, dy$$

$$E[X|Y=y] = \int_{-\infty}^{\infty} x\, f_{X|Y}(x|y)\, dx$$

$$E[X] = \int_{-\infty}^{\infty} f_Y(y)\, E[X|Y=y]\, dy$$

★ **Independence**

$$\boxed{f_{XY}(x,y) = f_X(x)\, f_Y(y)} \quad \forall\, x \text{ and } y$$

★ **Independent Standard normals**

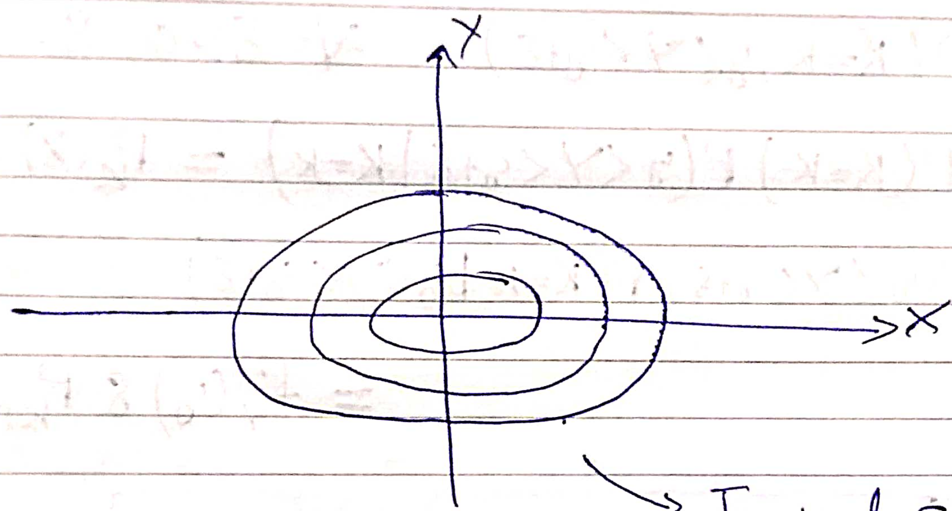
⇒ Let X and Y be two independent normal variable distributed normally.

$$f_X = \frac{1}{\sigma_X\sqrt{2\pi}}\, e^{\left(-\frac{1}{2}\frac{(x-\mu_X)^2}{\sigma_X^2}\right)}$$

$$f_Y = \frac{1}{\sigma_Y\sqrt{2\pi}}\, e^{\left(-\frac{1}{2}\frac{(y-\mu_Y)^2}{\sigma_Y^2}\right)}$$

So, $f_{XY}(x,y) = f_X(x)\, f_Y(y)$

$$\Rightarrow \frac{1}{2\pi\sigma_x\sigma_y} e^{\left\{-\frac{(x-\mu_x)^2}{2\sigma_x^2} - \frac{(y-\mu_y)^2}{2\sigma_y^2}\right\}}$$



→ Typical Contour plot of above distribution.

Note

→ The contours are elipse when major & minor axis are always aligned along either X or Y axis.

→ It can not make some arbitrary angle.

★ **Bayes rule**

$$f_{X|Y}(x|y) = \frac{f_X(x)\, f_{Y|X}(y|x)}{f_Y(y)}$$

{Prior}

{Posterior}

★ **The Bayes rule : One discrete and one continuous random variable**

K: discrete   Y: continuous

$\Rightarrow P(K=k, y \leq Y \leq y+\delta) \quad \forall \delta > 0, \delta \simeq 0$

$\Rightarrow P(K=k)\, P(y < Y < y+\delta \mid K=k) = P_K(k)\, f_{Y|K}(y|k)\delta$

$\Rightarrow P(y \leq Y < y+\delta)\, P(K=k \mid y \leq Y \leq y+\delta)$

$= f_Y(y)\, \delta\, P_{K|Y}(k|y)$

$$\Rightarrow \boxed{P_{K|Y}(k|y) = \frac{P_K(k)\, f_{Y|K}(y|k)}{f_Y(y)}}$$

$$\Rightarrow \boxed{f_{Y|K}(y|k) = \frac{f_Y(y)\, P_{K|Y}(k|y)}{P_K(k)}}$$