6 {From Introduction to linear Algebra}
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Eigenvalues & Eigenvectors

1. Introduction to Eigen Value

→ Eigenvalues have their greatest importance in dynamic problems.

Eigen Vectors

↳ Almost all vectors changes direction, when they are multiplied by A. Certain exceptional vector x are in the same direction as Ax. Those are the "eigen vectors"

$$\boxed{A x = \underset{\text{Eigen Value of A}}{\lambda} \underset{\text{Eigen Vector}}{x}}$$

All vectors are eigenvectors of I. {As $I x = x$}
→ & All eigen value of I is $\lambda = 1$.

⇒ Most of 2 by 2 matrices have two eigen vector directions & two eigen value.

$$A x - \lambda x = 0$$

$$(A - \lambda I) x = 0 \quad - ①$$

For a non zero eigen vector x eq ① is possible only if $A - \lambda I$ is singular.

$$\Rightarrow \boxed{|A - \lambda I| = 0} \longrightarrow \text{From here we can find } \lambda\text{'s}$$

#When A is Squared, the eigenvectors stay the Same. The eigenvalues are Squared.

$$Ax = \lambda x$$

$$\Rightarrow AAx = A(\lambda x)$$

$$\Rightarrow A^2x = \lambda(Ax)$$

$$\Rightarrow \boxed{A^2x = \lambda^2 x} \quad \{As, Ax = \lambda x\}$$

Eigen Value   Eigen Vector

#Every Vector is linear Combination of the eigen Vectors.

#The product of the n eigen values equals the determinant.

$$\lambda_1 + \lambda_2 + \cdots + \lambda_n = |A|$$

#The Sum of the n eigenValues equals the Sum of the n diagonal entries. (Trace)

$$\lambda_1 + \lambda_2 + \cdots \lambda_n = \text{trace} = a_{11} + a_{22} + \cdots + a_{nn}$$

2. Diagonalizing a Matrix

⇒ The matrix A turns into a diagonal matrix Λ when we use the eigenvectors Properly.

↳ This is matrix form of our Key idea.

Diagonalization ⇒ Suppose the n by n matrix A has n linearly independent eigenvectors $x_1 \cdots x_n$.

→ Put them into the columns of an eigenvector matrix S.

→ Then $S^{-1}AS$ is the eigen value matrix. (Λ)

$S \Rightarrow$ Eigenvector matrix

$\Lambda \Rightarrow$ Eigen value matrix

$$S = \begin{bmatrix} x_1 & x_2 & x_3 & \cdots & x_n \end{bmatrix} \qquad \Lambda = S^{-1}AS = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \lambda_3 & \\ & & & \ddots \lambda_n \end{bmatrix}$$

Eigen vectors

Capital lambda

Matrix A is diagonalized.

Proof

$$AS = A\begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} = \begin{bmatrix} Ax_1 & Ax_2 & \cdots & Ax_n \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \cdots & \lambda_n x_n \end{bmatrix}$$

⇒ The trick is to split the matrix AS into S times Λ.

$$\Rightarrow \begin{bmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \cdots & \lambda_n x_n \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \cdots \lambda_n \end{bmatrix}$$

$$\Rightarrow AS = S\Lambda$$

$$\Rightarrow \boxed{\Lambda = S^{-1}AS} \text{ or } \boxed{A = S\Lambda S^{-1}}$$

⇒ The matrix S has an inverse, because its Columns (the eigen vectors of A) were assumed to be linearly independent.

↳ Without n independent eigenvectors we cannot diagonalize.

$$\boxed{A^n = S\Lambda^n S^{-1}}$$

$$\left\{ \begin{aligned} A &= S\Lambda S^{-1} \\ A^2 &= (S\Lambda S^{-1})(S\Lambda S^{-1}) = S\Lambda(S^{-1}S)\Lambda S \\ A^2 &= S\Lambda^2 S \quad \text{Similarly} \end{aligned} \right\}$$

#Suppose the eigenvalues λ_1 --- λ_n are all different. Then it is automatic that the eigenvector x_1 --- x_n are independent. Any matrix that has no repeated eigenvalues Can be diagonalized

#The eigenvectors in S Come in the same order as the eigenvalues in Λ.

#Some matrix have too few eigenvectors. (repeated eigenvalues). Those matrix cannot be diagonalized.

* Solving difference Equation

{Fibonacci number F_{100}}

$$\boxed{F_{K+2} = F_{K+1} + F_K}$$

⇒ The Key is to begin with matrix equation $u_{K+1} = A U_K$.

Let $U_K = \begin{bmatrix} F_{K+1} \\ F_K \end{bmatrix} \Rightarrow U_{K+1} = \begin{bmatrix} F_{K+2} \\ F_{K+1} \end{bmatrix}$

$$\begin{bmatrix} F_{K+2} \\ F_{K+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{K+1} \\ F_K \end{bmatrix}$$

$$U_{K+1} = A U_K \quad \left\{ A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \right\}$$

$$U_1 = A U_0$$

$$U_2 = A^2 U_0$$

$$U_{100} = A^{100} U_0 = \begin{bmatrix} F_{101} \\ F_{100} \end{bmatrix}$$

$U_0 \rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ {Starting of Fibonacci number}

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \longrightarrow \left\{ \begin{aligned} \lambda_1 &= \frac{1+\sqrt{5}}{2} = 1.618 \\ \lambda_2 &= \frac{1-\sqrt{5}}{2} = -0.618 \end{aligned} \right\}$$

$$x_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}$$

$$x_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$$

$$U_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\lambda_1 - \lambda_2} \left(\begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} - \begin{bmatrix} \lambda_2 \\ 2 \end{bmatrix} \right)$$

$$U_0 = \left(\frac{x_1 - x_2}{\lambda_1 - \lambda_2} \right)$$

$$U_{100} = A^{100}\left(\frac{x_1 - x_2}{\lambda_1 - \lambda_2}\right)$$

$$\Rightarrow \frac{A^{99}}{\lambda_1 - \lambda_2}\left(\lambda_1 x_1 - \lambda_2 x_2\right)$$

$$\Rightarrow \frac{A^{99}}{\lambda_1 - \lambda_2}\left(\lambda_1^2 x_1 - \lambda_2^2 x_2\right)$$

$$\Rightarrow \frac{\lambda_1^{100} x_1 - \lambda_2^{100} x_2}{\lambda_1 - \lambda_2}$$

$$F_{100} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{100} - \left(\frac{1-\sqrt{5}}{2}\right)^{100}\right] \approx \underline{\underline{3.54 \times 10^{20}}}$$

So K^{th} term of Fibonacci number $= \underline{\underline{\dfrac{\lambda_1^K - \lambda_2^K}{\lambda_1 - \lambda_2}}}$

3. Application to Differential Equations

→ The whole point of this section is to convert constant-coefficient differential equation into linear algebra.

#n equations $\frac{du}{dt} = Au$ starting from the vector $U(0)$ at $t=0$.

$\Rightarrow$ $e^{\lambda_1 t}x_1$, $e^{\lambda_2 t}x_2$ .... $e^{\lambda_n t}x_n$ are n solutions.

$\Rightarrow$ So general solution will be linear combination of these solutions.

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$$\frac{du}{dt} = Au$$

$$\Downarrow$$

$$u = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2 + \cdots + c_n e^{\lambda_n t} x_n$$

Second Order equation

⇒ The most important equation in mechanics:-

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0 \quad \{\text{Let } m=1\}$$

Le $u = \begin{bmatrix} y \\ \dot{y} \end{bmatrix} \longrightarrow \dot{u} = \begin{bmatrix} \dot{y} \\ \ddot{y} \end{bmatrix}$

$$\begin{bmatrix} \dot{y} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k & -b \end{bmatrix} \begin{bmatrix} y \\ \dot{y} \end{bmatrix}$$

$\dot{u} = Au$

→ $\lambda_1\ \lambda_2$ be is eigen Value

→ $x_1 = \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} \quad x_2 = \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}$

$$\boxed{u(t) = c_1 e^{\lambda_1 t} \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} + c_2 e^{\lambda_2 t} \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}}$$

The Exponential Matrix

↳ We want to write the Solution $u(t)$ in a new form $e^{At}u(0)$.

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots$$

$$e^{At} = I + (At) + \frac{1}{2}(At)^2 + \frac{1}{6}(At)^3 + \cdots$$

Putting $A = S\Lambda S^{-1}$ we get

$$\boxed{e^{At} = S e^{\Lambda t} S^{-1}} \quad e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{bmatrix}$$