3

Convolution and the Solution of Ordinary Differential Equation

3.1) Introduction

"Mastery of the Convolution theorem greatly extends the power of Laplace Transforms to Solve ODEs"

3.2) Convolution

Defination 3.1: The Convolution of two given functions $f(t)$ and $g(t)$ is written $f*g$ and is defined by the integral

$$f*g = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

Note: The only Condition that is necessary to impose on the function f & g is that their behaviour be such that the integral on the right exists.

↳ Piecewise Continuity of both in the interval $[0,t]$ is certainly sufficient.

\# **Definition 3.2:** If an interval $[0, t_0]$ say can be partitioned into a finite number of subintervals $[0, t_1], [t_1, t_2], [t_2, t_3] \cdots, [t_n, t_0]$ with $0, t_1, t_2 \cdots t_n, t_0$ an increasing sequence of time and such that a given function $f(t)$ is continuous in each of these subintervals but not necessarily at the end points themselves, then $f(t)$ is piecewise continuous in the interval $[0, t_0]$.

\# **Theorem 3.3 (Symmetry):** $f * g = g * f$

\# **Theorem 3.4 (Convolution):** If $f(t)$ and $g(t)$ are two functions of exponential order, and writing $\mathcal{L}\{f\} = \bar{f}(s)$ and $\mathcal{L}\{g\} = \bar{g}(s)$ as the two Laplace Transforms then $\mathcal{L}^{-1}\{\bar{f}\bar{g}\} = f * g$ where $*$ is the convolution operator.

Proof

$$\mathcal{L}\{f(t) * g(t)\} = \int_0^\infty e^{-st} \int_0^t f(\tau) g(t-\tau) \, d\tau \, dt$$

$$\Rightarrow \int_0^\infty \int_0^t e^{-st} f(\tau) g(t-\tau) \, d\tau \, dt$$

Changing order of integration

$$\Rightarrow \boxed{\int_0^\infty \int_\tau^\infty e^{-st} f(\tau) g(t-\tau) \, dt \, d\tau}$$

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$$\Rightarrow \int_0^\infty f(\tau)\left\{\int_\tau^\infty e^{-st} g(t-\tau)\,dt\right\} d\tau$$

let $u = t-\tau$

$$\text{So } \int_\tau^\infty e^{-st} g(t-\tau)\,dt = \int_0^\infty e^{-s(u+\tau)} g(u)\,du$$

$$= e^{-s\tau}\int_0^\infty e^{-su} g(u)\,du$$

$$= e^{-s\tau}\bar{g}(s)$$

$$\text{So } \mathcal{L}\{f*g\} = \int_0^\infty f(\tau)\, e^{-s\tau}\bar{g}(s)\,d\tau$$

$$= \bar{g}(s)\int_0^\infty f(\tau)\, e^{-s\tau}\,d\tau$$

$$\mathcal{L}\{f*g\} = \bar{g}(s)\bar{f}(s)$$

$$\Rightarrow \boxed{f(t)*g(t) = \mathcal{L}^{-1}\{\bar{f}\bar{g}\}}$$

Borel's Theorem

Example 3.5 : find value of $\cos(t) * \sin(t)$.

$$\cos(t) * \sin(t) = \int_0^t \cos(\tau)\sin(t-\tau)\,d\tau$$

$$\Rightarrow \frac{1}{2}\int_0^t \sin(t) + \sin(t-2\tau)\,d\tau$$

$$\Rightarrow \frac{1}{2}\left\{\left[\tau\sin(t)\right]_0^t + \frac{1}{2}\left[\cos(t-2\tau)\right]_0^t\right\}$$

$$= \frac{1}{2}\left(t\sin(t)\right) + \frac{1}{4}\left(\cos(t) - \cos(t)\right)$$

$$= \frac{1}{2}t\sin t$$

Example 3.7 : Find the following Inverse Laplace Transforms:

a) $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}$

$\Rightarrow$

$$\mathcal{L}\{\cos(t)\}\,\mathcal{L}\{\sin(t)\} = \frac{s}{(s^2+1)^2}$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \cos(t) * \sin(t) = \frac{1}{2}t\sin t$$