I PROBABILITY

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Probability

"Mathematical Language for quantifying Uncertainity"

★ **Sample Space and Events**

→ It is set of possible outcomes of an experiment

→ Generally denoted by Ω

⇒ Subsets of Ω are called Events.

⇒ Point ω of Ω are called Sample outcome or realizations or elements ✓

Example: If we toss a coin twice then:

$$\Omega = \{HH, HT, TH, TT\}$$

→ One of the element of Sample Space

⇒ Event that first is heads $A = \{HT, HH\}$

⇒ Let $A^c = \{\omega \in \Omega \;\; \omega \notin A\}$ denotes the Complement of A.

↳ A^c can be read as "not A".

↳ Complement of Ω is the empty set ϕ.

⇒ The union of events A & B is defined

$$A \cup B = \{\omega \in \Omega : \omega \in A \text{ or } \omega \in B\}$$

⇒ If $A_1, A_2, \ldots$ is a sequence of sets then

$$\bigcup_{i=1}^{\infty} A_i = \{\omega \in \Omega : \omega \in A_i \text{ for at least one } i\}$$

⇒ The Intersection of A and B is

$$A \cap B = \{\omega \in \Omega : \omega \in A \;\&\; \omega \in B\}$$

{sometimes we write $A \cap B$ as AB}

⇒ If $A_1, A_2 \ldots$ is a sequence of set then

$$\bigcap_{i=1}^{\infty} A_i = \{\omega \in \Omega : \omega \in A_i \;\; \forall i\}$$

⇒ The set difference is defined by

$$A - B = \{\omega : \omega \in A, \omega \notin B\}$$

⇒ If every element of A is contained in B

$A \subset B$ or $B \supset A$.

(sub set) (super set)

⇒ If A is a finite set, let $|A|$ denote the number of elements in A.

⇒ We say $A_1, A_2 \cdots$ are disjoint or mutually exclusive if $A_i \cap A_j = \phi$ whene $i \neq j$.

⇒ A partition of Ω is a sequence of disjoint sets $A_1, A_2 \ldots$ such that $\bigcup_{i=1}^{\infty} A_i = \Omega$.

⇒ Given an event A, defined the indicator function of A by

$$I_A(\omega) = I(\omega \in A) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases}$$

⇒ A sequence of set $A_1, A_2, \ldots$ is monotone increasing if $A_1 \subset A_2 \subset \cdots$ and we define $\lim_{n\to\infty} A_n = \bigcup_{i=1}^{\infty} A_i$.

⇒ A sequence of set $A_1, A_2, \ldots$ is monotone decreasing if $A_1 \supset A_2 \supset \cdots$ and then we define $\lim_{n\to\infty} A_n = \bigcap_{i=1}^{\infty} A_i$.

⇒ In either case, we will write $A_n \to A$.

★ Probability

⇒ A function $\mathbb{P}$ that assigns a real number $\mathbb{P}(A)$ to each event A is a Probability distribution or a Probability measure if it satisfies the following three axioms:

Axiom 1: $\mathbb{P}(A) \geq 0 \quad \forall A$

Axiom 2: $\mathbb{P}(\Omega) = 1$

Axiom 3: If $A_1, A_2, \ldots$ are disjoint then

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$$

⇒ Properties of $\mathbb{P}$ from the axioms

$$\mathbb{P}(\phi) = 0$$

$$A \subset B = P(A) \leq P(B)$$

$$0 \leq \mathbb{P}(A) \leq 1$$

$$P(A^c) = 1 - P(A)$$

$$A \cap B = \phi \Rightarrow P(A \cup B) = P(A) + P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

★ Probability on Finite Sample Space

⇒ Suppose that the Sample space $\Omega = \{\omega_1, \omega_2 \cdots \omega_n\}$ is finite.

⇒ If Ω is finite and if each outcome is equally likely, then

$$\boxed{\mathbb{P}(A) = \frac{|A|}{|\Omega|}}$$

⇒ It is called uniform probability distribution.

★ Independent Events

⇒ Two events A and B are independent if

$$\mathbb{P}(AB) = \mathbb{P}(A) + \mathbb{P}(B)$$

→ $A \perp\!\!\!\perp B$ ⇒ A and B are Independent.

→ $A \not\perp\!\!\!\perp B$ ⇒ A and B are Not independent.

⇒ A set of events $\{A_i : i \in I\}$ is independent if

$$\mathbb{P}\left(\bigcap_{i \in J} A_i\right) = \prod_{i \in J} \mathbb{P}(A_i)$$

$$\forall \; J \subset I$$

⇒ Independence can arise in two distinct ways.

→ we explicitly assume that two events are independent.

→ we derive independence by verifying that $\mathbb{P}(AB) = \mathbb{P}(A)\,\mathbb{P}(B)$ holds.

⇒ There is no way to judge independence by looking at the sets in a venn diagram.

★ Conditional Probability

If $\mathbb{P}(B) > 0$ then the Conditional Probability of A given B is

$$\boxed{P(A|B) = \frac{\mathbb{P}(AB)}{\mathbb{P}(B)}}$$

"Think of $P(A|B)$ as fraction of times A occurs among those in which B occurs"

⇒ The rules of probability apply to events on the left of the bar.

⇒ If A and B are independent events then $\mathbb{P}(A|B) = P(A)$

⇒ For any pair of events A and B

$$P(AB) = P(A|B)\,P(B)$$

→ Another interpretation of independence is that knowing B doesn't change the probability of A.

* Bayes Theorem

The Law of Total Probability

↳ Let $A_1 \dots A_K$ be a partition of Ω

Then for any event B

$$\mathbb{P}(B) = \sum_{i=1}^{K} \mathbb{P}(B|A_i)\, P(A_i)$$

Bayes Theorem

↳ Let $A_1, \dots A_K$ be a partition of Ω

Such that $\mathbb{P}(A_i) > 0 \ \forall$ each i.

If $\mathbb{P}(B) > 0$ then, for each $i = 1 \dots K$

$$\boxed{\mathbb{P}(A_i|B) = \frac{\mathbb{P}(B|A_i)\, P(A_i)}{\sum_j \mathbb{P}(B|A_j) \cdot \mathbb{P}(A_j)}}$$

$P(A_i) \Rightarrow$ Prior probability of A.

$P(A_i|B) \Rightarrow$ Posterior probability of A.