# 6 Determinants

## ★ Permutations, Signature of permutation

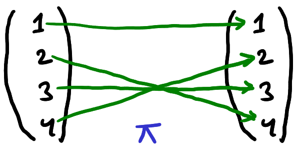
⇒ Let $[n] = \{1, 2 \ldots, n\}$, where $n \in \mathbb{N}$, and $n > 0$.

⇒ Definition: A permutation on n elements is a bijection $\pi : [n] \to [n]$.

↳ Example

If $n=4$

then, $[n] = \{1,2,3,4\}$




⇒ A permutation $\sigma$ on $n$ elements, say $\sigma(i) = k_i$ for $i = 1, \ldots, n$, can be represented in functional notation by the $2 \times n$ array

$$\begin{pmatrix} 1 & \cdots & i & \cdots & n \\ k_1 & \cdots & k_i & \cdots & k_n \end{pmatrix}$$

known as *Cauchy two-line notation*.

⇒ A more concise notation often used in computer science and in combinatorics is to represent a permutation by its image, namely by the sequence

$$\sigma(1) \ \sigma(2) \ \cdots \ \sigma(n)$$

↳ The above is known as the one-line notation.

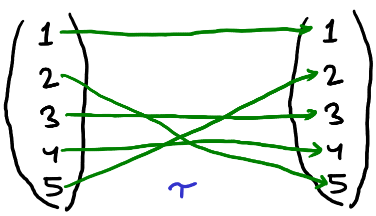
⇒ A transposition is a permutation $\tau : [n] \to [n]$ such that, for some $i < j$ (with $1 \leq i < j \leq n$), $\tau(i) = j$, $\tau(j) = i$, and $\tau(k) = k$, for all $k \in [n] - \{i, j\}$.

↳ If $\tau$ is a transposition, clearly, $\tau \circ \tau = \mathrm{id}$.

↳ Example

Let $n=5$ $i=2$, $j=5$

so $[n] = \{1,2,3,4,5\}$




⇒ set of permutations on $[n]$ is a *group* often denoted $\mathfrak{S}_n$ and called the *symmetric group* on $n$ elements.

It is easy to show by induction that the group $\mathfrak{S}_n$ has $n!$ elements.

**Proposition 6.1.** *For every $n \geq 2$, every permutation $\pi\colon [n] \to [n]$ can be written as a nonempty composition of transpositions.*

A transposition τ that exchanges two consecutive elements k and k + 1 of [n] ($1 \leq k \leq n-1$) may be called a basic transposition.

Every transposition can be written as a product of basic transpositions.

In fact, the transposition that exchanges k and k + p ($1 \leq p \leq n - k$) can be realized using 2p − 1 basic transpositions.

Therefore, the group of permutations $\mathfrak{S}_n$ is also generated by the basic transpositions.

**Definition 6.2.** For every $n \geq 2$, let $\Delta\colon \mathbb{Z}^n \to \mathbb{Z}$ be the function given by

$$\Delta(x_1, \ldots, x_n) = \prod_{1 \leq i < j \leq n} (x_i - x_j).$$

The expression $\Delta(x_1, \ldots, x_n)$ is often called the *discriminant* of $(x_1, \ldots, x_n)$.

The discriminant consists of $\binom{n}{2}$ factors. When $n = 3$,

$$\Delta(x_1, x_2, x_3) = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3).$$

More generally, for any permutation $\sigma \in \mathfrak{S}_n$, define $\Delta(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$ by

$$\Delta(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = \prod_{1 \leq i < j \leq n} (x_{\sigma(i)} - x_{\sigma(j)}).$$

**Proposition 6.2.** *For every basic transposition $\tau$ of $[n]$ $(n \geq 2)$, we have*

$$\Delta(x_{\tau(1)}, \ldots, x_{\tau(n)}) = -\Delta(x_1, \ldots, x_n).$$

*The above also holds for every transposition, and more generally, for every composition of transpositions $\sigma = \tau_p \circ \cdots \circ \tau_1$, we have*

$$\Delta(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = (-1)^p \Delta(x_1, \ldots, x_n).$$

*Consequently, for every permutation $\sigma$ of $[n]$, the parity of the number $p$ of transpositions involved in any decomposition of $\sigma$ as $\sigma = \tau_p \circ \cdots \circ \tau_1$ is an invariant (only depends on $\sigma$).*

**Definition 6.3.** For every permutation $\sigma$ of $[n]$, the parity $\epsilon(\sigma)$ (or $\mathrm{sgn}(\sigma)$) of the number of transpositions involved in any decomposition of $\sigma$ is called the *signature* (or *sign*) of $\sigma$.

Obviously $\epsilon(\tau) = -1$ for every transposition $\tau$ (since $(-1)^1 = -1$).

**Definition 6.4.** Given any permutation $\sigma$ on $n$ elements, we say that a pair $(i, j)$ of indices $i, j \in \{1, \ldots, n\}$ such that $i < j$ and $\sigma(i) > \sigma(j)$ is an *inversion* of the permutation $\sigma$.

For example, the permutation $\sigma$ given by

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 3 & 6 & 5 & 1 \end{pmatrix}$$

has seven inversions

$$(1,6),\ (2,3),\ (2,6),\ (3,6),\ (4,5),\ (4,6),\ (5,6).$$

**Proposition 6.3.** *The signature $\epsilon(\sigma)$ of any permutation $\sigma$ is equal to the parity $(-1)^{I(\sigma)}$ of the number $I(\sigma)$ of inversions in $\sigma$.*

$$\boxed{\epsilon(\pi' \circ \pi) = \epsilon(\pi')\epsilon(\pi)}$$

## ★ Alternating Multilinear Map

Let $E_1, \ldots, E_n$, and $F$, be vector spaces over a field $K$, where $n \geq 1$.

**Definition 6.5.** A function $f : E_1 \times \ldots \times E_n \to F$ is a *multilinear map* (or an *$n$-linear map*) if it is linear in each argument, holding the others fixed. More explicitly, for every $i$, $1 \leq i \leq n$, for all $x_1 \in E_1, \ldots, x_{i-1} \in E_{i-1}, x_{i+1} \in E_{i+1}, \ldots, x_n \in E_n$, for all $x, y \in E_i$, for all $\lambda \in K$,

$$\begin{aligned} f(x_1, \ldots, x_{i-1}, x + y, x_{i+1}, \ldots, x_n) &= f(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n) \\ &\quad + f(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n), \\ f(x_1, \ldots, x_{i-1}, \lambda x, x_{i+1}, \ldots, x_n) &= \lambda f(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n). \end{aligned}$$

When $F = K$, we call $f$ an *$n$-linear form* (or *multilinear form*).

If $n \geq 2$ and $E_1 = E_2 = \ldots = E_n$, an $n$-linear map $f : E \times \ldots \times E \to F$ is called *symmetric*, if $f(x_1, \ldots, x_n) = f(x_{\pi(1)}, \ldots, x_{\pi(n)})$ for every permutation $\pi$ on $\{1, \ldots, n\}$.

An $n$-linear map $f : E \times \ldots \times E \to F$ is called *alternating*, if $f(x_1, \ldots, x_n) = 0$ whenever $x_i = x_{i+1}$ for some $i$, $1 \leq i \leq n - 1$

When $n = 2$, a 2-linear map $f : E_1 \times E_2 \to F$ is called a *bilinear map*.

Multiplication · : K × K → K is a bilinear map, treating K as a vector space over itself.

Symmetric bilinear maps (and multilinear maps) play an important role in geometry (inner products, quadratic forms) and in differential calculus (partial derivatives).

A bilinear map is symmetric if f (u, v) = f (v, u), for all u, v ∈ E.

**Proposition 6.4.** *Let $f : E \times \ldots \times E \to F$ be an $n$-linear alternating map, with $n \geq 2$. The following properties hold:*

*(1)*

$$f(\ldots, x_i, x_{i+1}, \ldots) = -f(\ldots, x_{i+1}, x_i, \ldots)$$

*(2)*

$$f(\ldots, x_i, \ldots, x_j, \ldots) = 0,$$

*where $x_i = x_j$, and $1 \leq i < j \leq n$.*

(3)

$$f(\ldots, x_i, \ldots, x_j, \ldots) = -f(\ldots, x_j, \ldots, x_i, \ldots),$$

*where* $1 \leq i < j \leq n$.

(4)

$$f(\ldots, x_i, \ldots) = f(\ldots, x_i + \lambda x_j, \ldots),$$

*for any* $\lambda \in K$, *and where* $i \neq j$.

**Lemma 6.5.** *Let* $f \colon E \times \ldots \times E \to F$ *be an* $n$*-linear alternating map. Let* $(u_1, \ldots, u_n)$ *and* $(v_1, \ldots, v_n)$ *be two families of* $n$ *vectors, such that,*

$$v_1 = a_{1\,1} u_1 + \cdots + a_{n\,1} u_n,$$
$$\cdots$$
$$v_n = a_{1\,n} u_1 + \cdots + a_{n\,n} u_n.$$

*Equivalently, letting*

$$A = \begin{pmatrix} a_{1\,1} & a_{1\,2} & \ldots & a_{1\,n} \\ a_{2\,1} & a_{2\,2} & \ldots & a_{2\,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n\,1} & a_{n\,2} & \ldots & a_{n\,n} \end{pmatrix},$$

*assume that we have*

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = A^\top \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}.$$

*Then,*

$$f(v_1, \ldots, v_n) = \Big(\sum_{\pi \in \mathfrak{S}_n} \epsilon(\pi) a_{\pi(1)\,1} \cdots a_{\pi(n)\,n}\Big) f(u_1, \ldots, u_n),$$

*where the sum ranges over all permutations* $\pi$ *on* $\{1, \ldots, n\}$.

$$\left\{ \det(A) = \sum_{\pi \in \mathfrak{S}_n} \epsilon(\pi) a_{\pi(1)\,1} \cdots a_{\pi(n)\,n} \right\}$$

★ Defination of a Determinant

**Definition 6.6.** A *determinant* is defined as any map

$$D \colon \mathrm{M}_n(K) \to K,$$

which, when viewed as a map on $(K^n)^n$, i.e., a map of the $n$ columns of a matrix, is $n$-linear alternating and such that $D(I_n) = 1$ for the identity matrix $I_n$. Equivalently, we can consider a vector space $E$ of dimension $n$, some fixed basis $(e_1, \ldots, e_n)$, and define

$$D \colon E^n \to K$$

as an $n$-linear alternating map such that $D(e_1, \ldots, e_n) = 1$.

**Definition 6.7.** Given any $n \times n$ matrix with $n \geq 2$, for any two indices $i, j$ with $1 \leq i, j \leq n$, let $A_{ij}$ be the $(n-1) \times (n-1)$ matrix obtained by deleting Row $i$ and Column $j$ from $A$ and called a *minor*:

$$A_{ij} = \begin{pmatrix} & & & & \times & & \\ & & & & \times & & \\ \times & \times & \times & \times & \times & \times & \times \\ & & & & \times & & \\ & & & & \times & & \\ & & & & \times & & \\ & & & & \times & & \end{pmatrix}.$$

**Definition 6.8.** For every $n \geq 1$, we define a finite set $\mathcal{D}_n$ of maps $D\colon \mathrm{M}_n(K) \to K$

When $n = 1$, $\mathcal{D}_1$ consists of the single map $D$ such that, $D(A) = a$, where $A = (a)$, with $a \in K$.

Assume that $\mathcal{D}_{n-1}$ has been defined, where $n \geq 2$. Then $\mathcal{D}_n$ consists of all the maps $D$ such that, for some $i$, $1 \leq i \leq n$,

$$D(A) = (-1)^{i+1} a_{i\,1} D(A_{i\,1}) + \cdots + (-1)^{i+n} a_{i\,n} D(A_{i\,n}),$$

where for every $j$, $1 \leq j \leq n$, $D(A_{i\,j})$ is the result of applying any $D$ in $\mathcal{D}_{n-1}$ to the minor $A_{i\,j}$.

Each $(-1)^{i+j} D(A_{i\,j})$ is called the *cofactor* of $a_{i\,j}$,

This expression for D(A) is called a Laplace expansion of D according to the i-th Row

We can think of each member of D_n as an algorithm to evaluate “the” determinant of A.

given a $n \times n$-matrix $A = (a_{i\,j})$,

$$A = \begin{pmatrix} a_{1\,1} & a_{1\,2} & \ldots & a_{1\,n} \\ a_{2\,1} & a_{2\,2} & \ldots & a_{2\,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n\,1} & a_{n\,2} & \ldots & a_{n\,n} \end{pmatrix},$$

its determinant is denoted by $D(A)$ or $\det(A)$, or more explicitly by

$$\det(A) = \begin{vmatrix} a_{1\,1} & a_{1\,2} & \ldots & a_{1\,n} \\ a_{2\,1} & a_{2\,2} & \ldots & a_{2\,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n\,1} & a_{n\,2} & \ldots & a_{n\,n} \end{vmatrix}.$$

1. When $n = 2$, if

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then by expanding according to any row, we have

$$D(A) = ad - bc.$$

2. When $n = 3$, if

$$A = \begin{pmatrix} a_{1\,1} & a_{1\,2} & a_{1\,3} \\ a_{2\,1} & a_{2\,2} & a_{2\,3} \\ a_{3\,1} & a_{3\,2} & a_{3\,3} \end{pmatrix},$$

then by expanding according to the first row, we have

$$D(A) = a_{1\,1} \begin{vmatrix} a_{2\,2} & a_{2\,3} \\ a_{3\,2} & a_{3\,3} \end{vmatrix} - a_{1\,2} \begin{vmatrix} a_{2\,1} & a_{2\,3} \\ a_{3\,1} & a_{3\,3} \end{vmatrix} + a_{1\,3} \begin{vmatrix} a_{2\,1} & a_{2\,2} \\ a_{3\,1} & a_{3\,2} \end{vmatrix},$$

that is,

$$D(A) = a_{1\,1}(a_{2\,2}a_{3\,3} - a_{3\,2}a_{2\,3}) - a_{1\,2}(a_{2\,1}a_{3\,3} - a_{3\,1}a_{2\,3}) + a_{1\,3}(a_{2\,1}a_{3\,2} - a_{3\,1}a_{2\,2}),$$

which gives the explicit formula

$$D(A) = a_{1\,1}a_{2\,2}a_{3\,3} + a_{2\,1}a_{3\,2}a_{1\,3} + a_{3\,1}a_{1\,2}a_{2\,3} - a_{1\,1}a_{3\,2}a_{2\,3} - a_{2\,1}a_{1\,2}a_{3\,3} - a_{3\,1}a_{2\,2}a_{1\,3}.$$

We now show that each $D \in \mathcal{D}_n$ is a determinant (map).

**Lemma 6.6.** *For every $n \geq 1$, for every $D \in \mathcal{D}_n$ as defined in Definition 6.8, $D$ is an alternating multilinear map such that $D(I_n) = 1$.*

**Theorem 6.7.** *For every $n \geq 1$, for every $D \in \mathcal{D}_n$, for every matrix $A \in \mathrm{M}_n(K)$, we have*

$$D(A) = \sum_{\pi \in \mathfrak{S}_n} \epsilon(\pi) a_{\pi(1)\,1} \cdots a_{\pi(n)\,n},$$

*where the sum ranges over all permutations $\pi$ on $\{1, \ldots, n\}$. As a consequence, $\mathcal{D}_n$ consists of a single map for every $n \geq 1$, and this map is given by the above explicit formula.*

⇛ There is a geometric interpretation of determinants which we find quite illuminating. Given $n$ linearly independent vectors $(u_1, \ldots, u_n)$ in $\mathbb{R}^n$, the set

$$P_n = \{\lambda_1 u_1 + \cdots + \lambda_n u_n \mid 0 \leq \lambda_i \leq 1,\ 1 \leq i \leq n\}$$

is called a *parallelotope*.

→ If n = 2, then P 2 is a parallelogram and if n = 3, then P 3 is a parallelepiped

→ Then it turns out that $\det(u_1, \ldots, u_n)$ is the *signed volume* of the parallelotope $P_n$ (where volume means $n$-dimensional volume). The sign of this volume accounts for the orientation of $P_n$ in $\mathbb{R}^n$.

⇛ **Corollary 6.8.** *For every matrix $A \in \mathrm{M}_n(K)$, we have $\det(A) = \det(A^\top)$.*

⇛ **Example 6.2.** Consider the so-called *Vandermonde determinant*

$$V(x_1, \ldots, x_n) = \begin{vmatrix} 1 & 1 & \ldots & 1 \\ x_1 & x_2 & \ldots & x_n \\ x_1^2 & x_2^2 & \ldots & x_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \ldots & x_n^{n-1} \end{vmatrix}.$$

→ We claim that

$$\boxed{V(x_1, \ldots, x_n) = \prod_{1 \leq i < j \leq n} (x_j - x_i),}$$

→ with $V(x_1, \ldots, x_n) = 1$, when $n = 1$.

⇛ $\Delta(x_1, \ldots, x_n) = V(x_n, \ldots, x_1) = (-1)^{\binom{n}{2}} V(x_1, \ldots x_n),$

⇛ **Proposition 6.9.** *Let $f \colon E \times \ldots \times E \to F$ be an $n$-linear alternating map. Let $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$ be two families of $n$ vectors, such that*

$$v_1 = a_{1\,1} u_1 + \cdots + a_{1\,n} u_n,$$
$$\ldots$$
$$v_n = a_{n\,1} u_1 + \cdots + a_{n\,n} u_n.$$

*Equivalently, letting*

$$A = \begin{pmatrix} a_{1\,1} & a_{1\,2} & \ldots & a_{1\,n} \\ a_{2\,1} & a_{2\,2} & \ldots & a_{2\,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n\,1} & a_{n\,2} & \ldots & a_{n\,n} \end{pmatrix},$$

*assume that we have*

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = A \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}.$$

*Then,*

$$f(v_1, \ldots, v_n) = \det(A) f(u_1, \ldots, u_n).$$

**Proposition 6.10.** *For any two $n \times n$-matrices $A$ and $B$, we have* $\det(AB) = \det(A)\det(B)$.

★ Inverse Matrices and Determinants

**Definition 6.9.** Let $K$ be a commutative ring. Given a matrix $A \in \mathrm{M}_n(K)$, let $\widetilde{A} = (b_{i\,j})$ be the matrix defined such that

$$b_{i\,j} = (-1)^{i+j} \det(A_{j\,i}),$$

the cofactor of $a_{j\,i}$. The matrix $\widetilde{A}$ is called the *adjugate* of $A$, and each matrix $A_{j\,i}$ is called a *minor* of the matrix $A$.

**Proposition 6.11.** *Let $K$ be a commutative ring. For every matrix $A \in \mathrm{M}_n(K)$, we have*

$$A\widetilde{A} = \widetilde{A}A = \det(A) I_n.$$

*As a consequence, $A$ is invertible iff $\det(A)$ is invertible, and if so, $A^{-1} = (\det(A))^{-1}\widetilde{A}$.*

★ System of Linear Equation and determinant

⇒ Let $A$ be an $n \times n$-matrix, $x$ a column vectors of variables, and $b$ another column vector, and let $A^1, \ldots, A^n$ denote the columns of $A$. Observe that the system of equations $Ax = b$,

$$\begin{pmatrix} a_{1\,1} & a_{1\,2} & \ldots & a_{1\,n} \\ a_{2\,1} & a_{2\,2} & \ldots & a_{2\,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n\,1} & a_{n\,2} & \ldots & a_{n\,n} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

is equivalent to

$$x_1 A^1 + \cdots + x_j A^j + \cdots + x_n A^n = b,$$

since the equation corresponding to the $i$-th row is in both cases

$$a_{i\,1} x_1 + \cdots + a_{i\,j} x_j + \cdots + a_{i\,n} x_n = b_i.$$

**Proposition 6.12.** *Given an $n \times n$-matrix $A$ over a field $K$, the columns $A^1, \ldots, A^n$ of $A$ are linearly dependent iff* $\det(A) = \det(A^1, \ldots, A^n) = 0$. *Equivalently, $A$ has rank $n$ iff* $\det(A) \neq 0$.