(9)

Continuous Random Variable II

★ Conditional PDF

$$P(X \in B \mid A) = \int_B f_{X|A}(x)\,dx$$

$$f_{X|X\in A}(x) = \begin{cases} 0 & \text{if } x \notin A \\ \dfrac{f_X(x)}{P(A)} & \text{if } x \in A \end{cases}$$

★ Conditional Expectation

$$E[X|A] = \int x\, f_{X|A}(x)\,dx$$

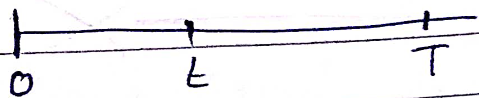
$$E[g(X)|A] = \int g(x)\, f_{X|A}(x)\,dx$$

{ Expected value rule }

★ Memorylessness of the exponential PDF

⇒ Let the life of a light bulb be given by:

$$P(T > x) = e^{-\lambda x}$$



⇒ We are told that $T > t$

⇒ Let X : remaining lifetime $= T - t$

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$P(X > x \mid T > t)$ {Probability distribution on remaining lifetime, given bulb has been used for t time}

$\Rightarrow P(T - t > x \mid T > t)$

$\Rightarrow P(T > t + x \mid T > t)$

$$\Rightarrow \frac{P(T > t+x,\ T > t)}{P(T > t)} = \frac{P(T > t+x)}{P(T > t)}$$

$$\Rightarrow \frac{e^{-\lambda(t+x)}}{e^{-\lambda t}} = e^{-\lambda x}$$ {Same PDF as if bulb was new}

$\Rightarrow$ This property is called Memorylessness.

★ <u>Total probability and expectation theorem</u>

$$P(X \leq x) = P(A_1)\, P(X \leq x \mid A_1) + \cdots + P(A_n)\, P(X \leq x \mid A_n)$$

$$F_X(x) = P(A_1)\, F_{X|A_1}(x) + \cdots + P(A_n)\, F_{X|A_n}(x)$$

$\rightarrow$ CDF of random variable X

$\Rightarrow$ Differentiating both side:

$$\boxed{f_X(x) = P(A_1)\, f_{X|A_1}(x) + \cdots + P(A_n)\, f_{X|A_n}(x)}$$

{Total probability theorem}

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⇒ Multiplying both side by x and integrating.

$$\int x f_X(x)\,dx = P(A_1)\int x f_{X|A_1}(x)\,dx + \dots + P(A_n)\int x f_{X|A_n}(x)\,dx$$

$$\Rightarrow \boxed{E[X] = P(A_1)E[X|A_1] + \dots + P(A_n)E[X|A_n]}$$

{Total Expectation theorem}

★ Mixed distribution

⇒ Let us consider the following random variable:

$$X = \begin{cases} \text{Uniform } [0,2] & \text{with probability } 1/2 \\ 1 & \text{with probability} \end{cases}$$

→ X is not discrete as X varies over a continuous range $[0,2]$

→ X is not continuous because there is no pdf associated.

⇒ These variables are called mixed random variables.

⇒ These can be described with help of CDF.

$$X = \begin{cases} Y, & \text{with probability } p \\ Z, & \text{with probability } 1-p \end{cases}$$

where, $Y \Rightarrow$ discrete

$Z \Rightarrow$ Continuous

$$\boxed{F_X(x) = pF_Y(x) + (1-p)F_Z(x)}$$

★ Joint PDFs

$$P((X,Y) \in B) = \iint_{(x,y)\in B} f_{XY}(x,y)\,dx\,dy$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = 1$$

$\Rightarrow$ Two random variable are jointly continuous if they can be described by a joint PDF.

Example: Let X be a continuous random variable and let $Y = g(X)$

$\Rightarrow$ The variable X and Y are not jointly continuous.

★ Marginal PDFs

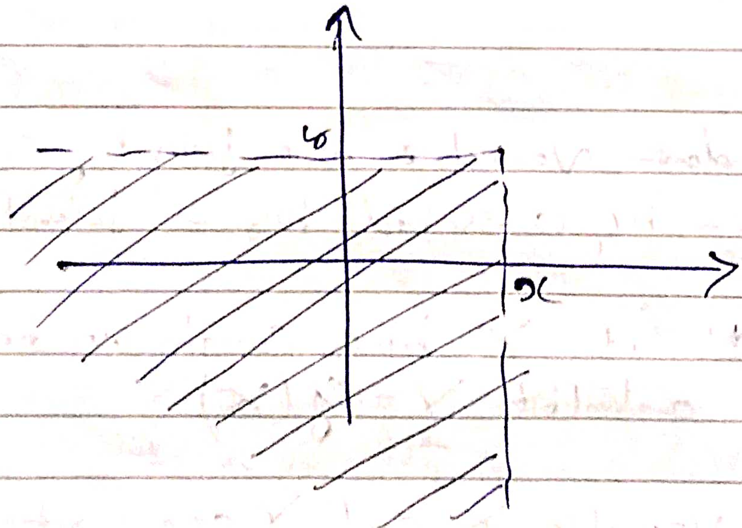
$$f_X(x) = \int f_{X,Y}(x,y)\,dy$$

$$f_Y(y) = \int f_{XY}(x,y)\,dx$$

★ Joint CDF

$$F_{XY}(x,y) = P(X \leq x, Y \leq y)$$

$$= \int_{-\infty}^{y}\int_{-\infty}^{x} f_{XY}(s,t)\,ds\,dt$$



$$\boxed{f_{XY}(x,y) = \frac{\delta^2 F_{XY}}{\delta x\,\delta y}(x,y)}$$