Appendix: Controllability & Observability

{Kalman Decomposition}

⇒ For CT LTI System:

$$\boxed{\begin{aligned} \dot{X}(t) &= AX(t) + BU(t) \\ Y(t) &= CX(t) + DU(t) \end{aligned}}$$

- $$O = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$ is Observability matrix.

- $\mathcal{C} = [B \;\; AB \;\; \cdots \;\; A^{n-1}B]$ is Controllability matrix

⇒ The System is fully observable if Observability matrix O has rank n.

⇒ The System is fully controllable if Controllability matrix $\mathcal{C}$ has rank n.

★ Transforming Controllable System into CCF (Controllable canonical form)

$\Rightarrow$ Let $x = M\tilde{x}$, where $M = [m_1\ m_2\ \cdots\ m_n]$, then

$$\dot{\tilde{x}} = M^{-1}\dot{x} = M^{-1}(Ax + Bu)$$

$$\vdots$$

$$\dot{\tilde{x}} = \underbrace{M^{-1}AM}_{\tilde{A}}\tilde{x} + \underbrace{M^{-1}B}_{\tilde{B}}u$$

$\Rightarrow$ Goal 1: $\tilde{B}$ be in controllable canonical form

$$M^{-1}B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \Rightarrow B = [m_1\ m_2\ \cdots\ m_n]\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} = m_n$$

$\Rightarrow$ Goal 2: $\tilde{A}$ be in controllable canonical form

$$\Rightarrow A[m_1\ m_2\ \cdots\ m_n] = [m_1\ m_2\ \cdots\ m_n]\underbrace{\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & \\ & & & & \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & & \cdots & -a_{n-1} \end{bmatrix}}_{\tilde{A}}$$

$$\Rightarrow [Am_1\ \ Am_2\ \cdots\ Am_n] = [\ \cdots\ \ (m_{n-2} - a_{n-2}m_n)\ \ (m_{n-1} - a_{n-1}m_n)]$$

$\Rightarrow$ Solving goals 1 & 2 yields:

$$m_n = B$$
$$m_{n-1} = A m_n + a_{n-1} m_n$$
$$m_{n-2} = A m_{n-1} + a_{n-2} m_n$$
$$\vdots \qquad \vdots$$

★ Transforming observable system in OCF (observable canonical form)

$\Rightarrow$ Let $x = R^{-1}\tilde{x}$ where $R = [r_1^T \; r_2^T \; \cdots \; r_n^T]^T$

{ r_i is a row vector }

$$\dot{\tilde{x}} = R\dot{x} = R(Ax + Bu) = RAR^{-1}\tilde{x} + RBu$$

$$y = Cx = CR^{-1}\tilde{x}$$

$\Rightarrow$ Goal 1: $\tilde{C}$ be in controllable canonical form

$$CR^{-1} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}^T \Rightarrow C = r_1$$

$\Rightarrow$ Goal 2: $\tilde{A}$ be in observable canonical form

$$\begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix} A = \begin{bmatrix} -a_{n-1} & 1 & 0 & \cdots & 0 \\ \vdots & 0 & 1 & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ -a_1 & \vdots & & & 1 \\ -a_0 & 0 & \cdots & \cdots & 0 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix}$$

⇒ Solving eqn 1 & 2 yield

$$r_1 = C$$
$$r_2 = r_1 A + a_{n-1} r_1$$
$$r_3 = r_2 A + a_{n-2} r_1$$
$$\vdots \qquad \vdots \qquad \vdots$$

* Controllable Subspace

⇒ Consider an uncontrollable LTI system

$$x(k+1) = Ax(k) + Bu(k)$$
$$y(k) = Cx(k) + Du(k)$$

⇒ Let the controllability matrix

$$P = [B \ AB \ A^2B - - A^{n-1}B]$$

has rank $n_1 < n$.

⇒ The Controllable subspace X_c is the set of vectors $x \in R^n$ that can be reached from the origin.

⇒ From,

$$x(n) - A^n x(n) = \underbrace{[B, AB \ A^2B - - A^{n-1}B]}_{P_d} \begin{bmatrix} u(n-1) \\ u(n-2) \\ \vdots \\ u(0) \end{bmatrix}$$

⇒ X_c is the range space of P_d.

★ Observable subspace

⇒ Consider an unobservable LTI System

$$x(k+1) = Ax(k) + Bu(k) \quad A \in R^{n \times n}$$
$$y(k) = Cx(k) + Du(k)$$

⇒ Let the observability matrix

$$Q = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

has rank $n_2 < n$.

⇒ The Unobservable subspace X_{uo} is the set of all nonzero initial conditions $x(0) \in R^n$ that produces a zero free response.

⇒ Form

$$\underbrace{\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(n-1) \end{bmatrix}}_{Y} = \underbrace{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}}_{Q} x(0)$$

⇒ X_{uo} is the null space of Q

* Separating the Un-controllable subspace

$$\boxed{\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) + Du(k) \end{aligned}}$$

Let $x(k) = Mx^*(k)$

$$\boxed{\begin{aligned} x^*(k+1) &= M^{-1}AMx^*(k) + M^{-1}Bu(k) \\ Y(k) &= CMx^*(k) + Du(k) \end{aligned}}$$

$\Rightarrow$ decoupled structure of generalized system:

$$\begin{bmatrix} \bar{X}_c(k+1) \\ \bar{X}_{uc}(k+1) \end{bmatrix} = \begin{bmatrix} \bar{A}_c & \bar{A}_{12} \\ 0 & \bar{A}_{uc} \end{bmatrix} \begin{bmatrix} \bar{X}_c(k) \\ \bar{X}_{uc}(k) \end{bmatrix} + \begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} \bar{C}_c & \bar{C}_{uc} \end{bmatrix} \begin{bmatrix} \bar{X}_c(k) \\ \bar{X}_{uc}(k) \end{bmatrix} + Du(k)$$

★ Theorem: Kalman canonical form (Controllability)

⇒ Let $x \in \mathbb{R}^n$, $\boxed{\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) + Du(k) \end{aligned}}$ be uncontrollable with rank of the controllability matrix

$$\boxed{\text{rank}(P) = n_1 < n}$$

⇒ Let $M = [M_c, M_{uc}]$, where $M_c = [m_1 \cdots m_{n_1}]$ consists of n_1 linearly independent columns of P, and $M_{uc} = [m_{n_1+1} \cdots m_n]$ are added columns to complete the basis and yield a nonsingular M.

⇒ Then $x = M\bar{x}$ transforms the system equation to

$$\begin{bmatrix} \bar{x}_c(k+1) \\ \bar{x}_{uc}(k+1) \end{bmatrix} = \begin{bmatrix} \bar{A}_c & \bar{A}_{12} \\ 0 & \bar{A}_{uc} \end{bmatrix} \begin{bmatrix} \bar{x}_c(k) \\ \bar{x}_{uc}(k) \end{bmatrix} + \begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} \bar{C}_c & \bar{C}_{uc} \end{bmatrix} \begin{bmatrix} \bar{x}_c(k) \\ \bar{x}_{uc}(k) \end{bmatrix} + Du(k)$$

⇒ Furthermore $(\bar{A}_c\ \bar{B}_c)$ is controllable, and

$$\boxed{C(zI-A)^{-1}B + D = \bar{C}_c(zI-\bar{A}_c)^{-1}\bar{B}_c + D}$$

$\left\{ \boxed{\text{Note: Proof in Lecture}} \right\}$

* Separating the unobservable subspace

$$\boxed{\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) + Du(k) \end{aligned}}$$

Let $x = O^{-1}x^*$

$$\boxed{\begin{aligned} x^*(k+1) &= OAO^{-1}x^*(k) + OBu(k) \\ y(k) &= CO^{-1}x^*(k) + Du(k) \end{aligned}}$$

⇒ Decoupled structure for generalized systems

$$\begin{bmatrix} \bar{X}_o(k+1) \\ \bar{X}_{uo}(k+1) \end{bmatrix} = \begin{bmatrix} \bar{A}_o & 0 \\ \bar{A}_{21} & \bar{A}_{uo} \end{bmatrix} \begin{bmatrix} \bar{X}_o(k) \\ \bar{X}_{uo}(k) \end{bmatrix} + \begin{bmatrix} \bar{B}_o \\ \bar{B}_{uo} \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} \bar{C}_o & 0 \end{bmatrix} \begin{bmatrix} \bar{X}_o(k) \\ \bar{X}_{uo}(k) \end{bmatrix} + Du(k)$$

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★ Theorem (Kalman canonical form (observability))

⇒ Let $x \in R^n$, $x(K+1) = Ax(K) + Bu(K)$ be unobservable
$Y(K+1) = Cx(K) + Du(k)$
with rank of the observability matrix,

$$\text{rank}(Q) = n_2 < n.$$

⇒ Let $O = \begin{bmatrix} O_o \\ O_{uo} \end{bmatrix}$ where O_o consists of n_2 linearly

independent rows of Q and $O_{uo} = [O_{n_1+1}^T, \cdots O_n^T]^T$ are added rows to complete the basis and yield a nonsingular O.

⇒ Then $\bar{x} = Ox$ transforms the system equation to

$$\begin{bmatrix} \bar{x}_o(K+1) \\ \bar{x}_{uo}(K+1) \end{bmatrix} = \begin{bmatrix} \bar{A}_o & 0 \\ \bar{A}_{21} & \bar{A}_{uo} \end{bmatrix} \begin{bmatrix} \bar{x}_o(K) \\ \bar{x}_{uo}(K) \end{bmatrix} + \begin{bmatrix} \bar{B}_o \\ \bar{B}_{uo} \end{bmatrix} u(K)$$

$$y(K) = [\bar{C}_o, 0] \begin{bmatrix} \bar{x}_o(K) \\ \bar{x}_{uo}(K) \end{bmatrix} + Du(K)$$

⇒ Furthermore $(\bar{A}_o, \bar{C}_o)$ is controllable and

$$C(zI-A)^{-1}B + D = \bar{C}_o(zI-\bar{A}_o)^{-1}\bar{B}_o + D$$