Projective Geometry

$\Rightarrow$ **Homogeneous Coordinates** are system of Coordinates used in projective Geometry.

x — In Euclidian Coordinates

X — In homogeneous Coordinates

"Representation X of a geometric object is homogeneous if X and λX represents the Same object $\forall\ \lambda \neq 0$".

$\Rightarrow$ H.C uses $n+1$ dimensional Vector to represent the Same n-dim point.

Point in $\mathbb{R}^2$	Point in $\mathbb{P}^2$
$\begin{bmatrix} x \\ y \end{bmatrix}$	$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$

$\boxed{X^T L = 0} \rightarrow$ If point X lies on line L.

2D
- line l
- Point x

3D
- Plane A
- line L
- Point X

① $x \cdot l = 0$

{if point x lies on line l}

① $X \cdot A = 0$

{Point X lies of plane A}

② $x = l \times m$

{x is point of intersection of line l and m}

★ <u>Transformation</u>

{Projective transformation is an invertable linear mapping}

$$X' = HX$$

$$\lambda \begin{bmatrix} R & t \\ 0^T & 1 \end{bmatrix}$$ {Rigid body transformation}

Camera Extrinsics and Intrinsics

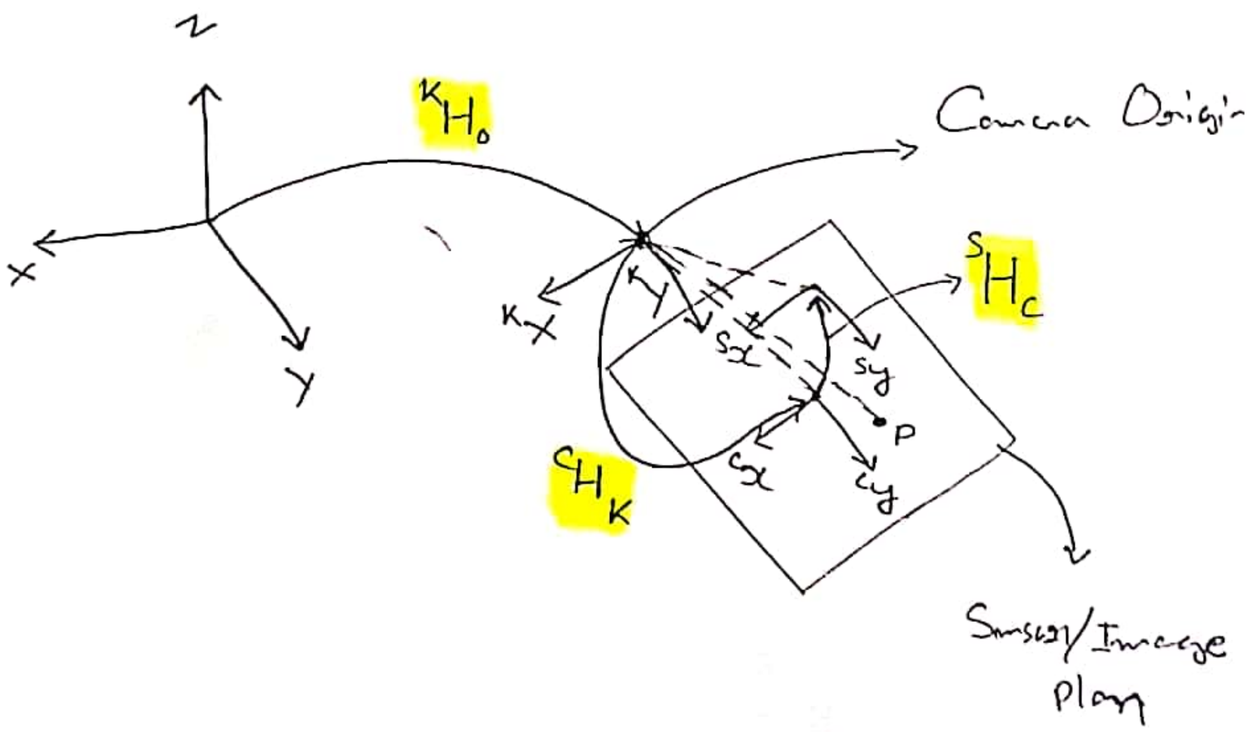
⇒ We need four coordinate system when working with a camera.

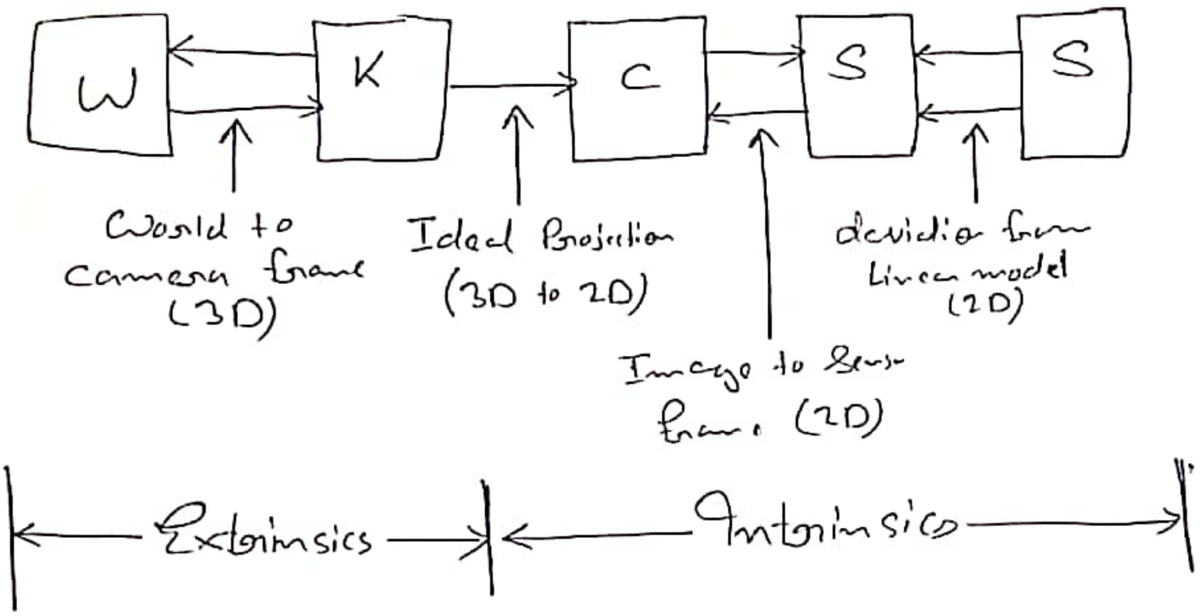
① World Coordinate System S_o $[X, Y, Z]^T$

② Camera Coordinate System S_K $[{}^kX, {}^kY, {}^kZ]^T$

③ Image Coordinate System S_c $[{}^cx, {}^cy]$

④ Sensor coordinate System S_s $[{}^sx, {}^sy]$





$$\begin{bmatrix} {}^{s}x \\ {}^{s}y \\ 1 \end{bmatrix} = {}^{S}H_{C}\, {}^{C}H_{K}\, {}^{K}H_{O} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

Coordinate of the point in Sensor frame

Image frame to Sensor frame

Camera frame to Image frame

World frame to Camera frame

{Coordinate of Point in world coordinate System}

⇒ Let X_0 be coordinate of the origin of camera frame in world coordinate frame.

$$X_0 = [X_0, Y_0, Z_0]^T$$

⇒ Let R be rotation matrix from frame S_0 to S_K'

$$\boxed{{}^{K}X_P = R(X_P - X_0)}$$

{In Cartesian Coordinate}

(Coordinate of point P in Camera Coordinate frame)

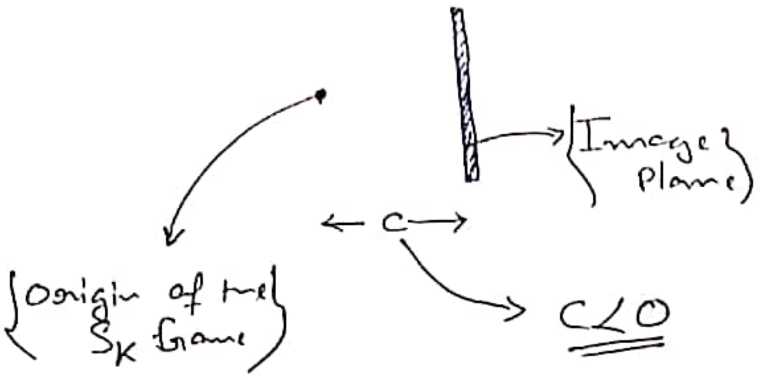
(Coordinate of point P in World coordinate frame)

$$\boxed{{}^{K}H_0 = \begin{bmatrix} R & -RX_0 \\ 0^T & 1 \end{bmatrix}}$$

{This Contains all the Extrinsic parameter}

★ Ideal perspective Projection

⇒ All rays are straight lines which passes through the Origin of S_K.



Principal point ⇒ Projection of Origin of S_K to Image plane.

$$\boxed{\begin{aligned} {}^c x_p &= c\,\frac{{}^K X_p}{{}^K Z_p} \\ {}^c y_p &= c\,\frac{{}^K Y_p}{{}^K Z_p} \end{aligned}}$$

$${}^c x_p = {}^c P_K \, {}^K X_p$$

$$\downarrow$$

$$\begin{bmatrix} c & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$ {}^{c}x_p = \underbrace{{}^{c}P_K \, {}^{K}H_o}\, X $$

$$ {}^{c}P = {}^{c}P_K \, {}^{K}H = \begin{bmatrix} c & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} R & -RX_0 \\ 0^T & 1 \end{bmatrix} $$

$$ {}^{c}P = {}^{c}K R [I_3 | -X_0] $$

Projection matrix

$$ \begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

{Calibration matrix for the ideal camera}

$$ \begin{bmatrix} 1 & 0 & 0 & -X_0 \\ 0 & 1 & 0 & -Y_0 \\ 0 & 0 & 1 & -Z_0 \end{bmatrix} $$

★ Image frame to Sensor frame

- → Location of principal point in Sensor frame.

  $(x_H \;\; y_H)^T$

- → Scale difference m
- → Shear Compensation s

  {For digital cameras, we typically have $s \simeq 0$}

$${}^{s}H_{c} = \begin{bmatrix} 1 & s & x_H \\ 0 & 1+m & y_H \\ 0 & 0 & 1 \end{bmatrix}$$

⇒ finally,

$${}^{s}x = \underbrace{{}^{s}H_{c}\,{}^{c}K}\, R[I_3 \,|\, -X_0]\, X$$

$$K = {}^{s}H_{c}\,{}^{c}K = \begin{bmatrix} c & cs & x_H \\ 0 & c(1+m) & y_H \\ 0 & 0 & 1 \end{bmatrix}$$

Calibration matrix

{This contain 5 parameters}

{It is actually an affine transformation}

⇒ The mapping $\boxed{{}^{s}x = PX}$ is called direct linear transform.

{This contains 11 parameters}

→ 6 extrinsic

→ 5 intrinsic

★ Non-Linear Effect

$${}^{a}x = {}^{s}x + \Delta x(x, a)$$ {Indivi shift of each pixel (Location dependent)}

$${}^{a}x = {}^{a}H_s(x)\,{}^{s}x$$

$$\begin{bmatrix} 1 & 0 & \Delta x(x,a) \\ 0 & 1 & \Delta y(x,a) \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow {}^{a}x = \underbrace{{}^{a}H_s(x)\,K}\,R\,[I_3 | -X_0]\,X$$

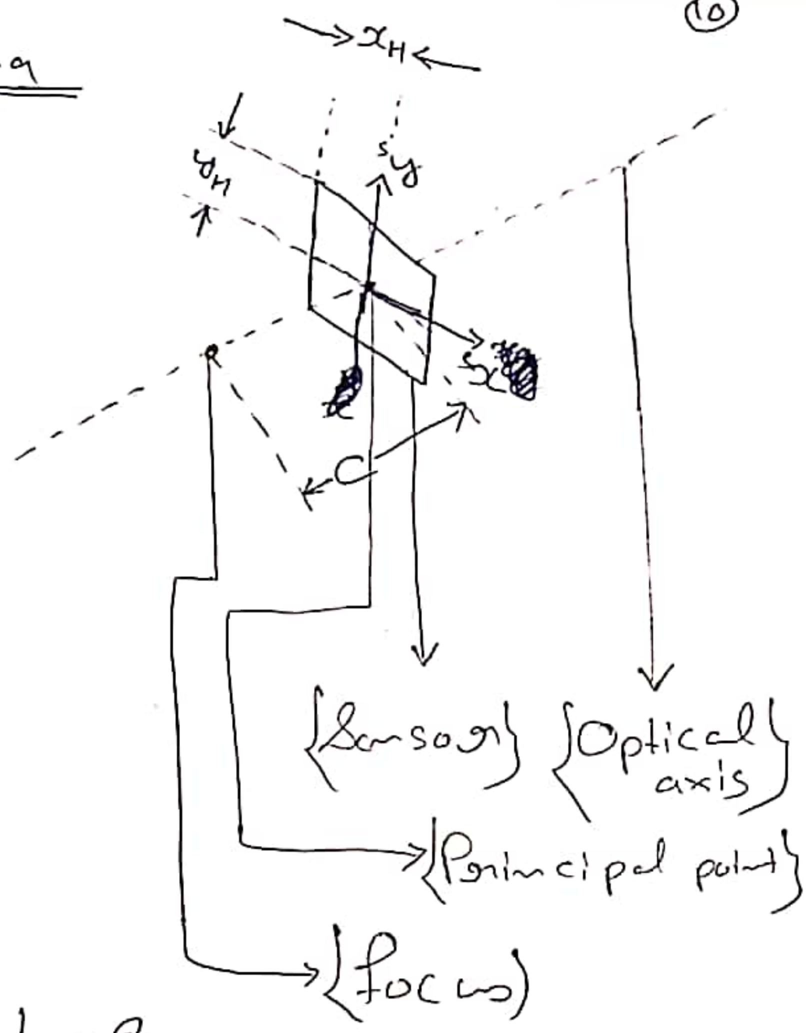
{Overall mapping}

$${}^{a}K(x,a) = \begin{bmatrix} c & cs & x_H + \Delta x(x,a) \\ 0 & c(1+m) & y_H + \Delta y(x,a) \\ 0 & 0 & 1 \end{bmatrix}$$

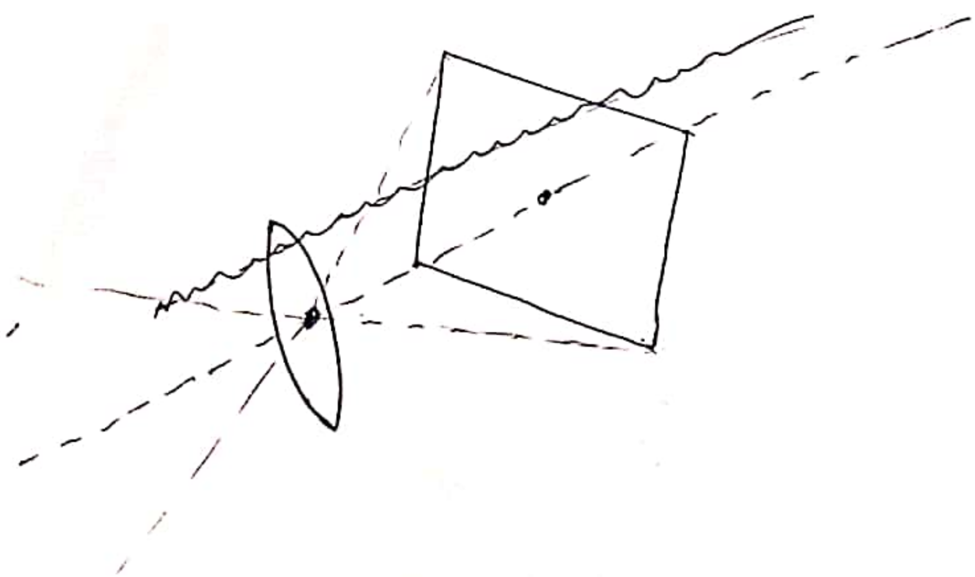
General Calibration Matrix

Pin hole Camera

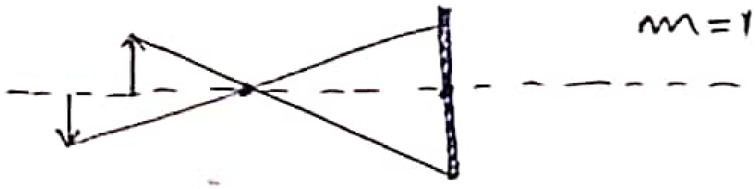
$m = 1$
$S = 0$
$\Delta x = 0$
$\Delta y = 0$



Camera With Lense

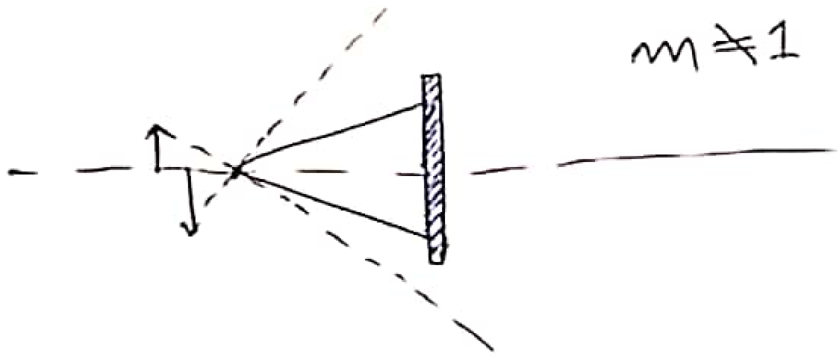


1D Camera (Pin hole)



$m = 1$

1D Camera (with lense)



$m \neq 1$

★ Inversion of Mapping

① ${}^{a}x \rightarrow {}^{s}x$ {Correcting distortion, no frame transformation happens here.}

② ${}^{s}x \rightarrow X$

{Requires iterative Solution.}

$$\boxed{X = X_0 + \lambda (KR)^{-1}\,{}^{s}x}$$

Equation of line in 3D Passing through point X_0 in direction $\lambda (KR)^{-1}\,{}^{s}x$.