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★ <u>Classical Control Theory</u> ⇒ It is a branch of Control theory that deals with the behavior of dynamic systems with inputs and how their behavior is modified by feedback, using the <u>laplace transform as a basic tool to model</u> such systems

★ <u>Modern Control theory</u> ⇒ Instead of changing domains to avoide the complexities of time domain ODE mathematics, converts the differential equations into a system of lower order time domain equations called <u>State equations</u>- which can then be manipulated using techniques from linear algebra.

* Robust Control Theory ⇒ In Control theory, robust Control is an approach to Controller design that explicitly deals with Uncertainity.

———— X ———— X ————

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CHAPTER 1

Mathematical Modeling of Control System

1

Mathematical Modeling of Control Systems

"A mathematical model of a dynamic system is defined as a set of equations that represents the dynamics of the System accurately or at least fairly well"

↳ Simplicity Vs Accuracy

⇒ We must be well aware that a linear lumped-parameter model, which may be valid in low-frequency operations, may not be valid at sufficiently high frequency.

★ Linear System ⇒ A system is called linear if the principle of Superposition applies.

⇓

It states that the response produced by the Simultaneous application of two different forcing function is the sum of the two individual responses

⇒ A differential equation is linear if the Coefficients are constants or function only of the independent variable.

[redacted]

Linear Time-Invariant System

System that are represented by differential equations whose coefficients are not a function of time

Linear Time-Varying System

System that are represented by differential equations whose coefficients are a function of time

★ Transfer Function

"The transfer function of a linear, time-invariant, differential equation system is defined as the ratio of the Laplace transform of the output (response function) to the Laplace transform of the input (driving function) under the assumption that all initial conditions are zero"

⇒ Consider a linear time-invariant system defined by the following differential equation:

$$a_0 \overset{n}{y} + a_1 \overset{n-1}{y} + \dots + a_{n-1}\dot{y} + a_n y$$
$$= b_0 \overset{m}{x} + b_1 \overset{m-1}{x} + \dots + b_{m-1}\dot{x} + b_m x$$

Where y is the output of the system and x is the input.

So, Transfer function, $G(s) = \left. \dfrac{\mathcal{L}[\text{Output}]}{\mathcal{L}[\text{Input}]} \right|_{\text{zero initial condition}}$

$$G(s) = \frac{Y(s)}{X(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1} s^1 + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}$$

If the highest power of S in the denominator of the transfer function is equal to n, the system is called an n^{th}-order system.

Note

"The applicability of the concept of the transfer function is limited to linear, time-invariant differential equation system"

★ Convolution Integral

$$G(s) = \frac{Y(s)}{X(s)} \Rightarrow Y(s) = G(s)\,X(s)$$

⇒ Multiplication in the complex domain is equivalent to convolution in the time domain, so the inverse Laplace transform of above equation is given by the following convolution integral

$$\boxed{\begin{aligned} y(t) &= \int_0^t x(\tau)\, g(t-\tau)\, d\tau \\ &= \int_0^t g(\tau)\, x(t-\tau)\, d\tau \end{aligned}}$$

where both $g(t)$ & $x(t)$ are 0 $\forall\ t<0$.

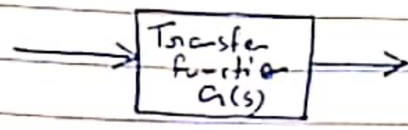
★ Automatic Control Systems

\# **Block Diagram**: A block diagram of a system is a pictorial representation of the functions performed by each component and of the flow of signals.

⇒ In a block diagram all system variables are linked to each other through functional block.

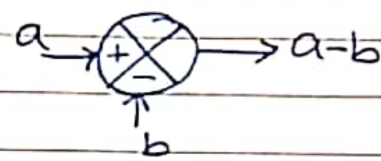
⇒ The functional block or simply block is a symbol for the mathematical operation on the input signal to the block that produces output.

↳ The transfer functions of the components are usually entered in the corresponding block, which are connected by arrow to indicate the direction of flow of signals.



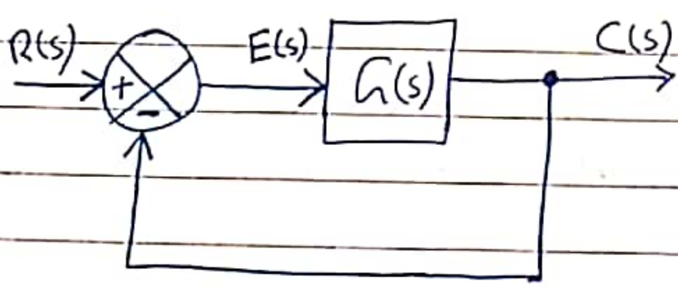
i) Summing point: Circle with a Cross is the Symbol that indicates a summing operation.

→ The plus and minus sign at each arrowhead indicates whether that signal is to be added or subtracted.



ii) Branch point: A branch point is a point from which the signal from a block goes concurrently to other blocks or summing point.

Block Diagram of a Closed-loop System

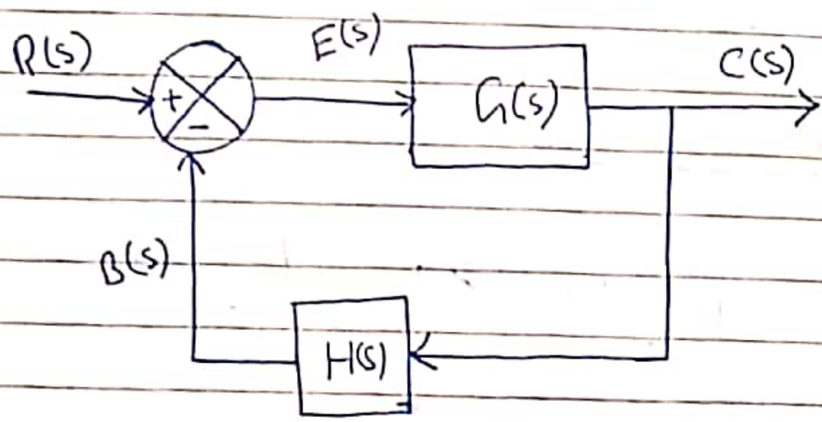


C(s) : Output
R(s) : Reference Input
E(s) : Error Signal

⇒ Any linear Control system may be represented by a block diagram consisting of block, Summing point and branch point.

⇒ When the output is fed back to the Summing point for comparision with the input, it is necessary to convert the form of the output signal ~~is would~~ to that of input signal.

↳ This Conversion is achived by the feedback element whose transfer function is H(s).



⇒ In most of the cases feedback element is a sensor that measures the output of the plant.

Transfer Function

Open loop Transfer function

$$\Rightarrow \boxed{\frac{B(s)}{E(s)} = G(s)\,H(s)}$$

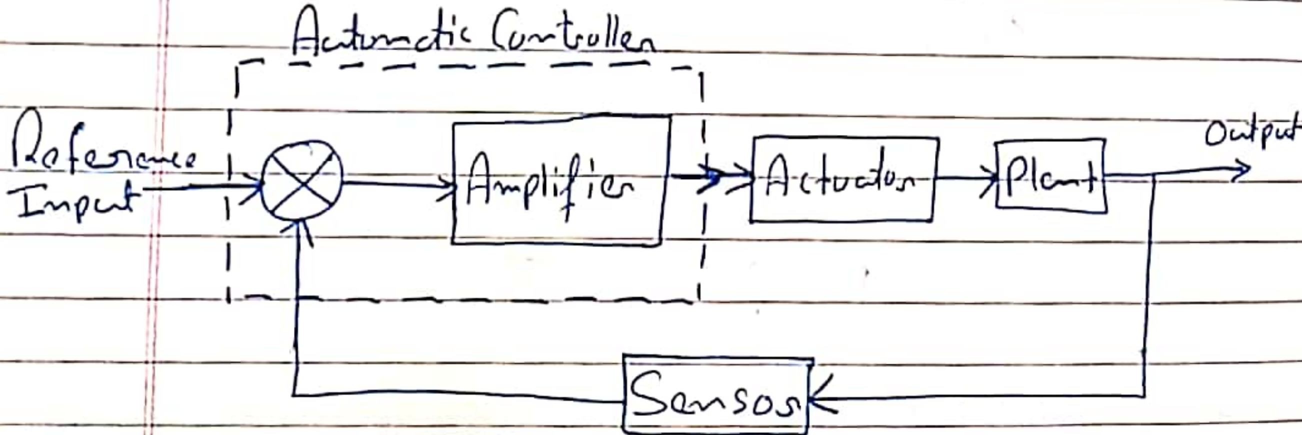
Feed forward Transfer function

$$\boxed{\frac{C(s)}{E(s)} = G(s)}$$

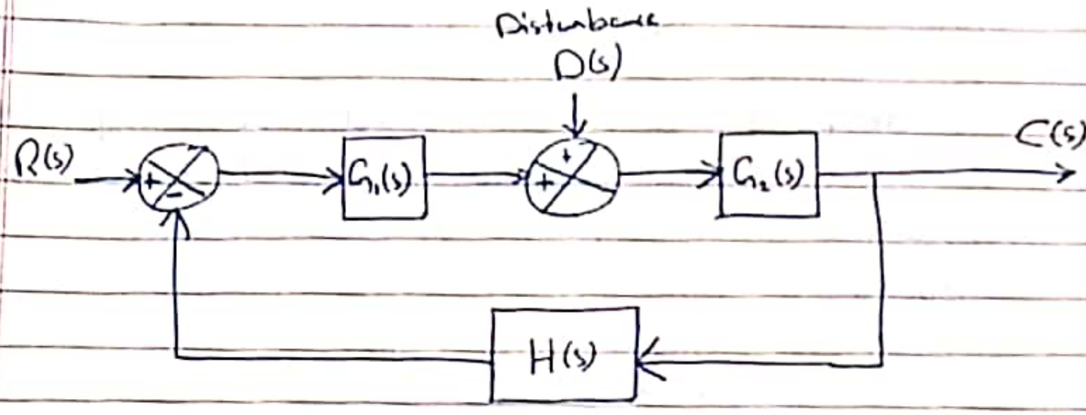
Closed loop Transfer function

$$\boxed{\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}}$$

★ **Automatic Controller**: An automatic Controller Compares the actual value of the plant output with the reference input (desired value), determines the deviation, and produces a control signal that will reduce the deviation to zero or to a small value.



* Closed-Loop System Subjected to Disturbance



⇒ When two inputs (the reference input and disturbance) are present in a linear time invariant system, each input can be treated independently of the other; and the output corresponding to each input alone can be added to give the complete output.

⇒ In examining the effect of the disturbance D(s), we may assume that the reference input is zero,

$$\frac{C_D(s)}{D(s)} = \frac{G_2(s)}{1 + G_1(s)\,G_2(s)\,H(s)} \quad \text{—— (1)}$$

⇒ On the other hand, in considering the response to the reference input R(s), we may assume that the disturbance is zero,

$$\frac{C_R(s)}{R(s)} = \frac{G_1(s)\,G_2(s)}{1+G_1(s)\,G_2(s)\,H(s)}$$

⇒ The response to the simultaneous application of the refference input and disturbance can be obtained by adding the two individual responses.

$$C(s) = C_R(s) + C_D(s)$$

$$\boxed{C(s) = \frac{G_2(s)}{1+G_1(s)\,G_2(s)\,H(s)}\left[G_1(s)\,R(s) + D(s)\right]}$$

★ Procedure for Drawing a Block Diagram

⇒ To draw a block diagram for a system, first write the equations that describe the dynamic behavior of each component.

⇒ Then take the laplace transform of these equations, assuming zero initial conditions and represent each Laplace-transformed equation individually in Block form.

⇒ Finally assembly of these elements into a Complete block diagram.

★ Linearization of Non-Linear Mathematical Models

Nonlinear System :- A system is nonlinear if the principle of superposition does not apply.

Linearization of Nonlinear System

⇒ In Control engineering a normal operation of the system may be around an equilibrium point, and the signal may be considered small signal around the equilibrium.

⇒ If the System operates around an equilibrium point and if the signal involved are small signals, then it is possible to approximate the non-linear system by a linear system.

⇒ The linearization procedure is based on the expantion of nonlinear function into a Taylor series about the operating point and the retention of only the linear term.

Linear approximation of Non linear Mathematical Models

⇒ Consider a system whose input is $x(t)$ and output is $y(t)$. The relationship between $y(t)$ and $x(t)$ is given by:-

$$y = f(x)$$

⇒ If the normal operating condition corresponds to $\bar{x}, \bar{y}$ then Equation may be expanded into a Taylor Series about this point as follows

$$y = f(\bar{x}) + \frac{df}{dx}(x-\bar{x}) + \frac{1}{2!}\frac{d^2f}{dx^2}(x-\bar{x})^2 + \cdots$$

where, the derivatives $\frac{df}{dx}, \frac{d^2f}{dx^2}$ are evaluated at $x = \bar{x}$.

⇒ If the variation $x-\bar{x}$ is small, we may neglect the higher-order terms in $x-\bar{x}$.

$$y = \bar{y} + K(x-\bar{x})$$

$$\left\{ \begin{array}{l} \bar{y} = f(\bar{x}) \\ K = \left.\frac{df}{dx}\right|_{x=\bar{x}} \end{array} \right.$$

$$\Rightarrow \boxed{(y-\bar{y}) = K(x-\bar{x})}$$

Equation above gives a linear mathematical model for a non-linear system about operating point $x=\bar{x}, y=\bar{y}$.

$\Rightarrow$ Next Consider a nonlinear system whose output y is a function of two input x_1 and x_2.

$$y = f(x_1, x_2)$$

$\Rightarrow$ To obtain a linear approximation to this nonlinear system, we may expand above into a Taylor Series about the normal operating point $\bar{x}_1, \bar{x}_2$.

$$y = f(\bar{x}_1, \bar{x}_2) + \left[\frac{\delta f}{\delta x_1}(x_1-\bar{x}_1) + \frac{\delta f}{\delta x_2}(x_2-\bar{x}_2)\right]$$

$$+ \frac{1}{2!}\left[\frac{\delta^2 f}{\delta x_1^2}(x_1-\bar{x}_1)^2 + 2\frac{\delta^2 f}{\delta x_1 \delta x_2}(x_1-\bar{x}_1)(x_2-\bar{x}_2)\right.$$

$$\left. + \frac{\delta^2 f}{\delta x_2^2}(x_2-\bar{x}_2)^2\right] + \cdots$$

where partial derivatives are evaluated at $x_1=\bar{x}_1$ & $x_2=\bar{x}_2$.

⇒ Near the normal operating point, the higher order terms may be neglected.

$$\boxed{y - \bar{y} = K_1(x_1 - \bar{x}_1) + K_2(x_2 - \bar{x}_1)}$$

$$\bar{y} = f(\bar{x}_1, \bar{x}_2)$$

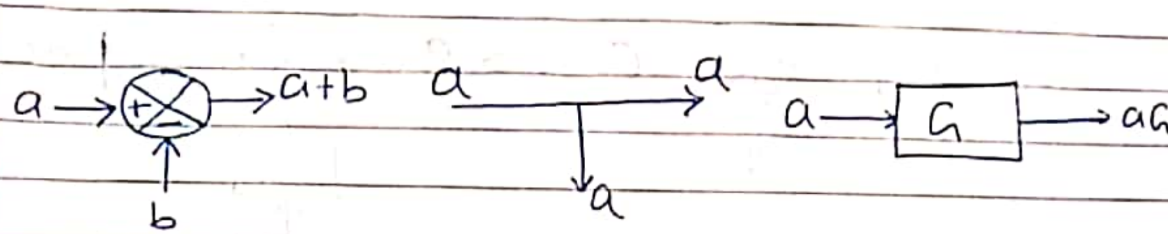
$$K_1 = \left.\frac{\delta f}{\delta x_1}\right|_{x_1 = \bar{x}_1,\, x_2 = \bar{x}_2}$$

$$K_2 = \left.\frac{\delta f}{\delta x_2}\right|_{x_1 = \bar{x}_1,\, x_2 = \bar{x}_2}$$

⇒ The linearization technique presented here is valid in the vicinity of the operating condition. If operating conditions vary widely, however, such linearization equations are not adequate and nonlinear equations must be delt with.

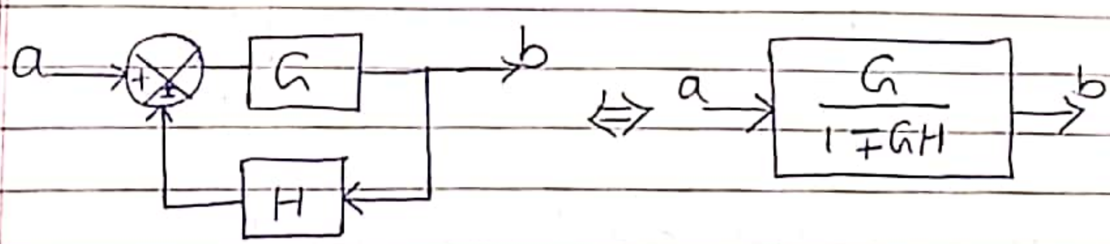
★ Block-Diagram Algebra

Elements of Block Algebra

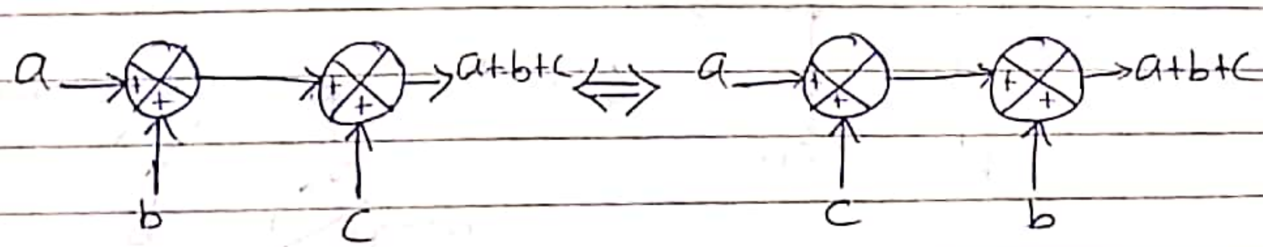


(i) (ii) (iii)

1. Feedback Loop

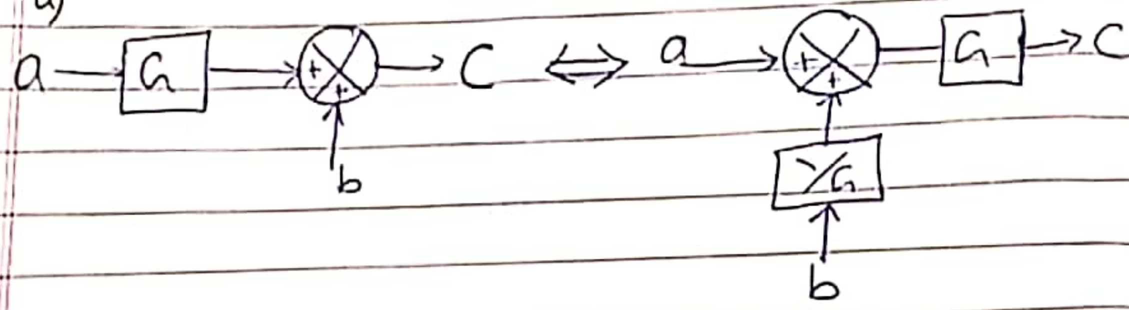


2. Two Summing points

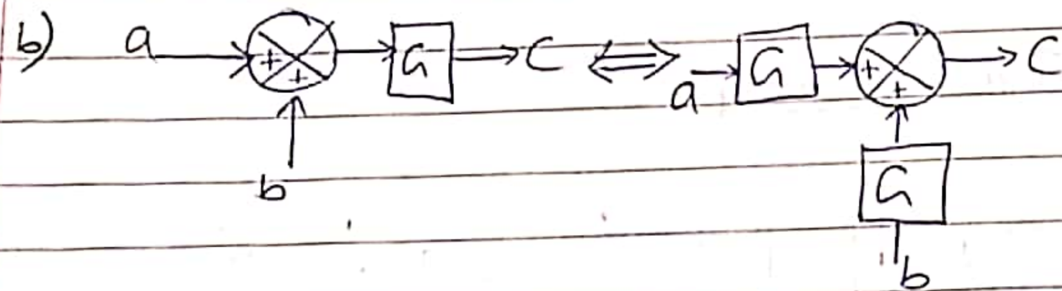


3. Summing point and block

a)

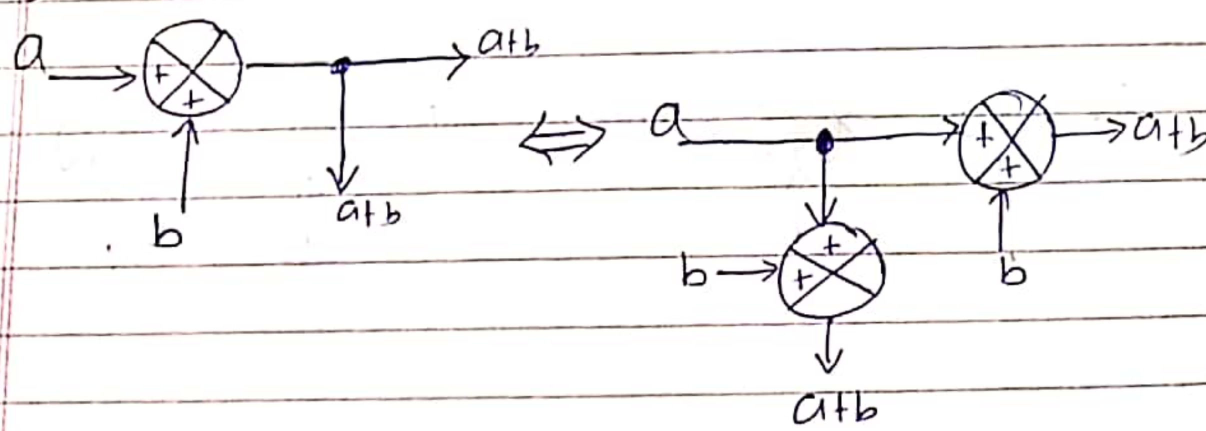


b)

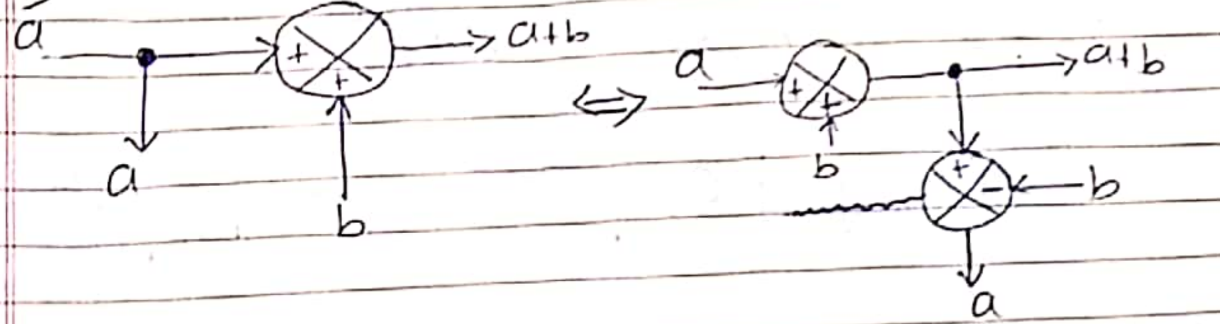


4. Summing point and branch point

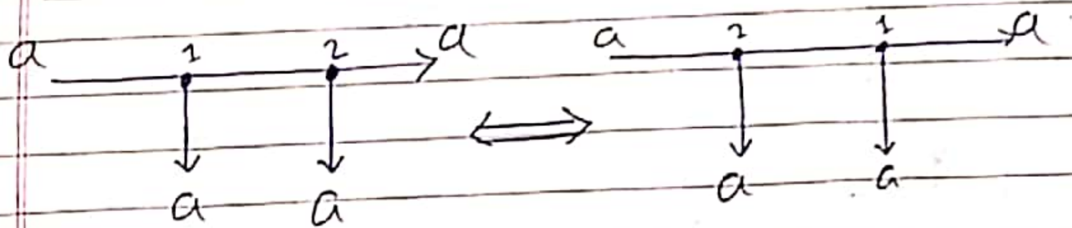
a)



b)

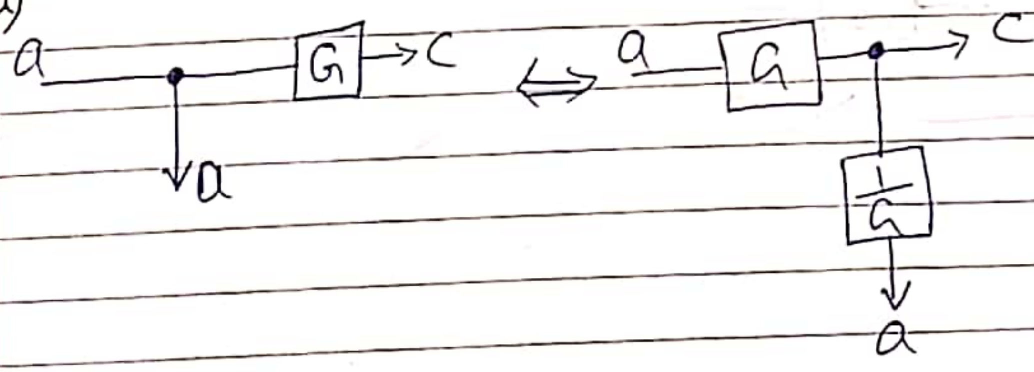


5. Two branch point

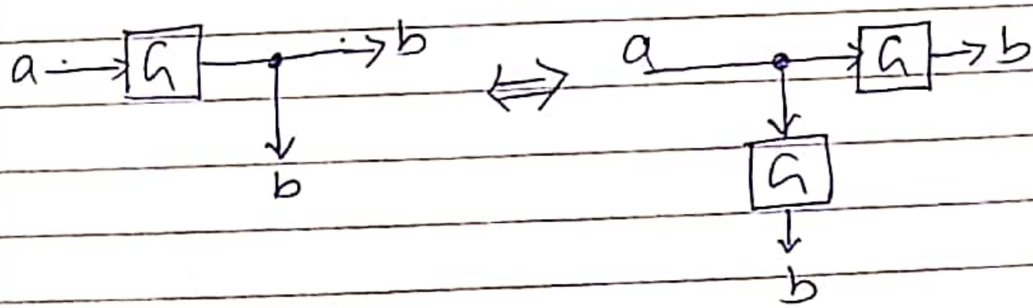


6. Branch point and Block

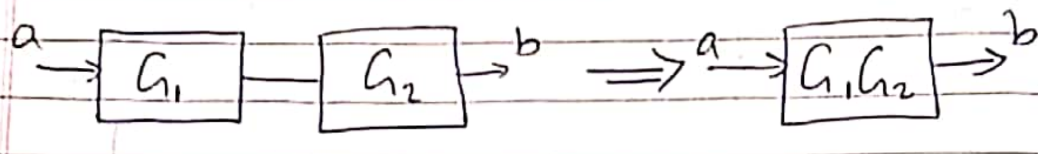
a)



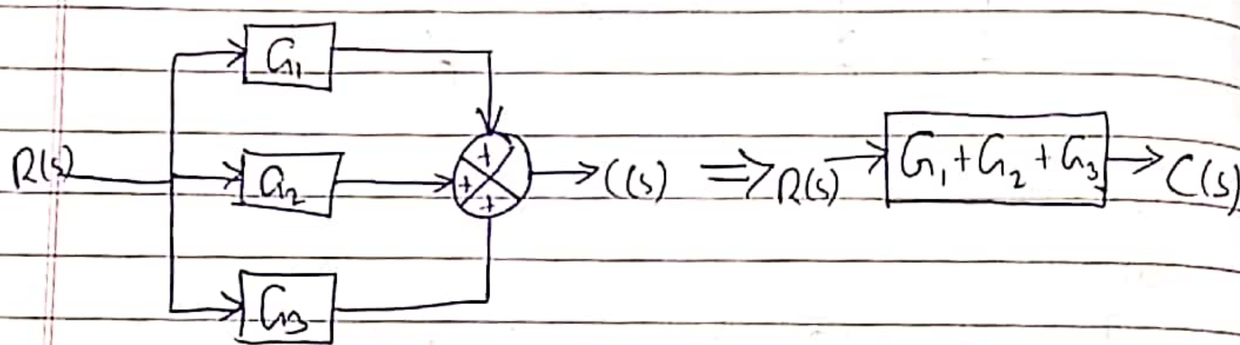
b)



7. Two Block (Cascaded)



8. Blocks (in parallel)



CHAPTER 2

Mathematical Modeling of Mechanical & Electrical System

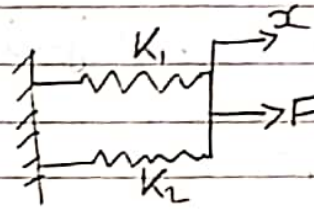
2

Mathematical Modeling of Mechanical & Electrical System

★ Mathematical Modeling of Mechanical System

Example 3.1

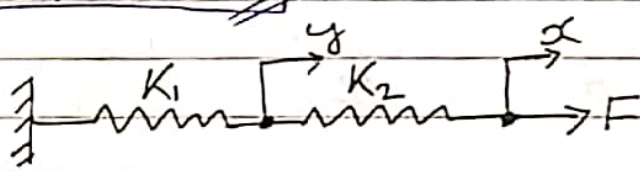
ⓐ Parallel



$$F = K_{eq}\,x = K_1 x + K_2 x$$

$$\Rightarrow \boxed{K_{eq} = K_1 + K_2}$$

ⓑ Series



$$F = K_{eq}\,x$$

$$F = K_1 y = K_2 (x - y)$$

$$\Rightarrow K_1 y = K_2 x - K_2 y$$

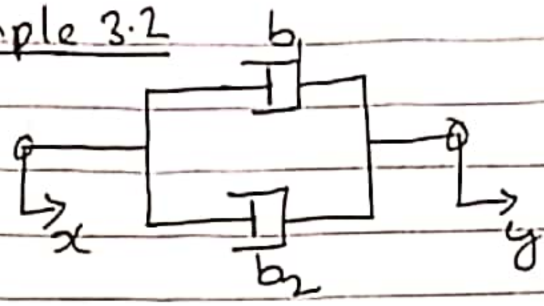
$$y = \frac{K_2 x}{K_1 + K_2}$$

$$\boxed{\frac{1}{K_{eq}} = \frac{1}{K_1} + \frac{1}{K_2}}$$

$$\frac{K_1 K_2 x}{K_1 + K_2} = K_{eq}\,x \quad \Rightarrow K_{eq} = \frac{K_1 K_2}{K_1 + K_2}$$

Example 3.2

(a)

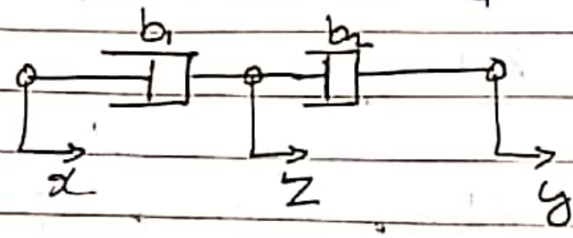


$$f = b_1(\dot{y}-\dot{x}) + b_2(\dot{y}-\dot{x}) = (b_1+b_2)(\dot{y}-\dot{x})$$

$$f = b_{eq}(\dot{y}-\dot{x})$$

So $\boxed{b_{eq} = b_1 + b_2}$

(c)



$$f = b_{eq}(\dot{y}-\dot{x})$$

$$f = b_1(\dot{z}-\dot{x}) = b_2(\dot{y}-\dot{z})$$

$$b_1\dot{z} - b_1\dot{x} = b_2\dot{y} - b_2\dot{z}$$

$$(b_1+b_2)\dot{z} = b_2\dot{y} + b_1\dot{x} - (b_1+b_2)\dot{x}$$

$$-(b_1+b_2)\dot{x}$$

$$(b_1+b_2)(\dot{z}-\dot{x}) = b_2\dot{y} - b_2\dot{x}$$

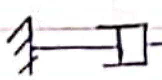
$$(\dot{z}-\dot{x}) = \frac{b_2}{b_1+b_2}(\dot{y}-\dot{x})$$

$$\frac{b_1 b_2}{b_1+b_2}(\dot{y}-\dot{x}) = b_{eq}(\dot{y}-\dot{x})$$

$$b_{eq} = \frac{b_1 b_2}{b_1+b_2} \Rightarrow \boxed{\frac{1}{b_{eq}} = \frac{1}{b_1}+\frac{1}{b_2}}$$

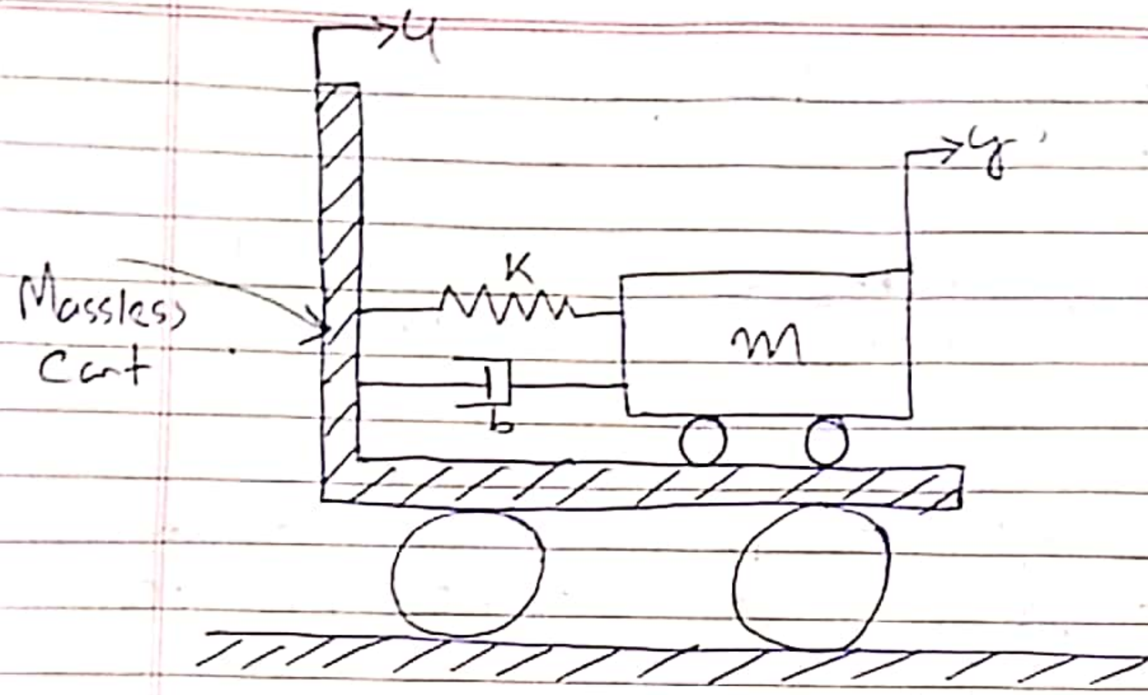
<u>Dashpot</u>: A dashpot is a mechanical device, a damper which resists motion via viscous friction.

→ The resulting force is proportional to the velocity, but acts in the opposite direction, slowing the motion and absorbing energy.

 $\rightarrow F = bV / b\dot{x}$

<u>Example 3.3:</u>

PTO



⇒ Cart is standing still at $t<0$

⇒ Spring-mass-dashpot system on the cart is also standing still $\forall\, t<0$

$u(t)$: Input {displacement of cart}

at $t=0$ $\quad \dot{u} =$ Constant.

$y(t)$: Output {displacement of mass relative to ground}

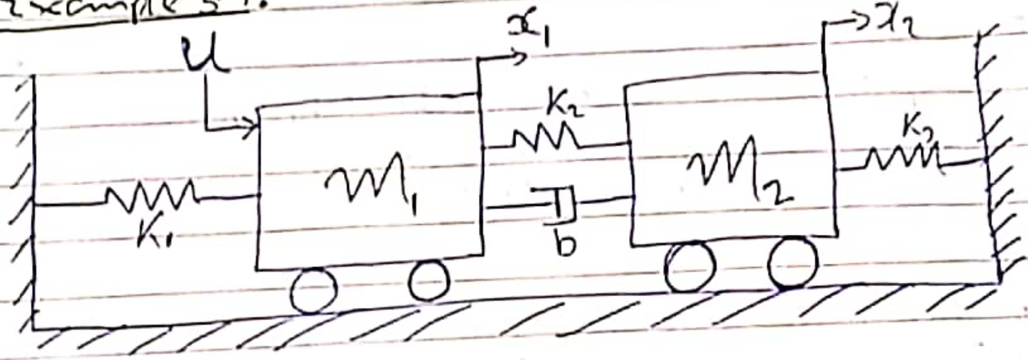
$$-K(y-u) - b\left(\frac{dy}{dt} - \frac{du}{dt}\right) = m\frac{d^2y}{dt^2}$$

$$\Rightarrow \quad m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + Ky = b\frac{du}{dt} + Ku$$

$$(ms^2+bs+K)Y(s) = (bs+K)U(s)$$

$$\boxed{G(s) = \frac{Y(s)}{U(s)} = \frac{bs+K}{ms^2+bs+K}}$$

Example 3.4:



u is the external force applied and x_1 & x_2 represents position of m_1 & m_2 respectively.

Input : u
Output : x_1 & x_2

$$u - K_1x_1 - K_2(x_1-x_2) - b(\dot{x}_1-\dot{x}_2) = m_1\ddot{x}_1 \quad \text{--- (1)}$$

$$-K_3x_2 - K_2(x_2-x_1) - b(\dot{x}_2-\dot{x}_1) = m_2\ddot{x}_2 \quad \text{--- (2)}$$

Equation (1) & (2) is the mathematical model of the given System.

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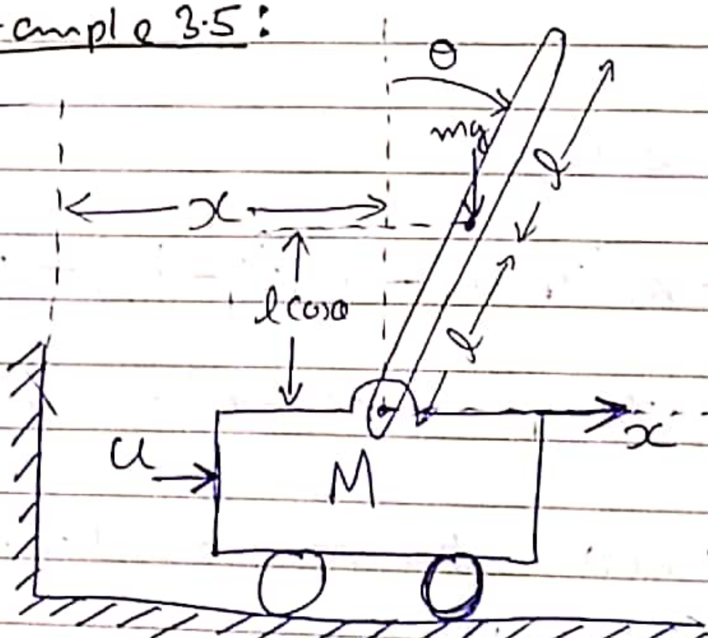
$$U(s) - K_1 X_1(s) - K_2\left(X_1(s) - X_2(s)\right) - bs\left(X_1(s) - X_2(s)\right)$$
$$= m_1 s^2 X_1(s) \quad \text{— (1)}$$

$$-K_3 X_2(s) - K_2\left(X_2(s) - X_1(s)\right) - bs\left(X_2(s) - X_1(s)\right)$$
$$= m_2 s^2 X_2(s) \quad \text{— (2)}$$

$$\boxed{\frac{X_1(s)}{U(s)} = \frac{m_2 s^2 + bs + K_2 + K_3}{(m_1 s^2 + bs + K_1 + K_2)(m_2 s^2 + bs + K_2 + K_3) - (bs + K_2)^2}}$$

$$\boxed{\frac{X_2(s)}{U(s)} = \frac{bs + K_2}{(m_1 s^2 + bs + K_1 + K_2)(m_2 s^2 + bs + K_2 + K_3) - (bs + k_2)^2}}$$

Example 3.5:



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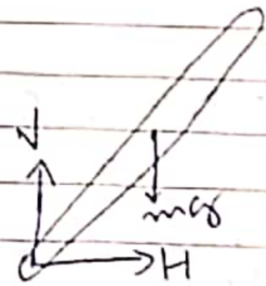
Inverted pendulum: Pendulum that has its Center of mass above its pivot point.

eg → human being

⇒ 2 DOF System (θ, x)

Input force $= u$

$$x_G = x + l\sin\theta$$
$$y_G = l\cos\theta$$



$$I\ddot{\theta} = Vl\sin\theta - Hl\cos\theta \quad \text{--- (1)}$$

$$H = m\frac{d^2}{dt^2}(x + l\sin\theta) \quad \text{--- (2)}$$

$$V - mg = m\frac{d^2}{dt^2}(l\cos\theta) \quad \text{--- (3)}$$

$$M\frac{d^2x}{dt^2} = u - H \quad \text{--- (4)}$$

The above four equation is the mathematical model of the given System.

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$\Rightarrow$ Since the inverted pendulum is kept vertical.

$\hookrightarrow$ θ can be assumed to be close to zero.

$\sin\theta \approx \theta$
$\cos\theta \approx 1$ {linearization}

$I\ddot{\theta} = Vl\theta - Hl$ — ①
$m(\ddot{x} + l\ddot{\theta}) = H$ — ②
$V = mg$ — ③
$M\ddot{x} = u - H$ — ④

{linearized Mathematical Model}

$\Downarrow$ Eliminating H & V

$(M+m)\ddot{x} + ml\ddot{\theta} = u$ — ⓐ

$(I+ml^2)\ddot{\theta} + ml\ddot{x} = mgl\theta$ — ⓑ

$(M+m)s^2X(s) + mls^2\theta(s) = U(s)$

$(I+ml^2)s^2\theta(s) + mls^2X(s) = mgl\,\theta(s)$

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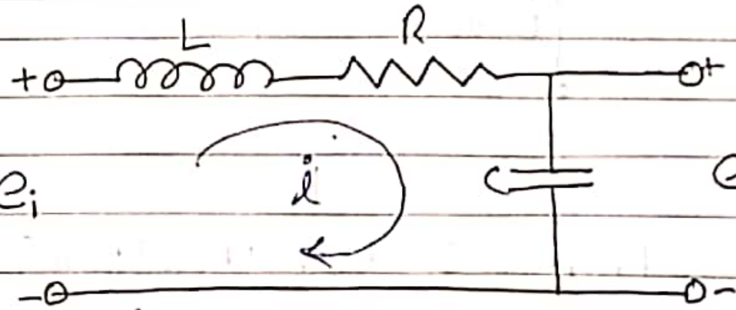
★ Mathematical Modeling of Electrical System

Basic laws governing electrical circuit are Kirchhoff's Law:

- Current Law: Algebraic sum of all currents entering and leaving a node is zero.
- Voltage Law: At any given instant the algebraic sum of the voltage around any loop in an electric circuit is zero.

LRC Circuit

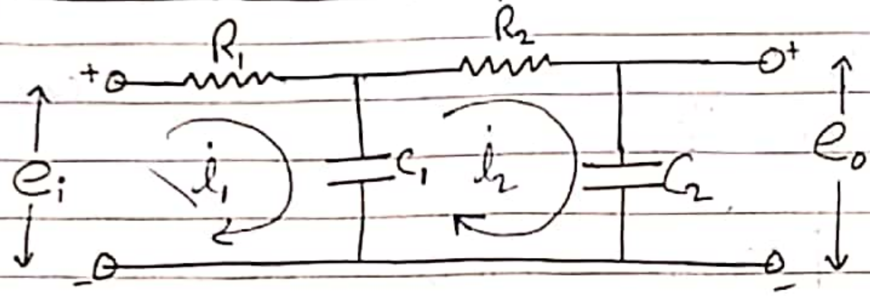


$$-e_i + L\frac{di}{dt} + iR + \frac{1}{C}\int_0^t i\,dt = 0 \quad \text{—— (a)}$$

$$e_o = \frac{1}{C}\int_0^t i\,dt \quad \text{—— (b)}$$

$$\boxed{\frac{E_o(s)}{E_i(s)} = \frac{1}{LCs^2 + RCs + 1}}$$

Transfer Function of Cascaded Element



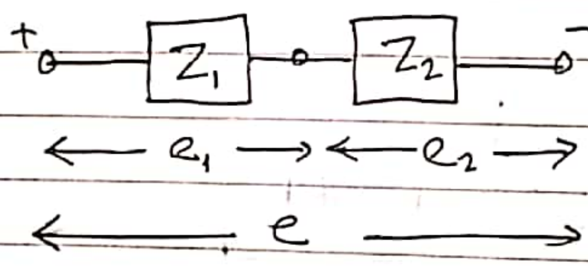
$$-e_i + i_1R_1 + \frac{1}{C_1}\int_0^t (i_1 - i_2)\,dt = 0 \quad \text{— (1)}$$

$$\frac{1}{C_1}\int_0^t (i_2 - i_1)\,dt + i_2R_2 + \frac{1}{C_2}\int_0^t i_2\,dt = 0 \quad \text{— (2)}$$

$$e_0 = \frac{1}{C_2}\int_0^t i_2\,dt \quad \text{— (3)}$$

$$\boxed{\frac{E_0(s)}{E_i(s)} = \frac{1}{R_1C_1R_2C_2S^2 + (R_1C_1 + R_2C_2 + R_1C_2)S + 1}}$$

Complex Impedance



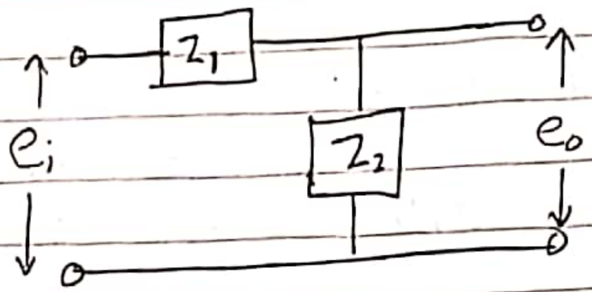
⇒ For electrical circuit, it is convenient to write the Laplace-transformed equation directly without writing the differential equations.

⇒ The Complex impedance Z(s) of a two-terminal circuit is the ratio of E(s), the Laplace transform of the voltage across the terminals to I(s), the Laplace transform of current through the element.

{Under the assumption initial condition is zero}

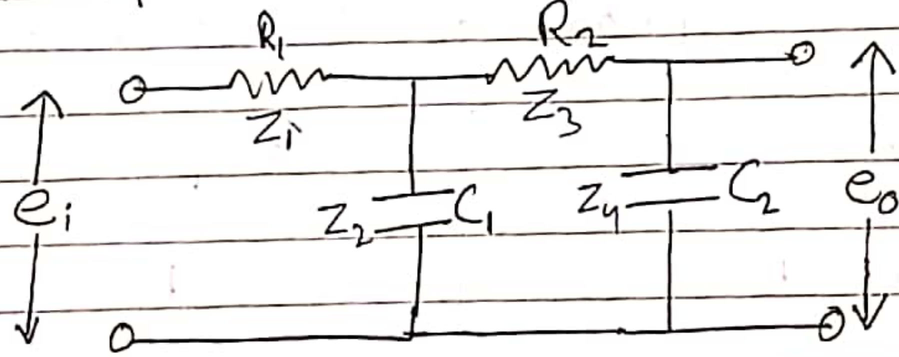
Element	Impedance
1) Resistance (R)	R
2) Capacitance (C) $$V(t) = \frac{1}{C}\int_0^t i\,dt$$	$\frac{1}{CS}$
3) Inductance (L) $$V(t) = L\frac{di}{dt}$$	LS

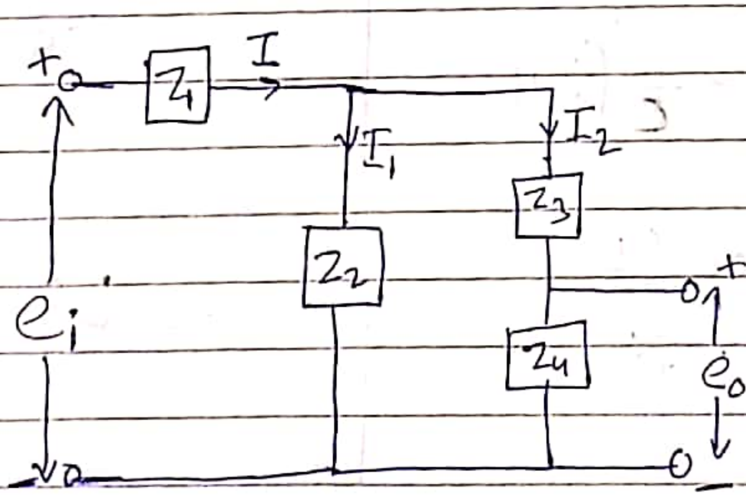
⇒ If complex impedance are connected in series, the total impedance is the sum of the individual complex impedances.



$$\frac{E_o(s)}{E_i(s)} = \frac{Z_2}{Z_1+Z_2}$$

Example 3.7





$$I_1 = \frac{Z_3 + Z_4}{Z_2 + Z_3 + Z_4} I \qquad I_2 = \frac{Z_2}{Z_2 + Z_3 + Z_4} I$$

$$E_i = Z_1 I + Z_2 I_1$$

$$= Z_1 I + \frac{Z_2 (Z_3 + Z_4)}{Z_2 + Z_3 + Z_4} I$$

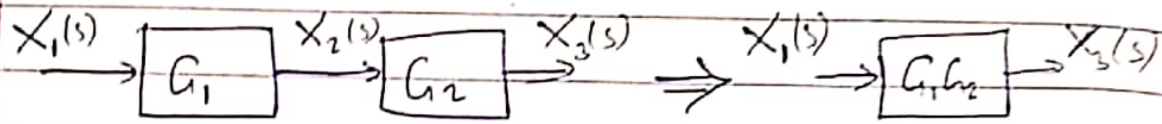
$$E_o = Z_4 I_2 = \frac{Z_4 Z_2}{Z_2 + Z_3 + Z_4} I$$

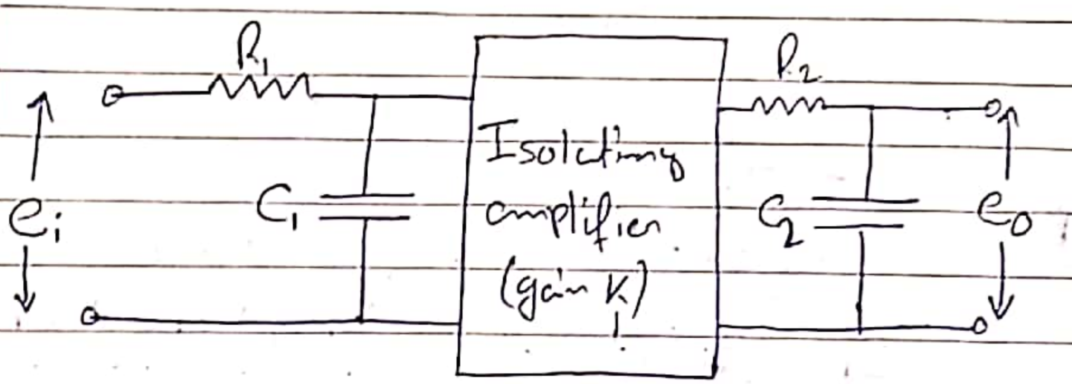
$$\frac{E_o}{E_i} = \frac{\dfrac{Z_4 Z_2}{Z_2 + Z_3 + Z_4}}{Z_1 + \dfrac{Z_2 (Z_3 + Z_4)}{Z_2 + Z_3 + Z_4}} = \frac{Z_4 Z_2}{Z_1 Z_2 + Z_1 Z_3 + Z_1 Z_4 + Z_2 Z_3 + Z_2 Z_4}$$

$$\boxed{\frac{E_o(s)}{E_i(s)} = \frac{1}{R_1 C_1 R_2 C_2 s^2 + (R_1 C_1 + R_2 C_2 + R_1 C_2) s + 1}}$$

Transfer Function of Nonloading Cascaded Element

⇒ The transfer function of a system consisting of two nonloading cascaded element can be obtained by eliminating the intermediate input & output





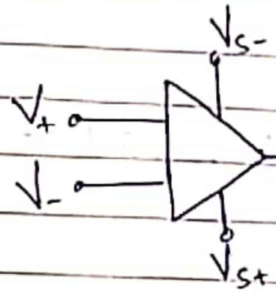
$$\frac{E_o(s)}{E_i(s)} = \left(\frac{1}{1+R_1C_1S}\right) K \left(\frac{1}{1+R_2C_2S}\right)$$

$$\frac{E_o(s)}{E_i(s)} = \frac{K}{(1+R_1C_1S)(1+R_2C_2S)}$$

→ Opamp has gain of about $10^5 \sim 10^6$ ∀ dc & ac signal (f < 10 Hz)

Electronic Controllers

* **Operational Amplifiers**: It is DC-coupled high-gain electronic voltage amplifier with a differential input and usually, a single-ended output.

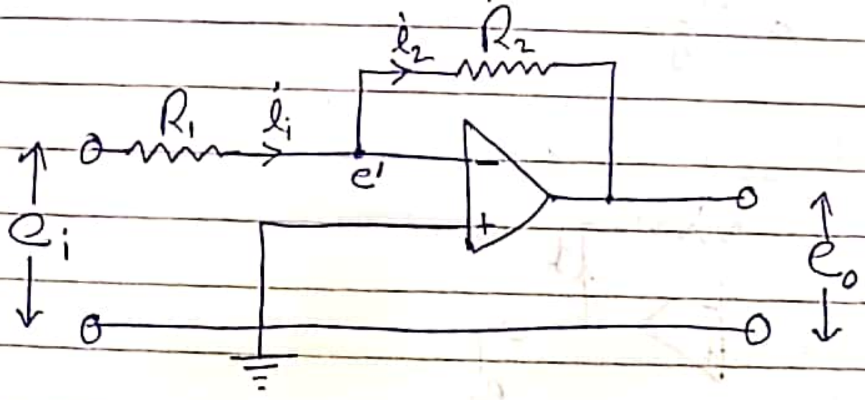


$V_{out} = A_{OL}(V_+ - V_-)$

→ It is frequently used to amplify signals in sensor circuit.

→ Also called differential amplifier / OpAmps

Inverting Amplifiers



$$i_1 = \frac{e_i - e'}{R_1} \qquad i_2 = \frac{e' - e_0}{R_2}$$

Since negligible current flows into the amplifier, the current $i_1 = i_2$

$$\frac{e_i - e'}{R_1} = \frac{e' - e_0}{R_2}$$

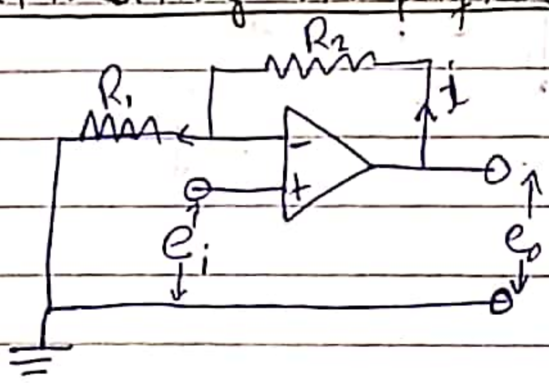
$$K(0 - e') = e_0 \quad \text{as } K \gg 1 \Rightarrow e' \approx 0$$

So $$\frac{e_i}{R_1} = -\frac{e_0}{R_2}$$

$$\boxed{e_0 = -\frac{R_2}{R_1} e_i}$$

If $R_1 = R_2$, then the opamp circuit shown acts as a sign inverter.

Noninverting Amplifier



$$e_0 = i(R_1 + R_2)$$

$$e_0 = K(e_i - iR_1)$$

$$e_0 = K\left(e_i - \frac{e_0 R_1}{R_1 + R_2}\right)$$

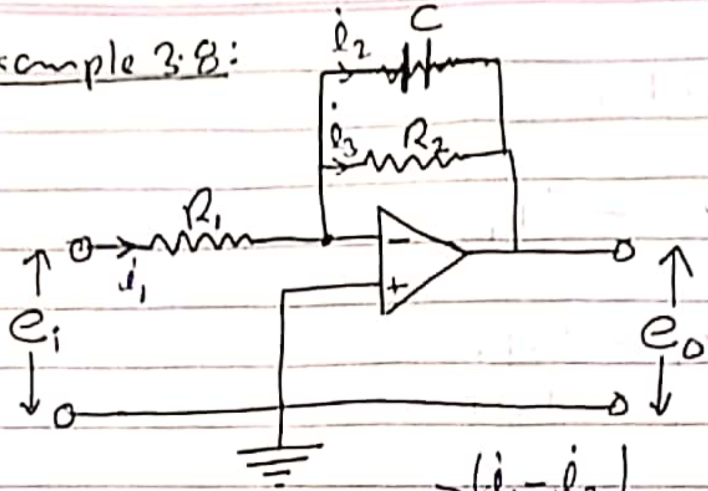
$$e_0\left(1 + \frac{K R_1}{R_1 + R_2}\right) = K e_i$$

$$\frac{e_0}{e_i} = \frac{K}{1 + \frac{K R_1}{R_1 + R_2}}$$

$$\Rightarrow \frac{e_i}{e_0} = \frac{1}{K} + \frac{R_1}{R_1 + R_2} \quad \left\{ \frac{1}{K} \ll \frac{R_1}{R_1 + R_2} \right\}$$

$$\boxed{\frac{e_0}{e_i} = 1 + \frac{R_2}{R_1}}$$

Example 3.8:



$$-e_i + i_1 R_1 + \overset{(i_1 - i_2)}{i_3} R_3 + e_o = 0 \quad \text{——} \ (1)$$

$$-e_i + i_1 R_1 + \frac{1}{C}\int_0^t i_2 \, dt = 0 \quad \text{——} \ (2)$$

$$-e_i + i_1 R_1 + e' = 0 \quad \text{——} \ (3)$$

$$K(0 - e') = e_o \quad \text{——} \ (4)$$

$$i_1 = i_2 + i_3$$

$$e_o = -K(e_i - i_1 R_1)$$

$$E_o(s) = -K\left(E_i(s) - I_1(s) R_1\right)$$

$$-E_i(s) + I_1(s) R_1 + \left(I_1(s) - I_2(s)\right) R_3 + E_o(s) = 0$$

$$-E_i(s) + I_1(s) R_1 + \frac{I_2(s)}{CS} = 0$$

$$I_2(s) = \left(-R_1 I_1(s) + E_i(s)\right) CS$$

$$-E_i(s) + I_1(s)R_1 + I_1(s)R_3 + R_1 CS I_1(s) R_3$$
$$- CS R_3 E_i(s) + E_o(s) = 0$$

$$= I_1(s)\left\{R_1 + R_3 + R_1R_3CS\right\} - E_i(s)\left\{1 + CSR_3\right\}$$
$$+ E_o(s) = 0$$

$$= E_o(s) = -KE_i(s) + KR_1\left\{\frac{E_i(s)\{1+CSR_3\} - E_o(s)}{R_1 + R_3 + R_1R_3CS}\right\}$$

$$\left(E_o(s) + \frac{KR_1}{R_1+R_3+R_1R_3C(s)} E_o(s)\right)$$

$$= -KE_i(s) + \frac{KR_1(1+CSR_3)}{R_1+R_3+R_1R_3CS}E_i(s)$$

$$E_o(s)\left\{R_1 + R_3 + R_1R_3CS + KR_1\right\}$$

$$= K E_i(s)\left\{-R_1 - R_3 - \cancel{R_1R_3}Cs + R_1 + \cancel{R_1R_3Cs}\right\}$$

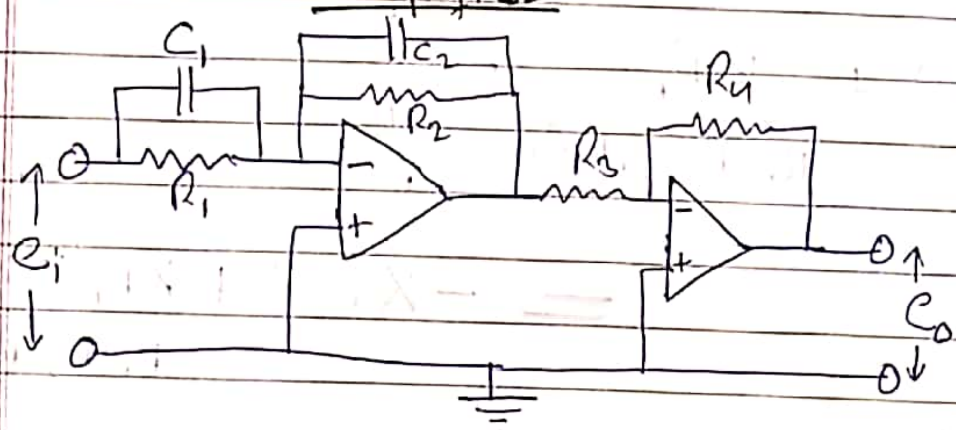
$$\boxed{\frac{E_o(s)}{E_i(s)} = K\frac{R_1 - \cancel{(R_1+R_3)}^{-R_3} + \cancel{(K-1)R_1R_3Cs}}{KR_1 + (R_1+R_3) + R_1R_3Cs}}$$

$$\frac{E_o(s)}{E_i(s)} = \frac{-KR_3}{K_1R_1 + (R_1+R_3) + R_1R_3Cs}$$

$$\Rightarrow \frac{-R_3}{R_1 + \frac{R_1+R_3}{K} + \frac{R_1R_3Cs}{K}}$$

$$\boxed{\frac{E_o(s)}{E_i(s)} = \frac{-R_3}{R_1}\left(\frac{1}{R_3Cs+1}\right)}$$

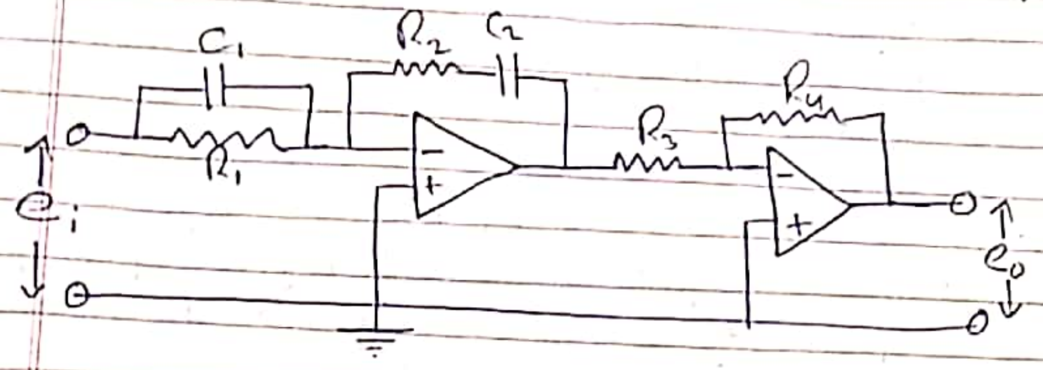
Lead and Lag Network using Operational Amplifiers



$$\boxed{\frac{E_o(s)}{E_i(s)} = \frac{R_4C_1}{R_3C_2}\frac{s+\frac{1}{R_1C_1}}{s+\frac{1}{R_2C_2}}}$$

If $R_1C_1 > R_2C_2 \Rightarrow$ Lead network.
$R_2C_2 > R_1C_1 \Rightarrow$ Lag network.

PID Controller Using Operational Amplifier



$$\boxed{\frac{E_o(s)}{E_i(s)} = \frac{R_4(R_1C_1+R_2C_2)}{R_3R_1C_1}\left[1+\frac{1}{(R_1C_1+R_2C_2)s}+\frac{R_1C_1R_2C_2}{R_1C_1+R_2C_2}s\right]}$$

When a PID Controller is expressed as :-

$$\boxed{\frac{E_o(s)}{E_i(s)} = K_p + \frac{K_i}{s} + K_d s}$$

$$\left\{\begin{aligned} K_p &= \frac{R_4(R_1C_1+R_2C_2)}{R_3R_1C_2} \\ K_i &= \frac{R_4}{R_3R_1C_2} \\ K_d &= \frac{R_4R_2C_1}{R_3} \end{aligned}\right.$$

CHAPTER 3

Transient and Steady State Response Analysis

3

Transient and Steady-State Responce Analysis

⇒ In analysing and designing control systems, we must have a basis of comparison of performance of various control systems.

⇒ This basis may be set up by specifying particular test input signals and by comparing the response ~~of such test signals~~ of various systems to these input signals.

★ Typical test Signals:

⇒ Common test signals are:-

(I) Step function

(II) Ramp function

(III) acceleration function

(IV) Impulse function

(V) Sinusoidal function

(VI) White noise.

⇒ Which of these test input signals to use for analysing system characteristics may be determined by the form of the input that the system will be subjected to most frequently under normal condition.

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★ Transient Responce and Steady-State Responce

\# Transient Responce ⇒ By transient responce, we mean that which goes from the initial state to the final state.

\# Steady-State Responce ⇒ By steady state responce, we mean the manner in which the System output behave as t approaches infinity.

★ Absolute Stability, Relative Stability and Steady-State Error

\# Absolute Stability ⇒ Whether the System is Stable or Unstable.

\# Equilibrium ⇒ A Control System is in equilibrium if, in the absence of any disturbance or input the Output stays in the Same State.

\# Stable ⇒ A System is stable if the output eventually Comes back to its equilibrium state when the System is Subjected to initial Condition.

\# Critically Stable ⇒ If oscillation of output Continue forever.

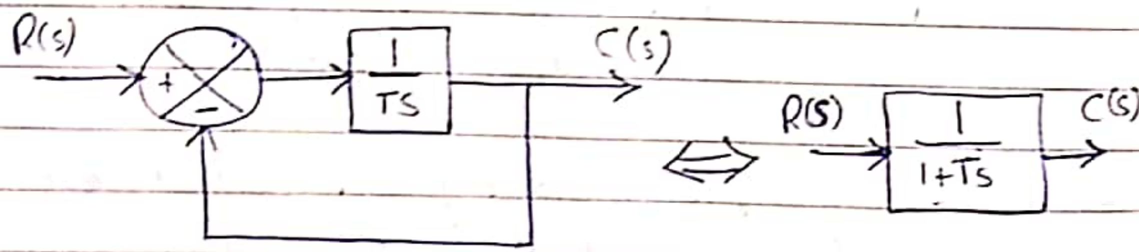
Unstable ⇒ The output diverges without bound from its equilibrium state when the System is Subjected to initial Condition.

Steady state Error ⇒ If output of a System at steady state does not exactly agree with the input, the System is said to have steady State error.

Relative Stability ⇒ It gives the degree of Stability or how Close it is to instability.

★ First-Order System {System whose closed loop Transfer function has one pole}

$$\boxed{\frac{C(s)}{R(s)} = \frac{1}{1+Ts}}$$



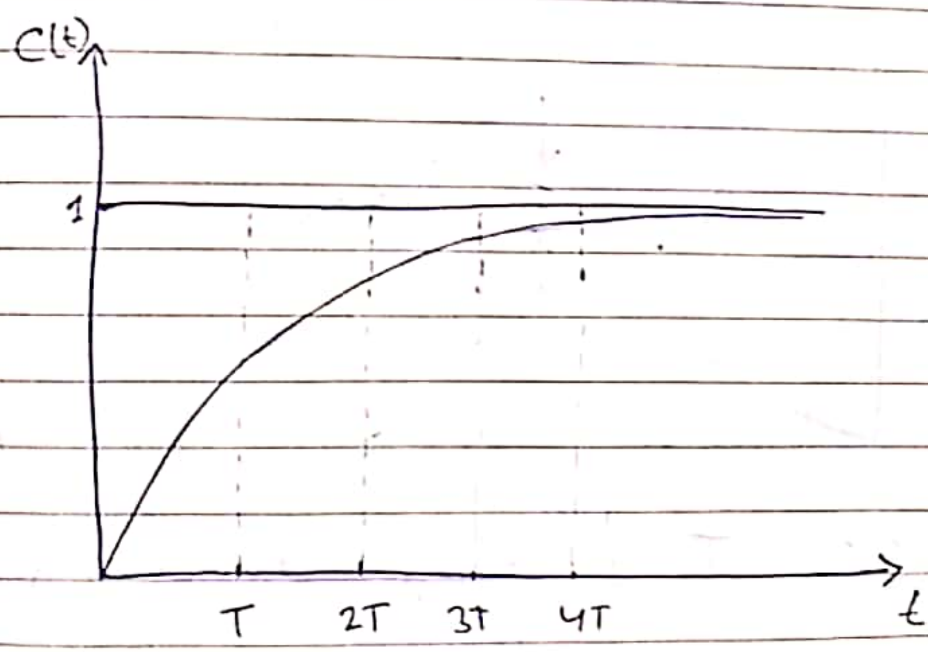
★ Unit-Step Responce of First Order System

$$\Rightarrow \quad r(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$

$$R(s) = \frac{1}{s}$$

$$C(s) = \frac{1}{s(Ts+1)} = \frac{1}{s} - \frac{1}{s+\frac{1}{T}}$$

$$\boxed{c(t) = 1 - e^{-\frac{t}{T}}} \quad \forall\, t \geq 0$$



★ Unit-Ramp Responce of First Order System

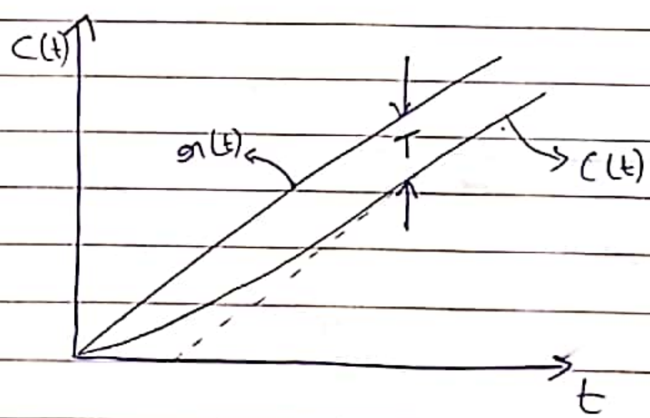
$$r(t) = \begin{cases} 0 & \forall t < 0 \\ t & \forall t > 0 \end{cases}$$

$$R(s) = \frac{1}{s^2}$$

$$C(s) = \frac{1}{s^2(Ts+1)} = \frac{1}{s^2} - \frac{T}{s} + \frac{T^2}{Ts+1}$$

$$\boxed{c(t) = t - T + Te^{-\frac{t}{T}}}$$

$$e(t) = r(t) - c(t) = T\left(1 - e^{-\frac{t}{T}}\right)$$



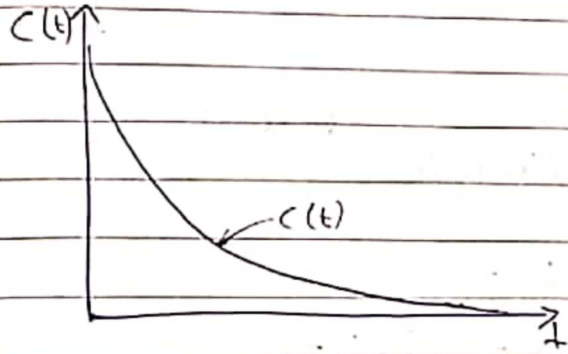
$e(\infty) = T$ {Steady State error}

★ Unit-Impulse Response of First Order System:

$r(t) = \delta(t)$ {dirac delta function}

$R(s) = 1$

$$C(s) = \frac{1}{Ts+1} \Rightarrow c(t) = \frac{1}{T} e^{-t/T} \quad \forall\ t \geq 0$$



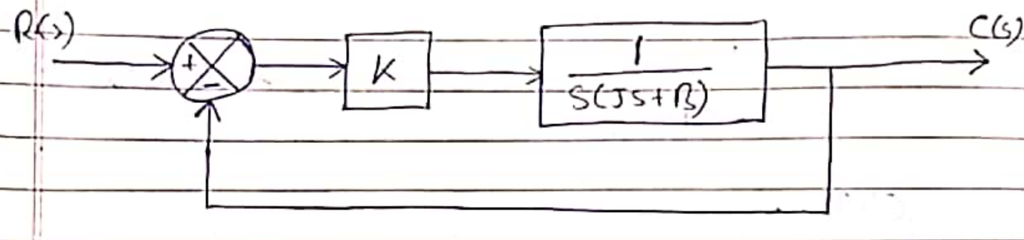
★ Summary of Response of First Order System

Test Input	C(t)
1. Unit Step Input	$C(t) = 1 - e^{-\frac{t}{T}}$
2. Unit Ramp Input	$C(t) = t - T + Te^{-t/T}$ $e(t) = T(1 - e^{-t/T})$
3. Unit-Impulse	$C(t) = \frac{1}{T} e^{-\frac{t}{T}}$

★ Second Order System

{System whose closed loop Transfer function have two Poles}

Servo System



Servo System with Proportional Controller

$$\frac{C(s)}{R(s)} = \frac{K/J}{s^2 + (B/J)s + (K/J)}$$

★ Step Responce of Second-Order System

$$\frac{C(s)}{R(s)} = \frac{K/J}{\left[S + \frac{B}{2J} + \sqrt{\left(\frac{B}{2J}\right)^2 - \frac{K}{J}}\right]\left[S + \frac{B}{2J} - \sqrt{\left(\frac{B}{2J}\right)^2 - \frac{K}{J}}\right]}$$

In the transient-responce analysis, it is Convenient to write.

$$\frac{K}{J} = \omega_n^2 \qquad \frac{B}{J} = 2\xi\omega_n = 2\sigma$$

Where, σ = attenuation
ω_n = Undamped natural frequency.
ξ = damping Ratio

So $$\boxed{\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}}$$

{Standard form of Second order System}

⇒ The dynamic behavior of the second order system can then be described in terms of two parameters ξ and ω_n.

<u>Case 1:</u> $0 \leq \xi < 1$ {Closed-loop poles are complex conjugates}

<u>Case 2:</u> $\xi = 1$ {Closed loop poles are Real and equal}

<u>Case 3:</u> $\xi > 1$ {Closed loop poles are Real and different}

Case 1: $0 \leq \xi < 1$ {Underdamped Case}

$$\boxed{\frac{C(s)}{R(s)} = \frac{\omega_n^2}{\left(s+\xi\omega_n+\omega_n\sqrt{\xi^2-1}\right)\left(s+\xi\omega_n-\omega_n\sqrt{\xi^2-1}\right)}}$$

Let $\omega_d = \omega_n\sqrt{1-\xi^2}$ {damped natural frequency}

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s+\xi\omega_n+j\omega_d)(s+\xi\omega_n-j\omega_d)}$$

for unit step input $R(s) = \frac{1}{s}$

$$C(s) = \frac{1}{s} - \frac{s+\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2} - \frac{\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2}$$

$$C(t) = 1 - e^{-\xi\omega_n t}\cos\omega_d t - \frac{\xi\omega_n}{\omega_d}e^{-\xi\omega_n t}\sin\omega_d t$$

$$\Rightarrow 1 - e^{-\xi\omega_n t}\left(\cos\omega_d t + \frac{\xi}{\sqrt{1-\xi^2}}\sin\omega_d t\right)$$

$$\Rightarrow 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin\left(\omega_d t + \tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right)\right)$$

$e(t) = r(t) - c(t)$

$$c(t) = \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin\left(\omega_d t + \tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right)\right)$$

→ damped Sinusoidal Oscillation

→ No Steady state error

⇒ If $\xi = 0$, the response becomes Undamped and oscillation continue indefinitely.

Case 2: {Critically damped Case}

$(\xi = 1)$

$$C(s) = \frac{\omega_n^2}{s(s+\omega_n)^2}$$

$$C(t) = 1 - e^{-\omega_n t}(1+\omega_n t) \quad \forall\ t \geq 0$$

Case 3: {Overdamped Case}

$\{\xi > 1\}$

$$C(s) = \frac{\omega_n^2}{s\left(s+\xi\omega_n+\omega_n\sqrt{\xi^2-1}\right)\left(s+\xi\omega_n-\omega_n\sqrt{\xi^2-1}\right)}$$

$$\Rightarrow C(t) = 1 + \frac{\omega_n}{2\sqrt{\xi^2-1}}\left(\frac{e^{-s_1 t_1}}{s_1} - \frac{e^{-s_2 t_2}}{s_2}\right)$$

$$\left\{ s_1 = \left(-\xi+\sqrt{\xi-1}\right)\omega_n \ \ \& \ \ s_2 = \left(\xi-\sqrt{\xi-1}\right)\omega_n \right\}$$

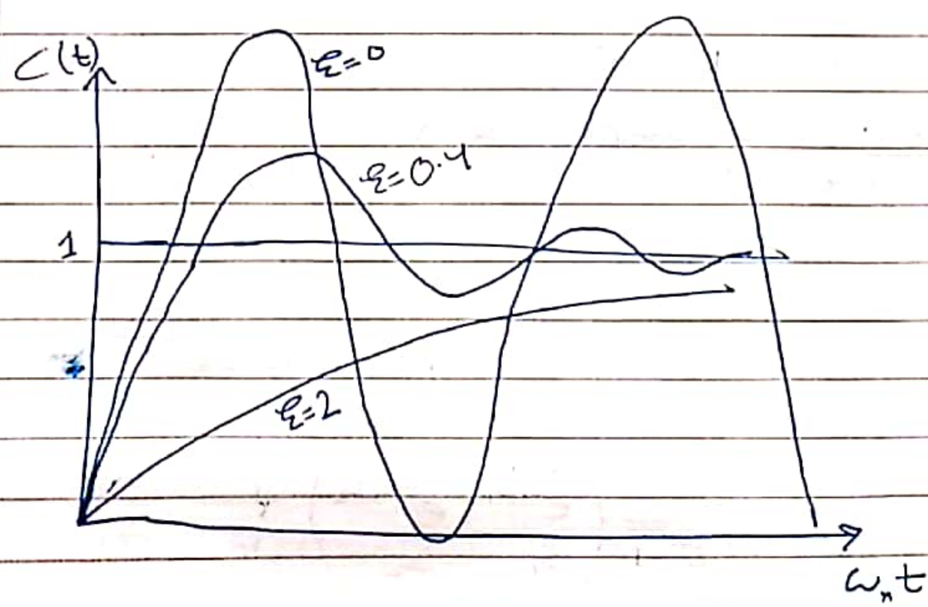
⇒ as $\xi > 1$, One of the two decaying exponentials decrease much faster than the other, so the faster-decaying exponentials term may be neglected.

⇒ So $\frac{C(s)}{R(s)}$ may be approximated as :-

$$\frac{C(s)}{R(s)} = \frac{(\xi - \sqrt{\xi^2-1})\omega_n}{s + (\xi - \sqrt{\xi^2-1})\omega_n} = \frac{s_2}{s+s_2}$$

⇒ The response is similar to that of first order system.

$$C(t) = 1 - e^{-s_2 t} \quad \forall\ t \geq 0$$



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Note

* Two 2^{nd} order system having same ξ but different ω_n will exhibit the same overshoot and the same oscillatory pattern. Such system is said to have same relative stability.

* Definitions of Transient Response Specification

It is common practice to use the standard initial condition that the system is at rest initially with the output and all time derivatives thereof zero.

1. Delay time (t_d): Time required for the response to reach half the final value the very first time.

2. Rise time (t_r): time required for the response to rise from (10% to 90%), (5% to 95%) or (0% to 100%) of its final value.

3. Peak time (t_p): Time required for the response to reach the first peak of the overshoot.

4. Maximum (%) overshoot (M_p): $\dfrac{C(t_p) - C(\infty)}{C(\infty)} \times 100$

5. Settling time: Time required for the response curve to reach and stay within a range about the final value. (Usually 2% to 5%)

★ Second Order System and Transient-Response Specification

1. Delay time (t_d)

$$\boxed{e^{-\xi\omega_n t_d}\sin\left(\omega_d t_d + \tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right)\right) = \frac{1}{2}\sqrt{1-\xi^2}}$$

2. Rise time (t_r)

$$\boxed{t_r = \frac{1}{\omega_d}\tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right)}$$

3. Peak time (t_p)

$$\sin \omega_d t_p = 0$$

or

$$\boxed{t_p = \frac{\pi}{\omega_d}}$$

4) Maximum overshoot (M_p)

$$\boxed{M_p = e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}}}$$

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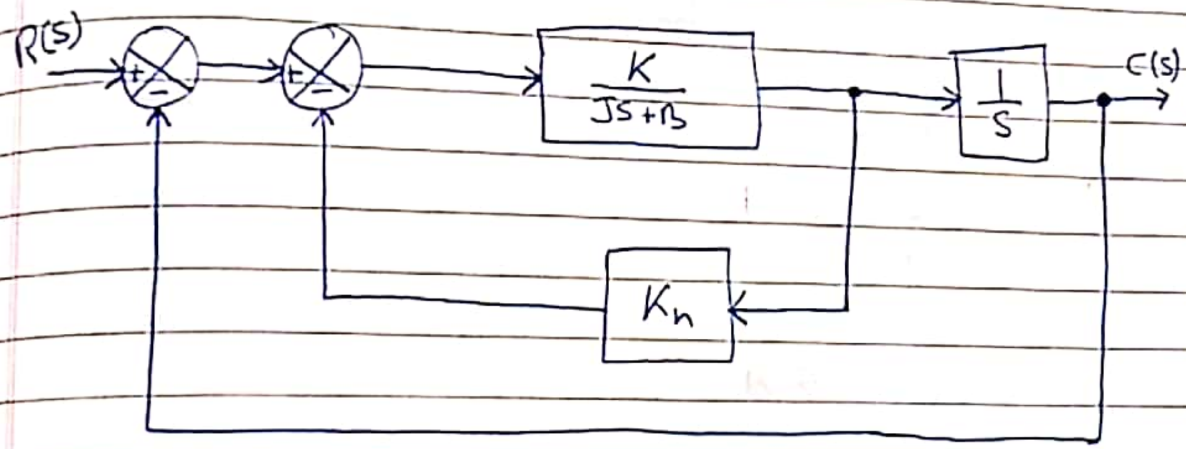
5) Setting time

$$\boxed{t_s = \frac{4}{\xi \omega_n}} \quad \{2\% \text{ Criterion}\}$$

★ Servo System with Velocity Feedback



$$\frac{C(s)}{R(s)} = \frac{K}{J(s^2) + (B+KK_n)s + K}$$

$$\xi = \frac{B + KK_n}{2\sqrt{KJ}}$$

⇒ Velocity feedback has the effect of increasing damping.

→ We can adjust K_n so that ξ is between 0.4 to 0.7.

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★ Impulse Response of Second-Order System

$$C(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

Case 1: $0 \le \xi < 1$

$$\boxed{C(t) = \frac{\omega_n}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t} \sin\sqrt{1-\xi^2}\, t}$$

Case 2: $\xi = 1$

$$\boxed{C(t) = \omega_n^2 t e^{-\omega_n t}}$$

Case 3: $\xi > 1$

$$C(t) = \frac{\omega_n}{2\sqrt{\xi^2-1}} e^{-C_1 t} - \frac{\omega_n}{2\sqrt{\xi^2-1}} e^{-C_2 t}$$

$$\Rightarrow \boxed{C(t) = \frac{\omega_n}{2\sqrt{\xi^2-1}} \left(e^{-C_1 t} - e^{-C_2 t} \right)}$$

$$\left\{ \begin{aligned} C_1 &= \left(-\xi - \sqrt{\xi^2-1}\right)\omega_n \\ C_2 &= \left(-\xi + \sqrt{\xi^2-1}\right)\omega_n \end{aligned} \right\}$$

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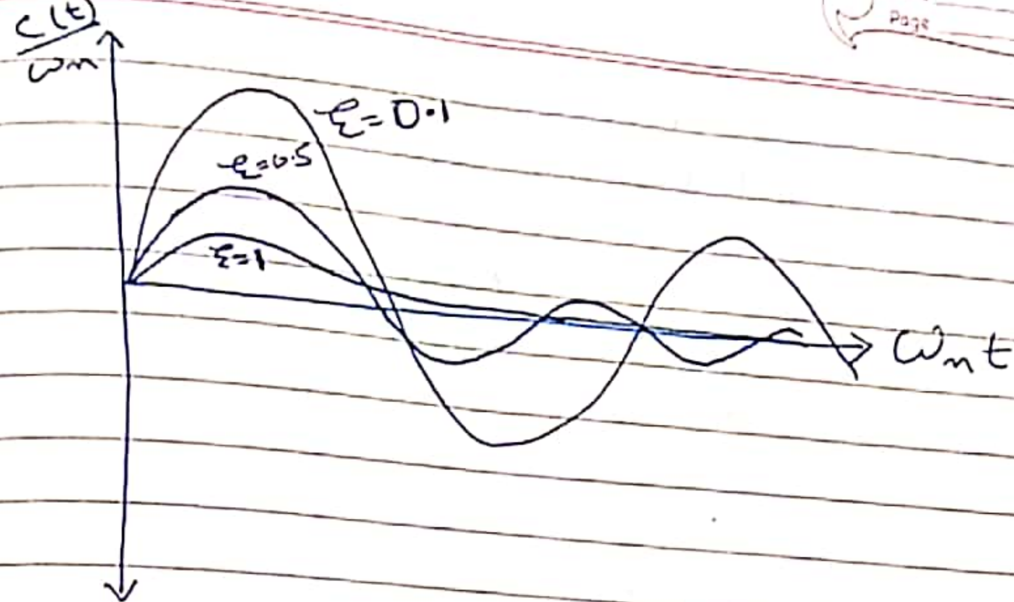
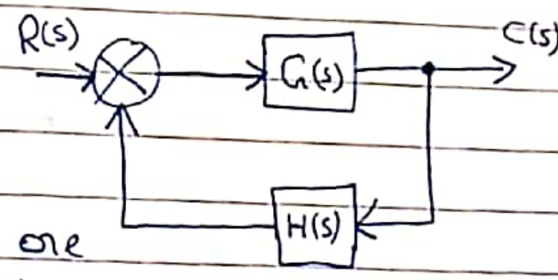


★ Higher Order Systems

Response of a higher-order system is the sum of the responses of first-order system & second-order systems.

★ Transient Response of Higher Order System

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$



In general $G(s)$ and $H(s)$ are given as ratios of polynomials in s

$$G(s) = \frac{P(s)}{q(s)} \quad \& \quad H(s) = \frac{n(s)}{d(s)}$$

$$\Rightarrow \frac{C(s)}{R(s)} = \frac{P(s)\,d(s)}{q(s)d(s) + P(s)n(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_n}$$

⇒ The transient response of this system to any given input can be obtained by a Computer Simulation.

⇒ If an analytical expression of the ~~transfer~~ transient response is desired, then it is necessary to factor the denominator polynomial.

$$\frac{C(s)}{R(s)} = \frac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$

⇒ Let us examine the response behavior of this system to a unit step input.

<u>Case 1:</u> Closed loop poles are real & distinct

$$C(s) = \frac{a}{s} + \sum_{i=1}^{n} \frac{a_i}{s+p_i}$$

{where a_i is the residue of the pole at $s=-p_i$}

$$\boxed{C(t) = a + \sum_{i=1}^{n} a_i e^{-p_i t}}$$

⇒ If all closed-loop poles lies in the left half s plane, then $C(\infty) = a$.

Case 2: Closed loop poles of $C(s)$ consist of real poles and pairs of complex-conjugate poles.

$$C(s) = \frac{a}{s} + \sum_{j=1}^{q} \frac{a_j}{s+p_j} + \sum_{k=1}^{r} \frac{b_k(s+\xi_k\omega_k) + c_k\omega_k\sqrt{1-\xi_k^2}}{s^2+2\xi_k\omega_k s+\omega_k^2}$$

$$\boxed{C(t) = a + \sum_{j=1}^{q} a_j e^{-p_j t} + \sum_{k=1}^{r} b_k e^{-\xi_k\omega_k t}\cos\omega_k\sqrt{1-\xi_k^2}\,t + \sum_{k=1}^{r} c_k e^{-\xi_k\omega_k t}\sin\omega_k\sqrt{1-\xi_k^2}\,t}$$

⇒ Thus the response curve of a stable higher order system is the sum of a number of exponential curves and damped sinusoidal curves.

⇒ If all closed-loop poles lie in the left half s-plane, then the exponential terms and the damped exponential term will approach zero as time t increases.

↳ The steady state output is then $C(\infty) = a$.

* Dominant Closed loop poles

⇒ The relative dominance of closed-loop poles is determined by the ratio of the real part of the closed-loop poles, as well as by the relative magnitudes of the residues evaluated at the closed loop poles.

→ {depends on both closed loop Poles & Zeros}

⇒ If the ratio of the real parts of the closed-loop poles exceed 5 and there are no zeros nearby, then the closed-loop pole nearest the $j\omega$ axis will dominate in the transient response behavior.

↳ Those closed loop poles that have dominant effects on the transient response behavior are called <u>dominant closed-loop poles</u>.

* Stability Analysis in Complex plane

⇒ The Stability of a linear closed-loop System can be determined from the location of the closed loop poles in the s plane.

<u>Closed loop poles on left half of s-plane</u> ⇒ Stable

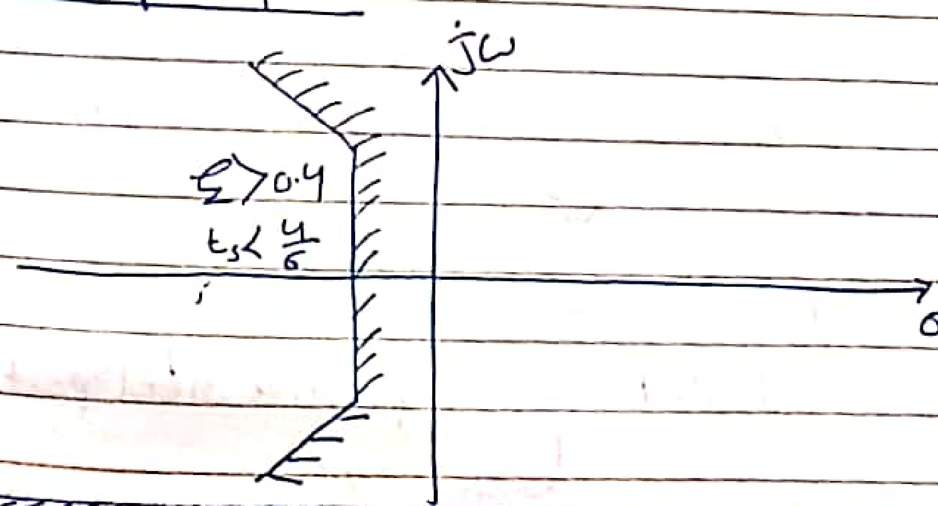
Closed loop poles on Right half of S-plane ⇒ Unstable

⇒ Whether a linear System is stable or Unstable is a property of the System itself and does not depend on the input or driving function of the System

↳ The poles of the input or driving function, do not affect the property of stability of the system, but they contribute only to steady-state response terms in the solution.

⇒ If dominant complex-Conjugate closed loop poles lies close to the jω axis, the transient response may exhibit excessive oscillations or may be very slow

↳ To guarantee fast yet well-damped, transient response characteristics, it is necessary that the closed-loop poles of the System lie in a particular region on the complex plane.



CHAPTER 4

Designing of a Control System

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Designing of a Control System

Root-Locus Method

Frequency Responce method

* Routh's stability Criterion

⇒ Routh's stability Criterion tells us whether or not there are unstable roots in a polynomial equation without actually solving them.

Procedure

1. Write the polynomial in s in the following form:

$$a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n = 0$$

{ Cofficients are real
  a_n & $a_0 \neq 0$ }

2. If any of the cofficient are zero or negative in the presence of atleast one positive cofficient

⇓

{ a root or root's exist that are imaginary and that have a positive real part }

→ { Therefor system is not stable }

"A polynomial in s having real coefficients can always be factored into linear and quadratic factor, such as $(s+a)$ and (s^2+bs+c) where a, b, c are real"

Proof: ⓐ for $(s+a)$ to yield negative roots a must be positive.

ⓑ The factor (s^2+bs+c) to yield roots having negative real parts only if b and c are both positive.

⇒ For all roots to have negative real part, the constant a, b, c and soon in all factor must be positive.

⇒ The product of any number of linear and quadratic factors containing only positive coefficients always yields a polynomial with positive coefficient.

⇒ It is important to note that the condition that all the coefficients be positive is not sufficient to assume stability. (It is necessary but not necessary).

Ex: ~~$(s+8)(s-3)(s-2)$~~ $(s^2-2s+10)(s+3)$

~~$\Rightarrow (s+6)(s^2-5s+6)$~~ $\Rightarrow s^3+(3-2)s^2+(10-6)s+30$

~~$\Rightarrow s^3$~~ $\Rightarrow s^3+s^2+4s+30$

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3. If all coefficients are positive, arrange the coefficients of the polynomial in rows and columns according to the following pattern.

$$
\begin{array}{lllll}
s^n & a_0 & a_2 & a_4 & a_6 \cdots \\
s^{n-1} & a_1 & a_3 & a_5 & a_7 \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & b_4 \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
s^2 & e_1 & e_2 & & \\
s^1 & f_1 & & & \\
s^0 & g_1 & & &
\end{array}
$$

⇒ The process of forming rows continues until we run out of element.

⇒ The coefficients $b_1, b_2 \cdots b_3$ and so on are evaluated as follows :-

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1}$$

$$b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1}$$

$$b_3 = \frac{a_1 a_6 - a_0 a_7}{a_1}$$

⇒ Routh's stability criterion states that the number of roots of equation above with positive real parts is equal to number of change in sign of the coefficients of the first column of the array.

Example

$$s^4 + 2s^3 + 3s^2 + 4s + 5 = 0$$

s^4	1	3	5
s^3	2	4	0
s^2	1	5	
s^1	-6	0	
s^0	5		

⇒ There are two roots with positive real part.

★ Special Case

⇒ If a first-column term in any row is zero, but the remaining terms are not zero or there is no remaining term, then the zero term is replaced by a very small positive number ϵ.

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<u>Example</u>

$$s^3 + 2s^2 + s + 2 = 0$$

s^3	1	1
s^2	2	2
s^1	ε	
s^0	2	

⇒ If the sign of the coefficient above the zero (ε) is same as that below it, it indicates that there is a pair of imaginary roots at $s = \pm j$.

⇒ If, however, the sign of the coefficients above the zero (ε) is opposite that below it, it indicates that there is one sign change.

If all the coefficients in any desired row are zero, it indicates that there are roots of equal magnitude <u>lying radially opposite in s-plane.</u>

↓

{ Two real roots with equal magnitude and opposite sign

and/or

Two conjugate imaginary roots }

Example

$$s^5 + 2s^4 + 24s^3 + 48s^2 - 25s - 50 = 0$$

s^5	1	24	-25
s^4	2	48	-50
s^3	0	0	

← Auxiliary polynomial. P(s)

$$P(s) = 2s^4 + 48s^2 - 50$$
$$P'(s) = 8s^3 + 96s$$

Indicates two pair of roots with real and opp sign

⇒ s^5	1	24	-25
s^4	2	48	-50
s^3	8	96	
s^2	24	-50	
s^1	112.7	0	
s^0	-50		

⇒ Clearly, the original equation have one root with a positive real part.

★ Relative stability analysis

⇒ Routh's stability criterion provides the answer to the question of absolute stability. This in many practical cases, is not sufficient.

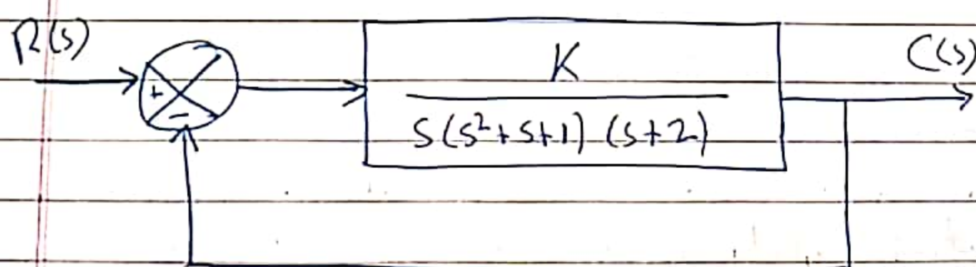
⇒ A useful approach for examining relative stability is to shift the s-plane axis and apply Routh's Stability Criterion.

$$S = \hat{S} - \sigma$$

⇒ Write polynomial in terms of $\hat{s}$.

⇒ One applying Routh's Stability Criterion. The number of change of sign indicates number of roots that are located to the right of the vertical line $s = -\sigma$.

★ Application of Routh's Stability Criterion to Control-System Analysis



$$\frac{C(s)}{R(s)} = \frac{K}{s(s^2+s+1)(s+2)+K}$$

⇒ The Characteristic equation is

$$s^4 + 3s^3 + 3s^2 + 2s + K = 0$$

s^4	1	3	K
s^3	3	2	0
s^2	7/3	K	
s^1	$2 - \frac{9}{7}K$		
s^0	K		

For stability $K > 0$ & $2 - \frac{9}{7}K > 0$

$$\Rightarrow \boxed{0 < K < \frac{14}{9}} \quad \{\text{for stability}\}$$

★ Effects of Integral and Derivative Control action on System peformance

★ Integral Control action

⇒ In proportional Control of a plant whose TF does not possess an integrator (1/s), there is a steady-state error, or offset, in the response to a step input.

↳ Such an offset can be eliminated if the integral control action is included in the controller.

⇒ Integral control action, while removing offset or steady-state error, may lead to oscillatory response of slowly decreasing amplitude or even increasing amplitude, both of which are usually undesirable.

★ **Proportional Control Systems**

⇒ Proportional control of system without an integrator will result in a steady-state error with a step input.

★ **Derivative Control Action**

⇒ Derivative control when added to a proportional controller, provides a means of obtaining a controller with high sensitivity.

⇒ Derivative control thus anticipates the actuating error, initiates an early corrective action and tends to increase the stability of the system.

CHAPTER 5

Designing of a Control System Using Frequency Response Method

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Designing of Control System Using Frequency Responce Method

★ Introduction

⇒ By the term Frequency response we mean the steady-state response of a system to a sinusoidal input.

↳ In frequency-response methods, we vary the frequency of the input signal over a certain range and study the resulting response.

⇒ The information we got from such analysis is different from what we get from root-locus analysis.

↳ ~~The information we get from such analysis is different from what we get from root locus analysis.~~ In Fact, the frequency response and root-locus approach complement each other.

⇒ In many practical designs of control systems both approaches are employed.

⇒ Frequency responce method was developed in 1930s and 1940s by Nyquist, Bode, Nichols and many others.

⇒ The Frequency-response methods are most powerful in Classical Control theory.

↳ They are indispensable to robust Control theory.

⇒ The Nyquist stability Criterion enables us to investigate both the absolute and relative stability of linear Closed loop Systems from a Knowledge of their open loop frequency response Characteristics.

⇒ An advantage of frequency response-approach is that frequency-response test are, in general Simple and can be made accurately by use of readily available Sinusoidal Signal generators and precise measurement equipment.

↳ TF of a Complicated Components can be determined experimentally by frequency-response tests.

⇒ In addition, the frequency-response approach has the advantages that a system may be designed so that the effects of undesired noise are neglected and that such analysis and design can be extended to certain nonlinear control system.

⇒ In designing a closed loop system, we adjust the frequency-response characteristics of the open loop ~~transient~~ transfer function by using several design criteria in order to obtain acceptable transient-response characteristics for the system.

★ Obtaining Steady State Output to Sinusoidal Input

⇒ Consider the stable, linear, time-invariant system as shown.

$$\xrightarrow[X(s)]{x(t)} \boxed{G(s)} \xrightarrow[Y(s)]{y(t)}$$

⇒ If input $x(t)$ is a sinusoidal signal, the steady-state output will also be a sinusoidal signal of the same frequency, but with possibly different magnitude and phase angle.

Let $x(t) = X \sin \omega t$

Let $G(s) = \dfrac{P(s)}{q(s)} = \dfrac{P(s)}{(s+s_1)(s+s_2)(s+s_3)\cdots(s+s_n)}$

So $Y(s) = G(s)X(s) = \dfrac{P(s)}{q(s)} X(s)$

⇒ If $Y(s)$ has only distinct poles then ÷

$$Y(s) = G(s)\frac{\omega X}{s^2+\omega^2}$$

$$= \frac{a}{s+j\omega} + \frac{\bar{a}}{s-j\omega} + \frac{b_1}{s+s_1} + \frac{b_2}{s+s_2} + \cdots \frac{b_n}{s+s_n}$$

Where a and b_i $(\forall i = 1, 2 \cdots n)$ are constant and $\bar{a}$ is the complex conjugate of a.

So $y(t) = a e^{j\omega t} + \bar{a} e^{j\omega t} + b_1 e^{-s_1 t} + b_2 e^{-s_2 t} + \cdots + b_n e^{-s_n t}$

$\{\forall\ t > 0\}$

⇒ For a stable system $-s_1, -s_2 \cdots -s_n$ have negative real parts. Therefore as $t \to \infty$ the term $e^{-s_1 t}$, $e^{-s_2 t}$ --- and $e^{-s_n t}$ approaches zero.

⇒ If $Y(s)$ involves multiple poles s_j of multiplicity m_j, then $y(t)$ will involve terms such as $t^{h_j} e^{-s_j t}$ $(h_j = 0, 1, 2 \cdots m_j - 1)$. For stable system, the terms $t^{h_j} e^{-s_j t}$ approaches zero as $t \to \infty$.

⇒ Thus regardless of whether the system is of the distinct-pole type or multiple-pole type, the steady state response becomes:

$$y_{ss}(t) = a e^{-j\omega t} + \bar{a} e^{j\omega t}$$

$$Y(s)\,(s+j\omega) = (s+j\omega)\left\{\frac{a}{s+j\omega} + \frac{\bar{a}}{s-j\omega} + \frac{b_1}{s+s_1} + \cdots + \frac{b_m}{s+s_m}\right\}$$

$$\Rightarrow a + \bar{a}\frac{(s+j\omega)}{s-j\omega} + b_1\left(\frac{s+j\omega}{s+s_1}\right) + \cdots b_m\left(\frac{s+j\omega}{s+s_m}\right)$$

$$Y(s)\,(s+j\omega)\Big|_{s=-j\omega} = a + 0 + 0 + \cdots + 0$$

$$\text{So } a = \frac{G(s)\,\omega X}{s^2+\omega^2}(s+j\omega)\Big|_{s=-j\omega} = -\frac{X\,G(-j\omega)}{2j}$$

$$\text{Similarly } \bar{a} = G(s)\frac{\omega X}{s^2+\omega^2}(s-j\omega)\Big|_{s=j\omega} = \frac{X\,G(j\omega)}{2j}$$

⇒ Since $G(j\omega)$ is complex quantity,

$$G(j\omega) = |G(j\omega)|\,e^{j\phi}$$

Magnitude

Angle

$$\phi = \angle G(j\omega) = \tan^{-1}\left[\frac{\text{Imaginary part of } G(j\omega)}{\text{Real part of } G(j\omega)}\right]$$

$$G(-j\omega) = |G(-j\omega)|\,e^{-j\phi} = |G(j\omega)|\,e^{-j\phi}$$

$$a = -\frac{X|G(j\omega)|e^{-j\phi}}{2j}$$

$$\bar{a} = +\frac{\cancel{X}|G(j\omega)|e^{j\phi}}{2j}$$

$$y_{ss}(t) = X|G(j\omega)|\,\frac{e^{j(\omega t+\phi)} - e^{-j(\omega t+\phi)}}{2j}$$

$$y_{ss}(t) = X|G(j\omega)|\sin(\omega t+\phi)$$

$$\boxed{y_{ss}(t) = Y\sin(\omega t+\phi)}$$

$$\left\{\begin{aligned} Y &= X|G(j\omega)| \\ \phi &= \angle G(j\omega) \end{aligned}\right\}$$

⇒ For stable, linear, time-invariant system subjected to a sinusoidal input will, at steady state, have a sinusoidal output of the same frequency as input.

⇒ Positive phase angle is called phase lead and negative phase angle is called phase lag.

→ A network of phase lead characteristics is called lead network

→ While a network of phase lag characteristics is called lag network.

★ Presenting Frequency Responce Characteristics in Graphical forms

⇒ $G(j\omega)$ is called Sinusoidal Transfer function.

→ It is characterized by its magnitude and phase angle with frequency as the parameter.

⇒ There are three Commonly used representation of Sinusoidal transfer functions:

1. Bode plot or Logarithmic polt

2. Nyquist plot or polar plot

3. Nichols plot or Log magnitude vs phase plot

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★ Bode Diagrams

⇒ A Bode diagram consists of two graphs:-

(i) Plot of log of magnitude of a Sinusoidal transfer function Vs frequency on a log Scale

(ii) Plot of phase angle Vs frequency on a logarithmic scale.

⇒ Standard representation of the logarithmic magnitude of $G(j\omega)$ is $20\log_{10}|G(j\omega)|$.

↳ The unit used in this representation of magnitude is the decibel. Usually abbreviated dB.

⇒ In the logarithmic representation, the curves are drawn on semilog paper using the log scale for frequency and linear scale for either magnitude (but in decibles) or phase angle (in degree).

⇒ The main advantage of using Bode diagram is that multiplication of magnitudes can be converted into addition.

⇒ Note: The experimental determination of a TF can be made simple if frequency-response data are presented in the form of a Bode diagram.

★ Basic Factors of $G(j\omega)H(j\omega)$

1. Gain K
2. Integral and derivative factor
3. First-Order factor
3. Quadratic factor

⇒ Once we become familiar with the logarithmic plots of these basic factors, it is possible to utilize them in constructing a composite logarithmic plot of any general form of $G(j\omega)H(j\omega)$.

1. Gain K

⇒ A number greater than unity has a positive value in decibels, while a number smaller than unity has a negative value.

⇒ The phase angle of gain K is zero.

"The effect of varying the gain K in the TF is that it raises or lowers the log-magnitude curve of the TF by the corresponding constant amount, but it has no effect on the phase curve"

2. Integral and Derivative Factors $(j\omega)^{\mp 1}$

$\frac{1}{j\omega}$ → Magnitude $\frac{1}{\omega}$

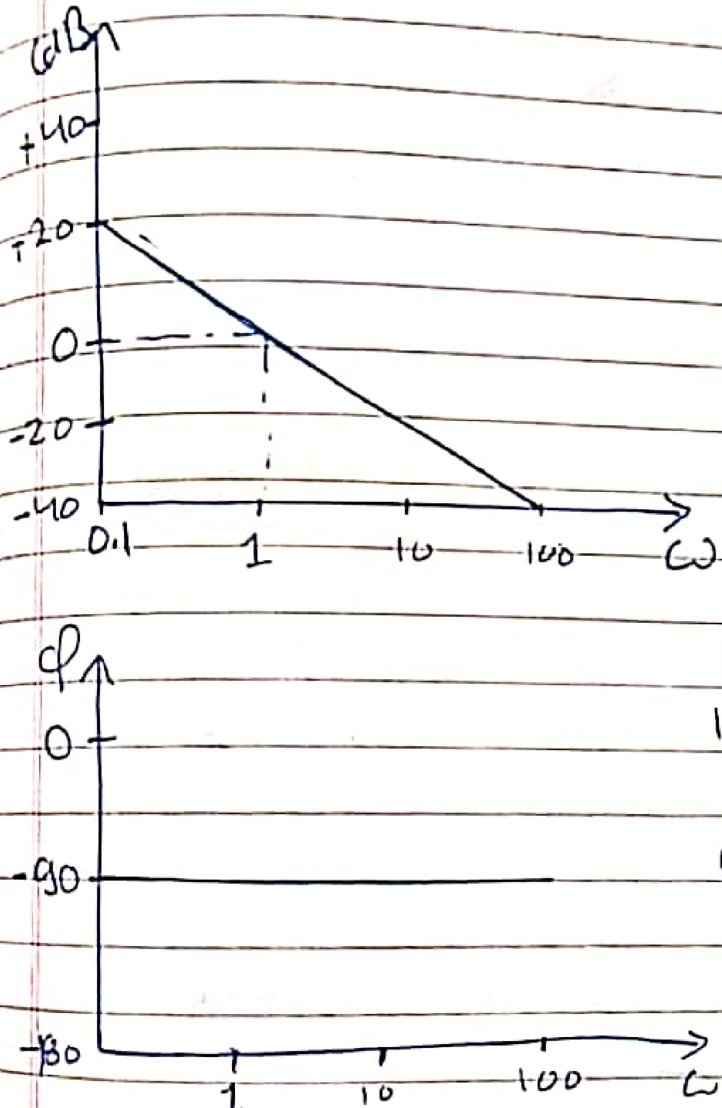
$\frac{1}{j\omega}$ → Angle $-\pi/2$

$$\left(20 \log \frac{1}{\omega}\right) = -20 \log \omega \text{ dB}$$

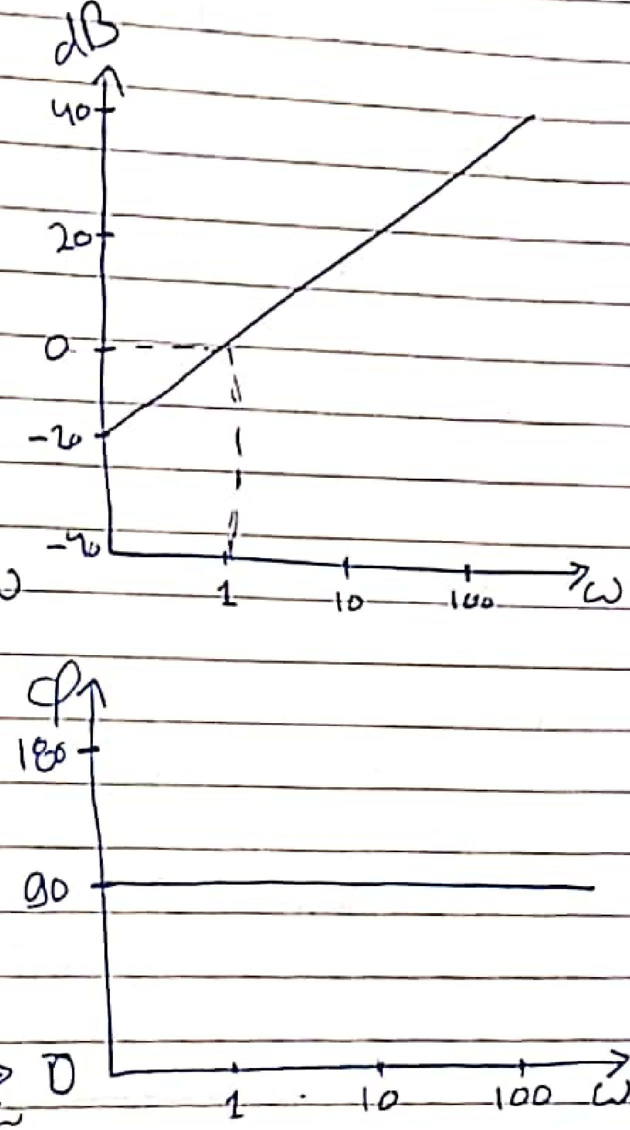
$j\omega \rightarrow$ Magnitude $\quad \omega$

$\quad\ \ \rightarrow$ Angle $\quad \pi/2$

$|20 \log \omega|$



Bode plot of $G(j\omega) = \frac{1}{j\omega}$



Bode plot of $G(j\omega) = j\omega$

3. First-Order Factors

$$\left(\frac{1}{1+j\omega T}\right) \Rightarrow 20\log\left|\frac{1}{1+j\omega T}\right| = -20\log\sqrt{1+\omega^2T^2}$$

// For low frequencies $\omega \ll 1/T$

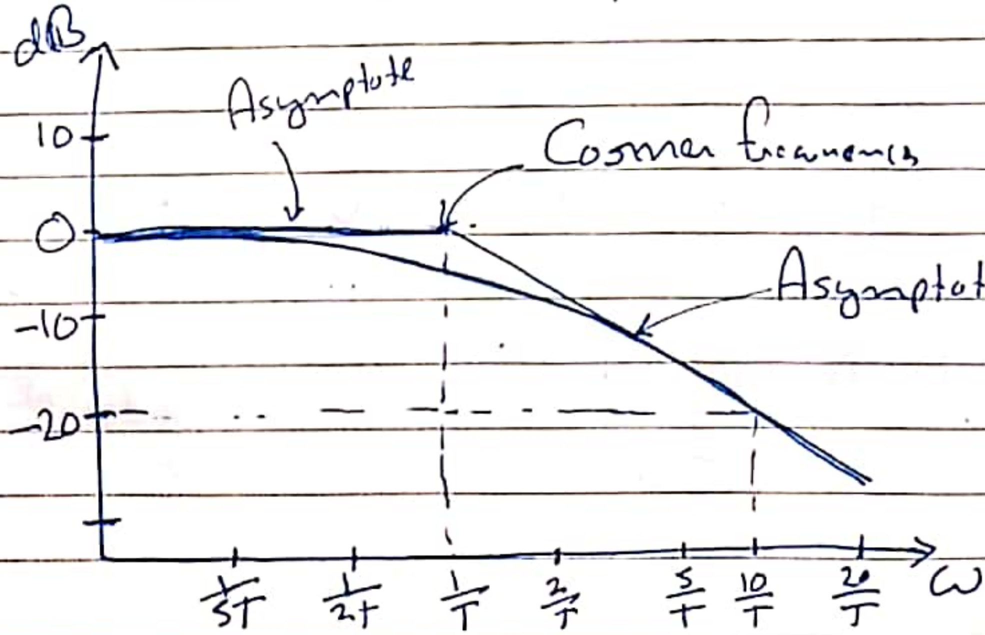
$$\Rightarrow -20\log\sqrt{1+\omega T^2} = -20\log 1 = 0\,dB$$

// For high frequencies $\omega \gg 1/T$

$$-20\log\sqrt{1+\omega^2T^2} = -20\log\omega T$$

⇒ At $\omega = 1/T$ log magnitude is equal to zero at $\omega = 10/T$ the log magnitude is $-20\,dB$.

⇒ Thus the value of $-20\log\omega T$ dB decrease by 20dB for every decade of ω.



⇒ The corner frequency divides the frequency-response curve into two regions :-

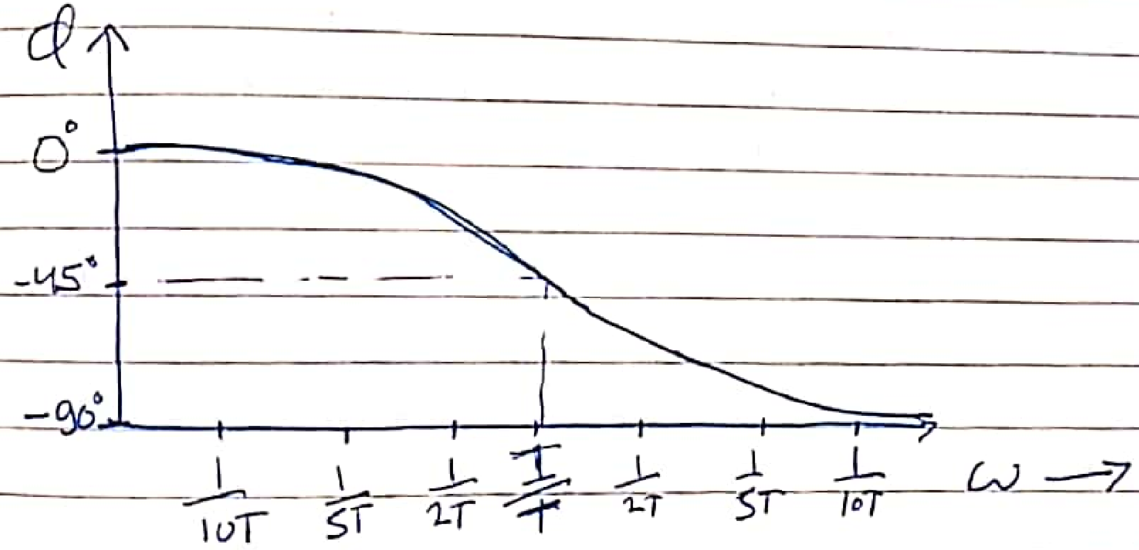
- ⇒ Curve for low frequency region
- ⇒ Curve for high frequency region.

$$\phi = -\tan^{-1}(\omega T)$$

\# $\omega = 0$ $\quad \phi = 0$

\# $\omega = 1/T$ $\quad \phi = -\pi/4$

\# $\omega = \infty$ $\quad \phi = -\pi/2$



⇒ The transfer function $1/(1+j\omega T)$ has the characteristic of a low-pass filter.

⇒ Bode plot of $(1+j\omega T)^{n}$ can be easily found out.

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4> Quadratic Factor

$$G(j\omega) = \frac{1}{1 + 2\xi\left(j\frac{\omega}{\omega_n}\right) + \left(j\frac{\omega}{\omega_n}\right)^2}$$

⇒ If $\xi > 1$, this quadratic factor can be expressed as a product of two first-order factors; with real poles.

⇒ If $0 < \xi < 1$, this quadratic factor is the product of two complex-conjugate factors.

⇒ Asymptotic approximations to the frequency-response curves are not accurate for a factor with low values of ξ.

↳ This is because the magnitude and phase of quadratic factor depends ~~on~~ on both the corner frequency and the damping ratio ξ.

$$\Rightarrow 20\log\left|\frac{1}{1 + 2\xi\left(j\frac{\omega}{\omega_n}\right) + \left(j\frac{\omega}{\omega_n}\right)^2}\right|$$

$$\Rightarrow -20\log\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+\left(2\xi\frac{\omega}{\omega_n}\right)^2}$$

$\forall$ low frequency $\omega \ll \omega_n$

$$-20\log 1 = 0 \quad \Rightarrow \text{Horizontal line at } 0\,dB$$

$\forall$ high frequency $\omega \gg \omega_n$

$$-20\log\frac{\omega^2}{\omega_n^2} = -40\log\frac{\omega}{\omega_n}$$

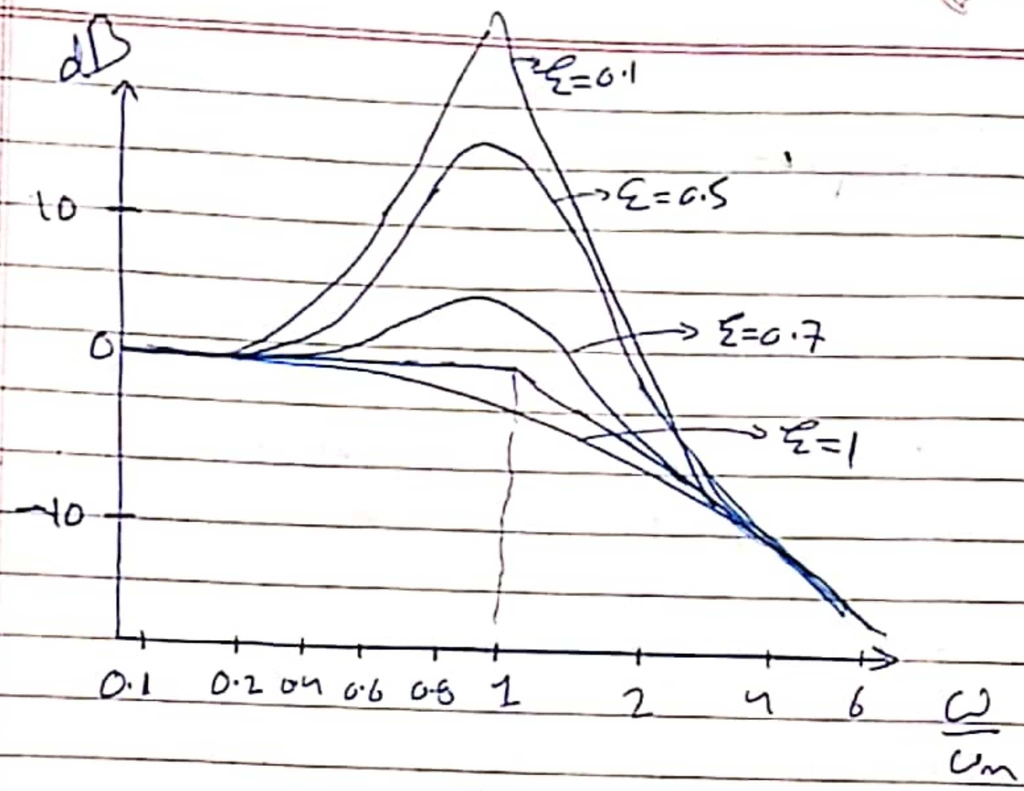
$\Rightarrow$ Straight line with slope $-40\,dB$

$\Rightarrow$ The high-frequency asymptote intersect the low frequency one at $\omega = \omega_n$.

ω_n = Corner frequency.

$\Rightarrow$ Near the frequency $\omega = \omega_n$, a resonant peak occurs, ~~as may b~~ and damping ratio ξ determines the magnitude of this resonant peak.

$\hookrightarrow$ Resonance peak is large for small value of ξ.



$$\phi = \angle \frac{1}{1 + 2\xi\left(j\frac{\omega}{\omega_n}\right) + \left(j\frac{\omega}{\omega_n}\right)^2}$$

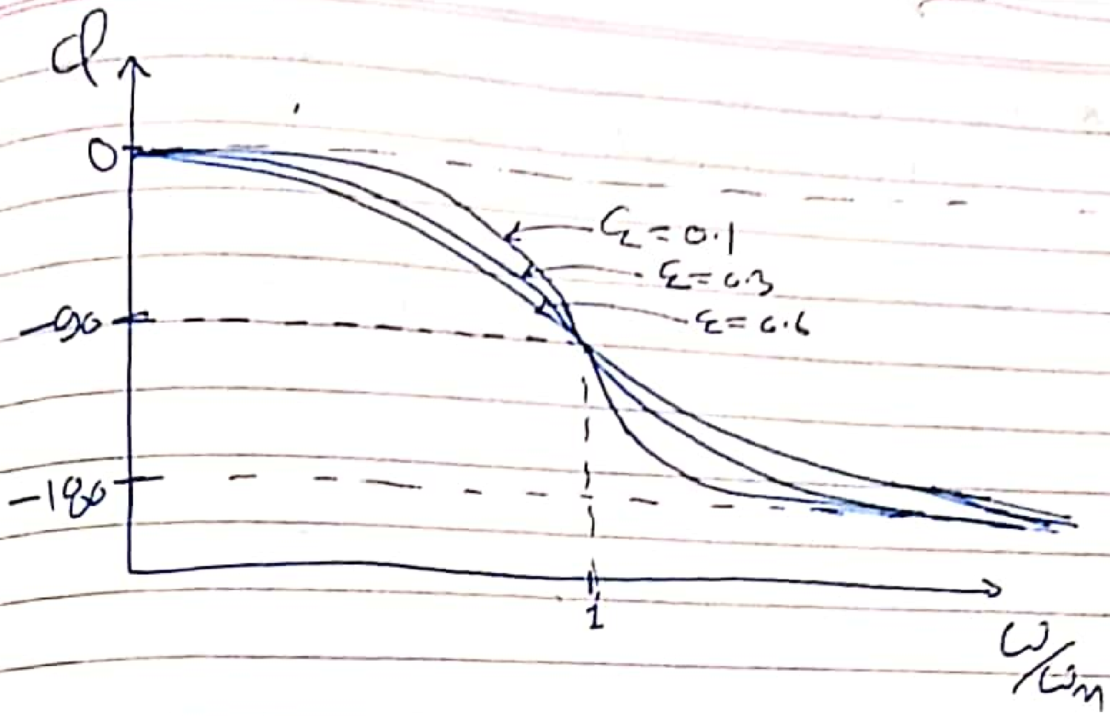
$$= -\tan^{-1}\left[\frac{2\xi\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right]$$

⇒ The phase angle is a function of both ω & ξ.

$\omega = 0 \quad \phi = 0$

$\omega = \omega_n \quad \phi = -\pi/2$

$\omega = \infty \quad \phi = -\pi$



⇒ The frequency-response curves for the factor:

$$1+2\xi\left(j\frac{\omega}{\omega_n}\right)+\left(j\frac{\omega}{\omega_n}\right)^2$$

Can easily be obtained by merely reversing the sign of the log magnitude and that of the phase angle of the factor.

$$\frac{1}{1+2\xi\left(j\frac{\omega}{\omega_n}\right)+\left(j\frac{\omega}{\omega_n}\right)^2}$$

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★ <u>The resonance frequency (ω_r) and the Resonant peak Value M_r</u>

$$G(j\omega) = \frac{1}{1 + 2\xi\left(j\frac{\omega}{\omega_n}\right) + \left(j\frac{\omega}{\omega_n}\right)^2}$$

$$|G(j\omega)| = \frac{1}{\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\xi\frac{\omega}{\omega_n}\right)^2}}$$

$|G(j\omega)|$ will be maximum if $g(\omega)$ is minimum.

$$g(\omega) = \left(1-\frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\xi\frac{\omega}{\omega_n}\right)^2$$

$$g(\omega) = \left[\frac{\omega^2 - \omega_n^2(1-2\xi^2)}{\omega_n^2}\right]^2 + 4\xi^2(1-\xi^2)$$

$g(\omega)$ is minimum when $\boxed{\omega = \omega_n\sqrt{1-2\xi^2}}$

So $\boxed{\omega_r = \omega_n\sqrt{1-2\xi^2}}$ $\left\{\forall\ 0 \leq \xi \leq \sqrt{2}\right\}$

$\Rightarrow$ for $\xi > \sqrt{2}$ there is no resonant peak.

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$$\boxed{M_r = |G(j\omega)|_{max} = \frac{1}{2\xi\sqrt{1-\xi^2}}}$$

$\Rightarrow$ As $\xi \to 0 \Rightarrow M_r \to \infty$

$\Rightarrow$ Phase angle at resonance frequency,

$$\boxed{\phi = -\tan\frac{\sqrt{1-2\xi^2}}{\xi} = -\frac{\pi}{2} + \sin^{-1}\left(\frac{\xi}{\sqrt{1-\xi^2}}\right)}$$

★ General Procedure of Plotting bode diagram

1) first write the Sinusoidal TF $G(j\omega)H(j\omega)$ as a product of basic factors.

2) Then identify Corner frequency associated with these basic factors.

3) Finally draw the asymptotic log magnitude curve with proper slope between the Corner frequencies.

4) Then exact curve that lies close to the asymptotic Curve, can be obtained by adding proper Corrections.

5) The phase-angle Curve of $G(j\omega)H(j\omega)$ can be drawn by adding the phase angle curves of individual factor.

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★ Minimum-Phase System and Nonminimum phase System

Minimum-phase System

⇒ System with minimum-phase transfer function.

⇒ TF having neither poles nor zero in the right half of S plane are minimum phase TF.

Non Minimum-Phase System.

⇒ System with non-minimum phase TF.

⇒ TF having poles or/and zeros in the right half of S plane are Non Minimum-Phase TF.

// For a System with the Same magnitude characteristic, the range in phase angle of the minimum-phase TF is minimum among all such System while the range in phase angle of any non-minimum phase TF is greater than this minimum.

★ **Transport Lag (dead time)**

↳ It is of nonminimum phase behavior and has an excessive phase lag with no attenuation at high frequencies.

⇒ Transport lag normally exist in thermal, hydraulic and pneumatic systems.

⇒ Consider a transport lag given by

$$G(j\omega) = e^{-j\omega T}$$

→ Magnitude is always equal to 1.

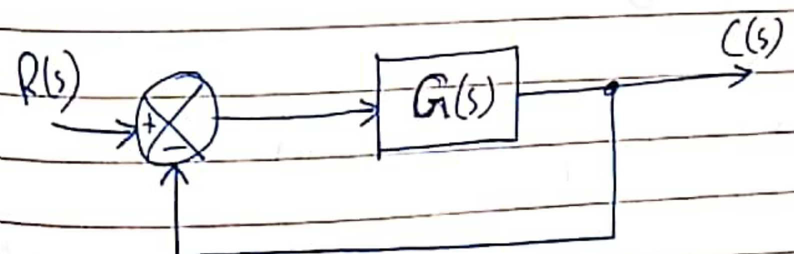
$$|e^{-j\omega T}| = 1$$

→ The phase angle,

$$\angle e^{-j\omega T} = -\omega T \quad \text{(radians)}$$

★ **Relationship between System Type & Log-Magnitude Curve**

i) **Determination of Static position error Constant**



Let $G(s) = \dfrac{K(T_a s+1)(T_b s+1)\cdots(T_m s+1)}{s^N (T_1 s+1)(T_2 s+1)\cdots(T_p s+1)}$

$$G(j\omega) = \frac{K(T_a j\omega+1)(T_b j\omega+1)\cdots(T_m j\omega+1)}{(j\omega)^N (T_1 j\omega+1)(T_2 j\omega+1)\cdots(T_p j\omega+1)}$$

$$\lim_{\omega\to 0} G(j\omega) = K = K_p \quad \left\{\begin{array}{c}\text{for type zero system} \\ N=0\end{array}\right\}$$

ii) Determination of Static Velocity Error Constant

K_v

(iii) Determination of static acceleration error Constant

$$K_a = \omega_a^2 \quad \{\text{for intersection}\}$$

For type 1 System

$$G(j\omega) = \frac{K_v}{j\omega} \quad \forall\ \omega \ll 1$$

$$20\log\left|\frac{K_v}{j\omega}\right|_{\omega=1} = 20\log K_v$$

for intersection $\left|\dfrac{K_v}{j\omega}\right| = 1 \Rightarrow K_v = \omega_1$

Polar plot {Nyquist plot}

"The polar plot of a sinusoidal transfer function $G(j\omega)$ is plot of the magnitude of $G(j\omega)$ vs phase angle of $G(j\omega)$ on polar coordinate. as ω is varied from zero to infinity"

→ Thus polar plot is the locus of vector $|G(j\omega)| \angle G(j\omega)$ as ω is varied from zero to infinity.

Note
→ Negative phase angle is measured clockwise.
→ Positive phase angle as anti-clockwise.

⇒ In the polar plot it is important to show the frequency graduation of the locus.

Advantage: It depicts the frequency-response characteristics of a system over the entire frequency range in a single plot.

Disadvantage: Dose not clearly indicates the contribution of each individual factor of the open-loop TF

Integral and Derivative factor

⇒ $\frac{1}{j\omega} = -\frac{1}{\omega}j = \frac{1}{\omega}\angle{-90}$

So polar plot is negative imaginary axis.

⇒ $j\omega = \omega\angle 90$

So polar plot is positive imaginary axis.

First Order factor $(1+j\omega T)^{\mp 1}$

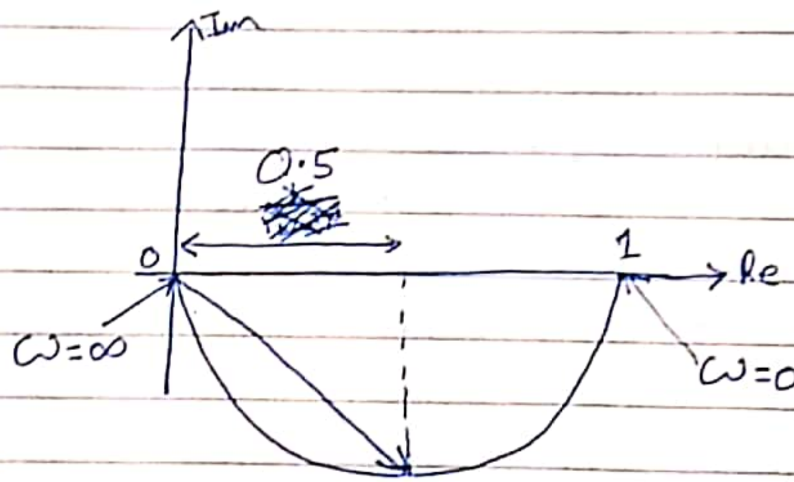
⇒ $G(j\omega) = \frac{1}{1+j\omega T} = \frac{1}{\sqrt{1+\omega^2T^2}}\angle{-\tan^{-1}\omega T}$

1) $\omega = 0$

$1\angle 0$

2) $\omega = \frac{1}{T}$

$\frac{1}{\sqrt{2}}\angle{-45}$



⇒ The polar plot of this Transfer function is a Semicircular as the frequency ω varies from zero to infinity.

⇒ To prove that polar plot of the first-order factor $G(j\omega) = 1/(1+j\omega T)$ is a semicircle define,

$$G(j\omega) = X + jY$$

$$X = \frac{1}{1+\omega^2T^2} \qquad Y = \frac{-\omega T}{1+\omega^2T^2}$$

$$\left(X - \frac{1}{2}\right)^2 + Y^2 = \left(\frac{1}{1+\omega^2T^2} - \frac{1}{2}\right)^2 + \left(\frac{-\omega T}{1+\omega^2T^2}\right)^2 = \left(\frac{1}{2}\right)^2$$

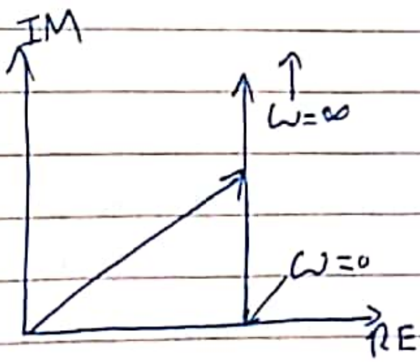
$$\Rightarrow \boxed{\left(X - \frac{1}{2}\right)^2 + Y^2 = \left(\frac{1}{2}\right)^2}$$

{Hence semicircular of center $x = \frac{1}{2}$ and radius $\frac{1}{2}$}

⇒ The lower semicircle corresponds to $0 \leq \omega \leq \infty$ and the upper semicircle corresponds to $-\infty \leq \omega \leq 0$.

$$1 + j\omega T = \sqrt{1+\omega^2T^2} \angle \tan^{-1}\omega T$$

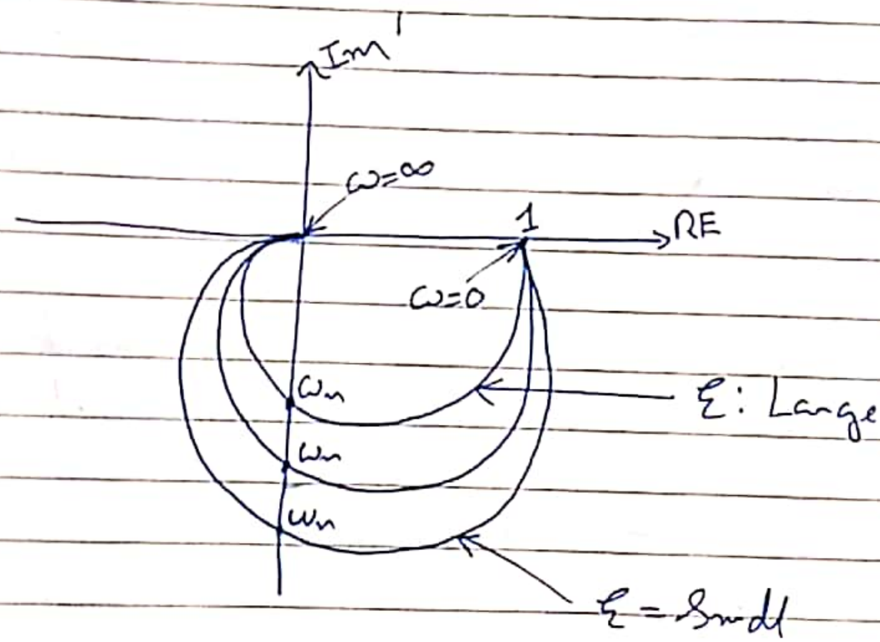
⇒ The polar plot of the transfer function $1+j\omega T$ is simply the upper half of the straight line passing through point $(1,0)$ in the complex plane and ∥ to imaginary axis as shown.



Quadratic Factor $\left[1 + 2\xi\left(j\omega/\omega_n\right) + \left(j\omega/\omega_n\right)^2\right]^{\mp 1}$

$G(j\omega) = \dfrac{1}{1 + 2\xi\left(j\frac{\omega}{\omega_n}\right) + \left(j\frac{\omega}{\omega_n}\right)^2} \quad \forall\ \xi > 0$

$\lim_{\omega \to \infty} G(j\omega) = 1\angle 0 \qquad \lim_{\omega \to \infty} G(j\omega) = 0\ \angle -180$



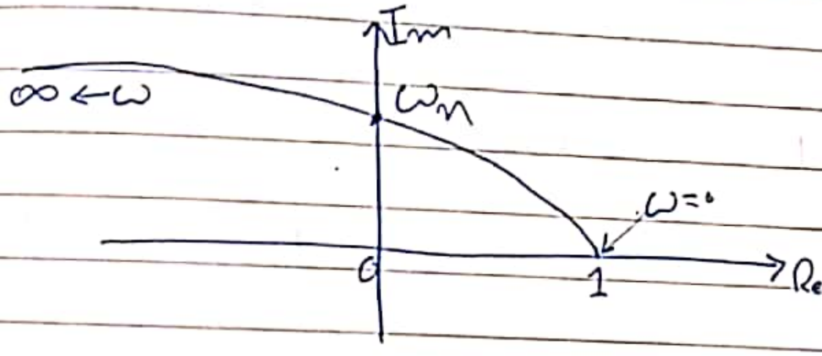
⇒ The exact shape of the polar plot depends on the value of ξ but the general shape of the plot is the same for both the underdamped case $(1 > \xi > 0)$ and overdamped case $(\xi > 1)$.

⇒ In the polar plot, the frequency point whose distance from the origin is maximum corresponds to the resonant frequency ω_r.

⇒ For the overdamped case, as ξ increases well beyond unity, the $G(j\omega)$ locus approaches a semicircle.

$G(j\omega) = 1 + 2\xi\left(j\frac{\omega}{\omega_n}\right) + \left(j\frac{\omega}{\omega_n}\right)^2$

$\lim_{\omega \to 0} G(j\omega) = 1\angle 0$ $\qquad$ $\lim_{\omega \to \infty} G(j\omega) = \infty \angle 180$



★ **General Shape of Polar plot**

$$G(j\omega) = \frac{K(1+j\omega T_a)(1+j\omega T_b)\cdots}{(j\omega)^{\lambda}(1+j\omega T_1)(1+j\omega T_2)\cdots}$$

$$= \frac{b_0(j\omega)^m + b_1(j\omega)^{m-1} + \cdots}{a_0(j\omega)^n + a_1(j\omega)^{n-1} + \cdots}$$

{where $n > m$}

1. For $\lambda = 0$ or type 0 System

→ Starting point finite k is on positive real axis.

→ Tangent to polar plot at $\omega = 0$ is $\perp$ to real axis.

→ The terminal point, which Corresponds to $\omega = \infty$, is at origin, and the Curve is tangent to one of the axis.

2. For $\lambda = 1$ or type 1 System.

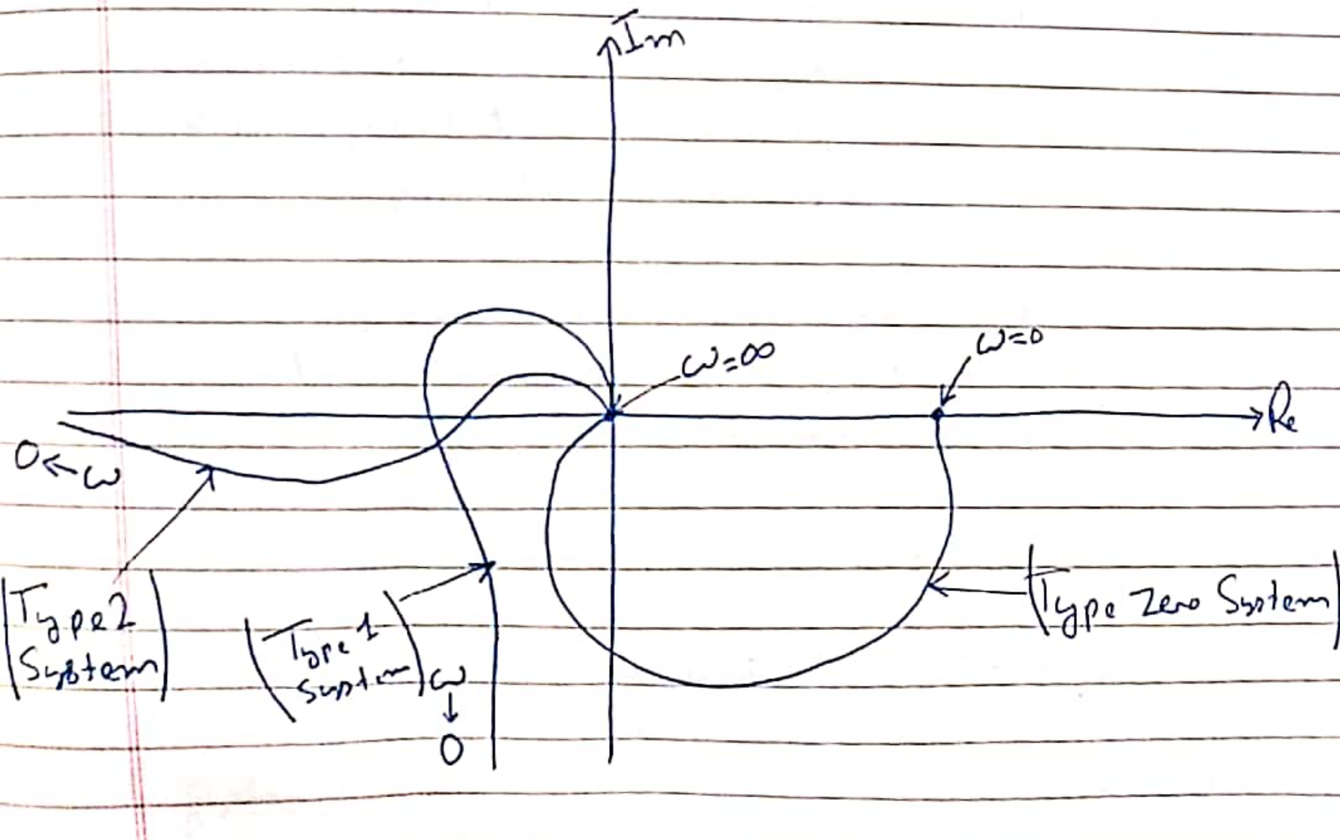
→ At $\omega = 0$ the magnitude of $G(j\omega)$ is ∞ and the phase angle become $-90°$.

→ At low frequency, the polar plot is asymptotic to a line parallel to the negative Imaginary axis.

→ At $\omega = \infty$, the magnitude becomes zero, and the curve Converges to the origin and is tangent to one of the axis.

3. For $\lambda = 2$ or type 2 System

- At $\omega = 0$, the magnitude of $G(j\omega)$ is ∞ and the phase angle is equal to $-180°$.
- At low frequency, the polar plot may be asymptotic to the negative real axis.
- At $\omega = \infty$, the magnitude becomes zero, and the curve is tangent to one of the axes.



★ Log Magnitude-Versus Phase plot (Nichols plot)

↳ Plot of the logarithm magnitude in decibels verses the phase angle or phase margin for a frequency range of interest.

⇒ Phase margin is the difference between actual phase angle φ and -180°; that is $\varphi-(-180) = 180+\varphi$.

⇒ The curve is graduated in terms of the frequency ω.

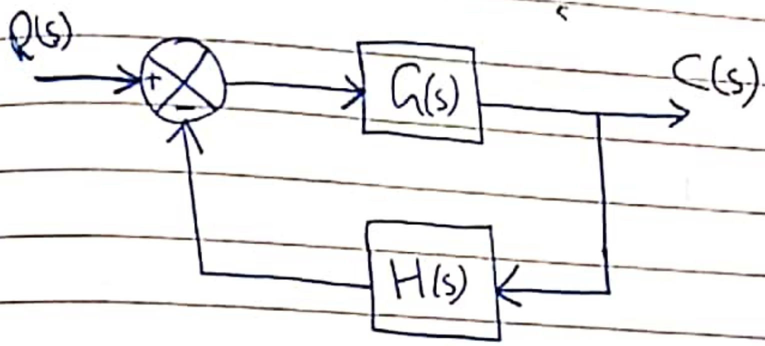
Advantage ⇒ The relative stability of the closed loop system can be determined quickly and that compensation can be worked out easily.

⇒ The log-magnitude-versus phase plot for the sinusoidal transfer function $G(j\omega)$ and that for $1/G(j\omega)$ are skew symmetrical about the origin.

$$\left|\frac{1}{G(j\omega)}\right| = -|G(j\omega)|$$

★ Nyquist Stability Criterion

⇒ The Nyquist Stability Criterion determines the stability of a closed-loop system from its frequency response and open loop poles.



$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

For stability all roots of $\{1 + G(s)H(s) = 0\}$ must lie in the left-half s plane.

⇒ The Nyquist Stability Criterion relates the open-loop frequency response $G(j\omega)H(j\omega)$ to the number of zeros and poles of $1 + G(s)H(s)$ that lie in the right half s plane.

⇒ This criterion is useful in control engineering because the absolute stability of the closed-loop system can be determined graphically from open-loop frequency response curve, and there is no need for actually determining the closed-loop poles.

⇒ The Nyquist stability criterion is based on a theorem from the theory of complex variable. To understand the criterion, we shall first discuss mappings of contours in the complex plane.

★ **Preliminary Study**

$$F(s) = 1 + G(s)H(s) = 0$$

⇒ For a given continuous closed path in the s-plane that does not go through any singular points, there corresponds a closed curve in F(s) plane.

⇒ The number and direction of encirclement of the origin of ~~the origin of~~ the F(s) plane by the closed curve plays an important role in determining stability of system

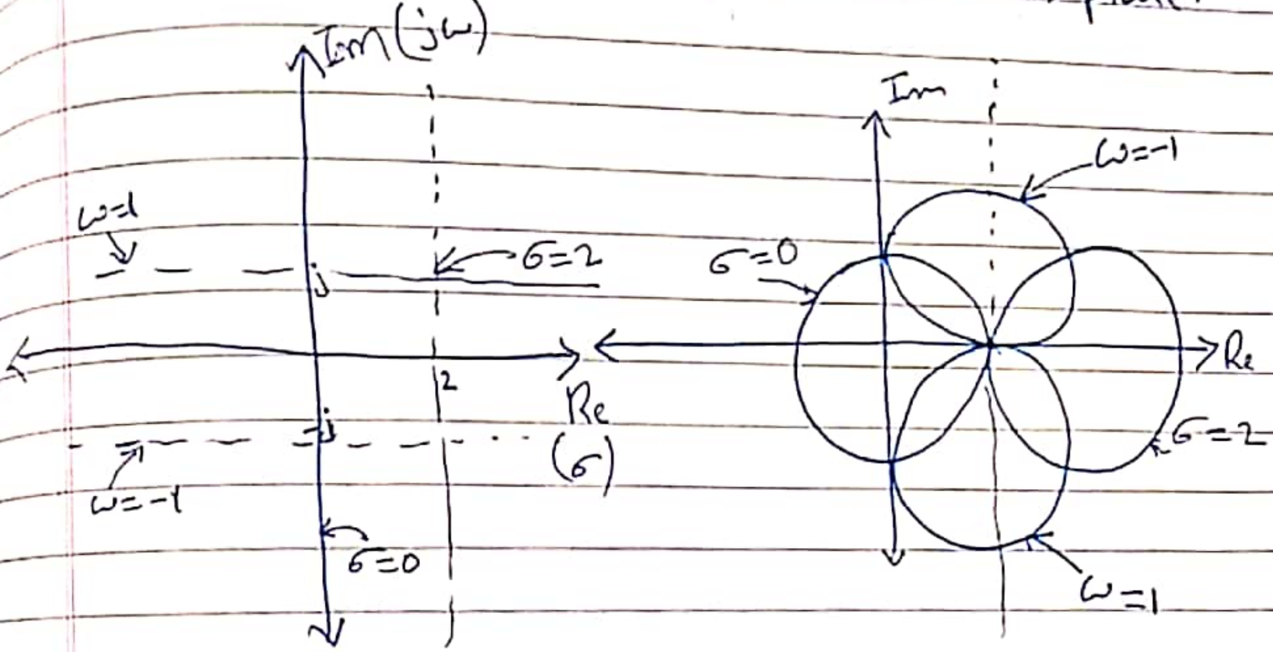
⇒ Consider, for example, the following openloop transfer function:

$$G(s)H(s) = \frac{2}{s-1}$$

So $F(s) = 1 + \frac{2}{s-1} = \frac{s+1}{s-1} = 0$

⇒ The function F(s) is analytical everywhere in the s plane except at its singular points.

↳ For each point of analyticity in the s plane, there corresponds a point in the F(s) plane.



Splane   F(s) plane

$$\text{For } F(s) = \frac{s+1}{s-1}$$

⇒ Suppose that representative point s trace out a contour in the s plane in the clockwise direction.

↳ If the contour in the s plane encloses the pole of F(s), there is one encirclement of the origin of the F(s) plane by the locus of F(s) in counterclockwise direction.

Mapping Theorem

→ If the Contour in the S plane encloses the zero of F(s), there is one encirclement of the origin of the F(s) plane by the locus of F(s) in the clockwise direction.

→ If the contour in the S plane encloses both the zero and the pole or if the contour encloses neither the zero nor the pole, then there is no encirclement of the origin of F(s) plane by the locus of F(s).

Note: Location of a pole or zero in the S plane, whether in the right-half or left half s plane, does not make any difference, but the enclosure of a pole or zero does.

★ Mapping Theorem {Cauchy's argument principle}

Let F(s) be a ratio of two polynomials in s. Let P be the number of poles and Z be the number of zeros of F(s) that lie inside some closed contour in the s plane, with multiplicity of poles and zeros accounted for. Let the contour be such that it does not pass through any poles or zeros of F(s).

This closed contour in the s plane is then mapped into the F(s) plane as a closed curve. The total number N of clockwise encirclements of the origin of the F(s) plane, as a representative point s traces out the entire contour in the clockwise direction is equal to Z-P //

⇒ In mapping theorem, the numbers of zeros and of poles cannot be found - only their difference //.

★ <u>Application of the Mapping Theorem to the Stability Analysis of Closed loop System</u>

⇒ For analyzing the stability of linear control systems, we let the closed contour in the s plane enclosing the entire right half s-plane.

↳ The contour consists of the entire $j\omega$ axis from $\omega = -\infty$ to $+\infty$ and a semicircular path of infinite radius in the right half s-plane.

↳ Such contour is called Nyquist path.

⇒ It is necessary that the closed contour on the Nyquist path, not pass through any zero and poles.

$$\lim_{s\to\infty}\left[1+G(s)H(s)\right] = \text{Constant.}$$

⇒ The function $1+G(s)H(s)$ remains constant as 'S' traverse the semicircle of infinite radius.

⇒ Because of this whether the locus of $1+G(s)H(s)$ encircles the origin of the $1+G(s)H(s)$ plane can be determined by considering only a part of the closed contour in the S plane — that is the $j\omega$ axis.

↳ Encirclement of the origin, if there are any occur only while a representative point moves from $-j\infty$ to $+j\infty$ along the $j\omega$ axis, provided that no zeros or poles lie on the $j\omega$ axis.

⇒ Note that the portion of the $1+G(s)H(s)$ contour from $\omega=-\infty$ to $\omega=\infty$ is simply $1+G(j\omega)H(j\omega)$

⇒ Since $1+G(j\omega)H(j\omega)$ is the vector sum of the unit vector and vector $G(j\omega)H(j\omega)$, $1+G(j\omega)H(j\omega)$ is identical to the vector drawn from the $-1+j0$ point to the terminal point of the vector $G(j\omega)H(j\omega)$.

⇒ Encirclement of the origin by the graph of $1 + G(j\omega)H(j\omega)$ is equivalent to encirclement of the $-1 + 0j$ point by just the $G(j\omega)H(j\omega)$ locus.

↳ Thus, the stability of a closed loop system can be investigated by examining encirclement of the $-1 + 0j$ point by the locus of $G(j\omega)H(j\omega)$

⇒ Plotting $G(j\omega)H(j\omega)$ for the Nyquist path is straight forward.

↳ Plot of $G(j\omega)H(j\omega)$ and the plot of $G(-j\omega)H(-j\omega)$ are symmetrical with each other about the real axis.

* <u>Nyquist Stability Criterion</u> {when $G(s)H(s)$ has neither poles nor zeros on $j\omega$ axis}

"In a System if the open loop transfer function $G(s)H(s)$ has K poles in the right-half 's'-plane and $\lim_{s \to \infty} G(s)H(s) = \text{Constant}$, then for stability, the $G(s)H(s)$ locus, as ω varies from $-\infty$ to ∞, must encircle the $-1 + j0$ point K times in the Counterclockwise direction"

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★ Remarks on Nyquist Stability Criterion

1. The Criterion can be expressed as

$$Z = N + P$$

Where Z = Number of zeros of $1+G(s)H(s)$ in Right half s plane
N = number of clockwise encirclements of the $-1+0j$ point
P = number of poles of $G(s)H(s)$ in the right half s plane.

If P is not zero, for a stable control system, we must have $Z=0$, or $N=-P$, which means that we must have P counterclockwise encirclement of the $-1+0j$ point.

2. If the locus of $G(j\omega)H(j\omega)$ passes through the $-1+j0$ point, then zeros of the characteristics equation, or closed-loop poles, are located on the $j\omega$ axis. This is not desirable for practical control system. For a well-designed closed-loop system none of the roots of the characteristics equation should lie on the $j\omega$ axis.

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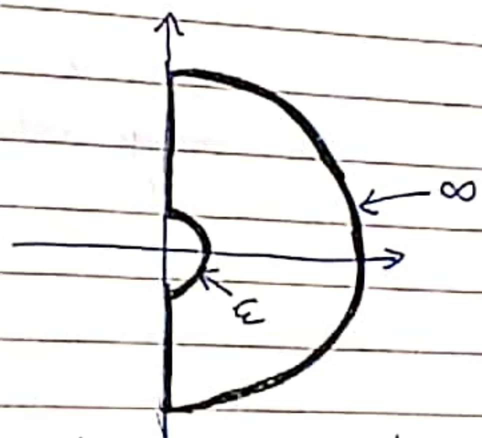
★ Special Case when $G(s)H(s)$ Involves Poles and/or Zeros on the $j\omega$ Axis

⇒ If function $G(s)H(s)$ has poles or zeros at the origin (or on $j\omega$ axis at points other than origin), the contour in the S plane must be modified.

↳ The usual way of modifying the contour near the origin is to use a semicircle with the infinitesimal radius ε.



⇒ The semicircle may lie in the right half S-plane or in the left half S-plane.

⇒ Consider for example, a closed-loop system whose open-loop transfer function is given by

$$G(s)H(s) = \frac{K}{s(Ts+1)}$$

⇒ The points corresponding to $s = j0_+$ and $s = j0_-$ on the locus of $G(s)H(s)$ in the $G(s)H(s)$ plane are $-j\infty$ and $+\infty j$ respectively.

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⇒ On the Semicircular path with radius ε (where $\varepsilon \ll 1$), the Complex Variable s can be written

$$S = \varepsilon e^{j\theta} \quad \left\{ \text{where } \theta \text{ varies from } -\frac{\pi}{2} \text{ to } +\frac{\pi}{2}. \right\}$$

⇒ Then G(s) H(s) becomes

$$G(\varepsilon e^{j\theta}) H(\varepsilon e^{j\theta}) = \frac{K}{\varepsilon e^{j\theta}} = \frac{K}{\varepsilon} e^{-j\theta}$$

⇒ The infinitesimal Semicircle detour around the origin in the S plane maps into the GH plane as a semicircle of infinite radius.

⇒ For an open-loop transfer function G(s) H(s) involving $1/s^n$ factor, the plot of G(s) H(s) has n clockwise Semicircle of infinite radius about the Origin as a representative point S moves along the Semicircle of radius ε.

★ **Nyquist Stability Criterion** {for general case when G(s) H(s) has poles and/or zeros on $j\omega$ axis}

// In a system if the open-loop transfer function G(s) H(s) has K poles in the right-half s plane, then for stability the G(s) H(s) locus, as a representative point S traces on the modified Nyquist path in the clockwise direction must encircle the $-1+j0$ point K times in the counter clockwise direction//

★ Stability Analysis

⇒ In examining the stability of linear control system using the Nyquist stability criterion, we see that three possibilities can occur:-

1. There is no encirclement of the $-1+j0$ point. This implies that the system is stable if there are no poles of $G(s)H(s)$ in the Right half s-plane; otherwise, the system is unstable.

2. There is one or more counterclockwise encirclements of the $-1+j0$ point. In this case the system is stable if the number of counterclockwise encirclements is the same as the number of poles of $G(s)H(s)$ in the right -half s plane; otherwise the system is unstable.

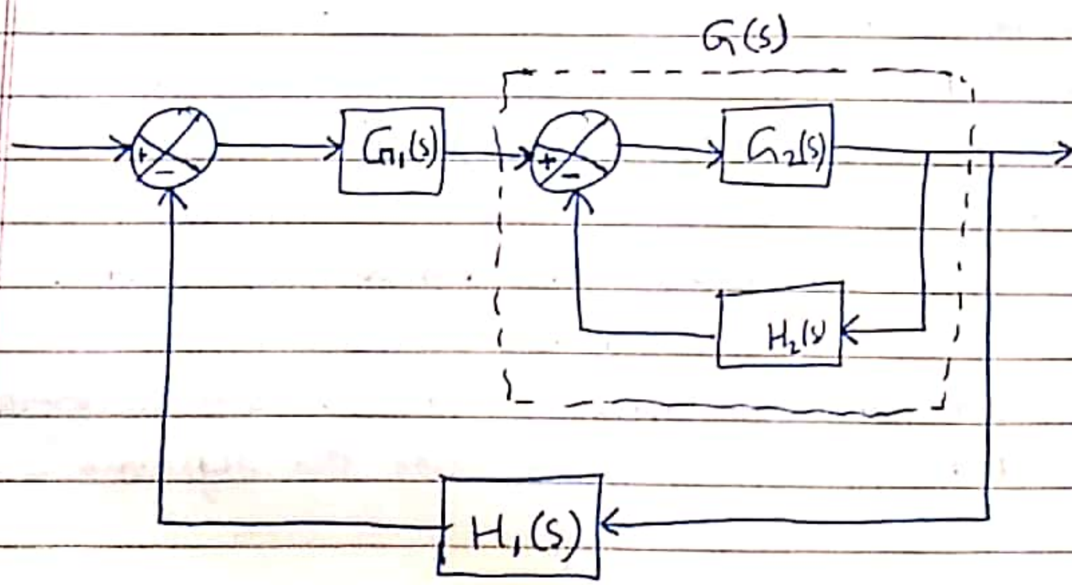
3. There are one or more clockwise encirclements of the $-1+0j$ point. In this case the system is unstable

★ Conditionally Stable System

⇒ A conditionally stable System is stable for the value of the open-loop gain lying between critical values, but it is unstable if the open loop gain is either increased or decreased sufficiently.

↳ Such system becomes unstable when large input signals are applied, since a large signal may cause saturation, which in turn reduces the open loop gain of the System. It is advisable to avoid such a situation.

★ Multiple-Loop System



$$G(s) = \frac{G_2(s)}{1 + G_2(s)H_2(s)}$$

⇒ Since the open-loop transfer function of the entire System is given by $G_1(s)\, G(s)\, H_1(s)$, the stability of this closed loop system can be found from the Nyquist plot of $G_1(s)\, G(s)\, H_1(s)$ and Knowledge of the right-half plane poles of $G_1(s)\, G(s)\, H_1(s)$.

★ **Nyquist Stability Criterion Applied to Inverse Polar plot**

⇒ In analysing multiple-loop system, the inverse transfer function may sometimes be used in order to permit graphical analysis; this avoids much of the numerical calculation.

⇒ The inverse polar plot of $G(j\omega)H(j\omega)$ is a graph of $1/[G(j\omega)H(j\omega)]$ as a function of ω.

Nyquist Stability Criterion {Applied to inverse plots}

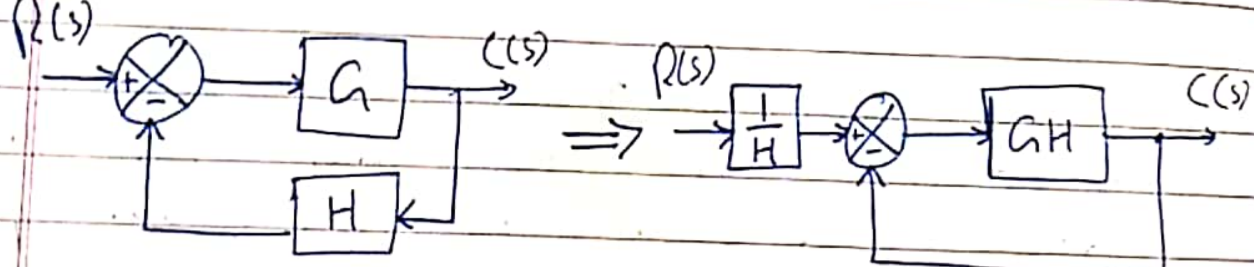
" For a closed-loop System to be stable, the encirclement, if any of the $-1+0j$ point by the $1/G(s)H(s)$ locus must be Counterclockwise and the number of such encirclements must be equal to number of poles of $1/G(s)H(s)$ that lie in the right-half s plane."

★ Relative Stability Analysis

Relative stability ⇒ In designing a Control System, we require that the system be stable. Furthermore, it is necessary that the system have adequate relative stability !!

⇒ In analysis we shall assume that the systems considered have unity feedback.

↳ It is always possible to reduce a system with feedback element to a unit feedback system.



⇒ We shall also assume that, unless otherwise stated, the systems are minimum-phase systems; that is, the open-loop transfer function have neither poles nor zeros in the right half s plane.

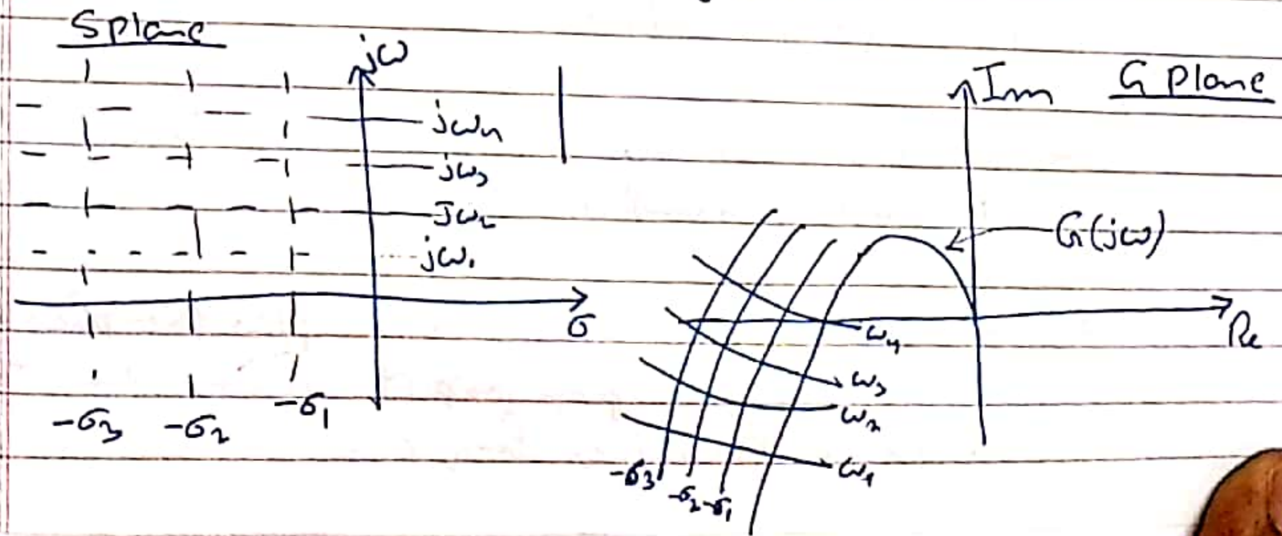
★ Relativity stability ~~ad~~ analysis by Conformal Mapping

⇒ One of the important problems in analysing a control system is to find all closed-loop poles or atleast those closest to the jω axis.

⇒ If the open loop frequency response characteristics of a system are known, it may be possible to estimate the closed loop poles closest to the jω axis.

⇒ Consider the Conformal mapping of constant-σ lines (lines σ+jω, where σ is constant and ω varies) and constant-ω lines (lines s = σ+jω, where ω = constant and σ varies) in the s plane.

⇒ The constant σ lines in the s plane map into curves that are similar to the Nyquist plot and in a sense parallel to the Nyquist plot.



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⇒ The Closeness of approach of the $G(j\omega)$ locus to the $-1 + j0$ point is an indication of the relative stability of a stable system.

↳ The closer the $G(j\omega)$ locus is to the $-1 + j0$ point, the larger the maximum overshoot is in the step transient response and the longer it takes to damp out.

★ Phase and Gain Margins

⇒ In general, the closer the $G(j\omega)$ locus comes to encircling the $-1 + j0$ point, the more oscillatory is the system response.

Phase margin: The phase margin is that amount of additional phase lag at the gain crossover frequency required to bring the system to the verge of instability.

⇒ The gain crossover frequency is the frequency at which $|G(j\omega)|$ is unity.

⇒ The phase margin γ is 180° plus the phase angle ϕ of the open loop transfer function at the gain crossover frequency.

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$$\boxed{\gamma = 180^\circ + \phi}$$

⇒ For a minimum phase system to be stable, the phase margin must be positive.

Gain margin: The gain margin is the reciprocal of the magnitude of the frequency at which phase angle is -180°.

$$K_g = \frac{1}{|G(j\omega_1)|}$$

→ Phase Crossover frequency

⇒ In terms of decibels,

$$K_g\, dB = 20 \log K_g = -20 \log |G(j\omega_1)|$$

⇒ Positive gain margin (in decibels) means that the System is stable, and a negative gain margin (in decibels) means that the System is unstable.

⇒ For a stable Minimum-phase system, the gain margin indicates how much the gain can be increased before the System becomes unstable.

⇒ For an unstable system the gain margin indicates how much gain must be decreased to make the System stable.

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⇒ The gain margin of a first or second order system is infinite since the polar plot for such system do not cross the negative real axis.

For a nonminimum-phase system with unstable open loop the stability condition will not be satisfied unless the $G(j\omega)$ plot encircles the $-1+0j$ point.

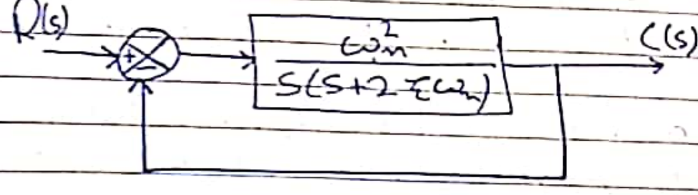
↳ Hence, such a stable nonminimum-phase system will have negative phase & gain margins.

Conditionally stable system and some higher-order system with complicated numerator dynamics may also have two or more gain crossover frequencies, the phase margin is measured at the highest gain crossover frequency.

⇒ Either the gain margin alone or the phase margin alone does not give a sufficient indication of the relative stability. Both should be given in the determination of relative stability.

⇒ For satisfactory performance, the phase margin should be between 30° and 60° and the gain margin should be greater than 6dB.

★ Resonant Peak Magnitude M_r and Resonant Frequency ω_r



$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2} = Me^{j\alpha}$$

$$M = \frac{1}{\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+\left(2\xi\frac{\omega}{\omega_n}\right)^2}} \qquad \alpha = -\tan^{-1}\frac{2\xi\frac{\omega}{\omega_n}}{1-\frac{\omega^2}{\omega_n^2}}$$

$$\omega_r = \omega_n\sqrt{1-2\xi^2}$$

resonant frequency

Resonant Peak Magnitude

$$M = \frac{1}{2\xi\sqrt{1-\xi^2}}$$

⇒ The magnitude of the resonant peak gives an indication of the relative stability of the system.

↳ A large resonant peak indicates the presence of a pair of dominant closed-loop poles with small damping ratio, which will yield an undesirable transient response.

⇒ In pratical design problems the phasemargin and gain margin are more frequently specified than the resonant peak magnitude to indicate the degree of damping ~~of~~ in a system.

★ <u>Correlation between $\boxed{\text{Step transient Response}}$ and $\boxed{\text{Frequency Response}}$ in the Standard Second Order System</u>

⇒ The <u>maximum overshoot</u> in the unit-step response of the Standard Second Order System Can be exactly Correlated with the <u>resonant peak magnitude</u> in the frequency response.

$$C(t) = 1 - e^{-\xi\omega_n t}\left(\cos\omega_d t + \frac{\xi}{\sqrt{1-\xi^2}}\sin\omega_d t\right)$$

$$\omega_d = \omega_n\sqrt{1-\xi^2}$$

$$M_p = e^{-(\xi/\sqrt{1-\xi^2})\pi} \quad \{\text{Maximum overshoot}\}$$

⇒ This maximum overshoot occurs in the transient response that has the damped natural frequency $\omega_d = \omega_n\sqrt{1-\xi^2}$

$$G(s) = \frac{\omega_n^2}{s(s+2\xi\omega_n)} \quad \left\{\begin{array}{l}\text{Open-loop transfer function}\\ \text{of Second order system}\end{array}\right\}$$

for sinusoidal operation, the magnitude of $G(j\omega)$ become Unity when

[redacted]

$$\omega = \omega_n\sqrt{\sqrt{1+4\xi^4}-2\xi^2}$$

At this frequency phase angle of $G(j\omega)$ is

$$\angle G(j\omega) = -90 - \tan^{-1}\frac{\sqrt{\sqrt{1+4\xi^4}-2\xi^2}}{2\xi}$$

So $\gamma = 180 + \angle G(j\omega)$

$$\boxed{\gamma = \tan^{-1}\frac{2\xi}{\sqrt{\sqrt{1+4\xi^4}-2\xi^2}}}$$

★ Correlation between step transient Responce and Frequency Response in General System

⇒ The design of control system is very often carried out on the basis of the frequency response.

↳ The main reason for this is the relative simplicity of this approach compared with others.

1. The value of M_r is indicative of the relative stability.

   ↳ Satisfactory transient performance is usually obtained if the value of M_r is in the range $1.0 < M_r < 1.4$ ($0\,dB < M_r < 3\,dB$), which corresponds to an effective damping ratio $0.4 < \xi < 0.7$.

2. The magnitude of the resonant frequency ω_r is indicative of the speed of the transient response.

   ↳ Rise time is inversely with ω_r.

3. The resonant peak frequency ω_r and the damped natural frequency ω_d for the step transient response are very close to each other for lightly damped system.

★ <u>Cutoff Frequency and Bandwidth</u>

The frequency ω_b at which the magnitude of the closed-loop frequency response is 3dB below its zero-frequency value is called the <u>Cutoff frequency.</u>

⇒ The closed-loop system filters out the signal components whose frequencies are greater than the cutoff frequency and transmits those signal components with frequencies lower than the cutoff frequency.

The frequency range $0 \leq \omega \leq \omega_b$ in which the magnitude of $C(j\omega)/R(j\omega)$ is greater than $-3dB$ is called the Bandwidth of the System.

↳ Bandwidth indicates the frequency where the gain starts to fall off from its low-frequency value.

⇒ Rise time & the bandwidth ~~decreases with the increasing~~ are inversely proportional to each other.

Cutoff Rate ⇒ It is the slope of the log-magnitude curve near the cutoff frequency.

↳ The cutoff rate indicates the ability of a system to distinguish the signal from noise.

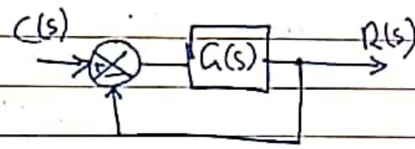
★ **Closed-Loop Frequency Responce of unity feedback System**

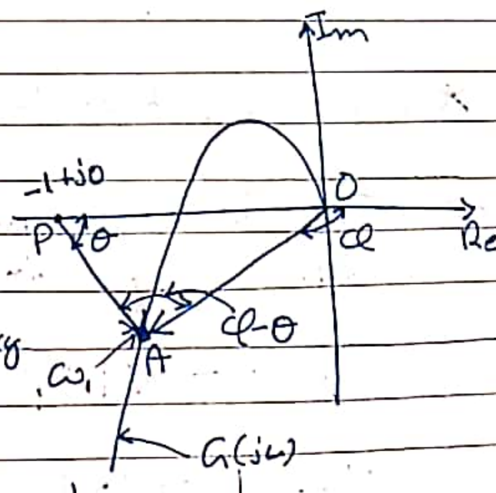
★ **Closed loop Frequency Response**

⇒ For a stable, unity-feedback closed-loop system, the closed-loop frequency response can be obtained easily from that of the open-loop frequency responce.

⇒ Consider a unity-feedback system :-

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$





⇒ The vector $\overrightarrow{OA}$ represents $G(j\omega_1)$, where ω_1 is the frequency at point A.

⇒ The vector $\overrightarrow{PA}$, the vector from the $-1+j0$ point to the Nyquist locus, represents $1+G(j\omega_1)$.

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⇒ Therefore ratio of $\overrightarrow{OA}$ to $\overrightarrow{PA}$ represents the closed loop frequency response.

$$\frac{\overrightarrow{OA}}{\overrightarrow{PA}} = \frac{G(j\omega_1)}{1+G(j\omega_1)} = \frac{C(j\omega_1)}{R(j\omega_1)}$$

⇒ The magnitude of the closed loop transfer function at $\omega=\omega_1$ is the ratio of the magnitudes of $\overrightarrow{OA}$ to $\overrightarrow{PA}$.

↳ The phase angle of the closed loop transfer function at $\omega=\omega_1$ is the angle formed by the vectors $\overrightarrow{OA}$ to $\overrightarrow{PA}$ (ie $\phi-\theta$).

⇒ Let us define the magnitude of the closed loop frequency response as M and the phase angle as α.

$$\frac{C(j\omega)}{R(j\omega)} = Me^{j\alpha}$$

★ Constant-Magnitude Loci (M circle)

⇒ $G(j\omega)$ is a complex quantity and can be written as following:

$$G(j\omega) = X + jY \quad \{X \,\&\, Y \text{ are real quantity}\}$$

$$M = \frac{|X+jY|}{|1+X+jY|} \Rightarrow M^2 = \frac{X^2+Y^2}{(1+X)^2+Y^2}$$

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$$\Rightarrow X^2(1-M^2) - 2M^2X - M^2 + (1-M^2)Y^2 = 0$$

If $M=1 \Rightarrow X = -\frac{1}{2}$ {Straight line parallel to Y-axis passing through the Point $(-\frac{1}{2}, 0)$}

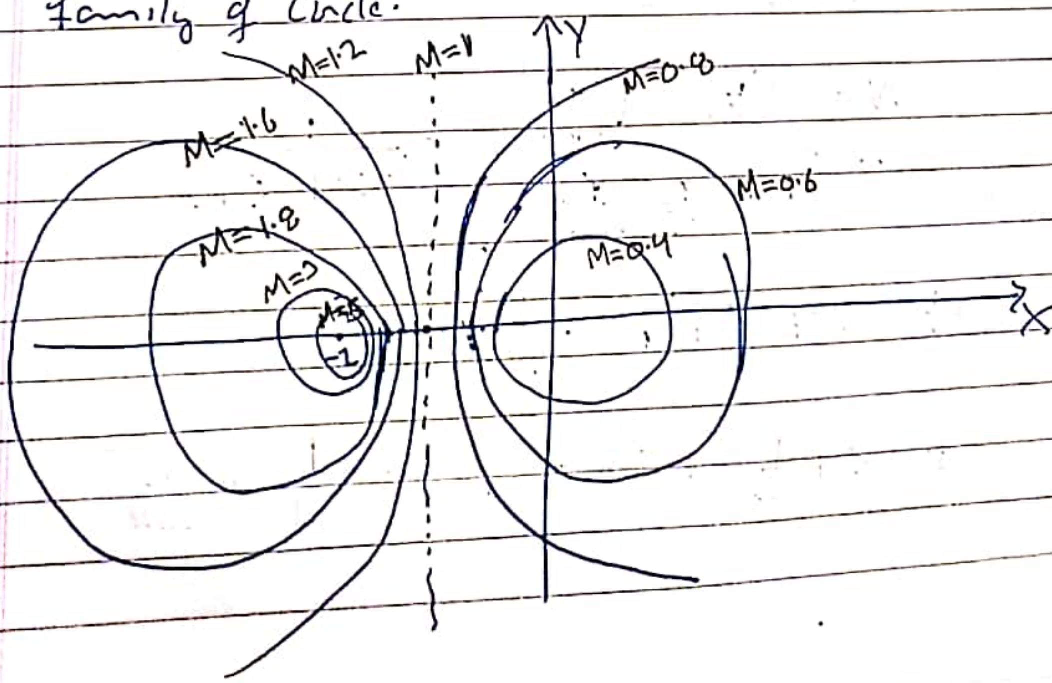
If $M \neq 1$

$$X^2 + \frac{2M^2}{M^2-1}X + \frac{M^2}{M^2-1} + Y^2 = 0$$

$$\Rightarrow \boxed{\left(X + \frac{M^2}{M^2-1}\right)^2 + Y^2 = \frac{M^2}{(M^2-1)^2}}$$

$\Rightarrow$ Above equation is of a Circle with center at $\left(\frac{-M^2}{M^2-1}, 0\right)$ and radius $\left|\frac{M}{M^2-1}\right|$.

$\Rightarrow$ The Constant M loci on the G(s) plane are thus a family of Circle.



⇒ It is seen that as M become larger compared with 1, the M circles ~~lies to the left of the -1+j0 point~~ become smaller and converge to the $-1+j0$ point.

⇒ Similarly as M become smaller compared with 1, the M circles ~~lie to~~ becomes smaller and converges to the origin.

⇒ The M circles are symmetrical with respect to the straight line corresponding to $M=1$ and with respect to real axis.

★ Constant-Phase-Angle Loci (N circle)

$$\angle e^{j\alpha} = \angle \frac{X+jY}{1+X+jY}$$

$$\alpha = \tan^{-1}\left(\frac{Y}{X}\right) - \tan^{-1}\left(\frac{Y}{1+X}\right)$$

Let $N = \tan\alpha$

$$N = \frac{\left(\frac{Y}{X}\right) - \left(\frac{Y}{1+X}\right)}{1 + \frac{Y}{X}\left(\frac{Y}{1+X}\right)} = \frac{Y}{X^2+X+Y^2}$$

$$\Rightarrow X^2+X+Y^2-\frac{1}{N}Y = 0.$$

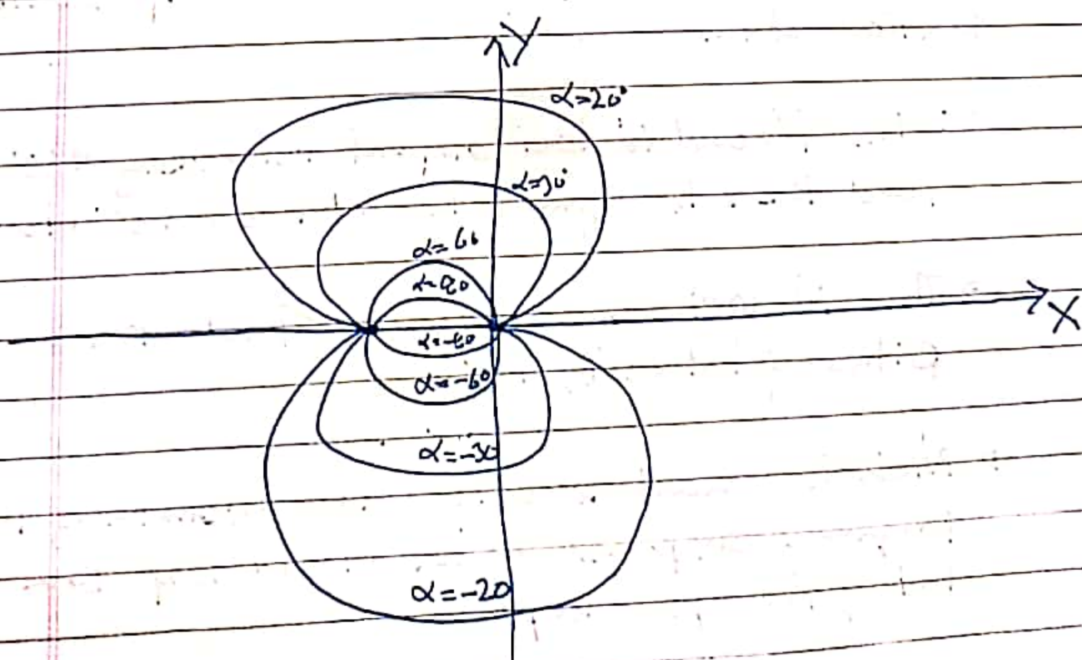
$$\Rightarrow \left(X+\frac{1}{2}\right)^2 + \left(Y-\frac{1}{2N}\right)^2 = \frac{1}{4} + \left(\frac{1}{2N}\right)^2$$

⇒ This is an equation of circle with center at $\left(-\frac{1}{2}, \frac{1}{2N}\right)$ and radius $\sqrt{\frac{1}{4}+\frac{1}{(2N)^2}}$.

⇒ Above equation is satisfied when $X=Y=0$ and $X=-1, Y=0$ regardless of the value of N, each circle passes through the origin and the $-1+0j$ point.

⇒ Constant N locus for a given value of α is not actually the entire circle but only an arc.
{ ie $\alpha = 30$ & $\alpha = -150$ are an part of same circle }

⇒ The use of the M & N circle enables us to find the entire closed-loop frequency response from the open-loop frequency responce $G(j\omega)$ without calculating the magnitude and phase of the closed-loop transfer function at each frequency.



⇒ Graphically, the intersections of the $G(j\omega)$ locus and M circle give the values of M at the frequencies denoted on the $G(j\omega)$ locus.

⇒ The resonant peak value is the value of M corresponding to the M circle of smallest radius that is tangent to the $G(j\omega)$ locus.

★ Nichols Chart

⇒ In dealing with design problems, we find it convenient to construct the M & N loci in the Nichols plot.

↳ The chart is called Nichols chart.

⇒ The $G(j\omega)$ locus drawn on the Nichols chart gives both the gain characteristic & phase characteristics of the closed-loop transfer function at the same time.

→ The M and N loci repeat for every 360°, and there is symmetry at every 180° interval.

→ The M loci are centered about the critical point (0dB, −180°)

→ Nichols chart is useful for determining the frequency response of the closed loop from that of the open loop.

⇒ If the open-loop frequency response curve is superimposed on the Nichols chart, the intersection of the open-loop frequency-response curve $G(j\omega)$ and M and N loci give the value of the magnitude (M) and phase angle α of the closed-loop frequency response at each frequency point.

★ Experimental determination of Transfer Function

⇒ First step in the analysis and design of a Control System is to derive its mathematical model.

↳ Obtaining a model analytically may be quite difficult. We may have to obtain it by means of experimental analysis.

⇒ If the amplitude ratio and phase shift have been measured at a sufficient number of frequencies within the frequency range of interest, they may be plotted on the Bode diagram.

↳ Then the transfer function can be determined by asymptotic approximations.

Sinusoidal signal generators

↳ The Signal may have to be in mechanical, electrical or pneumatic form.

⇒ Frequency range needed:-

- Large time Constant: 0·001 Hz to 10 Hz
- Small time Constant: 0·1 Hz to 1000 Hz

Determination of Minimum-Phase Transfer function from Bode Diagrams

⇒ To determine the transfer function experimentally, we first draw asymptotes to the experimentally obtained log magnitude curve.

→ If the slope of the experimentally obtained log-magnitude curve changes from −20 dB to −40 dB/decade at $\omega = \omega_1$, it is clear that a factor $1/[1 + j\omega/\omega_1]$ exist in the transfer function.

→ If the slope changes by −40 dB/decade at $\omega = \omega_2$, there must be a quadratic factor of the form

$$\frac{1}{1 + 2\xi\left(j\frac{\omega}{\omega_2}\right) + \left(j\frac{\omega}{\omega_2}\right)^2}$$

in the transfer function.

⇒ The undamped natural frequency of this quadratic factor is equal to the corner frequency.

⇒ The damping ratio ξ can be determined from the experimentally obtained log-magnitude curve by measuring the amount of resonant peak near the corner frequency ω_2.

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⇒ Once the factors of the transfer function $G(j\omega)$ have been determined, the gain can be determined from the low-frequency portion of the log-magnitude curve.

↳ Since terms $[1+j(\omega/\omega_1)]$ and $[1+2\xi(j\omega/\omega_2)+(\frac{j\omega}{\omega_2})^2]$ become unity at $\omega \to 0$, the sinusoidal transfer function $G(j\omega)$ can be written as:

$$\lim_{\omega \to 0} G(j\omega) = \frac{K}{(j\omega)^{\lambda}}$$

⇒ In many practical system $\lambda = 1, 2$ or 0.

1. $\lambda = 0$ {Type 0 System}

$$\lim_{\omega \to 0} G(j\omega) = K$$

⇒ Value of K can thus be found from this horizontal asymptote.

2. $\lambda = 1$ {Type 1 System}

$$G(j\omega) = \frac{K}{j\omega}$$

$$20 \log |G(j\omega)| = 20 \log K - 20 \log \omega$$

⇒ The frequency at which the low frequency asymptote intersects the 0-dB line is numerically equal to K.

3. For $\lambda = 2$ {Type 2 System}

$$G(j\omega) = \frac{K}{(j\omega)^2}$$

⇒ The frequency at which this asymptote intersects the 0 dB line is numerically equal to $\sqrt{K}$.

⇒ The experimentally obtained phase-angle curve provides a means of checking the transfer function obtained from the log-magnitude curve.

⇒ If the experimentally obtained phase angle at very high frequencies is not equal to $-90(q-p)$, where p & q are the degrees of the numerator and denominator polynomial of the transfer function, respectively then the transfer function must be a nonminimum phase transfer function.

Nonminimum-Phase transfer function

⇒ If at the high frequency end, the computed phase lag is 180° less than the experimentally obtained phase lag, then one of the zeros of the transfer function should have been in the right-half S plane instead of the left half S plane.

⇒ I
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⇒ If the computed phase lag differed from the experimentally obtained phase lag by a constant rate of change of phase, then transport lag or dead time, is present.

↳ We assume the TF of the form

$$G(s)e^{-TS}$$

where $G(s)$ is a ratio of two polynomials.

5.1 Appendix 1: Steady State Error in Unity Feedback Control System

★ Steady-State Error In Unity feedback Control System

⇒ Any physical Control System inherently suffers steady-state error in response to certain type of input.

↳ A System may have no steady state error to a step input, but the same system may exhibit non zero steady-state error to a ramp input.

Classification of Control System

⇒ Control System may be classified according to their ability to follow step input, ramp input, parabolic inputs, and so on.

↳ The magnitudes of the steady-state errors due to these individual inputs are indicative of the goodness of the system.

⇒ Consider the unity-feed Control System with the following open loop transfer function $G(s)$:-

$$G(s) = \frac{K\,(T_a s+1)(T_b s+1)\cdots(T_m s+1)}{s^N (T_1 s+1)(T_2 s+1)\cdots(T_p s+1)}$$

→ s^N in the denominator, representing a pole of multiplicity N at the Origin.

→ A System is called type 0, type 1, type 2 if $N=0$, $N=1$ & $N=2$ respectively.

⇒ As the type number is increased, accuracy is improved, however increasing the type number aggravates the stability problem.

↳ Compromise between steady state accuracy and relative stability is always necessary.

Static e

Steady State Error

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} \qquad \frac{E(s)}{R(s)} = 1 - \frac{C(s)}{R(s)} = \frac{1}{1+G(s)}$$

⇒ The Final Value theorem provides a convenient way to find the steady state performance of a stable system.

$$E(s) = \frac{1}{1+G(s)} R(s)$$

$$e_{ss} = \boxed{\lim_{t \to \infty} e(t) = \lim_{s \to 0} s E(s)} = \lim_{s \to 0} \frac{s R(s)}{1+G(s)}$$

⇒ In a given system, the output may be the position, velocity, pressure, temperature etc...

⇒ The physical form of output, however is immaterial to the present analysis. Therefore what follows, we shall call the output "position", the rate of change of output "velocity" and so on.

Static position error Constant K_p

$$e_{ss} = \lim_{s\to 0} \frac{s}{1+G(s)} \cdot \frac{1}{s} \quad \{\text{Steady-state error of unit step input}\}$$

$$= \frac{1}{1+G(0)}$$

⇒ The static position error constant K_p is defined by:

$$K_p = \lim_{s\to 0} G(s) = G(0)$$

So $$\boxed{e_{ss} = \frac{1}{1+K_p}}$$

// For a type 0 System

$$K_p = \lim_{s\to 0} \frac{K(T_a s+1)(T_b s+1)\cdots}{(T_1 s+1)(T_2 s+1)\cdots} = K$$

$$\rightarrow e_{ss} = \frac{1}{1+K}$$

// For higher order system

$$K_p = \infty$$

$$\rightarrow e_{ss} = 0$$

Static Velocity Error Constant K_v

⇒ The steady-state error of the system with a unit-ramp input is given by

$$e_{ss} = \lim_{s\to 0} \frac{s}{1+G(s)} \frac{1}{s^2} = \lim_{s\to 0} \frac{1}{sG(s)}$$

So, Static velocity error Constant K_p is defined by

$$K_v = \lim_{s\to 0} sG(s) \implies \boxed{e_{ss} = \frac{1}{K_v}}$$

⇒ The term velocity error is used here to express the steady-state error for a ramp input.

System type	K_v	e_{ss}
0^{th} type	0	∞
1^{st} type	K	$1/K$
2^{nd} type & higher	∞	0

Similarly for Static Acceleration Error Constant K_a

CHAPTER 6

Control System Analysis and Design by Root Locus Methods

4B

Control System Analysis and Design by the Root Locus method

★ Introduction

⇒ The basic characteristics of the transient response of a closed-loop System is closely related to the location of the closed loop poles.

→ If the System has a variable loop gain, then the location of the Closed loop poles depends on the value of the loop gain Chosen.

→ So the designer must Know how the closed loop poles move in the s plane as the loop gain is varied.

If the gain adjustment alone does not yield a desired result, addition of a Compensator to the system will become necessary.

Root-Locus method

→ by W.R Evans

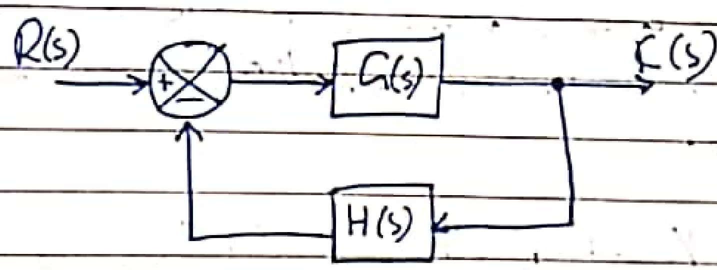
→ Roots of the characteristic equation are plotted for all values of a System Parameter.

⇒ Note: The parameter that varies is usually the gain, but any other variable of the open-loop transfer function may be used.

⇒ Root loci by use of MATLAB is very simple, one may think sketching the root loci by hand is a waste of time and effort. However experience in sketching the root loci by hand is invaluable for interpreting computer generated root loci, as well as for getting a rough idea of the root loci very quickly.

★ Root Locus plot

Angle and Magnitude Condition



$$\frac{R(s)}{C(s)} = \frac{G(s)}{1+G(s)H(s)}$$

⇒ $1+G(s)H(s)$ is the characteristic equation.

⇒ $G(s)H(s) = -1$

⇒ We assume that $G(s)H(s)$ is ratio of polynomial fns.

⇒ Since G(s)H(s) is a complex quantity, the above equation can be split into two equations by equating the angles and magnitudes on both side.

Angle Condition

$$\angle G(s)\,H(s) = \angle -1 = \pm(2K+1)180 \quad \{K = 0, 1, 2 \dots\}$$

Magnitude Condition

$$|G(s)\,H(s)| = 1$$

⇒ The values of s that fulfill both the angle & magnitude conditions are the roots of the characteristic equation, or the closed loop poles.

⇒ In many cases, G(s)H(s) involves a gain K, and the characteristic equation may be written as,

$$1 + \frac{K\,(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)} = 0$$

⇒ Then the root loci for the system are the loci of the closed loop poles as the gain K is varied from zero to infinity.

⇒ <u>Note</u>: To begin sketching the root loci of a system by the root locus method we must know the location of the poles and zeros of G(s) H(s)

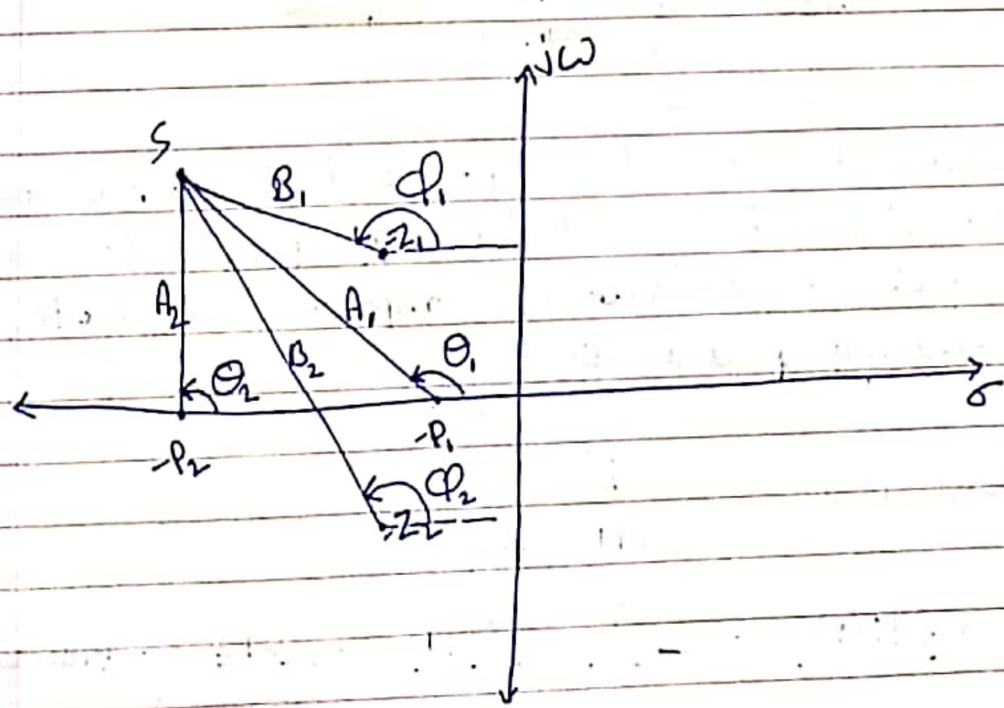
⇒ Angle of the complex quantity originating from the open loop poles and open loop zeros to the test point s are measured in the counterclockwise direction.

Let $G(s)H(s) = \dfrac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$

$$\angle G(s)H(s) = \angle K(s+z_1)(s+z_2)\cdots(s+z_m) - \angle (s+p_1)(s+p_2)\cdots(s+p_n)$$

$$= \left\{\angle s+z_1 + \angle s+z_2 + \cdots + \angle s+z_m\right\} - \left\{\angle s+p_1 + \angle s+p_2 + \cdots \angle s+p_n\right\}$$

$$\Rightarrow \boxed{\angle G(s)H(s) = (\phi_1 + \phi_2 + \cdots \phi_m) - (\theta_1 + \theta_2 + \cdots \theta_n)}$$



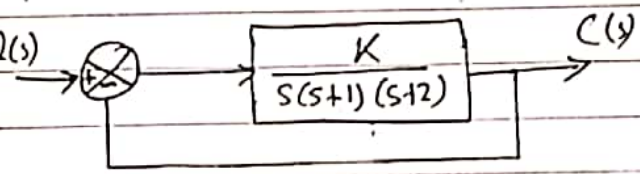
$$|G(s)H(s)| = |K(s+z_1)(s+z_2)\cdots(s+z_m)| \,/\, |(s+P_1)(s+P_2)\cdots(s+P_n)|$$

$$= \frac{K|s+z_1||s+z_2|\cdots|s+z_m|}{|s+P_1||s+P_2|\cdots|s+P_n|}$$

$$\Rightarrow \boxed{|G(s)H(s)| = \frac{KB_1B_2\cdots B_m}{A_1A_2\cdots A_n}}$$

⇒ Open-loop Complex Conjugate poles and Complex Conjugate zeros if any are always located Symmetrically about the real axis, the root locus are always symmetrical with respect to this axis.

Example 6.1:



Sketch the root-locus plot and then determine the value of K such that the damping ratio ξ of a pair of dominant complex conjugate closed loop-poles is 0.5.

$$\Rightarrow G(s)H(s) = \frac{K}{s(s+1)(s+2)}$$

$$\angle G(s)H(s) = -\angle s - \angle s+1 - \angle s+2 = \pm 180(2k+1)$$

$$\{k = 0, 1, 2 \cdots\}$$

$$\left|\frac{K}{S(s+1)(s+2)}\right| = 1$$

// A typical procedure for sketching the root-locus Plot is as follows.

1. <u>Determine the root locus on the real axis.</u>

<u>Note</u>

→ The location of open loop poles are indicated by crosses.

→ The location of open loop zeros are indicated by small circle.

⇒ Starting point of the root loci are open loop poles. (the point corresponding to ~~open loop poles~~ $K=0$.)

⇒ The number of individual root loci for this system is three, which is the same as the number of open loop poles.

⇒ To determine the root loci on the real axis, we Select a test points, s.

If the test point is on the positive real axis then,

$$\angle s = \angle s+1 = \angle s+2 = 0$$

→ Angle Condition <u>cannot be satisfied.</u>

If test point is between $\boxed{0 \ \& -1}$:

$\angle s = 180 \quad \angle(s+1) = \angle s+2 = 0$

→ The angle condition is satisfied.

If test point is between -1 & -2

$\angle s = \angle s+1 = 180 \ ; \ \angle s+2 = 0$

→ The angle condition is not satisfied.

If test point is between $\boxed{-2 \text{ and } -\infty}$

$\angle s = \angle s+1 = \angle s+2 = 180$

→ The angle condition is satisfied.

2. Determine the asymptotes of the root loci.

The asymptotes of the root loci as s approaches infinity can be determined as follow

$$\lim_{s\to\infty} G(s)H(s) = \lim_{s\to\infty} \frac{K}{s(s+1)(s+2)} = \lim_{s\to\infty} \frac{K}{s^3}$$

Angle condition: $-3\angle s = \pm 180^\circ (2K+1)$.

$$\Rightarrow \boxed{\angle s = \pm 60^\circ (2K+1) \quad (K = 0, 1, 2 \cdots)}$$

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⇒ Since the angles repeats itself as K is varied, the distinct angles for the asymptotes are determined as 60°, −60° and 180°.

↳ Thus there are three asymptotes.

⇒ Before we can draw these asymptotes in the Complex plane, we must find the point where they intersect the real axis.

⇒ As $s \to \infty$ ⇒ $\angle s \cong \angle s+1 \cong \angle s+2$

So lets assume the asymptote to be at the avg of these three points.

$$\frac{(-2)+(-1)+0}{3} = -1$$

→ Thus the abscissa of the intersection of the asymptote and the real axis is −1.

3. <u>Determining the breakaway points</u>

⇒ Breakaway points are points where the root-locus branches originating from the poles at 0 and −1 break away from the real axis and move into the complex plane.

?

Let us write the characteristic equation as :-

$$f(s) = B(s) + KA(s) = 0$$

{A(s) & B(s) dose not contain K}

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$f(s) = 0$ has multiple roots at point
where,

$$\frac{df(s)}{ds} = 0$$

"The breakaway point corresponds to a point in s-plane where multiple roots of the characteristic equation occur"

Suppose that $f(s)$ has multiple roots of order r, where $r \geq 2$.

Then, $f(s) = (s-s_1)^r (s-s_2) \cdots (s-s_n)$

$$\left.\frac{df(s)}{ds}\right|_{s=s_1} = 0$$

$$\frac{df(s)}{ds} = B'(s) + K A'(s) = 0$$

$$\Rightarrow K = -\frac{B'(s)}{A'(s)}$$

So, $f(s) = B(s) - \frac{B'(s)}{A'(s)} A(s) = 0$

$$\boxed{B(s) A'(s) - B'(s) A(s) = 0}$$

If equation above is solved for s, the points where multiple roots occur can be obtained.

$$K = -\frac{B(s)}{A(s)}$$

$$\frac{dK}{ds} = \frac{B'(s)A(s) - B(s)A'(s)}{A^2(s)}$$

If dK/ds is set equal to zero, we get the same equation.

⇒ Therefore, the break-away points can be simply determined from the roots of

$$\boxed{\frac{dK}{ds} = 0}$$

If at a point at which $dK/ds = 0$ the value of K takes a real positive value, then the point is an actual breakaway point.

$$1 + \frac{K}{s(s+1)(s+2)} = 0$$

$$K = -(s^3 + 3s^2 + 2s)$$

$$\frac{dK}{ds} = -(3s^2 + 6s + 2) = 0$$

$$\Rightarrow \quad s = -0.4226, \quad s = -1.5774$$

⇒ Since the breakaway point must lie on a root locus between 0 and -1, it is clear that $S = -0.4226$ corresponds to the actual breakaway point.

$$S = -0.4226 \Rightarrow K = 0.3849$$

4. Determine the points where the root loci cross the imaginary axis.

⇒ These point can be found out by using Routh's Stability Criterion.

$$S^3 + 3S^2 + 2S + K = 0$$

S^3	1	2
S^2	3	K
S^1	$(6-K)/3$	0
S^0	K	

⇒ The value of K that makes S^1 term in the first column zero is 6

⇒ The crossing points on the imaginary axis can then be found by solving the auxilary equation obtained from the S^2 row;

$$3S^2 + K = 3S^2 + 6 = 0$$

$$S = \pm\sqrt{2}j$$

So $\omega = \pm\sqrt{2}$ for $K=6$.

$\Rightarrow$ An alternate approach is to let $s = j\omega$ in the characteristic equation, equate both real and imaginary part to zero, and then solve for ω and K.

$$(j\omega)^3 + 3(j\omega)^2 + 2(j\omega) + K = 0$$

$$\Rightarrow (K - 3\omega^2) + j(2\omega - \omega^3) = 0$$

$$K - 3\omega^2 = 0 \qquad 2\omega - \omega^3 = 0$$

So $\omega = \pm\sqrt{2} \quad \forall\ K = 6$

$\omega = 0 \quad \forall\ K_1 = 0$

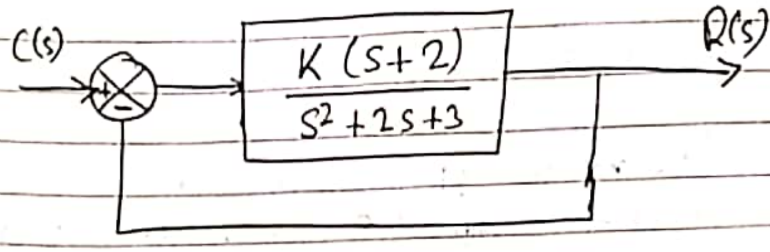
5. Choose a test point in the broad neighborhood of the $j\omega$ axis and the origin

6. Draw the root loci based on the information obtained in the foregoing steps.

7. Determine a pair of dominant complex-conjugate closed-loop poles such that the damping ratio ξ is 0.5.

$\Rightarrow$ Closed-loop poles with $\xi = 0.5$ lie on the lines passing through the origin and making the angle $\pm\cos^{-1}\xi = \pm\cos^{-1}0.5 = \pm 60°$ with the negative real axis.

Example 6.2:



$$G(s) = \frac{K(s+2)}{s^2+2s+3} \qquad H(s) = 1$$

$$G(s)H(s) = K\frac{s+2}{s^2+2s+3}$$

$$\boxed{1 + K\frac{s+2}{s^2+2s+3} = 0} \quad \{\text{Characteristic equation}\}$$

Poles: $s = -1 \pm j\sqrt{2}$

Zeros: $s = -2$

1. Determine the root loci on the real axis.

$\# \ \forall s \in (-1 \ \infty)$

$$\phi_1 + \phi_2 = 0 \quad \& \quad \theta = 0$$

$$-(\phi_1 + \phi_2) + \theta = 0$$

hence not possible.

$\# \ \forall s \in (-2 \ -1)$

$$\phi_1 + \phi_2 = 0 \quad \& \quad \phi = 0$$

$$\phi - (\theta_1 + \theta_2) = 0$$

hence not possible.

$\forall\ S \in (-\infty, -2)$

$\theta_1 + \theta_2 = 0 \quad \phi = 180$

$\phi - (\theta_1 + \theta_2) = 180$

Hence possible.

$\lim_{s \to \infty} G(s)H(s) = \dfrac{K}{S}$

Angle Condition.

$\angle S - 2\angle S = \pm 180\,(2K+1)$

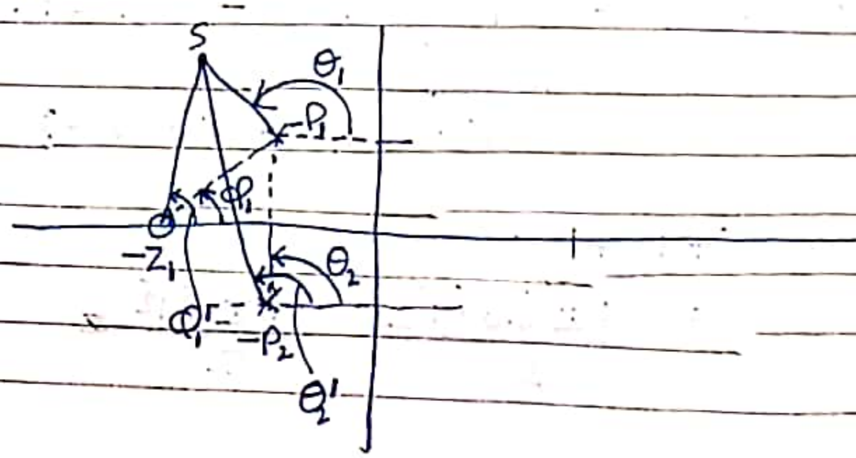
$\{K = 0, 1, 2 \cdots\}$

$\Rightarrow \angle S = \pm 180\,(2K+1)$

Asymptote at $\theta = -180°$

2. Determine the angle of departure from the complex-conjugate open-loop poles.

$\Rightarrow$ The presence of pair of complex conjugate open loop poles requires the determination of the angle of departure from these poles.



⇒ If we choose a test point and move it in the very vicinity of the complex open-loop pole at $s=-P_1$:

↳ We find that the sum of the angular contributions from the pole at $s=-P_2$ and zero at $s=-z_1$ to the test point can be considered remaining the same

$$\varphi_1' - (\theta_1 + \theta_2') = \pm 180(2K+1)$$

$$\theta_1 = 180 - \theta_2' + \varphi_1' = 180 - \theta_2 + \varphi_1$$

$$\theta_1 = 180 \underbrace{- 90 + 55}_{-35} = 145°.$$

⇒ Since the root locus is symmetric about the real axis, the angle of departure from the pole at $s=-P_2$ is $-145°$.

3. <u>Determine the break in point</u>

$$K = -\frac{s^2+2s+3}{s+2}$$

$$\frac{dK}{ds} = 0 = (s+2)(2s+2) - (s^2+2s+3)$$

$$2s^2+6s+4 - s^2 - 2s - 3$$

$$\Rightarrow s^2+4s+1 = 0$$

$$s = \frac{-4 \pm \sqrt{16-4}}{2} = \frac{-4 \pm \sqrt{12}}{2}$$

$S = -3.732 \;\; \forall \;\; K = 5.4641$

★ General Rules for Constructing Root Loci

First obtain the characteristic equation

$$1 + G(s)H(s) = 0$$

Then rearrange this equation so that the parameter of interest appears as the multiplying factor in the form:-

$$1 + K\frac{(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)} = 0$$

In present discussion, we assume that the parameter of interest is the gain K, where $K>0$. (If $K<0$, which corresponds to the positive-feedback case).

1. Locate the poles and zeros of $G(s)H(s)$ on the s plane. The root-locus branches start from open-loop poles and terminate at zeros (finite zeros or zeros at infinity).

Note that the root loci are symmetrical about the real axis of the s plane, because the complex poles and complex zeros occur only in conjugate pairs.

2. Determine the root loci on the real axis.

Root loci on the real axis are determined by open-loop poles & zeros lying on it.

The complex-conjugate poles & zeros of the open loop transfer function have no effect on the location of the root loci on the real axis because the angle contribution of a pair of complex-conjugate poles or complex-conjugate zeros is 360° on the real axis.

3. Determine the asymptotes of root loci

If the test point s is located far from the origin, then the angle of each complex quantity may be considered same.

→ Open loop poles & zeros cancel each other effect.

$$\text{Angle of asymptotes} = \frac{\pm 180(2K+1)}{n-m} \quad (K=0,1,2\ldots)$$

n = Number of finite poles of $G(s)H(s)$

m = number of finite zeros of $G(s)H(s)$

⇒ All the asymptotes intersect at a point on the real axis.

$$G(s)H(s) = \frac{K[s^m + (z_1+z_2+z_3+\cdots+z_m)s^{m-1} + \cdots + z_1 z_2 \cdots z_m]}{s^n + (P_1+P_2+\cdots P_n)s^{n-1} + \cdots + P_1 P_2 \cdots P_n}$$

⇒ If a test point is located very far from the origin, then by dividing the denominator by the numerator, it is possible to write $G(s)H(s)$ as

$$G(s)H(s) = \frac{K}{s^{n-m} + [(P_1+P_2+\cdots P_n)-(z_1+z_2+\cdots z_m)]s^{n-m-1} + \cdots}$$

$$= \frac{K}{\left[s + \frac{(P_1+P_2+\cdots P_n)-(z_1+z_2+\cdots+z_m)}{n-m}\right]^{n-m}}$$

⇒ The abscissa of the intersection of the asymptote and the real axis is then obtained by setting the denominator of the right-hand side of above equation equal to zero and solving for s.

$$s = -\frac{(P_1+P_2+\cdots+P_n)-(z_1+z_2+\cdots+z_m)}{n-m}$$

4. Find the breakaway and break in point

⇒ Breakaway and break in points can be determined from the roots of

$$\frac{dK}{ds} = 0$$

⇒ The breakaway points and break in points must be the roots of above equation, but not all roots are breakaway or breakin point.

⇒ If the two roots $s = s_1$ & $s = -s_1$ of above equation are a complex-conjugate pair and if it is not certain whether they are on root loci, then it is necessary to check the corresponding K value.

5. Determine the angle of departure (angle of arrival) of the root locus from a complex pole (at a complex zero)

6. Find the points where the root loci may cross the imaginary axis.

7. Taking a series of test points in the broad neighborhood of the origin of the s plane sketch the root loci.

8. Determine closed loop-poles.

★ Cancellation of poles of G(s) with Zeros of H(s)

⇒ If the denominator of G(s) and the numerator of H(s) involve common factors, then the corresponding open-loop poles & zeros will cancel each other, reducing the degree of the characteristic equation by one or more.

↳ The root-locus plot of G(s) H(s) does not show all the roots of the characteristic equation, only the roots of the reduced equation.

★ Constant ξ Loci and Constant ω_n Loci

$$\xi = \cos\phi$$

lines of constant damping ratio ξ are radial lines passing through the origin.

Distance of the pole from the origin is determined by undamped natural frequency ω_n. The constant ω_n loci are circles.

CHAPTER 7

Root Locus Approach to Control System Design

5

Compensator Design

Root-Locus Approach | Frequency-Response Approach

5A

Root Locus Approach to Control System design

* **Preliminary design consideration**

⇒ In practice root-locus plot of a system may indicate that the desired performance cannot be achieved just by the adjustment of gain (or some other adjustable parameter).

→ Then it is necessary to reshape the rootlocus to meet the performance Specifications.

→ The design problems, therefore, become those of improving system performance by insertion of a Compensator.

The design problems, therefore become those of improving system performance by insertion of a compensator. Compensation of a control System is reduced to the design of a filter whose characteristics tends to Compensate

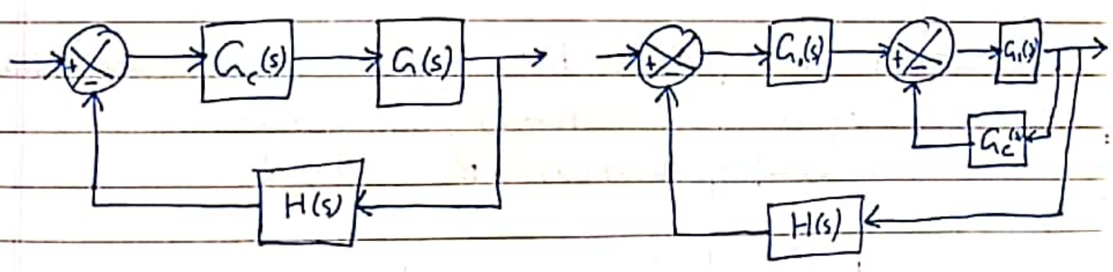
for the undesirable and unalterable characteristics of the plant.

★ Design by Root-Locus Method

"The design by the root locus method is based on reshaping the root locus of the system by adding poles and zeros to the system's open loop transfer function and forcing the root loci to pass through desired closed-loop poles in the s-plane"

⇒ The characteristic of the root locus design is its being based on the assumption that the closed loop system has a pair of dominant closed-loop poles.

★ Series Compensation and Parallel (or Feedback) Compensation



Series Compensation

Parallel Compensation

$G_c(s)$ is the compensator.

⇒ In general, Series compensation may be simpler than parallel compensation; however, Series compensation frequently requires additional amplifiers to increase the gain and/or to provide isolation.

★ <u>Commonly used Compensators</u>

<u>Lead network</u> ⇒ Output lead the Input

<u>Lag network</u> ⇒ Output lag behind the output.

<u>Lead-Lag network</u> ⇒ Output both lead and lag the input depending on frequency.

↳ Phase lag occurs at low frequency region and phase lead occurs at high frequency region.

⇒ A Compensation having a characteristics of a lead network, Lag network or lead-lag network is called :-

(i) Lead Compensator

(ii) Lag Compensator

(iii) Lag-Lead Compensator

Velocity feedback Compensator

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★ Effect of the addition of poles (Adding Integral Control)

⇒ The addition of a pole to the openloop transfer function has the effect of pulling the root locus to the right, tending to lower the System's relative stability and to slow down the Settling of the response.

★ Effect of addition of Zeros (Adding derivative Control)

⇒ The addition of a zero to the openloop transfer function has the effect of pulling the root locus to the left, tending to make the System more Stable and to Speed up the Settling of the response.

★ Lead Compensation

⇒ In carrying out a control-system design, we place a compensator in series with the unalterable transfer function G(s) to obtain desirable behavior.

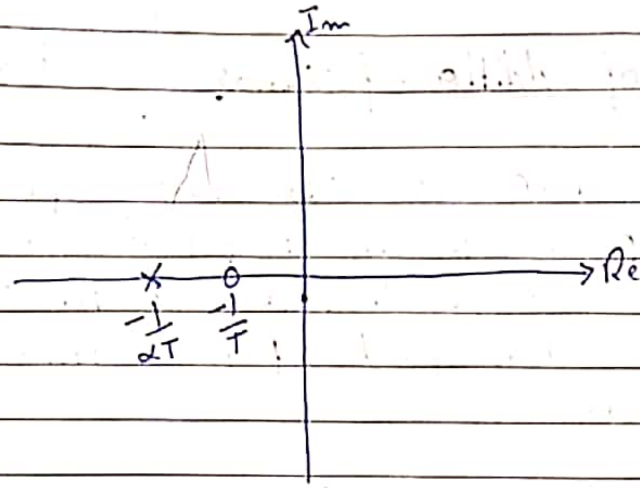
→ The main problem then involves the judicious choice of the poles & zeros of Compensator $G_c(s)$ to ~~obtain~~ have the dominant closed-loop poles at the desired location in s plane.

Lead & Lag Compensator (Reff M2 end)

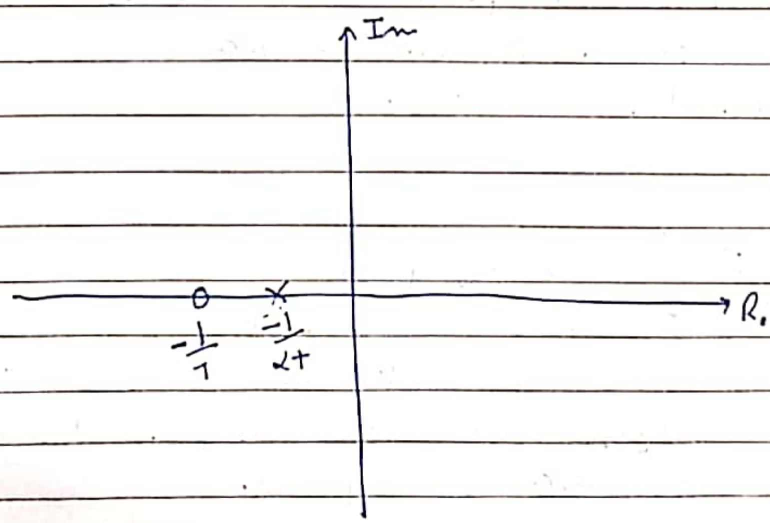
$$\frac{E_o(s)}{E_i(s)} = K_c \frac{s+\frac{1}{T}}{s+\frac{1}{\alpha T}}$$

Lead network $\Rightarrow$ $\alpha < \phi$

Lag network $\Rightarrow$ $\alpha > \phi$



Lead network

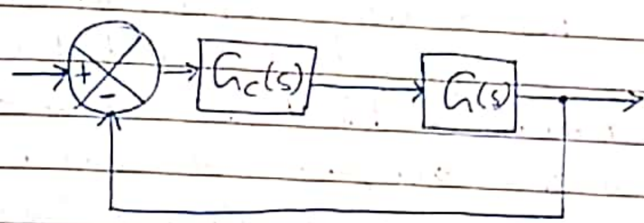


Lag network

Lead Compensation Technique based on the Root Locus Approach

⇒ Root locus approach to design is very powerful when the specifications are given in terms of time-domain quantities, such as the damping ratio and undamped natural frequency of the desired dominant closed-loop poles, maximum overshoot, rise time & settling time.

The procedure for designing a lead compensator for the system by the root locus method may be stated as follows:-



1. From the performance specifications, determine the desired location for the dominant closed loop poles.

2. By drawing the root-locus plot of the uncompensated system ascertain whether or not the gain adjustment done can yield the desired closed-loop poles. If not calculate the angular deficiency ϕ.

↳ The angle must be contributed by the lead compensator if the new root locus is to pass through the desired locations for the dominant closed-loop poles.

3. Assume the lead Compensator $G_c(s)$ to be

$$G_c(s) = K_c \alpha \frac{Ts+1}{\alpha Ts+1} \quad (0 < \alpha < 1)$$

⇒ α & T are determined from angle deficiency.

⇒ K_c is determined from the requirement of the open loop gain.

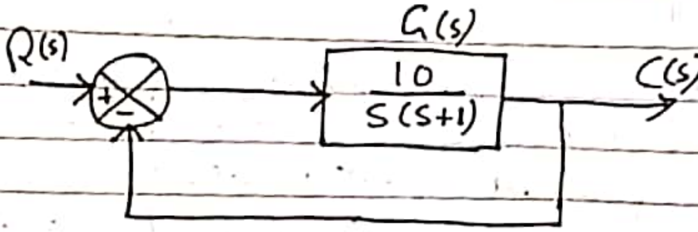
4. If static error constants are not specified, determine the location of the poles and zero of the lead compensator so that the lead compensator will contribute the necessary angle ϕ. If no other requirements are imposed on the system, try to make the value of α as large as possible.

↳ A larger value of α generally results in a large value of K_v, which is desirable.

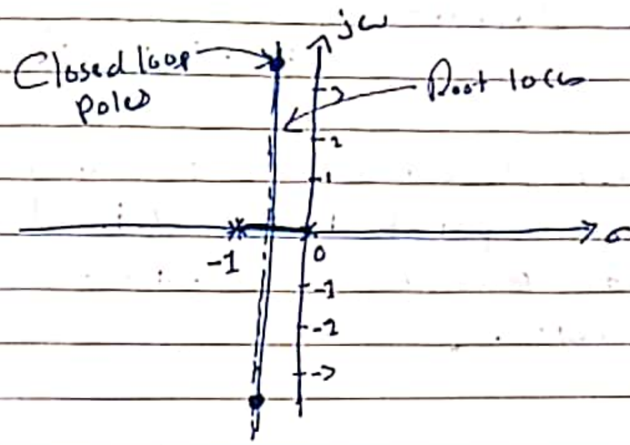
$$K_v = \lim_{s \to 0} s G_c(s) G(s) = K_c \alpha \lim_{s \to 0} s G_0(s)$$

5. Determine the value of K_c of the lead compensator from the magnitude condition.

Example 6.6:



$$\frac{C(s)}{R(s)} = \frac{10}{s^2+s+10} = \frac{10}{(s+0.5+j3.1225)(s+0.5-j3.1225)}$$

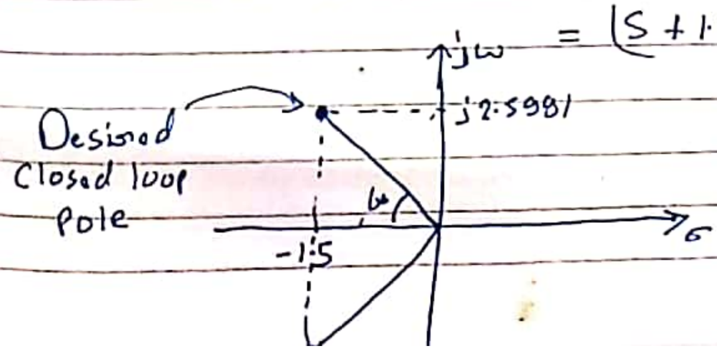


$\xi = \frac{1}{2\sqrt{10}} = 0.158$ {Present Values}
$\omega_n = \sqrt{10} = 3.162$ rad/sec

$\xi = 0.5$ {desired Values}
$\omega_n = 3$

⇒ The desired location of the dominant closed loop poles can be determined from

$$s^2 + 2\xi\omega_n s + \omega_n^2 = s^2 + 3s + 9$$
$$= (s+1.5+j2.5981)(s+1.5-j2.5981)$$

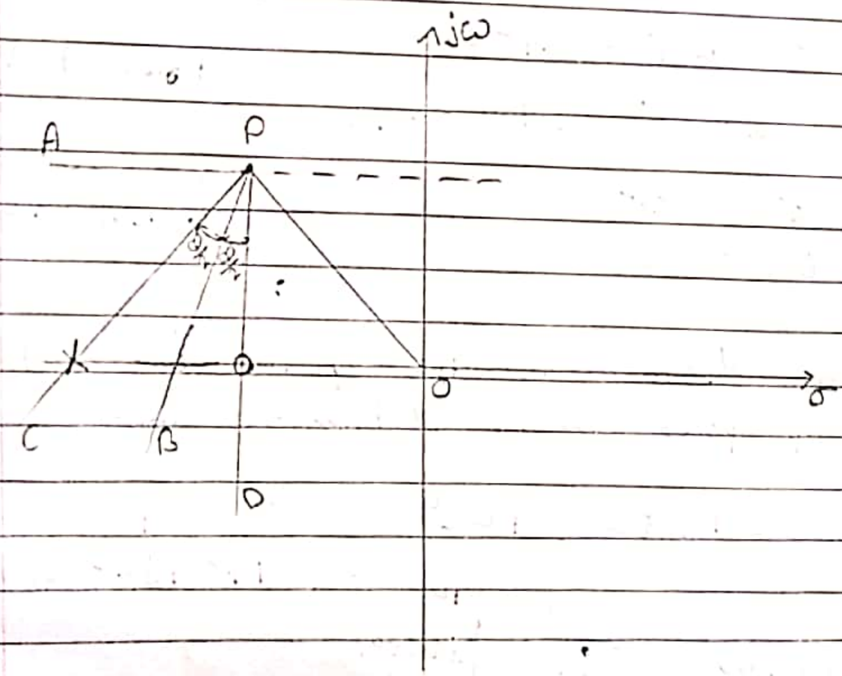


⇒ The angle from the pole at the origin to the desired dominant closed-loop pole at $s=-1.5+j2.5981$ is $120°$. The angle from the pole at $s=-1$ to the desired closed loop pole is 100.894.

Hence, Angle of deficiency $= 180-120-100.894$

$= -40.894°$

<u>Method 1:</u> # First draw horizontal line passing through point P, the desired location for one of the dominant closed-loop poles.



Draw also a line connecting point P & the origin.

Bisect angle between line PA & PO by PB.

Draw two line PC & PD that makes angle $\pm \phi/2$ with the bisector PB.

The intersections of PC and PD with the negative real axis give the necessary locations for the poles & zero of the lead network.

⇒ The Compensator thus designed will make point P a point on the root locus of the Compensated System.

⇒ The locations of the zeros and poles are found as follows.

Zero at $s = -1.9432$

Pole at $s = -4.6458$

$$G_c(s) = K_c \frac{s + \frac{1}{T}}{s + \frac{1}{\alpha T}} = K_c \frac{s + 1.9432}{s + 4.6458}$$

$$\left.\begin{aligned} \alpha &= 0.418 \\ T &= 0.5146 \end{aligned}\right\}$$

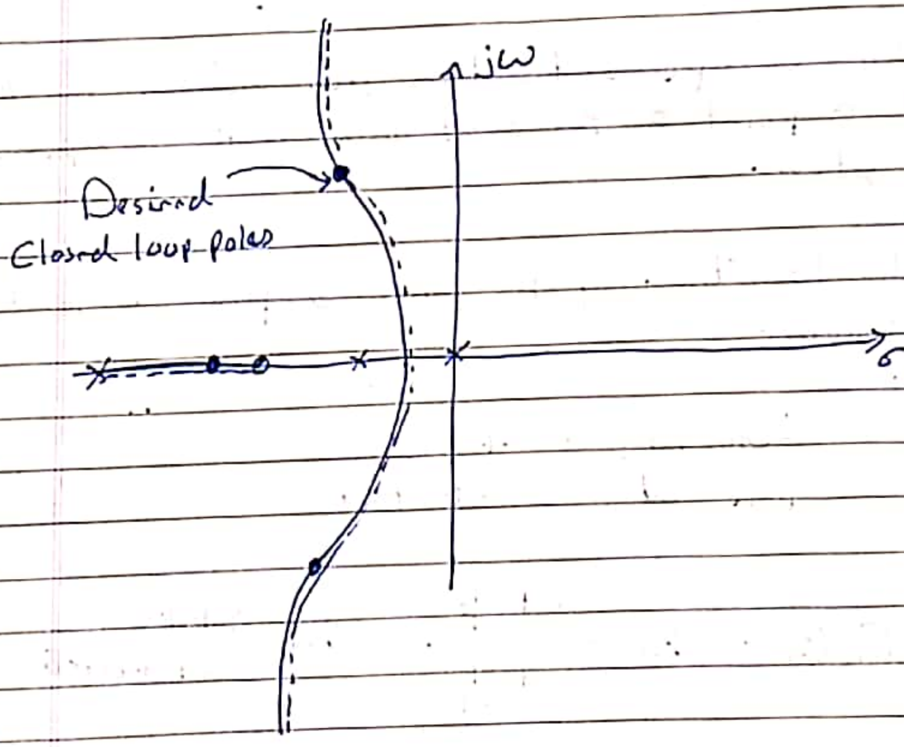
⇒ The Value of K_c can be determined by use of the magnitude Condition ÷

$$\left| K_c \frac{s+1.9432}{s+4.6458} \times \frac{10}{s(s+1)} \right|_{s=-1.5+j2.5981} = 1$$

$$K_c = 1.2287$$

Hence lead compensator $G_c(s)$ is given by :-

$$G_c(s) = 1.2287 \, \frac{s+1.9432}{s+4.6458}$$



⇒ It is worthwhile to check the static velocity error constant K_v for the system just designed.

$$K_v = \lim_{s \to 0} s\,G_c(s)\,G(s) = 5.139$$

Method 2: If we choose the zero of the lead Compensator at $s=-1$, so that it will cancel the plant pole at $s=-1$, then the Compensator pole must be located at $s=-3$.

$$G_c(s) = K_c \frac{s+1}{s+3}$$

$$\left| K_c \frac{s+1}{s+3} \frac{10}{s(s+1)} \right| = 1 \Rightarrow K_c = 0.9$$

$$\Rightarrow \boxed{G_c = 0.9 \frac{s+1}{s+3}}$$

⇒ The static velocity error constant for the present case is obtained as follows.

$$K_v = \lim_{s \to 0} s G_c(s) G(s) = 3$$

\# For different combination of a zero and pole of the Compensator that Contributes $40.894°$, the value of K_v will be different.

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* Lag Compensation

$$\frac{E_o(s)}{E_i(s)} = \hat{K}_c \frac{s+\frac{1}{T}}{s+\frac{1}{\beta T}} \quad \{\beta > 1\}$$

⇒ Consider the problem of finding a suitable compensation network for the case where the system exhibits satisfactory transient response characteristics but unsatisfactory steady-state characteristics.

⇒ Compensation in this case essentially consists of increasing the open loop gain without appreciably changing the transient response characteristic.

↳ This can be accomplished by Lag Compensator.

⇒ To avoid an appreciable change in the root loci, the angle contribution of the lag network should be limited to a small amount, say less than 5°.

↳ To assure this, we place the pole & zero of the lag network relatively close together and near the origin of s plane.

↳ Then the closed-loop poles of the compensated system will be shifted only slightly from original location.

↳ Hence the transient-response characteristics will be changed only slightly.

⇒ If we place the zero & pole of the lag Compensator very close to each other then at $s = s_1$ (one of the dominant closed-loop poles)

$$s_1 + \left(\frac{1}{T}\right) \approx s_1 + \frac{1}{\beta T}$$

So, $$|G_c(s_1)| = \left| \hat{K}_c \frac{s_1 + \frac{1}{T}}{s_1 + \frac{1}{\beta T}} \right| = \hat{K}_c$$

⇒ To make the angle Contribution of the Lag portion of the Compensator small, we require ÷

$$-5^\circ < \angle \frac{s_1 + \frac{1}{T}}{s_1 + \frac{1}{\beta T}} < 0$$

⇒ If gain $\hat{K}_c$ of Lag Compensator is set equal to 1, the alteration in the transient-response Characteristics will be very small.

⇒ The Static velocity error Constant $\hat{K}_v$ ÷

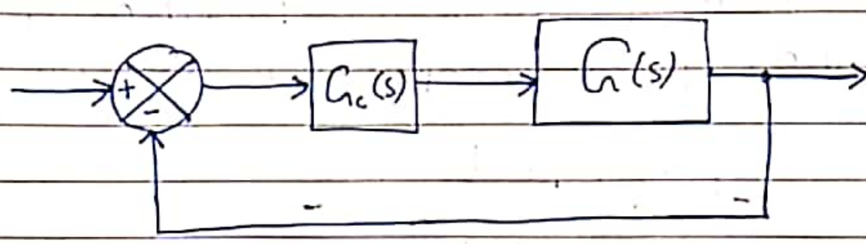
$$\hat{K}_v = \lim_{s \to 0} s G_c(s) G(s) = \lim_{s \to 0} G_c(s) K_v = \beta K_v \hat{K}_v$$

⇒ β should be chosen as high as possible.

⇒ The main negative effect of the lag Compensation is that the Compensator zero that will be generated near the origin creates a closed loop pole near the origin.

↳ This closed loop pole & Compensator zero will generate a long tail of small amplitude in the step response, thus increasing the settling time.

Design procedures for lag Compensation by root Locus Method



⇒ We assume that the uncompensated system meets the transient-response specification by simple gain adjustment.

1. Draw the root-locus plot for the uncompensated system whose open-loop transfer function is $G(s)$. Based on the transient-response specifications, locate the dominant closed loop poles on the root locus.

2. Assume the transfer function of the lag compensator to be given by :-

$$G_c(s) = \hat{K}_c \beta \frac{Ts+1}{\beta Ts+1} = \hat{K}_c \frac{s+\frac{1}{T}}{s+\frac{1}{\beta T}}$$

→ Then the openloop transfer function of the compensated system becomes: $G_c(s)\,\hat{G}(s)$.

3. Evaluate the particular static error constant specified in the problem.

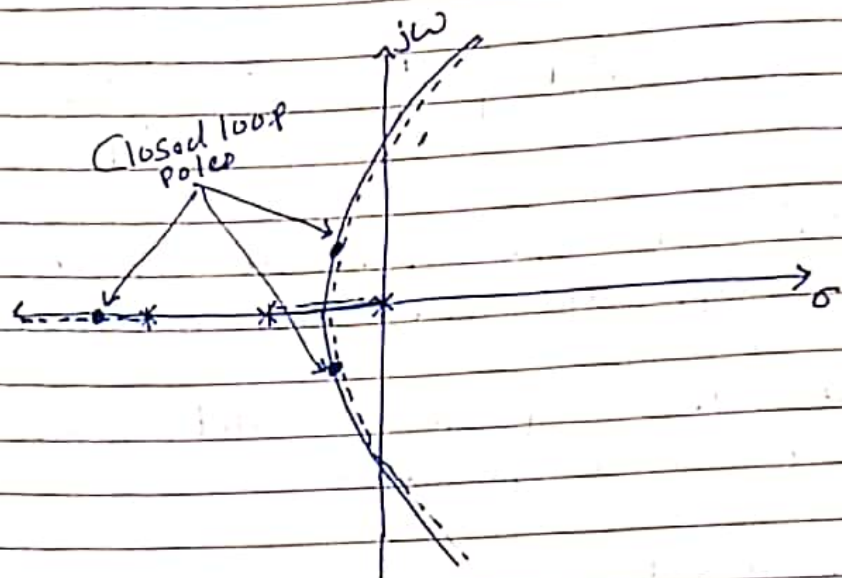
4. Determine the amount of increase in the static error constant necessary to satisfy the specification.

5. Determine the pole and zero of the lag compensator that produce the necessary increase in the particular static error constant without appreciably altering the original root loci.

6. Draw a new root locus plot for the compensated system. Locate the desired dominant closed-loop poles on the root locus.

7. Adjust gain $\hat{K}_c$ of the compensator from the magnitude condition so that the dominant closed-loop poles lie at the desired location.

Example 6.7: $G(s) = \frac{1.06}{s(s+1)(s+2)}$



⇒ The Closed loop transfer function becomes:-

$$\frac{C(s)}{R(s)} = \frac{1.06}{s(s+1)(s+2) + 1.06}$$

$$= \frac{1.06}{(s+0.3307-j0.5864)(s+0.3307+j0.5864)(s+2.3386)}$$

⇒ The dominant, closed-loop poles are

$$s = -0.3307 \pm j0.5864$$

$\xi = 0.491$

$\omega_n = 0.673$ rad/sec

$K_v = 0.53$ 1/sec

⇒ It is desired to increase the static velocity error constant K_v to about $5\ sec^{-1}$ without appreciably changing the location of the dominant closed-loop poles.

⇒ To increase the static velocity error constant by a factor of about 10, let us choose $\beta = 10$ and place the zero and pole of the lag compensator at $s = -0.05$ and $s = -0.005$ respectively.

↳ The transfer function of the lag compensator becomes

$$G_c(s) = \hat{K}_c \frac{s+0.05}{s+0.005}$$

⇒ The angle contribution of this lag network near a dominant closed-loop pole is about 4°.

⇒ The open-loop transfer function of the compensated system then become.

$$G_c(s)G(s) = \frac{K(s+0.05)}{s(s+0.005)(s+1)(s+2)} \quad \{K = 1.06\hat{K}_c\}$$

⇒ If the damping ratio of the new dominant closed loop poles is kept the same, then these poles are obtained from the new root locus plot as follows.

$$s = -0.31 \pm j0.55$$

⇒ The open loop gain K is determined by magnitude condition :-

$$K = \left| \frac{s(s+0.005)(s+1)(s+2)}{(s+0.005)} \right|_{s=-0.31+j0.55}$$

$$K = 1.0235$$

$$\hat{K}_c = \frac{K}{1.06} = 0.9656$$

So $G_c(s) = 0.9656 \dfrac{s+0.05}{s+0.005}$

⇒ The static velocity error constant K_v is :-

$$K_v = \lim_{s \to 0} s\,G_c(s)\,G(s) = 5.12\ \text{sec}^{-1}$$

⇒ The steady-state error with ramp input has decreased to about 10% of the original system.

* Lead-Lag Compensation

Lead Compensation basically speeds up the response and increases the stability of the system.

Lag Compensation improves the steady-state accuracy of the system, but reduces the speed of the response.

⇒ If improvements in both transient response and steady state response are desired, then both a lead compensator and a lag compensator may be used simultaneously.

↳ Rather than introducing both a lead compensator and a lag compensator as separate units, how-ever it is economical to use a single lead-lag compensator.

$$\frac{E_o(s)}{E_i(s)} = K_c \frac{\left(s+\frac{1}{T_1}\right)\left(s+\frac{1}{T_2}\right)}{\left(s+\frac{\gamma}{T_1}\right)\left(s+\frac{1}{\beta T_2}\right)} \quad \{\beta > 1 \quad \gamma > 1\}$$

LEAD → LAG

⇒ In designing lag-lead compensators, we consider two cases where $\gamma \neq \beta$ & $\gamma = \beta$.

Case 1: ($\gamma \neq \beta$) The design process is combination of the design of the lead compensator and that of the lag compensator.

1. From the given performance specifications, determine the desired location of the dominant closed loop poles.

2. Using the uncompensated open loop transfer function $G(s)$, determine the angle deficiency ϕ if the dominant closed-loop poles are to be at the desired location. The phase-lead portion of the lag-lead compensator must contribute this angle ϕ.

3. Assume that we later choose T_2 sufficiently large so that the magnitude of the lag portion is approximately unity, where $s = s_1$ is one of the dominant closed-loop poles, choose the values of T_1 & γ from the requirement that

$$\angle \frac{s_1 + \frac{1}{T_1}}{s_1 + \frac{\gamma}{T_1}} = \phi$$

The choice of T_1 & γ is not unique. Then determine the value of K_c from the magnitude condition.

$$\left| K_c \frac{s_1 + \frac{1}{T_1}}{s_1 + \frac{\gamma}{T_1}} G(s_1) \right| = 1$$

4. If the Static Velocity error Constant K_v is specified, determine the value of B to Satisfy the requirement for K_v.

$$K_v = \lim_{s\to 0} s\, G_c(s)\, G(s)$$

$$K_v = \lim_{s\to 0} s K_c \frac{B}{\gamma} G(s)$$

Hence given the value of K_v, the value of B Can be determined. Then, using the value of B thus determined, choose the value of T_2 such that.

$$\left|\frac{s_1+\frac{1}{T_2}}{s_1+\frac{1}{BT_2}}\right| = 1 \qquad -5^\circ < \angle \frac{s+\frac{1}{T_2}}{s+\frac{1}{BT_2}} < 0^\circ$$

Case 2: ($\gamma = B$)

1. From the given performance specification, determine the desired location for the dominant closed-loop poles.

2. If Static velocity error Constant K_v is specified, determine the value of Constant K_c from the following equation:-

$$K_v = \lim_{s\to 0} s\, G_c(s)\, G(s)$$

$$= \lim_{s\to 0} s K_c\, G(s)$$

3

3. To have the dominant closed-loop poles at the desired location, calculate the angle contribution ϕ needed from the phase lead portion of the lag lead compensator.

4. For the lag-lead compensator we later choose T_2 sufficiently large so that

$$\left| \frac{s + \frac{1}{T_2}}{s_1 + \frac{1}{\beta T_2}} \right|$$

is approximately unity, where $s = s_1$ is one of the dominant closed loop poles. Determine the values of T_1 & β from the magnitude and angle condition.

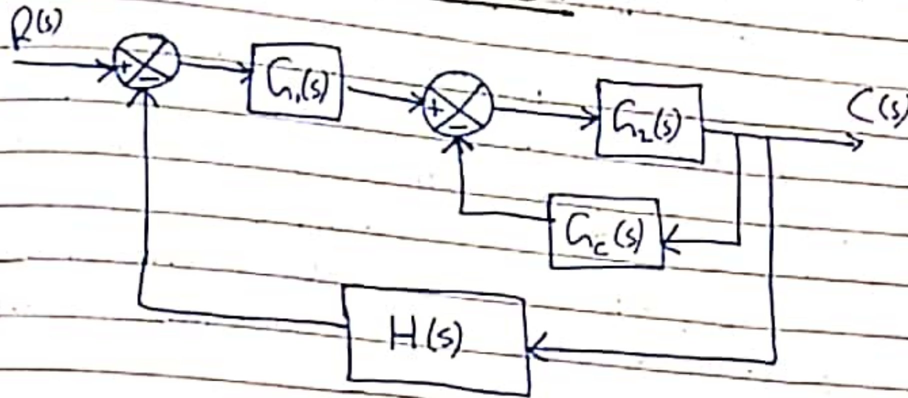
$$\left| K \left(\frac{s_1 + \frac{1}{T_1}}{s_1 + \frac{\beta}{T_1}} \right) G(s_1) \right| = 1$$

$$\angle \frac{s_1 + \frac{1}{T_1}}{s_1 + \frac{\beta}{T_1}} = \phi$$

5. Usualy the value of β just determined, choose T_2 so that

$$\left| \frac{s_1 + \frac{1}{T_2}}{s_1 + \frac{1}{\beta T_2}} \right| \doteq 1 \qquad -5^\circ < \angle \frac{s_1 + \frac{1}{T_2}}{s_1 + \frac{1}{\beta T_2}} < 0^\circ$$

★ Parallel Compensation



$$\frac{C(s)}{R(s)} = \frac{G_1 G_2}{1 + G_2 G_c + G_1 G_2 H}$$

⇒ The characteristic equation is

$$1 + G_1 G_2 H + G_2 G_c = 0$$

$$\Rightarrow \quad 1 + \frac{G_c G_2}{1 + G_1 G_2 H} = 0$$

Let $G_f = \dfrac{G_2}{1 + G_1 G_2 H}$

the equation become

$$1 + G_c G_f = 0$$

fixed TF

⇒ Hence the same design approach applies to the parallel compensated system.

ex ⇒ Velocity feedback system.

CHAPTER 8

Frequency Response Approach to Control System Design

5B

Frequency responce approach to Control System design *

⇒ Because of difficulty in deriving the equations governing certain Components, such as pneumatic & hydraulic Component, the dynamic characteristics of such Components are usually determined experimentally through frequency response test.

⇒ In dealing with high frequency noise, we find that the frequency-response approach is more Convenient than other approaches.

⇒ There are basically two approaches in the frequency domain design.

- Nyquist plot Approach
- Bode plot Approach

⇒ For the design purposes, therefore it is best to work with the Bode diagram.

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★ Lead Compensation

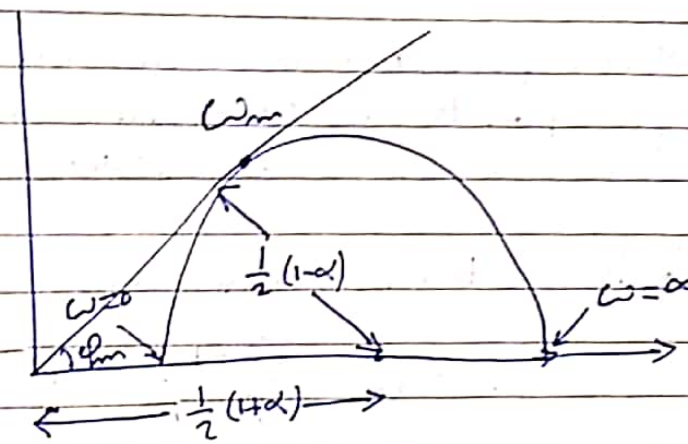
Characteristics of Lead Compensator

$$G_c(s) = K_c\alpha \frac{Ts+1}{\alpha Ts+1} = K_c \frac{s+\frac{1}{T}}{s+\frac{1}{\alpha T}} \qquad (0<\alpha<1)$$

α = attenuation factor

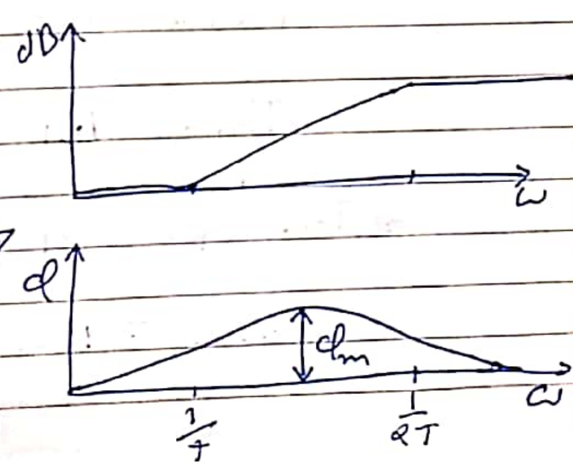
⇒ Polar plot of $K_c\alpha \dfrac{j\omega T+1}{j\omega\alpha T+1}$ $(0<\alpha<1)$

with $K_c = 1$



$$\sin\phi_m = \frac{1-\alpha}{1+\alpha} \qquad \omega_m = \frac{1}{\sqrt{\alpha}T}$$

{max phase lead angle}



Bode Diagram

Lead Compensation Technique Based on Frequency-Response Approach

⇒ Assume that the performance specifications are given in terms of phase margin, gain margin, static velocity error constant & so on.

⇒ The primary function of the Lead Compensator is to reshape the frequency-response curve to provide sufficient phase lead angle to offset the excessive phase lag associated with the components of the fixed system.

⇒ The procedure for designing a lead Compensator by the frequency-response approach may be stated as follows.

1. Assume the following lead Compensator:-

$$G_c(s) = K_c\alpha \frac{Ts+1}{\alpha Ts+1} = K_c \frac{s+\frac{1}{T}}{s+\frac{1}{\alpha T}}$$

$$\{0 < \alpha < 1\}$$

Define $K_c\alpha = K$

Then, $G_c(s) = K\frac{Ts+1}{\alpha Ts+1}$

⇒ The open loop TF of Compensated System:-

$$G_c(s)G(s) = KG(s)\frac{Ts+1}{\alpha Ts+1} = G_1(s)\frac{Ts+1}{\alpha Ts+1}$$

$$\{\text{where } G_1(s) = KG(s)\}$$

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1. Determine gain K to satisfy steady state requirement.

2. Using the gain K thus determined, draw a Bode diagram of $G_1(j\omega)$, the gain-adjusted but uncompensated system. Evaluate the phase margin.

3. Determine the necessary phase-lead angle to be added to the system. Add an additional 5° to 12° to the phase lead angle required, because the addition of the lead compensator shifts the gain crossover frequency to the right and decreases the phase margin.

4. Determine the attenuation factor α by use of following equation:

$$\sin \phi_m = \frac{1-\alpha}{1+\alpha}$$

Determine the frequency where the magnitude of the uncompensated system $G_1(j\omega)$ is equal to $-20\log(1/\sqrt{\alpha})$. Select this frequency as the new gain crossover frequency. This frequency corresponds to $\omega_m = 1/\sqrt{\alpha}T$, and the maximum phase shift ϕ_m occurs at this frequency.

5. Determine the corner frequencies of the lead compensator as follows:

Zero of lead compensator $\Rightarrow$ $\omega = \frac{1}{T}$

Pole of lead compensator $\Rightarrow$ $\omega = \frac{1}{\alpha T}$

6. Using the value of K determined in step 1 and that of α determined in step 4, Calculate Constant K_c from

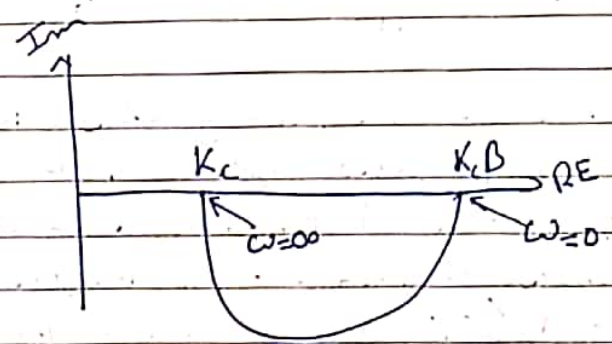
$$K_c = \frac{K}{\alpha}$$

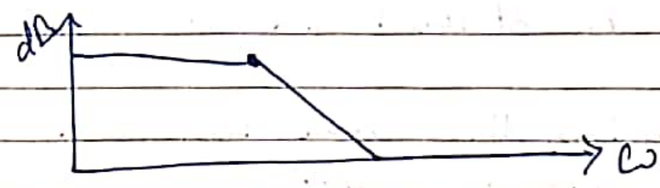
7. Check the gain margin to be sure it is satisfactory. If not, repeat the design process by modifying the pole-zero location of the compensator until a satisfactory result is obtained.

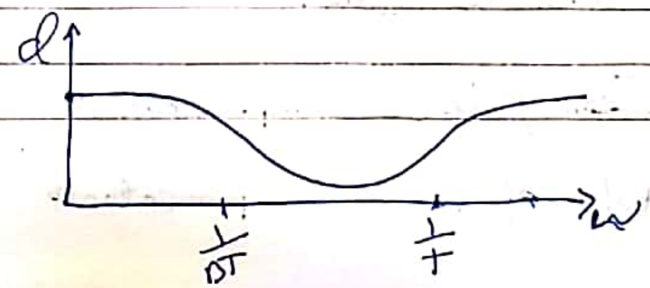
★ Lag Compensation

Characteristics of Lag Compensators

$$G_c(s) = K_c B \frac{Ts+1}{BTs+1} = K_c \frac{s+\frac{1}{T}}{s+\frac{1}{BT}} \quad \{B>1\}$$







Lag Compensation Technique based on the Frequency-Response Approach

⇒ The primary function of a lag compensator is to provide attenuation in the high-frequency range to give a system sufficient phase margin.

⇒ The procedure for designing lag compensators for the system shown by the frequency-response approach may be stated as follows:

1. Assume the following lag compensator:-

$$G_c(s) = K_c\beta \frac{Ts+1}{\beta Ts+1} = K_c \frac{s+\frac{1}{T}}{s+\frac{1}{\beta T}} \quad (\beta > 1)$$

Define $K_c\beta = K$

Then, $G_c(s) = K\frac{Ts+1}{\beta Ts+1}$;

⇒ The open loop TF of the compensated system is

$$G_c(s)\,G(s) = K\frac{Ts+1}{\beta Ts+1}G(s) = \frac{Ts+1}{\beta Ts+1}KG(s)$$

$$= \frac{Ts+1}{\beta Ts+1}G_1(s) \quad \left\{ \text{where } G_1(s) = KG(s) \right\}$$

Determine gain K to satisfy the requirement on the given static velocity error constant.

2. If the gain adjusted but uncompensated System $G_1(j\omega) = KG(j\omega)$ does not satisfy the Specifications on the phase and gain margins, then find the frequency point where the phase angle of the open loop TF is equal to $-180°$, plus the required phase margin.

↳ The required phase margin is the specified phase margin plus $5°$ to $12°$.

⇒ Choose this frequency as the new gain cross over frequency.

3. To prevent detrimental effects of phase lag due to the lag compensator, the pole and zero of the lag compensator must be located substantially lower than the new gain crossover frequency.

↳ Choose the compensator pole and zero sufficiently small. Thus the phase lag occurs at the low-frequency region so that it will not affect the phase margin.

4. Determine the attenuation necessary to bring the magnitude curve down to 0dB at the new gain crossover frequency. Noting that this attenuation is $-20\log\beta$, determine the value of β. Then the other corner frequency is determined from $\omega = \frac{1}{\beta T}$.

5. Using the value of K determined in Step 1 and that of β determined in Step 4, Calculate Constant K_c from

$$K_c = \frac{K}{\beta}$$

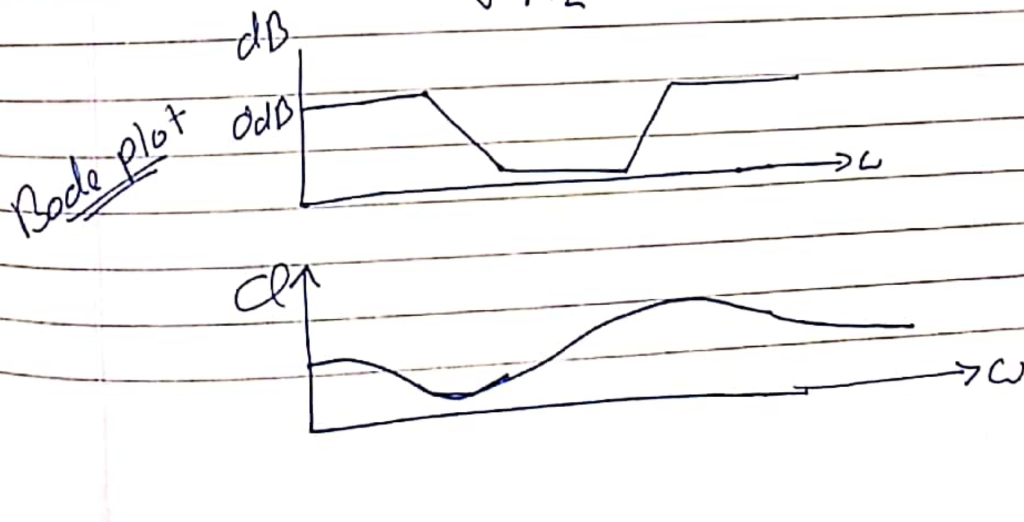
★ Lead Lag Compensation

Characteristic of Lead-Lag Compensator

$$G_c(s) = K_c \left(\frac{s + \frac{1}{T_1}}{s + \frac{\gamma}{T_1}} \right) \left(\frac{s + \frac{1}{T_2}}{s + \frac{1}{\beta T_2}} \right)$$

When $\gamma > 1$ & $\beta > 1$

$$\omega_1 = \frac{1}{\sqrt{T_1 T_2}}$$



⇒ The design of a lag-lead Compensator by the frequency-response approach is based on the Combination of the design techniques discussed under lead Compensation and lag Compensation.

The phase-lead portion of the leag-lag Compensator alters the frequency-response curve by adding phase-lead angle and increasing the phase margin at the gain Crossover frequency.

The phase-lag portion provides attenuation near and above the gain Cross over frequency and there be allows an increase of gain at the low frequency range to improve the Steady state performance.

CHAPTER 9

PID Controller and Modified PID Controller

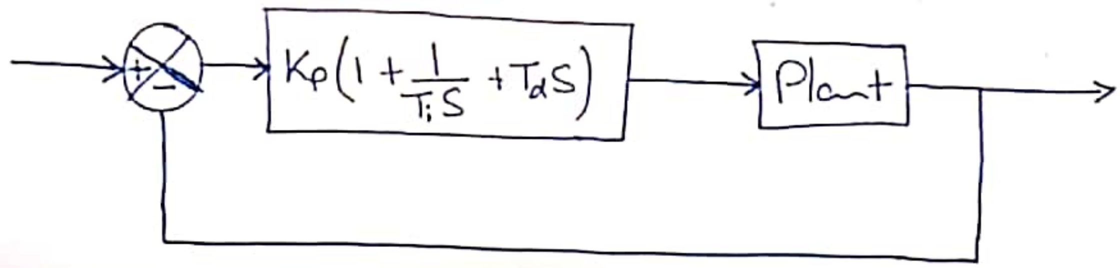
6

PID Controllers and Modified PID Controllers

★ Introduction

⇒ It is interesting to note that more than half of the industrial controllers in use today are PID Controller or modified PID controllers.

★ Ziegler–Nichols rules for Tuning PID Controllers



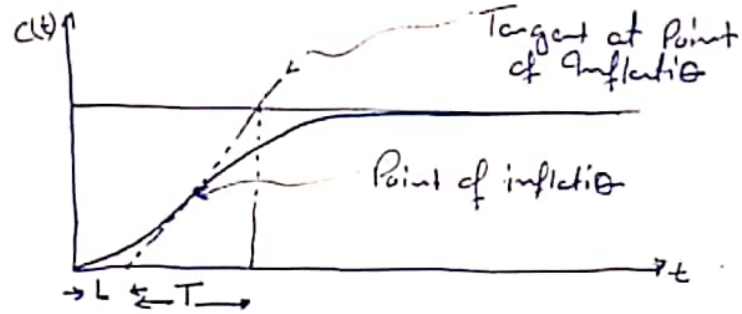
⇒ If mathematical model of the plant can be derived, then it is possible to apply various design techniques for determining parameters of the controller that will meet the transient and steady-state. Specification of the the closed loop system.

↳ If the plant is so complicated that its mathematical model cannot be easily obtained, then we must resort to experimental approaches to the tuning of PID controller.

⇒ There are two methods called Ziegler–Nichols tuning rules: the first method and the second method.

First Method

⇒ First we obtain experimentally the response of the Plant to a unit-step input.



→ This method applies if the response to a step input exhibits an S-shaped Curve.

Type of Controller	K_p	T_i	T_d
P	T/L	∞	0
PI	$0.9\frac{T}{L}$	$\frac{L}{0.3}$	0
PID	$1.2\frac{T}{L}$	$2L$	$0.5L$

$$\boxed{G_c(s) = 0.6T\,\frac{\left(s+\frac{1}{L}\right)^2}{s}}$$

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Second Method

⇒ In the second method, we first set $T_i = \infty$ and $T_d = 0$. Using proportional control action only, increase K_p from 0 to a critical value K_{cr} at which the output first exhibits sustained oscillations.

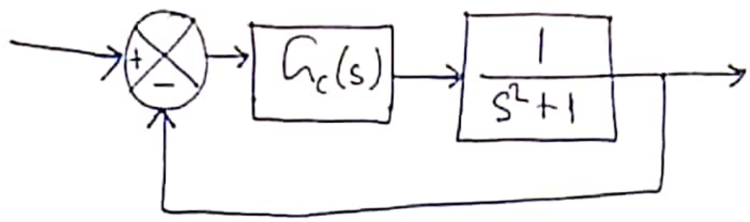
↳ If the output does not exhibit sustained oscillations for whatever value K_p may take, then this method does not apply.

⇒ Thus, the critical gain K_{cr} and the corresponding period P_{cr} are experimentally determined.

Type of Controller	K_p	T_i	T_d
P	$0.5 K_{cr}$	∞	0
PI	$0.45 K_{cr}$	$\frac{1}{1.2} P_{cr}$	0
PID	$0.6 K_{cr}$	$0.5 P_{cr}$	$0.125 P_{cr}$

$$G_c(s) = 0.075\, K_{cr} P_{cr} \frac{\left(s + \frac{4}{P_{cr}}\right)^2}{s}$$

★ Design of PID Controllers With Frequency Response Approach



\# Using a frequency-response approach, design a PID Controller such that the static velocity error constant is 4 sec⁻¹; Phase margin is 50° or more and gain margin 10dB or more.

Let us choose the PID Controller to be

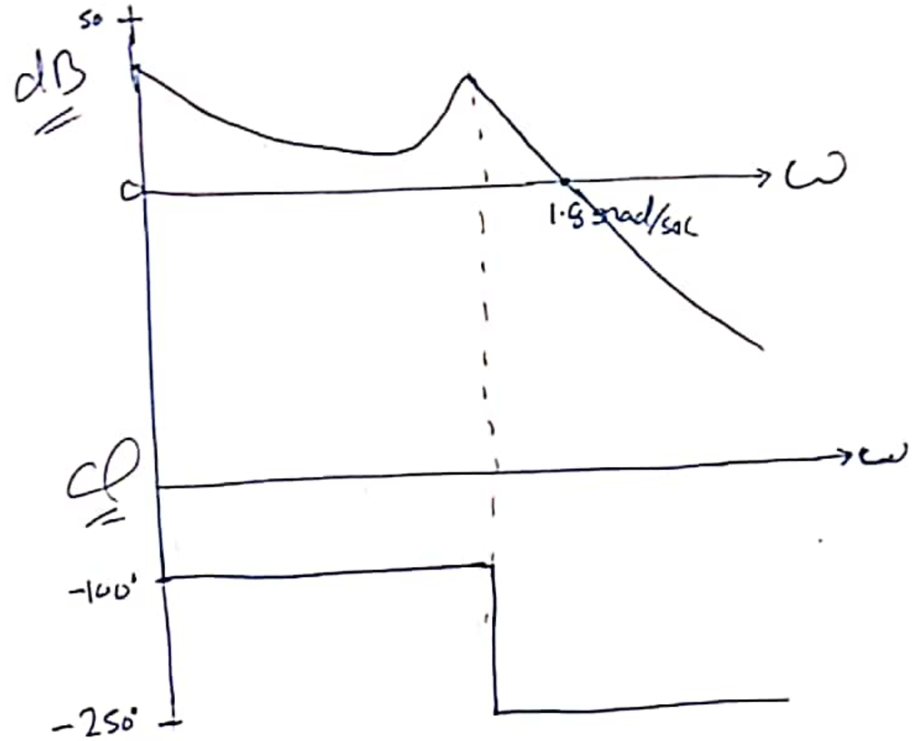
$$G_c = \frac{K(as+1)(bs+1)}{s}$$

$$K_v = \lim_{s\to 0} G_c(s)\frac{1}{s^2+1} = \lim_{s\to 0} s\frac{K(as+1)(bs+1)}{s(s^2+1)} = K = 4$$

So $$G_c(s) = \frac{4(as+1)(bs+1)}{s}$$

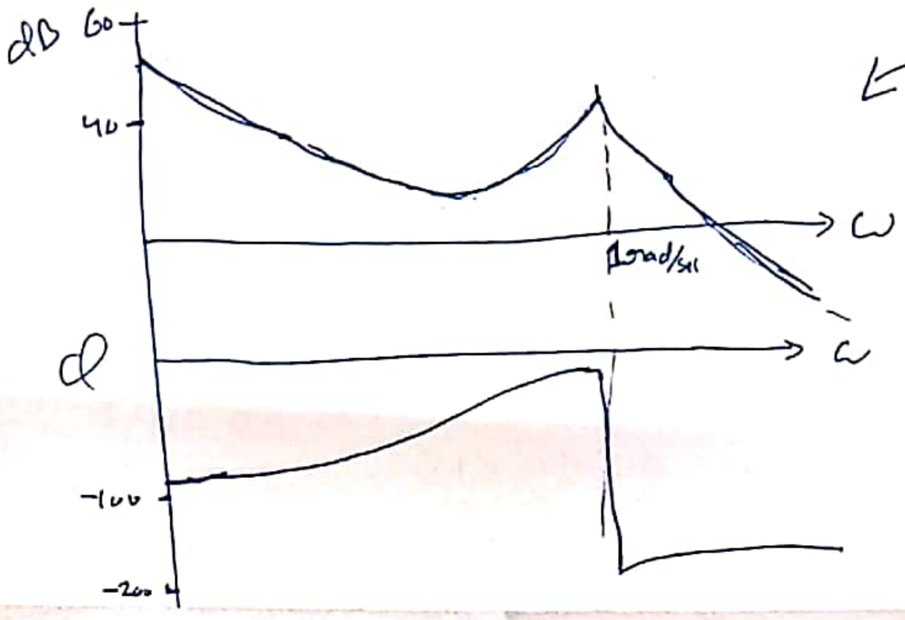
Let us plot a Bode diagram of

$$G(s) = \frac{4}{s(s^2+1)}$$



⇒ Let us assume the gain crossover frequency of the compensated system to be somewhere between $\omega=1$ and $\omega=10$ rad/s.

↳ We choose $a=5$. Then $(as+1)$ will contribute up to $90°$ phase lead in the high frequency region



$$\leftarrow \left(\frac{4(5s+1)}{s(s^2+1)} \right)$$

⇒ The term $(bs+1)$ needs to give phase margin of at least $50°$. We find $b = 0.25$ give phase margin $50°$ & gain margin $+\infty$.

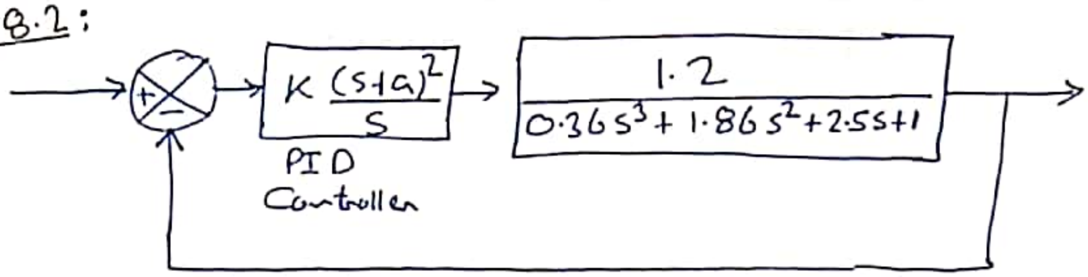
So $$G_c(s) = \frac{4(5s+1)(0.25s+1)}{s}$$

$$\text{Open loop TF of designed System} = \frac{4(5s+1)(0.25s+1)}{s} \times \frac{1}{s^2+1}$$

$$= \frac{5s^2+21s+4}{s^3+s}$$

★ Design of PID Controllers with Computational Optimization Approach

Example 8.2:



It is desired to find a Combination of K & a such that the closed-loop system will have 10% maximum overshoot in the unit step response.

⇒ To solve this problem, we first specify the region to search for appropriate K and a.

↳ Assume that the region to search for K & a is

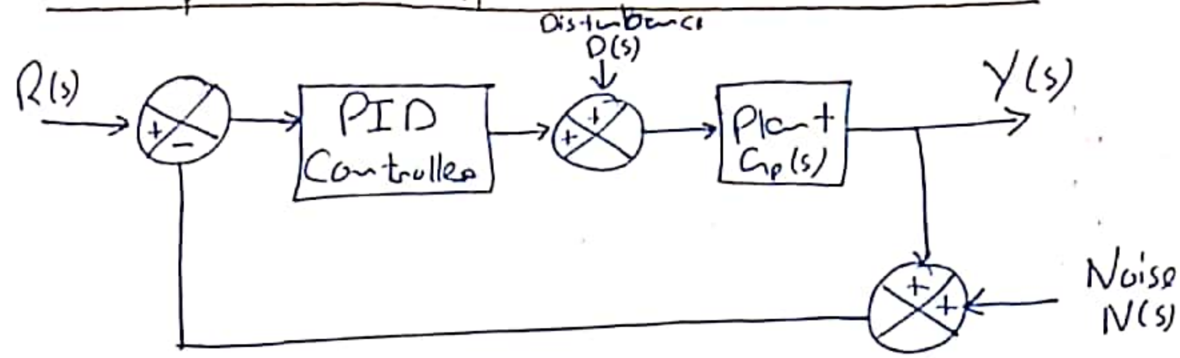
$$2 \leq K \leq 3 \quad \& \quad 0.5 \leq a \leq 1.5$$

⇒ If solution does not exist in the region, then we need to expand it.

⇒ In the Computational approach we need to determine the step size for each of K and a.

$\boxed{\text{Solution found } K=2 \quad a=0.9}$

★ Modifications of PID Control Schemes



PID-Controlled System

⇒ If the reference input is a step function, then; because of the presence of the derivative term in the control action, the manipulated variable u(t) will involve impulse function.

⇒ In an actual PID Controller, instead of the pure derivative term $T_d s$, we employ.

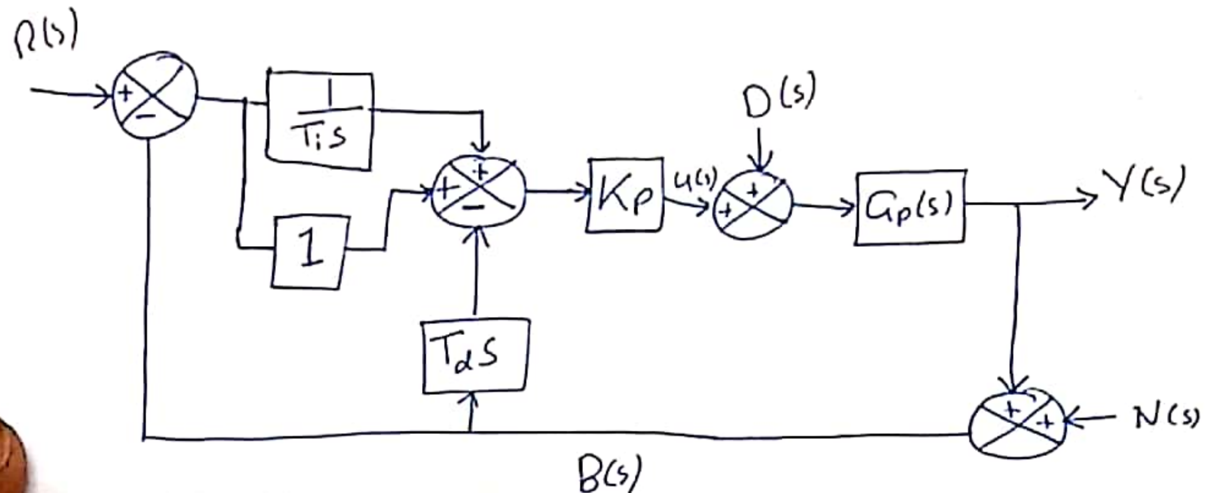
$$\frac{T_d s}{1+\gamma T_d s} \quad \{\text{where } \gamma \text{ is around } 0.1\}$$

⇒ Now, when the reference input is a step function the manipulated variable u(t) will not involve an impulse function, but will involve a sharp pulse function.

↳ Such a phenomenon is called Set-point Kick.

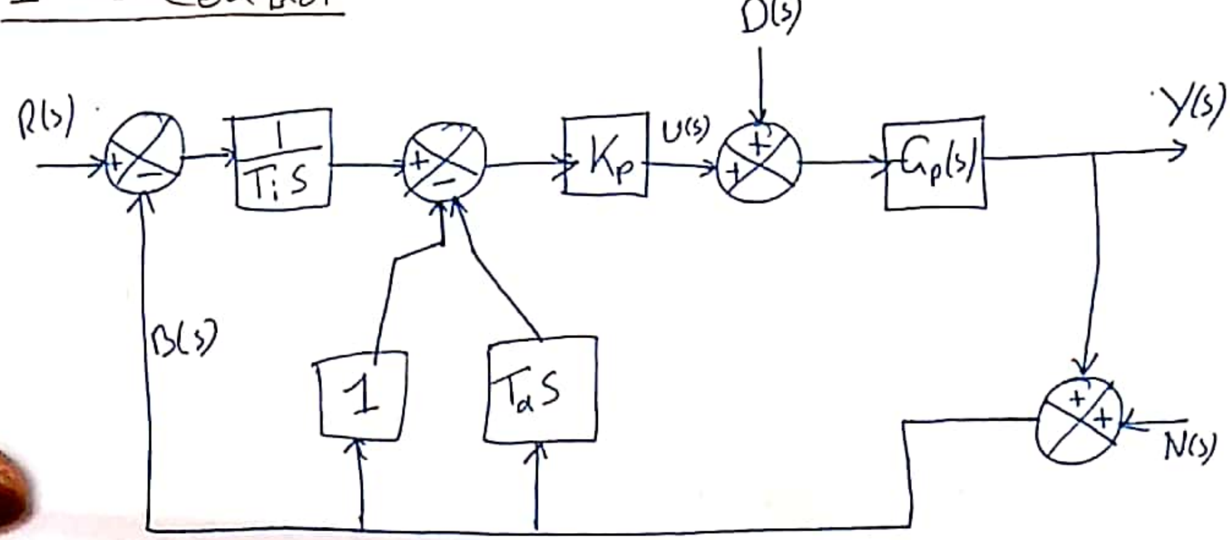
PI-D Control

To avoide the set-point kick phenomenon, we may wish to operate the derivative action only in the feedback path so the differentiation occurs only on the feedback signal and not on the reference signal.



$$U(s) = K_p\left(1 + \frac{1}{T_i s}\right)R(s) - K_p\left(1 + \frac{1}{T_i s} + T_d s\right)B(s)$$

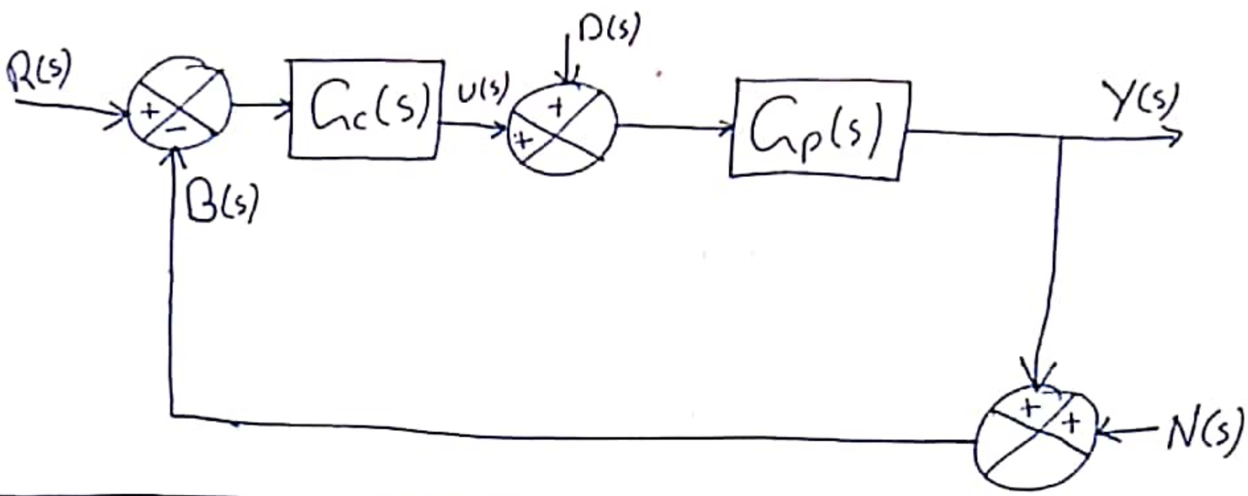
I-PD Control



$$U(s) = \frac{K_p}{T_i s}R(s) - K_p\left(1 + \frac{1}{T_i s} + T_d s\right)B(s)$$

★ Two degrees of Freedom Control

⇒ Consider the System, where the System is subjected to the disturbance input $D(s)$ and noise input $N(s)$, in addition to the reference input $R(s)$.



$$\boxed{Y(s) = \frac{G_c G_p}{1+G_c G_p} R(s) + \frac{G_p}{1+G_c G_p} D(s) + \frac{G_c G_p}{1+G_c G_p} N(s)}$$

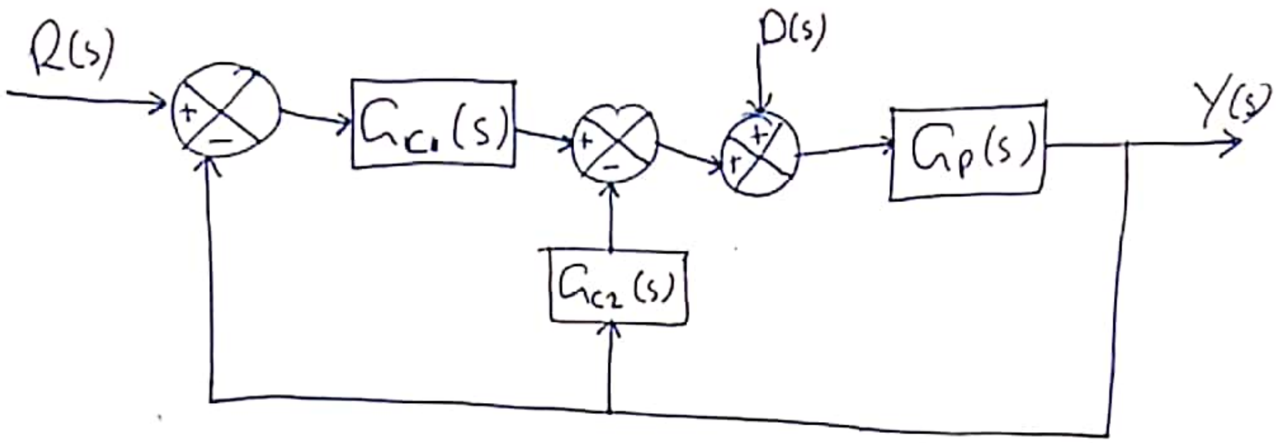
★ Zero-Placement Approach to Improve Response Characteristics

⇒ In high-performance Control System it is always desired that the System output follow the changing input with minimum error.

↳ For Step, ramp and acceleration input, it is desired that the System output exhibit no steady-state error.

⇒ Consider the two DOF Control System shown:

$$G_p = K \frac{A(s)}{B(s)}$$



$$A(s) = (s+z_1)(s+z_2)\cdots(s+z_m)$$
$$B(s) = s^N(s+P_{N+1})(s+P_{N+2})\cdots(s+P_n)$$

⇒ Let us Assume G_{c1} is a PID Controller followed by a filter $1/A(s)$.

$$G_{c1}(s) = \frac{\alpha_1 s + \beta_1 + \gamma_1 s^2}{s}\,\frac{1}{A(s)}$$

and G_{c2} is a PID, PI, PD, I, D or P Controller followed by a $1/A(s)$ filter.

$$G_{c2}(s) = \frac{\alpha_2 s + \beta_2 + \gamma_2 s^2}{s}\,\frac{1}{A(s)}$$

⇒ Then $G_{c1}(s) + G_{c2}(s) = \dfrac{\alpha s + \beta + \gamma s^2}{s}\,\dfrac{1}{A(s)}$

$$\frac{Y(s)}{D(s)} = \frac{G_p}{1+(G_{c1}+G_{c2})G_p} = \frac{sKA(s)}{sB(s) + (\alpha s+\beta+\gamma s^2)K}$$

$$\frac{Y(s)}{R(s)} = \frac{G_{c1}G_p}{1+(G_{c1}+G_{c2})G_p}$$

Zero placement

Consider the system

$$\frac{Y(s)}{R(s)} = \frac{P(s)}{s^{n+1} + a_n s^n + a_{n-1} s^{n-1} + \cdots + a_2 s^2 + a_1 s + a_0}$$

If we choose $P(s)$ as

$$P(s) = a_2 s^2 + a_1 s + a_0 = a_2 (s+s_1)(s+s_2)$$

→ The numerator polynomial $P(s)$ is equal to the sum of the last three terms of the denominator polynomial - then the system will exhibit no steady state error in response to the step input, ramp input & acceleration input.