3

Configuration Space

⇒ To create motion plans for robots, we must be able to give location of every point on the robot.

↳ Since we need to ensure that no point on the robot collides with an obstacle.

3.1) Specifying the Robot's Configuration

Configuration

→ Complete specification of the position of every point of that system.

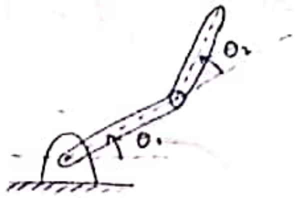
Configuration space {Q}

→ Space of all possible configuration of the system.

degree of freedom ⇒ {Dimension of Configuration space}

⇒ Robots move in a 2D or 3D Euclidean ambient space represented by R^2 and R^3 respectively.

↳ We sometimes refer this as Workspace



⇒ Each joint angle θ_i corresponds to a point on the unit circle S^1, and the configuration space is $S^1 \times S^1 = T^2$, the two-dimensional torus.

3.2> Obstacle and the Configuration Space

⇒ $c : [0,1] \rightarrow Q$ $\quad \{q(0) = \text{start} \quad q(1) = \text{goal}\}$

such that no configuration in the path causes a collision between the robot & the obstacle.

$$QO_i = \{q \in Q \mid R(q) \cap WO_i \neq \emptyset\}$$

{Configuration space obstacle}

$$Q_{free} = Q \setminus \cup_i QO_i$$

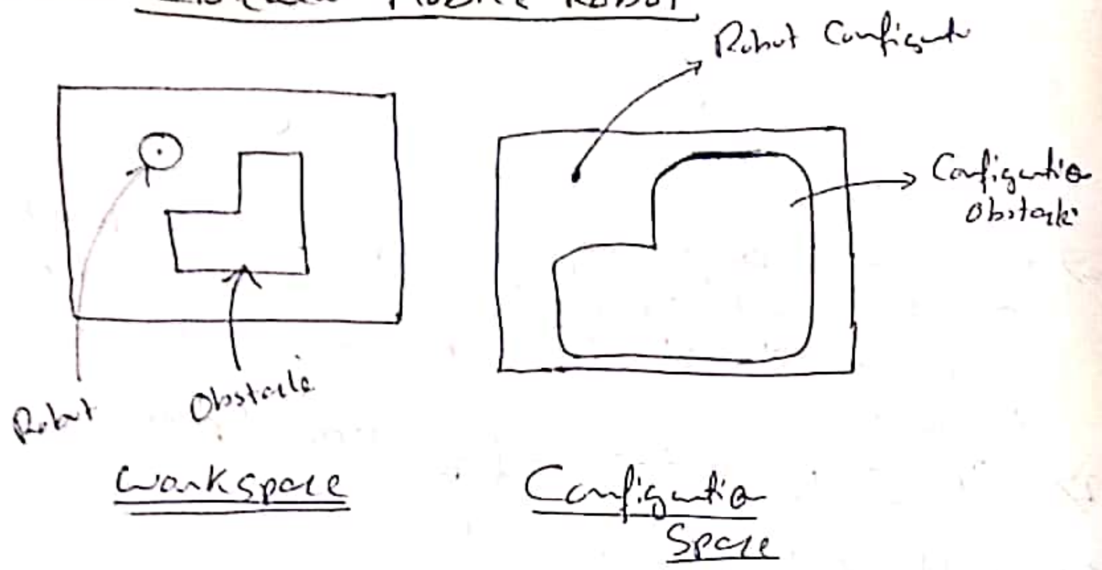
{Free configuration space}

$$c : [0,1] \rightarrow Q_{free}$$

Free path ⇒ Does not allow contact between the robot and obstacle.

Semifree path ⇒ Allows the robot to contact the boundary of an obstacle.

3.2.1 Circular Mobile Robot



⇒ The Configuration Space and workspace are different spaces, and the transformation from workspace obstacles to Configuration Space obstacles is not always so simple.

3.2.2 Two Joint Planar arm

⇒ For two-joint planar arm $Q = T^2$.

{ For this reason, grid-based representations of the Configuration space are sometimes used. }

⇒ We can define a grid on the surface of the torus, and for each point on this grid we can perform a fairly simple test to see if the corresponding Configuration causes a Collision between the arm and any obstacle in the workspace.

3.3) The Dimension of Configuration Space

⇒ Real robots are typically modeled as a set of rigid bodies connected by joints, not a set of points that are free to move independently.

⇒ Let us consider a robot which is planar rigid body that can both translate and rotate in the plane.

↳ This body has three degrees of freedom (x, y, θ), and its configuration space is $R^2 \times S^1$.

Holonomic Constraint

↳ A holonomic constraint is one that can be expressed purely as a function g of the configuration variables (and possibly time)

$$\boxed{g(q,t)=0}$$

⇒ Each linearly independent holonomic constraint on a system reduces the dimension of the system's configuration space by one.

⇒ Nonholonomic constraints are velocity constraints of the form:

$$\boxed{g(q,\dot{q},t)=0}$$

⇒ Nonholonomic constraints do not reduce the dimension of the configuration space

⇒ A closed-chain robot, also known as a Parallel mechanism, is one where the link form one or more closed loops.

↳ If the mechanism has K links, then one is designed as a Stationary "ground" link and $K-1$ links are movable.

⇒ Therefore the System has $N(K-1)$ dof before the joints are taken into account.

⇒ Now each of the n joints between the links place $N-f_i$ Constraints on the feasible motions of the links, where f_i is the number of dof at joint i.

$f_i = 1$ ∀ revolute joint

$f_i = 3$ ∀ Spherical joint

$N = 6$ ∀ Spatid mechanism

$N = 3$ ∀ Planar mechanism

⇒ The number of DOF M of the mechanism is given by:

$$M = N(K-1) - \sum_{i=1}^{M} (N - f_i)$$

$$\boxed{M = N(K-M-1) + \sum_{i=1}^{M} f_i}$$

↳ This is known as Grublers formula for closed chains.

⇒ Grubler's formula for closed chains, is only valid if the constraints due to the joints are independent.

3.4> The topology of Configuration Space

{ In mathematics, topology is concerned with the properties of a geometric object that are preserved under continuous deformation (or transformation) }

⇒ Two spaces are topologically different if cutting or pasting is required to turn one into the other.

↳ As cutting and pasting are not continuous transformation.

<u>Topologically</u>

T^2 ⟶ Rubber doughnut

R^2 ⟶ Rubber sheet

⇒ To a topologist, all rubber doughnuts are the same, regardless of how they are stretched or deformed, it can never be a rubber sheet.

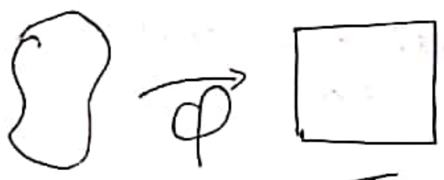
⇒ One reason that we care about the topology of Configuration Space is that it will affect our representation of the space.

3.4.1) Homeomorphisms and Diffeomorphisms

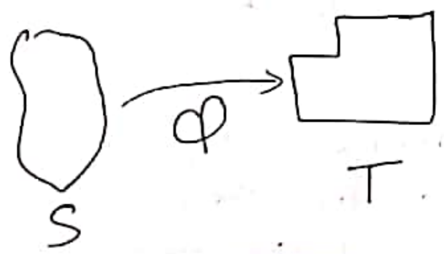
⇒ A mapping $\phi : S \to T$ is a rule that places element of S into ~~corresponding~~ correspondence with elements of T.

⇒ If $\phi(S) = T$, then we say that ϕ is surjective.

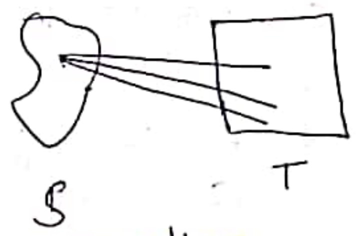
⇒ If ϕ puts each element of S into correspondence with at most one element of T, then we say ϕ is injective.



Surjective



Not Surjective



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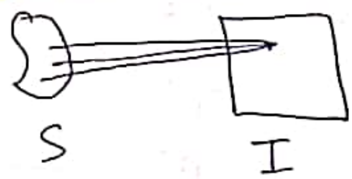
not injective



not injective

⇒ Maps that are both surjective and injective are said to be bijective.

⇒ Bijective map have the property that their inverse exists all all points.

homeomorphic

→ If $\phi: S \to T$ is a bijection, both ϕ and ϕ^{-1} are continuous, then ϕ is a homeomorphism

→ When such a ϕ exists, S and T are said to be <u>homeomorphic</u>.

⇒ A mapping $\phi: U \to V$ is said to be smooth if all partial derivatives of ϕ, of all orders are well defined (i.e. ϕ is of class C^∞)

Diffeomorphic

→ A smooth map $\phi: U \to V$ is a diffeomorphism if ϕ is bijective and ϕ^{-1} is smooth.

→ When such a ϕ exists, U and V are said to be diffeomorphic.

⇒ All diffeomorphisms are homeomorphisms.

⇒ We are often concerned about the local properties of configuration space.

→ Local properties are defined on neighborhoods.

→ Neighborhood are most easily defined in terms of open balls.

⇒ For a point P of some manifold M, we define an open ball of radius ϵ by:

$$B_\epsilon(P) = \{P' \in M \mid d(P, P') < \epsilon\}$$

3.4.2> Differentiable Manifolds

Manifold

→ A set S is a K-dimensional manifold if it is locally homeomorphic to $\mathbb{R}^K$, meaning that each point in S possesses a neighborhood that is homeomorphic to an open set in $\mathbb{R}^K$.

Chart

→ A pair (U, φ) such that U is an open set in an K dimensional manifold and φ is a diffeomorphism from U to some open set in $\mathbb{R}^K$ is called a chart.

C^∞-related

→ Let (U, φ) and (V, ψ) be two charts on a K-dimensional manifold. Let X be the image of $U \cap V$ under φ and Y be the image of $U \cap V$ under ψ.

$$X = \{ \varphi(x) \in \mathbb{R}^K \mid x \in U \cap V \}$$

$$Y = \{ \psi(y) \in \mathbb{R}^K \mid y \in U \cap V \}$$

⇒ These two charts are said to be C^∞-related if both of the composite functions

$$\psi \circ \varphi^{-1} : X \to Y$$

$$\varphi \circ \psi^{-1} : Y \to X$$

are C^∞.

⇒ If two charts are C^∞ related, we can switch back and forth between them in a smooth way when their domain overlap.

⇒ For many interesting configuration space, it will be the case that we cannot construct a single chart whose domain contains the entire configuration space.

↳ In these cases, we construct a collection of charts that covers the configuration space.

⇒ We are not free to choose these charts arbitrarily, charts should be C^∞-related.

⇒ A set of chart that are C^∞-related, and whose domains cover the entire configuration space Q, form an atlas for Q.

⇒ Together, the atlas and Q comprise a differentiable manifold.

3.4.3) Connectedness and Compactness

↓

{If there exists a path between any two points of the manifold}

⇒ A space is compact if it resembles a closed, bounded subset of $\mathbb{R}^n$.

↳ R^2 Not compact

↳ T^2 Compact

⇒ The product of compact configuration spaces is also compact.

3.5) Embeddings of Manifolds in R^n

⇒ When we are confronted with a configuration space that does not permits a single global coordinate chart, we are faced with a choice:-

- → Use single set of parameters and suffer the consequences of singularities and discontinuities in representation.
- → Use multiple charts to construct an atlas
- → Use single global representation by embedding the configuration space in a higher-dimensional space.

3.5.1) Matrix Representations of Rigid-Body Configuration

⇒ It is often convenient to represent the position and orientation of a rigid body using $m \times m$ matrix of real number.

⇒ We describe the orientation of a rigid-body in n-dimensional space by the matrix group SO(n), and the position and orientation by the matrix groups SE(n).

{Special orthogonal group} {Special Euclidean Group}

⇒ The matrix groups SO(n) and SE(n) can be used to

1. Represent rigid body configuration.
2. Change the reference frame for the representation of a configuration or a point
3. display (move) a configuration or a point

⇒ When matrix is used for representing configuration, we often call it a <u>frame</u>.

⇒ When used for coordinate change, we often call it a <u>transform</u>.

3.7) <u>Example of Configuration Space</u>

⇒ In most cases, we can model robots as rigid bodies, articulated chains or combinations of these two.

⇒ When designing a motion planner it is often important to understand the underlying structure of the robot's configuration space.

(Mobile robot translating and rotating in the plane) → SE(2) or $\mathbb{R}^2 \times S^1$

3.8) Transforming Configuration and Velocity Representation

Forward Kinematic map $\varphi : Q \rightarrow M$ *

Inverse Kinematic map $\varphi^{-1} M \rightarrow Q$

$\Rightarrow$ As the robot system moves the time derivative x is related to the time derivative $\dot{q}$ by:

$$\boxed{\dot{x} = J(q)\dot{q}}$$

where J is the Jacobian of map φ.

$\Rightarrow$ The Jacobian of the map φ, is also useful for transforming forces expressed in one set of coordinates to another.

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* We often need to transform from one representation of the configuration of a robot q ∈ Q to some other representation x ∈ M.

{ example, when q represents the joint angles of a robot arm and x represents the configuration of the end effector as a rigid body in the ambient space }