4

Design of discrete-time Control System by Conventional method

4.1) Introduction

⇒ Conventionally three different design methods for SISO discrete-time control system ÷

i) Root Locus technique using pole-Zero Configurations in the z plane

ii) Frequency response method in the ω plane.

iii) Analytical method in which we attempt to obtain a desired behavior of the closed-loop System by manipulating the pulse transfer function of digital Controller.

4.2) Mapping between the s plane and the z plane

⇒ The absolute stability and relative stability of a linear time-Invariant Continuous time closed-loop Control System are determined by the locations of the Closed-loop poles in the s plane.

⇒ Since the Complex variables z and s are related by $z = e^{Ts}$, the pole and zero locations in the z plane are related to the pole & zero locations in the s plane.

⇒ Therefore, the Stability of the linear time-invariant discrete-time closed loop System can be determined in terms of the locations of the poles of the closed loop pulse transfer function.

⇒ Dynamic behavior of the discrete-time control system depends on the sampling period T. Therefore change in the sampling period T modifies the pole and zero locations in the z plane and causes the response behavior to change.

★ Mapping of the Left half of S plane into the Z plane

⇒ Pole in S plane can be located in the Z plane through the transformation $Z = e^{TS}$.

$$S = \sigma + j\omega$$

$$\Rightarrow Z = e^{T(\sigma + j\omega)} = e^{T\sigma} e^{jT\omega} = e^{T\sigma} * e^{j(T\omega + 2\pi K)}$$

⇒ From above we see that poles and zeros in the s plane, where frequencies differ in integral multiple of the sampling frequency $2\pi/T$, are mapped into the same locations in the z plane.

↳ This means that there are infinite many values of s for each value of z.

⇒ Since σ is negative in the left half of the s plane, the left half of the s plane corresponds to

$$|Z| = e^{T\sigma} < 1$$

$\Rightarrow$ The $j\omega$ axis in the S plane Corresponds to $|z|=1$. That is, the imaginary axis in the S plane (the line $\sigma=0$) Corresponds to the unit Circle in the z plane, and the interior of the unit Circle Corresponds to the left half of the S plane.

★ **Primary Strip and Complementary strips**

$$\boxed{\angle z = T\omega}$$

$\Rightarrow$ Consider a representative point on the $j\omega$ axis in the S plane. As this point moves from $-\frac{1}{2}\omega_s j$ to $j\frac{1}{2}\omega_s$ on the $j\omega$ axis where ω_s is the sampling frequency, we have $|z|=1$, and $\angle z$ Varies from $-\pi$ to π in the Counterclockwise direction in the z plane.

$\Rightarrow$ As the representative point moves from $j\frac{1}{2}\omega_s$ to $j\frac{3}{2}\omega_s$ on the $j\omega$ axis, the Corresponding point in the z plane traces out the unit Circle once in the Counterclockwise direction.

$\Rightarrow$ From this analysis, it is clear that each strip of width ω_s in the left half of the S plane maps into the inside of the unit Circle in the z plane.

$\Rightarrow$ This implies that the left half of the S plane may be divided into an infinite number of periodic Strips.

* Primary strip $\Rightarrow$ $j\omega = \left(-j\frac{1}{2}\omega_s \text{ to } j\frac{1}{2}\omega_s\right)$

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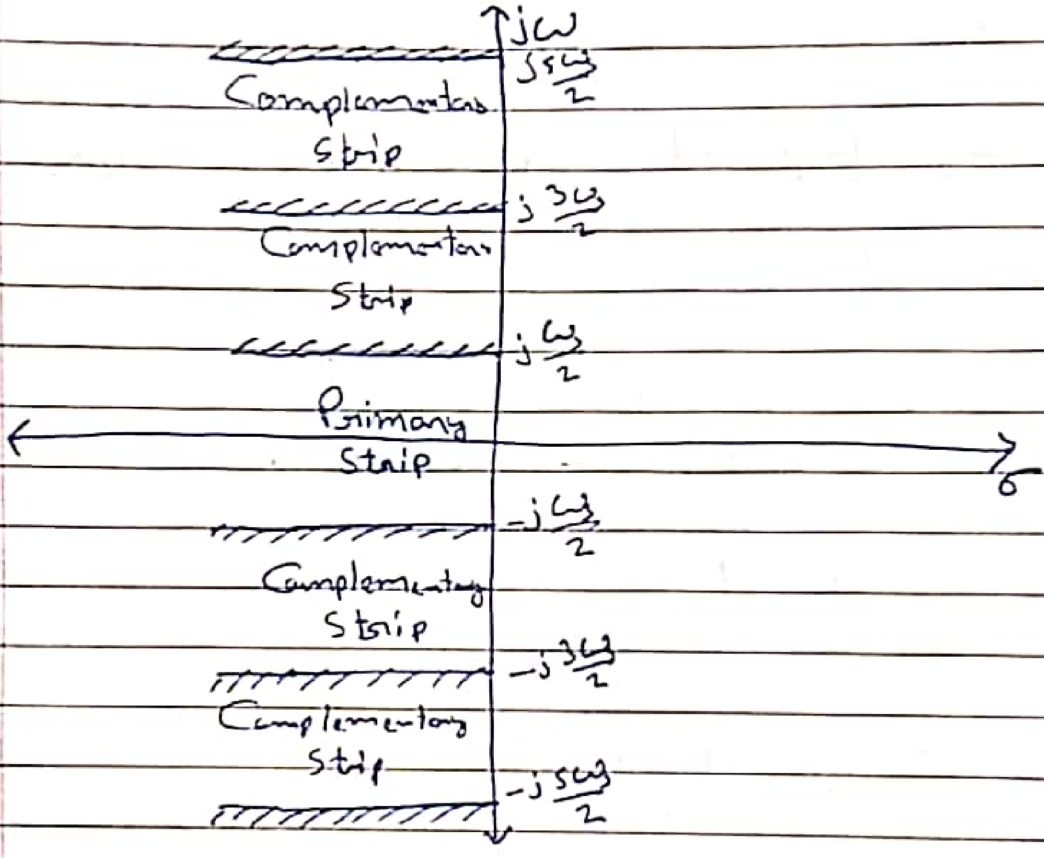
$\ldots \left(-j\frac{1}{2}\omega_s \text{ to } -j\frac{3}{2}\omega_s\right),$

* Complementary strips $\Rightarrow \left(j\frac{1}{2}\omega_s \text{ to } j\frac{3}{2}\omega_s\right)$

$, \left(j\frac{3}{2}\omega_s \text{ to } j\frac{5}{2}\omega_s\right) \ldots$



$\Rightarrow$ A point in the Z plane corresponds to an infinite number of points in the S plane, although a point in the S plane corresponds to a single point in the Z plane.

$\Rightarrow$ If the sampling frequency is at least twice as fast as the highest-frequency component involved in the system, then every point in the unit circle in the Z plane represents frequencies between $-\frac{1}{2}\omega_s$ and $\frac{1}{2}\omega_s$.

★ Constant-Attenuation Loci

A Constant-attenuation line (σ = Constant) in the S plane maps into a Circle of radius $Z = e^{T\sigma}$ Centered at the Origin in the Z plane.

★ Settling time t_s

⇒ The settling time is determined by the value of attenuation σ of the dominant Closed loop poles.

⇒ If the settling time is specified, it is possible to draw a line $\sigma = \sigma_1$ in the S plane Corresponds to a given Settling time.

⇒ The region to the left of the line $\sigma = \sigma_1$ in the S plane Corresponds to the inside of a Circle with radius $e^{-\sigma_1 T}$ in the Z plane.

★ Constant-Frequency Loci

⇒ A Constant-frequency locus $\omega = \omega_1$ in the S plane is mapped into a radial line of Constant angle $T\omega_1$ (in radians) in the Z plane.

★ Constant-Damping Ratio Loci

⇒ A Constant damping ratio line (a radial line) in the S plane is mapped into a Spiral in the Z plane.

⇒ This can be seen as follows :-

$$S = -\xi\omega_n + j\omega_n\sqrt{1-\xi^2} = -\xi\omega_n + j\omega_d$$

$$\{\omega_d = \omega_n\sqrt{1-\xi^2}\}$$

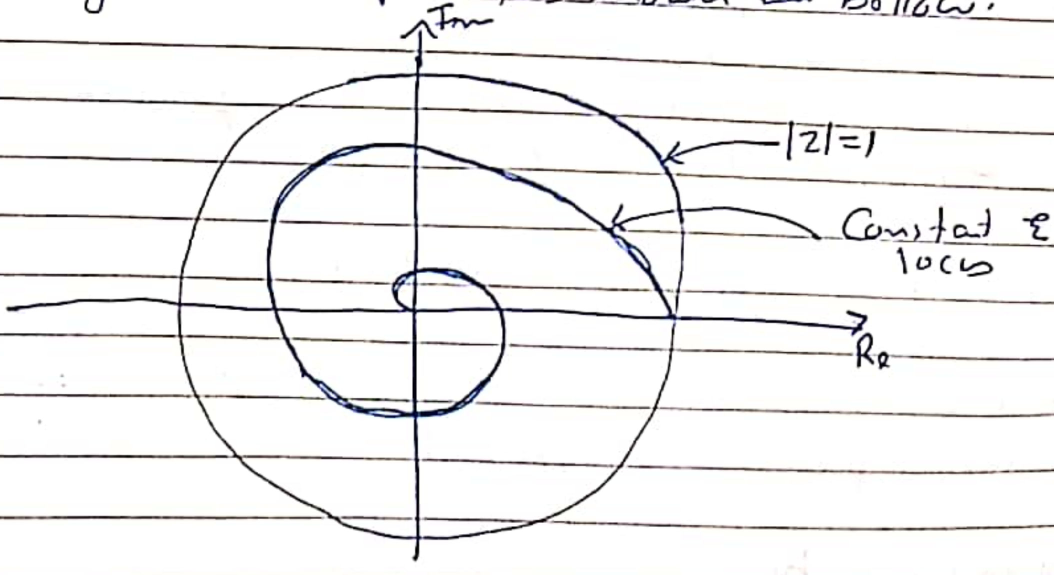
$$Z = e^{TS} = \exp(-\xi\omega_n T + j\omega_d T)$$

$$= \exp\left(\frac{-2\pi\xi\omega_d}{\omega_s\sqrt{1-\xi^2}} + j2\pi\frac{\omega_d}{\omega_s}\right)$$

$$|Z| = \exp\left(\frac{-2\pi\xi}{\omega_s\sqrt{1-\xi^2}}\omega_d\right)$$

$$\angle Z = 2\pi\frac{\omega_d}{\omega_s}$$

⇒ Thus, the magnitude of z decreases and the angle of z increases linearly as ω_d increases, and the locus in the Z plane becomes a logarithmic spiral, as shown bellow.



⇒ Thus the Spiral can be graduated in terms of a normalized frequency ω_d/ω_s.

⇒ Constant ξ loci are normal to the constant ω_n loci in the s plane.

↳ In the z plane mapping, constant ω_n loci intersect constant ξ spirals at right angles.

⇒ A mapping such as this, which preserves both the size and the sense of angles, is called a Conformal mapping.

4.3) Stability Analysis of Closed-loop System in the Z-plane

⇒ Consider the following closed-loop pulse-transfer function system:

$$\frac{C(z)}{R(z)} = \frac{G(z)}{1 + GH(z)}$$

⇒ The stability of the system defined above as well as of other types of discrete-time control systems, may be determined from the location of the closed loop poles in the z plane, or the roots of the characteristic equation

$$\boxed{P(z) = 1 + GH(z) = 0}$$

as follows :-

1. For the system to be stable, the closed-loop poles or the roots of the characteristic equation must lie within the unit circle in the z plane.
   ↳ Any closed-loop pole outside the unit circle makes the system unstable.

2. If a simple pole lie at $z=1$, then the system becomes critically stable. Also the system becoms critically stable if a single pair of conjugate complex poles lies on the unit circle in the z plane.
   ↳ Any multiple closed-loop pole on the unit circle makes the system unstable.

3. Closed loop zeros donot affect the absolute stability and therefor may be located anywhere in the z plane.

★ Methods for Testing Absolute Stability

⇒ Three stability test can be applied directly to the characteristic equation $P(z)=0$ without solving for the roots.

1. Schur-Cohn stability test
2. Jury stability test.

3. Bilinear transformation coupled with the Routh stability criterion.

1) The Jury Stability test

Let the Characteristic equation $P(z)$ is a polynomial in z as follows:

$$P(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$$

$$\{a_0 > 0\}$$

⇒ Elements in the first row consist of the coefficients in $P(z)$ arranged in the ascending order of powers of z.

⇒ The elements in the second row consist of the coefficients of $P(z)$ arranged in the descending order of power of z.

⇒ The elements for rows 3 through $2n-3$ are given by the following determinants.

$$b_K = \begin{vmatrix} a_n & a_{n-1-K} \\ a_0 & a_{K+1} \end{vmatrix} \quad K = 0, 1, 2 \cdots n-1$$

$$C_K = \begin{vmatrix} b_{n-1} & b_{n-2-K} \\ b_0 & b_{K+1} \end{vmatrix} \quad K = 0, 1, 2 \cdots n-2$$

$$\vdots$$

$$q_K = \begin{vmatrix} P_3 & P_{2-K} \\ P_0 & P_{K+1} \end{vmatrix} \quad K = 0, 1, 2$$

⇒ Element of any even-numbered row are simply the reverse of the immediately preceding odd-numbered row.

Stability Criterion by the Jury test

1. $|a_n| < a_0$

2. $P(z)\big|_{z=1} > 0$

3. $P(z)\big|_{z=-1} \begin{cases} >0 \ \forall n \text{ even} \\ <0 \ \forall n \text{ odd} \end{cases}$

4. $|b_{n-1}| > |b_0|$
   $|c_{n-2}| > |c_0|$
   ⋮ ⋮
   $|a_2| > |a_0|$

Example 4.3: $P(z) = a_0 z^4 + a_1 z^3 + a_2 z^2 + a_3 z + a_4$

Row	z^0	z^1	z^2	z^3	z^4
1	a_4	a_3	a_2	a_1	a_0
2	a_0	a_1	a_2	a_3	a_4
3	b_3	b_2	b_1	b_0	
4	b_0	b_1	b_2	b_3	
5	C_2	C_1	C_0		

$$P(1) = a_0 + a_1 + a_2 + a_3 + a_4 > 0 \quad \text{——} \quad (a)$$

$$P(-1) = a_0 - a_1 + a_2 - a_3 + a_4 > 0 \quad \text{——} \quad (b)$$

Example 4.4: $P(z) = z^4 - 1.2z^3 + 0.07z^2 + 0.3z - 0.08 = 0$

	z^0	z^1	z^2	z^3	z^4
1	1	−1.2	0.07	0.3	−0.08
2	−0.08	0.3	0.07	−1.2	1
3	~~0.204~~ −0.99	~~0.0756~~ −1.176	~~−1.176~~ 0.076	~~−0.99~~ 0.204	
4	~~−0.99~~ 0.204	~~−1.176~~ 0.0756	~~0.0756~~ −1.176	~~0.204~~ −0.55	
5	0.938	1.148	0.16506		
6	0.16506	1.148	0.938		

$P(1) = 0.09$ ✓

$P(-1) = 1.89$ ✓

		z^0	z^{-1}	z^2	z^3	z^4
a	1	−0.08	0.3	0.07	−1.2	1
	2	1	−1.2	0.07	0.3	−0.08
b	3	−0.99	1.176	−0.0756	−0.204	
	4	−0.204	−0.0756	1.176	−0.99	
c	5	0.94	−1.18	0.3147		
	6	0.3147	−1.18	0.94		

2) Stability Analysis by Use of the Bilinear Transformation and Routh Stability

⇒ The method requires transformation from the Z plane to another complex plane, the ω plane.

↳ The amount of computation required is much more than that required in the Jury stability criterion.

⇒ The bilinear transformation defined by

$$Z = \frac{\omega+1}{\omega-1} \quad \text{or} \quad \omega = \frac{Z+1}{Z-1}$$

⇒ This transformation maps the unit circle in the z plane into the left half of the ω plane.

$$P(z) = a_0 Z^n + a_1 Z^{n-1} + \cdots a_{n-1} Z + a_n = 0$$

$$\Rightarrow a_0\left(\frac{\omega+1}{\omega-1}\right)^n + a_1\left(\frac{\omega+1}{\omega-1}\right)^{n-1} + \cdots + a_{n-1}\left(\frac{\omega+1}{\omega-1}\right) + a_n = 0$$

⇒ Multiplying both side by $(\omega-1)^n$ we obtain.

$$Q(\omega) = b_0\omega^n + b_1\omega^{n-1} + \cdots + b_{n-1}\omega + b_n = 0$$

⇒ Once we transform $P(z)=0$ into $Q(\omega)=0$, it is possible to apply the Routh stability criterion in the Same manner as in Continuous-time system.

4.4) Transient and Steady-State Response Analysis

"Absolute stability is a basic requirement of all Control system. In addition, good relative stability and steady-state accuracy are also required of any Control system, whether Continuous time or discrete time"

★ Transient Response Specifications

⇒ The transient response of a System to a unit-step input depends on the initial Conditions. For Convenience in Comparing transient responses of various system, it is a common practice to use the standard initial condition: the system is at rest initially and the output and all its time-derivatives are zero.

1. Delay time t_d
2. Rise time t_r
3. Peak time t_p
4. Maximum overshoot M_p
5. Settling time t_s

⇒ Let us assume that the Sampling theorem is satisfied and no frequency folding occurs.

⇒ Consider the discrete-time Control System defined by:-

$$\frac{C(z)}{R(z)} = \frac{b_0 z^n + b_1 z^{n-1} + \cdots + b_n}{z^n + a_1 z^{n-1} + \cdots + a_n}$$

Where $R(z)$ is the Z transform of the input and $C(z)$ is the Z transform of the output.

⇒ The transient response of such a system to to the Kronecker delta input, step input, ramp input, and so on, can be obtained easily by use of Matlab.

★ Steady-State Error Analysis

⇒ The steady-state performance of a stable Control System is generally judged by the Steady-State error due to step, ramp, and acceleration input.

⇒ Consider the Continuous time Control System whose open-loop transfer function $G(s)H(s)$ is given by

$$G(s)\,H(s) = \frac{K(T_a s+1)(T_b s+1)\cdots(T_m s+1)}{s^N (T_1 s+1)(T_2 s+1)\cdots(T_p s+1)}$$

⇒ It is Customary to classify the System according to the number of integrators in the open-loop transfer function.

↳ A System is said to be of type 0, type 1, type 2 ··· if $N=0$, $N=1$, $N=2\cdots$ respectively.

System	Steady state Error to		
	Step Input	Ramp input	acceleration input
Type 0	finite	∞	∞
Type 1	0	finite	∞
Type 2	0	0	finite

⇒ However, increasing the type number aggravates the stability problem.

↳ A Compromise between steady-state accuracy and relative stability is always necessary.

⇒ The Concepts of static error Constants can be extended to the discrete-time Control System, as discussed in what follows.

⇒ Discrete-time control system can be classified according to the number of open-loop poles at $z=1$.

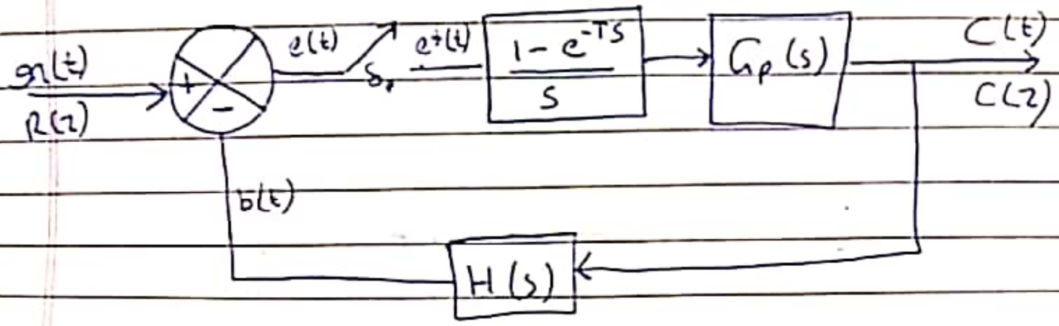
⇒ Suppose open-loop transfer function is given by

$$\text{Open-loop pulse transfer function} = \frac{1}{(z-1)^N}\frac{B(z)}{A(z)}$$

Where $B(z)/A(z)$ Contains neither a pole nor a zero at $z=1$.

⇒ The N is the type of the System.

$\Rightarrow$ Consider the discrete-time control system shown below:-



$$e(t) = r(t) - b(t)$$

$$\lim_{K\to\infty} e(KT) = \lim_{z\to 1}\left[(1-z^{-1})E(z)\right]$$

$$G(z) = (1-z^{-1})\, Z\left[\frac{G_p(s)}{s}\right]$$

$$GH(z) = (1-z^{-1})\, Z\left[\frac{G_p(s)H(s)}{s}\right]$$

$$\frac{C(z)}{R(z)} = \frac{G(z)}{1+GH(z)}$$

$$\Rightarrow E(z) = R(z) - B(z) = R(z) - GH(z)E(z)$$

$$\Rightarrow E(z) = \frac{R(z)}{1+GH(z)}$$

$$e_{ss} = \lim_{z\to 1}\left[(1-z^{-1})\frac{R(z)}{1+GH(z)}\right]$$

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★ Static Position Error Constant

For unit-step input $r(t) = H(t)$

$$R(z) = \frac{1}{1-z^{-1}}$$

$$e_{ss} = \lim_{z \to 1}\left[(1-z^{-1})\frac{1}{1+GH(z)}\frac{1}{1-z^{-1}}\right] = \lim_{z \to 1}\frac{1}{1+GH(z)}$$

⇒ We define the static position error constant K_p as

$$\boxed{K_p = \lim_{z \to 1} GH(z)}$$

$$\boxed{e_{ss} = \frac{1}{1+K_p}}$$

★ Static velocity Error constant

$r(t) = tH(t)$

$$\Rightarrow R(z) = \frac{Tz^{-1}}{(1-z^{-1})^2}$$

$$e_{ss} = \lim_{z \to 1}\left[(1-z^{-1})\frac{1}{1+GH(z)}\frac{Tz^{-1}}{(1-z^{-1})^2}\right] = \lim_{z \to 1}\frac{T}{(1-z^{-1})GH(z)}$$

Let $\boxed{K_v = \lim_{z \to 1}\frac{(1-z^{-1})GH(z)}{T}}$

$$\boxed{e_{ss} = \frac{1}{K_v}}$$

★ Static Acceleration Error Constant

$$r(t) = \frac{1}{2} t^2 H(t)$$

$$R(z) = \frac{T^2 (1 + z^{-1}) z^{-1}}{2 (1 - z^{-1})^3}$$

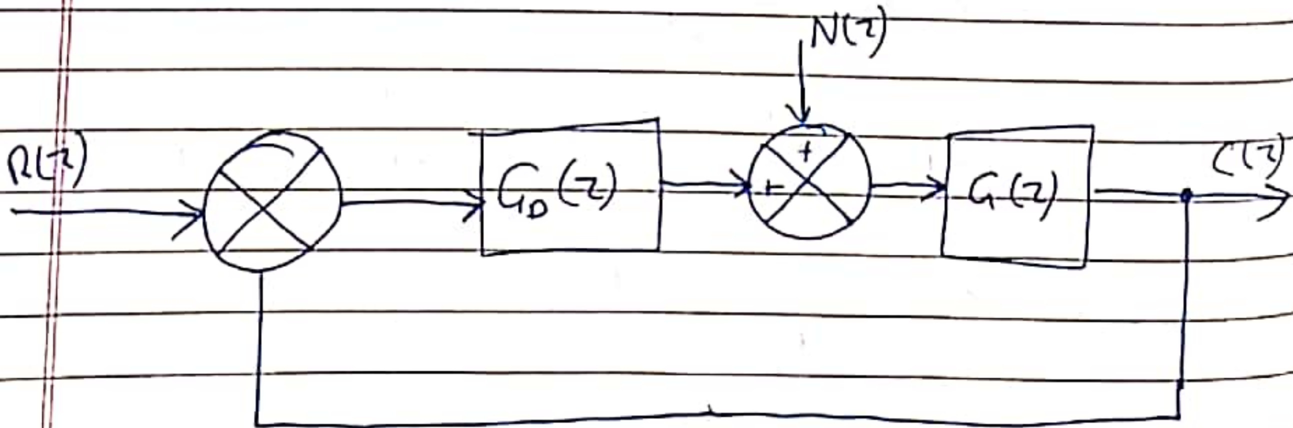
$$e_{ss} = \lim_{z \to 1} \left[(1 - z^{-1}) \frac{1}{1 + GH(z)} \frac{T^2 (1 + z^{-1}) z^{-1}}{2 (1 - z^{-1})^3} \right]$$

$$= \lim_{z \to 1} \frac{T^2}{(1 - z^{-1})^2 GH(z)}$$

Let $\boxed{K_a = \frac{(1 - z^{-1})^2 GH(z)}{T^2}}$

$$\boxed{e_{ss} = \frac{1}{K_a}}$$

★ Response to Disturbances



$$\frac{C(z)}{N(z)} = \frac{G(z)}{1 + G_D(z)G(z)} \quad \{\text{assuming } R(z)=0\}$$

If $|G_D(z)\,G(z)| \gg 1$

$$\Rightarrow \frac{C(z)}{N(z)} \approx \frac{1}{G_D(z)}$$

$$E(z) = R(z) - C(z) = -C(z) = -\frac{N(z)}{G_D(z)}$$

⇒ Thus the larger the gain $G_D(z)$ is, the smaller the error $E(z)$.

⇒ Note that the point where the disturbance enters the System is very important in adjusting the gain of $G_D(z)\,G(z)$.

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4.5> Design based on the Root locus Method

⇒ In addition to the transient response characteristics of a given System, it is often necessary to investigate the effects of the System gain and/or Sampling period on the absolute and relative stability of the Closed-loop System.

↳ For such purpose the root-locus method proves to be very useful.

⇒ The root-locus method developed for Continuous-time systems can be extended to discrete-time System without modifications, except that the stability boundary is changed from the jω axis in the S plane to the unit circle in the Z plane.

* **Angle and Magnitude Condition**

$1 + G(z)H(z) = 0$ or $1 + GH(z) = 0$

$\Rightarrow 1 + F(z) = 0$

$F(z)$ → $G(z)H(z)$ or $GH(z)$

→ It is open-loop pulse transfer function.

$\Rightarrow \boxed{F(z) = -1}$ → $|F(z)| = 1$

→ $\angle F(z) = \pm 180^\circ (2K+1) \quad \forall K = 0, 1, 2 \cdots$

$\Rightarrow$ The value of z that fulfill both the angle and magnitude conditions are the roots of the characteristic equation, or the closed-loop poles.

General Procedure for Constructing Root Loci

1. Obtain the characteristic equation

$$1 + F(z) = 0$$

and then rearrange this equation as given bellow.

$$1 + K\frac{(z+z_1)(z+z_2)\cdots(z+z_m)}{(z+p_1)(z+p_n)\cdots(z+p_n)} = 0$$

↳ Locate open loop poles and zeros in z plane.

2. Find the starting points and terminating points of the root loci.

↳ Starting point $\Rightarrow$ Open loop poles $(K=0)$

↳ Terminating point $(K=\infty)$ $\Rightarrow$ (Open loop zero) or (open loop zero at ∞)

3. Determine the root loci on the real axis.

↳ In constructing the root loci on the real axis, choose a test point on it and try to satisfy the angle condition.

$$\angle F(z) = \pm 180(2K+1) \quad \forall K = 0, 1, 2 \ldots$$

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4. Determine the asymptotes of the root loci.

→ If the test point z is located far from the origin, then the angles of all the complex quantities may be considered the same.

→ One open-loop zero and one open-loop pole then each cancel the effects of the other.

$$\text{Angle of asymptote} = \frac{\pm 180^\circ (2N+1)}{n-m} \quad N = 0, 1, 2 \cdots$$

n → {Number of finite Poles}

m → {number of finite zeros}

⇒ All the asymptotes intersect on the real axis. The point at which they do so is obtained as follows.

$$F(z) = \frac{K\left[z^m + (z_1 + z_2 + \cdots z_m)z^{m-1} + \cdots + z_1 z_2 z_3 \cdots z_m\right]}{z^n + (P_1 + P_2 + \cdots P_m)z^{n-1} + \cdots + P_1 P_2 \cdots P_n}$$

$$= K \times \frac{1}{z^{n-m} + \left[(P_1 + P_2 \cdots + P_n) - (z_1 + z_2 \cdots z_n)\right] z^{n-m-1} + \cdots}$$

for a large value of z above equation can be approximated as :-

$$F(z) \simeq \frac{K}{\left[z + \frac{(P_1 + P_2 + \cdots P_n) - (z_1 + z_2 + \cdots z_m)}{n-m}\right]^{n-m}}$$

⇒ If the abscissa of the intersection of the asymptotes and the real axis is denoted by $-\sigma_a$ then

$$-\sigma_a = -\frac{(P_1+P_2+\cdots P_n)-(Z_1+Z_2+\cdots+Z_m)}{n-m}$$

5. Find the breakaway points and break in points.

→ If a root locus lies between two adjacent open-loop poles on the real axis, then there exists at least one breakaway point between the two poles.

→ If the root locus lies between two adjacent zeros, on the real axis (one zero may be at ∞), then there always exists at least one break in point between the two zeros.

If the characteristic equation is

$$1 + K\frac{B(z)}{A(z)} = 0$$

$$\Rightarrow K = -\frac{A(z)}{B(z)}$$

⇒ The breakaway and breakin point can be determined from the roots of

$$\boxed{\frac{dK}{dz} = 0}$$

6. Determine the angle of departure (or angle of arrival) of the root loci from the complex poles (or at the complex zeros).

↳ Angle can be found out by applying angle condition at that particular pole or zero.

7. Find the point where the root loci crosses the imaginary axis.

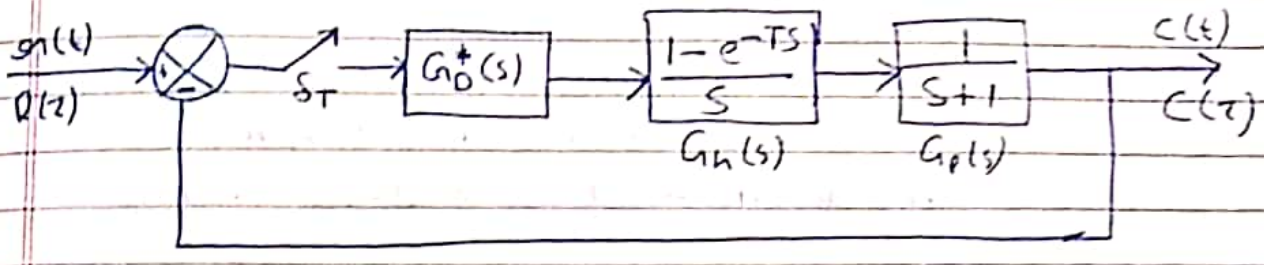
8. Any point on the root loci is a possible closed-loop pole.

↳ Value of K can be found by satisfying magnitude condition.

★ Cancellation of Poles of $G(z)$ With Zeros of $H(z)$

⇒ If $F(z) = G(z)H(z)$ and the denominator of $G(z)$ and the numerator of $H(z)$ involve common factor then the corresponding open loop poles and zeros will cancel each other, reducing the degree of the characteristic equation by one or more.

★ Root locus Diagrams of Digital Control System



Let $G_D(z)$ be an integral type controller.

$$G_D(z) = \frac{K}{1-z^{-1}} = K\frac{z}{z-1}$$

$$Z[G_h(s)\,G_p(s)] = Z\left[\frac{1-e^{-TS}}{S}\,\frac{1}{S+1}\right]$$

$$\boxed{Z[G_h(s)\,G_p(s)] = \frac{1-e^{-T}}{z-e^{-T}}}$$

$$G(z) = G_D(z)\,Z[G_h(s)\,G_p(s)] = \frac{Kz}{z-1}\,\frac{1-e^{-T}}{z-e^{-T}}$$

$\Rightarrow$ $1 + G(z) = 0$ {characteristic equation}

$$\Rightarrow \boxed{1 + \frac{Kz\,(1-e^{-T})}{(z-1)(z-e^{-T})} = 0}$$

★ Effect of Sampling period T on transient Response Characteristics

"For a given value of gain K, increasing the sampling period T will make the discrete-time control system less stable and eventually make it ~~stable~~ unstable.

⇒ A Rule of thumb:-

→ Sample eight to ten times during a cycle of the damped sinusoidal oscillations of the output of the closed-loop system, if it is under-damped.

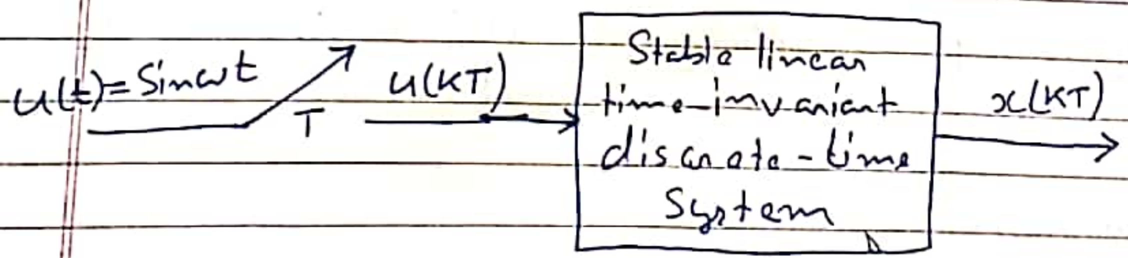
→ For overdamped system, sample eight to ten times during the rise time in the step response.

4.6) Design based on the Frequency-Response Method

⇒ In performing frequency-response tests on a discrete-time system, it is important that the system have a low-pass filter before the sampler so that sidebands are filtered out.

★ Response of a Linear Time-Invariant Discrete-Time System to a Sinusoidal Input

⇒ Consider the stable linear time-invariant discrete-time control system as shown.



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$u(t) = \sin \omega t$

$u(kT) = \sin k\omega T$

$U(z) = Z[\sin k\omega T]$

$$U(z) = \frac{z \sin \omega T}{(z-e^{j\omega T})(z-e^{-j\omega T})}$$

$$X(z) = G(z)\,U(z) = G(z) \frac{z \sin \omega T}{(z-e^{j\omega T})(z-e^{-j\omega T})}$$

$$= \frac{a z}{z-e^{j\omega T}} + \frac{\bar{a} z}{z-e^{-j\omega T}} + [\text{terms due to poles of } G(z)]$$

$$a = G(z) \frac{\sin \omega T}{z-e^{-j\omega T}} \Bigg|_{z=e^{j\omega T}} = \frac{G(e^{j\omega T})}{2j}$$

$$\bar{a} = -\frac{G(e^{-j\omega T})}{2j}$$

Let us define, $G(e^{j\omega T}) = M e^{j\theta}$

& $G(e^{-j\omega T}) = M e^{-j\theta}$

So $$X(z) = \frac{M e^{j\theta}}{2j}\left(\frac{z}{z-e^{j\omega T}}\right) - \frac{M e^{-j\theta}}{2j}\left(\frac{z}{z-e^{-j\omega T}}\right)$$

$$+ [\text{terms due to poles of } G(z)]$$

$$x(kT) = \frac{M}{2j}\left[e^{jk\omega T} e^{j\theta} - e^{-jk\omega T} e^{-j\theta}\right]$$

$$+ Z^{-1}[\text{terms due to poles of } G(z)]$$

$\Rightarrow$ Since the system $G(s)$ has been assumed to be stable, all transient response terms will disappear at steady state and will get the following steady-state response $x_{ss}(kT)$

$$x_{ss}(kT) = \frac{M}{2j}\left[e^{j(k\omega T+\theta)} - e^{-j(k\omega T+\theta)}\right]$$

$$\boxed{x_{ss}(kT) = M \sin(k\omega T+\theta)}$$

$$M = |G(e^{j\omega T})|$$

$$\theta = \angle G(e^{j\omega T})$$

$\Rightarrow$ The function $G(e^{j\omega T})$ is commonly called the Sinusoidal pulse transfer function.

$$e^{j(\omega + \frac{2\pi}{T})T} = e^{j\omega T}$$

↳ Sinusoidal pulse transfer function $G(e^{j\omega T})$ is periodic, with the period equal to T.

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★ Bilinear Transformation and the w plane

⇒ Before we can advantageously apply our well developed frequency response methods to the analysis and design of discrete-time Control Systems, Certain modifications in the z plane approach are necessary.

→ We transform the pulse transfer function in the z plane into that in the w plane.

→ The transformation, commonly called the w transformation, a bilinear transformation, is defined by

$$\left\{ w = \frac{2}{T}\frac{z-1}{z+1} \right\} \quad z = \frac{1+(T/2)w}{1-(T/2)w}$$

$T \rightarrow$ {Sampling period}

⇒ Through the z transform and the w transformation, the primary strip of the left half of the s plane is first mapped into the inside of the unit Circle in the z plane and then mapped into the entire left half of the w plane.

⇒ Once the pulse transfer function G(z) is transformed into G(w) by means of w transformation.

→ It may be treated as a conventional TF in w.

→ Conventional frequency response techniques can then be used in w plane.

⇒ As noted earlier, ν represents the fictitious frequency. By replacing ω by $j\nu$ conventional frequency-response techniques may be used to draw the Bode diagram. for the transfer function in ω.

⇒ Although the ω plane resembles the S plane geometrically, the frequency axis in the ω plane is distorted.

$$\omega\Big|_{\omega=j\nu} = j\nu = \frac{2}{T}\frac{z-1}{z+1}\Big|_{z=e^{j\omega T}} = \frac{2}{T} j \tan\frac{\omega T}{2}$$

$$\nu = \frac{2}{T}\tan\frac{\omega T}{2}$$

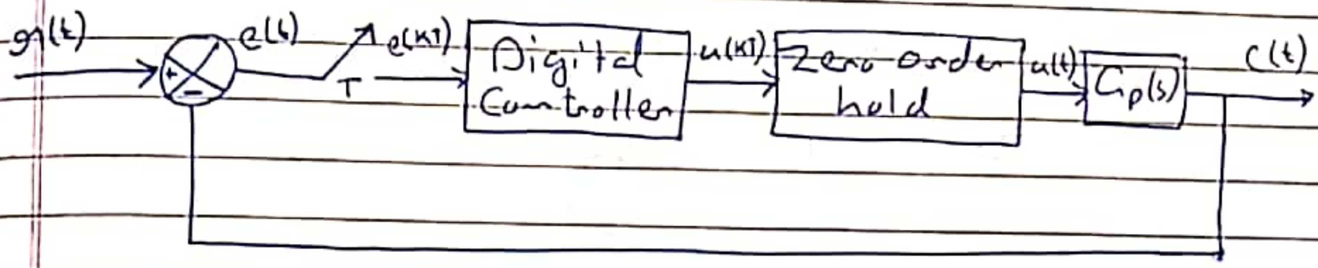
4.7 ⇒ Analytical Design Method

⇒ The main reason why the control actions of analog controllers are limited is that there are physical limitations in pneumatic, hydraulic, and electronic components.

↳ Such limitations may be completely ignored in designing digital controllers.

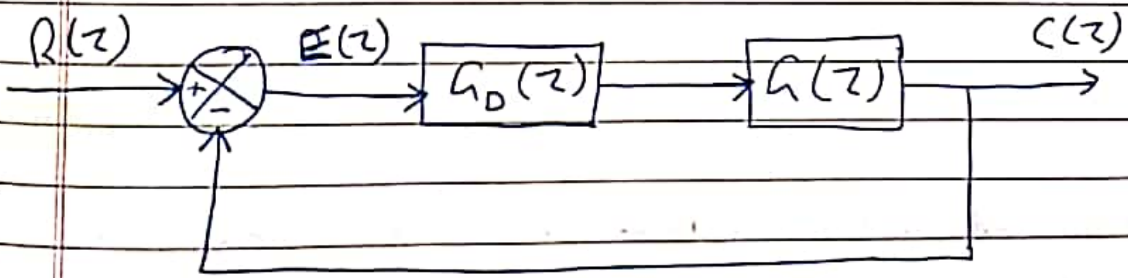
⇒ If the response of a closed-loop control system to a step input exhibits the minimum possible settling time, no steady-state error, and no ripples between the sampling instants, then this type of response is commonly called a deadbeat response.

★ Design of Digital Controllers for Minimum Settling time with Zero Steady State error



Let us define the Z transform of the plant that is preceded by the zero-order hold as $G(z)$

$$G(z) = Z\left[\frac{1-e^{-Ts}}{s} G_p(s)\right]$$



Let the desired closed-loop pulse transfer function as $F(z)$.

$$F(z) = \frac{C(z)}{R(z)} = \frac{G_D(z)G(z)}{1 + G_D(z)G(z)}$$

⇒ Since it is required that the System exhibit a finite settling time with zero steady-state error, the System must exhibit a finite impulse response.

↳ Hence, the desired closed loop pulse transfer function must be of the following form:-

$$F(z) = \frac{a_0 z^N + a_1 z^{N-1} + \cdots + a_N}{z^N}$$

$$\Rightarrow a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots a_N z^{-N}$$

$$\{N \geq n\}$$

Order of the System

$$\Rightarrow G_D(z) = \frac{F(z)}{G(z)[1-F(z)]}$$

⇒ The designed System must be physically realizable.

↳ The conditions for physical realizability may be stated as follows:-

1. The order of the numerator of $G_D(z)$ must be equal to or lower than the order of the denominator.

2. If the plant $G_p(s)$ involves a transportation lag e^{-Ls}, then the designed closed loop System

must involve at least the same magnitude of the transportation Lag.

3. If $G(z)$ is expended into a Series in z^{-1}, the lowest-power term of the Series expansion of $F(z)$ in z^{-1} must be at least as larger as that of $G(z)$.

⇒ In addition to the physical realizability Conditions we must pay attention to the Stability aspect of the System.

↳ Specifically, we must avoide canceling an unstable pole of the plant by a zero of the digital Controller.

↳ If such a Cancellation is attempted any error in the pole-zero cancellation will diverge as time elapses and the System will become Unstable.

⇒ Let us investigate what will happen to the Closed-loop pulse transfer function $F(z)$ if $G(z)$ involves an Unstable pole:

$$G(z) = \frac{G_1(z)}{z-\alpha} \quad \{|\alpha| > 1\}$$

→ It dose not involves terms that canil with $(z-\alpha)$

So $$\frac{C(z)}{R(z)} = \frac{G_D(z)G(z)}{1+G_D(z)G(z)} = \frac{G_D(z)\,\frac{G_1(z)}{z-\alpha}}{1+G_D(z)\,\frac{G_1(z)}{z-\alpha}} = F(z)$$

$$1-F(z) = \frac{1}{1+G_D(z)\,\frac{G_1(z)}{z-\alpha}} = \frac{(z-\alpha)}{(z-\alpha)+G_D(z)G_1(z)}$$

⇒ Since we require that no zero of $G_D(z)$ cancel the unstable pole of $G(z)$ at $z=\alpha$.

↳ We must have that $1-F(z)$ must have $z=\alpha$ as a zero.

Summarize

1. Since the digital controller $G_D(z)$ should not cancel unstable poles of $G(z)$, all unstable poles of $G(z)$ must be included in $1-F(z)$ as zeros.

2. Zeros of $G(z)$ that lie inside the unit circle may be canceled with poles of $G_D(z)$. However, zeros of $G(z)$ that lie on or outside the unit circle must not be cancled with poles of $G_D(z)$. Hence, all zeros of $G(z)$ that lies on or outside the unit circle must be included in $F(z)$ as zero.

$$\Rightarrow E(z) = R(z) - C(z)$$
$$= R(z)\{1 - F(z)\}$$

$$R(z) = \frac{1}{1-z^{-1}} \quad \{\text{if } r(t) = H(t)\}$$

$$R(z) = \frac{T z^{-1}}{(1-z^{-1})^2} \quad \{\text{if } r(t) = t H(t)\}$$

$$R(z) = \frac{T^2 z^{-1}(1+z^{-1})}{2(1-z^{-1})^3} \quad \{\text{if } r(t) = \tfrac{1}{2} t^2 H(t)\}$$

⇒ Thus in general z transform of such time domain polynomial inputs may be written as

$$R(z) = \frac{P(z)}{(1-z^{-1})^{q+1}} \quad P(z) \rightarrow \text{Polynomial in } z^{-1}$$

$$\Rightarrow E(z) = \frac{P(z)[1-F(z)]}{(1-z^{-1})^{q+1}}$$

⇒ To ensure that the system reaches steady state in finite number of sampling periods and maintain zero steady-state error, $E(z)$ must be a polynomial in z^{-1} with finite number of term.

$$1 - F(z) = (1-z^{-1})^{q+1} N(z) \quad N(z) \rightarrow \text{Polynomial in } z^{-1}$$

$$E(z) = P(z)\, N(z)$$

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⇒ From the preceding analysis, the pulse transfer function of the digital controller can be determined as follows:-

↳ By letting F(z) satisfy the physical realizability and stability conditions.

$$G_D(z) = \frac{F(z)}{G(z)\,(1 - z^{-1})^{a+1}\,N(z)}$$