3

Control System design in State Space

10.1> Introduction

- → Pole-placement method
- → Observers
- → Quadratic optimal regulator.
- → Robust Control {Introduction}

{Somewhat similar to the root-locus method}

The basic difference is that in the root-locus design we place only the dominant closed-loop poles at the desired locations while in the pole placement design we place all closed-loop poles at desired location.

10.2> Pole Placement (or Pole-assignment technique)

⇒ We assume all state variables are measurable and are available for feedback.

⇒ If the System Considered is completely state Controllable, then poles of the closed-loop system may be placed at any desired locations by means of state feedback through an appropriate state feedback gain matrix.

⇒ The present design technique begins with a determination of the desired closed-loop poles based on the transient response & frequency - response requirements.
eg → speed, damping ratio, bandwidth, as well as steady-state requirement.

⇒ Let us assume that we decided that the desired Closed-loop poles are to be at $s = \mu_1$, $s = \mu_2$ ... $s = \mu_n$.

↳ By choosing an appropriate gain matrix for state feedback it is possible to force the System to have Closed-loop poles at the desired locations, Provided the original System is Completely State Controllable.

> In this Chapter we limit our discussions to Single-input, Single-output Systems.

→ When Control Signal is a Vector quantity, the state feedback gain matrix is not unique.

★ Design by pole Placement

⇒ There is a Cost associated with placing all closed-loop poles, however, because placing all closed loop poles requires successful measurements of all State Variables or else requires the inclusions of a state observer in the System.

⇒ There is also requirement that the System is Completely state Controllable.

⇒ Consider a Control System be:

$$\dot{X} = AX + Bu$$

(1) $$y = CX + Du$$

X = State Vector (n-Vector)
y = output signal (scalar)
u = Control signal (scalar)
A = $n \times n$ Constant matrix
B = $n \times 1$ Constant matrix
C = $1 \times n$ Constant matrix
D = Constant (scalar)

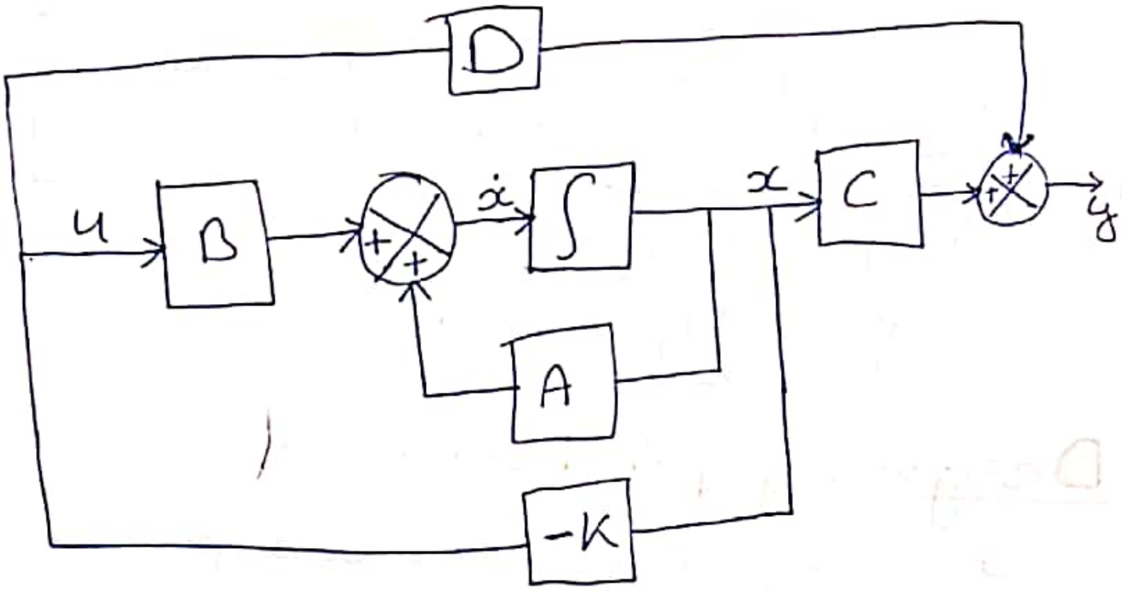
⇒ We choose the control signal to be:

$$\boxed{u = -\vec{K}\vec{x}} \quad \text{——} \ (2)$$

$(1 \times n)$ State feedback gain matrix

$(n \times 1)$ State vector

⇒ We assume
(1) all state variable are available for feedback.
(2) u is unconstrained



⇒ This closed-loop system has no input. Its objective is to maintain the zero output.

⇒ Such system where the reference input is always zero (or non zero constant) is called a regulator system.

⇒ Substituting (2) in (1) we get:-

$$\dot{X} = AX + B(-KX)$$

$$\Rightarrow \dot{X} = (A - KB)X$$

(Solution) $$\boxed{X(t) = e^{(A-BK)t} X(0)} \quad \text{——} \ (3)$$

Initial state caused by external disturbances

⇒ The stability and transient response characteristics are determined by the eigenvalue of the matrix $(A-BK)$.

↳ If matrix K is chosen properly, the matrix $A-BK$ can be made an asymptotically stable matrix for all $x(0)$.

↳ (i.e. $x(t)$ approaches 0 as $t \to \infty$)

⇒ The eigen-values of matrix $A-BK$ are called the <u>regulator poles</u>.

* If these regulator poles are placed in the left half s-plane, then $x(t) \to 0$ at $t \to \infty$.
* The problem of placing the regulator poles at the desired location is called a pole-placement problem.

★ <u>Necessary and Sufficient Condition for Arbitrary Pole Placement</u>

⇒ Suppose the system of Equation ① is not completely state controllable.

⇒ Then the rank of the controllability matrix is less than n

$$\text{rank}\,[B \,\vdots\, AB \,\vdots\, \cdots \,|\, A^{n-1}B] = q < n$$

⇒ This means that there are q linearly independent column vectors in the controllability matrix.

↳ Let q linearly independent column vectors be $f_1\ f_2 \cdots f_q$

↳ Let the remaining column vectors be $V_{q+1}, V_{q+2} \cdots V_n$

Let, $P = [f_1 \mid f_2 \mid \cdots \mid f_q \mid V_{q+1} \mid V_{q+2} \mid \cdots \mid V_n]$

⇒ By using matrix P as the transformation matrix define

$$\hat{A} = P^{-1}AP \quad , \quad \hat{B} = P^{-1}B$$

⇒ It can be proved that :-

$$\underbrace{\hat{A}}_{n\times n} = \left[\begin{array}{c|c} \underbrace{A_{11}}_{q\times q} & \underbrace{A_{12}}_{q\times(n-q)} \\ \hline \underbrace{0}_{(n-q)\times q} & \underbrace{A_{22}}_{(n-q)\times(n-q)} \end{array}\right] \qquad \underbrace{\hat{B}}_{n\times 1} = \left[\begin{array}{c} \underbrace{B_{11}}_{q\times 1} \\ \hline \underbrace{0}_{(n-q)\times 1} \end{array}\right]$$

Let $\hat{K} = KP = [\underbrace{K_1}_{1\times q} \mid \underbrace{K_2}_{1\times(n-q)}]$

Then we have

$$|sI - A + BK| = |P^{-1}(sI - A + BK)P|$$

$$\Rightarrow |sP^{-1}P - P^{-1}AP + P^{-1}BKP|$$

$$\Rightarrow \left|sI - \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline 0 & A_{22}\end{array}\right] + \begin{bmatrix} B_{11} \\ 0 \end{bmatrix}[K_1 \mid K_2]\right|$$

$$\Rightarrow \begin{vmatrix} sI_q - A_{11} + B_{11}K_1 & -A_{12} + B_{11}K_2 \\ 0 & sI_{n-q} - A_{22} \end{vmatrix}$$

$$\Rightarrow |sI_q - A_{11} + B_{11}K_1| \, |sI_{n-q} - A_{22}| = 0$$

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⇒ Notice that the eigenvalues of A_{22} do not depend on K.

⇒ Thus if the System is not Completely State Controllable, then there are eigenvalues of matrix A that can not be arbitrarily placed.

⇒ To place the eigenvalues of matrix A − BK arbitrarily, the System must be Completely State Controllable. (Necessary Condition)

Sufficient Condition

{If the System is completely state controllable, then all eigenvalues of matrix A can be arbitrarily placed.}

⇒ In proving a Sufficient Condition, it is Convenient to transform the state equation given by eq ① into Controllable Canonical form.

⇒ Define a transformation matrix T by

$$T = MW$$

Controllability Matrix

$$M = [B \,\vdots\, AB \,\vdots\, \cdots \,|\, A^{n-1}B]$$

$$W = \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & 1 \\ a_{n-2} & a_{n-3} & \cdots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ a_1 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

{a_i are Coefficient of the characteristic polynomial}

$$|SI - A| = s^n + a_1 s^{n-1} + \cdots + s\, a_{n-1} + a_n$$

Let us define new state vector $\hat{x}$ by

$$x = T\hat{x}$$

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⇒ If the rank of the controllability matrix M is n, then the inverse of matrix T exists.

$$\Rightarrow \dot{\hat{x}} = T^{-1}AT\hat{x} + T^{-1}Bu \quad \text{--- (9)}$$

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \cdots & 1 \\ -a_n & -a_{n-1} & \cdots & \cdots & -a_1 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

⇒ Thus given a state equation, it can be transformed into the controllable canonical form if the system is completely state controllable.

⇒ Let us choose a set of the desired eigenvalues as $\mu_1\ \mu_2 \cdots \mu_n$.

⇒ Then the desired characteristic equation becomes:-

$$(s-\mu_1)(s-\mu_2)\cdots(s-\mu_n) = s^n + \alpha_1 s^{n-1} + \cdots \alpha_{n-1}s + \alpha_n = 0$$

⇒ Let us write,

$$KT = [\delta_n\ \delta_{n-1}\ \cdots\ \delta_1]$$

⇒ When $u = -KT\hat{x}$ is used to control the system, eq (9) becomes:-

$$\dot{\hat{x}} = T^{-1}AT\hat{x} - T^{-1}BKT\hat{x}$$

⇒ The characteristic equation is

$$|SI - T^{-1}AT + T^{-1}BKT| = 0$$

⇒ Now let us simplify the characteristic equation of the system in the controllable canonical form.

$$\left| SI - \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & 1 \\ -a_n & -a_{n-1} & & -a_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} \delta_n & \delta_{n-1} & \cdots & \delta_1 \end{bmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} S & -1 & \cdots & 0 \\ 0 & S & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ a_n+\delta_n & a_{n-1}+\delta_{n-1} & \cdots & S+a_1+\delta_1 \end{vmatrix} = 0$$

$$\Rightarrow S^n + (a_1+\delta_1)S^{n-1} + \cdots + (a_{n-1}+\delta_{n-1})S + (a_n+\delta_n) = 0$$

So $\alpha_1 = (a_1 + \delta_1)$

$\alpha_2 = (a_2 + \delta_2)$

$\vdots$

$\alpha_n = (a_n + \delta_n)$

$$\Rightarrow K = [\alpha_n - a_n \,|\, \alpha_{n-1} - a_{n-1} \,|\, \cdots \,|\, \alpha_1 - a_1] T^{-1}$$

⇒ Thus, if the system is completely state controllable, all eigenvalues can be arbitrarily placed by choosing suitable matrix K. {Sufficient Condition}

★ Determination of Matrix K Using Transformation Matrix T

Let the System be defined by:-

$$\dot{x} = Ax + Bu$$

and the Control Signal is given by:-

$$u = -Kx$$

⇒ The feedback gain matrix K forces the eigen value of A−BK to be $\mu_1, \mu_2 \ldots \mu_n$ (desired values)

{may be real or Complex}

Step 1: Check the controllability condition for the System. If the System is completely state Controllable, then use the following steps.

Step 2: Form the Characteristic polynomial of matrix A, that is

$$|sI-A| = s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n$$

determine the values of $a_1, a_2 \ldots a_n$.

Step 3: Determine the transformation matrix T that transforms the system state equation into the Controllable Canonical form.

$$T = MW$$ {M & W are as shown as previous}

Step 4: Using the desired eigen value, write the desired Characteristic polynomial.

$$(s-\mu_1)(s-\mu_2)\ldots(s-\mu_n) = s^n + \alpha_1 s^{n-1} + \ldots + \alpha_{n-1} s + \alpha_n$$

and determine the values of $\alpha_1, \alpha_2 \ldots \alpha_n$.

Step 5: The required state feedback gain matrix K can be determined as:-

$$K = [\alpha_n - a_n \mid \alpha_{n-1} - a_{n-1} \mid \cdots \mid \alpha_1 - a_1] T^{-1}$$

★ Determination of Matrix K Using Direct Substitution Method

⇒ If the system is of low order ($n \leq 3$), direct substitution of matrix K into the desired characteristic polynomial may be simpler.

⇒ For example if $n=3$

$$K = [K_1 , K_2 \; K_3]$$

⇒ Substitute this K matrix into the desired characteristic polynomial and equate it to $(s-\mu_1)(s-\mu_2)--(s-\mu_3)$.

$$|sI - A + BK| = (s-\mu_1)(s-\mu_2)(s-\mu_3)$$

⇒ By equating the coefficients of the like powers of s on both sides, it is possible to determine the values of $K_1 K_2$ & K_3.

{If system is not completely controllable, matrix K cannot be determined.}

{no solution exists}

★ Determination of Matrix K Using Ackermann's Formula

⇒ Consider the System:-

$$\dot{x} = Ax + Bu \quad \text{—— (1)}$$

⇒ where we use the state feedback control
$u = -Kx$. —— (2)

⇒ We assume that the System is completely State Controllable.

⇒ We also assume that the desired Closed-loop Poles are at $S = \mu_1$ $S = \mu_2$ - - - $S = \mu_n$.

⇒ Using (2) in (1) we get:-

$$\dot{x} = (A - BK)x$$

⇒ Let us define,

$$\tilde{A} = A - BK$$

⇒ The desired Characteristic equation is

$$|SI - A + BK| = |SI - \tilde{A}| = (s-\mu_1)(s-\mu_1)' \cdots (s-\mu_n)$$

$$= S^n + \alpha_1 S^{n-1} + \cdots \alpha_{n-1} S + \alpha_n = 0$$

⇒ Using Cayley-Hamilton theorem we get.

$$\phi(\tilde{A}) = \tilde{A}^n + \alpha_1 \tilde{A}^{n-1} + \cdots \alpha_{n-1}\tilde{A} + \alpha_n I = 0 \text{—(3)}$$

⇒ To Simplify the derivation, we consider the Case when $n=3$. The following derivation can easily be extended for any value of n.

$\Rightarrow$ Consider the following:-

- $I = I$
- $\tilde{A} = A - BK$
- $\tilde{A}^2 = (A-BK)^2 = A^2 - 2ABK + (BK)^2$

$\Rightarrow A^2 - ABK - BK(A-BK)$

$\Rightarrow A^2 - ABK - BK\tilde{A}$

- $\tilde{A}^3 = (A-BK)^3 = A^3 - A^2BK - ABK\tilde{A} - BK\tilde{A}^2$

$\Rightarrow$ Multiplying the preceding equations in order by $\alpha_3\ \alpha_2\ \alpha_1$ & $\alpha_0\ (\alpha_0 = 1)$ and adding the result.

$$\alpha_3 I + \alpha_2 \tilde{A} + \alpha_1 \tilde{A}^2 + \tilde{A}^3.$$

$$= \alpha_3 I + \alpha_2 (A-BK) + \alpha_1 (A^2 - ABK - BK\tilde{A}) + A^3 - A^2BK - ABK\tilde{A} - BK\tilde{A}^2$$

$$= \alpha_3 I + \alpha_2 A + \alpha_1 A^2 + A^3 - \alpha_2 BK - \alpha_1 ABK - \alpha_1 BK\tilde{A} - A^2BK - ABK\tilde{A} - BK\tilde{A}^2$$

We know, $\phi(A) \neq 0$ & $\phi(\tilde{A}) = 0$

$$\phi(\tilde{A}) = \phi(A) - \alpha_2 BK - \alpha_1 BK\tilde{A} - BK\tilde{A} - \alpha_1 ABK - ABK\tilde{A} - A^2BK$$

(3)

$$\Rightarrow \phi(A) = B(\alpha_2 K + \alpha_1 K\tilde{A} + K\tilde{A}^2) + AB(\alpha_1 K + K\tilde{A}) + A^2BK$$

$$= [B \mid AB \mid A^2B] \begin{bmatrix} \alpha_2 K + \alpha_1 K\tilde{A} + K\tilde{A}^2 \\ \alpha_1 K + K\tilde{A} \\ K \end{bmatrix}$$

$\Rightarrow$ Since the system is completely state controllable, the inverse of the controllability matrix exists.

$$[B \mid AB \mid A^2B]^{-1}\,\phi(A) = \begin{bmatrix} \alpha_2 K + \alpha_1 K\tilde{A} + K\tilde{A}^2 \\ \alpha_1 K + K\tilde{A} \\ K \end{bmatrix}$$

⇒ Premultiplying both side of this last equation ~~of~~ by [0 0 1] we obtain:-

$$K = [0\ 0\ 1]\,[B \mid AB \mid A^2B]\,\phi(A)$$

⇒ For any arbitrary positive integer n we have:-

$$\boxed{K = [0\ 0 \cdots 1]\,[B \mid AB \mid \cdots \mid A^{n-1}B]^{-1}\,\phi(A)}$$

{Ackermann's formula for the determination of the state feedback matrix K}

- Controllers
  - Regulator system
    {Reference input is constant including zero}
  - Control system
    {when reference input is time varying}

★ Choosing the Locations of Desired Closed-loop Poles

The most frequently used approach is to choose such poles based on experience in the root-locus design.

→ **Note**: If we place the dominant closed loop poles far from the jω axis, so that the

System response become very fast, the signal in the system become very large, with the result that the system may become nonlinear. This should be avoided.

Another approach is based on the quadratic optimal control approach.

→ This approaches will determine the desired closed loop poles such that it balances between the acceptable response and the amount of control energy required.

10.4) Design of Servo System

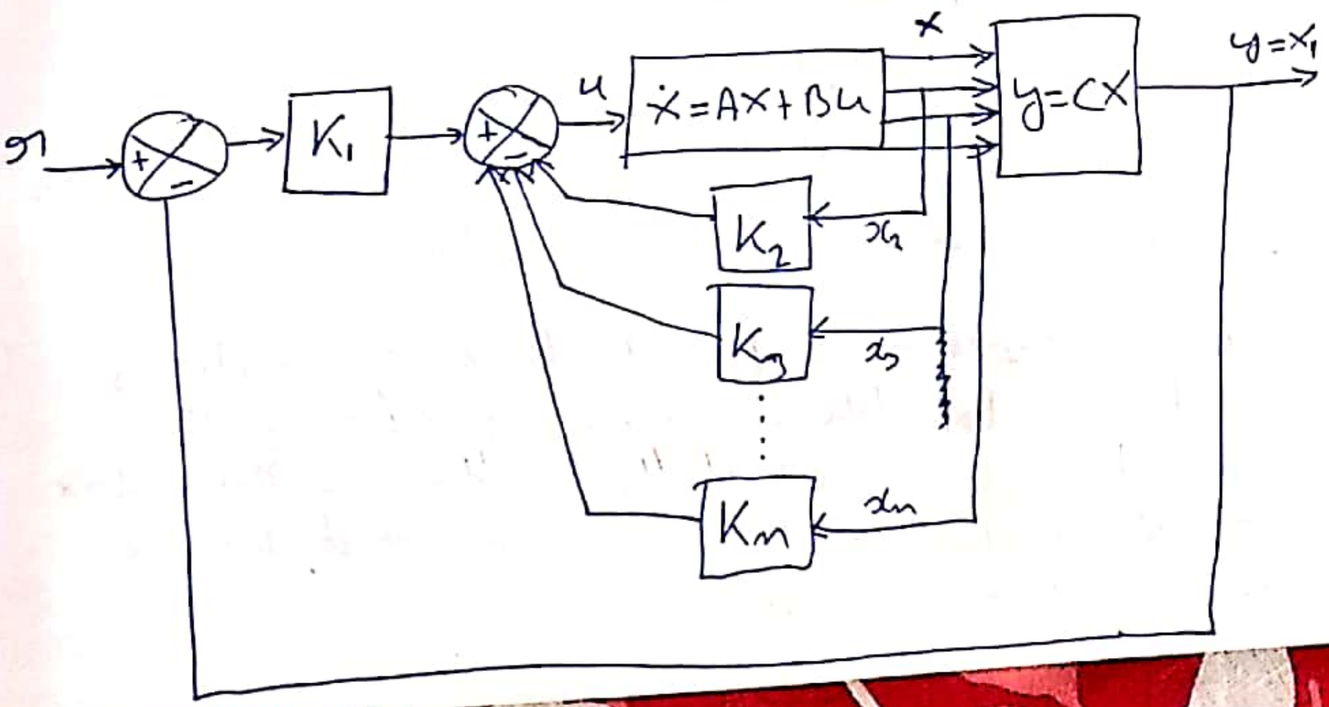
★ Design of Type 1 Servo System when the Plant has an Integrator

Let the plant be defined by:-

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

⇒ By proper choice of a set of state variables ; it is possible to choose the output to be equal to one of the state variable.

↳ Let us assume $y = x_1$



⇒ In present analysis we assume that the reference input r is a step function.

$$u = -\begin{bmatrix} 0 & K_2 & K_3 & \cdots & K_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + K_1(r - x_1)$$

$$u = -Kx + K_1 r \quad \rightarrow \quad [K_1 \; K_2 \; \cdots \; K_n]$$

So, ⇒ $\dot{x} = Ax + Bu = \cancel{Ax}\ (A-BK)x + BK_1 r$

$$\boxed{\dot{x} = (A-BK)x + BK_1 r}$$

⇒ $y(\infty) \rightarrow r$ ~~as~~ $u(\infty) \rightarrow 0$ {r = step input}

at steady state: $\dot{x}(\infty) = (A-BK)x(\infty) + BK_1 r(\infty)$

$\downarrow$ r (constant)

$\dot{x}(t) - \dot{x}(\infty) = (A-BK)[x(t) - x(\infty)]$ $\quad$ $r(t)\ \forall\, t>0$

Let $e(t) = x(t) - x(\infty)$

So, above condition becomes:

$$\boxed{\dot{e} = (A-BK)e} \quad \text{—— (1)}$$

eq (1) describes the error dynamics.

⇒ If the system defined above is completely state controllable then by specifying the desired eigenvalues $\mu_1, \mu_2 \ldots \mu_n$ for the matrix $A-BK$ matrix K can be determined by the pole placement technique.

⇒ The steady state values of $x(t)$ & $u(t)$ can be found out as follows:-

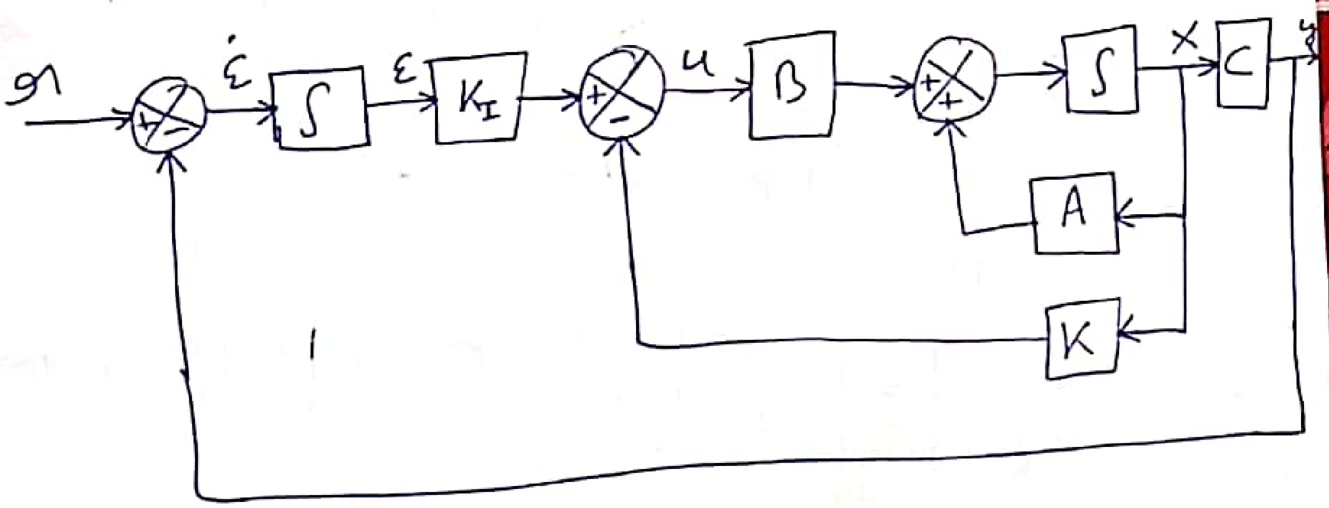
$$\dot{x}(\infty) = 0 = (A-BK)x(\infty) + BK_1 r$$

$$\Rightarrow \boxed{x(\infty) = -(A-BK)^{-1} BK_1 r}$$

$$\boxed{u(\infty) = -Kx(\infty) + K_1 r}$$

* Design of Type 1 Servo System when the plant has no Integrator.

⇒ If the plant has no integrator (type 0 plant), the basic principle of the design of a type 1 Servo System is to insert an integrator in the feed forward path between the error Compensator and the plant as shown below.



⇒ From diagram we obtain:

$$\boxed{\dot{x} = Ax + Bu} \quad \text{--- (1)}$$

$$y = Cx$$

$$u = -Kx + K_I \varepsilon$$

$$\dot{\varepsilon} = r - y = r - Cx \quad \text{--- (2)}$$

⇒ Assume that the reference input (Step function) is applied at $t=0$. Then for $t>0$, the System dynamics can be determined by an equation that is Combination of eq ① & ②

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\varepsilon}(t) \end{bmatrix} = \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ \varepsilon(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r(t) \text{ -③}$$

⇒ We shall design an asymptotically stable System such that $x(\infty)$, $\varepsilon(\infty)$ & $u(\infty)$ approache Constant values, respectively

→ At Steady state $\dot{\varepsilon}(t)=0$ & $y(\infty)=r$

⇒ At steady state we have:-

$$\begin{bmatrix} \dot{x}(\infty) \\ \dot{\varepsilon}(\infty) \end{bmatrix} = \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix} \begin{bmatrix} x(\infty) \\ \varepsilon(\infty) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(\infty) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r(\infty) \text{ -④}$$

⇒ as $r(t)$ ⇒ Step Input ⇒ $r(\infty) = r(t) = r$ (const)

⇒ Subtracting eq ④ from eq ③ we get

$$\begin{bmatrix} \dot{x}(t) - \dot{x}(\infty) \\ \dot{\varepsilon}(t) - \dot{\varepsilon}(\infty) \end{bmatrix} = \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix} \begin{bmatrix} x(t) - x(\infty) \\ \varepsilon(t) - \varepsilon(\infty) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} [u(t) - u(\infty)]$$

Let us define; $x(t) - x(\infty) = x_e(t)$

$\varepsilon(t) - \varepsilon(\infty) = \varepsilon_e(t)$

$u(t) - u(\infty) = u_e(t)$

$$\Rightarrow \begin{bmatrix} \dot{x}_e(t) \\ \dot{\varepsilon}_e(t) \end{bmatrix} = \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix} \begin{bmatrix} x_e(t) \\ \varepsilon_e(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u_e(t)$$

where, $u_e(t) = -K x_e(t) + K_I \varepsilon_e(t)$

Let us define :-

$$e(t) = \begin{bmatrix} x_e(t) \\ \varepsilon_e(t) \end{bmatrix} \quad \hat{A} = \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix} \quad \hat{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}$$

⇒ So previous equation become:-

$$\dot{e} = \hat{A}e + \hat{B}u_e \quad — (5)$$

$$u_e = -\hat{K}e \quad — (6), \quad \hat{K} = [K \,\vdots\, -K_I]$$

⇒ The State error equation can be obtained by substituting eq (6) in (5)

$$\dot{e} = (\hat{A} - \hat{B}\hat{K})e$$

⇒ If the desired eigenvalues of matrix $\hat{A} - \hat{B}\hat{K}$ are specified as $\mu_1\ \mu_2 \cdots \mu_{n+1}$, then the state feedback gain matrix K and the integral gain constant K_I can easily be determined by the pole-placement technique.

⇒ Not all State Variables Can be measured directly. If this is the case, we need to use a <u>State Observer</u>.

10.5) State Observers

⇒ In the pole-placement approach to the design of control system, we assumed that all state variables are feedback. ~~Especially~~

⇒ In practice however, not all state variables are available for feedback.

⇒ Estimation of unmeasurable state variable is commonly called observation.

↳ A device (or computer program) that estimates or observes the state variables is called a state observer.

Full order state observer → If the state observer observes all state variables of the system.

Reduced order state observer → If the state observer observes not all state variables of the system.

Minimum order state observer → If the order of the reduced order state observer is the minimum possible it is called minimum order state observer.

★ State Observer

↳ State observer estimates the state variables based on the measurements of the output and control variables.

⇒ We shall use $\tilde{x}$ to designate the observed state vector.

⇒ Consider the plant defined by

$$\dot{x} = Ax + Bu \quad \text{—} (1)$$
$$y = Cx \quad \text{—} (2)$$

⇒ The mathematical model of the observer is basically the same as that of the plant, except that we include additional term that include the estimation error to compensate for inaccuracy in Matrix A & B & the lack of the initial error.

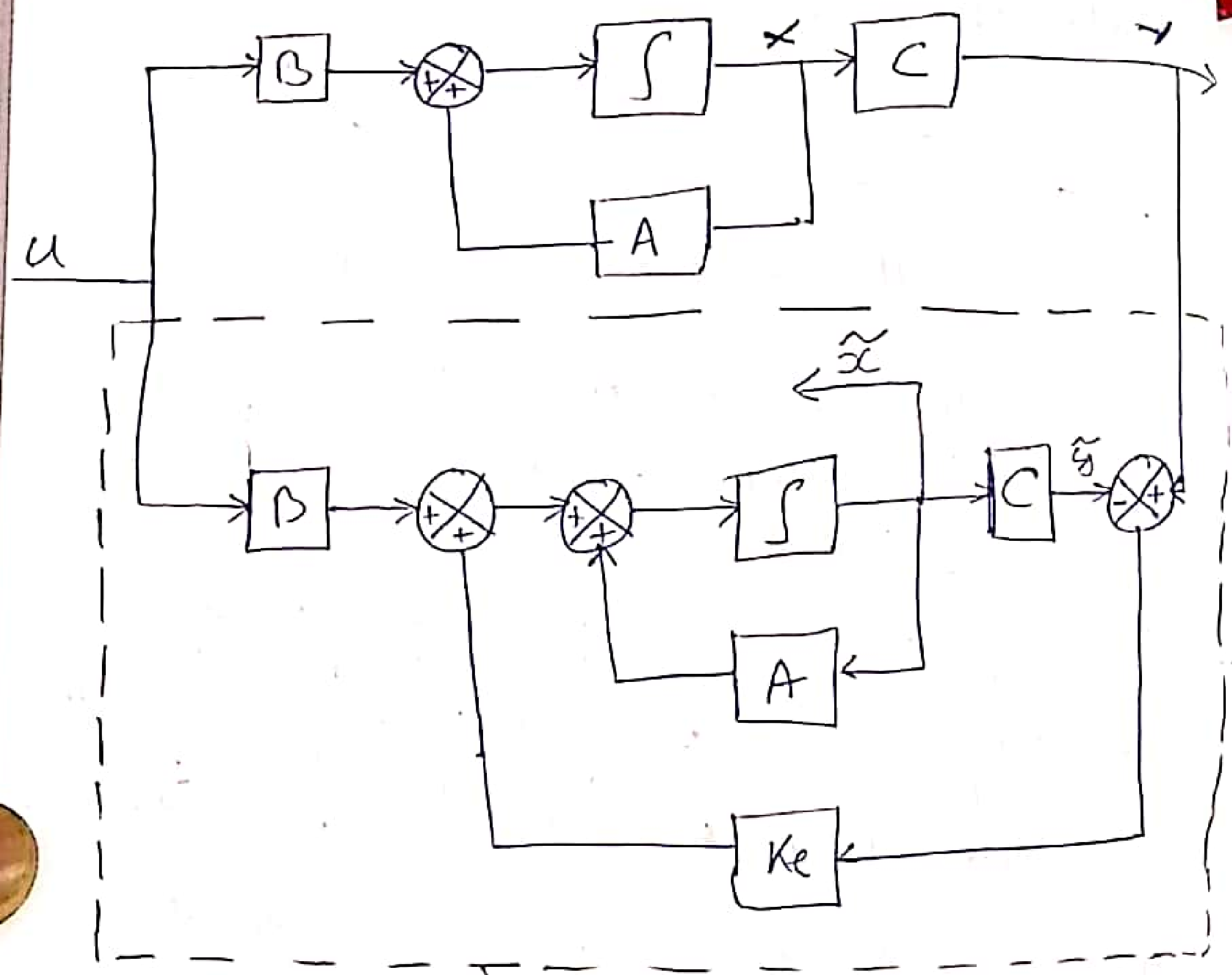
⇒ Thus we define the mathematical model of the observer to be:

$$\dot{\tilde{x}} = A\tilde{x} + Bu + K_e(y - C\tilde{x}) \quad \text{—} (3)$$
$$= (A - K_eC)\tilde{x} + Bu + K_ey$$

$\tilde{x} \rightarrow$ estimated state
$C\tilde{x} \rightarrow$ estimated output
$y \rightarrow$ actual output
$u \rightarrow$ actual input.

⇒ Matrix K_e is called observer gain matrix.

↳ This term continuously corrects the model output & improves the performance of observer.



{Full-order state observer}

★ <u>Full-order State observer</u>

⇒ Consider plant is discribed by eq ① & ② & Observer by eq ③.

⇒ Substituting eq ① & ③ we get:-

$$\dot{x} - \dot{\tilde{x}} = Ax - A\tilde{x} - K_e(Cx - C\tilde{x})$$
$$= (A - K_eC)(x - \tilde{x})$$

Let $e = x - \tilde{x}$

$$\Rightarrow \quad \dot{e} = (A - K_eC)e \quad — ④$$

⇒ From eq (b) we see that dynamic behavior of the error vector is determined by the eigen-values of matrix $(A - K_eC)$.

↳ If matrix $(A - K_eC)$ is a stable matrix the error vector will converge to zero for any initial error vector $e(0)$.

↳ If the eigenvalues of matrix $A - K_eC$ are chosen in such a way that the dynamic behaviour of the error vector is asymptotically stable and is adequately fast, then any error vector will tend to zero with an adequate speed.

⇒ If the plant is completely observable, then it can be proved that it is possible to choose matrix K_e such that $A - K_eC$ has arbitrarily desired eigen value.

★ Dual Problem

⇒ The problem of designing a full-order observer becomes that of determining the observer gain matrix K_e such that the error dynamics are asymptotically stable with sufficient speed of response.

↳ (ie determining an appropriate K_e such that $A - K_eC$ has desired eigen value)

⇒ In designing the full-order state observer, we may solve the dual problem, that is, solve the pole-placement problem for the dual system.

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$$\dot{Z} = A^T Z + C^T V$$
$$\eta = B^T Z$$

assume the Control Signal V to be

$$V = -KZ$$

⇒ If the dual system is completely state controllable, then the state feedback gain matrix K can be determined such that matrix $A^T - C^T K$ will yield a set of desired eigen values.

⇒ If $\mu_1\ \mu_2 \cdots \mu_n$ are the desired eigen values of the state observer matrix, then by taking the same μ_i's as the desired eigen values of the state-feedback matrix of the dual system, we obtain:-

$$|SI - (A^T - C^T K)| = (S-\mu_1)(S-\mu_2)\cdots(S-\mu_n)$$

⇒ eigen values of $A^T - C^T K$ are same as eigen values of $A - K^T C$.

We find $\boxed{K_e = K^T}$

⇒ Thus, using the matrix K determined by the pole-placement approach in the dual system, the observer gain matrix K_e for the original system can be determined by using the relationship $K_e = K^T$.

*Transformation Approach to Obtain State Observer Gain Matrix Ke

By following the same approach as we used in deriving the equation for the state feedback gain matrix K, we can obtain

$$K_e = Q \begin{bmatrix} \alpha_n - a_n \\ \alpha_{n-1} - a_{n-1} \\ \vdots \\ \alpha_1 - a_1 \end{bmatrix} = (WN^T)^{-1} \begin{bmatrix} \alpha_n - a_n \\ \alpha_{n-1} - a_{n-1} \\ \vdots \\ \alpha_1 - a_1 \end{bmatrix}$$

$$Q = (WN^T)^{-1}$$

$$N = [C^T \vdots A^TC^T \vdots \cdots (A^T)^{n-1}C^T]$$

$$W = \begin{bmatrix} a_{n-1} & a_{n-1} & \cdots & a_1 & 1 \\ a_{n-2} & a_{n-1} & & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ a_1 & 1 & & 0 & 0 \\ 1 & 0 & & 0 & 0 \end{bmatrix}$$

* Direct-Sustitution Approach to obtain State Observer gain Matrix Ke

If the System is of low order, then direct substitution of matrix Ke into the desired charateristic Polynomial may be simpler.

ex ⇒ If x is a 3-Vector

$$K_e = \begin{bmatrix} K_{e1} \\ K_{e2} \\ K_{e3} \end{bmatrix}$$

$$|SI - (A - K_eC)| = (S-\mu_1)(s-\mu_1)(s-\mu_3)$$

⇒ By equating the coefficients of the like power of s on both side, we get K_{e1} K_{e2} & K_{e3}.

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★ Ackermann's Formula

⇒ Consider the System defined by

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

$$K = [0\,0 \cdots 0\,1]\left[B \,\vdots\, AB \,\vdots\, \cdots \,\vdots\, A^{n-1}B\right]^{-1} \phi(A)$$

{from pole placement}

⇒ For the dual of the System:

$$\dot{z} = A^T z + C^T v$$
$$n = B^T z$$

⇒ The preceding Ackermann's formula for Pole placement can be modified to

$$K = [0\,0 \cdots 1]\left[C^T \,\vdots\, A^T C^T \,\vdots\, \cdots \,\vdots\, (A^T)^{n-1} C^T\right]^{-1} \phi(A^T)$$

⇒ As stated earlier $K_e = K^T$

$$K_e = K^T = \phi(A^T)^T \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-2} \\ CA^{n-1} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$\boxed{K_e = \phi(A) \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-2} \\ CA^{n-1} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}$$

where $\phi(s)$ is the desired characteristic polynomial for the state observer.

$$\phi(s) = (s-\mu_1)(s-\mu_2)\cdots(s-\mu_n)$$

★ Comment on Selecting best K_e

⇒ The choice of a set of $\mu_1, \mu_2 \cdots \mu_n$ is in many instances, not unique.

↳ As a general rule however, the observer poles must be two to five times faster than the controller poles to make sure the observation error converges to zero quickly.

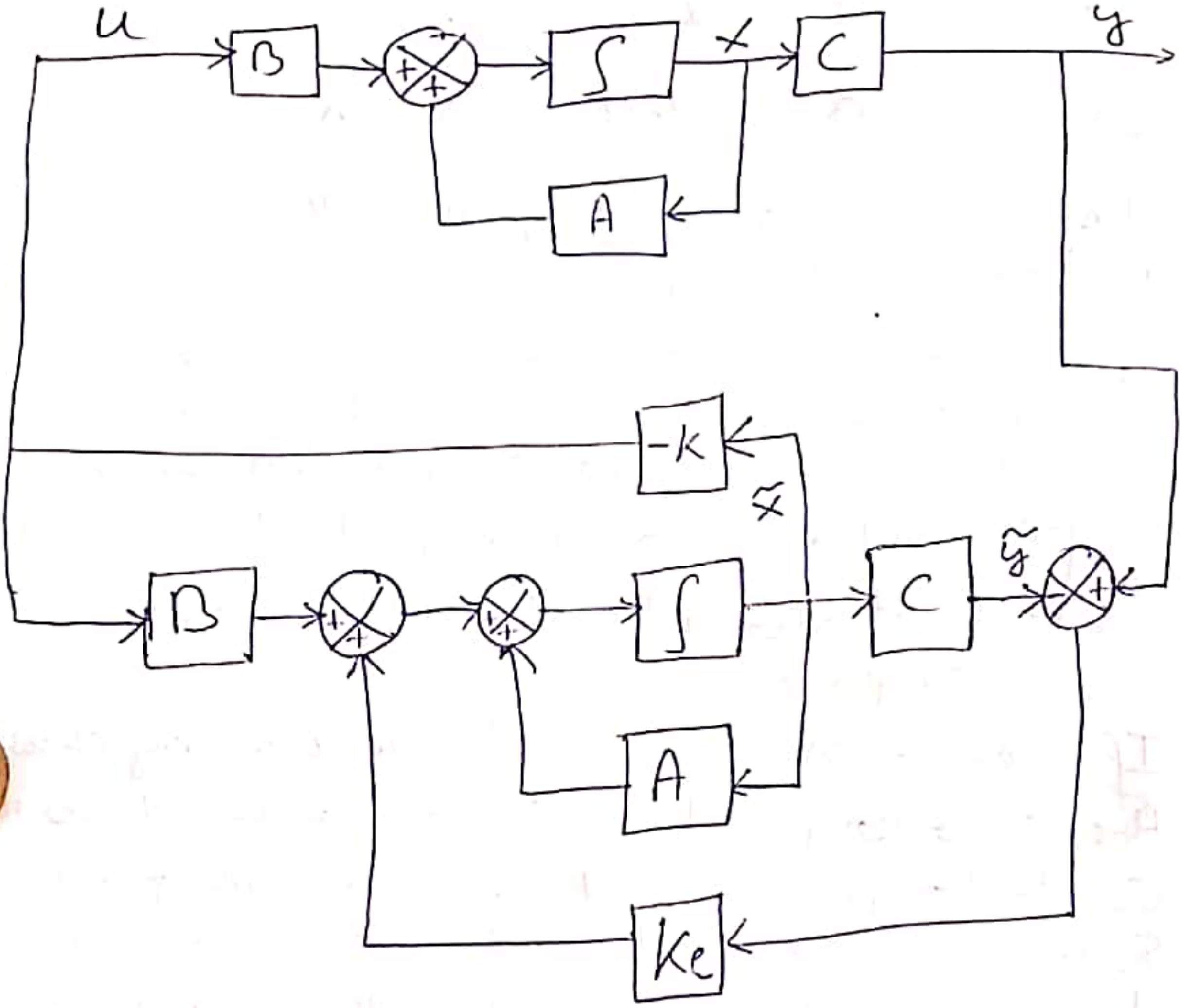
↳ Controller poles dominate the system response.

⇒ If sensor noise is considerable, we may choose the observer poles to be slower than two times the controller poles, so that the bandwidth of the system will become lower & smooth the noise.

↳ Response will be strongly influenced by the observer poles.

⇒ In many practical cases, the selection of the best matrix K_e boils down to a compromise between speedy response & sensitivity to disturbance & noises.

★ Effects of the Addition of the Observer on a Closed-Loop System



Observed-state feedback Control System

⇒ The design process, therefore, becomes a two stage process, i.e.

1) The first stage being the determination of the feedback gain matrix K to yield the desired characteristic equation.

2) Second stage being the determination of the observer gain matrix Ke to yield the desired observer characteristic equation.

⇒ Let us now investigate the effects of the use of the observed state $\tilde{x}(t)$, rather than the actual state $x(t)$, on the characteristic equation of a closed-loop control system.

⇒ Consider the completely state controllable & completely observable system defined by equations:

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

⇒ For the state-feedback control based on the observed state $\tilde{x}$.

$$u = -K\tilde{x}$$

⇒ with the control, the state equation become

$$\dot{x} = Ax - BK\tilde{x} = (A-BK)x + BK(x-\tilde{x})$$

Let $e(t) = x(t) - \tilde{x}(t)$

$$\Rightarrow \dot{x} = (A-BK)x + BKe \quad \text{— (1)}$$

⇒ Observer error is given by equation as:

$$\dot{e} = (A - K_e C)e \quad \text{— (2)}$$

Combining eq (1) & (2) we get:-

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A-BK & BK \\ 0 & A-K_eC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} \quad \text{— (3)}$$

⇒ Equation (3) describes the dynamics of the observed-state feedback control system.

⇒ The characteristic equation for the system is:-

$$\begin{vmatrix} SI-A+BK & -BK \\ 0 & SI-A+K_eC \end{vmatrix} = 0$$

$$\Rightarrow |SI-A+BK||SI-A+K_eC| = 0$$

⇒ <u>Note</u>: Closed-loop poles of the observed state feedback control system consists of the poles due to the:-

→ Pole-placement design done

→ Pole - due to the observed design done.

⇒ This means that the pole-placement design and the observer design are independent of each other.

↳ They can be designed separately & combined.

★ <u>Transfer Function of the Observer-Based Controller</u>

⇒ Consider the plant defined by:-

$$\dot{x} = Ax + Bu \quad \text{——} (1)$$

$$y = Cx \quad \text{——} (2)$$

⇒ Use observed-state feedback control

$$u = -K\tilde{x}. \quad \text{——} (3)$$

⇒ Then equations for the observer are given by:-

$$\dot{\tilde{x}} = (A - K_e C - BK)\tilde{x} + K_e y \quad \text{— (4)}$$

$$u = -K\tilde{x} \quad \text{— (5)}$$

⇒ By taking laplace transform of eq (4) with zero initial condition we get:-

$$\tilde{X}(s) = (sI - A + K_e C + BK)^{-1} K_e Y(s)$$

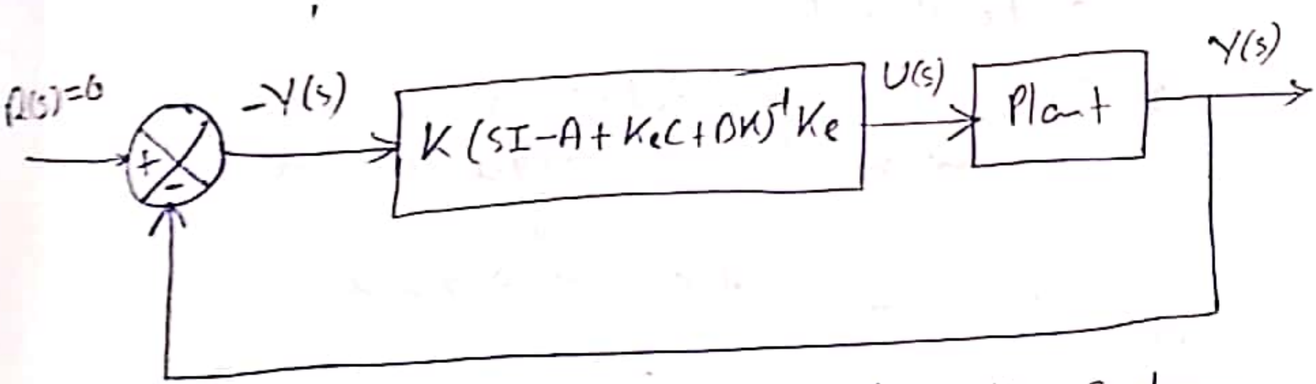
⇒ By substituting this $\tilde{X}(s)$ from Laplace transform of Eq (5) we obtain:-

$$U(s) = -K(sI - A + K_e C + BK)^{-1} K_e Y(s)$$

⇒ Then the TF $U(s)/Y(s)$ can be obtained as:-

$$\frac{U(s)}{Y(s)} = -K(sI - A + K_e C + BK)^{-1} K_e$$

⇒ Block diagram representation of the system.



⇒ The TF as as Controller ~~of~~ for the System.

★ Minimum-Order Observer

⇒ The Observers discussed so far are designed to reconstruct all the state variables.

↳ In practice, some of the state variables may be accurately measured.

↳ Such accurately measured state variable need not be estimated.

⇒ Suppose

→ State vector x is an n-vector

→ Output vector y is m-vector

→ Since m output variables are linear combination of the state variable

↓

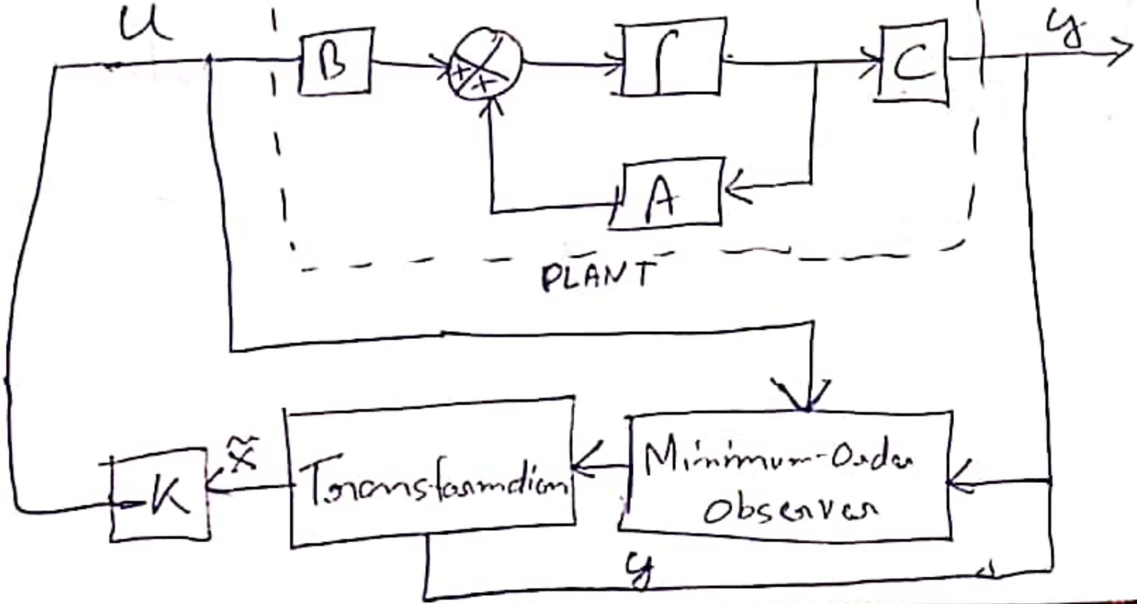
(m state variable need not be estimated)

→ We need to estimate only $n-m$ state variables.

→ Then the reduced-order observer becomes an $(n-m)^{th}$ order observer.

↓

(Minimum-order observer)



⇒ To present the basic idea of the minimum-order observer, without ~~undue~~ mathematical complications, we shall present the case where the output is scalar.

↳ Consider the system

$$\dot{X} = AX + Bu$$
$$y = CX$$

⇒ Let state vector X be partitioned into two part as X_a (scalar) & X_b [an (n-1) vector].

↳ Mean state variable x_a is equal to the output y and thus can be directly measured.

{ Unmeasurable Portion of State Vector }

⇒ Then the partitioned state & output equation become:-

$$\begin{bmatrix} \dot{x}_a \\ \dot{x}_b \end{bmatrix} = \left[\begin{array}{c:c} A_{aa} & A_{ab} \\ \hdashline A_{ba} & A_{bb} \end{array}\right] \begin{bmatrix} X_a \\ \hdashline X_b \end{bmatrix} + \begin{bmatrix} B_a \\ \hdashline B_b \end{bmatrix} u$$

$$y = [1 \,\vdots\, 0] \begin{bmatrix} X_a \\ \hdashline X_b \end{bmatrix}$$

$$\dot{x}_a - A_{aa} x_a - B_a u = A_{ab} X_b \quad \text{(a)}$$ { Measured Portion }

$$\dot{x}_b = A_{ba} x_a + A_{bb} x_b + B_b u \quad \text{(b)}$$ { Unmeasured Portion }

⇒ Let us compare the state equation for the full-order observer with that for the minimum-order observer

⇒ The state equation for the full-order observer is

$$\dot{x} = Ax + Bu \quad \text{—— ①}$$

⇒ and the state equation for the minimum order observer is

$$\dot{x}_b = A_{bb}x_b + A_{ba}x_a + B_bu \quad \text{—— ②}$$

⇒ The output equation for the full-order observer is

$$y = Cx \quad \text{—— ③}$$

⇒ and the "output equation" for the minimum-order observer is

$$\dot{x}_a - A_{aa}x_a - B_au = A_{ab}x_b \quad \text{—— ④}$$

Full-order State Observer	Minimum-Order State Observer
$\tilde{x}$	$\tilde{x}_b$
A	A_{bb}
Bu	$A_{ba}x_a + B_bu$
y	$\dot{x}_a - A_{aa}x_a - B_au$
C	A_{ab}
K_e [$n \times 1$ matrix]	K_e [$(n-1) \times 1$ matrix]

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⇒ Observer equation for full order observer:

$$\dot{\tilde{x}} = (A - K_e C)\tilde{x} + Bu + K_e y \quad \text{—(5)}$$

⇒ By making substitutions from table we get:-

$$\dot{\tilde{x}}_b = (A_{bb} - K_e A_{ab})\tilde{x}_b + A_{ba} x_a + B_b u + K_e(\dot{x}_a - A_{aa}x_a - B_a u) \quad \text{—(6)}$$

⇒ If x_a is noisy the use of $\dot{x}_a$ is unacceptable.

⇒ To avoide this difficulty, we eliminate $\dot{x}_a$ in the following way:-

$$\dot{\tilde{x}}_b - K_e \dot{x}_a = (A_{bb} - K_e A_{ab})\tilde{x}_b + (A_{ba} - K_e A_{aa})y + (B_b - K_e B_a)u$$

$$\Rightarrow (A_{bb} - K_e A_{ab})(\tilde{x}_b - K_e y) + \left[(A_{bb} - K_e A_{ab})K_e + A_{ba} - K_e A_{aa}\right]y + (B_b - K_e B_a)u$$

Let us define $x_b - K_e y = x_b - K_e x_a = \eta$

$$\tilde{x}_b - K_e y = \tilde{x}_b - K_e x_a = \tilde{\eta}$$

$$\Rightarrow \dot{\tilde{\eta}} = (A_{bb} - K_e A_{ab})\tilde{\eta} + \left[(A_{bb} - K_e A_{ab})K_e + A_{ba} - K_e A_{aa}\right]y + (B_b - K_e B_a)u$$

Let us define $\hat{A} = A_{bb} - K_e A_{ab}$

$$\hat{B} = \hat{A}K_e + A_{ba} - K_e A_{aa}$$

$$\hat{F} = B_b - K_e B_a$$

$$\boxed{\dot{\tilde{\eta}} = \hat{A}\tilde{\eta} + \hat{B}y + \hat{F}u} \quad \text{—} (8)$$

$$\tilde{x} = \begin{bmatrix} x_a \\ \tilde{x}_b \end{bmatrix} = \begin{bmatrix} y \\ \tilde{x}_b \end{bmatrix} = \begin{bmatrix} 0 \\ I_{n-1} \end{bmatrix} [\tilde{x}_b - K_e y] + \begin{bmatrix} 1 \\ K_e \end{bmatrix} y$$

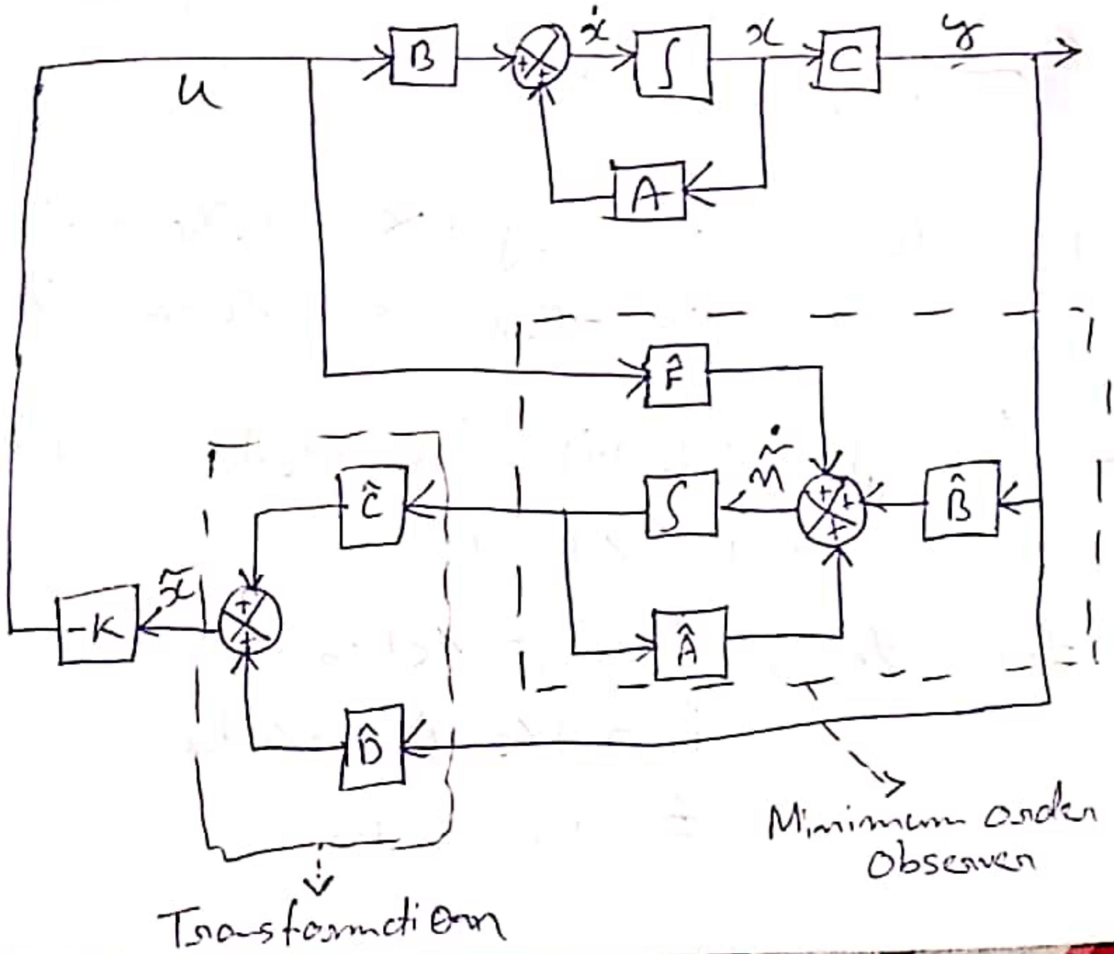
0 is row vector Consisting of $(n-1)$ zeros.

Let $\hat{C} = \begin{bmatrix} 0 \\ I_{n-1} \end{bmatrix}$ $\quad \hat{D} = \begin{bmatrix} 1 \\ K_e \end{bmatrix}$

⇒ then $\tilde{x}$ in terms of $\tilde{\eta}$ & y:-

$$\boxed{\tilde{x} = \hat{C}\tilde{\eta} + \hat{D}y}$$

⇒ This equation gives the transformation from $\tilde{\eta}$ to $\tilde{x}$.



$\Rightarrow$ Next we shall derive the observer error equation.

{using eq 8 eq 6 can be modified as}

$$\dot{\tilde{x}}_b = (A_{bb} - K_e A_{ab})\tilde{x}_b + A_{ba}x_a + B_b u + K_e A_{ab} x_b \quad (9)$$

$\Rightarrow$ By subtracting eq 9 & 6 we obtain:-

$$\dot{x}_b - \dot{\tilde{x}}_b = (A_{bb} - K_e A_{ab})(x_b - \tilde{x}_b)$$

$$\text{Let } e = x_b - \tilde{x}_b = \eta - \hat{\eta}$$

$$\Rightarrow \boxed{\dot{e} = (A_{bb} - K_e A_{ab})e} \quad (10)$$

$\Rightarrow$ Provided the rank of matrix

$$\begin{bmatrix} A_{ab} \\ A_{ab}A_{bb} \\ \vdots \\ A_{ab}A_{bb}^{n-2} \end{bmatrix}$$

is $n-1$.

{This is the complete observability condition applicable to the minimum order observer}

$\Rightarrow$ The characteristic equation for the minimum order observer is obtained from eq 10 as:-

$$|sI - A_{bb} + K_e A_{ab}| = (s-\mu_1)(s-\mu_2)\cdots(s-\mu_{n-1})$$

$$= s^{n-1} + \hat{\alpha}_1 s^{n-2} + \cdots \hat{\alpha}_{n-2} s + \hat{\alpha}_{n-1} = 0$$

where $\mu_1, \mu_2 \ldots \mu_{n-1}$ are desired eigenvalue for the minimum-order observer.

$$K_e = \hat{Q}\begin{bmatrix} \hat{\alpha}_{n-1} - \hat{a}_{n-1} \\ \hat{\alpha}_{n-2} - \hat{a}_{n-2} \\ \vdots \\ \hat{\alpha}_1 - \hat{a}_1 \end{bmatrix} = (\hat{W}\hat{N}^T)^{-1}\begin{bmatrix} \hat{\alpha}_{n-1} - \hat{a}_{n-1} \\ \hat{\alpha}_{n-2} - \hat{a}_{n-2} \\ \vdots \\ \hat{\alpha}_1 - \hat{a}_1 \end{bmatrix}$$

$$\hat{N} = \left[A_{ab}^T \,\vdots\, A_{bb}^T A_{ab}^T \,\vdots\, \cdots \,\vdots\, A_{bb}^{T\,n-2} A_{ab}^T\right]$$

$$\hat{W} = \begin{bmatrix} \hat{a}_{n-2} & \hat{a}_{n-3} & \cdots & \hat{a}_1 & 1 \\ \hat{a}_{n-1} & \hat{a}_{n-4} & \cdots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ \hat{a}_1 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ $\hat{a}_1, \hat{a}_2 \cdots \hat{a}_{n-2}$ are coefficients in the characteristics equation for the state eqution

$$|IS - A_{bb}| = S^{n-1} + \hat{a}_1 S^{n-2} + \cdots + \hat{a}_{n-2}S + \hat{a}_{n-1} = 0$$

$\Rightarrow$ Ackermann's formula can be modified to :-

$$K_e = \phi(A)\begin{bmatrix} A_{ab} \\ A_{ab}A_{bb} \\ \vdots \\ A_{ab}A_{bb}^{n-3} \\ A_{ab}A_{bb}^{n-2} \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Where $\phi(A_{bb}) = A_{bb}^{n-1} + \hat{\alpha}_1 A_{bb}^{n-2} + \cdots + \hat{\alpha}_{n-2} A_{bb} + \hat{\alpha}_{n-1} I$

* Observed-State feedback Control System with Minimum-Order Observer

⇒ The closed-loop poles of the observed-state feedback control system with a minimum order ~~Poles of~~ observer comprise of :-

1) Closed loop poles due to pole placement

2) Closed loop poles due to the minimum order observer

⇒ The system characteristic equation can be derived as :-

$$|sI - A + BK|\,|sI - A_{bb} + K_e A_{ab}| = 0$$

⇒ Therefore, the pole-placement design and the minimum order observer are independent of each other.

* Transfer function of Minimum-order observer based Controller

⇒ minimum-order observer equation is given by

$$\dot{\tilde{\eta}} = (A_{bb} - K_e A_{ab})\tilde{\eta} + [(A_{bb} - K_e A_{ab})K_e + A_{ba} - K_e A_{aa}]y + (B_b - K_e B_a)u$$

⇒ Let us define
$$\hat{A} = A_{bb} - K_e A_{ab}$$
$$\hat{B} = \hat{A}K_e + A_{ba} - K_e A_{aa}$$
$$\hat{F} = B_b - K_e B_a$$

⇒ Then the following three equations define the minimum-order observer:

$$\dot{\tilde{\eta}} = \hat{A}\tilde{\eta} + \hat{B}y + \hat{F}u \quad \text{—— (1)}$$
$$\tilde{\eta} = \tilde{x}_b - K_e y \quad \text{—— (2)}$$
$$u = -K\tilde{x} \quad \text{—— (3)}$$

$\Rightarrow$ Eq (3) Can be rewritten as:-

$$u = -K\tilde{x} = -[K_a \; K_b]\begin{bmatrix} y \\ \tilde{x}_b \end{bmatrix} = -K_a y - K_b \tilde{x}_b$$

$$u = -K_b \tilde{\eta} - (K_a + K_b K_e) y \quad \text{--- (4)}$$

$\Rightarrow$ Substituting eq (4) in (1) we get

$$\dot{\tilde{\eta}} = \hat{A}\tilde{\eta} + \hat{B}y + \hat{F}\left[-K_b\tilde{\eta} - (K_a + K_b K_e)y\right]$$

$$\dot{\tilde{\eta}} = (\hat{A} - \hat{F}K_b)\tilde{\eta} + [\hat{B} - \hat{F}(K_a + K_b K_e)]y \quad \text{--- (5)}$$

Let we define, $\tilde{A} = \hat{A} - \hat{F}K_b$

$\tilde{B} = \hat{B} - \hat{F}(K_a + K_b K_e)$

$\tilde{C} = -K_b$

$\tilde{D} = -(K_a + K_b K_e)$

Thus eq (4) & (5) become:-

$$\dot{\tilde{\eta}} = \tilde{A}\tilde{\eta} + \tilde{B}y$$
$$u = \tilde{C}\tilde{\eta} + \tilde{D}y$$

$$\Rightarrow \boxed{\frac{U(s)}{-Y(s)} = -\left[\tilde{C}(sI - \tilde{A})^{-1}\tilde{B} + \tilde{D}\right]}$$

$\left\{ \text{Since input to the observer controller is } -Y(s) \text{ rather than } Y(s) \right\}$

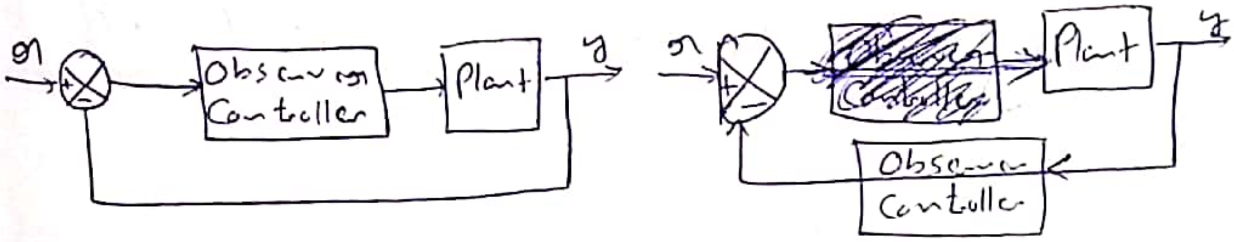
10.6> **Design of Regulator System with Observer**

1. Derive a state-space model of the plant.
2. Choose the desired closed-loop poles for pole placement. Choose the desired observer poles.
3. Determine the state feedback gain matrix K and the observer gain matrix.
4. Using the gain matrices K and Ke obtained in Step 3, derive the transfer function of the observer controller. If it is a stable controller, check the response to the given initial condition. If the response is not acceptable, adjust the closed-loop pole location and/or observer pole location until an acceptable response is obtained.

10.7> **Design of Control System with Observer**

⇒ In this section we consider the design of control system with observer when the systems have reference inputs.

↳ Output of control system must follow the input that is time varying.



Configuration 1 Configuration 2

10.8) Quadratic Optimal Regulator System

An advantage of the Quadratic optimal control method over the pole-placement method is that the former provides a Systematic way of Computing the State feedback Control gain matrix

★ Quadratic Optimal Regulator Problem

Let System be given by :-

$$\dot{x} = Ax + Bu \quad \text{—— (1)}$$

We have to determine matrix K of the optimal control vector

$$u(t) = -K\,x(t) \quad \text{—— (2)}$$

⇒ So as to minimize the performance index

$$\boxed{J = \int_0^{\infty} (\underbrace{x^T Q x}_{\text{accounts for error}} + \underbrace{u^T R u}_{\text{accounts for energy expenditure of control signal}})\,dt} \quad \text{—— (3)}$$

Where Q is a positive-definite Hermitian matrix

& R is a positive-definite Hermitian matrix

⇒ The matrix Q & R determines the relative importance of the error & the expenditure of this energy.

⇒ Therefore, the unknown elements of the matrix K are determined so as to minimize the performance index.

⇒ Substituting eq (2) in (1) we get

$$\dot{x} = Ax - BKx = (A-BK)x \quad \text{—— (4)}$$

{We assume matrix (A−BK) is stable or eigenvalues of A−BK have negative real part}

⇒ Substituting eq (2) in (3) we get:-

$$J = \int_0^\infty (x^TQx + x^TK^TRKx)\,dt$$

$$J = \int_0^\infty x^T(Q + K^TRK)x\,dt$$

Let us set:

$$x^T(Q+K^TRK)x = -\frac{d}{dt}(x^TPx)$$

{where P is a positive-definite Hermitian matrix}

$$x^T(Q+K^TRK)x = -\dot{x}^TPx - x^TP\dot{x}$$

$$= -x^T\left[(A-BK)^TP + P(A-BK)\right]x$$

{using eq (4)}

⇒ Comparing both side of last eqn we get

$$-(Q+K^TRK) = (A-BK)^TP + P(A-BK) \quad \text{—— (5)}$$

⇒ It can be proved that if (A−BK) is stable, there exists a positive definite matrix P that satisfies eq (5).

⇒ Hence our Procedure is to determine the elements P from the eq ⑤ & see if it is positive definite.

→ More than one matrix P may satisfy this equation.

→ If the System is stable, there always exist one positive definite matrix P to satisfy this equation.

→ This means that, if we solve this equation & find one positive definite matrix P, the System is stable.

⇒ The performance index J can be evaluated as:-

$$J = \int_0^{\infty} x^T(Q + K^T R K)x \, dt = -x^T P x \Big|_0^{\infty}$$

$$J = -x^T(\infty) P x(\infty) + x^T(0) P x(0)$$

⇒ Since all eigen value of A-BK are assumed to have negative real parts we have $x(\infty) \to 0$

So $\boxed{J = x^T(0) P x(0)}$ — ⑥

⇒ To obtain the solution to the quadratic Optimal Control problem, we proceed as follows:

⇒ Since R has been assumed to be a positive definite Hermitian matrix

$R = T^T T$ {where T is not singular matrix}

⇒ Then equation ⑥ can be ~~...~~ rewritten as

$$(A^T - K^T B^T)P + P(A - BK) + Q + K^T T^T T K = 0$$

$$\Rightarrow A^T P + PA + [TK - (T^T)^{-1} B^T P]^T [TK - (T^T)^{-1} B^T P] - PBR^{-1}B^T P + Q = 0$$

⇒ Minimization of J with respect to K requires minimization of

$$X^T [TK - (T^T)^{-1} B^T P]^T [TK - (T^T)^{-1} B^T P] X$$

with respect to K.

⇒ Since this last expression is nonnegative, the minimum occurs when it is zero.

$$TK = (T^T)^{-1} B^T P$$

$$\Rightarrow K = T^{-1}(T^T)^{-1} B^T P = R^{-1} B^T P$$

$$\boxed{K = R^{-1} B^T P} \quad \{\text{optimal } K\} \quad \text{—⑦}$$

⇒ Thus the optimal control law to the quadratic optimal control problem is given by:

$$u(t) = -Kx(t) = -R^{-1} B^T P x(t)$$

⇒ The matrix P must satisfy the following reduced equation:-

$$\boxed{A^T P + PA - PBR^{-1} B^T P + Q = 0} \quad \text{—⑧}$$

{Reduced-matrix Riccati equation}

⇒ The design step:-

1. Solve, the reduced-matrix Riccati equation, for the matrix P

$$A^TP + PA - PBR^{-1}B^TP + Q = 0$$

2. Substitute this matrix P into equation

$$K = P^{-1}B^TP$$

The resulting matrix K is the optimal matrix.