Lecture 3: Time Response

⇒ Let us consider a LTI first order system:

$$\boxed{\begin{aligned} \dot{x}(t) &= a x(t) + b u(t) \\ y(t) &= c x(t) + d u(t) \end{aligned}}$$

here, $x(t), y(t), u(t) \in R$
& $a, b, c, d \in R$

⇒ Let use denote the above system by Σ.

So $\boxed{\Sigma u = y}$

⇒ Time Response: $y(t)$ given $u(t)$ an $x(0)$ $\forall t \geq 0$.

⇒ Let use break u into two parts:

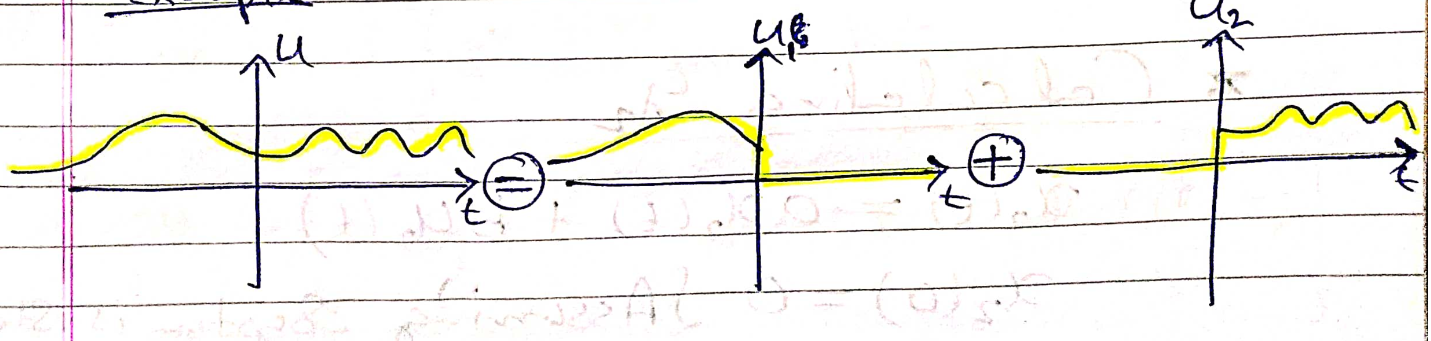
$$\cancel{u = (\sigma_0 u) + (u - \sigma_0 u)}$$

$$u = (P_0 u) + (u - P_0 u)$$

⇒ $u = u_1 + u_2$

{ P_T is an operator which truncate u after time T }

Example



$\Rightarrow \quad \Sigma u = \Sigma(u_1 + u_2)$

$\Rightarrow \quad y = \Sigma u_1 + \Sigma u_2$ {As the System is Linear}

$y = y_1 + y_2$

★ <u>Calculating y_1</u>

$\dot{x}_1(t) = a x_1(t) + b u_1(t)$

$\forall\, t \geq 0$

$\dot{x}_1(t) = a x_1(t)$

$$\frac{dx_1(t)}{dt} = a\, x_1(t) \Rightarrow \int_0^t \frac{dx_1(t)}{x_1(t)} = \int_0^t a\,dt$$

$\Rightarrow \quad x_1(t) = x_1(0) e^{at}$

$\Rightarrow \quad x_1(t) = x(0) e^{at}$

$\Rightarrow \quad \boxed{y_1(t) = c\, x(0)\, e^{at}}$

{As Σ is causal i.e. $\Sigma P_T u(t) = P_T \Sigma u(t) \quad \forall\, t \leq T$}

★ <u>Calculating y_2</u>

$\dot{x}_2(t) = a x_2(t) + b u_2(t)$

$x_2(0) = 0$ {Assuming System is stable}

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$\Rightarrow$ Multiplying both side by e^{-at}.

$\Rightarrow$ $e^{-at}\dot{x}_2(t)\,dt = e^{-at}a x_2(t)\,dt + e^{-at}bu_2(t)\,dt$

$\Rightarrow$ $\int_0^t e^{-a\tau}\dot{x}_2(\tau)\,d\tau = \int_0^t e^{-a\tau}a x_2(\tau)\,d\tau + \int_0^t e^{-a\tau}bu_2(\tau)\,d\tau$

$\Rightarrow$ LHS $= e^{-a\tau}x_2(\tau)\Big|_0^t - \int_0^t -a e^{-a\tau}x_2(\tau)\,d\tau$

Integration by parts
$\int_0^t u(\tau)\dot{v}(\tau)\,d\tau$
$= u(\tau)v(\tau)\Big|_0^t - \int_0^t \dot{u}(\tau)v(\tau)\,d\tau$

$\Rightarrow$ LHS $= e^{-at}x_2(t) + a\int_0^t e^{-a\tau}x_2(\tau)\,d\tau$ —— ①

$\Rightarrow$ RHS $= a\int_0^t e^{-a\tau}x_2(\tau)\,d\tau + \int_0^t e^{-a\tau}bu_2(\tau)\,d\tau$ —— ②

$\Rightarrow$ $x_2(t) = \int_0^t e^{a(t-\tau)}bu_2(\tau)\,d\tau \quad \forall\, t \geq 0$

$\Rightarrow$ $x_2(t) = \int_0^t e^{a(t-\tau)}bu(\tau)\,d\tau$

$\Rightarrow$ $\boxed{y_2(t) = c\int_0^t e^{a(t-\tau)}bu(\tau)\,d\tau + du(t)}$

★ Calculating y

$\Rightarrow$ $y = y_1 + y_2$.

$\Rightarrow$ $y(t) = c\,x(0)\,e^{at} + c\int_0^t e^{a(t-\tau)}\,b\,u(\tau)\,d\tau + d\,u(t)$

Let $\phi(t) = e^{at}$

So $e^{a(t-\tau)} = \phi(t-\tau)$

$\Rightarrow$ $$\boxed{y(t) = c\,\phi(t)\,x(0) + c\int_0^t \phi(t-\tau)\,b\,u(\tau)\,d\tau + d\,u(t)}$$

★ The matrix exponential

$\Rightarrow$ For a matrix A let

$$\boxed{e^{At} = I + At + \frac{1}{2}(At)^2 + \cdots + \frac{1}{n!}(At)^n + \cdots}$$

{ This overloading of the exponential operation is inspired by taylor expansion of e^{at} $\forall a \in \mathbb{R}$

$$e^{at} = 1 + at + \frac{1}{2}(at)^2 + \cdots + \frac{1}{n!}(at)^n + \cdots$$ }

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$$\Rightarrow \frac{de^{At}}{dt} = A + \frac{1}{2!}2A^2t + \frac{1}{3!}3A^3t^2 + \cdots$$

$$\Rightarrow A\left(I + At + \frac{1}{2!}(At)^2 + \cdots\right)$$

$$\boxed{\frac{de^{At}}{dt} = A\,e^{At}}$$ {Similar to e^{at} $\forall a \in R$}

$\Rightarrow$ For n^{th} order LTI System ~~x_1~~ will be

$$\dot{\cancel{x}}_1(t) = A\cancel{x}_1(t)$$

$\Rightarrow$ $X_1(t) = e^{At}K$ is solution of the above differential equation.

$$X_1(0) = e^{A0}K = K$$

$$\{K \in R^{n\times 1}\}$$

$\Rightarrow$ Verifying the Solution.

$$\cancel{\frac{d}{dt}e^{A(t)}X_1(0)} \quad \frac{d}{dt}e^{At}X_1(0) = Ae^{At}X_1(0)$$

$$\Rightarrow \frac{d}{dt}X_1(t) = AX_1(t)$$

So $\boxed{X_1(t) = e^{At}X(0) = \Phi(t)X(0)}$

So $\boxed{Y_1(t) = Ce^{At}X(0)}$

⇒ Similar Arguments can be followed for $Y_2(t)$ as of first order system.

$$\Rightarrow \quad \dot{X}_2(t) = AX_2(t) + BU_2(t)$$

$$\Rightarrow \quad e^{-At}\dot{X}_2(t) = e^{-At}AX_2(t) + e^{-At}BU_2(t)$$

$$\Rightarrow \quad \int_0^t e^{-A\tau}\dot{X}_2(\tau)\,d\tau = \int_0^t e^{-A\tau}AX_2(\tau)\,d\tau + \int_0^t e^{-A\tau}BU_2(\tau)\,d\tau$$

$$\Rightarrow \quad LHS = e^{-A\tau}x_2(\tau)\Big|_0^t + \cancel{\int e^{-A\tau}AX_2(\tau)\,d\tau}$$

$$\Rightarrow \quad RHS = \cancel{\int_0^t e^{-A\tau}AX_2(\tau)\,d\tau} + \int_0^t e^{-A\tau}BU_2(\tau)\,d\tau$$

$$\Rightarrow \quad e^{-At}x_2(t) = \int_0^t e^{-A\tau}BU_2(\tau)\,d\tau$$

$$\Rightarrow \quad e^{At}e^{-At}x_2(t) = e^{At}\int_0^t e^{-A\tau}BU_2(\tau)\,d\tau$$

$$\left\{ e^{A}e^{B} = e^{A+B} \text{ iff } AB = BA \right\}$$

$$\Rightarrow X_2(t) = \int_0^t e^{A(t-\tau)} B U(\tau)\, d\tau$$

$$\Rightarrow Y_2(t) = C \int_0^t e^{A(t-\tau)} B U(\tau)\, d\tau + D U(t)$$

$$\Rightarrow \text{So } Y(t) = Y_1(t) + Y_2(t)$$

$$Y(t) = C e^{At} X(0) + C \int_0^t e^{A(t-\tau)} B U(\tau)\, d\tau + D U(\tau)$$

$$\boxed{Y(t) = C \Phi(t) X(0) + C \int_0^t \Phi(t-\tau) B U(\tau)\, d\tau + D U(\tau)}$$

★ <u>How to compute a matrix exponential</u>

1. <u>Brute force</u>: Compute many terms of Taylor expansion.

2. Find a realization of the system (i.e. a special choice of the state vector x) such that the matrix A is either:
   - → Diagonal $\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$
   - → Jordan form

★ Jordan Form

"A square matrix is said to be in Jordan form if it is block diagonal where each block is a Jordan block."

⇒ A Jordan block of size ~~[illegible]~~ K is a $K \times K$ matrix of form:

$$\begin{pmatrix} \lambda & 1 & & & 0 \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ 0 & & & & \lambda \end{pmatrix}$$

★ Qualitative behavior of a LTI System

1) **Lyapunov stability:** A system is called Lyapunov stable if, for any bounded initial condition, and zero input, the state remains bounded.

$$\|x(0)\| < \epsilon \text{ and } u = 0 \Rightarrow \|x(t)\| < \delta \quad \forall\, t \geq 0$$

2) **Asymptotic stability:** A system is called asymptotically stable if, for any bounded initial condition and zero input, the state converges to zero.

$$\|x(0)\| < \epsilon \;\&\; u = 0 \rightarrow \lim_{t \to +\infty} \|x(t)\| = 0$$

3) Bounded-Input Bounded-Output Stability (BIBO)

A System is called BIBO-stable if, for any bounded input, the output remains bounded

$$\|u(t)\| < \epsilon \quad \forall t \geq 0 \ \& \ x(0)=0$$

$$\Downarrow$$

$$\| y(t) \| < \delta \quad \forall t \geq 0$$

4) Unstable: A system is called unstable if it is not stable by any of the above criteria.

★ Stability analysis of a general Linear time-invariant System

⇒ For Lyapunov and Asymptotic stability check $u=0$.

$$\boxed{\begin{aligned} \dot{X} &= AX \quad \text{—(a)} \\ Y &= CX \quad \text{—(b)} \end{aligned}}$$

① A has distinct Eigen values

⇒ Let $V_1, V_2 \cdots V_n$ be the eigen vectors corresponding to the eigen values $\lambda_1, \lambda_2 \cdots \lambda_n$ of A.

⇒ Let $n \times n$ matrix $T = [V_1 \ V_2 \cdots V_n]$ —①

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$\Rightarrow$ ~~Ax~~ $AT = T\Lambda$

where, $\Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$

$\Rightarrow A = T\Lambda T^{-1}$ or $\Lambda = T^{-1}AT$

$\Rightarrow$ Let $X = T\tilde{X}$ — (2)

$\Rightarrow$ Substituting eq (2) in eq (1) we get:

$\Rightarrow T\dot{\tilde{X}} = AT\tilde{X}$

$\Rightarrow \dot{\tilde{X}} = T^{-1}AT\tilde{X} = \Lambda\tilde{X}$

$\Rightarrow \dot{\tilde{X}} = \Lambda\tilde{X}$

$\Rightarrow$ Solution of the above equation is

$\boxed{\tilde{X} = e^{\Lambda t}\tilde{X}(0)}$ — (3)

$\Rightarrow$ Substituting eq (2) in eq (3) to eliminate $\tilde{X}$

$\Rightarrow T^{-1}X = e^{\Lambda t}T^{-1}X(0)$

$\Rightarrow \boxed{X = Te^{\Lambda t}T^{-1}X(0)}$

$\Rightarrow$ Let $T^{-1} = \begin{bmatrix} \omega_1^T \\ \omega_2^T \\ \vdots \\ \omega_n^T \end{bmatrix}$

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$\Rightarrow \omega_1^T, \omega_2^T \cdots \omega_n^T$ are the left eigen vector corresponding to $\lambda_1, \lambda_2 \cdots \lambda_n$.

$$AT = T\Lambda$$

$$\Rightarrow T^{-1}AT = \Lambda$$

$$\Rightarrow T^{-1}A = \Lambda T^{-1}$$

$$\Rightarrow \begin{bmatrix} \omega_1^T \\ \vdots \\ \omega_n^T \end{bmatrix} A = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \; \lambda_n \end{bmatrix} \begin{bmatrix} \omega_1^T \\ \vdots \\ \omega_n^T \end{bmatrix}$$

$$\Rightarrow \omega_1^T A = \lambda_1 \omega_1$$

$$\vdots \qquad \vdots$$

$$\omega_n^T A = \lambda_n \omega_n$$

$$\Rightarrow X(t) = [V_1 V_2 \cdots V_n] \begin{bmatrix} e^{\lambda_1 t} & & \\ & e^{\lambda_2 t} & \\ & & \ddots \; e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} \omega_1^T \\ \omega_2^T \\ \vdots \\ \omega_n^T \end{bmatrix} X(0)$$

$$= [V_1 V_2 \cdots V_n] \begin{bmatrix} e^{\lambda_1 t} \omega_1^T \\ e^{\lambda_2 t} \omega_2^T \\ \vdots \\ e^{\lambda_n t} \omega_n^T \end{bmatrix} X(0)$$

$$= \left(V_1 e^{\lambda_1 t} \omega_1^T + V_2 e^{\lambda_2 t} \omega_2^T + \cdots + V_n e^{\lambda_n t} \omega_n^T \right) X(0)$$

$$\Rightarrow X(t) = \left(e^{\lambda_1 t} v_1 \omega_1^T + e^{\lambda_2 t} v_2 \omega_2^T + \cdots e^{\lambda_n t} v_n \omega_n^T\right) X(0)$$

⇒ If all the eigen values are real, then all the terms of $X(t)$ are linear combination of form $e^{\lambda_i t}$ $\forall t \geq 0$ and $i = \{1, 2, \dots n\}$.

⇒ As Y is just a linear combination of X, it also has the same form.

⇒ So for Lyapunov stability:

$$\boxed{\lambda_i \leq 0 \quad \forall i \in \{1, 2, \dots n\}}$$

⇒ For Asymptotic stability:

$$\boxed{\lambda_i < 0 \quad \forall i \in \{1, 2, \dots n\}}$$

⇒ If some of the eigen value are complex conjugate pairs:

⇒ Let λ and $\bar{\lambda}$ be two eigen values.

⇒ Let V and ω^T be the right and left eigen vector corresponding to λ.

$$AV = \lambda V \implies A\bar{V} = \bar{\lambda}\bar{V}$$

$$\omega^T A = \lambda \omega^T \implies \bar{\omega}^T A = \bar{\lambda}\,\bar{\omega}^T$$

$\Rightarrow$ So $\bar{V}$ and $\bar{\omega}^T$ are the right and left eigen vectors corresponding to $\bar{\lambda}$.

$\Rightarrow$ $e^{\lambda t} V\omega^T + e^{\bar{\lambda} t}\bar{V}\bar{\omega}^T$ is the term contributed in the solution by λ and $\bar{\lambda}$.

$\Rightarrow$ Let, $\lambda = \sigma + j\omega \implies \bar{\lambda} = \sigma - j\omega$

$, V\omega^T = E + jF, \implies \bar{V}\bar{\omega}^T = E - jF$

$$\Rightarrow e^{(\sigma+j\omega)t}(E+jF) + e^{(\sigma-j\omega)t}(E-jF)$$

$$= e^{\sigma t} e^{j\omega t}(E+jF) + e^{\sigma t} e^{-j\omega t}(E-jF)$$

$$\Rightarrow e^{\sigma t}\Big(\big(\cos(\omega t) + j\sin(\omega t)\big)(E+jF) + \big(\cos(\omega t) - j\sin(\omega t)\big)(E-jF)\Big)$$

$$\Rightarrow e^{\sigma t}\Big(\cos(\omega t)E - F\sin(\omega t) + E\cos(\omega t) - F\sin(\omega t) \cancel{+ j\sin(\omega t)F} \cancel{+ j\cos(\omega t)E} \cancel{- j\sin(\omega t)E} \cancel{- j\cos(\omega t)F}\Big)$$

$$\Rightarrow e^{\sigma t}\big(2\cos(\omega t)E - 2\sin(\omega t)F\big)$$

$\Rightarrow$ So this contains sinusoid terms with exponential terms.

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⇒ So, if complex conjugate eigen values are available, it will introduce terms of type $e^{\sigma t}\sin(\omega t+\phi)$ in the solution (Y).

⇒ For Lyapunov stability:

$$\boxed{Re(\lambda_i) \leq 0 \quad \forall i \in \{1,2,\dots n\}}$$

⇒ For Asymptotic stability:

$$\boxed{Re(\lambda_i) < 0 \quad \forall i \in \{1,2,\dots n\}}$$

② A has repeated Eigen values

⇒ If the matrix A is not diagonalizable (i.e. algebraic multiplicity of eigen values is not equal to its geometric multiplicity) then it can be converted to jordan form.

⇒ Let T be the similarity transform that will convert matrix A into jordan form

$$T^{-1}AT = J \quad \text{where, } J = \begin{bmatrix} J_1 & & \\ & J_2 & \\ & & \ddots & \\ & & & J_p \end{bmatrix}$$

where $J_1, J_2 \dots J_p$ are Jordan blocks.

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$\Rightarrow$ From previous we know solution is

$$\boxed{X(t) = T e^{Jt} T^{-1} X(0)}$$

$$e^{Jt} = \begin{bmatrix} e^{J_1 t} & & & \\ & e^{J_2 t} & & \\ & & \ddots & \\ & & & e^{J_p t} \end{bmatrix}$$

$\Rightarrow$ If

$$J_i = \begin{pmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix}$$

then

$$e^{J_i t} = e^{\lambda t} \begin{pmatrix} 1 & \frac{t}{1!} & \frac{t^2}{2!} & \cdots \\ & 1 & \frac{t}{1!} & \cdots \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

$\Rightarrow$ So from above we know that e^{Jt} contains terms of form

$$\left\{ e^{\lambda_i t} \quad t e^{\lambda_i t} \quad t^2 e^{\lambda_i t} \quad \cdots \right\}$$

$\Rightarrow$ So solution will also terms of above form.

$\Rightarrow$ If some of the eigen values are complex then term of form given below will appear.

$$t^m e^{\lambda_i t} \sin(\omega t + \phi)$$

⇒ For asymptotic stability:

$$\boxed{\lambda_i < 0 \quad \forall\, i \in \{1, \dots n\}}$$

⇒ FACT: For linear system asymptotic stability = BIBO stability.