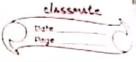
2

The Z Toransfoom



2.1> Introduction

- => The mole of Z tonoms form in discrete-time Systems is Similar to that of the Laplace transform, in Continuous-time System.
- = In a linear discrete -time Control System, a linear difference eignation Characterizes the dynamice of the System.

* Discrete - Time Signal .

The Z transform applies to the Continuous time. Signed of (KT), Sampled Signed of (KT) and the number sequence of (K).

2.27 The z transform

In Considering the 2 transform of a time function x(t), we consider only the Sampled values of x(t), that is x(t), x(t)

$$\chi(z) = Z[\alpha(k)] = Z[\alpha(kT)] = \sum_{K=0}^{\infty} \alpha(kT)Z^{-K}$$

=> For a Seamence of number a(K), the Z transform is defined by:

$$X(z) = Z[\alpha(\kappa)] = \sum_{k=0}^{\infty} \alpha(k) z^{-k}$$



7 The z transform defined dove to referred to as

The the ans-Sided z transim, we assume I(1) = 0 + E < 0 on x(x) = 0 + x < 0.

Lo Z is a Complex Variable.

> The z transform of O(L), when -oo L t < 00 on of o(K), when K takes integer values (K=0, ±1, ±1...) is defined by

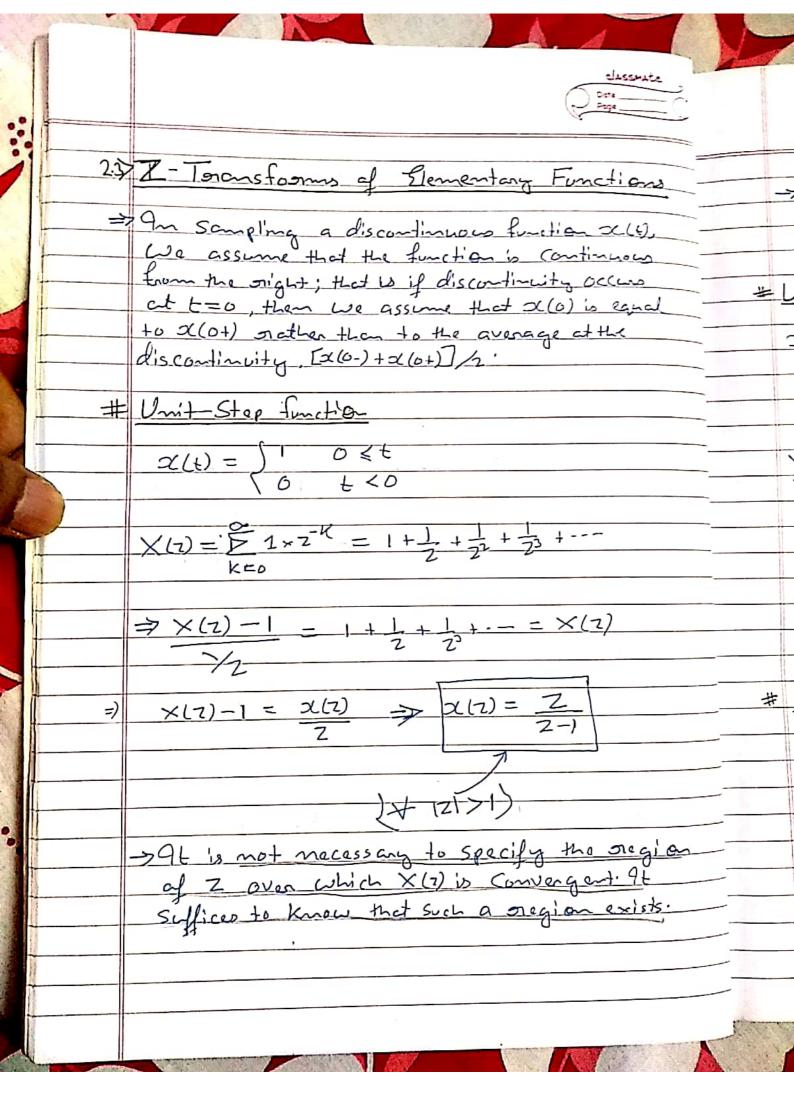
 $\chi(z) = Z[\chi(E)] = Z[\chi(KT)] = \sum_{K=-\infty} \chi(KT) z^{-K}$

$$\chi(z) = Z[\alpha(x)] = \sum_{k=-\infty}^{\infty} \alpha(x) z^{-k}$$

=> The z transform defined above is referred to as

Note: In this book, only the one-sided 2 transforms

 $\chi(z) = \chi(0) + \chi(T)z^{-1} + \chi(2T)z^{-2} + \dots + \chi(kT)z^{-k} + \dots$





The z transform X(z) of a time function x(t)

Obtained In this way is valid throughout the z place

axcept at poles of X(z)

$$X(v)=Z(v)=\sum_{k=0}^{\infty} o(kT) z^{-k}=T\sum_{k=0}^{\infty} kz^{-k}$$

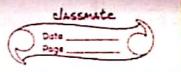
$$\Rightarrow T\left(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots \right) \times (2) = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$$

$$\frac{1}{2} = \frac{1}{2^{2}} + \frac{1}{2^{3}} + \frac{1}$$

$$X(z) = \sum_{k=1}^{\infty} \chi(k) z^{k} = 1 + \frac{2}{2} + \frac{2}{2} + \frac{2}{2} + \cdots$$

$$\Rightarrow$$
 $\times(z) = Z$
 $z-a$

	classmate Date Page
#	Exponential Function OLLE) = e at O \le E O & E < O
	$\chi(x_T) = e^{-\alpha kT} \forall K = 0,1,2$ $\chi(z) = \sum_{K=0}^{\infty} \frac{e^{-\alpha kT}}{z^K}$
	$= 1 + \underbrace{\frac{e^{-2aT}}{z} + \underbrace{\frac{e^{-3aT}}{z^3}}_{+ \cdot \cdot \cdot}$ $\times (2) = Z$ $Z - e^{-aT}$
#	Sincooldal Function $ \alpha(t) = \int \sin \omega t dt $ $ \alpha(t) = \int \sin \omega t dt $
	ejut = Cosut + j Smut e-jut = Cosut - j Simut
	$\Rightarrow \mathcal{S}_{in}\omega t = \frac{1}{2i} \left(e^{i\omega t} e^{-j\omega t} \right)$ $\times (z) = Z \left[\frac{1}{2i} \left(e^{j\omega t} + e^{-j\omega t} \right) \right]$



$$\Rightarrow \frac{1}{2^{j}} \left(\frac{1}{1 - e^{j\omega T_{2^{-j}}}} - \frac{1}{1 - e^{-j\omega T_{2^{-j}}}} \right)$$

$$\Rightarrow \frac{1 - e^{-j\omega t} z^{-1} - 1 + e^{j\omega t} z^{-1}}{z^{-1} + 1 - (e^{j\omega t} + e^{-j\omega t}) z^{-1}}$$

$$= 7 Z [Sim \omega t] = \frac{Z Sim \omega T}{Z^2 - 2z Cos \omega T + 1}$$

Similarly
$$Z[\cos\omega t] = \frac{Z^2 - Z\cos\omega T}{Z^2 - 2Z\cos\omega T + 1}$$

$$\times (7) = 7[1-e^{-t}] = \frac{1}{1-7} - \frac{1}{1-e^{-7}}$$

$$\Rightarrow \times (z) = (1 - e^{-T}) Z$$

$$(2-1) (2-e^{-T})$$

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	Page	
1	$Z[x(t-nT)] = z^{-n} \times (z)$	
) Where ×(2) = Z[X(1)] }	
	- R - 7	
47	$Z\left[2(L+1)\right] = Z^{n}\left[\times(z) - \sum_{k=0}^{m-1} \alpha(k\tau)z^{-k}\right]$	
	Lihere n'is zero on a positive integer?	
	200	-
	$\frac{ \nabla g_{100} }{ \nabla g_{100} } = \sum_{r=1}^{\infty} g_{r}(kT-mT) z^{-r}$	
*	K=0	
7	$= Z^{-n} \sum_{k=0}^{\infty} \mathcal{I}(kT-mT) Z^{-(k-n)}$	
	Let m= K-M	-
	$= Z[x(t-nT)] = Z^{m} \left[\sum_{m=-m}^{\infty} x(mT) z^{m} \right]$	
		-
<i></i>	Since oc(mT)=0 + m lo so we may chage lower limit from m=-n to m=0.	-
	$Z[x(t-nT)] = z^{-n} \sum_{m=0}^{\infty} x(mT)z^{-m} = z^{-m} \times (z)$	-
-	$\frac{m=0}{Z\left[\zeta(t+n\tau)\right]} = \frac{\omega}{z} \chi(k\tau+n\tau) z^{-k}$	 対
X	W = 0	
<u>; </u>	$= Z^{N-D} \propto (KT + mT) Z^{-(K+m)}$	
	K=0	

bleserala () tota () tota

$$\Rightarrow Z^{n} \begin{bmatrix} \sum_{K=0}^{\infty} cc(\kappa T_{1001}) \frac{1}{2} c(\kappa m) & \sum_{K=0}^{\infty-1} cc(\kappa T) \frac{1}{2} ck \\ \vdots & \vdots & \vdots & \vdots \\ & K=0 \end{bmatrix}$$

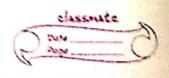
$$= Z^{m} \left[\times (z) - \sum_{k=0}^{k=0} c(\kappa r) z^{-k} \right]$$

Similalo

$$Z[D((K+m)] = Z^m \times (z) - Z^m \times (0) - Z^{m-1} \times (1)$$

(where mis a positive number)

Note: Multiplication of the 2 tours from X(1)
by 2 how the effect of advanting the
Signed d(KT) by one stap and that wellflinden
of the 2 tours form X(1) by 2' how the effect



of delaying the signed x(KT) by one step.

$$Z[1(t-1T)] = Z^{-7}Z[1(t)] = Z^{-7}$$

$$\overline{1-z^{-7}}$$

$$\Rightarrow$$
 $Z[f(y)] = \frac{1}{z-q}$

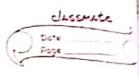
$$\Rightarrow Z[f(k)] = \frac{1}{2-a}$$
Example 2.5: $y(k) = \sum_{h=0}^{K} \chi(h) \quad K = 0, 1, 2 \cdots$

$$y(k) = 0 \quad \forall \quad K \neq 0$$

$$y(k) = 0 \quad \forall \quad k < 0$$

$$\Rightarrow y(k) = \chi(0) + \chi(1) + \chi(2) + \cdots + \chi(K)$$

$$y(K-1) = \chi(0) + \chi(1) + \chi(2) + \cdots + \chi(K-1)$$



$$\Rightarrow \forall (z) = \times (z) \qquad \text{where } \times (z) = Z[x(x)]$$

* Complex Tononslation Theorem

If o(t) has the 2 transform X(z), then the 2 transform of $e^{-at}xlt$) can be given by $X(ze^{at}) \cdot Then is known as the complex translation theorem.$

$$\frac{P_{500}f}{Z\left(e^{-at}Sc(t)\right)} = \sum_{K=0}^{\infty} \chi(KT)e^{-aKT} - K$$

$$= \sum_{K=0}^{\infty} \mathfrak{I}(KT) \left(\mathbb{Z}e^{aT} \right)^{-K}$$

$$= \times (Ze^{aT})$$



Example 2-6: Given the 2 transforms of Sincet and Coswt, Obtain the 2 transform of e-at sincet and e-at Coswt, onespectively by using the Complex translation theorem.

 $\frac{Z[sin\omegat] = Z^{-1}Sin\omega T}{1-2Z^{-1}Cos\omega T + Z^{-2}}$

80 Z[e-at sin w] = e-at z-1 sin w] 1-2e-at z-1 (0) wt +e-2at z-2

Similary 7 [e-at (05W+] = 1 - eatz-1 (05 WT 1-2e-atz-1 CoswT + e-2atz-2

Example 2-7: Z transform of te-at

 $\frac{Z[te^{-at}] = \overline{1e^{-aT}z^{-1}}}{(1-e^{-at}z^{-1})^2}$

* Anitid value theorem: If o(t) has the z transform

X(z) and if lim X(z) exists, then the initial

Value o((0) of x(t) on x(k) is given by:

 $\chi(0) = \lim_{z \to \infty} \chi(z)$

Proof:

 $\frac{1}{(2)} = \sum_{k=0}^{\infty} 2((k)2^{-k} = \chi(0) + \chi(1)z^{-1} + \chi(2)z^{-2} + \dots$

80 11m ×(2) = 2((0)

Final Value Theorem: Suppose that x (K), where
X(K)=0 for K(O, has the 2 transform X(Z)
and that all the poles of X(7) lie inside the
Unit circle, with the possible exception of
a Simple pole at 2=1.
·

Then the final value of $\chi(k)$, that is the value of $\chi(k)$ as k approaches infinity (and be given by:

$$\lim_{K\to\infty}\chi(K)=\lim_{Z\to 1}\left[\left(1-Z^{-1}\right)\chi(Z)\right]$$

Proof

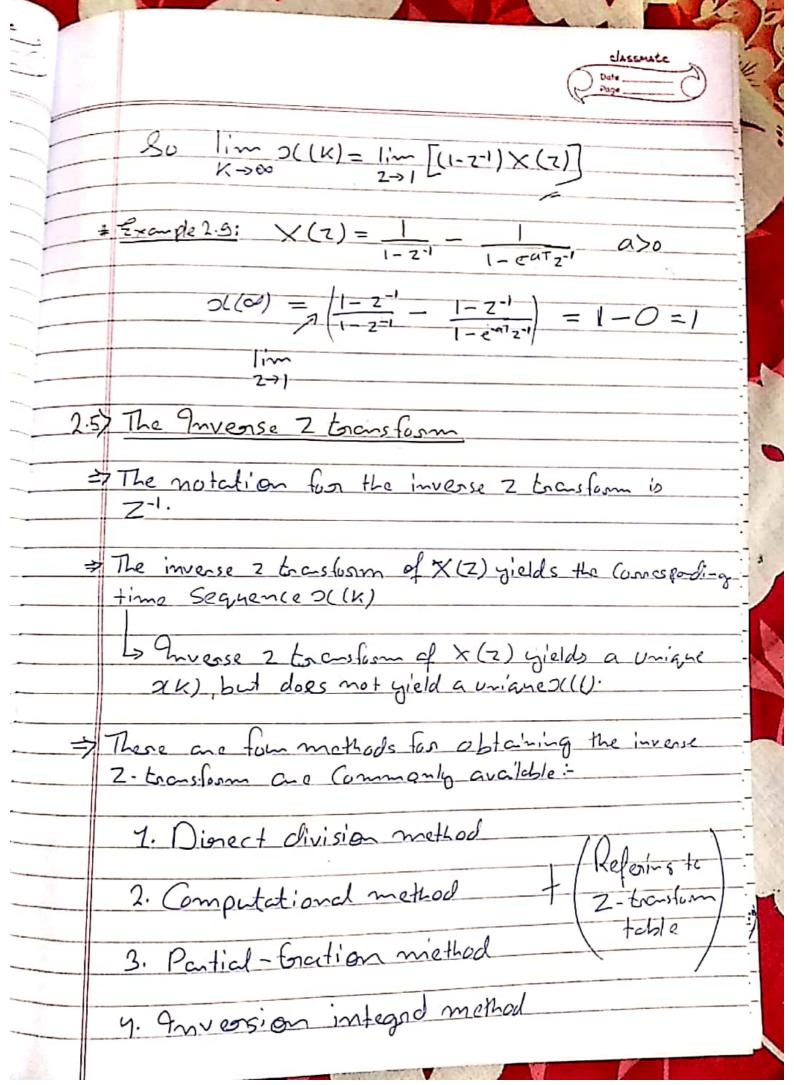
$$\frac{Z[\alpha(\kappa)] = \chi(z) = \sum_{k=0}^{\infty} \alpha(k) z^{-k}}{k}$$

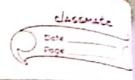
$$Z[\chi(\kappa-1)] = Z^{-1} \times (Z) = \sum_{k=0}^{\infty} \chi(\kappa-1) Z^{-k}$$

$$\sum_{k=0}^{\infty} \mathcal{D}((k) z^{-k} - \sum_{k=0}^{\infty} \chi(k-1) z^{-k} = (1-z^{-1}) \times (z)$$

$$\lim_{z \to \infty} (1-z^{-1}) \times (z) = [2(yy) - 2(x^{-1})] + [2(y) - 2(y^{-1})]$$

$$= 2((0))$$





assume on would, that the time Sequence I(KT) on X(K) is zero for K10.

* Poles and Zeros in the z plane

In angineering applications of the Z-transform method, X(2) may have the form:

 $X(z) = \frac{b_0 z^m + b_1 z^{m_1} + \dots + b_m}{z^n + a_1 z^{m_1} + \dots + a_m}$ (m < n)

 $091 \\ \times (z) = \frac{b_0(2-7,)(2-7,-)\cdots(2-2m)}{(2-p_1)(2-p_2)\cdots(2-p_m)}$

The localion of the poles and zeros of x(2) determine the chanceteristic of x(k), the Sequence of values as numbers.

=> In Control engineering and Signal processing ×(z) to frequently expressed as a ratio of polynomials in z' or follows:-

 $X(z) = b_0 z^{-(n-m)} + b_1 z^{-(n-m+1)} + \cdots + b_m z^{-n}$ $1 + a_n z^{-1} + a_2 z^{-2} + \cdots + a_m z^{-n}$

Operation.

(1) Digect division Method

2. transform by expanding X(Z) into an infinite pure series in Z-1.

-> This method is useful when it is difficult to Oblain the Clused-form expossession for the inverse z transform.

Several terms of o((K).

$$\times (7) = \sum_{k=0}^{\infty} \chi(kT) 2^{-k}$$

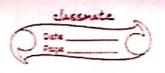
$$= \chi(0) + \chi(1)z' + \chi(1)z^{2} + \cdots + \chi(kr)z^{k} + \cdots$$

$$\frac{091}{\times (2)} = \frac{\infty}{\sum_{\kappa=0}^{\infty} \chi(\kappa) 2^{-\kappa}}$$

$$= 2(0) + 2((1)2^{-1} + 2(2)2^{-2} + \cdots + 2((\kappa 0)2^{k} + \cdots + 2((\kappa 0)2^$$

If X(z) is given in the form of a stational function, the expension into an infinite power series in increasing powers of z-1 (as be accomplished by simply dividing the numerator by the denominator, where both the numerator k denominator, where both the numerator k denominator of X(z) are worlden in increasing fower of z-1.

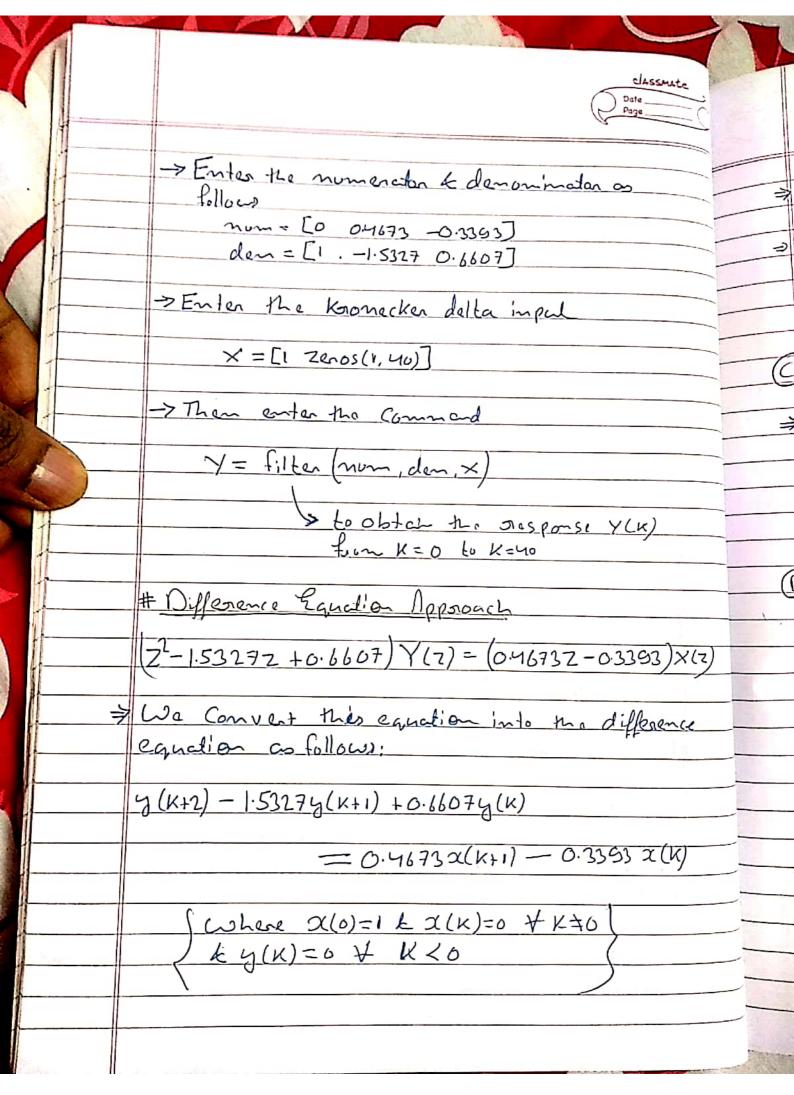
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	# 825-212 2 100
•	# Example 2.10: X(Z) = 102+5
	(Z-1)(Z-6.2)
	$\Rightarrow \times (z) = 0z^{-1} + 5z^{-2}$
8	$\frac{1 - 1.2 z^{-1} + 0.2 z^{-2}}{1 - 1.2 z^{-1} + 0.2 z^{-2}}$
V	102-1+172-2+18.42-3+18.682-1+
	1-1.22-1+022-2 102-1+5Z-2
	$\int \frac{10z^{-1} - 12z^{-2} + 2z^{-3}}{10z^{-1} - 12z^{-2} + 2z^{-3}}$
	17 2-2 - 22-3
	172-2-20:7-3+3:42-4
	1+2 -10-2 +3-2
1	18.423 -3.42-4 18.423 -22 282-4 +3.68 2-5
	18.68 z-4-3.68Z=5
	18.6824-22.4162-5+2726
111	18.02 2 - 22116 2 +112
	Thus, X(z)=102+172-2+18.42-3+18.682-4+.
	1265, 262) 10 2 1112 110 12 110 12
fi	$\chi(b) = 0$
<u> </u>	
1	$\alpha(1) = 10$
1	$\chi(n)=17$
	$\alpha(3) = 18.4$
4	2(M) = 18.68
(3)	Computational Method
	Two Computational approache to obtain inverse
	Z-transform +
1	
	1. Motlab approach
	2 Difference exaction approach.

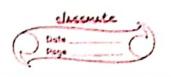


$$G(z) = \frac{0.4473 z^{-1} - 0.3343 z^{-2}}{1 - 1.5327 z' + 0.4607 z^{2}} - 0$$

$$\times$$
 (2) = 1.

$$G(7) = \frac{1}{2} - \frac{1.53177 + 6.6607}{2}$$





> Putting K=-2 we get +

= Pitti-5 K=-1 U = g16 =

Similarly all values can be found!

(C) Partial-Facition-Expasion Method

The partial-Forcetion expansion mothed presented hage, which is parallel to the partial-fraction expansion method used in Laplace transformation, is widely used in souther problem involving

1 Inversion Intogral Mathod.

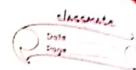
 $Z^{-1}[X(z)] = \chi(\kappa \tau) = \chi(\kappa) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \chi(z) z^{\kappa-1} dz$

where C is a circle with center at the origin of the 7 plane such that all poles of X(2) 2 k-1 are inside it.

=> The equation for giving the inverse z transform
in terms of masidnes can be derived by using
theory of complex variables. 96 can be obtained
on follows:

 $\chi(KT) = \chi(K) = K_1 + K_2 + K_3 + \cdots + K_m$

 $= \sum_{i=1}^{m} \left[\text{sidue of } \times (z) z^{k-i} \text{ at } = Z; \text{ of } \times (z) z^{k-i} \right]$



Where K, K2 -- . Kn denote the nesidue of X(2) zk-1 of poles 7, 72 -- . Zm nespectively.

To the denominator of X(2)Z K-1 Contains a simple pole Z=Z; then the Consuspending onesidue K is given by

K = 1/m [(z-zi) ×(z) zk-1]

Lif X(z) ZK-1 Contains a multiple pole Zi of order q, then the mesidue K is given by

 $K = \frac{1}{(\alpha - 1)!} \lim_{z \to z_{i}} \frac{d^{\alpha - 1}}{dz^{\alpha - 1}} [(z - z_{i})^{\alpha} \times (z)z^{\kappa - 1}]$

It should be noted that the inversion integral method, when evaluated by orasidue, is a very Simple technique for obtaining the inverse 2 transform, Provided med X(VZK-1 has no poles at the origin 2=0.

If however X(V) 2^{k-1} has a Simple pole on a multiple pole of Z=6, then Calculations may become Cumbersome and the partial -fration-expension method may prove to be Simple to the Simple apply.



$$\times (7) e^{K-1} = Z^{K} (1 - e^{-\alpha T})$$

$$= \overline{(2-1)(2-e^{-\alpha T})}$$

for K=0,1,2- X(z)ZK-1 has two single poles Z=7,=1 L Z=Z=eat.

$$K_1 = \lim_{z \to 1} [(z-1)] = 1$$
 $(z-1)(z-e^{-a_1}) = 1$

$$K_2 = 1 \text{im} \left[(2 - e^{-\alpha 7}) \times Z^{\kappa} (1 - e^{\alpha 7}) \right] = -e^{-\alpha \kappa T}$$
 $Z \Rightarrow e^{-\alpha 7} \left[(2 - e^{-\alpha 7}) \times Z^{\kappa} (1 - e^{\alpha 7}) \right]$

- 2.6) Z A Tonon sturm method for Solving difference Equation
 - of a digital computer, provided the numerical values of all coefficients and parameters are given.
 - However, closed-form expressions of x(x) cannot be obtained from the Computer solution, except for very special case.
 - The wefultness of the z-transform method is that it enables up to obtain the closed-form exposession of x(K).



time System Characterized by the following linear difference equation:

$$\chi(k) + a_1 \chi(k-1) + \cdots + a_m \chi(k-n)$$

= $b_0 u(k) + b_1 u(k-1) + \cdots + b_n u(k-n)$

When U(K) & O(K) as the System's impet & Output nespectively, at the km it enation.

$$x(k+2) + 3x(k+1) + 2x(k) = 0$$

$$\left\{ \chi(0)=0, \chi(1)=1 \right\}$$

$$\times(2) = \frac{2}{2+1} - \frac{2}{2+2} = \frac{1}{1+2} \cdot \frac{1}{1+22}$$

$$\chi(K) = (-1)^{K} - (-2)^{K} + K = 0, 1, 2 - -$$

