6

Pole Placement and Observer Design

6.1 Introduction

Controllability

→ Concerned with the problem of whether it is possible to steer a system from a given initial state to an arbitrary state.

Observability

→ Concerned with the problem of determining the state of a dynamic system from observations of the output and control vectors in a finite number of sampling period.

⇒ The concept of controllability and observability were introduced by R.E. Kalman.

⇒ The design approach of placing the closed loop poles in the desired locations in the Z plane is called the pole placement design technique.

→ In this technique we feed back all state variables so that all poles of the closed-loop system are placed at desired location.

⇒ The pole placement design process of control systems may be separated into two phases:-

First Phase: We design the System assuming that all state variables are available for feed back.

Second Phase: We design the state observer that estimates all state variables (or only those that are not directly measurable) that are required for feedback to complete design.

⇒ Assumptions in this chapter:

- Disturbance are impulses that take place randomly.
- Effect of such impulses is to change the System state. So disturbance mabe be represented as an initial state.
- Spacing between adjacent disturbances is sufficiently wide that any response to such a disturbance settels down before the next disturbance takes place.

Regulator Problem

Control Problem

⇒ Both Control & Regulator problem boils down to the determination of the desired state feedback matrix.

* <u>Regulator</u> ⇒ We desire to transfer nonzero error vector to the origin

* <u>Servo</u> ⇒ We require output to follow the command input.

→ Servo System must follow the command input and at the same time must solve any regulator problem.

6.2) Controllability

"A Control System is said to be completely state controllable if it is possible to transfer the system from any arbitrary initial state to any desired state in a finite time period"

⇒ If any state variable is independent of the control signal, then it is impossible to control this state variable and therefore the system is uncontrollable.

★ Complete State Controllability for a linear time-invariant Discrete time System

⇒ Consider the discrete-time control system defined by:-

$$\overline{X}((K+1)T) = \overline{\overline{G}}\,\overline{X}(KT) + \overline{H}\,\overline{U}(KT) \quad \text{—— (1)}$$

State Vector

Control Signal

⇒ We assume that $u(KT)$ is constant for $KT \leq t \leq (K+1)T$

⇒ The discrete-time control system given by eq ① is said to be state controllable if

↳ There exist a piece wise constant control signal $u(KT)$ defined over a finite number of sampling period that,

↳ Starting from any initial state, the state $\overline{X}(KT)$ can be transferred to the desired state $\overline{X}_f$ in at most n sampling periods.

{In this discussion $\overline{X}_f = \overline{0}$}

⇒ Solution of eq ① is given by :-

$$\overline{X}(nT) = \bar{\bar{G}}^n \overline{X}(0) + \sum_{j=0}^{n-1} \bar{\bar{G}}^{n-j-1} \overline{H} u(jT)$$

$$\Rightarrow \overline{X}(nT) - \bar{\bar{G}}^n \overline{X}(0) = \left[\overline{H} \,\vdots\, \bar{\bar{G}}\overline{H} \,\vdots\, \cdots \,\vdots\, \bar{\bar{G}}^{n-1}\overline{H} \right] \begin{bmatrix} u((n-1)T) \\ u((n-1)T) \\ \vdots \\ u(0) \end{bmatrix} \quad ②$$

⇒ If $\text{Rank}\left[\overline{H} \,\vdots\, \bar{\bar{G}}\overline{H} \,\vdots\, \cdots \,\vdots\, \bar{\bar{G}}^{n-1}\overline{H} \right] = n$ ③ then

the n vectors $\overline{H}, \bar{\bar{G}}\overline{H} \cdots \bar{\bar{G}}^{n-1}\overline{H}$ can span the n dimensional space.

⇒ The matrix $[\bar{H} \vdots \bar{G}\bar{H} \vdots --- \vdots \bar{G}^{n-1}\bar{H}]$ is commonly called the Controllability matrix.

⇒ If Rank of Controllability matrix is n, then for an arbitrary state $\bar{x}(nT) = \bar{x}_f$ there exists ~~a~~ sequence of unbounded control signals $u(0), u(T) --- u((n-1)T)$ that satisfies equation ②.

↳ Hence rank of the Controllability matrix be n gives a sufficient condition for complete state Controllability.

⇒ To prove that eq ③ it is also a necessary condition for complete state Controllability, let us assume that,

$$\text{Rank}\,[\bar{H} \vdots \bar{G}\bar{H} \vdots --- \vdots \bar{G}^{n-1}\bar{H}] < n$$

⇒ By use of Cayley-Hamilton theorem, it can be shown that, for an arbitrary i $\bar{G}^i\bar{H}$ can be expressed as a linear combination of $\bar{H}, \bar{G}\bar{H}$ --- $\bar{G}^{n-1}\bar{H}$.

↳ Consequently for any i

$$\text{Rank}\,[\bar{H} \vdots \bar{G}\bar{H} \vdots --- \vdots \bar{G}^{i-1}\bar{H}] < n$$

So vectors $\bar{H}, \bar{G}\bar{H}, \ldots \bar{G}^{i-1}\bar{H}$ cannot span the n-dimensional space.

⇒ Therefor, for some $\overline{X}_f$, it is not possible to have $\overline{X}(iT) = \overline{X}_f \; \forall \; i$.

↳ Thus the Condition given by eq ③ is necessary.

★ <u>Complete State Controbality in the Case where $\overline{u}(KT)$ is a Vector</u>

⇒ If the System is defined by :-

$$\overline{X}((K+1)T) = \overline{\overline{G}}\,\overline{X}(KT) + \overline{\overline{H}}\,\overline{u}(KT)$$

$\overline{\overline{G}}$: $n \times n$; $\overline{X}(KT)$: n-State Vector; $\overline{\overline{H}}$: $n \times r$; $\overline{u}(KT)$: r-Control Vector

⇒ It can be Proved that the Condition for Complete State Controllability is :-

$$\text{Rank}\underbrace{\left[\overline{\overline{H}} \;\vdots\; \overline{\overline{G}}\,\overline{\overline{H}} \;\vdots\; \cdots \;\vdots\; \overline{\overline{G}}^{n-1}\overline{\overline{H}}\right]}_{n \times nr} = n$$

★ <u>Determining Control Sequence to Bring the initial State to a Desired State</u>

⇒ If Controbality matrix is of rank n and $u(KT)$ is a Scalar, then, it is possible to find n linearly independend Scalar equations

↳ From which a Sequence of unbounded Control signal $u(KT)$ $\forall K = 0\, 1 \cdots n-1$ Can be uniquely determined.

↳ Such that any initial state $\overline{X}(0)$ is transferred to the desired state in n sampling periods. (eq ②)

⇒ If the control signal is not a scalar but a vector, then the sequence of $\overline{u}(KT)$ is not unique.

★ <u>Alternative form of the Condition for Complete State Controllability</u>

⇒ Consider the System defined by

$$\overline{X}((K+1)T) = \overline{G}\,\overline{X}(KT) + \overline{\overline{H}}\,\overline{u}(KT)$$

⇒ If the eigenvectors of $\overline{G}$ are distinct, then it is possible to find a transformation matrix $\overline{P}$ such that

$$\overline{\overline{P}}^{-1}\,\overline{\overline{G}}\,\overline{\overline{P}} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

⇒ <u>Note</u>: i^{th} Column of the $\overline{\overline{P}}$ matrix is an eigenvector of $\overline{G}$ associated with i^{th} eigenvalue λ_i.

Let us define,

$$\bar{X}(KT) = \bar{\bar{P}}\,\hat{X}(KT) \quad \text{— (5)}$$

So,

$$\hat{X}((K+1)T) = \bar{\bar{P}}^{-1}\bar{\bar{G}}\,\bar{\bar{P}}\,\hat{X}(KT) + \bar{\bar{P}}^{-1}\bar{\bar{H}}\,\bar{u}(KT) \quad \text{— (6)}$$

Let $\bar{\bar{F}} = \bar{\bar{P}}^{-1}\bar{\bar{H}}$

⇒ Then eq (6) may be written as :-

$$\hat{X}_1((K+1)T) = \lambda_1 \hat{X}_1(KT) + f_{11} u_1(KT) + f_{12} u_2(KT) + \cdots + f_{1r} u_r(KT)$$

$$\hat{X}_2((K+1)T) = \lambda_2 \hat{X}_2(KT) + f_{21} u_1(KT) + f_{22} u_2(KT) + \cdots f_{2r} u_r(KT)$$

$$\vdots$$

$$\hat{X}_n((K+1)T) = \lambda_n \hat{X}_n(KT) + f_{n1} u_1(KT) + f_{n2} u_2(KT) + \cdots + f_{nr} u_r(KT)$$

⇒ If the element of any one row of the $n \times r$ matrix $\bar{\bar{F}}$ are all zero, then the corresponding state variables cannot be controlled by any of the $u_i(KT)$.

↳ Hence, the condition for complete state controllability is that, if the eigenvectors of $\bar{\bar{G}}$ are distinct, then the system is completely state controllable if and only if no row of $\bar{\bar{P}}^{-1}\bar{\bar{H}}$ has all zero elements.

⇒ In case $\bar{\bar{G}}$ is not diagonalizable then we may transform $\bar{\bar{G}}$ into a Jordan canonical form.

⇒ Suppose it is possible to find a transformation matrix $\bar{\bar{S}}$ such that

$$\bar{\bar{S}}^{-1}\bar{\bar{G}}\bar{\bar{S}} = \bar{\bar{J}}$$

⇒ If we define a new state vector $\hat{x}$ by

$$\bar{X}(KT) = \bar{\bar{S}}\hat{x}(KT) \quad \text{—} \ (7)$$

So, $\hat{x}((K+1)T) = \bar{\bar{S}}^{-1}\bar{\bar{G}}\bar{\bar{S}}\hat{x}(KT) + \bar{\bar{S}}^{-1}\bar{\bar{H}}\bar{u}(KT)$

$$= \bar{\bar{J}}\hat{x}(KT) + \bar{\bar{S}}^{-1}\bar{\bar{H}}\bar{u}(KT) \quad \text{—} \ (8)$$

⇒ The system given by eq. (8) is completely state controllable if and only if :-

(1) No two Jordan blocks in $\bar{\bar{J}}$ are associated with the same eigen value.

(2) Elements of any row of $\bar{\bar{S}}^{-1}\bar{\bar{H}}$ that corresponds to the last row of each Jordan block are not all zeros.

(3) Elements of each row of $\bar{\bar{S}}^{-1}\bar{\bar{H}}$ that corresponds to distinct eigenvalues are not zero.

★ Condition for Complete State Controllability in the Z-plane

"A necessary and sufficient condition for complete state controllability is that no cancellation occur in the pulse transfer function.

↳ If cancellation occurs, the system cannot be controlled in the direction of the cancelled mode.

★ Complete Output Controllability

⇒ If practical design of a control system we may want to control the output rather than the state of the system.

↳ Complete state controllability is neither necessary nor sufficient for controlling the output of the system.

⇒ Consider the system defined by the equations

$$\bar{X}((K+1)T) = \bar{G}\bar{x}(KT) + \bar{H}u(KT) \quad \text{— (9)}$$

$$\bar{Y}(KT) = \bar{C}\bar{x}(KT) \quad \text{— (10)}$$

⇒ The system defined by eq (9) & (10) is said to be completely output controllable, if it is possible to construct an unconstrained control signal $u(KT)$ defined over a finite number of sampling periods $0 \leq KT \leq nT$ such that,

↳ Starting from any initial output $\bar{y}(0)$, the output $\bar{y}(KT)$ can be transferred to the desired point $\bar{y}_f$ in the output space in at most n sampling periods.

⇒ Solution of eq (9) is :-

$$\bar{x}(nT) = \bar{\bar{G}}^{n}\bar{x}(0) + \sum_{j=0}^{n-1} \bar{\bar{G}}^{n-j-1}\bar{H}u(jT)$$

We have $\bar{y}(nT) = \bar{C}\bar{x}(nT)$

$$\bar{y}(nT) = \bar{\bar{C}}\bar{\bar{G}}^{n}\bar{x}(0) + \sum_{j=0}^{n-1} \bar{\bar{C}}\bar{\bar{G}}^{n-j-1}\bar{H}u(jT)$$

$$\bar{y}(nT) - \bar{C}\bar{\bar{G}}^{n}\bar{x}(0) = \left[\bar{\bar{C}}\bar{H} \,\vdots\, \bar{\bar{C}}\bar{\bar{G}}\bar{H} \,\vdots\, \cdots \,\vdots\, \bar{\bar{C}}\bar{\bar{G}}^{n-1}\bar{H}\right]\begin{bmatrix} u((n-1)T) \\ u((n-2)T) \\ \vdots \\ u(0) \end{bmatrix}$$

⇒ A necessary and sufficient condition for the system to be completely output controllable is that vectors $\bar{C}\bar{H}, \bar{C}\bar{\bar{G}}\bar{H} \cdots \bar{C}\bar{\bar{G}}^{n-1}\bar{H}$ spans m-D output space.

$$\text{Rank}\left[\bar{\bar{C}}\bar{H} \,\vdots\, \bar{\bar{C}}\bar{\bar{G}}\bar{H} \,\vdots\, \cdots\cdots \,\vdots\, \bar{C}\bar{\bar{G}}^{n-1}\bar{H}\right] = m \quad \text{—— (11)}$$

⇒ Next, Consider the system defined by the equations:-

$$\bar{x}((k+1)T) = \bar{G}\bar{x}(kT) + \bar{H}\bar{u}(kT) \quad \text{—— (12)}$$

$$\bar{y}(kT) = \bar{C}\bar{x}(kT) + \bar{D}\bar{u}(kT) \quad \text{—— (13)}$$

A necessary & Sufficient Condition for the System defined by eq (12) & (13) to be Completely output Controllable if :-

$$\text{Rank}\left[\bar{D} \;\vdots\; \bar{C}\bar{H} \;\vdots\; \bar{C}\bar{G}\bar{H} \;\vdots\; \cdots \;\vdots\; \bar{C}\bar{G}^{n-1}\bar{H}\right] = m$$

6-3) Observability