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Mathematical Modeling of Control Systems

"A mathematical model of a dynamic system is defined as a set of equations that represents the dynamics of the System accurately or at least fairly well"

→ Simplicity Vs Accuracy

⇒ We must be well aware that a linear lumped-Parameter model, which may be valid in low-frequency operations, may not be valid at sufficiently high frequency.

★ Linear System ⇒ A System is called linear if the principle of Superposition applies.

⇓

It States that the response produced by the Simultaneous application of two different forcing function is the sum of the two individual responces

⇒ A differential equation is linear if the Coefficients are constants or function only of the independent Variable.

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Linear Time-Invariant System

System that are represented by differential equations whose coefficients are not a function of time

Linear Time-Varying System

System that are represented by differential equations whose coefficients are a function of time

★ Transfer Function

"The transfer function of a Linear, time-invariant, differential equation system is defined as the ratio of the Laplace transform of the output (response function) to the Laplace transform of the input (driving function) under the assumption that all initial conditions are zero"

⇒ Consider a linear time-invariant system defined by the following differential equation:

$$a_0 \overset{n}{y} + a_1 \overset{n-1}{y} + \cdots + a_{n-1} \dot{y} + a_n y$$
$$= b_0 \overset{m}{x} + b_1 \overset{m-1}{x} + \cdots + b_{m-1} \dot{x} + b_m x$$

Where y is the output of the system and x is the input

So, Transfer function, $G(s) = \left.\dfrac{\mathcal{L}[\text{Output}]}{\mathcal{L}[\text{Input}]}\right|_{\text{zero initial condition}}$

$$G(s) = \frac{Y(s)}{X(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1} s^1 + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}$$

If the highest power of S in the denominator of the transfer function is equal to n, the System is Called an n^{th}-order system.

<u>Note</u>

"The applicability of the concept of the transfer function is <u>limited to linear, time-invariant differential equation System</u>"

★ Convolution Integral

$$G(s) = \frac{Y(s)}{X(s)} \Rightarrow Y(s) = G(s)\,X(s)$$

⇒ Multiplication in the Complex domain is equivalent to Convolution in the time domain, so the inverse Laplace transform of above equation is given by the following Convolution integral

$$y(t) = \int_0^t x(\tau)\, g(t-\tau)\, d\tau$$
$$= \int_0^t g(\tau)\, x(t-\tau)\, d\tau$$

where both $g(t)$ & $x(t)$ are 0 $\forall$ $t<0$.

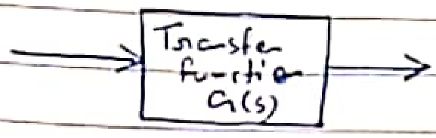
★ Automatic Control Systems

Block Diagram: A block diagram of a system is a pictorial representation of the functions performed by each component and of the flow of signals.

⇒ In a block diagram all system variables are linked to each other through functional block.

⇒ The functional block or simply block is a symbol for the mathematical operation on the input signal to the block that produces output.

↳ The transfer functions of the components are usually entered in the corresponding block, which are connected by arrow to indicate the direction of flow of signals.

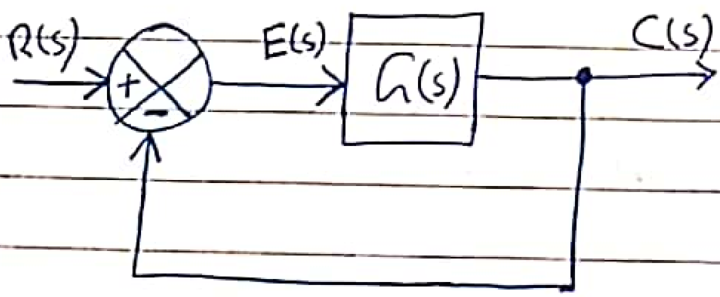


i) <u>Summing point</u>: Circle with a Cross is the Symbol that indicates a summing operation.

→ The plus and minus sign at each arrowhead indicates whether the signal is to be added or subtracted.

ii) <u>Branch point</u>: A branch point is a point from which the signal from a block goes concurrently to other blocks or summing point.

<u>Block Diagram of a Closed-loop System</u>

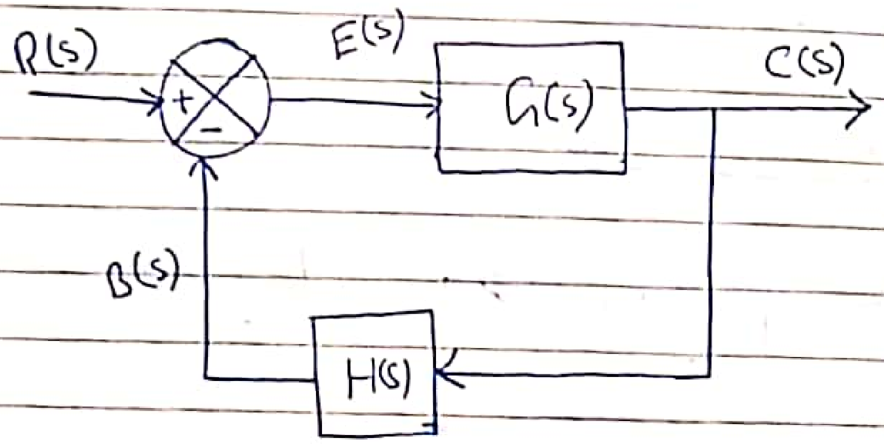


C(s) : Output
R(s) : Reference Input
E(s) : Error Signal

⇒ Any linear control system may be represented by a block diagram consisting of block, summing point and branch point.

⇒ When the output is fed back to the summing point for comparision with the input, it is necessary to convert the form of the output signal ~~is would~~ to that of input signal.

↳ This conversion is achived by the feedback element whose transfer function is H(s).



⇒ In most of the cases feedback element is a sensor that measures the output of the plant.

Transfer Function

Open loop Transfer function

$$\Rightarrow \frac{B(s)}{E(s)} = G(s)\,H(s)$$

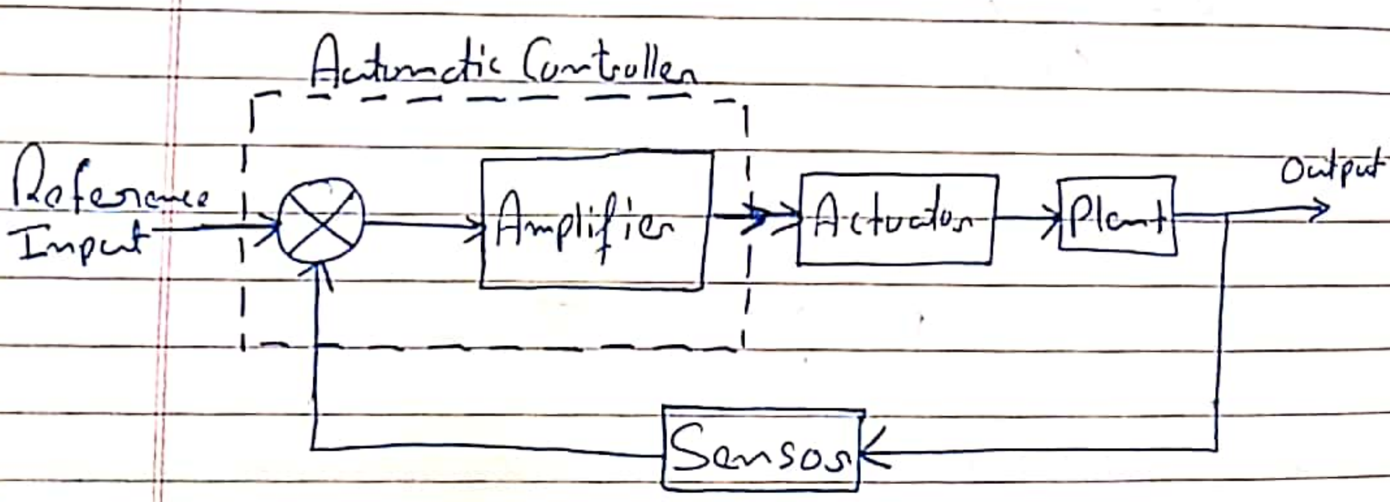
Food forward Transfer function

$$\frac{C(s)}{E(s)} = G(s)$$

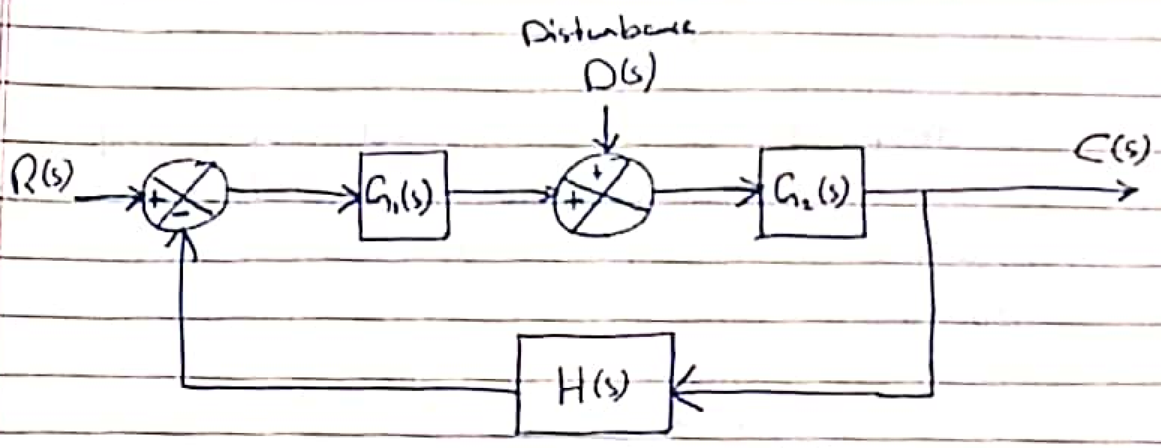
Closed loop Transfer function

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

★ **Automatic Controller:** An automatic controller compares the actual value of the plant output with the reference input (desired value), determines the deviation, and produces a control signal that will reduce the deviation to zero or to a small value.



★ Closed-Loop System Subjected to Disturbance



⇒ When two inputs (the reference input and disturbance) are present in a linear time invariant system, each input can be treated independently of the other; and the output corresponding to each input alone can be added to give the complete output.

⇒ In examining the effect of the disturbance D(s), we may assume that the reference input is zero,

$$\frac{C_D(s)}{D(s)} = \frac{G_2(s)}{1 + G_1(s)G_2(s)H(s)} \quad \text{——— (1)}$$

⇒ On the other hand, in considering the response to the reference input R(s), we may assume that the disturbance is zero,

$$\frac{C_R(s)}{R(s)} = \frac{G_1(s)\,G_2(s)}{1 + G_1(s)\,G_2(s)\,H(s)}$$

⇒ The response to the simultaneous application of the refference input and disturbance can be obtained by adding the two individual responses.

$$C(s) = C_R(s) + C_D(s)$$

$$\boxed{C(s) = \frac{G_2(s)}{1 + G_1(s)\,G_2(s)\,H(s)}\left[G_1(s)\,R(s) + D(s)\right]}$$

★ Procedure for Drawing a Block Diagram

⇒ To draw a block diagram for a system, first write the equations that describe the dynamic behavior of each component.

⇒ Then take the laplace transform of these equations, assuming zero initial conditions and represent each Laplace-transformed eqution individually in Block form.

⇒ Finally assembly of these elements into a complete block diagram.

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★ Linearization of Non-Linear Mathematical Models

Nonlinear System :- A system is nonlinear if the principle of superposition does not apply.

Linearization of Nonlinear System

⇒ In Control engineering a normal operation of the system may be around an equilibrium point, and the signal may be considered small signal around the equilibrium.

⇒ If the system operates around an equilibrium point and if the signal involved are small signals, then it is possible to approximate the non-linear system by a linear system.

⇒ The linearization procedure is based on the expantion of nonlinear function into a Taylor series about the operating point and the retention of only the linear term.

Linear approximation of Non linear Mathematical Models

⇒ Consider a system whose input is $x(t)$ and output is $y(t)$. The relationship between $y(t)$ and $x(t)$ is given by:-

$$y = f(x)$$

⇒ If the normal operating condition corresponds to $\bar{x}, \bar{y}$ then Equation may be expanded into a Taylor series about this point as follows

$$y = f(\bar{x}) + \frac{df}{dx}(x-\bar{x}) + \frac{1}{2!}\frac{d^2f}{dx^2}(x-\bar{x})^2 + \cdots$$

where, the derivatives $\frac{df}{dx}, \frac{d^2f}{dx^2}$ are evaluated at $x = \bar{x}$.

⇒ If the variation $x-\bar{x}$ is small, we may neglect the higher-order terms in $x-\bar{x}$.

$$y = \bar{y} + K(x-\bar{x})$$

$$\left\{ \begin{aligned} \bar{y} &= f(\bar{x}) \\ K &= \left.\frac{df}{dx}\right|_{x=\bar{x}} \end{aligned} \right.$$

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$$\Rightarrow \boxed{(y-\bar{y}) = k(x-\bar{x})}$$

Equation above gives a linear mathematical model for a non-linear system about operating point $x=\bar{x}$, $y=\bar{y}$.

$\Rightarrow$ Next Consider a nonlinear system whose output y is a function of two input x_1 and x_2.

$$y = f(x_1, x_2)$$

$\Rightarrow$ To obtain a linear approximation to this nonlinear system, we may expand above into a Taylor Series about the normal operating point $\bar{x}_1, \bar{x}_2$.

$$y = f(\bar{x}_1, \bar{x}_2) + \left[\frac{\delta f}{\delta x_1}(x_1-\bar{x}_1) + \frac{\delta f}{\delta x_2}(x_2-\bar{x}_2)\right]$$

$$+ \frac{1}{2!}\left[\frac{\delta^2 f}{\delta x_1^2}(x_1-\bar{x}_1)^2 + 2\frac{\delta^2 f}{\delta x_1 \delta x_2}(x_1-\bar{x}_1)(x_2-\bar{x}_2)\right.$$

$$\left. + \frac{\delta^2 f}{\delta x_2^2}(x_2-\bar{x}_2)^2\right] + \cdots$$

where partial derivatives are evaluated at $x_1=\bar{x}_1$ & $x_2=\bar{x}_2$.

⇒ Near the normal operating point, the higher order terms may be neglected.

$$\boxed{y - \bar{y} = K_1(x_1 - \bar{x}_1) + K_2(x_2 - \bar{x}_1)}$$

$$\bar{y} = f(\bar{x}_1, \bar{x}_2)$$

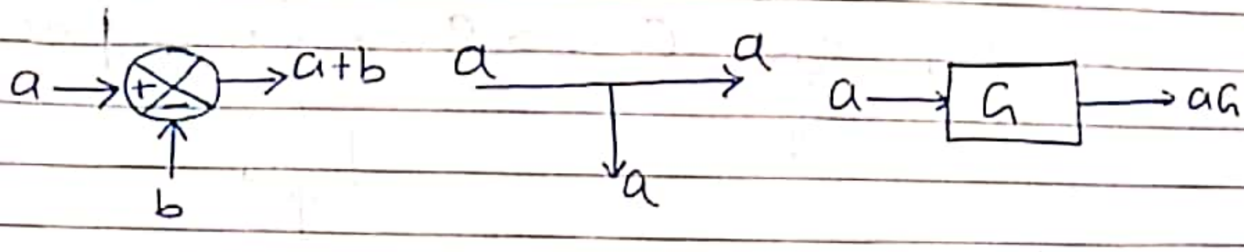
$$K_1 = \left.\frac{\delta f}{\delta x_1}\right|_{x_1 = \bar{x}_1,\ x_2 = \bar{x}_2}$$

$$K_2 = \left.\frac{\delta f}{\delta x_2}\right|_{x_1 = \bar{x}_1,\ x_2 = \bar{x}_2}$$

⇒ The linearization technique presented here is valid in the vicinity of the operating condition. If operating conditions vary widely, however, such linearization equations are not adequate and nonlinear equations must be delt with.

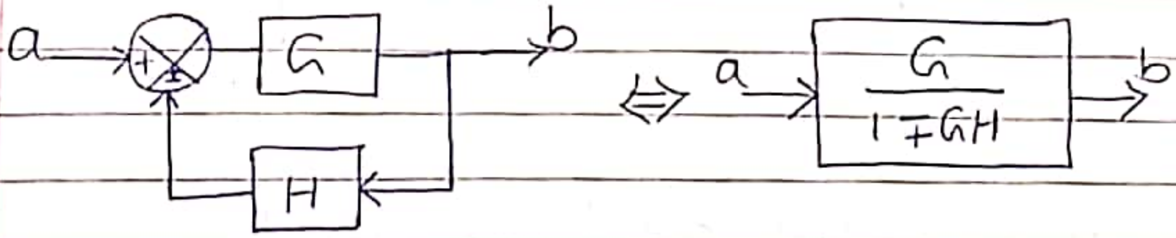
★ Block-Diagram Algebra

Elements of Block Algebra

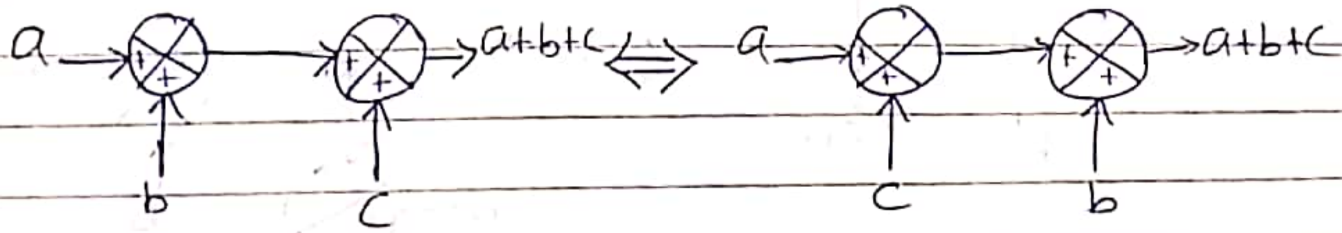


(i) (ii) (iii)

1. Feedback Loop

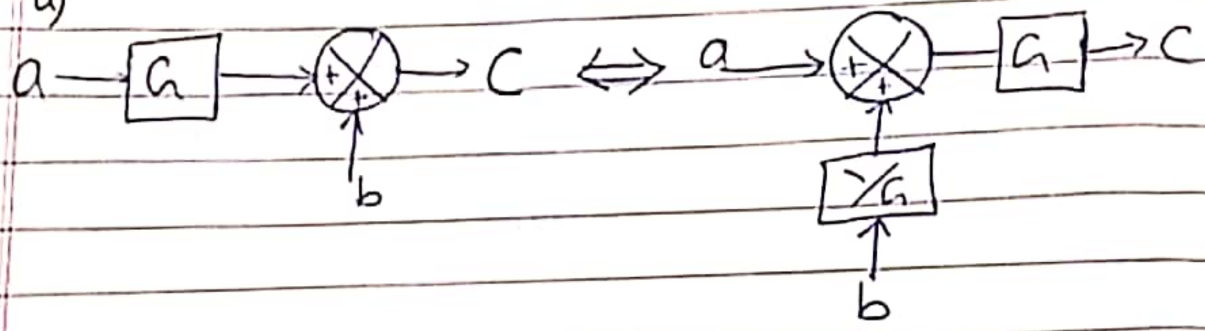


2. Two Summing points

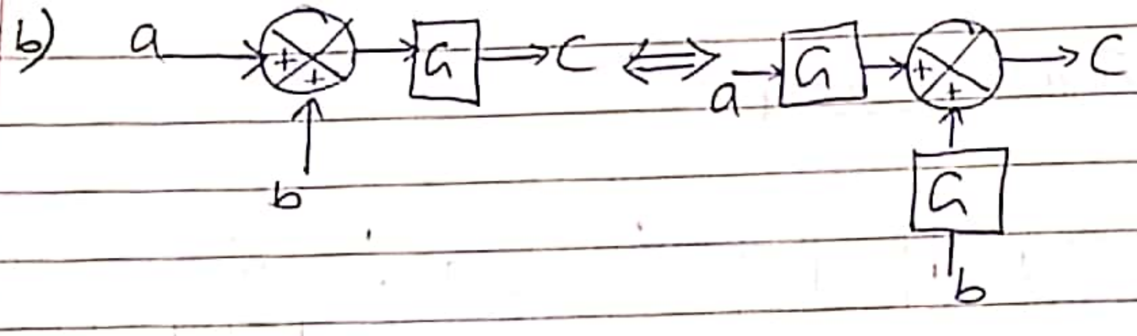


3. Summing point and block

a)

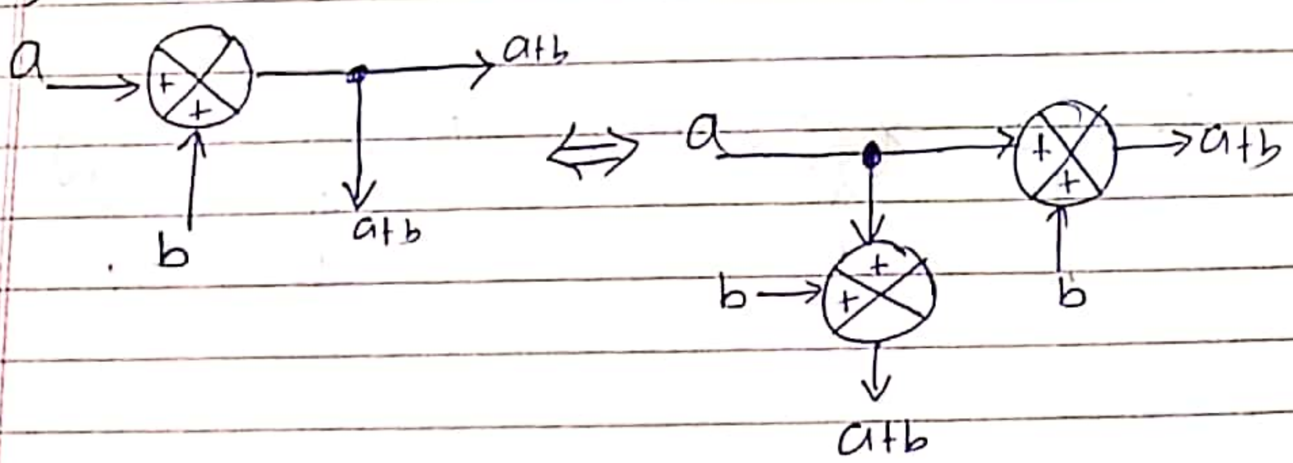


b)



4. Summing point and branch point

a)



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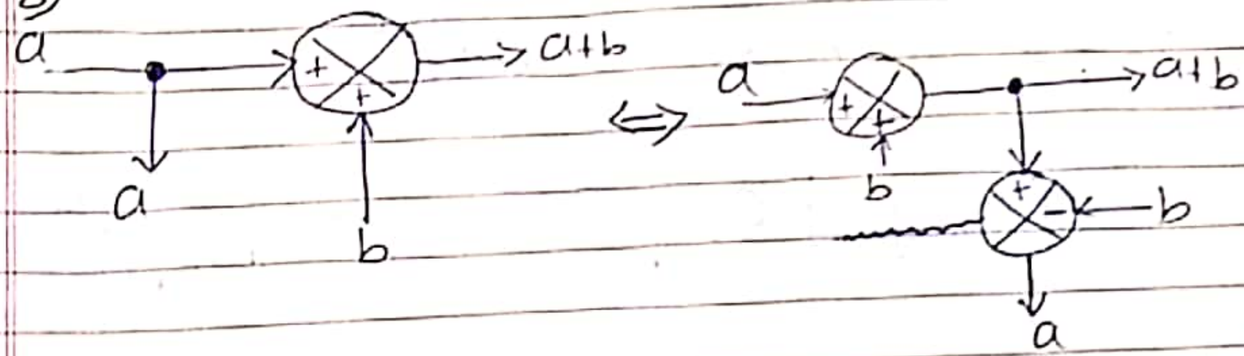
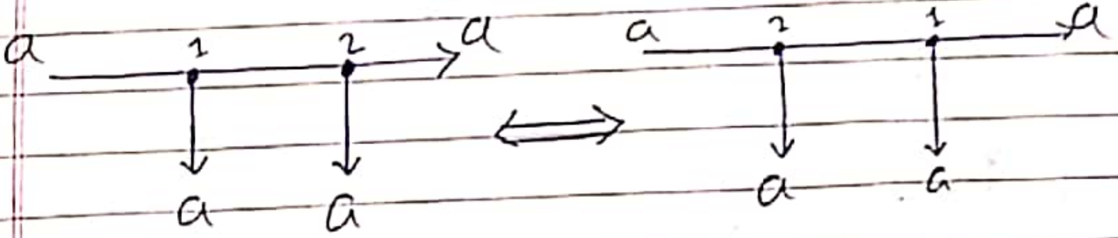
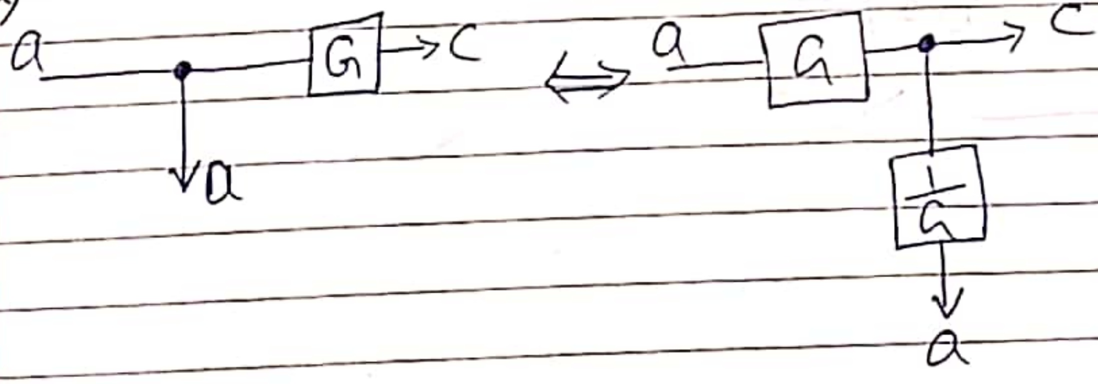


5. Two branch point

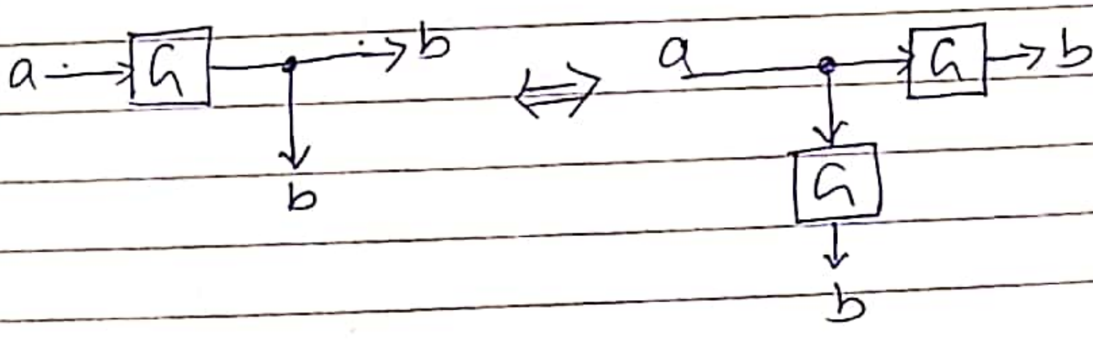


6. Branch point and Block

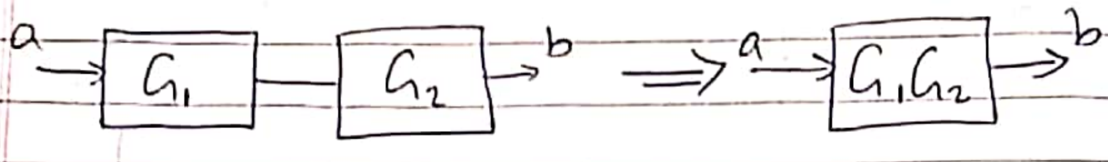
a)



b)



7. Two Block (Cascaded)



8. Blocks (in parallel)