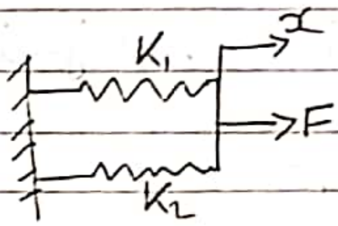
2

Mathematical Modeling of Mechanical & Electrical System

★ Mathematical Modeling of Mechanical System

Example 3.1

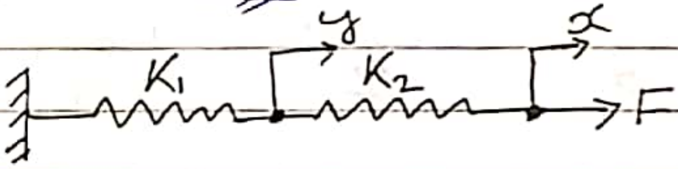
ⓐ Parallel



$$F = K_{eq}\,x = K_1 x + K_2 x$$

$$\Rightarrow \boxed{K_{eq} = K_1 + K_2}$$

ⓑ Series



$$F = K_{eq}\,x$$

$$F = K_1 y = K_2 (x - y)$$

$$\Rightarrow K_1 y = K_2 x - K_2 y$$

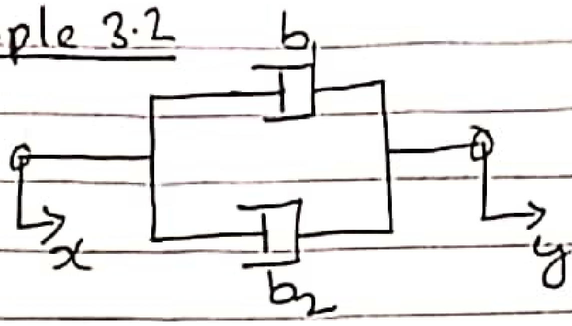
$$y = \frac{K_2 x}{K_1 + K_2}$$

$$\boxed{\frac{1}{K_{eq}} = \frac{1}{K_1} + \frac{1}{K_2}}$$

$$\frac{K_1 K_2 x}{K_1 + K_2} = K_{eq}\,x \Rightarrow K_{eq} = \frac{K_1 K_2}{K_1 + K_2}$$

Example 3.2

(a)

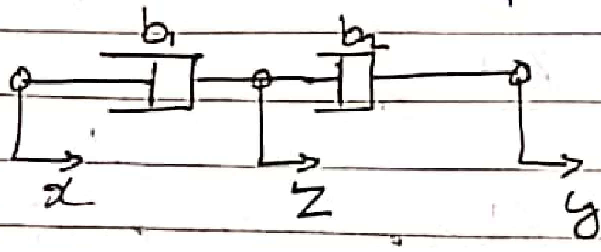


$$f = b_1(\dot{y}-\dot{x}) + b_2(\dot{y}-\dot{x}) = (b_1+b_2)(\dot{y}-\dot{x})$$

$$f = b_{eq}(\dot{y}-\dot{x})$$

So $\boxed{b_{eq} = b_1 + b_2}$

(c)



$$f = b_{eq}(\dot{y}-\dot{x})$$

$$f = b_1(\dot{z}-\dot{x}) = b_2(\dot{y}-\dot{z})$$

$$b_1\dot{z} - b_1\dot{x} = b_2\dot{y} - b_2\dot{z}$$

$$(b_1+b_2)\dot{z} = b_2\dot{y} + b_1\dot{x} - (b_1+b_2)\dot{x}$$

$$-(b_1+b_2)\dot{x}$$

$$(b_1+b_2)(\dot{z}-\dot{x}) = b_2\dot{y} - b_2\dot{x}$$

$$(\dot{z}-\dot{x}) = \frac{b_2}{b_1+b_2}(\dot{y}-\dot{x})$$

$$\frac{b_1 b_2}{b_1 + b_2}(\dot{y} - \dot{x}) = b_{eq}(\dot{y} - \dot{x})$$

$$b_{eq} = \frac{b_1 b_2}{b_1 + b_2} \Rightarrow \boxed{\frac{1}{b_{eq}} = \frac{1}{b_1} + \frac{1}{b_2}}$$

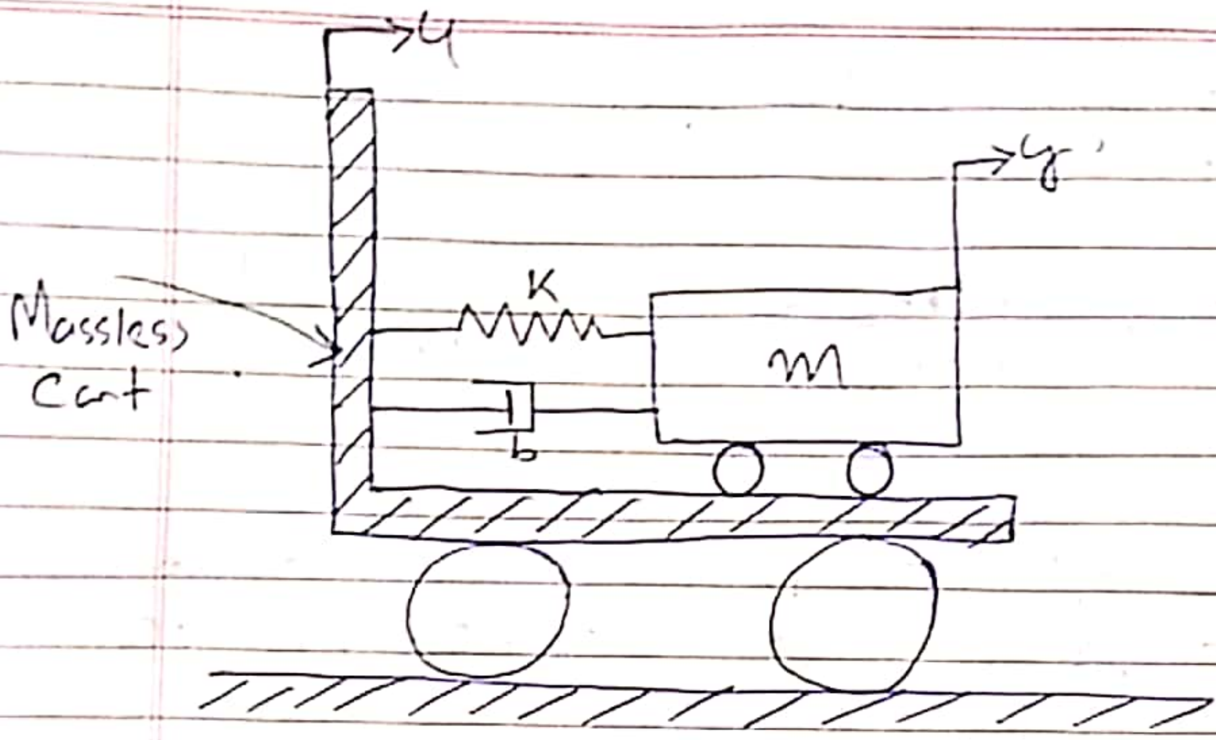
<u>Dashpot</u>: A dashpot is a mechanical device, a damper which resists motion via viscous friction.

→ The resulting force is proportional to the velocity, but acts in the opposite direction, slowing the motion and absorbing energy.

 $\rightarrow F = bV/b\dot{x}$

<u>Example 3.3:</u>

PTO



⇒ Cart is standing still at $t<0$

⇒ Spring-mass-dashpot system on the cart is also standing still $\forall\, t<0$

$u(t)$: Input {displacement of cart}

at $t=0$ $\quad \dot{u}$ = Constant.

$y(t)$: Output {displacement of mass relative to ground}

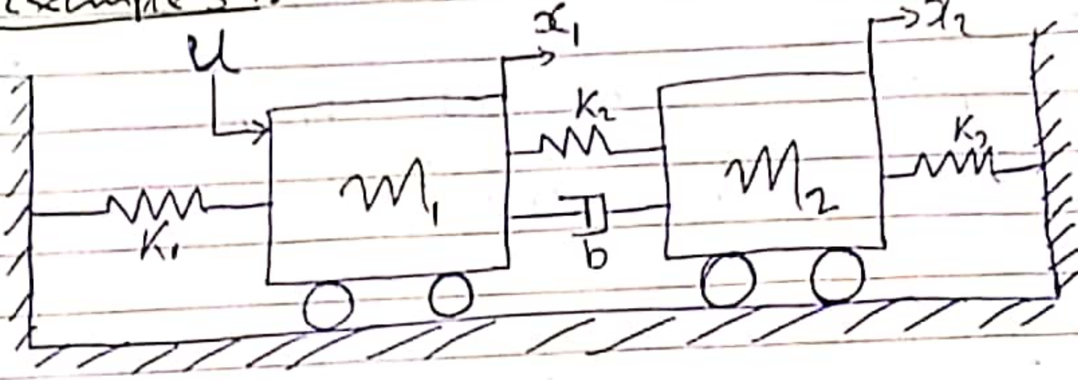
$$-K(y-u) - b\left(\frac{dy}{dt} - \frac{du}{dt}\right) = m\frac{d^2y}{dt^2}$$

$$\Rightarrow \quad m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + Ky = b\frac{du}{dt} + Ku$$

$$(ms^2 + bs + K)Y(s) = (bs + K)U(s)$$

$$\boxed{G(s) = \frac{Y(s)}{U(s)} = \frac{bs + K}{ms^2 + bs + K}}$$

Example 3.4:



u is the external force applied and x_1 & x_2 represents position of m_1 & m_2 respectively.

Input : u
Output : x_1 & x_2

$$u - K_1 x_1 - K_2(x_1 - x_2) - b(\dot{x}_1 - \dot{x}_2) = m_1 \ddot{x}_1 \quad \text{--- (1)}$$

$$-K_3 x_2 - K_2(x_2 - x_1) - b(\dot{x}_2 - \dot{x}_1) = m_2 \ddot{x}_2 \quad \text{--- (2)}$$

Equation (1) & (2) is the mathematical model of the given System.

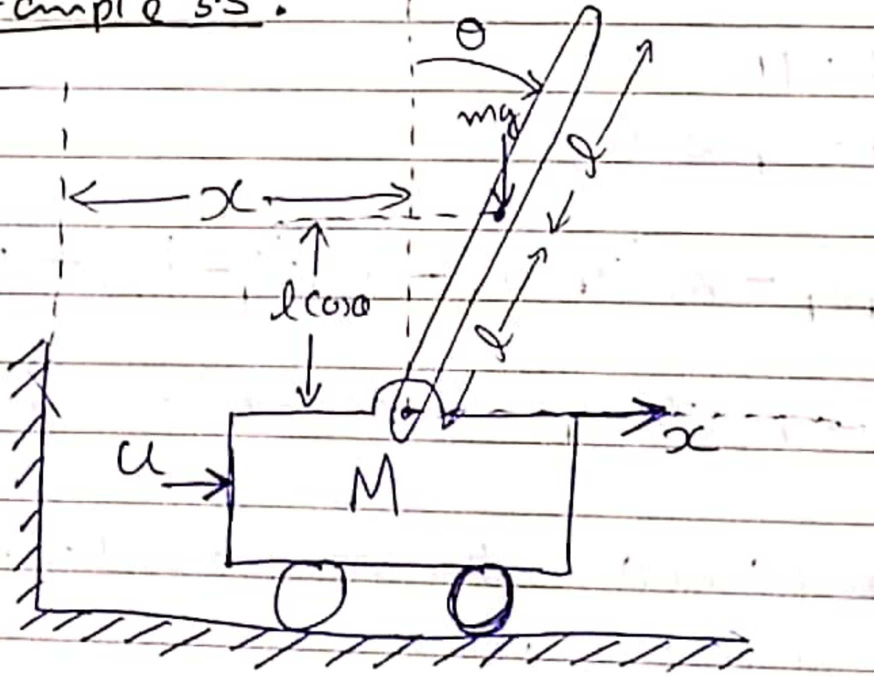
$$U(s) - K_1 X_1(s) - K_2\left(X_1(s) - X_2(s)\right) - bs\left(X_1(s) - X_2(s)\right)$$
$$= m_1 s^2 X_1(s) \quad \text{--- (1)}$$

$$-K_3 X_2(s) - K_2\left(X_2(s) - X_1(s)\right) - bs\left(X_2(s) - X_1(s)\right)$$
$$= m_2 s^2 X_2(s) \quad \text{--- (2)}$$

$$\boxed{\frac{X_1(s)}{U(s)} = \frac{m_2 s^2 + bs + K_2 + K_3}{(m_1 s^2 + bs + K_1 + K_1)(m_2 s^2 + bs + K_2 + K_3) - (bs + K_2)^2}}$$

$$\boxed{\frac{X_2(s)}{U(s)} = \frac{bs + K_2}{(m_1 s^2 + bs + K_1 + K_2)(m_2 s^2 + bs + K_2 + K_3) - (bs + K_2)^2}}$$

Example 3.5:



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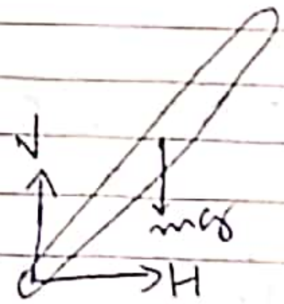
<u>Inverted pendulum</u>: Pendulum that has its Center of mass above its pivot point.

eg → human being

⇒ 2 DOF System (θ, x)

Input force = u

$$x_G = x + l\sin\theta$$
$$y_G = l\cos\theta$$



$$I\ddot{\theta} = Vl\sin\theta - Hl\cos\theta \quad \text{—— (1)}$$

$$H = m\frac{d^2}{dt^2}(x + l\sin\theta) \quad \text{—— (2)}$$

$$V - mg = m\frac{d^2}{dt^2}(l\cos\theta) \quad \text{—— (3)}$$

$$M\frac{d^2x}{dt^2} = u - H \quad \text{—— (4)}$$

The above four equation is the mathematical model of the given System.

$\Rightarrow$ Since the inverted pendulum is kept vertical.

$\hookrightarrow$ θ can be assumed to be close to zero.

$$\sin\theta \approx \theta \qquad \cos\theta \approx 1 \qquad \{\text{linearization}\}$$

$$I\ddot{\theta} = Vl\theta - Hl \quad \text{--- (1)}$$
$$m(\ddot{x} + l\ddot{\theta}) = H \quad \text{--- (2)}$$
$$V = mg \quad \text{--- (3)}$$
$$M\ddot{x} = u - H \quad \text{--- (4)}$$

{linearized Mathematical Model}

$\Downarrow$ Eliminating H & V

$$(M+m)\ddot{x} + ml\ddot{\theta} = u \quad \text{--- (a)}$$

$$(I+ml^2)\ddot{\theta} + ml\ddot{x} = mgl\theta \quad \text{--- (b)}$$

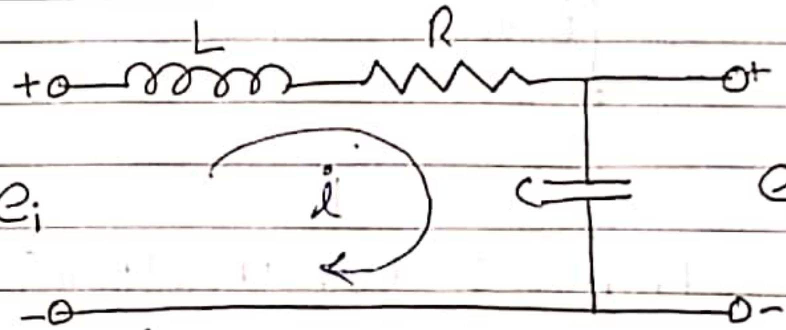
$$(M+m)s^2X(s) + mls^2\theta(s) = U(s)$$
$$(I+ml^2)s^2\theta(s) + mls^2X(s) = mgl\theta(s)$$

★ Mathematical Modeling of Electrical System

Basic laws governing electrical circuit are Kirchhoff's Law:

- Current Law: Algebraic Sum of all currents entering and leaving a node is zero.
- Voltage Law: At any given instant the algebraic sum of the voltage around any loop in an electric circuit is zero.

LRC Circuit

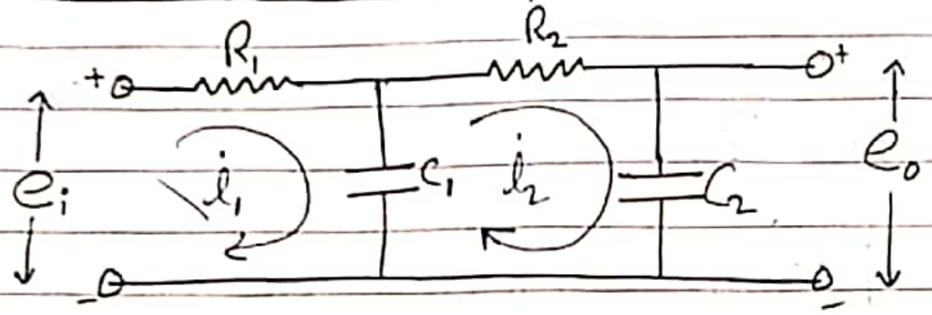


$$-e_i + L\frac{di}{dt} + iR + \frac{1}{C}\int_0^t i\,dt = 0 \quad \text{——} (a)$$

$$e_o = \frac{1}{C}\int_0^t i\,dt \quad \text{——} (b)$$

$$\boxed{\frac{E_o(s)}{E_i(s)} = \frac{1}{LCs^2 + RCs + 1}}$$

Transfer Function of Cascaded Element



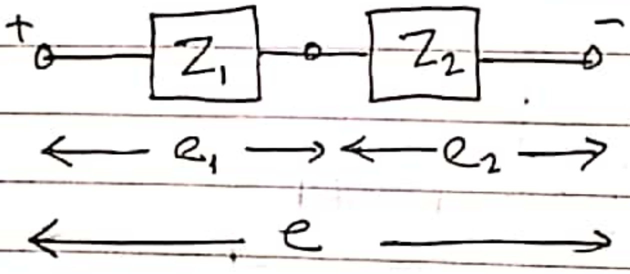
$$-e_i + i_1 R_1 + \frac{1}{C_1}\int_0^t (i_1 - i_2)\,dt = 0 \quad \text{—(1)}$$

$$+\frac{1}{C_1}\int_0^t (i_2 - i_1)\,dt + i_2 R_2 + \frac{1}{C_2}\int_0^t i_2\,dt = 0 \quad \text{—(2)}$$

$$e_o = \frac{1}{C_2}\int_0^t i_2\,dt \quad \text{—(3)}$$

$$\boxed{\frac{E_o(s)}{E_i(s)} = \frac{1}{R_1C_1R_2C_2 s^2 + (R_1C_1 + R_2C_2 + R_1C_2)s + 1}}$$

Complex Impedance



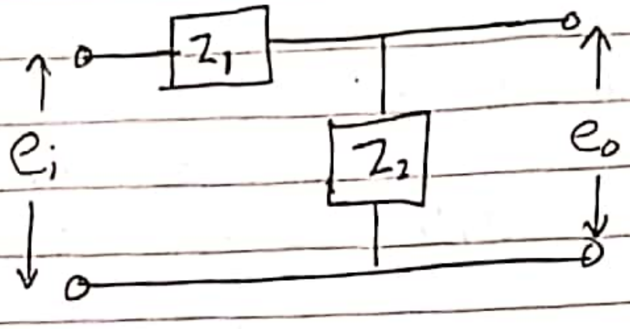
⇒ For electrical circuit, it is convenient to write the Laplace-transformed equation directly without writing the differential equations.

⇒ The complex impedance $Z(s)$ of a two-terminal circuit is the ratio of $E(s)$, the Laplace transform of the voltage across the terminals to $I(s)$, the Laplace transform of current through the element.

{Under the assumption initial condition is zero}

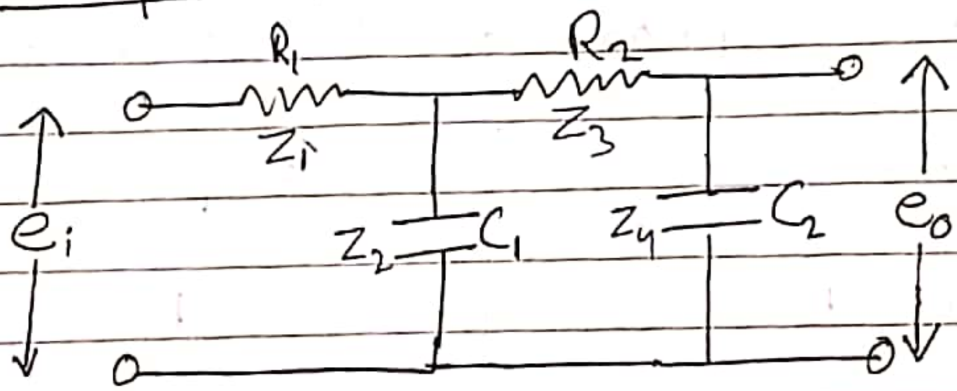
Element	Impedance
1) Resistance (R)	R
2) Capacitance (C) $V(t) = \frac{1}{C}\int_0^t i\,dt$	$\frac{1}{CS}$
3) Inductance (L) $V(t) = L\frac{di}{dt}$	LS

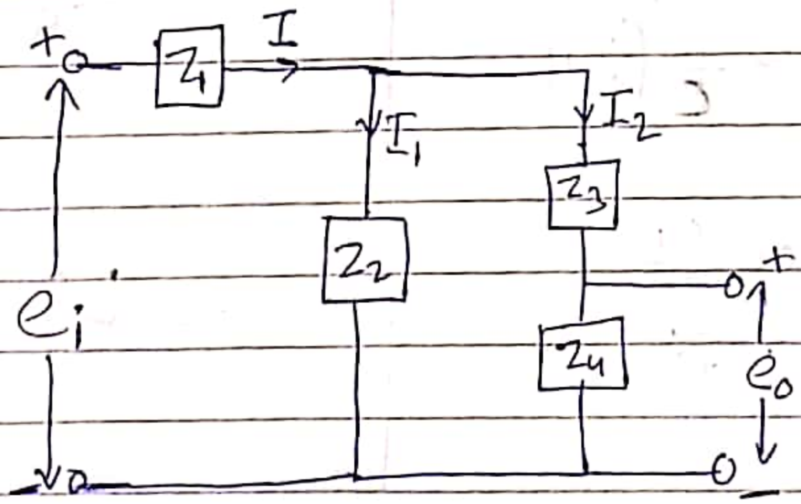
⇒ If complex impedance are connected in series, the total impedance is the sum of the individual complex impedances.



$$\frac{E_o(s)}{E_i(s)} = \frac{Z_2}{Z_1+Z_2}$$

Example 3.7





$$I_1 = \frac{Z_3 + Z_4}{Z_2 + Z_3 + Z_4} I \qquad I_2 = \frac{Z_2}{Z_2 + Z_3 + Z_4} I$$

$$E_i = Z_1 I + Z_2 I_1$$

$$= Z_1 I + \frac{Z_2 (Z_3 + Z_4)}{Z_2 + Z_3 + Z_4} I$$

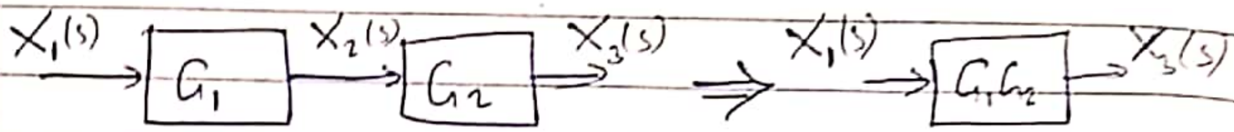
$$E_o = Z_4 I_2 = \frac{Z_4 Z_2}{Z_2 + Z_3 + Z_4} I$$

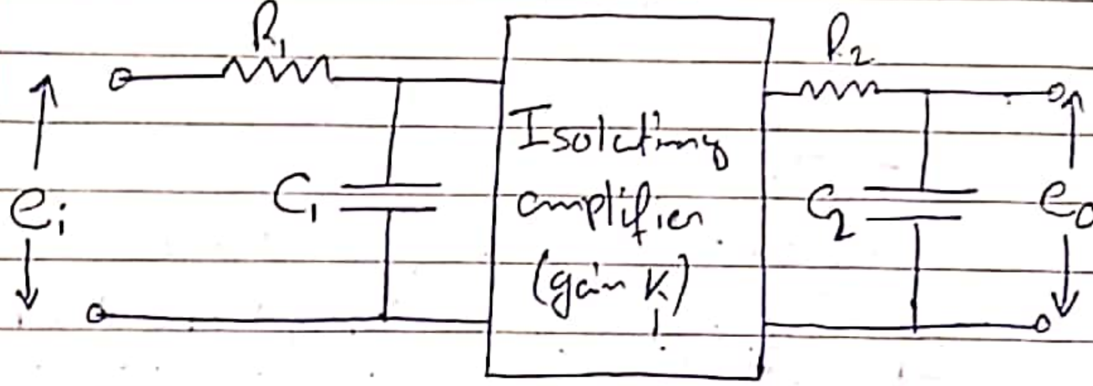
$$\frac{E_o}{E_i} = \frac{\dfrac{Z_4 Z_2}{Z_2 + Z_3 + Z_4}}{Z_1 + \dfrac{Z_2 (Z_3 + Z_4)}{Z_2 + Z_3 + Z_4}} = \frac{Z_4 Z_2}{Z_1 Z_2 + Z_1 Z_3 + Z_1 Z_4 + Z_2 Z_3 + Z_2 Z_4}$$

$$\boxed{\frac{E_o(s)}{E_i(s)} = \frac{1}{R_1 C_1 R_2 C_2 s^2 + (R_1 C_1 + R_2 C_2 + R_1 C_2) s + 1}}$$

Transfer Function of Nonloading Cascaded Element

⇒ The transfer function of a system consisting of two nonloading cascaded element can be obtained by eliminating the intermediate input & output





$$\frac{E_o(s)}{E_i(s)} = \left(\frac{1}{1+R_1C_1S}\right) K \left(\frac{1}{1+R_2C_2S}\right)$$

$$\frac{E_o(s)}{E_i(s)} = \frac{K}{(1+R_1C_1S)(1+R_2C_2S)}$$

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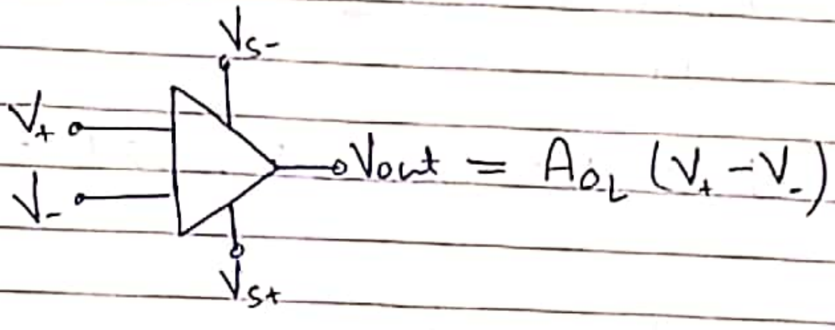
→ Opamp has gain of about $10^5 \sim 10^6$ ∀ dc & ac signal ($f < 10kHz$)

Electronic Controllers

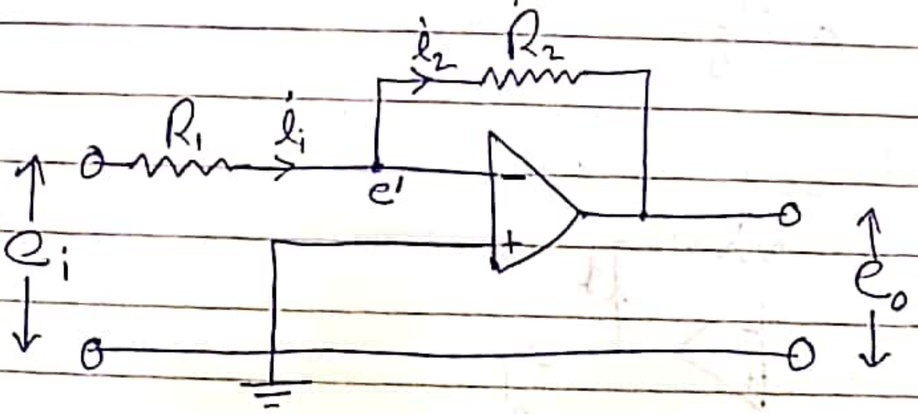
* **Operational Amplifiers**: It is DC-coupled high-gain electronic voltage amplifier with a differential input and usually, a single-ended output.



→ It is frequently used to amplify signals in sensor circuit.

→ Also called differential amplifier / opamps

Inverting Amplifiers



$$i_1 = \frac{e_i - e'}{R_1} \qquad i_2 = \frac{e' - e_0}{R_2}$$

Since negligible current flows into the amplifier, the current $i_1 = i_2$

$$\frac{e_i - e'}{R_1} = \frac{e' - e_0}{R_2}$$

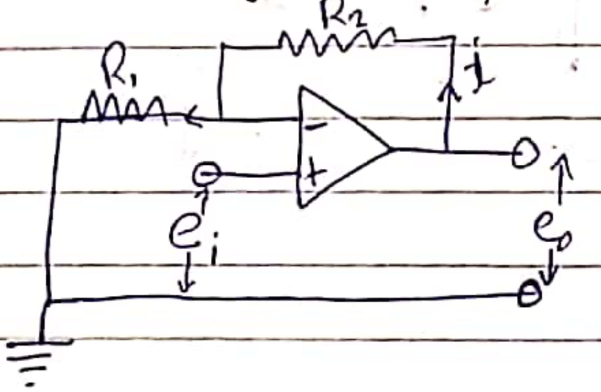
$$K(0 - e') = e_0 \quad \text{as } K \gg 1 \Rightarrow e' \simeq 0$$

So $$\frac{e_i}{R_1} = -\frac{e_0}{R_2}$$

$$\boxed{e_0 = -\frac{R_2}{R_1} e_i}$$

If $R_1 = R_2$, then the opamp circuit shown acts as a sign inverter.

Noninverting Amplifier



$$e_o = i(R_1 + R_2)$$

$$e_o = K(e_i - iR_1)$$

$$e_o = K\left(e_i - \frac{e_o R_1}{R_1 + R_2}\right)$$

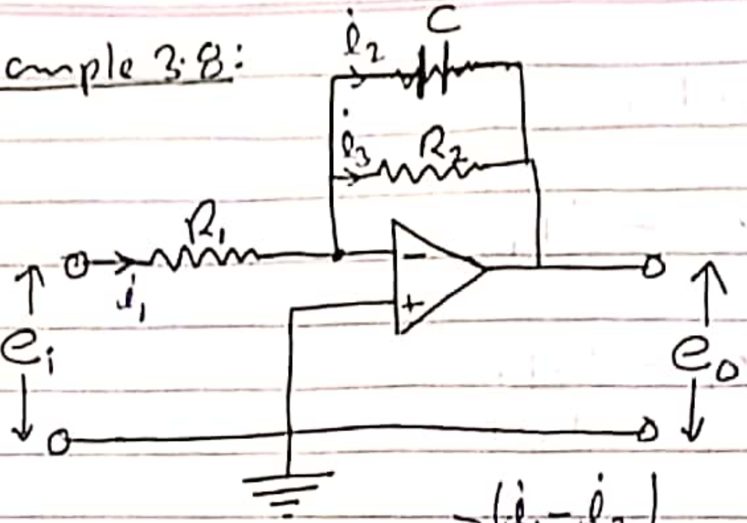
$$e_o\left(1 + \frac{K R_1}{R_1 + R_2}\right) = K e_i$$

$$\frac{e_o}{e_i} = \frac{K}{1 + \frac{K R_1}{R_1 + R_2}}$$

$$\Rightarrow \frac{e_i}{e_o} = \frac{1}{K} + \frac{R_1}{R_1 + R_2} \quad \left\{ \frac{1}{K} \ll \frac{R_1}{R_1 + R_2} \right\}$$

$$\boxed{\frac{e_o}{e_i} = 1 + \frac{R_2}{R_1}}$$

Example 3.8:



$$-e_i + i_1 R_1 + \overset{(i_1 - i_2)}{i_3} R_3 + e_0 = 0 \qquad \text{--- (1)}$$

$$-e_i + i_1 R_1 + \frac{1}{C}\int_0^t i_2\,dt = 0 \qquad \text{--- (2)}$$

$$-e_i + i_1 R_1 + e' = 0 \qquad \text{--- (3)}$$

$$K(0 - e') = e_0 \qquad \text{--- (4)}$$

$$i_1 = i_2 + i_3$$

$$e_0 = -K(e_i - i_1 R_1)$$

$$E_0(s) = -K(E_i(s) - I_1(s) R_1)$$

$$-E_i(s) + I_1(s) R_1 + (I_1(s) - I_2(s)) R_3 + E_0(s) = 0$$

$$-E_i(s) + I_1(s) R_1 + \frac{I_2(s)}{CS} = 0$$

$$I_2(s) = (-R_1 I_1(s) + E_i(s))\, CS$$

$$-E_i(s) + I_1(s)R_1 + I_1(s)R_3 + R_1CsI_1(s)R_3 - CsR_3E_i(s) + E_0(s) = 0$$

$$I_1(s)\left\{R_1 + R_3 + R_1R_3Cs\right\} - E_i(s)\left\{1 + CsR_3\right\} + E_0(s) = 0$$

$$E_0(s) = -KE_i(s) + KR_1\left\{\frac{E_i(s)\{1+CsR_3\} - E_0(s)}{R_1 + R_3 + R_1R_3Cs}\right\}$$

$$\left(E_0(s) + \frac{KR_1}{R_1+R_3+R_1R_3Cs}E_0(s)\right) = -KE_i(s) + \frac{KR_1(1+CsR_3)}{R_1+R_3+R_1R_3Cs}E_i(s)$$

$$E_0(s)\left\{R_1+R_3+R_1R_3Cs+KR_1\right\} = KE_i(s)\left\{-R_1 - R_3 - \cancel{R_1R_3Cs} + R_1 + \cancel{R_1R_3Cs}\right\}$$

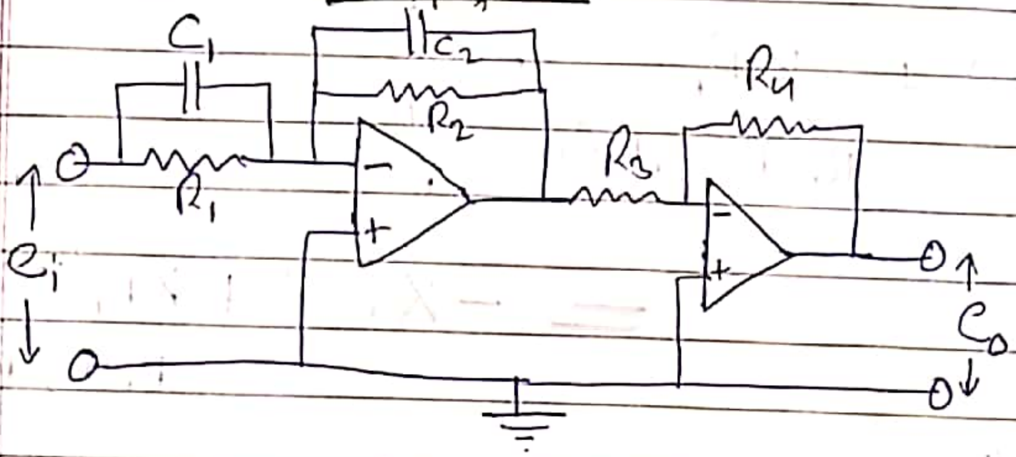
$$\boxed{\frac{E_0(s)}{E_i(s)} = K\,\frac{\cancel{KR_1R_3}\;R_1 \overset{-R_3}{\cancel{-(R_1+R_3)}} + \cancel{(K-1)R_1R_3Cs}}{KR_1 + (R_1+R_3) + R_1R_3Cs}}$$

$$\frac{E_0(s)}{E_i(s)} = \frac{-KR_3}{K_1R_1 + (R_1+R_3) + R_1R_3Cs}$$

$$\Rightarrow \frac{-R_3}{R_1 + \frac{R_1+R_3}{K} + \frac{R_1R_3Cs}{K}}$$

$$\boxed{\frac{E_0(s)}{E_i(s)} = \frac{-R_3}{R_1}\left(\frac{1}{R_3Cs+1}\right)}$$

Lead and Lag Network using Operational Amplifiers

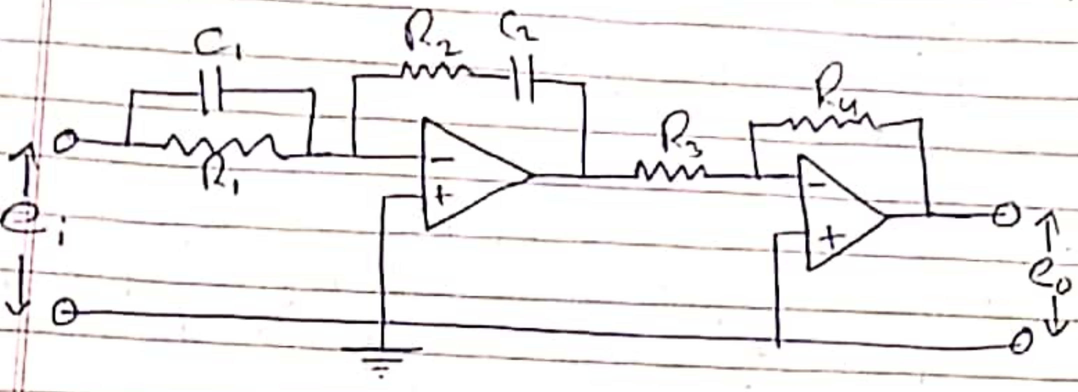


$$\boxed{\frac{E_0(s)}{E_i(s)} = \frac{R_4C_1}{R_3C_2}\,\frac{s+\frac{1}{R_1C_1}}{s+\frac{1}{R_2C_2}}}$$

If $R_1C_1 > R_2C_2 \Rightarrow$ Lead network.

$R_2C_2 > R_1C_1 \Rightarrow$ Lag network.

PID Controller Using Operational Amplifier



$$\boxed{\frac{E_o(s)}{E_i(s)} = \frac{R_4(R_1C_1+R_2C_2)}{R_3R_1C_1}\left[1+\frac{1}{(R_1C_1+R_2C_2)s}+\frac{R_1C_1R_2C_2}{R_1C_1+R_2C_2}s\right]}$$

When a PID Controller is expressed as :-

$$\boxed{\frac{E_o(s)}{E_i(s)} = K_p + \frac{K_i}{s} + K_d s}$$

$$K_p = \frac{R_4(R_1C_1+R_2C_2)}{R_3R_1C_2}$$

$$K_i = \frac{R_4}{R_3R_1C_2}$$

$$K_d = \frac{R_4R_2C_1}{R_3}$$