2

The Z Transform

2.1> Introduction

⇒ The role of z transform in discrete-time systems is similar to that of the Laplace transform in continuous-time system.

⇒ In a linear discrete-time control system, a linear difference equation characterizes the dynamics of the system.

★ **Discrete-Time Signal**

→ The z transform applies to the continuous time signal $x(t)$, sampled signal $x(KT)$ and the number sequence $x(K)$.

2.2> The z transform

⇒ In considering the z transform of a time function $x(t)$, we consider only the sampled values of $x(t)$, that is $x(0), x(T), x(2T)$ ---
where T is the sampling period.

$$X(z) = Z[x(t)] = Z[x(KT)] = \sum_{K=0}^{\infty} x(KT)z^{-K}$$

⇒ For a sequence of number $x(K)$, the z transform is defined by ÷

$$X(z) = Z[x(K)] = \sum_{K=0}^{\infty} x(K)z^{-K}$$

⇒ The z transform defined above is referred to as the one-sided z-transform.

→ For the one-sided z transform, we assume $x(t)=0 \; \forall \; t<0$ or $x(k)=0 \; \forall \; k<0$.

→ Z is a Complex Variable.

⇒ The z transform of $x(t)$, where $-\infty < t < \infty$ or of $x(K)$, where K takes integer values $(K=0, \pm 1, \pm 2 \cdots)$ is defined by

$$X(z) = Z[x(t)] = Z[x(KT)] = \sum_{K=-\infty}^{\infty} x(KT) z^{-K}$$

$$X(z) = Z[x(k)] = \sum_{K=-\infty}^{\infty} x(k) z^{-K}$$

⇒ The z transform defined above is referred to as the two-sided z transform.

(Note: In this book, only the one-sided z transform is considered in detail.)

$$X(z) = x(0) + x(T) z^{-1} + x(2T) z^{-2} + \cdots + x(KT) z^{-K} + \cdots$$

2.3) Z - Transforms of Elementary Functions

⇒ In sampling a discontinuous function $x(t)$, we assume that the function is continuous from the right; that is if discontinuity occurs at $t=0$, then we assume that $x(0)$ is equal to $x(0+)$ rather than to the average at the discontinuity, $[x(0-)+x(0+)]/2$.

Unit Step function

$$x(t) = \begin{cases} 1 & 0 \leq t \\ 0 & t < 0 \end{cases}$$

$$X(z) = \sum_{k=0}^{\infty} 1 \times z^{-k} = 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots$$

$$\Rightarrow \frac{X(z) - 1}{1/z} = 1 + \frac{1}{z} + \frac{1}{z^2} + \cdots = X(z)$$

$$\Rightarrow X(z) - 1 = \frac{X(z)}{z} \Rightarrow \boxed{X(z) = \frac{z}{z-1}}$$

$\{\forall\ |z| > 1\}$

→ It is not necessary to specify the region of z over which X(z) is convergent. It suffices to know that such a region exists.

→ The z transform $X(z)$ of a time function $x(t)$ obtained in this way is valid throughout the z plane except at poles of $X(z)$

Unit-Ramp Function

$$x(t) = \begin{cases} t & 0 \leq t \\ 0 & t < 0 \end{cases}$$

$$x(KT) = KT \quad \forall\, t = 0, 1, 2 \cdots$$

$$X(z) = Z[t] = \sum_{K=0}^{\infty} x(KT) z^{-K} = T \sum_{K=0}^{\infty} K z^{-K}$$

$$\Rightarrow T\left(\frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \cdots\right)$$

$$X(z) = \frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \cdots$$

$$\Rightarrow \frac{X(z)}{z} = \frac{1}{z^2} + \frac{2}{z^3} + \frac{3}{z^4} + \cdots$$

$$\Rightarrow X(z) - z^{-1}X(z) = z^{-1} + z^{-2} + z^{-3} + \cdots$$

$$\Rightarrow X(z)(1 - z^{-1}) = \frac{1}{1 - z^{-1}}$$

$$\Rightarrow \boxed{X(z) = \frac{1}{(1 - z^{-1})^2}}$$

$$\boxed{X(z) = \frac{Tz}{(z-1)^2}}$$

Polynomial Function a^K

$$x(k) = \begin{cases} a^K & K = 0, 1, 2 \cdots \\ 0 & K < 0 \end{cases}$$

$$X(z) = \sum_{K=0}^{\infty} x(k) z^{-K} = 1 + \frac{a}{z} + \frac{a^2}{z^2} + \frac{a^3}{z^3} + \cdots$$

$$\Rightarrow \boxed{X(z) = \frac{z}{z-a}}$$

Exponential Function

$$x(t) = \begin{cases} e^{-at} & 0 \leq t \\ 0 & t < 0 \end{cases}$$

$$x(KT) = e^{-aKT} \quad \forall K = 0, 1, 2 \ldots$$

$$X(z) = \sum_{K=0}^{\infty} \frac{e^{-aKT}}{z^K}$$

$$= 1 + \frac{e^{-aT}}{z} + \frac{e^{-2aT}}{z^2} + \frac{e^{-3aT}}{z^3} + \cdots$$

$$\boxed{X(z) = \frac{z}{z - e^{-aT}}}$$

Sinusoidal Function

$$x(t) = \begin{cases} \sin\omega t & 0 \leq t \\ 0 & t < 0 \end{cases}$$

$$e^{j\omega t} = \cos\omega t + j\sin\omega t$$

$$e^{-j\omega t} = \cos\omega t - j\sin\omega t$$

$$\Rightarrow \sin\omega t = \frac{1}{2j}\left(e^{j\omega t} - e^{-j\omega t}\right)$$

$$X(z) = Z[\sin\omega t] = Z\left[\frac{1}{2j}\left(e^{j\omega t} + e^{-j\omega t}\right)\right]$$

$$\Rightarrow \frac{1}{2j}\left(\frac{1}{1-e^{j\omega T}z^{-1}} - \frac{1}{1-e^{-j\omega T}z^{-1}}\right)$$

$$\Rightarrow \frac{1}{2j} \times \frac{1-e^{-j\omega T}z^{-1} - 1 + e^{j\omega T}z^{-1}}{z^{-2}+1-(e^{j\omega T}+e^{-j\omega T})z^{-1}}$$

$$\Rightarrow \frac{z^{-1}\sin\omega T}{1-2z^{-1}\cos\omega T + z^{-2}}$$

$$\Rightarrow \boxed{Z[\sin\omega t] = \frac{z\sin\omega T}{z^2-2z\cos\omega T+1}}$$

Similarly $\boxed{Z[\cos\omega t] = \frac{z^2 - z\cos\omega T}{z^2-2z\cos\omega T+1}}$

<u>Example 2.2:</u> $X(s) = \frac{1}{s(s+1)}$

$$\Rightarrow x(t) = 1-e^{-t} \qquad 0 \leq t$$

$$X(z) = Z[1-e^{-t}] = \frac{1}{1-z^{-1}} - \frac{1}{1-e^{-T}z^{-1}}$$

$$\Rightarrow X(z) = \frac{(1-e^{-T})z}{(z-1)(z-e^{-T})}$$

2.4) Important Properties and Theorems of the z transform

Multiplication by a Constant

$$Z[a x(t)] = a Z[x(t)] = a X(z)$$

Linearity of the z Transform

If $x(k) = \alpha f(k) + \beta g(k)$

then, $X(z) = \alpha F(z) + \beta G(z)$

Multiplication by a^k

$$Z[a^k x(k)] = X(a^{-1}z)$$

Proof

$$Z[a^k x(k)] = \sum_{k=0}^{\infty} a^k x(k) z^{-k}$$

$$= \sum_{k=0}^{\infty} x(k) (a^{-1}z)^{-k}$$

$$= X(a^{-1}z)$$

Shifting Theorem (also reffered to as Real translation theorem)

$$\Rightarrow \boxed{Z[x(t-nT)] = z^{-n}X(z)}$$

{where $X(z) = Z[x(t)]$}

&

$$\Rightarrow \boxed{Z[x(t+nT)] = z^{n}\left[X(z) - \sum_{k=0}^{n-1} x(kT)z^{-k}\right]}$$

{where n is zero or a positive integer}

Proof

★ $$Z[x(t-nT)] = \sum_{k=0}^{\infty} x(kT-nT)z^{-k}$$

$$= z^{-n}\sum_{k=0}^{\infty} x(kT-nT)\, z^{-(k-n)}$$

Let $m = k-n$

$$\Rightarrow Z[x(t-nT)] = z^{-n}\left[\sum_{m=-n}^{\infty} x(mT)z^{-m}\right]$$

Since $x(mT)=0 \;\forall\; m<0$ so we may change lower limit from $m=-n$ to $m=0$.

$$Z[x(t-nT)] = z^{-n}\sum_{m=0}^{\infty} x(mT)z^{-m} = z^{-n}X(z)$$

★ $$Z[x(t+nT)] = \sum_{k=0}^{\infty} x(kT+nT)\, z^{-k}$$

$$= z^{n}\sum_{k=0}^{\infty} x(kT+nT)\, z^{-(k+n)}$$

$$\Rightarrow z^n \left[\sum_{k=0}^{\infty} x(kT+nT)\, z^{-(k+n)} + \sum_{k=0}^{n-1} x(kT)\, z^{-k} - \sum_{k=0}^{n-1} x(kT)\, z^{-k} \right]$$

$$\Rightarrow z^n \left[\sum_{k=0}^{\infty} x(kT)\, z^{-k} - \sum_{k=0}^{n-1} x(kT)\, z^{-k} \right]$$

$$= z^n \left[X(z) - \sum_{k=0}^{n-1} x(kT)\, z^{-k} \right]$$

$$Z[x(k+1)] = z\left[X(z) - \sum_{k=0}^{0} x(k)\, z^{-k} \right]$$

$$\Rightarrow Z[x(k+1)] = zX(z) + zx(0)$$

$$Z[x(k+2)] = z^2 X(z) - z^2 x(0) - z x(1)$$

Similarly,

$$\boxed{Z[x(k+n)] = z^n X(z) - z^n x(0) - z^{n-1} x(1) \;\ldots\; - z x(n-1)}$$

{where n is a positive number}

Note: Multiplication of the z transform $X(z)$ by z has the effect of advancing the signal $x(kT)$ by one step and that multiplication of the z transform $X(z)$ by z^{-1} has the effect of delaying the signal $x(KT)$ by one step.

Example 2.3:

$$Z[1(t-T)] = z^{-1} Z[1(t)] = z^{-1} \times \frac{1}{1-z^{-1}} = \frac{1}{z-1}$$

$$Z[1(t-nT)] = z^{-n} Z[1(t)] = \frac{z^{-n}}{1-z^{-1}}$$

Example 2.4:

$$f(k) = \begin{cases} a^{k-1} & K = 1, 2, 3 \dots \\ 0 & K \leq 0 \end{cases}$$

let $g(K) = \begin{cases} a^{K} & K = 0, 1, 2 \dots \\ 0 & K < 0 \end{cases}$

$$f(K) = g(K-1)$$

$$Z[f(K)] = Z[g(K-1)] = z^{-1} Z[g(K)]$$

$$= z^{-1} \times \frac{z}{z-a}$$

$$\Rightarrow Z[f(K)] = \frac{1}{z-a}$$

Example 2.5:

$$y(K) = \sum_{h=0}^{K} x(h) \qquad K = 0, 1, 2 \dots$$

{$x(h)$ is a function}

$$y(K) = 0 \qquad \forall\, K < 0$$

$$\Rightarrow \quad y(K) = x(0) + x(1) + x(2) + \dots x(K)$$

$$y(K-1) = x(0) + x(1) + x(2) + \dots x(K-1)$$

$$y(K) - y(K-1) = x(K)$$

$$Z[y(k) - y(k-1)] = Z[x(k)]$$

$$\cancel{Y(z) - Y(k-1) = X(k)}$$

$$\Rightarrow \cancel{Y(z) - z^{-1}Y(z)}$$

$$\Rightarrow Y(z) - z^{-1}Y(z) = X(z)$$

$$\Rightarrow Y(z)\{1 - z^{-1}\} = X(z)$$

$$\Rightarrow Y(z) = \frac{X(z)}{1 - z^{-1}} \quad \{\text{where } X(z) = Z[x(k)]\}$$

★ <u>Complex Translation Theorem</u>

If $x(t)$ has the z transform $X(z)$, then the z transform of $e^{-at}x(t)$ can be given by $X(ze^{aT})$. This is known as the complex translation theorem.

<u>Proof</u>

$$Z\{e^{-at}x(t)\} = \sum_{k=0}^{\infty} x(kT)e^{-akT}z^{-k}$$

$$= \sum_{k=0}^{\infty} x(kT)\,(ze^{aT})^{-k}$$

$$= X(ze^{aT})$$

Example 2-6: Given the z transforms of $\sin\omega t$ and $\cos\omega t$, obtain the z transform of $e^{-at}\sin\omega t$ and $e^{-at}\cos\omega t$, respectively by using the Complex translation theorem.

$$\Rightarrow \quad Z[\sin\omega t] = \frac{z^{-1}\sin\omega T}{1 - 2z^{-1}\cos\omega T + z^{-2}}$$

$$\text{So } Z[e^{-at}\sin\omega t] = \frac{e^{-aT}z^{-1}\sin\omega T}{1 - 2e^{-aT}z^{-1}\cos\omega T + e^{-2aT}z^{-2}}$$

$$\text{Similarly } Z[e^{-at}\cos\omega t] = \frac{1 - e^{-aT}z^{-1}\cos\omega T}{1 - 2e^{-aT}z^{-1}\cos\omega T + e^{-2aT}z^{-2}}$$

Example 2-7: Z transform of te^{-at}

$$Z[te^{-at}] = \frac{Te^{-aT}z^{-1}}{(1 - e^{-aT}z^{-1})^2}$$

★ **Initial value theorem**: If $x(t)$ has the z transform $X(z)$ and if $\lim_{z\to\infty} X(z)$ exists, then the initial value $x(0)$ of $x(t)$ or $x(k)$ is given by:

$$\boxed{x(0) = \lim_{z\to\infty} X(z)}$$

Proof:

$$X(z) = \sum_{k=0}^{\infty} x(k)z^{-k} = x(0) + x(1)z^{-1} + x(2)z^{-2} + \cdots$$

$$\text{So } \lim_{z\to\infty} X(z) = x(0)$$

Scanned by CamScanner

★ Final Value Theorem: Suppose that $x(k)$, where $x(k)=0$ for $k<0$, has the z transform $X(z)$ and that all the poles of $X(z)$ lie inside the unit circle, with the possible exception of a simple pole at $z=1$.

⇒ Then the final value of $x(k)$, that is, the value of $x(k)$ as k approaches infinity can be given by:

$$\boxed{\lim_{k\to\infty} x(k) = \lim_{z\to 1}\left[(1-z^{-1})X(z)\right]}$$

Proof

$$Z[x(k)] = X(z) = \sum_{k=0}^{\infty} x(k)z^{-k}$$

$$Z[x(k-1)] = z^{-1}X(z) = \sum_{k=0}^{\infty} x(k-1)z^{-k}$$

$$\sum_{k=0}^{\infty} x(k)z^{-k} - \sum_{k=0}^{\infty} x(k-1)z^{-k} = (1-z^{-1})X(z)$$

→ Taking limit as $z\to 1$.

$$\lim_{z\to 1}(1-z^{-1})X(z) = [\cancel{x(0)} - \cancel{x(-1)}^{\,0}] + [\cancel{x(1)} - \cancel{x(0)}]$$

$$[\cancel{x(2)} - \cancel{x(1)}] + \dots [x(\infty) - \cancel{x(\infty-1)}]$$

$$= x(\infty)$$

So $\lim_{k\to\infty} x(k) = \lim_{z\to 1} \left[(1-z^{-1}) X(z)\right]$

* Example 2.9: $X(z) = \frac{1}{1-z^{-1}} - \frac{1}{1-e^{-aT}z^{-1}}$ $a>0$

$$x(\infty) = \lim_{z\to 1}\left(\frac{1-z^{-1}}{1-z^{-1}} - \frac{1-z^{-1}}{1-e^{-aT}z^{-1}}\right) = 1-0 = 1$$

2.5) The Inverse Z transform

⇒ The notation for the inverse z transform is Z^{-1}.

⇒ The inverse z transform of $X(z)$ yields the corresponding time sequence $x(k)$

↳ Inverse z transform of $X(z)$ yields a unique $x(k)$, but does not yield a unique $x(t)$.

⇒ There are four methods for obtaining the inverse Z-transform are commonly available :-

1. Direct division method
2. Computational method
3. Partial-fraction method
4. Inversion integral method

+ (Refering to Z-transform table)

⇒ In obtaining the inverse z transform, we assume, as usual, that the time sequence $x(kT)$ or $x(k)$ is zero for $k<0$.

★ **Poles and Zeros in the z plane**

In engineering applications of the z-transform method, $X(z)$ may have the form:-

$$X(z) = \frac{b_0 z^m + b_1 z^{m-1} + \cdots + b_m}{z^n + a_1 z^{n-1} + \cdots a_n} \quad (m \leq n)$$

or

$$X(z) = \frac{b_0 (z-z_1)(z-z_2)\cdots(z-z_m)}{(z-P_1)(z-P_2)\cdots(z-P_n)}$$

⇒ The location of the poles and zeros of $X(z)$ determine the characteristics of $x(k)$, the sequence of values or numbers.

⇒ In control engineering and signal processing $X(z)$ is frequently expressed as a ratio of polynomials in z^{-1} as follows:-

$$X(z) = \frac{b_0 z^{-(n-m)} + b_1 z^{-(n-m+1)} + \cdots + b_m z^{-n}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_n z^{-n}}$$

Where z^{-1} is interpreted as the unit delay operator.

classmate
Date
Page

(1) Direct division Method

⇒ In the direct division method we obtain the inverse z-transform by expanding $X(z)$ into an infinite power series in z^{-1}.

→ This method is useful when it is difficult to obtain the closed-form expression for the inverse z transform.

or

→ It is ~~diff~~ desired to find only the first several terms of $x(k)$.

$$X(z) = \sum_{k=0}^{\infty} x(kT) z^{-k}$$

$$= x(0) + x(T)z^{-1} + x(2T)z^{-2} + \dots + x(kT)z^{-k} + \dots$$

or

$$X(z) = \sum_{k=0}^{\infty} x(k) z^{-k}$$

$$= x(0) + x(1)z^{-1} + x(2)z^{-2} + \dots + x(k)z^{-k} + \dots$$

⇒ If $X(z)$ is given in the form of a rational function, the expansion into an infinite power series in increasing powers of z^{-1} can be accomplished by simply dividing the numerator by the denominator, where both the numerator & denominator of $X(z)$ are written in increasing power of z^{-1}.

Example 2.10: $X(z) = \frac{10z + 5}{(z-1)(z-0.2)}$

$$\Rightarrow X(z) = \frac{10z^{-1} + 5z^{-2}}{1 - 1.2z^{-1} + 0.2z^{-2}}$$

$$
\begin{array}{r|l}
 & 10z^{-1} + 17z^{-2} + 18.4z^{-3} + 18.68z^{-4} + \cdots \\
\hline
1 - 1.2z^{-1} + 0.2z^{-2} & 10z^{-1} + 5z^{-2} \\
 & 10z^{-1} - 12z^{-2} + 2z^{-3} \\
\cline{2-2}
 & 17z^{-2} - 2z^{-3} \\
 & 17z^{-2} - 20.4z^{-3} + 3.4z^{-4} \\
\cline{2-2}
 & 18.4z^{-3} - 3.4z^{-4} \\
 & 18.4z^{-3} - 22.08z^{-4} + 3.68z^{-5} \\
\cline{2-2}
 & 18.68z^{-4} - 3.68z^{-5} \\
\cline{2-2}
 & 18.68z^{-4} - 22.416z^{-5} + 3.7z^{-6}
\end{array}
$$

Thus, $X(z) = 10z^{-1} + 17z^{-2} + 18.4z^{-3} + 18.68z^{-4} + \cdots$

$x(0) = 0$
$x(1) = 10$
$x(2) = 17$
$x(3) = 18.4$
$x(4) = 18.68$
$\vdots \quad \vdots$

(B) Computational Method

Two Computational approaches to obtain inverse z-transform:

1. Matlab approach
2. Difference equation approach.

⇒ Consider a System $G(z)$ defined by

$$G(z) = \frac{0.4673 z^{-1} - 0.3393 z^{-2}}{1 - 1.5327 z^{-1} + 0.6607 z^{-2}} \quad \text{--- (1)}$$

⇒ For finding the inverse z transform we utilize the Kronecker delta function $\delta_0(KT)$

$$\delta_0(KT) = 1, \ \forall K = 0$$
$$= 0 \ \forall K \neq 0$$

⇒ Assume that $x(K)$, the input to the System $G(z)$, is the Kronecker delta input.

$$x(K) = \begin{cases} 1 & \forall K = 0 \\ 0 & \forall K \neq 0 \end{cases}$$

The z transform of the Kronecker delta input is

$$X(z) = 1.$$

⇒ Using the Kronecker delta input Eq (1) can be written as

$$G(z) = \frac{Y(z)}{X(z)} = \frac{0.4673z - 0.3353}{z^2 - 1.5327z + 0.6607}$$

Matlab Approach

⇒ To obtain the inverse z transform of $G(z)$ with matlab, we proceed as follows:

→ Enter the numerator & denominator as follows

num = [0 0.4673 -0.3393]
den = [1 . -1.5327 0.6607]

→ Enter the Kronecker delta input

x = [1 zeros(1, 40)]

→ Then enter the Command

Y = filter(num, den, x)

→ to obtain the response Y(k) from k = 0 to k = 40

Difference Equation Approach

$$(z^2 - 1.5327z + 0.6607)Y(z) = (0.4673z - 0.3393)X(z)$$

⇒ We Convert this equation into the difference equation as follows:

$$y(k+2) - 1.5327y(k+1) + 0.6607y(k)$$
$$= 0.4673x(k+1) - 0.3393x(k)$$

$$\left\{ \begin{array}{l} \text{where } x(0)=1 \text{ \& } x(k)=0 \ \forall\, k \neq 0 \\ \text{\& } y(k)=0 \ \forall\, k<0 \end{array} \right\}$$

⇒ Putting $K = -2$ we get :-

$y(0) = 0$

⇒ Putting $K = -1$ we get :-

$y(1) = 0.4673$

Similarly all values can be found!

(C) Partial-Fraction-Expansion Method

⇒ The partial-Fraction expansion method presented here, which is parallel to the partial-fraction expansion method used in Laplace transformation, is widely used in routine problem involving Z transforms.

(D) Inversion Integral Method

$$Z^{-1}[X(z)] = x(KT) = x(K) = \frac{1}{2\pi j}\oint_C X(z)\, z^{K-1}\, dz$$

where C is a circle with the center at the origin of the z plane such that all poles of $X(z)z^{K-1}$ are inside it.

⇒ The equation for giving the inverse z transform in terms of residues can be derived by using theory of complex variables. It can be obtained as follows:-

$$x(KT) = x(K) = K_1 + K_2 + K_3 + \dots + K_m$$

$$= \sum_{i=1}^{m}\left[\text{residue of } X(z)z^{K-1} \text{ at pole } z = z_i \text{ of } X(z)z^{K-1}\right]$$

Scanned by CamScanner

Where $K_1, K_2 \cdots K_m$ denote the residues of $X(z)z^{k-1}$ at poles $z_1, z_2 \cdots z_m$ respectively.

→ If the denominator of $X(z)z^{k-1}$ contains a simple pole $z = z_i$ then the corresponding residue K is given by

$$K = \lim_{z \to z_i} \left[(z - z_i) X(z) z^{k-1} \right]$$

→ If $X(z)z^{k-1}$ contains a multiple pole z_i of order q, then the residue K is given by

$$K = \frac{1}{(q-1)!} \lim_{z \to z_i} \frac{d^{q-1}}{dz^{q-1}} \left[(z - z_i)^q X(z) z^{k-1} \right]$$

⇒ It should be noted that the inversion integral method, when evaluated by residues, is a very simple technique for obtaining the inverse z transform, provided that $X(z)z^{k-1}$ has no poles at the origin $z=0$.

⇒ If however $X(z)z^{k-1}$ has a simple pole or a multiple pole at $z=0$, then calculations may become cumbersome and the partial-fraction-expansion method may prove to be simple to ~~be simpler~~ apply.

Example 2-16: $X(z) = \frac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})}$

$$X(z)\, z^{k-1} = \frac{z^k(1-e^{-aT})}{(z-1)(z-e^{-aT})}$$

For $k=0,1,2 \ldots$ $X(z)z^{k-1}$ has two simple poles $z=z_1=1$ & $z=z_2=e^{-aT}$.

$$K_1 = \lim_{z\to 1}\left[(z-1)\frac{z^k(1-e^{-aT})}{(z-1)(z-e^{-aT})}\right] = 1$$

$$K_2 = \lim_{z\to e^{-aT}}\left[(z-e^{-aT}) \times \frac{z^k(1-e^{aT})}{(z-1)(z-e^{-aT})}\right] = -e^{-akT}$$

Hence, $\boxed{x(kT) = K_1 + K_2 = 1 - e^{-akT}}$

2.6) Z Transform method for Solving difference Equation

⇒ Difference equations can be solved easily by use of a digital computer, provided the numerical values of all coefficients and parameters are given.

⇒ However, closed-form expressions of $x(k)$ cannot be obtained from the computer solution, except for very special case.

⇒ The usefulness of the z-transform method is that it enables us to obtain the closed-form expression of $x(k)$.

Consider the linear time-invariant discrete time system characterized by the following linear difference equation:

$$x(k) + a_1 x(k-1) + \cdots + a_n x(k-n)$$
$$= b_0 u(k) + b_1 u(k-1) + \cdots + b_n u(k-n)$$

Where $u(k)$ & $x(k)$ are the system's input & output respectively, at the k^{th} iteration.

Let $Z[x(k)] = X(z)$

Example 2-18

$$x(k+2) + 3x(k+1) + 2x(k) = 0$$
$$\{x(0) = 0,\ x(1) = 1\}$$

$$\Rightarrow Z[x(k+2)] + 3Z[x(k+1)] + 2Z[x(k)] = 0$$

$$\Rightarrow \left(z^2 X(z) - z^2 x(0) - z x(1)\right) + 3\left(zX(z) - zx(0)\right)$$
$$+ 2X(z) = 0$$

$$X(z) = \frac{z}{z+1} - \frac{z}{z+2} = \frac{1}{1+z^{-1}} - \frac{1}{1+2z^{-1}}$$

$$x(k) = (-1)^k - (-2)^k \quad \forall\ k = 0, 1, 2 \ldots$$

2.7) Concluding Comments

"With the Z transform method, linear time-invariant difference equations can be transformed into algebraic equations. This facilitates the transient response analysis of the digital control system."