Minimal polynomial of an $n \times n$ matrix A is defined as the polynomial $\phi(\lambda)$ of least degree,

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$$\phi(\lambda) = \lambda^m + a_1 \lambda^{m-1} + \cdots + a_{m-1}\lambda + a_m$$

$$m \leq n$$

Such that

$$\phi(\bar{A}) = \bar{O}$$

Let $d(\lambda)$, a polynomial in λ, is the greatest common divisor of all the elements of $adj(\lambda I - A)$.

If the coefficient of the highest-degree term in λ of $d(\lambda)$ is chosen as 1, then the minimum polynomial $\phi(\lambda)$ is given by

$$\phi(\lambda) = \left|\frac{\lambda\bar{\bar{I}} - \bar{\bar{A}}}{d(\lambda)}\right|$$

$$\Rightarrow \quad adj(\lambda\bar{\bar{I}} - \bar{\bar{A}}) = d(\lambda)\,\bar{\bar{B}}(\lambda) \quad \left\{ \begin{array}{l} \text{as } d(\lambda) \text{ is greatest common} \\ \text{divisor of the matrix } adj(\lambda I - A) \end{array} \right\}$$

$$\Rightarrow (\lambda\bar{\bar{I}} - \bar{\bar{A}})\, adj(\lambda\bar{\bar{I}} - \bar{\bar{A}}) = |\lambda\bar{\bar{I}} - \bar{\bar{A}}|\,\bar{\bar{I}}$$

$$\Rightarrow d(\lambda)(\lambda\bar{\bar{I}} - \bar{\bar{A}})\,\bar{\bar{B}}(\lambda) = |\lambda\bar{\bar{I}} - \bar{\bar{A}}|\,\bar{\bar{I}}$$

From above we find that $|\lambda\bar{\bar{I}} - \bar{\bar{A}}|$ is divisible by $d(\lambda)$.

$$\Rightarrow |\lambda\bar{\bar{I}} - \bar{\bar{A}}| = d(\lambda)\,\psi(\lambda)$$

Because the coefficient of the higher degree term in λ of $d(\lambda)$ has been chosen as 1, the coefficient of the higher-degree term in λ of $\psi(\lambda)$ is also 1.

$$\Rightarrow (\lambda\bar{\bar{I}} - \bar{\bar{A}})\,\bar{\bar{B}}(\lambda) = \psi(\lambda)\,\bar{\bar{I}}$$

hence, $\psi(A) = 0$

So $\boxed{\psi(\lambda) = \dfrac{|\lambda I - A|}{d(\lambda)}}$

<u>Minimal</u>
<u>Polynomial</u>