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Fundamentals of Robotic Mechanical System

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CHAPTER 1

AN Overview Of Robot Mechanical System

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An Overview of Robotic Mechanical System

1.1> Introduction

System → "Combinations of Parts forming a Complex whole"

dynamic System → "It is a System in which one can distinguish three elements, namely, a State, an Input and an Output.

& Rule of transition from One Current State to a Future one."

Mathematical modle of the dynamic System ← (Rule of transition from One Current State to a Future one.)

⇒ State of a dynamic System at a Certain instant is determined not only by the Value of the input at that instant, but also by the past history of the input.

↳ Due to this dynamic Systems are said to have memory.

Static System → System whose state at a given instant is only function of the input at the current time.

↳ They are said to have no memory.

Programmable robot

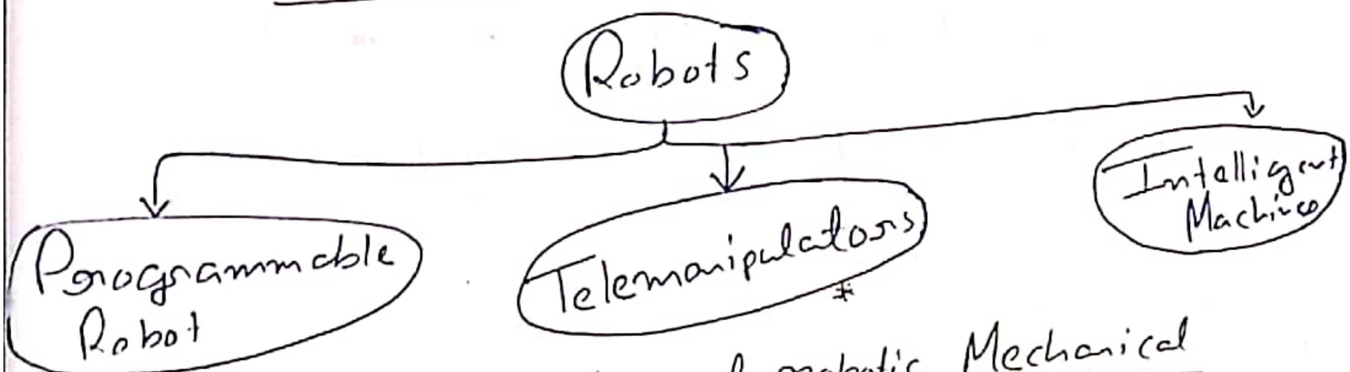
→ Operation is limited to structured environments.

Intelligent Robot

→ Capable of reacting to Unpredictable Changes in an un-structured environment.

→ They are expected to perceive their environment and draw conclusions based on this perception.

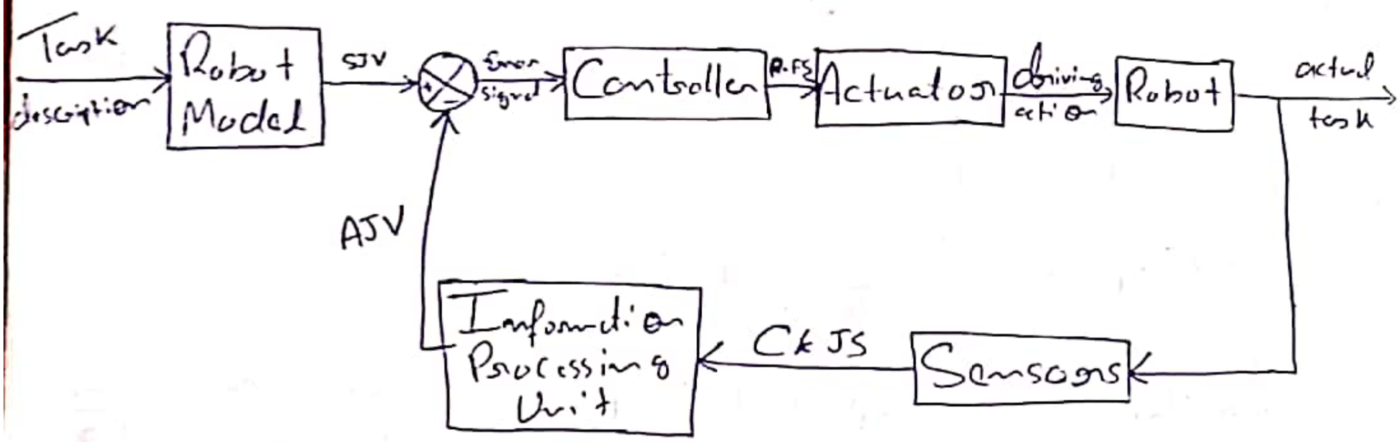
→ By far most Complex of perception tasks is Visual.



1.2) The general Structure of robotic Mechanical System

⇒ A robotic mechanical system is composed of a few subsystem:-

(i) Rigid & deformable bodies.
(ii) Sensing subsystem.
(iii) Actuation subsystem.
(iv) Controller
(v) Information-processing subsystem.



SJV ⇒ Synthesized joint Variables (angles & torque)

P&FS ⇒ Position & force signals.

C&JS ⇒ Cartesian & Joint signal

AJV ⇒ Actual joint Variables (angles & torques)

Block diagram of a general robotic mechanical System

Telemanipulators → Robotic mechanical System with a human being in their Control loop.

1.3> Serial Manipulator

↳ They are simplest of all robotic mechanical System.

⇒ A manipulator, in general is a mechanical System aimed at manipulating objects.

1.4> Parallel Manipulator

⇒ In a parallel manipulator, we distinguish one base platform, one moving platform, and various legs.

⇒ Contrary to serial manipulators, all of whose joints are actuated, parallel manipulators contain unactuated joints.

↳ The presence of unactuated joints makes the analysis of parallel manipulators, in general, more complex than that of their serial counterparts.

Example

1. Delta robot
2. Hexa robot
3. Star robot
4. Truss arm

1.5> Robotic Hand

Manipulation

→ Simple manipulation

→ Dextrous Manipulation

→ In dextrous manipulation, the grasped object is required to move with respect to the palm of the grasping hand

1.6> Walking Machines

⇒ In walking machines, Stability is the main issue.

Stability

- → **Static Stability** → { Ability of sustaining a Configuration from reaction forces only. }
- → **Dynamic Stability** → { Refers to ability of Sustaining a Configuration from both reaction & inertial force }

1.7> Rolling Robot

⇒ Parallel or Serial manipulator, they all have very limited workspace.

⇒ Manipulators with limited workspaces can be enhanced by mounting them on rolling robots.

⇒ Rolling robots are Systems evolved from earlier systems called automatic guided Vehicles. (AGV)

→ { These Vehicles are usually limited to motion along Predefined tracks that are either railways or magnetic Strips glued to the ground }

⇒ As a means to supply rolling robots with 3-dof Capabilities, omnidirectional wheels have been Proposed.

ex ⇒ Mekanum wheel

⇒ Recent developments in the technology of rolling robots have been reported that incorporation of alternate type of ODW.

↳ Orthogonal ball wheels

CHAPTER 2

Mathematical Background

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Mathematical Background

2.1> Preamble

⇒ Study of motions undergone by robotic mechanical systems, requires a suitable <u>motion representation</u>.

⇒ The most general <u>kind of</u> rigid body motion consists of both translation and rotation.

{Translation can be studied simply with the aid of 3-D Vector Calculus}

{Rigid body rotation requires introduction of Tensor}

{Entities mapping Vector space into Vector space}

Invariant → does not depends on the choice of frame.

→ {Coordinate frame is required to compute but the final result will be independent of the choice of frame}

⇒ Right and Left hand side quantities are not the same.

Vector	1-D array
Tensor	2-D array

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2.2) Linear Transformation

⇒ 3D Space is particular case of a vector space.

Vector Space → A vector space is a set of object called vectors, that follows certain (algebraic rules)

→ Every linear combination of vectors in the vector space lies in the same vector space.

* **Vector** ⇒ will be denoted by lower case letter with a bar on its top.

  eg ⇒ $\bar{a}$, $\bar{b}$

Tensor ⇒ Will be denoted by upper case letter with two bars on its top

  eg ⇒ $\bar{\bar{A}}$, $\bar{\bar{B}}$

⇒ In elementary mechanics, the dimension of the vector space needed is usually three. but when studying multibody system, an arbitrary finite dimension will be required.

⇒ **Linear transformation**

→ Represented by operator $\bar{\bar{L}}$ of a vector space $\mathcal{U}$ into a vector space $\mathcal{V}$, is a rule that assigns to every vector $\bar{u}$ of $\mathcal{U}$ at least one vector $\bar{v}$ of $\mathcal{V}$, represented as $\bar{v} = \bar{\bar{L}}\bar{u}$.

Example of Linear Transformation ⇒
* Projection
* Reflections
* Rotations

Example of Non linear Transformation

→ affine transformation

Kernel

Kernel of $\bar{L}$ is the set of vectors $\bar{u}_N$ of $\mathcal{U}$ that are mapped by $\bar{L}$ into the zero vector $\bar{0} \in \mathcal{V}$

Range

The range of a linear transformation $\bar{L}$ of $\mathcal{U}$ into $\mathcal{V}$ is the set of vectors $\bar{v}$ of $\mathcal{V}$ into which some vector $\bar{u}$ of $\mathcal{U}$ is mapped.

⇒ It can be proved that Kernel & Range, they are themselves a vector space.

⇒ Kernel of a linear transformation is often called the null space.

⇒ 3-d Euclidean space is denoted by $\mathcal{E}^3$.

⇒ A matrix $\bar{P}$ is said to be idempotent of degree l then.

$$\bar{P}^l = \bar{P}$$

Projection of a position vector $\bar{P}$ denoted by $\bar{P'}$, onto a plane π of unit normal $\bar{n}$ is

$$\bar{P'} = \bar{P} - \bar{n}(\bar{n}^T \bar{P})$$

dot product

$$\left\{ \bar{a} \cdot \bar{b} = \bar{a}^T \bar{b} = \bar{b}^T \bar{a} \right\}$$
$$\bar{a} \cdot \bar{b} = \bar{b} \cdot \bar{a}$$

$$\Rightarrow \quad \bar{p}' = (\bar{\bar{I}} - \bar{n}\bar{n}^T)\bar{p}$$

So $\bar{\bar{P}} = (\bar{\bar{I}} - \bar{n}\bar{n}^T)$ → Orthogonal Projection

A reflection $\bar{\bar{R}}$ of $\mathcal{E}^3$ on to plane Π passing through the origin and having a unit normal $\bar{n}$ is a linear transformation of the said space into itself such that a position vector $\bar{p}$ is mapped by $\bar{\bar{R}}$ in a vector $\bar{p}'$ given by:-

$$\bar{p}' = (\bar{p} - \bar{n}(\bar{n}^T\bar{p})) - \bar{n}(\bar{n}^T\bar{p})$$

$$\bar{p}' = (\bar{p} - 2\bar{n}(\bar{n}^T\bar{p}))$$

$$\bar{p}' = (\bar{\bar{I}} - 2\bar{n}\bar{n}^T)\bar{p}$$

$$\boxed{\bar{\bar{R}} = \bar{\bar{I}} - 2\bar{n}\bar{n}^T}$$

→ Reflection

(i) Symmetric $\bar{\bar{R}}^T = \bar{\bar{R}}$

(ii) Has idempotent of degree 3 $\bar{\bar{R}}^3 = \bar{\bar{R}}$ or $\bar{\bar{R}}^2 = \bar{\bar{I}}$

$\bar{\bar{R}}^{-1} = \bar{\bar{R}}$ "Self-inverse"

Basis of a vector space V is a set of linearly independent vectors of V, $\{\bar{v}_i\}_1^n$, in terms of which any vector $\bar{v}$ of V can be expressed as:-

$$\bar{v} = \alpha_1\bar{v}_1 + \alpha_2\bar{v}_2 + \dots \alpha_n\bar{v}_n$$

⇒ The number n is called dimension of the space.

Note: Any set of n linearly independent vectors of V can play the role of a basis of this space.

⇒ Let U and V be two vector space of dimensions m and n respectively and $\bar{L}$ a linear transformation of U into V, and define B_U and B_V basis for U and V.

$$B_U = \{\bar{U}_j\}_1^m \quad \& \quad B_V = \{\bar{V}_i\}_1^n$$

$$\bar{L}\bar{U}_j = l_{1j}\bar{V}_1 + l_{2j}\bar{V}_2 + \cdots l_{nj}\bar{V}_n$$

$$[L]_{Bu}^{Bv} = \begin{bmatrix} l_{11} & l_{12} & \cdots & l_{1m} \\ l_{21} & l_{22} & \cdots & l_{2m} \\ \vdots & \vdots & & \vdots \\ l_{n1} & l_{n2} & \cdots & l_{nm} \end{bmatrix}$$

⇒ The foregiven array is thus called the matrix representation of $\bar{L}$ with respect to B_u and B_v.

⇒ When mapping $\bar{L}$ is a maping U onto itself, then single basis suffices to represent L in matrix form.

A mapping $\bar{L}$ of an n-dimensional vector space V, a U into the n-dimensional vector space that is mapped by $\bar{L}$ into non zero vector $\bar{e}$ a multiple of itself, $\lambda\bar{e}$ is called an eigen vector

of $\overline{\overline{L}}$, the scalar λ being called Eigen values of $\overline{\overline{L}}$.

⇒ Eigen values λ can be determined by

$$|\lambda \overline{\overline{I}} - \overline{\overline{L}}| = 0$$

→ This is called characteristics Polynomial of $\overline{\overline{L}}$.

2.3) Rigid body rotations

⇒ A linear isomorphism (one to one linear transformation mapping a space V onto itself) is called isometry if it preserves distance between any two points of V.

⇒ Let $\overline{P}$ be the position vector of any point of $\mathcal{E}^3$, its image under a rotation $\overline{\overline{Q}}$ be $\overline{P'}$.

→ Distance preservation requires that

$$\overline{P}^T \overline{P} = \overline{P'}^T \overline{P'} \quad \text{—①}$$

where, $$\overline{P'} = \overline{\overline{Q}}\,\overline{P} \quad \text{—②}$$

Putting ② in ① we get

$$\overline{P}^T \overline{P} = \overline{P}^T \overline{\overline{Q}}^T \overline{\overline{Q}}\,\overline{P}$$

So, $\boxed{\overline{\overline{Q}}^T \overline{\overline{Q}} = \overline{\overline{I}}}$

⇒ So $\overline{\overline{Q}}$ is orthogonal matrix.

Let $T = [\overline{u}\ \overline{v}\ \overline{w}]$ $\quad$ $T' = [\overline{u'}\ \overline{v'}\ \overline{w'}]$

⇒ From above it is clear that:-

$$\boxed{\bar{\bar{T}}' = \bar{\bar{Q}}\,\bar{\bar{T}}}$$

⇒ For isometry to represent rotation

$$|\bar{\bar{T}}| = |\bar{\bar{T}}'|$$

Reff1
else it represent reflection

So $\boxed{|\bar{\bar{Q}}| = 1}$

⇒ Therefore $\bar{\bar{Q}}$ is a <u>proper orthogonal matrix</u>.

<u>Theorem 1</u>: The eigen values of a proper orthogonal matrix $\bar{\bar{Q}}$ lie on the unit circle centered at the origin of the complex plane.

<u>Proof</u>: Let λ be one of the eigen values of $\bar{\bar{Q}}$ and $\bar{e}$ the corresponding eigenvector, so that

$$\bar{\bar{Q}}\,\bar{e} = \lambda \bar{e} \quad \text{—— (1)}$$

⇒ The transpose conjugate of the above equation takes form

$$\bar{e}^* \bar{\bar{Q}}^* = \lambda^* \bar{e}^* \quad \text{—— (2)}$$

⇒ Multiplying eq (1) & (2) we get:-

$$\bar{e}^* \bar{\bar{Q}}^* \bar{\bar{Q}}\,\bar{e} = \lambda \lambda^* \bar{e}^* \bar{e}$$

$$\Rightarrow \bar{e}^* \bar{\bar{Q}}^T \bar{\bar{Q}}\,\bar{e} = \lambda \lambda^* \bar{e}^* \bar{e}$$ $\left\{ \bar{\bar{Q}} \text{ is assumed to be real.} \right\}$

$\Rightarrow \overline{e}^{*}\,\overline{e} = |\lambda|^2 \overline{e}^{*}\overline{e}$ {as Q is orthogonal}

$\Rightarrow \boxed{|\lambda|^2 = 1}$

Theorem 3: (Cayley-Hamilton) Let $P(\lambda)$ be the characteristic polynomial of a $n\times n$ matrix A

$$P(\lambda) = |\lambda\overline{I} - \overline{A}| = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots a_1\lambda + a_0$$

Then $\overline{A}$ satisfies its characteristic equation:-

$$\overline{A}^n + a_{n-1}\overline{A}^{n-1} + \cdots + a\overline{A} + a_0\overline{I} = \overline{\overline{O}}$$

{$n\times n$ zero matrix}

Theorem 2: (Euler 1776) A rigid-body motion about a point O leaves fixed set of point lying on a line $\mathcal{L}$ that passes through O and is parallel to the eigenvector $\overline{e}$ of $\overline{Q}$ associated with the eigen value $+1$.

@Analytic function $\Rightarrow$ function that is locally given by a convergent power series.

$\Rightarrow$ An important consequence of Cayley-Hamilton is that any analytic matrix function of A can be expressed not as an infinite series, but as a sum namely, a linear combination of first n powers of $\overline{A}$.

example $\Rightarrow$ Exponential function.

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2.3.1) The Cross-Product Matrix

⇒ Let $\bar{u}$ and $\bar{v}$ be vectors of space U and V, of dimensions m and n respectively.

⇒ Let t be a real variable and f be real-valued functions of t, with $f = f(\bar{u}, \bar{v})$

⇒ The partial derivative of f with respect to $\bar{u}$ is a ~~m~~-dimensional vector whose i^{th} component is partial derivative of f with respect to u_i.

↳ Similarly for partial derivative of f with respect to $\bar{v}$.

$$\frac{\delta f}{\delta \bar{u}} = \begin{bmatrix} \delta f/\delta u_1 \\ \delta f/\delta u_2 \\ \vdots \\ \delta f/\delta u_m \end{bmatrix} \qquad \frac{\delta f}{\delta \bar{v}} = \begin{bmatrix} \delta f/\delta v_1 \\ \delta f/\delta v_2 \\ \vdots \\ \delta f/\delta v_n \end{bmatrix}$$

⇒ Furthermore, the partial derivative of $\bar{v}$ with respect to $\bar{u}$ in an $n \times m$ array whose (i, j) entry is defined as $\delta v_i / \delta v_j$.

$$\frac{\delta \bar{v}}{\delta \bar{u}} = \begin{bmatrix} \delta u_1/\delta u_1 & \delta v_1/\delta u_2 & \cdots & \delta v_1/\delta u_m \\ \delta u_2/\delta u_1 & \delta v_2/\delta u_2 & \cdots & \delta v_2/\delta u_m \\ \vdots & \vdots & & \vdots \\ \delta v_m/\delta u_1 & \delta v_n/\delta u_2 & \cdots & \delta v_n/\delta u_m \end{bmatrix}$$

⇒ Hence, the total derivative of f with respect to $\bar{u}$ can be written as:-

$$\boxed{\frac{df}{d\bar{u}} = \frac{\delta f}{\delta \bar{u}} + \left(\frac{\delta \bar{v}}{\delta \bar{u}}\right)^T \frac{\delta f}{\delta \bar{v}}}$$

& $\bar{v} = \bar{v}(\bar{u}, t)$

⇒ If f is an explicit function (ie $f = f(\bar{u}, \bar{v}, t)$) then one can write total derivative of f w.r.t. t as:-

$$\boxed{\frac{df}{dt} = \frac{\delta f}{\delta t} + \left(\frac{\delta f}{\delta \bar{u}}\right)^T \frac{d\bar{u}}{dt} + \left(\frac{\delta f}{\delta \bar{v}}\right)^T \frac{\delta \bar{v}}{\delta t} + \left(\frac{\delta f}{\delta \bar{v}}\right)^T \frac{\delta \bar{v}}{\delta \bar{u}} \frac{d\bar{u}}{dt}}$$

⇒ The total derivative of $\bar{v}$ with respect to t can be written, likewise as:-

$$\boxed{\frac{d\bar{v}}{dt} = \frac{\delta \bar{v}}{\delta t} + \frac{\delta \bar{v}}{\delta \bar{u}} \frac{d\bar{u}}{dt}}$$

Let the Components of $\bar{v}$ and $\bar{x}$ in a certain reference frame F is given by

$$[\bar{v}]_F = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \quad ; \quad [\bar{x}]_F = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

$$[\bar{v} \times \bar{x}]_F = \begin{bmatrix} v_2 x_3 - v_3 x_2 \\ v_3 x_1 - v_1 x_3 \\ v_1 x_2 - v_2 x_1 \end{bmatrix}$$

Hence,

$$\left[\frac{d\,[\bar{v} \times \bar{x}]}{d\bar{x}}\right]_F = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}$$

⇒ The partial derivative of the Cross product of any 3-dimensional vector $\bar{v}$ and $\bar{x}$ will be denoted by the 3×3 matrix $\bar{\bar{V}}$.

↳ $\bar{\bar{V}}$ is termed Cross product matrix of $\bar{v}$.

⇒ The Cross product can be alternative represented as:-

$$\bar{v} \times \bar{x} = \bar{\bar{V}}\bar{x}$$

Theorem 4: The Cross product matrix $\bar{A}$ of any 3D vector $\bar{a}$ is Skew Symmetric :-

$$A^T = -A$$

and as a consequence

$$\bar{a} \times (\bar{a} \times \bar{b}) = \bar{A}^2 \bar{b}$$

$\bar{A}^2$ can be readily proven to be

$$\bar{A}^2 = -\|a\|^2 \bar{I} + \bar{a}\bar{a}^T$$

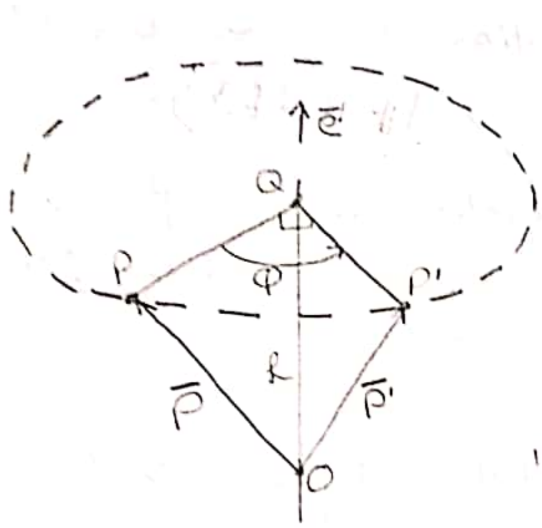
Reff 2 {using Vector triple product expansion}

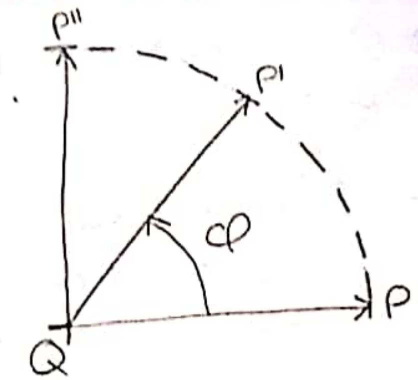
↳ Euclidean norm

2.3.2) The rotation matrix

⇒ Line $\mathcal{L}$ mentioned in Euler's Theorem, is termed the axis of rotation of the motion of interest.

⇒ Consider the rotation depicted in figure bellow of angle φ about line $\mathcal{L}$.





$$\overline{P'} = \overline{OQ} + \overline{QP'} \quad \text{— (1)}$$

$\overline{OQ} = \bar{e}\bar{e}^T\bar{P}$

$$\overline{QP'} = \cos\phi\,\overline{QP} + \sin\phi\,\overline{QP''}$$

normal component of $\bar{P}$ with respect to $\bar{e}$

$$\overline{QP} = (1-\bar{e}\bar{e}^T)\bar{P}$$

$$\overline{QP''} = \bar{e}\times\bar{P} = \bar{\bar{E}}\bar{P}$$

So $\overline{P'} = \bar{e}\bar{e}^T\bar{P} + \cos\phi(1-\bar{e}\bar{e}^T)\bar{P} + \sin\phi\,\bar{\bar{E}}\bar{P}$

$$\overline{P'} = \left[\bar{e}\bar{e}^T + \cos\phi(1-\bar{e}\bar{e}^T) + \sin\phi\,\bar{\bar{E}}\right]\bar{P}$$

Rotation matrix $\bar{\bar{Q}}$ is given by

$$\boxed{\bar{\bar{Q}} = \bar{e}\bar{e}^T + \cos\phi(1-\bar{e}\bar{e}^T) + \sin\phi\,\bar{\bar{E}}} \quad \text{— (1)}$$

One more representation of $\overline{\overline{Q}}$ in terms of $\overline{e}$ and φ is given:

{# Ref 3}

⇒ For a fixed axis rotation (ie for a fixed value of $\overline{e}$), the rotation matrix $\overline{\overline{Q}}$ is a function of the angle of rotation φ only.

So $\overline{\overline{Q}}(\varphi) = \overline{\overline{Q}}(0) + \overline{\overline{Q}}'(0)\varphi + \frac{1}{2!}\overline{\overline{Q}}''(0)\varphi^2 + \cdots + \frac{1}{K!}\overline{\overline{Q}}^K \varphi^K + \cdots$

⇒ Now, from the definition of $\overline{\overline{E}}$ one can readily prove the relations below:-

$$\overline{\overline{E}}^{2K+1} = (-1)^K \overline{\overline{E}} \quad ; \quad \overline{\overline{E}}^{2K} = (-1)^K (\overline{\overline{I}} - \overline{e}\overline{e}^T) \qquad (2)$$

⇒ Using eq (1) and (2) one can readily show that

$$\overline{\overline{Q}}^K(0) = \overline{\overline{E}}^K$$

So, $\overline{\overline{Q}}(\varphi) = \overline{\overline{I}} + \overline{\overline{E}}\varphi + \frac{1}{2!}\overline{\overline{E}}^2\varphi^2 + \cdots \frac{1}{K!}\overline{\overline{E}}^K\varphi^K + \cdots$

$$\Rightarrow \boxed{\overline{\overline{Q}}(\varphi) = e^{\overline{\overline{E}}\varphi}}$$

{Exponential representation of the rotation matrix in terms of its natural invariants $\overline{e}$ and φ}

⇒ The foregoing parameters are termed invariant because they are clearly independent of the coordinate axis chosen to represent the rotation under study.

⇒ Alternative representation of $\bar{\bar{Q}}$:-

$$\boxed{\bar{\bar{Q}} = \bar{\bar{I}} + \sin\phi\,\bar{\bar{E}} + (1-\cos\phi)\,\bar{\bar{E}}^2}$$

→ Expected result in view of Cayley - Hamilton theorem.

Canonical forms of the Rotation Matrix

⇒ The rotation matrix takes on especially simple form if the axis of rotation coincides with one of the coordinate axis.

eg ⇒ x-axis ∥ to axis of rotation :-

$$[\bar{e}]_x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad [E]_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \quad [E^2]_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\text{So } [\bar{\bar{Q}}]_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \quad \text{— (a)}$$

Similarly,

$$[\bar{\bar{Q}}]_y = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix} \quad \text{(b)} \qquad [\bar{\bar{Q}}]_x = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{(c)}$$

⇒ The representation of eq ⓐ ⓑ & ⓒ can be called the X- Y- and Z- Canonical forms of the rotation matrix.

⇒ An application of Canonical forms lies in the Parameterization of rotations by means of Euler angle consisting of three successive rotations ϕ, θ and ψ about the axis of coordinate frame.

2.3.3> <u>The Linear Invariants of a 3×3 Matrix</u>

⇒ Any 3×3 matrix $\bar{\bar{A}}$ Consists of the Sum of its Symmetric part $\bar{\bar{A}}_s$, and its Skew-Symmetric part $\bar{\bar{A}}_{ss}$.

$$\bar{\bar{A}} = \bar{\bar{A}}_s + \bar{\bar{A}}_{ss} \qquad \begin{cases} \bar{\bar{A}}_{s} = \frac{1}{2}(\bar{A} + \bar{A}^T) \\ \bar{\bar{A}}_{ss} = \frac{1}{2}(\bar{\bar{A}} - \bar{\bar{A}}^T) \end{cases}$$

⇒ Let $\bar{\bar{A}}_{ss}$ has Vector $\bar{a}$ So,

$$\bar{a} \times \bar{v} = \bar{\bar{A}}_{ss}\bar{v}$$

for any 3D Vector $\bar{v}$.

⇒ Let us assume that the entries of matrix $\bar{\bar{A}}$ in a Certain Coordinate frame are given by the array of real numbers a_{ij} $\forall$ $i,j = 1,2,3$.

Let $\bar{a}$ has Components a_i $\forall$ $i = 1,2,3$. In the same frame.

$$\mathrm{Vec}(\bar{A}_{ss}) = \bar{a} = \frac{1}{2}\begin{bmatrix} a_{32} - a_{23} \\ a_{13} - a_{31} \\ a_{21} - a_{12} \end{bmatrix}$$

$$\mathrm{tr}(\bar{A}) = a_{11} + a_{22} + a_{33}$$

Let us call Vector of $\bar{A}$ as $\bar{a}$. where $\bar{a}$ is Vector of $\bar{\bar{A}}_{ss}$.

So $$\mathrm{Vec}(\bar{A}) = \bar{a} = \frac{1}{2}\begin{bmatrix} a_{32} - a_{23} \\ a_{13} - a_{31} \\ a_{21} - a_{12} \end{bmatrix}$$

$$\mathrm{tr}(\bar{A}) = a_{11} + a_{22} + a_{33}.$$

Theorem 3: The Vector of a 3×3 matrix Vanishes if and only if it is Symmetric, whereas the trace of an n×n matrix Vanishes if the matrix is skew Symmetric.

⇒ Other useful relations are given bellow. For any 3-D Vector $\bar{a}$ and $\bar{b}$.

$$\mathrm{Vec}(\bar{a}\bar{b}^T) = -\frac{1}{2}(\bar{a}\times\bar{b})$$

& $\mathrm{tr}(\bar{a}\bar{b}^T) = \bar{a}^T\bar{b}$ {Straight forward}

Proof

Let W denote $\mathrm{Vec}(\bar{a}\bar{b}^T)$, for and 3D Vector $\bar{V}$

$$\bar{\omega}\times\bar{V} = \bar{\bar{W}}\bar{V}$$

where $\overline{\overline{W}}$ is the skew symmetric component of $\bar{a}\bar{b}^T$.

$$\overline{\overline{W}} = \frac{1}{2}\left(\bar{a}\bar{b}^T - \bar{b}\bar{a}^T\right)$$

$$\overline{\overline{W}}\bar{v} = \bar{\omega}\times\bar{v} = \frac{1}{2}\left[\bar{a}\left(\bar{b}^T\bar{v}\right) - \bar{b}\left(\bar{a}^T\bar{v}\right)\right]$$

$$= \frac{1}{2}\,\bar{b}\times\bar{a}\times\bar{v}$$

So $\boxed{\bar{\omega} = \frac{1}{2}\bar{b}\times\bar{a}}$

2.3.4> The Linear Invariants of a Rotation

$$\overline{\overline{Q}} = \bar{e}\bar{e}^T + \cos\varphi\left(\overline{\overline{I}} - \bar{e}\bar{e}^T\right) + \sin\varphi\,\overline{\overline{E}}$$

$\Rightarrow$ From above it is clear that first two terms of $\overline{\overline{Q}}$, $\bar{e}\bar{e}^T$ and $\cos\varphi\left(\overline{\overline{I}} - \bar{e}\bar{e}^T\right)$ are symmetric and third one, $\sin\varphi\,\overline{\overline{E}}$ is skew-symmetric.

$$\operatorname{tr}(\overline{\overline{Q}}) = \operatorname{tr}\left[\bar{e}\bar{e}^T + \cos\varphi\left(\overline{\overline{I}} - \bar{e}\bar{e}^T\right)\right]$$

$$= \bar{e}^T\bar{e} + \cos\varphi\left(3 - \bar{e}^T\bar{e}\right) = 1 + 2\cos\varphi$$

So $\boxed{\cos\varphi = \dfrac{\operatorname{tr}(\overline{\overline{Q}}) - 1}{2}}$

$\Rightarrow$ Henceforth, the vector of $\overline{\overline{Q}}$ will be denoted by $\bar{q}$ and its components in a given coordinate frame by $q_1\ q_2\ q_3$.

⇒ Rather than $tr(\bar{Q})$ as the other linear invariant $q_0 = \cos\phi$ will be linear invariants of the rotation matrix

⇒ The rotation matrix is fully defined by four scalar parameters, namely $\{q_i\}_0^3$, which will be conveniently stored in the 4D array $\bar{\lambda}$ defined as:-

$$\bar{\lambda} = [q_1\ q_2\ q_3\ q_0]^T$$

⇒ However, that the four components of $\bar{\lambda}$ are not independent as:

$$\bar{q} = \sin\phi\,\bar{e} \Rightarrow |\bar{q}|^2 = \sin^2\phi$$
$$q_0 = \cos\phi \Rightarrow q_0^2 = \cos^2\phi$$

$$\Rightarrow \|\bar{q}\|^2 + q_0^2 = 1$$

$$\Rightarrow q_1^2 + q_2^2 + q_3^2 + q_0^2 = 1$$

$$\Rightarrow \boxed{\|\bar{\lambda}\|^2 = 1}$$

Geometric Interpretation of above

"As a body rotates about a fixed point, its motion can be described in a 4D space by the motion of a point of vector $\bar{\lambda}$ that moves on the surface of the unit sphere centered at the origin"

⇒ Given the dependence of the four components of vector $\bar{\lambda}$, one might be temped to solve for q_0.

$$\Rightarrow q_0 = \pm\sqrt{1-(q_1^2+q_2^2+q_3^2)}$$

Sign ambiguity leaves angle ϕ undefined

⇒ Moreover, the three components of vector $\bar{q}$ alone, do not suffice to define the rotation represented by $\bar{Q}$.

$$\bar{e} = \frac{\bar{q}}{\sin\phi} \; ; \; \sin\phi = \pm\|\bar{q}\|$$

angle ϕ undefined.

In terms of $\bar{q}$ and q_0 rotation matrix can be written as :-

$$\bar{\bar{Q}} = \frac{\bar{q}\bar{q}^T}{\|\bar{q}\|^2} + q_0\left(\bar{\bar{I}} - \frac{\bar{q}\bar{q}^T}{\|\bar{q}\|^2}\right) + \bar{\bar{Q}}'$$

where $\bar{\bar{Q}}' = \dfrac{\delta(\bar{q}\times\bar{x})}{\delta x}$

$\forall$ any vector $\bar{x}$

$$\bar{\bar{Q}} = q_0\bar{\bar{I}} + \bar{\bar{Q}}' + \frac{\bar{q}\bar{q}^T}{1+q_0}$$

⇒ From above equation it is clear that linear invariants are not suitable to represent a rotation when the associated angle is either π or close to it.

⇒ Rotation through an angle α about an axis given by vector $\bar{e}$ is identical to a rotation through an angle $-\alpha$ about an axis given by vector $-\bar{e}$.

↳ Henceforth we will choose the sign of the components of $\bar{e}$ so that $\sin\alpha \geq 0$

↓

equivalent to $0 \leq \alpha \leq \pi$

Thus $\sin\alpha = \|a\|$ → { $\bar{e}$ is simply normalized $\bar{a}$ }

$\cos\alpha = a_0$.

$$\downarrow$$

$$\frac{\mathrm{tr}(\bar{\bar{Q}})-1}{2}.$$

2.3.6) The Euler-Rodrigues Parameters

Natural Invariant ⇒ $\mathrm{tr}(\bar{\bar{A}})$, $\mathrm{Vec}(\bar{\bar{A}})$

Linear Invariant ⇒ a_0 $\quad a_1\ a_2\ a_3$

⇒ Natural and the linear invariants of a rotation matrix, are not the only one that are used in kinematics.

↳ Additionally one has Euler-Rodrigues Parameter, represented here as $\bar{r}$ and r_0.

$$\bar{r} = \sin\left(\frac{\alpha}{2}\right)\bar{e} \qquad r_0 = \cos\left(\frac{\alpha}{2}\right)$$

⇒ One can readily show that $\bar{\bar{Q}}$ takes on a quite simple form in terms of Euler-Rodrigues Parameters.

$$\overline{\overline{Q}} = (r_0^2 - \bar{r}\cdot\bar{r})\overline{\overline{I}} + 2\bar{r}\bar{r}^T + 2r_0\overline{\overline{R}}$$

where $\overline{\overline{R}} = \dfrac{\delta(\bar{r}\times\bar{x})}{\delta\bar{x}}$ $\forall$ arbitrary $\bar{x}$

⇒ The relationship between the linear invariants and the Euler-Rodrigues parameters can be readily derived.

$$r_0 = \pm\sqrt{\frac{1+q_0}{2}} \qquad \bar{r} = \frac{\bar{q}}{2r_0} \qquad \varphi \neq \pi$$

$$\bar{r} = \bar{e} \quad \& \quad r_0 = 0 \quad \forall\ \varphi = \pi$$

⇒ We now derive invariant relations between the rotation matrix and the Euler-Rodrigues Parameters.

↳ To do this we resort to the Concept of Matrix Square root.

⇒ From the geometric meaning of a rotation through the angle φ about an axis ∥ to the unit vector $\bar{e}$,

↳ It is apparent that the Square of the matrix representing the foregoing rotation it itself a rotation about the same axis, but through the angle 2φ.

↳ By the same token, the square root of the same matrix is again a rotation matrix about the same axis but through an angle $\phi/2$.

⇒ While the square of a matrix is unique, its square root is not.

↳ Of these square roots, nevertheless, there is one that is proper orthogonal, the one representing a rotation of $\phi/2$

⇒ We will denote the particular square root of $\bar{\bar{Q}}$ by $\sqrt{\bar{\bar{Q}}}$.

⇒ The Euler-Rodrigues parameters of $\bar{\bar{Q}}$ can thus be expressed as the linear invariants of $\sqrt{\bar{\bar{Q}}}$ namely:-

$$\bar{r} = \text{Vect}\left(\sqrt{\bar{\bar{Q}}}\right) \qquad r_0 = \frac{\text{tr}\left(\sqrt{\bar{\bar{Q}}}\right) - 1}{2}$$

⇒ It is important to recognize the basic difference between

Linear Invariants

↳ Easily derived from matrix representation of rotation.

↳ fails to provide axis of rotation when angle is π.

Euler-Rodrigues Parameters

↳ Requires square root & involves sign ambiguities.

↳ Gives axis of rotation for any value of the angle of rotation.

⇒ The Euler-Rodrigues Parameters are nothing but the quaternions invented by Sir William Rowan Hamilton (1844).

2.4> Composition of Reflections and Rotation

Let $\bar{\bar{R}}$ ⇒ Pure reflection $\left\{\begin{array}{l} \Rightarrow \text{Symmetric} \\ \Rightarrow \text{Self inverse} \end{array}\right\}$

$\bar{\bar{Q}}$ ⇒ arbitrary rotation {Proper Orthogonal}

⇒ The product of these two transformations $\bar{\bar{Q}}\bar{\bar{R}}$, denoted by $\bar{\bar{T}}$ is apparently neither Symmetric nor Self-inverse.

↳ Same true for reverse order

⇒ As a Consequence, of an improper orthogonal transformation that is not Symmetric can always be decomposed into the product of a rotation and a pure reflection.

⇒ If we want to decompose $\bar{\bar{T}}$ in the above Paragraph into the product $\bar{\bar{Q}}\bar{\bar{R}}$,

↳ then we can freely chose the unit normal Vector $\bar{n}$ of the plane of reflection and write

$$\bar{\bar{R}} = \bar{\bar{I}} - 2\bar{n}\bar{n}^T$$

⇒ Hence the factor $\bar{\bar{Q}}$ is obtained as:-

$$\bar{\bar{Q}} = \bar{\bar{T}}\bar{\bar{R}}^{-1} = \bar{\bar{T}}\bar{\bar{R}} = \bar{\bar{T}} - 2(\bar{\bar{T}}\bar{n})\bar{n}^T$$

2.5) Coordinate Transformations and Homogeneous Coordinates

⇒ Crucial to robotics is the unambiguous description of the geometrical relation among the various bodies in the environment surrounding a robot.

↳ These relations are established by means of coordinate frame
↓
{attached to each rigid body in the scene, including the robot links.}

⇒ The origin of these frames, moreover, are set at landmark points and orientations defined by key geometric entities like lines and planes.

↳ Z-axis of each frame is defined, according to the Denavit-Hartenberg notation.

2.5.1) Coordinate transformation between frames With a Common Origin

Let two coordinate frame be $A = \{X\,Y\,Z\}$ & $B = \{x\,y\,z\}$.

Let $\bar{Q}$ be the rotation carrying A into B.

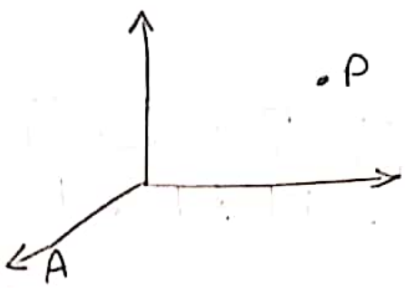
$$Q : A \rightarrow B$$

Let us denote position vector of a point P in A and B, denoted by $[P]_A$ and $[P]_B$ respectively.

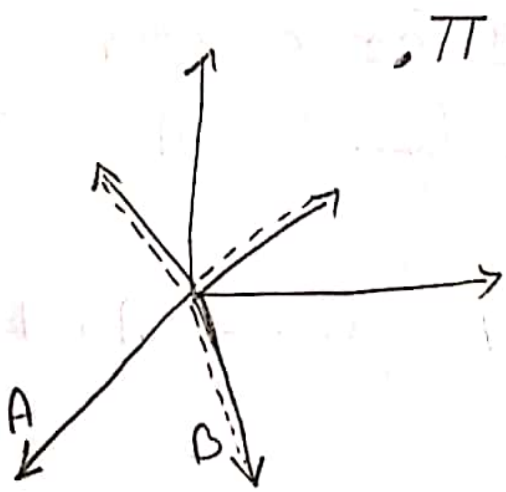
Let
$$[P]_A = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

⇒ We want to find $[P]_B$ in terms of $[P]_A$ and $\bar{Q}$.

⇒ Suppose P is attached to fram A.



⇒ Now frame A undergo rotation $\bar{Q}$ about its origin, that carries it to new configuration, that of fram B.



⇒ Point P in its rotated position is labeled Π, of position vector $\bar{\pi}$.

$$\bar{r} = \bar{Q}\bar{P}$$

& $$[\bar{r}]_B = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Let $$[\bar{r}]_A = \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix}$$

Theorem 2.5.1: The representations of the position vector $\bar{r}$ of any point in two fram. A & B, denoted by $[\bar{r}]_A$ and $[\bar{r}]_B$, respectively are related by

$$[\bar{r}]_A = [\bar{Q}]_A [\bar{r}]_B$$

Proof:

$$[\bar{r}]_A = [\bar{Q}]_A [\bar{P}]_A \quad \text{—— (1)}$$

& $$[\bar{r}]_B = [\bar{P}]_A \quad \text{—— (2)}$$

So using (1) & (2)

$$\boxed{[\bar{r}]_A = [\bar{Q}]_A [\bar{r}]_B}$$

Theorem 2.5.2: The representations of $\bar{Q}$ carrying A and B in these two frames are identical.

$$[\bar{Q}]_A = [\bar{Q}]_B$$

Proof

$$[\bar{r}]_A = [\bar{\bar{Q}}]_A [\bar{r}]_B \quad \text{--- ①}$$

$$\bar{r} = \bar{\bar{Q}}\bar{P} \quad \text{--- ②}$$

$\Rightarrow$ Using ① & ② we get:-

$$[\bar{\bar{Q}}\bar{P}]_A = [\bar{\bar{Q}}]_A [\bar{\bar{Q}}\bar{P}]_B$$

$$\Rightarrow [\bar{\bar{Q}}]_A [\bar{P}]_A = [\bar{\bar{Q}}]_A [\bar{\bar{Q}}]_B [\bar{P}]_B$$

$$\Rightarrow [\bar{P}]_A = [\bar{\bar{Q}}]_B [\bar{P}]_B$$

~~we know $[\bar{P}]_A = \bar{P}$~~

$\Rightarrow$ By Virtue of Theorem 2.5.1, the two representations of P observe the relation

$$[P]_A = [Q]_A [P]_B$$

Hence $[Q]_A = [Q_B]$

2.5.2) Coordinate transformation with Origin Shift

Frame A : Origin O_A

Frame B : Origin O_B

Let $\overline{O_A O_B} = \bar{b}$

& $\bar{\bar{Q}} : A \to B$

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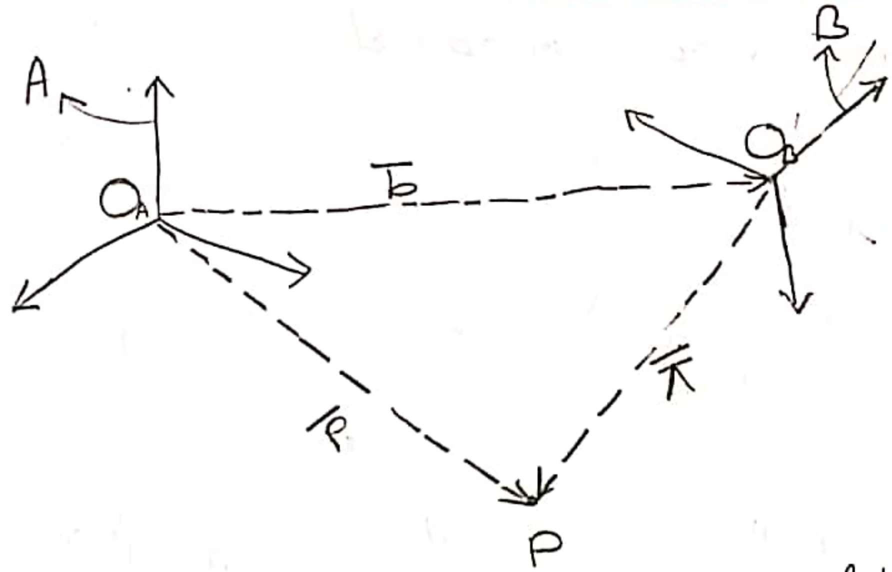

Theorem 2.5.4 : The representation of the position vector $\bar{p}$ of point P of Euclidean 3D Space in two frames A and B are related by

$$[\bar{p}]_A = [\bar{b}]_A + [\bar{Q}]_A [\bar{r}]_B$$

Proof

$$\bar{p} = \bar{b} + \bar{r}$$

$$\Rightarrow [\bar{p}]_A = [\bar{b}]_A + [\bar{r}]_A \quad \text{--- (1)}$$

$$[\bar{r}]_A = [\bar{Q}]_A [\bar{r}]_B \quad \text{--- (2)}$$

Using eq (1) & (2) we get :-

$$\boxed{[\bar{p}]_A = [\bar{b}]_A + [\bar{Q}]_A [\bar{r}]_B}$$

2.5.3) Homogeneous Coordinates

⇒ The general coordinate transformation, involving a shift of the origin, is not linear, in general, as can be readily realized by virtue of the nonhomogeneous term involved.

⇒ Let $[\bar{P}]_M$ be the coordinate array of a finite point P in reference frame M.

{finite points are one whose coordinates are all finite}

⇒ The homogeneous coordinates of P are those in the 4-dimensional array $\{P\}_M$ defined as:-

$$\{\bar{P}\}_M = \begin{bmatrix} [\bar{P}]_M \\ 1 \end{bmatrix}$$

We know

$$[\bar{P}]_A = [\bar{b}]_A + [\bar{\bar{Q}}]_A [\bar{A}]_B$$

⇒ The affine transformation of above equation can be rewritten in homogeneous-coordinate form as:

$\{\bar{P}\}_A = \{\bar{\bar{T}}\}_A \langle \bar{\pi} \rangle_B$

where,

$$\{\bar{\bar{T}}\}_A = \begin{bmatrix} [\bar{\bar{Q}}]_A & [b]_A \\ [0^T]_A & 1 \end{bmatrix}$$

$$\{\bar{\bar{T}}^{-1}\}_B = \begin{bmatrix} [\bar{\bar{Q}}]_B & [-b]_B \\ [0^T]_B & 1 \end{bmatrix}$$

⇒ Furthermore, homogeneous transformations can be <u>Concatenated</u>

{Link together in chain}

⇒ Let F_K $\forall$ $K = i-1, i, i+1$, denote three coordinate frames with origin at O_K.

⇒ Let $\bar{\bar{Q}}_{i-1}$ be the rotation carrying F_{i-1} into orientation coinciding with that of F_i.

Similarly, $\bar{\bar{Q}}_i : F_i \rightarrow F_{i+1}$

⇒ First the case in which all the three origins coincide is considered.

$$[\bar{P}]_i = [\bar{\bar{Q}}_{i-1}^T]_{i-1} [\bar{P}]_{i-1}$$

$$[\bar{P}]_{i+1} = [\bar{\bar{Q}}_i^T]_i [\bar{P}]_i \Rightarrow [\bar{\bar{Q}}_i^T]_i [\bar{\bar{Q}}_{i-1}^T]_{i-1} [\bar{P}]_{i-1}$$

⇒ If now the origins do not coincide, let $\bar{a}_{i-1}$ and $\bar{a}_i$ denote the vectors $\overline{O_{i-1}O_i}$ and $\overline{O_i O_{i+1}}$ respectively.

⇒ The homogeneous-coordinate transformation $\{\bar{\bar{T}}_{i-1}\}_{i-1}$ and $\{\bar{\bar{T}}_i\}_i$, thus arising are, obviously

$$\{\bar{\bar{T}}_{i-1}\}_{i-1} = \begin{bmatrix} [\bar{\bar{Q}}_{i-1}]_{i-1} & [a_{i-1}]_{i-1} \\ [O^T]_{i-1} & 1 \end{bmatrix}$$

$$\{\bar{\bar{T}}_i\}_i = \begin{bmatrix} [\bar{\bar{Q}}_i]_i & [\bar{a}_i]_i \\ [O^T]_i & 1 \end{bmatrix}$$

let

$$\{T_{i-1}^{-1}\}_i = \begin{bmatrix} [\bar{\bar{Q}}_{i-1}^T]_i & [\bar{\bar{Q}}_{i-1}^T]_i [-\bar{a}_{i-1}]_{i-1} \\ [O^T]_i & 1 \end{bmatrix}$$

$$\{T_i^{-1}\}_{i+1} = \begin{bmatrix} [\bar{\bar{Q}}_i^T]_{i+1} & [\bar{\bar{Q}}_i^T]_{i+1} [-\bar{a}_i]_i \\ [O^T]_{i+1} & 1 \end{bmatrix}$$

⇒ Hence the coordinate transformations involved are:

$$\{\bar{P}\}_{i-1} = \{\bar{\bar{T}}_{i-1}\}_{i-1} \{\bar{P}\}_i$$

$$\{\bar{P}\}_{i-1} = \{\bar{\bar{T}}_{i-1}\}_{i-1} \{\bar{\bar{T}}_i\}_i \{\bar{P}\}_{i+1}$$

$$\{\bar{P}\}_i = \{\bar{\bar{T}}_{i-1}^{-1}\}_i \{\bar{P}\}_{i-1}$$

$$\{\bar{P}\}_{i+1} = \{\bar{\bar{T}}_i^{-1}\}_{i+1} \{\bar{\bar{T}}_{i-1}^{-1}\}_i \{\bar{P}\}_{i-1}$$

$\Rightarrow$ Now, if P lies at infinity, we can express its homogeneous coordinates in a simple form

$$\{\bar{P}\}_M = \begin{bmatrix} [\bar{P}]_M \\ 1 \end{bmatrix}$$

{Unit vector of $[\bar{P}]_M$}

$$\Rightarrow \{\bar{P}_\infty\}_M = \|\bar{P}\| \begin{bmatrix} [e]_M \\ 1/\|\bar{P}\| \end{bmatrix}$$

{Length of $\bar{P}$}

$$\lim_{\|\bar{P}\|\to\infty} \{\bar{P}\}_M = \left(\lim_{\|\bar{P}\|\to\infty} \|\bar{P}\|\right) \begin{bmatrix} [e]_M \\ 0 \end{bmatrix}$$

$\Rightarrow$ We now define the homogeneous coordinates of a point P lying at infinity as the 4-D array:

$$\{\bar{P}_\infty\}_M \equiv \begin{bmatrix} [e]_M \\ 0 \end{bmatrix}$$

{Point at infinity in homogeneous coordinates, has only direction given by unit vector $\bar{e}$ but an undefined length}

If $[\bar{\bar{Q}}]_A = [\bar{e}_1 \; \bar{e}_2 \; \bar{e}_3]$

$$\{\bar{\bar{T}}\}_A = \begin{bmatrix} \bar{e}_1 & \bar{e}_2 & \bar{e}_3 & \bar{b} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2.6) Similarity Transformation

⇒ Transformations of the position vector of points under a change of coordinate frame involving both a translation of the origin and a rotation of the coordinate axis. was in Section 2.5.

⇒ In this Section, we study the transformations of components of vector other than the position vectors, while extending the concept to the transformation of matrix entries.

⇒ Let $A = \{\bar{a}_i\}_1^n$ and $B = \{\bar{b}_i\}_1^n$ be two different bases of the same space V.

⇒ Hence any vector $\bar{v}$ of V can be expressed in either two ways,

$$\bar{V} = \alpha_1 \bar{a}_1 + \alpha_2 \bar{a}_2 + \cdots \alpha_n \bar{a}_n$$

$$\bar{V} = \beta_1 \bar{b}_1 + \beta_2 \bar{b}_2 + \cdots \beta_n \bar{b}_n$$

$$[\bar{V}]_A = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} \qquad [\bar{V}]_B = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix}$$

$\Rightarrow$ Let the two foregoing bases be related by

$$b_j = a_{1j}\bar{a}_1 + a_{2j}\bar{a}_2 + \cdots a_{nj}\bar{a}_n \qquad j = 1 \cdots n$$

$\Rightarrow$ In order to find the relationship between the two representation:-

$$\bar{V} = \beta_1 (a_{11}\bar{a}_1 + a_{21}\bar{a}_2 + \cdots a_{n1}\bar{a}_n)$$
$$+ \beta_2 (a_{12}\bar{a}_1 + a_{22}\bar{a}_2 + \cdots + a_{n2}\bar{a}_n)$$
$$\vdots$$
$$+ \beta_n (a_{1n}\bar{a}_1 + a_{2n}\bar{a}_2 + \cdots + a_{nn}\bar{a}_n)$$

This can be written as:

$$\boxed{[\bar{V}]_A = [\bar{A}]_A [\bar{V}]_B} \quad \text{— ①}$$

$$[\bar{A}]_A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

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So

$$\boxed{[\bar{V}]_B = [\bar{\bar{A}}^{-1}]_A \, [\bar{V}]_A} \quad \text{—— (2)}$$

⇒ Next, Let $\bar{\bar{L}}$ have the representation in A given below:

$$[\bar{\bar{L}}]_A = \begin{bmatrix} l_{11} & l_{12} & \cdots & l_{1n} \\ l_{21} & l_{22} & \cdots & l_{2n} \\ \vdots & \vdots & & \vdots \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{bmatrix}$$

⇒ Now we aim at finding the relationship between $[\bar{\bar{L}}]_A$ and $[\bar{\bar{L}}]_B$.

⇒ Let $\bar{\omega}$ be the image of $\bar{V}$ under $\bar{\bar{L}}$

$$\bar{\bar{L}}\bar{V} = \bar{\omega}$$

Which can be expressed in terms of either A or B as:-

$$[\bar{\bar{L}}]_A [\bar{V}]_A = [\bar{\omega}]_A$$

$$[\bar{\bar{L}}]_B [\bar{V}]_B = [\bar{\omega}]_B$$

We Know, $[\bar{\omega}]_A = [\bar{\bar{A}}]_A [\bar{\omega}]_B$ & $[\bar{V}_A] = [\bar{\bar{A}}]_A [\bar{V}]_B$

$$\Rightarrow [\bar{\bar{A}}]_A [\bar{\omega}]_B = [\bar{\bar{L}}]_A [\bar{V}]_A = [\bar{\bar{L}}]_A [\bar{\bar{A}}]_A [\bar{V}]_B$$

$$[\bar{\omega}]_B = [\bar{A}^{-1}]_A [\bar{L}]_A [\bar{A}]_A [\bar{V}]_B$$

So $$\boxed{[\bar{L}]_B = [A^{-1}]_A [\bar{L}]_A [\bar{A}]_A} \quad \text{--- ③}$$

& $$\boxed{[\bar{L}]_A = [A]_A [\bar{L}]_B [A^{-1}]_A} \quad \text{--- ④}$$

⇒ Relations ①, ②, ③ & ④ Constitute what are called Similarity Transformation.

These are important because they Preserve invariant qualities.
{eg ⇒ eigen values, eigen vectors, magnitude, angle between Vectors etc....}

Theorem 7.6.1: The characteristic polynomial of a given $n \times n$ matrix remains unchanged under a Similarity transformation.

Theorem 7.6.2: If $[\bar{L}]_A$ and $[\bar{L}]_B$ are related by the Similarity transformation then,

$$[\bar{L}^K]_B = [\bar{A}^{-1}]_A [\bar{L}^K]_A [\bar{A}]_A$$

for any integer K.

Theorem 7.6.3: The trace of an $n \times n$ matrix does not change under a Similarity transformation.

2.7) Invariance Concept

Let a Scalar, Vector, and matrix function of the position Vector $\bar{P}$ be denoted by $f(\bar{P})$, $\bar{f}(\bar{P})$ and $\bar{\bar{F}}(\bar{P})$ respectively.

⇒ Representations of $\bar{f}(\bar{P})$ in two different Coordinate frames, labelled A & B, will be indicated as $[\bar{f}(\bar{P})]_A$ and $[\bar{f}(\bar{P})]_B$ respectively.

↳ Similar for $f(\bar{P})$ and $\bar{\bar{F}}(\bar{P})$

Let $[\bar{\bar{Q}}]_A$ denote rotation of coordinate frame A into B.

⇒ The Scalar function $f(\bar{P})$ is said to be frame invariant if:-

$$\boxed{f([\bar{P}]_B) = f([\bar{P}]_A)}$$

⇒ Vector quantity $\bar{f}$ is said to be invariant if:

$$\boxed{[\bar{f}]_A = [\bar{\bar{Q}}]_A [\bar{f}]_B}$$

⇒ Matrix quantity $\bar{\bar{F}}$ is said to be invariant if:-

$$\boxed{[\bar{\bar{F}}]_A = [\bar{\bar{Q}}]_A [\bar{\bar{F}}]_B [\bar{\bar{Q}}^T]_A}$$

⇒ The k^{th} moment of an $n \times n$ matrix $\bar{\bar{T}}$, denoted by I_k is defined as (Leigh 1968)

$$\boxed{I_k = tr(\bar{\bar{T}}^k)}$$

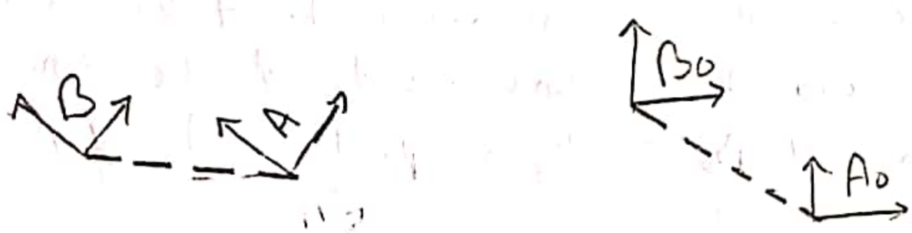
$$\left\{ \text{where } I_0 = tr(\bar{\bar{I}}) = n \right\}$$

2.7.1) Applications to Redundant Sensing

⇒ A sensor, such as a camera or a range finder, is often mounted on a robotic end-effector to determine the pose. → (Position and Orientation)

⇒ If redundant sensors are introduced, and we attach frames A and B to each of these, then each sensor can be used to determine the orientation of the end-effector with respect to a reference configuration.

⇒ This is a simple task, for all that is needed is to measure the rotation $\bar{R}$ that each of the foregoing frames undergo from the reference configuration.



⇒ With this information we would like to determine the relative orientation $\overline{\overline{Q}}$ of frame B with respect to frame.

↓

This problem is called <u>Instrument Calibration</u>

Let $\overline{\overline{A}} \equiv [\overline{\overline{R}}]_A \qquad \overline{\overline{B}} \equiv [\overline{\overline{R}}]_B$

$$\Rightarrow \overline{\overline{A}} = [\overline{\overline{Q}}]_A \overline{\overline{B}} [\overline{\overline{Q}}^T]_A$$

$$\Rightarrow \overline{\overline{A}} [\overline{\overline{Q}}]_A = [\overline{\overline{Q}}]_A \overline{\overline{B}}$$

⇒ This problem can be solved if we have three invariant vectors associated with each of the two matrices $\overline{\overline{A}}$ and $\overline{\overline{B}}$.

⇒ However, since $\overline{\overline{A}}$ and $\overline{\overline{B}}$ are orthogonal matrices, they admit only one real invariant vector, namely, their axial vector.

↳ we are short of two vector equation.

⇒ One way of obtaining one additional vector in each frame is to take not one, but two measurements of the orientation of A_0 and B_0 with respect to A and B respectively.

⇒ Let the matrices representing these orientation be given, in each of the two coordinate frames, by $\overline{A}_i$ and $\overline{B}_i$ $\forall i = 1, 2$.

↳ Let $\overline{a}_i$ and $\overline{b}_i$ $\forall i = 1, 2$ be the axial vectors of the matrices $\overline{A}_i$ and $\overline{B}_i$ respectively.

⇒ Now we have two possibilities :-

(i) Neither $\overline{a}_1$ and $\overline{a}_2$ and, consequently neither of b_1 and b_2 is zero.

⇒ Hence nothing prevents us from computing a third vector of each set, namely

$$\overline{a}_3 = \overline{a}_1 \times \overline{a}_2 \qquad \overline{b}_3 = \overline{b}_1 \times \overline{b}_2$$

(ii) At least one of $\overline{a}_1$ and $\overline{a}_2$ and consequently, the corresponding vector of the $\{\overline{b}_1, \overline{b}_2\}$ pair vanishes.

ⓐ The angle of rotation of the orthogonal matrix $\overline{A}_1$ or $\overline{A}_2$, whose axial vector vanishes is either 0 or π.

⇒ A underwent a pure translation from A_0 ; the same holding of course for B and B_0.

⇒ This means new measurements are needed involving rotation.

ⓑ If angle is π, then the associated rotation is symmetric and the unit vector $\overline{e}$ ∥ to its axis can be determined.

$$\bar{a}_i = [\bar{Q}]_A \bar{b}_i \quad \text{for } i = 1,2,3$$

$$\Rightarrow \quad \bar{\bar{E}} = [\bar{Q}]_A \bar{\bar{F}}$$

$$\left\{ \bar{\bar{E}} = [\bar{a}_1 \ \bar{a}_2 \ \bar{a}_3] \ , \ \bar{\bar{F}} = [\bar{b}_1 \ \bar{b}_2 \ \bar{b}_3] \right\}$$

$$\text{so } [\bar{\bar{Q}}]_A = \bar{\bar{E}} \, \bar{\bar{F}}^{-1}$$

$$\bar{\bar{F}}^{-1} = \frac{1}{\Delta} \begin{bmatrix} (\bar{b}_2 \times \bar{b}_3)^T \\ (\bar{b}_3 \times \bar{b}_1)^T \\ (\bar{b}_1 \times \bar{b}_2)^T \end{bmatrix} \qquad \Delta \equiv \bar{b}_1 \times \bar{b}_2 \cdot \bar{b}_3$$

Therefore,

$$\boxed{[\bar{\bar{Q}}]_A = \frac{1}{\Delta} \left[a_1 (b_2 \times b_3)^T + a_2 (b_3 \times b_1)^T + a_3 (b_1 \times b_2)^T \right]}$$

CHAPTER 3

Fundamental of Rigid Body Mechanics

3

Fundamentals of Rigid-Body Mechanics

3.1) Introduction

⇒ This chapter lay down the foundations of the Kinetostatics and dynamics of rigid body.

3.2) General Rigid-Body Motion and its associated Screw

Let A and P be two points of the same rigid body B.

(Particular reference Point) (Arbitrary Point)

⇒ Let the position vector of point A in the original configuration is $\bar{a}$ and the position vector of the same point in the ~~displacement~~ displaced configuration, denoted by A', is $\bar{a'}$.

⇒ Similarly $P \rightarrow \bar{P}$
$P' \rightarrow \bar{P'}$

⇒ Furthermore $\bar{P'}$ is to be determined, while $\bar{a}$, $\bar{a'}$ and $\bar{P}$ are given along with the rotation matrix $\bar{\bar{Q}}$.

$\Rightarrow$ Vector $\bar{p}-\bar{a}$ can be considered to undergo a rotation $\bar{\bar{Q}}$ about point A.

$\Rightarrow$ Since vector $\bar{p}-\bar{a}$ is mapped into $\bar{p}'-\bar{a}'$ under the rotation one can write

$$\bar{p}'-\bar{a}' = \bar{\bar{Q}}(\bar{p}-\bar{a})$$

$$\Rightarrow \bar{p}' = \bar{a}' + \bar{\bar{Q}}(\bar{p}-\bar{a}) \quad \text{--- (1)}$$

$\Rightarrow$ Moreover, let $\bar{d}_A$ and $\bar{d}_p$ denote the displacements of A and P respectively

$$\bar{d}_A = \bar{a}'-\bar{a} \quad \& \quad \bar{d}_p = \bar{p}'-\bar{p} \quad \text{--- (2)}$$

$\Rightarrow$ from eq (1),

$$\bar{p}'-\bar{p} = \bar{a}'-\bar{p} + \bar{\bar{Q}}(\bar{p}-\bar{a})$$

$$= \bar{a}'-\bar{p}-\bar{a}+\bar{a} + \bar{\bar{Q}}(\bar{p}-\bar{a})$$

$$= \bar{d}_A + (\bar{\bar{Q}}-\bar{\bar{I}})(\bar{p}-\bar{a})$$

$$\Rightarrow \boxed{\bar{d}_p = \bar{d}_A + (\bar{\bar{Q}}-\bar{\bar{I}})(\bar{p}-\bar{a})} \quad \text{--- (3)}$$

Displacement of an arbitrary Point P of a rigid body ~~whose position~~ ~~vector~~ is determined by displacement of one certain point A and the concomitant rotation $\bar{\bar{Q}}$

Theorem 3.2.1: The component of the displacements of all the points of a rigid body undergoing a general motion along the axis of the underlying rotation is a constant.

Proof: Multiply both side of eq ⊗ by $\bar{e}^T$.

{ where $\bar{e}$ is unit vector parallel to the axis of rotation represented by $\bar{\bar{Q}}$ }

units

$$\Rightarrow \bar{e}^T \bar{d}_P = \bar{e}^T \bar{d}_A + e^T(\bar{\bar{Q}} - \bar{\bar{I}})(\bar{P} - \bar{a})$$

⇒ Second term of the right-hand side of the above equation vanishes because $\bar{\bar{Q}}\bar{e} = \bar{e}$ and hence $\bar{\bar{Q}}^T \bar{e} = \bar{e}$.

$$\longrightarrow \bar{e}^T = \bar{e}^T \bar{\bar{Q}}$$

$$\bar{e}^T \bar{\bar{Q}} - \bar{e}^T = \bar{e}^T - \bar{e}^T = \bar{0}^T$$

$$\text{So } \bar{e}^T \bar{d}_P = \bar{e}^T \bar{d}_A \equiv d_0$$

Theorem 3.2.2: (Mozzi 1763; Chasles 1830)

Given a rigid body undergoing a general motion, a set of its points located on a line $\mathcal{L}$ undergo indentical displacements of minimum magnitude. Moreover, line $\mathcal{L}$ and the minimum-magnitude displacement are parallel to the axis of the rotation involved.

Proof: We can express the displacement of an arbitrary point P as the sum of two orthogonal components.

→ $d_{\parallel}$ ⇒ ∥ to axis of rotation

→ $d_{\perp}$ ⇒ ⊥ to axis of rotation

$$\bar{d}_p = \bar{d}_{\parallel} + \bar{d}_{\perp}$$

$$\bar{e}\bar{e}^T\bar{d}_p = d_0\bar{e} \qquad (\bar{\bar{I}} - \bar{e}\bar{e}^T)\bar{d}_p$$

$$\|\bar{d}_p\|^2 = \|\bar{d}_{\parallel}\|^2 + \|\bar{d}_{\perp}\|^2 = d_0^2 + \|d_{\perp}\|^2$$

⇒ In order to minimize $\|\bar{d}_p\|$ we have to make $\|\bar{d}_{\perp}\|$ equal to zero.

↳ So $\bar{d}_p$ must be $\parallel$ to $\bar{e}$

$$\bar{d}_p = \alpha\bar{e} \quad \{\forall \text{ Certain Scalar } \alpha\}$$

⇒ If P^* is a point of minimum magnitude of position vector $\bar{P^*}$, its component $\perp$ to the axis of rotation must vanish.

$$d_{\perp}^* = (\bar{\bar{I}} - \bar{e}\bar{e}^T)\bar{d}_p^*$$

$$\Rightarrow (\bar{\bar{I}} - \bar{e}\bar{e}^T)\bar{d}_A + (\bar{\bar{I}} - \bar{e}\bar{e}^T)(\bar{\bar{Q}} - \bar{\bar{I}})(\bar{P^*} - \bar{a}) = 0$$

$$\Rightarrow (\bar{\bar{I}} - \bar{e}\bar{e}^T)\bar{d}_A + (\bar{\bar{Q}} - \bar{\bar{I}})(\bar{P^*} - \bar{a}) = 0$$

$$\left\{ \begin{aligned} &(\bar{\bar{I}} - \bar{e}\bar{e}^T)(\bar{\bar{Q}} - I) \\ &= \bar{\bar{Q}} - \bar{e}\bar{e}^T\bar{\bar{Q}} - \bar{\bar{I}} + \bar{e}\bar{e}^T \\ &= (\bar{\bar{Q}} - \bar{\bar{I}}) \end{aligned} \right. \quad \{\text{as } \bar{e}\bar{e}^T = \bar{e}\bar{e}^T\bar{\bar{Q}}\}$$

$$\left\{ \begin{aligned} &\bar{\bar{Q}}\bar{e} = \bar{e} \\ &\Rightarrow \bar{e} = \bar{\bar{Q}}^T\bar{e} \\ &\Rightarrow \bar{e}^T = e^T\bar{\bar{Q}} \\ &= \boxed{\bar{e}\bar{e}^T = \bar{e}\bar{e}^T\bar{\bar{Q}}} \end{aligned} \right\}$$

⇒ Now, if we define a line $\mathcal{L}$ passing through p^* and ∥ to $\bar{e}$, then the position vector $p^* + \lambda\bar{e}$ of any points P satisfies the foregoing equation.

↳ As a consequence, all points of minimum magnitude lie in a line ∥ to the axis of rotation of $\bar{Q}$.

⇒ A rigid body can attain an arbitrary configuration from a given original one, following a screw-like motion of axis $\mathcal{L}$ and pitch P.

↳ It seems appropriate to call $\mathcal{L}$ the screw axis of the rigid body motion.

$$\text{Pitch, } P = \frac{d_0}{\phi} = \frac{\bar{d}_P^T \bar{e}}{\phi}, \quad \text{or} \quad P = \frac{2\pi d_0}{\phi}$$

{Rotation involved} m/rad m/turn

⇒ The angle ϕ is called <u>amplitude</u> associated with the said motion.

3.2.1) The Screw of a rigid body Motion

⇒ The Screw axis $\mathcal{L}$ is totally Specified by a given point P_0 of $\mathcal{L}$ and a unit vector $\bar{e}$ defining its direction.

{Can be chosen that lying closest to Origin}

⇒ Expression for the position vector P_0, $\bar{P_0}$, in terms of $\bar{a}$ $\bar{a}'$ and $\bar{Q}$ are derived below:-

⇒ If P_0 is defined as the point of $\mathcal{L}$ lying closest to the origin, then obviously $\bar{P_0}$ is $\perp$ to $\bar{e}$.

$$\bar{e}^T \bar{P_0} = 0 \quad \text{—— (1)}$$

⇒ Displacement $\bar{d_0}$ of P_0 (assuming P_0 is attach. to the Body) is $\parallel$ to $\bar{e}$ and hence is identical to $d_{\parallel}$ given by:-

$$\bar{d_{\parallel}} = \bar{e}\,\bar{e}^T \bar{d_p}$$

{$\forall$ arbitrary Point P on the body; $\rightarrow \bar{d_p}$ displacement Vector of Point P}

Where $\bar{d_0}$ is given by

$$\bar{d_0} = \bar{d_A} + (\bar{Q} - \bar{I})(\bar{P_0} - \bar{a})$$

⇒ Now since $\overline{d_0}$ is identical to $\overline{d_{11}}$ we have

$$\overline{d_A} + (\bar{\bar{Q}} - \bar{\bar{I}})(\overline{P_0} - \bar{a}) = \overline{d_{11}} = \bar{e}\bar{e}^T \overline{d_0}$$

⇒ But from Theorem 3.1.1 we have

$$e^T d_0 = e^T d_A$$

So also

$$\overline{d_A} + (\bar{\bar{Q}} - \bar{\bar{I}})(\overline{P_0} - \bar{a}) = \bar{e}\bar{e}^T \overline{d_A}$$

$$\Rightarrow \boxed{(\bar{\bar{Q}} - \bar{\bar{I}})\overline{P_0} = (\bar{\bar{Q}} - \bar{\bar{I}})\bar{a} - (1 - \bar{e}\bar{e}^T)\overline{d_A}} \text{ — (2)}$$

⇒ In order to find an expression for $\overline{P_0}$ then by obtaining:- eq (1) is adjoined to eq (2)

$$\boxed{\bar{\bar{A}}\overline{P_0} = \bar{b}} \text{ — (3)}$$

where,

$$\bar{\bar{A}} = \begin{bmatrix} \bar{\bar{Q}} - \bar{\bar{I}} \\ \bar{e}^T \end{bmatrix}_{4\times 3} \qquad \bar{b} = \begin{bmatrix} (\bar{\bar{Q}} - \bar{\bar{I}})\bar{a} - (\bar{\bar{I}} - \bar{e}\bar{e}^T)\overline{d_A} \\ 0 \end{bmatrix}$$

⇒ Equation (3) cannot be solved for $\overline{P_0}$ directly, because $\bar{\bar{A}}$ is not a square matrix.

↳ Equation represents overdeterminant system of four equation and three unknown.

⇒ In fact if both sides of eq ① are multiplied from the left by A^T, we have:

$$\boxed{\bar{A}^T \bar{A} \bar{P}_0 = \bar{A}^T \bar{b}} \quad \text{—④}$$

⇒ If the product $\bar{A}^T\bar{A}$ which is 3×3 matrix, is invertible, then $\bar{P}_0$ can be computed from eq ④.

⇒ In fact, the said product is not only invertible, but also admits an inverse that is rather simple to derive.

⇒ Now the rotation matrix $\bar{\bar{Q}}$ is recalled in terms of its natural invariants.

$$\boxed{\bar{\bar{Q}} = \bar{e}\bar{e}^T + \cos\phi(\bar{\bar{I}} - \bar{e}\bar{e}^T) + \sin\phi\,\bar{\bar{E}}} \quad \text{—⑤}$$

(cross. Product Matrix of $\bar{e}$)

⇒ further eq ⑤ can be substituted in $\bar{\bar{A}}$ from eq ③ we get:—

$$\boxed{\bar{\bar{A}}^T\bar{\bar{A}} = 2(1-\cos\phi)\bar{\bar{I}} - (1-2\cos\phi)\bar{e}\bar{e}^T} \quad \text{—⑥}$$

Proof

$$\bar{\bar{A}} = \begin{bmatrix} \bar{\bar{Q}} - \bar{\bar{I}} \\ \bar{e}^T \end{bmatrix} \qquad \bar{\bar{A}}^T = [\bar{\bar{Q}}^T - \bar{\bar{I}}^T,\ \bar{e}]$$

$$\bar{\bar{A}}^T\bar{\bar{A}} = (\bar{\bar{Q}}^T - \bar{\bar{I}}^T)(\bar{\bar{Q}} - \bar{\bar{I}}) + \bar{e}\bar{e}^T$$

$$\Rightarrow \left(\bar{Q}^T\bar{Q} - \bar{I}^T\bar{Q} - \bar{Q}^T\bar{I} + \bar{I}^T\bar{I}\right) + ee^T$$

$$\Rightarrow \left(\bar{\bar{I}} - \bar{Q} - \bar{Q}^T + \bar{\bar{I}}\right) + ee^T$$

$\{\text{as } \bar{Q}^{-1} = \bar{Q}^T\}$ $\{\bar{I}^T = \bar{I}\}$

$$\Rightarrow 2\bar{\bar{I}} + ee^T - (\bar{Q} + \bar{Q}^T)$$

$$\Rightarrow 2\bar{\bar{I}} + ee^T - \left(2ee^T + 2\cos\phi\,(\bar{\bar{I}} - ee^T)\right)$$

$$\Rightarrow \boxed{\bar{A}^T\bar{A} = 2(1-\cos\phi)\bar{\bar{I}} - (1-2\cos\phi)\,ee^T}$$

$\Rightarrow$ $\bar{A}^T\bar{A}$ is linear Combination of ee^T and $\bar{\bar{I}}$.

↳ This suggest that its inverse is very likely to be a linear Combination of these two matrices as well

$\Rightarrow$ If this is in fact true, then on can write

$$(\bar{A}^T\bar{A})^{-1} = \alpha\bar{\bar{I}} + \beta ee^T$$

$\Rightarrow$ The Coefficients α and β being determined from the Condition that the product of $\bar{A}^T\bar{A}$ by its inverse should be $\bar{\bar{I}}$, which yields

$$\alpha = \frac{1}{2(1-\cos\phi)} \qquad \beta = \frac{1-2\cos\phi}{2(1-\cos\phi)}$$

Hence,

$$\left(\bar{A}^T\bar{A}\right)^{-1} = \frac{\bar{\bar{I}}}{2(1-\cos\phi)} + \frac{1-2\cos\phi}{2(1-\cos\phi)}\,ee^T$$

⇒ On the other hand,

$$\bar{A}^T \bar{b} = (\bar{\bar{Q}} - \bar{\bar{I}})^T \left[(\bar{\bar{Q}} - \bar{\bar{I}}) \bar{a} - \bar{d}_A \right]$$

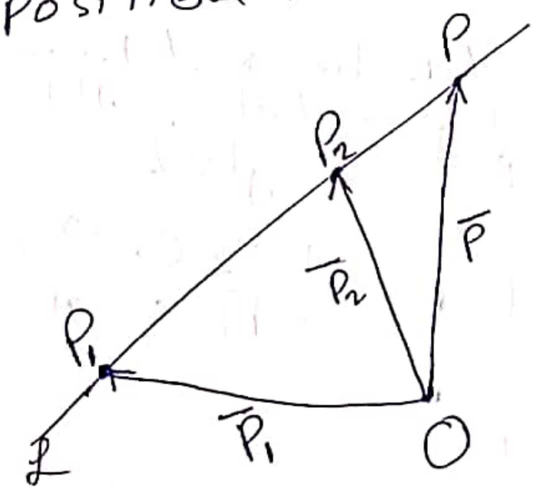
So finaly we get:-

$$\bar{P}_0 = \frac{(\bar{\bar{Q}} - \bar{\bar{I}})^T (\bar{\bar{Q}} \bar{a} - \bar{a}')}{2(1 - \cos\phi)} \quad \forall \phi \neq 0$$

3.2.2> <u>The Plücker Coordinates of a Line</u>

⇒ Alternatively, the Screw axis, and any line for that matter, can be defined more conveniently by its <u>Plucker Coordinates</u>.

⇒ Lets recall the equation of a line $\mathcal{L}$ passing through two points P_1 and P_2 of position vectors $\bar{P}_1$ and $\bar{P}_2$.



$$(\bar{P}_2 - \bar{P}_1) \times (\bar{P} - \bar{P}_1) = 0$$

$$\Rightarrow (\bar{P}_2 - \bar{P}_1) \times \bar{P} + \bar{P}_1 \times (\bar{P}_2 - \bar{P}_1) = 0$$

⇒ If we introduce the Cross-Product matrix $\bar{\bar{P_1}}$ and $\bar{\bar{P_2}}$ of Vector $\bar{P_1}$ & $\bar{P_2}$ then,

$$(\bar{\bar{P_2}} - \bar{\bar{P_1}})\bar{P} + \bar{P_1} \times (\bar{P_2} - \bar{P_1}) = 0$$

$$\Rightarrow \left[\bar{\bar{P_2}} - \bar{\bar{P_1}},\ \bar{P_1} \times (\bar{P_2} - \bar{P_1})\right] \begin{bmatrix} \bar{P} \\ 1 \end{bmatrix} = 0 \quad \text{—— (1)}$$

⇒ ~~...~~ It is apparent that the line is defined Completly by two Vectors, the $\bar{P_2} - \bar{P_1}$ and the Cross product $\bar{P_1} \times (\bar{P_2} - \bar{P_1})$.

↳ We will thus define a 6-dimensional array $\bar{\gamma}_{\mathcal{L}}$ Containing these two Vectors;

$$\bar{\gamma}_{\mathcal{L}} = \begin{bmatrix} \bar{P_2} - \bar{P_1} \\ \bar{P_1} \times (\bar{P_2} - \bar{P_1}) \end{bmatrix} \quad \text{—— (2)}$$

whose Six Scalar entries are the Plucker Coordinates of $\mathcal{L}$.

if $\bar{e} = \dfrac{\bar{P_2} - \bar{P_1}}{\|\bar{P_2} - \bar{P_1}\|}$ & $\bar{n} = \bar{P_1} \times \bar{e}$

$$\text{then, } \bar{\gamma}_L = \|P_2 - P_1\| \begin{bmatrix} \bar{e} \\ \bar{n} \end{bmatrix} \quad \text{—— (3)}$$

⇒ The Six scalar entries of the above array are the normalized Plücker Coordinates.

$\bar{e}$ determines direction of $\mathcal{L}$
$\bar{n}$ determines its location
→ also called moment of $\mathcal{L}$

⇒ The Plücker coordinate thus defined will be thus stored in a plücker array $\bar{K}_L$ in the form

$$\bar{K}_{\mathcal{L}} = \begin{bmatrix} \bar{e} \\ \bar{n} \end{bmatrix} \quad — (4)$$

⇒ However, six components of plücker array are not independent.

$$\bar{e}\cdot\bar{e} = 1 \quad , \quad \bar{n}\cdot\bar{e} = 0 \quad — (5)$$

⇒ Hence, any line $\mathcal{L}$ has only four independent Plucker coordinates.

⇒ The set of Plucker array is array of real number not constituting a vector space.

⇒ Let $\bar{K}_A$ and $\bar{K}_B$ denote the Plücker array of the same line $\mathcal{L}$ when its moment is measured at points A and B, respectively.

⇒ This line passes through a point P of position vector $\bar{P}$ for a particular origin O.

$$n_A \equiv (\bar{P} - \bar{a}) \times \bar{e} \quad n_B = (\bar{P} - \bar{b}) \times \bar{e} \quad — (6)$$

Hence, $\overline{K}_A = \begin{bmatrix} \overline{e} \\ \overline{\eta}_A \end{bmatrix}$ $\quad \overline{K}_B = \begin{bmatrix} \overline{e} \\ \overline{\eta}_B \end{bmatrix}$

Obviously, $\eta_B - \eta_A = (\overline{a}-\overline{b}) \times \overline{e}$ — (6*)

$$\Rightarrow \overline{K}_B = \begin{bmatrix} \overline{e} \\ \overline{\eta}_A + (\overline{a}-\overline{b}) \times \overline{e} \end{bmatrix} \quad — (7)$$

Which can be re-written as

$$\boxed{\overline{K}_B = \overline{\overline{U}}\, \overline{K}_A} \quad — (8)$$

$$\left\{ \overline{\overline{U}} = \begin{bmatrix} 1 & 0 \\ \overline{\overline{A}} - \overline{\overline{B}} & 1 \end{bmatrix} \right.$$

$\left. \begin{matrix} \overline{\overline{A}} = \text{Cross product Matrix of } \overline{a} \\ \overline{\overline{B}} = \text{" \quad " \quad " } \overline{b} \end{matrix} \right\}$

⇒ Given the lower-triangular structure of matrix $\overline{\overline{U}}$, its determinant is simply the product of its diagonal entries, which are all unity; hence

$$\det(\overline{\overline{U}}) = 1 \quad — (9)$$

⇒ $\overline{\overline{U}}$ thus belongs to the Unimodular group of 6×6 matrices. These matrices are rather simple to invert.

$$\overline{\overline{U}}^{-1} = \begin{bmatrix} 1 & 0 \\ \overline{\overline{B}} - \overline{\overline{A}} & 1 \end{bmatrix} \quad — (10)$$

⇒ Equation (8) can then be called the Plücker-coordinate transform formula.

⇒ Multiplying both side of eq (6^A) by $(\bar{a}-\bar{b})^T$

$$(\bar{a}-\bar{b})^T \bar{n}_B = (\bar{a}-\bar{b})^T \bar{n}_A$$

⇒ Hence, the moments of the same line $\mathcal{L}$ with respect to two points are not independent.

↳ They have same component along the line joining the two points.

⇒ A special case of a line of interest in Kinematics is a line at infinity.

$$\bar{K} = \|\bar{n}\| \begin{bmatrix} e/\|n\| \\ n/\|n\| \end{bmatrix}$$

$$\lim_{\|n\| \to \infty} \bar{K} = \left(\lim_{\|n\| \to \infty} \|n\| \right) \begin{bmatrix} \bar{0} \\ \bar{f} \end{bmatrix}$$

$\bar{f} \to \lim_{\|n\| \to \infty} \frac{n}{\|n\|}$

$$\bar{K}_\infty = \begin{bmatrix} \bar{0} \\ \bar{f} \end{bmatrix}$$

⇒ Line at infinity of unit moment $\bar{f}$ can be thought of as being a line lying in a plane ⊥ to the unit vector $\bar{f}$, but otherwise with an indefinite location in the plane, except that it is an infinitely large distance from the origin.

3.2

3.2.3> The Pose of a Rigid Body

⇒ A possible form of describing a general rigid body motion, then is through a Set of 8 real numbers:-

(i) Six Plücker coordinates of Screw axis

(ii) Its pitch

(iii) Its amplitude (ie angle)

⇒ Rigid-body motion is fully described by six independent parameters.

⇒ Alternatively, a rigid body motion can be described by seven dependent parameters as follows.

(i) four invariants of the Concomitant Rotation
- Natural Invariants
- Linear Invariants
- Euler-Rodrigues parameters.

(ii) Three Components of the displacement of a arbitrary point.

⇒ Let a rigid body undergo a general motion of rotation $\bar{\bar{Q}}$ and displacement $\bar{d}$ from a reference Configuration C_0.

⇒ If in the new Configuration C a landmark Point A of the body has a position vector $\bar{a}$.

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⇒ Then the pose array or simply the pose $\bar{S}$ of the body, is defined as a 7D array namely

$$\bar{S} \equiv \begin{bmatrix} \bar{q} \\ q_0 \\ \bar{d}_A \end{bmatrix}$$

$\bar{q}, q_0$ are four invariants of $\bar{\bar{Q}}$

$\bar{d}_A$ displacem of point $\bar{A}$

⇒ An important problem in Kinematics is the Computation of the $\underbrace{\text{Screw parameters}}_{\{\text{Components of } \bar{S}\}}$

↳ from coordinate measurement over a certain finite set of points.

⇒ This problem is known as Pose estimation.

{Position and Orientation}

3.3) Rotation of a Rigid body about a fixed Point

⇒ This motion is fully described by a rotation matrix $\bar{\bar{Q}}$ that is proper orthogonal.

⇒ The position vector of point P in an original configuration, denoted here by $\bar{P}_0$, is mapped smoothly into a new vector $\bar{P}(t)$, namely

$$\bar{P}(t) = \bar{\bar{Q}}(t)\,\bar{P}_0 \quad \text{——} \; (1)$$

⇒ The Velocity of P is computed by differentiating both side of eq (1)

$$\dot{\bar{P}}(t) = \dot{\bar{\bar{Q}}}(t)\,\bar{P}_0 \quad \text{——} \; (2)$$

→ Not very useful expression, because it requires knowledge of the original Position of P.

$$\Rightarrow \quad \boxed{\dot{\bar{P}} = \dot{\bar{\bar{Q}}}\,\bar{\bar{Q}}^T\,\bar{P}} \quad \text{——} \; (3)$$

⇒ The product $\dot{\bar{\bar{Q}}}\bar{\bar{Q}}^T$ is known as the angular-Velocity matrix of the rigid-body motion and is denoted by $\bar{\bar{\Omega}}$.

$$\bar{\bar{\Omega}} = \dot{\bar{\bar{Q}}}\,\bar{\bar{Q}}^T$$

Theorem 3.3.1: The angular-Velocity matrix is a Skew Symmetric matrix.

angular Velocity Vector $\quad \bar{\omega} = \mathrm{Vec}(\bar{\bar{\Omega}})$

⇒ Hence eq ⑤ can be written as:-

$$\dot{\bar{P}} = \bar{\Omega}\bar{P} = \bar{\omega} \times \bar{P}$$

⇒ So, Velocity of any point P of a body moving with a point O fixed is perpendicular to line OP.

3.4) General Instantaneous Motion of a Rigid body

⇒ If a rigid body now undergoes the most general motion, none of its points remains fixed, and the position vector of any of these P in a displaced configuration is given by:-

$$\bar{P}(t) = \bar{a}(t) + \bar{\bar{Q}}(t)\,(\bar{P}_0 - \bar{a}_0) \quad — ①$$

⇒ $\bar{a}_0$, $\bar{P}_0$ are position vectors of point A and P in reference configuration C_0

⇒ $\bar{a}(t)$ and $\bar{P}(t)$ being the position vector of the same point in displaced configuration C.

⇒ $\bar{\bar{Q}}(t)$ denotes rotation matrix

⇒ Now the Velocity of P is computed by differentiating both side of eq ①

$$\dot{\bar{P}}(t) = \dot{\bar{a}}(t) + \dot{\bar{\bar{Q}}}(t)\,(\bar{P}_0 - \bar{a}_0) \quad — ②$$

$$\Rightarrow \boxed{\dot{\bar{P}}(t) = \dot{\bar{a}}(t) + \bar{\Omega}\,(\bar{P}(t) - \bar{a}(t))} \quad \text{--- (3)}$$

or

$$\boxed{\dot{\bar{P}} = \dot{\bar{a}} + \bar{\omega} \times (\bar{P} - \bar{a})} \quad \text{--- (4)}$$

Theorem 3.4.1: The relative velocity of two Points of the Same rigid body is ⊥ to the line joining them.

Corollary 3.4.1: The Projections of all velocities of all the points of a rigid body onto the angular-velocity Vector are identical.

Theorem 3.4.2: Given a rigid body under general motion, a set of its points located on a line $\mathcal{L}'$ undergoes the identical minimum-magnitude velocity $\bar{V}_0$ parallel to the angular velocity.

Defination 3.4.1: The line Containing the points of a rigid body undergoing minimum-magnitude velocities is Called the instant Screw axis (ISA) of the body under the given motion.

3.4.1> The Instant Screw of a Rigid Body Motion

⇒ The instantaneous motion of a body is equivalent to that of the bolt of a Screw of axis $\mathcal{L}'$, the ISA.

⇒ As the body moves the ISA changes, and the motion of the body is called instantaneous Screw.

⇒ Since $\overline{V_0}$ is $\parallel$ to $\overline{\omega}$, it can be written in the form

$$\overline{V_0} = v_0 \frac{\overline{\omega}}{\|\overline{\omega}\|}$$

{where V_0 is scalar denoting the signed magnitude of $\overline{V_0}$ and bears sign of $\overline{V_0}\cdot\overline{\omega}$}

⇒ Further more the pitch of the instantaneous Screw p' is defined as

$$p' \equiv \frac{V_0}{\|\overline{\omega}\|} \equiv \frac{\dot{\overline{p}}\cdot\overline{\omega}}{\|\overline{\omega}\|^2} \quad \text{or} \quad p' \equiv : \frac{2\pi V_0}{\|\overline{\omega}\|}$$

(m/rad) (m/turn)

⇒ Again ISA $\mathcal{L}'$ can be specified uniquely through its plücker coordinates, stored in $\overline{P}_{\mathcal{L}'}$ array:-

$$\overline{P}_{\mathcal{L}'} = \begin{bmatrix} \overline{e}' \\ \overline{n}' \end{bmatrix} \begin{matrix} \rightarrow \text{direction of } \mathcal{L}' \\ \rightarrow \text{its moment about origin.} \end{matrix}$$

$$\overline{e}' = \frac{\overline{\omega}}{\|\overline{\omega}\|} \qquad n' = \overline{P} \times \overline{e}'.$$

{Position vector of any point of the ISA}

⇒ $\overline{e}'$ is defined uniquely but become trivial when the rigid body instantaneously undergoes a pure translation

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↳ In this case $\hat{e}'$ is defined as the unit vector ∥ to the associated displacement field.

⇒ Instantaneous rigid body motion is defined by a line $\mathcal{L}'$, a pitch p' and an amplitude $\|\bar{\omega}\|$

↳ Such a motion is, then fully determined by six independent parameters:-

↳ 4 Independent Plücker Coordinate of $\mathcal{L}'$

↳ Its pitch

↳ Its amplitude.

⇒ A line supplied with a pitch ⇒ Screw

⇒ Screw supplied with amplitude ⇒ Twist.

⇒ The ISA can be alternatively described in terms of the position vector $\bar{P}_0'$. → {Its point lying closest to the origin}

⇒ Expression for $\bar{P}_0'$ in terms of the position and the velocity of a arbitrary body-point and the angular velocity are derived below.

⇒ We decompose $\dot{\bar{P}}$ into two orthogonal components, $\dot{\bar{P}}_{\parallel}$ & $\dot{\bar{P}}_{\perp}$ along and transverse respectively to the angular-velocity vectors

⇒ $\dot{\bar{a}}$ is first decomposed into two such orthogonal components $\dot{\bar{a}}_{\parallel}$ and $\dot{\bar{a}}_{\perp}$

$$\dot{\vec{a}} = \dot{\vec{a}}_{\parallel} + \dot{\vec{a}}_{\perp}$$

$$\dot{\vec{a}}_{\parallel} = \left(\dot{\vec{a}} \cdot \frac{\bar{\omega}}{\|\bar{\omega}\|}\right)\frac{\bar{\omega}}{\|\bar{\omega}\|} = (\dot{\vec{a}} \cdot \bar{\omega})\frac{\bar{\omega}}{\|\bar{\omega}\|^2} = \frac{\bar{\omega}\,\bar{\omega}^T\dot{\vec{a}}}{\|\bar{\omega}\|^2}$$

$$\dot{\vec{a}}_{\perp} = \left(\bar{\bar{I}} - \frac{\bar{\omega}\bar{\omega}^T}{\|\bar{\omega}\|}\right)\dot{\vec{a}} = -\frac{1}{\|\bar{\omega}\|^2}\bar{\bar{\Omega}}^2\dot{\vec{a}}$$

$$\left\{\text{as } \Omega^2 = \bar{\omega}\bar{\omega}^T - \|\bar{\omega}\|^2\bar{\bar{I}}\right\}$$

$$\dot{\vec{P}} = \underbrace{\frac{\bar{\omega}\bar{\omega}^T}{\|\bar{\omega}\|^2}\dot{\vec{a}}}_{\dot{P}_{\parallel}} \underbrace{- \frac{1}{\|\bar{\omega}\|^2}\bar{\bar{\Omega}}^2\dot{\vec{a}} + \bar{\bar{\Omega}}(\bar{P} - \bar{a})}_{\dot{P}_{\perp}} \quad \text{—— (1)}$$

⇒ Vanishing of the normal component obviously implies the minimization of the magnitude of $\dot{\vec{P}}$.

$$\dot{P}_{\perp} = 0$$

$$\Rightarrow \bar{\bar{\Omega}}(\bar{P} - \bar{a}) - \frac{1}{\|\omega\|}\bar{\bar{\Omega}}^2\dot{\vec{a}} = 0$$

$$\Rightarrow \bar{\bar{\Omega}}\,\bar{P} = \bar{\bar{\Omega}}\left(a + \frac{1}{\|\omega\|^2}\bar{\bar{\Omega}}\dot{\vec{a}}\right)$$

$$\Rightarrow \bar{\bar{\Omega}}(\bar{P} - \bar{\sigma}) = 0 \quad \text{—— (2)}$$

$$\left\{\text{where, } \bar{\sigma} = \bar{a} + \frac{1}{\|\bar{\omega}\|^2}\Omega\dot{\vec{a}}\right\}$$

⇒ A possible solution of the foregoing problem is

$$\bar{P} = \bar{\pi} = \bar{a} + \frac{1}{\|\bar{\omega}\|^2}\bar{\Omega}\dot{\bar{a}} \quad \text{— (3)}$$

⇒ This solution is not unique.

↳ eq (2) does not require that $\bar{P} - \bar{\pi}$ be zero, only that the difference lie in the null space of $\bar{\Omega}$.

↓

$\bar{P} - \bar{\pi}$ is linearly dependent with $\bar{\omega}$.

⇒ In other words if a vector $\alpha\bar{\omega}$ is added to $\bar{P}$ then the sum also satisfy the equation.

1

⇒ Let $\bar{P}_0'$ be position vector of point of ISA lying closest to origin:

↳ This vector is obviously perpendicular to $\bar{\omega}$

$$\bar{\omega}^T \bar{P}_0' = 0 \quad \text{— (4)}$$

⇒ eq (2) in conjunction with (4) can be written as:-

$$\boxed{\bar{A}\,\bar{P}_0' = \bar{b}} \quad \text{— (5)}$$

where,

$$A = \begin{bmatrix} \bar{\Omega} \\ \bar{\omega}^T \end{bmatrix}_{4\times 3} \qquad \bar{b} = \begin{bmatrix} \bar{\Omega}\bar{a} + \frac{1}{\|\bar{\omega}\|^2}\bar{\Omega}^2\dot{\bar{a}} \\ 0 \end{bmatrix}_{4\times 1}$$

⇒ Multiplying both side by A^T

$$\bar{\bar{A}}^T \bar{\bar{A}}\, \bar{P}_0' = \bar{\bar{A}}^T \bar{b}$$

Where $A^T A = \Omega^T \Omega + \omega \omega^T = -\Omega^2 + \omega\omega^T$

⇒ $\bar{\bar{A}}^T \bar{\bar{A}} = \|\bar{\omega}\|^2 \bar{\bar{I}}$ $\quad \bar{\bar{A}}^T \bar{b} = \bar{\bar{\Omega}}(\dot{\bar{a}} - \bar{\bar{\Omega}}\bar{a})$

So $\boxed{\bar{P}_0' = \dfrac{\bar{\bar{\Omega}}(\dot{\bar{a}} - \bar{\bar{\Omega}}\bar{a})}{\|\omega\|^2} = \dfrac{\bar{\omega} \times (\dot{\bar{a}} - \bar{\omega} \times \bar{a})}{\|\omega\|^2}}$

3.4.2) The twist of a rigid body

⇒ A line, as we saw earlier, is fully defined by its 6-dimensional Plücker array, which contains only four independent components.

⇒ Now if a pitch is added as a fifth feature to the line or correspondingle to its plücker array we obtain a screw $\bar{S}$.

$$\bar{S} = \begin{bmatrix} \bar{e} \\ \bar{P} \times \bar{e} + p\bar{e} \end{bmatrix} \quad \text{—(1)}$$

⇒ An amplitude is any Scalar A multiplying the foregoing Screw.

↳ The amplitude produces a twist or a wrench.

⇒ The twist or the wrench thus defined can be regarded as an eight-parameter array.

⇒ Twist can be regarded as a 6-dimensional array defining completely the velocity field of a rigid body, and it comprises :-

(i) Three Components of angular Velocity

(ii) Three Components of Velocity

of any of the points of the body.

$$\text{twist}, \quad \bar{t} = \begin{bmatrix} \bar{A}\bar{e} \\ \bar{p} \times (A\bar{e}) + p(A\bar{e}) \end{bmatrix} \quad \text{—} ②$$

⇒ $A\bar{e}$ can be readily identified as the angular velocity $\bar{\omega}$ parallel to vector $\bar{e}$, of magnitude A.

⇒ If we regard the line $\mathcal{L}$ and point O as set of ~~point P moves~~ a rigid body B moving with an angular velocity $\bar{\omega}$ and such that point P moves with velocity $p\bar{\omega}$ ∥ to the angular velocity, then the lower vector of $\bar{t}$ denoted by $\bar{v}$, represents velocity of point O, i.e.

$$\bar{V} = -\bar{\omega} \times \bar{P} + p\bar{\omega}$$

We can thus express the twist $\bar{t}$ as

$$\bar{t} = \begin{bmatrix} \bar{\omega} \\ \bar{v} \end{bmatrix} \quad \text{—③}$$

⇒ A special case of great interest in kinematics is the screw of infinitely large pitch.

$$\lim_{p\to\infty} \begin{bmatrix} \bar{e} \\ \bar{P}\times\bar{e} + p\bar{e} \end{bmatrix} = \lim_{p\to\infty} p \begin{bmatrix} e/p \\ (\bar{P}\times\bar{e})/p + \bar{e} \end{bmatrix}$$

$$= \left(\lim_{p\to\infty} p\right) \underbrace{\begin{bmatrix} \bar{0} \\ \bar{e} \end{bmatrix}}_{\bar{S}_\infty}$$

{Screw of infinite pitch}

⇒ The twist array, as defined in eq ③, with $\bar{\omega}$ on top, represents the ray coordinates of the twist.

↳ An exchange of the order of the two Cartesian vectors of this array, in turn, gives rise to the <u>axis coordinates of the twist</u>.

⇒ The relationships between the angular-velocity vector and the time derivatives of the invariants of the associated rotation are linear.

⇒ Let the three sets of four invariants of rotation, namely, the natural invariants, the linear invariants and the Euler-Rodrigues parameters be grouped in the 4 dimensional array $\bar{V}$ $\bar{\lambda}$ $\bar{\eta}$ respectively.

$$\bar{V} = \begin{bmatrix} \bar{e} \\ \phi \end{bmatrix} \quad \bar{\lambda} = \begin{bmatrix} (\sin\phi)\bar{e} \\ \cos\phi \end{bmatrix} \quad \bar{\eta} = \begin{bmatrix} (\sin \phi/2)\,\bar{e} \\ \cos \phi/2 \end{bmatrix}$$

$$\dot{\bar{V}} = \bar{\bar{N}}\,\bar{\omega} \quad ; \quad \dot{\bar{\lambda}} = \bar{\bar{L}}\,\bar{\omega} \quad \dot{\bar{\eta}} = \bar{\bar{H}}\,\bar{\omega}$$

With

$$\bar{\bar{N}} = \begin{bmatrix} [\sin\phi/(2(1-\cos\phi))](1-\bar{e}\bar{e}^T) - \frac{1}{2}\bar{\bar{E}} \\ \bar{e}^T \end{bmatrix}$$

$$\bar{\bar{L}} = \begin{bmatrix} \frac{1}{2}[\operatorname{tr}(\bar{Q})\bar{\bar{I}} - \bar{Q}] \\ -(\sin\phi)\,\bar{e}^T \end{bmatrix}$$

$$\bar{\bar{H}} = \frac{1}{2}\begin{bmatrix} \cos(\phi/2)\bar{\bar{I}} - \sin(\phi/2)\bar{\bar{E}} \\ -\sin(\phi/2)\,\bar{e}^T \end{bmatrix}$$

⇒ The inverse relations:-

$$\cancel{\bar{\omega} = \tilde{\bar{N}}\dot{\bar{V}} = \tilde{\bar{L}}\dot{\bar{\lambda}} = \tilde{\bar{H}}\dot{\bar{\eta}}}$$

$$\boxed{\bar{\omega} = \tilde{\bar{N}}\dot{\bar{V}} = \tilde{\bar{L}}\dot{\bar{\lambda}} = \tilde{\bar{H}}\dot{\bar{\eta}}}$$

Where,

$$\tilde{\bar{\bar{N}}} = \left[\sin\phi \, \bar{\bar{I}} + (1-\cos\phi)\bar{\bar{E}} \quad \bar{e} \right]$$

$$\tilde{\bar{\bar{L}}} = \left[\bar{\bar{I}} + \frac{\sin}{1+\cos\phi}\bar{\bar{E}} \quad - \frac{\sin\phi}{1+\cos\phi}\bar{e} \right]$$

$$\tilde{\bar{\bar{H}}} = 2\left[\cos\frac{\phi}{2}\bar{\bar{I}} + \sin\frac{\phi}{2}\bar{\bar{E}} \quad -\sin\phi/_2 \, \bar{e} \right]$$

⇒ Now we can write the relationship between the twist and the time-rate of change of the 7-dimensional pose array namely

$$\boxed{\dot{\bar{S}} = \bar{\bar{T}} \, t_i}$$

Where, $\bar{\bar{T}} \equiv \begin{bmatrix} \bar{\bar{F}} & \bar{\bar{O}}_{43} \\ \bar{\bar{O}} & \bar{\bar{I}} \end{bmatrix}$

$\bar{\bar{O}}$ = 3×3 ~~Identity~~ zero matrix

$\bar{\bar{O}}_{43}$ = 4×3 zero matrix

$\bar{\bar{I}}$ = 3×3 Identity matrix

$\bar{\bar{F}} = \bar{\bar{N}}, \bar{\bar{L}}$ or $\bar{\bar{H}}$ depending upon the invariant representation chosen for the rotation

⇒ The inverse relationship takes form:

$$\bar{t} = \bar{\bar{S}}\,\dot{\bar{s}}$$

$$\text{when } \bar{\bar{S}} = \begin{bmatrix} \bar{\bar{E}} & \bar{\bar{O}} \\ \bar{\bar{O}}_{34} & \bar{\bar{I}} \end{bmatrix}$$

⇒ A formula that relates the twist of the same rigid body at two different points is now derived.

⇒ Let A and P be two arbitrary points of a rigid body.

↳ The twist at each of these points is defined as:-

$$\bar{t}_A = \begin{bmatrix} \bar{\omega} \\ \bar{V}_A \end{bmatrix}, \quad \bar{t}_P = \begin{bmatrix} \bar{\omega} \\ \bar{V}_P \end{bmatrix}$$

$$\bar{V}_P = \bar{V}_A + (\bar{a} - \bar{P}) \times \bar{\omega}$$

⇒ Combining above eqution we get

$$\bar{t}_P = \bar{\bar{U}}\,\bar{t}_A \quad \left\{ \text{where } \bar{\bar{U}} = \begin{bmatrix} \bar{\bar{I}} & \bar{\bar{O}} \\ \bar{\bar{A}} - \bar{\bar{P}} & \bar{\bar{I}} \end{bmatrix} \right\}$$

{Cross product matrix of vector $\bar{a}$ and $\bar{P}$}

tion This is Called the Twist transfer formula.

3.5) Acceleration analysis of Rigid body Motions

$$\dot{\bar{p}} = \dot{\bar{a}} + \bar{\bar{\Omega}}(\bar{p}-\bar{a})$$

$\Rightarrow$ upon differentiating both side with time:-

$$\Rightarrow \ddot{\bar{p}} = \ddot{\bar{a}} + \dot{\bar{\bar{\Omega}}}(\bar{p}-\bar{a}) + \bar{\bar{\Omega}}(\dot{\bar{p}}-\dot{\bar{a}})$$

$$\Rightarrow \ddot{\bar{p}} = \ddot{\bar{a}} + (\dot{\bar{\bar{\Omega}}} + \bar{\bar{\Omega}}^2)(\bar{p}-\bar{a})$$

$$\ddot{\bar{p}} = \ddot{\bar{a}} + \bar{\bar{W}}(\bar{p}-\bar{a}) \quad \text{--- (1)}$$

Where, $\bar{\bar{W}} = \dot{\bar{\bar{\Omega}}} + \Omega^2$ is termed as angular acceleration matrix of the rigid body ~~both~~ motion.

$\Rightarrow$ Clearly first term is Skew-Symmetric where as second one is Symmetric.

So $\text{Vect}(\bar{\bar{W}}) = \text{Vect}(\dot{\bar{\bar{\Omega}}}) = \dot{\bar{\omega}}$

{Termed as angular acceleration vector}

$\Rightarrow$ Eq (1) Can be rewritten as:-

$$\ddot{\bar{p}} = \ddot{\bar{a}} + \dot{\bar{\omega}} \times (\bar{p}-a) + \bar{\omega} \times [\bar{\omega} \times (\bar{p}-\bar{a})] \quad \text{--- (2)}$$

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⇒ On the other hand, the time derivative of t, referred as twist rate is:-

$$\dot{t} = \begin{bmatrix} \dot{\bar{\omega}} \\ \dot{\bar{v}} \end{bmatrix} \longrightarrow \text{③}$$

$\dot{\bar{v}}$ → acceleration of a point of the body.

⇒ The relationship between the twist rate and the second time derivative of the screw is derived by differentiating

$$\dot{S} = \bar{\bar{T}} t$$

$$\ddot{S} = \bar{\bar{T}} \dot{t} + \dot{\bar{\bar{T}}} t$$

$$\text{where, } \dot{\bar{\bar{T}}} = \begin{bmatrix} \dot{\bar{\bar{F}}} & \bar{\bar{O}}_{43} \\ \bar{\bar{O}} & \bar{\bar{O}} \end{bmatrix}$$

Similarly, inverse relationship:-

$$\dot{t} = \bar{\bar{S}} \ddot{s} + \dot{\bar{\bar{S}}} \dot{s}$$

$$\text{where } \dot{\bar{\bar{S}}} = \begin{bmatrix} \dot{\bar{\bar{F}}} & \bar{\bar{O}} \\ \bar{\bar{O}}_{34} & \bar{\bar{O}} \end{bmatrix}$$

3.6) Rigid-Body Motion referred to Moving Coordinate Axis

⇒ Although in Kinematics no "preferred" coordinate system exists.

↳ In dynamics the governing equation of rigid-body motion are valid only in inertial frames.

⇒ Consider the fixed coordinate frame X Y Z which we lable F, and the moving coordinate frame x y and z which will be labeled M.

⇒ Let $\bar{\bar{Q}}$ be the rotation matrix taking from F into the orientation M.

⇒ Let $\bar{O}$ be the position vector of the origin of M from the origin of F.

⇒ Further let $\bar{P}$ be the position vector of point P from the origin of F and $\bar{\rho}$ be position vector of the same point from the origin of M.

$$[\bar{P}]_F = [\bar{O}]_F + [\bar{\rho}]_F$$

We assume $\bar{\rho}$ is not available in frame F but in M

So $[\bar{\rho}]_F = [\bar{\bar{Q}}]_F [\bar{\rho}]_M$

So $[\bar{P}]_F = [\bar{O}]_F + [\bar{\bar{Q}}]_F [\bar{\rho}]_M$

⇒ Now in order to compute the velocity of $\bar{P}$, both side are differentiated.

$$[\dot{\bar{P}}]_F = [\dot{\bar{O}}]_F + [\dot{\bar{\bar{Q}}}]_F [\bar{\rho}]_M + [\bar{\bar{Q}}]_F [\dot{\bar{\rho}}]_M$$

$$[\dot{\bar{Q}}]_F = [\Omega]_F [\bar{Q}]_F$$

So

$$\boxed{[\dot{\bar{P}}]_F = [\dot{\bar{O}}]_F + [\bar{\Omega}]_F [\bar{Q}]_F [\bar{\rho}]_M + [\bar{Q}]_F [\dot{\bar{\rho}}]_M}$$

⇒ The above expression gives Velocity of P in F in terms of Velocity of P in M and the twist of M with respect to F.

⇒ Similarly for acceleration :-

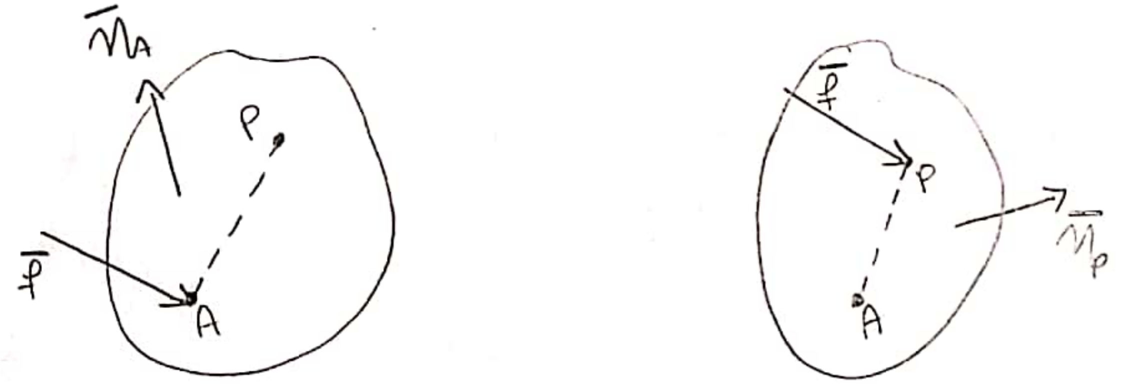
$$\boxed{\begin{aligned}[\ddot{\bar{P}}]_F = \underbrace{[\ddot{\bar{O}}]_F + \left([\dot{\bar{\Omega}}]_F + [\bar{\Omega}^2]_F\right) [\bar{Q}]_F [\bar{\rho}]_M}_{\text{Acceleration of P as point wrt M}} \\ + \underbrace{2 [\bar{\Omega}]_F [\bar{Q}]_F [\dot{\bar{\rho}}]_M}_{\text{Coriolis acceleration}} + \underbrace{[\bar{Q}]_F [\ddot{\bar{\rho}}]_M}_{\text{Acceleration of P Measured from M}}\end{aligned}}$$

3.7) Static Analysis of Rigid Body

⇒ From elementary statics it is known that resultant of all external forces & moments, exerted on a body can be reduced to a force $\overline{f}$ acting at a point, say A and a moment $\overline{M}_A$.

↳ Alternatively, the aforementioned force f can be defined as acting at an arbitrary point P of the body but then the resultant moment $\overline{M}_P$ changes corresponding.

⇒ In order to establish a relationship between $\overline{M}_A$ and $\overline{M}_P$, the moment of the first system of forces and moment with respect to point P is equated to the moment about the same point of the second system thus obtaining:-



(a) (b)

$$\overline{M}_P = \overline{M}_A + (\overline{a}-\overline{P}) \times \overline{f}$$

$$\Rightarrow \overline{M}_P = \overline{M}_A + \overline{f} \times (\overline{P}-\overline{a})$$

Theorem 3.7.1: For a given System of forces & moment acting on a rigid body, if the resultant force is applied at any point of a particular line $\mathcal{L}''$, then the resultant moment is of minimum magnitude. Moreover that minimum magnitude moment is parallel to the resultant force.

$\Rightarrow$ Let $\overline{M}_0$ be the minimum magnitude moment.

$$\overline{M}_0 = M_0 \frac{\overline{f}}{\|f\|}, \quad M_0 = \frac{\overline{M}_P \cdot f}{\|f\|}$$

Moreover, the pitch of the wrench P'' is defined as:-

$$P'' = \frac{M_0}{\|f\|} = \frac{\overline{M}_P \cdot \overline{f}}{\|\overline{f}\|^2} \quad \text{or} \quad P'' = \frac{2\pi \overline{M}_P \cdot \overline{f}}{\|\overline{f}\|^2}$$

$\Rightarrow$ Of Course the wrench axis can be defined by its Plücker array $\overline{P}_{L''}$

$$\overline{P}_{L''} = \begin{bmatrix} \overline{e}'' \\ \overline{M}'' \end{bmatrix}, \quad \overline{e}'' = \frac{\overline{f}}{\|\overline{f}\|}, \quad \overline{M}'' = \overline{P} \times \overline{e}''$$

{Unit vector ∥ to $\mathcal{L}''$}

{Position vector of any point of $\mathcal{L}''$}

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⇒ Let P_0'' be point on L'' lying closest to origin with position vector $\bar{P}_0''$.

$$\boxed{\bar{P}_0'' = \frac{1}{\|f\|^2}\bar{f}\times(\bar{n}_A - \bar{f}\times\bar{a})}$$

Theorem 3.7.2: The projection of the resultant moment of a system of moments and forces acting on a rigid body that arises when the resultant force is applied to an arbitrary point of the body onto the wrench axis is constant.

⇒ Upon the multiplication of the screw by an amplitude A with the unit of force, what we will obtain would be a wrench $\bar{W}$.

- first 3 component Unit of force
- last 3 Component unit of Moment

⇒ We would like to be able to obtain the power developed by the wrench on the body moving with the twist $\bar{t}$ by a simple inner product of two array.

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⇒ This can be done ~~as~~ if we redefine wrench not simply as the product of a screw by an amplitude, but as a linear transformation of that screw involving the 6×6 array $\overline{\overline{\Gamma}}$ defined as:-

$$\overline{\overline{\Gamma}} = \begin{bmatrix} \overline{\overline{O}} & \overline{\overline{I}} \\ \overline{\overline{I}} & \overline{\overline{O}} \end{bmatrix}$$

$\overline{\overline{I}}$ → 3×3 Identity Matrix

$\overline{\overline{O}}$ → 3×3 Zero Matrix

⇒ Now we define the wrench as a linear transformation of the screw S defined as:-

$$\overline{W} = \bar{A}\overline{\overline{\Gamma}}\,\overline{S} = \begin{bmatrix} \overline{P}\times(A\overline{e}) + p(A\overline{e}) \\ A\overline{e} \end{bmatrix}$$

A ⇒ Unit of force

First 3 components of the foregoing array can be readily identified as the moment of force of magnitude A acting along a line of action, with respect to a point P, to which moment ∥ to that line and of magnitude pA is added.

Last 3 components of that array is a force of magnitude A and ∥ to the same line.

⇒ So, Wrench has been defined so that the inner product $\bar{t}^T \bar{w}$ will product the Power Π.

$$\boxed{\Pi = \bar{t}^T \bar{w}}$$

⇒ Let the wrench and the twist be given in terms of their respective screws $\bar{S}_w$ and $\bar{S}_t$ as

$$\bar{w} = W \bar{\bar{\Gamma}} \bar{S}_w \qquad \bar{t} = T \bar{S}_t$$

{ where W and T are the amplitudes of the wrench and the twist respectively }

⇒ We say wrench and the twist are reciprocal to each other if zero power is developed.

$$(\bar{\bar{\Gamma}} \bar{S}_w)^T \bar{S}_t = \bar{S}_w^T \bar{\bar{\Gamma}}^T \bar{S}_t = 0$$

$$\Rightarrow \bar{S}_w^T \bar{\bar{\Gamma}} \bar{S}_t = 0 \quad \text{or} \quad \bar{S}_t^T \bar{\bar{\Gamma}} \bar{S}_w = 0$$

{as $\bar{\bar{\Gamma}}$ is symmetric}

⇒ Now if A and P are arbitrary points of a rigid body, we define the wrench at these points as

$$\overline{W}_A = \begin{bmatrix} \overline{n}_A \\ \overline{f} \end{bmatrix} \qquad \overline{W}_P = \begin{bmatrix} \overline{n}_P \\ \overline{f} \end{bmatrix}$$

So $\boxed{\overline{W}_P = \overline{\overline{V}}\,\overline{W}_A}$ {wrench-transfer formula}

where $\overline{\overline{V}} = \begin{bmatrix} \overline{\overline{I}} & \overline{\overline{A}}-\overline{\overline{P}} \\ \overline{\overline{0}} & \overline{\overline{I}} \end{bmatrix}$

{ $\overline{\overline{A}}$ & $\overline{\overline{P}}$ are cross product matrix of $\overline{a}$ & $\overline{p}$ }

3.8) Dynamics of Rigid Body

⇒ If a rigid body has a mass density ρ then its mass m is defined as:-

$$m = \int_B \rho \, dB \quad \text{——— (1)}$$

where B denotes the region of the 3D space occupied by the body.

⇒ If $\overline{p}$ denotes position vector of an arbitrary point of the body, from a previously defined origin O, the mass first moment of the body with respect to O, $\overline{q}_o$ is defined as:-

$$\overline{q}_o = \int_B \rho \, \overline{p} \, dB \quad \text{——— (2)}$$

⇒ Furthermore, the mass second moment of the body with respect to O is defined as:-

$$\bar{\bar{I}}_O = \int_B \rho \left[(\bar{p}\cdot\bar{p})\bar{\bar{I}} - \bar{p}\bar{p}^T \right] dB \quad — (3)$$

$\downarrow$

It is clearly a <u>Symmetric matrix</u> and is called the <u>moment of Inertia matrix</u> of the body under study with respect to O.

<u>Theorem 3.8.1</u>: The moment of inertia of a rigid body with respect to a point O is positive definite.

<u>Proof</u>: for any vector $\bar{\omega}$, the quadratic form

$$\bar{\omega}^T \bar{\bar{I}}_O \bar{\omega} > 0$$

$$\bar{\omega}^T \bar{\bar{I}}_O \bar{\omega} = \int_B \rho \left[\|\bar{p}\|^2 \|\bar{\omega}\|^2 - (\bar{p}\cdot\bar{\omega})^2 \right] dB$$

Now $\bar{p}\cdot\bar{\omega} = \|\bar{p}\| \, \|\bar{\omega}\| \cos(\bar{p},\bar{\omega})$

So, $$\bar{\omega}^T \bar{\bar{I}}_O \bar{\omega} = \int_B \rho \left[\|\bar{p}\|^2 \|\bar{\omega}\|^2 - \|\bar{p}\|^2 \|\bar{\omega}\|^2 \cos^2(\bar{p},\bar{\omega}) \right] dB$$

$$\Rightarrow \int_B \rho \|\bar{P}\|^2 \|\bar{\omega}\|^2 \sin^2(\bar{P},\bar{\omega})\, dB$$

⇒ which is positive Quatity, that vanishes only in the ideal case of <u>Slender body</u> having all its mass concentrated along a line passing through O and $\parallel$ to $\bar{\omega}$.

⇒ If Vector $\bar{\omega}$ of the porevious discussion is the angular valocity of the rigid body, then the quadratic form turns out to be twice the Kinitic energy of the body.

$$T = \int_B \frac{1}{2} \rho \|\dot{\bar{P}}\|^2 dB$$

→ for this purpose it is assumed that point O is instantaneously at rest.

$$\dot{\bar{P}} = \bar{\omega} \times \bar{P} = -\bar{\bar{P}}\bar{\omega}$$

{Cross product matrix of $\bar{P}$}

$$\|\dot{\bar{P}}\|^2 = (\bar{\bar{P}}\bar{\omega})^T(\bar{\bar{P}}\bar{\omega}) = \bar{\omega}^T \bar{\bar{P}}^T \bar{\bar{P}} \bar{\omega} = -\bar{\omega}^T \bar{\bar{P}}^2 \bar{\omega}$$

we know $\bar{\bar{P}}^2 = \bar{P}\bar{P}^T - \|\bar{P}\|^2 I$

$$\text{so } \|\dot{\bar{P}}\|^2 = \bar{\omega}^T \left(\|\bar{P}\|^2 \bar{\bar{I}} - \bar{P}\bar{P}^T\right)\bar{\omega}$$

$$T = \frac{1}{2}\int_B \rho\, \bar{\omega}^T \left(\|\bar{P}\|^2 \bar{\bar{I}} - \bar{P}\bar{P}^T\right)\bar{\omega}\, dB$$

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$$\boxed{T = \frac{1}{2}\bar{\omega}^T \bar{\bar{I}}_0 \bar{\omega}} \quad \text{—— (4)}$$

⇒ The mass center of a rigid body, measured from O, is defined as a point C, not necessarily within the body, of position vector $\bar{c}$ given by:

$$\boxed{\bar{c} = \frac{\bar{q}_0}{m}} \quad \text{—— (5)}$$

⇒ Mass moment of inertia of a body with respect to its centroid is defined as:-

$$\boxed{\bar{\bar{I}}_c = \int_B \rho \left[\|\bar{r}\|^2 \bar{\bar{I}} - \bar{r}\bar{r}^T \right] dB} \quad \text{—— (6)}$$

where r is defined as $\boxed{\bar{r} = \bar{p} - \bar{c}}$

{Called Centroidal mass moment of Inertia}

⇒ The three eigenvalues are positive and are referred to as the principal moments of inertia of the body.

⇒ The eigen vector of the inertia matrix are furthermore mutually orthogonal and defined the principal axis of inertia of the body.

⇒ Let $\bar{\bar{I}}_o$ & $\bar{\bar{I}}_c$ be defined as in ⑤ & ⑥.
It is possible to show that:

$$\boxed{\bar{\bar{I}}_o = \bar{\bar{I}}_c + m\left(\|\bar{c}\|^2 \bar{\bar{I}} - \bar{c}\bar{c}^T\right)} \quad —⑦$$

→ Theorem of ∥ axis.

⇒ Smallest principal moment of inertia of a rigid body attains its minimum value at the mass centre of the body.

⇒ <u>Newton - Euler Equation governing the Motion of a rigid body</u>

Let the body at hand be acted upon by a wrench of force $\bar{f}$ applied at its centre, and a moment $\bar{n}_c$.

\# <u>Newtons Equation:</u> $\boxed{\bar{f} = m\ddot{\bar{c}}}$ —⑧

\# <u>Euler Equation:</u> $\boxed{\bar{n}_c = \bar{\bar{I}}_c\dot{\bar{\omega}} + \bar{\omega} \times \bar{\bar{I}}_c\bar{\omega}}$ —⑨

⇒ The momentum $\bar{m}$

$$\boxed{\bar{m} = m\dot{\bar{c}}} \quad —⑩$$

⇒ Angular momentum $\bar{h}_c$ of rigid body with respect to mass centre..

$$\boxed{\bar{h}_c = \bar{\bar{I}}_c\bar{\omega}} \quad —⑪$$

So $\dot{\bar{m}} = m\ddot{\bar{c}}$ & $\dot{\bar{h}}_c = \bar{\bar{I}}_c\dot{\bar{\omega}} + \bar{\omega}\times\bar{\bar{I}}_c\bar{\omega}$

Hence Law of Motion take form:-

$$\boxed{\bar{f} = \dot{\bar{m}}} \quad \text{—— } (8^*)$$

$$\boxed{M_c = \dot{\bar{h}}_c} \quad \text{—— } (9^*)$$

⇒ Above equation can be written in a more compact form as:-

⇒ Let us introduce a 6×6 matrix $\bar{\bar{M}}$:

$$\bar{\bar{M}} = \begin{bmatrix} \bar{\bar{I}}_c & \bar{\bar{O}} \\ \bar{\bar{O}} & m\bar{\bar{I}} \end{bmatrix}$$

⇒ Now Newton-Euler equations can be written as

$$\boxed{\bar{\bar{M}}\dot{\bar{t}} + \bar{\bar{W}}\bar{\bar{M}}\bar{t} = \bar{w}}$$

$$\bar{\bar{W}} = \begin{bmatrix} \bar{\bar{\Omega}} & \bar{\bar{O}} \\ \bar{\bar{O}} & \bar{\bar{O}} \end{bmatrix}$$

⇒ The momentum screw of the rigid body about the mass center is the 6-dimensional vector $\bar{\mu}$ defined as

$$\bar{\mu} = \begin{bmatrix} \bar{\bar{I}}_c\bar{\omega} \\ m\dot{\bar{c}} \end{bmatrix} = \bar{\bar{M}}\bar{t}$$

$$\dot{\mathcal{M}} = \bar{\bar{M}}\dot{E} + \bar{\bar{W}}\mathcal{M} = \bar{\bar{M}}\dot{E} + \bar{\bar{W}}\bar{\bar{M}}E$$

$\Rightarrow$ Kinetic energy can be written in Compact form as:

$$\boxed{T = \frac{1}{2} E^T \bar{\bar{M}} E}$$

$\Rightarrow$ Finaly Newton-Euler equations can be written in an even more Compact form as:-

$$\boxed{\dot{\mathcal{M}} = \bar{W}}$$

CHAPTER 4

Geometry of Decoupled Serial Robot

4

Geometry of Decoupled Serial Robot

4.1> Introduction

Kinematic State ⇒ Position & Orientation

Dynamic state ⇒ Velocities

Statics ⇒ Bodies are at Rest

4.2> The Denavit-Hartenberg Notation

⇒ The first task of a robotics engineer is the Kinematic modeling of a robotic manipulator.

⇒ The Simplest way to Kinematically modeling a robotic manipulator is by means of the Concept of Kinematic chain.

Set of rigid body (links), coupled by Kinematic Pairs.

Coupling of two rigid bodies so as to Constrain their relative motion

Kinematic Pair

Upper Kinematic Pair

"Contact takes place along" a line or at a point

eg = Cam & follower ; gear trains, roller bearing.

Lower Kinematic Pair

"Contact takes place along a Common Surface"

Rotating Pair (Revolute)

R

- Common surface of Contact is Circular Cylinder.
- The two rigid body can rotate relative to each other about the axis of Common Cylinder. (Axis of Revolute)

Sliding Pair (Prismatic)

P

- Common Surface is Prism of arbitrary Cross-section.
- Can move only in a pure translation motion along a direction ll to the Axis of Prism

Simple Kinematic Chain

- Kinematic chain with each link connected to at most two other links.

Simple Kinematic chain

Closed

"every link is coupled to two other links."

⇒ This chain is called <u>Linkage</u>.

Open

"Contains exactly two links, coupled to only one other link"

⇒ In Simple Open Kinematic Chain the first link is called <u>Manipulator base</u>, whereas last link is termed the <u>End-effector (EE)</u>.

Manipulator base — Binary links — End Effector

⇒ In order to uniquely describe the architecture of a Kinematic chain (Relative location and Orientation of its neighbouring links), the Denavit-Hartenberg notation (1955) is introduced.

⇒ links are numbered 0, 1, ... n.
⇒ The i^{th} pair being defined as the coupling the $(i-1)^{st}$ link to the i^{th} link.

Hence the manipulator is assumed to be composed of n+1 links and n pairs.

⇒ link 0 is fixed Base.
⇒ each pair can be either R or P.
⇒ Link n is the End-Effector.

⇒ Next, a coordinate frame F_i is defined with origin O_i and axes X_i Y_i Z_i is attached to $(i-1)$st link $\forall\, i = 1, 2, \ldots n+1$.

⇒ For the first n frames, this is done following the rules given below:-

1) Z_i is the axis of the i^{th} pair.
↳ There are two possibilities of defining the positive direction of this axis, since each pair axis is only a line, not a directed segment.

2) X_i is defined as the common perpendicular to Z_{i-1} and Z_i, directed from the former to the latter.

If these two axis intersect the positive direction of X_i is undefined and hence can be freely assigned.

↳ Henceforth we will follow the right-hand rule in this case.

⇒ If Unit Vector $\overline{i_i}$, $\overline{K_{i-1}}$ & $\overline{K_i}$ are attached to X_i, Z_{i-1} & Z_i then $\overline{i_i} = \overline{K_{i-1}} \times \overline{K_i}$.

⇒ Point of intersection is defined the Origin.

If Z_{i-1} & Z_i are ∥, the location of X_i is Undefined.

↳ In order to define it uniquely we will specify X_i as passing through the Origin of the $(i+1)$st frame.

3> Distance between Z_i and Z_{i+1} is defined as a_i, which is thus non negative.

4> The Z_i-Coordinate of the intersection O_i' of Z_i with X_{i+1} is denoted by b_i

→ It can be either positive or Negative.

→ It is also called Offset between common perpendiculars.

5> The angle between Z_i and Z_{i+1} is defined as α_i and is measured about the positive direction of X_{i+1}.

↳ Twist angle between Successive Pair axes.

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6.) The angle between X_i and X_{i+1} is defined as θ_i and is measured about the positive direction of Z_i.

7) The (n+1) st Coordinate frame is attached to the far end of the nth link.

↳ We have the freedom to define this frame as it best suits the task in hand.

⇒ All quantities involved in these definitions are constant, except θ_i, which is variable and is thus termed the joint variable of the ith pair.

↳ The other quantities ie a_i, b_i & α_i are the joint parameters of the said pair.

⇒ Alternatively, the ith pair is P, then b_i is Variable and the other quantities are constant.

↳ Joint Parameter: a_i α_i & θ_i
↳ Joint Variable: b_i

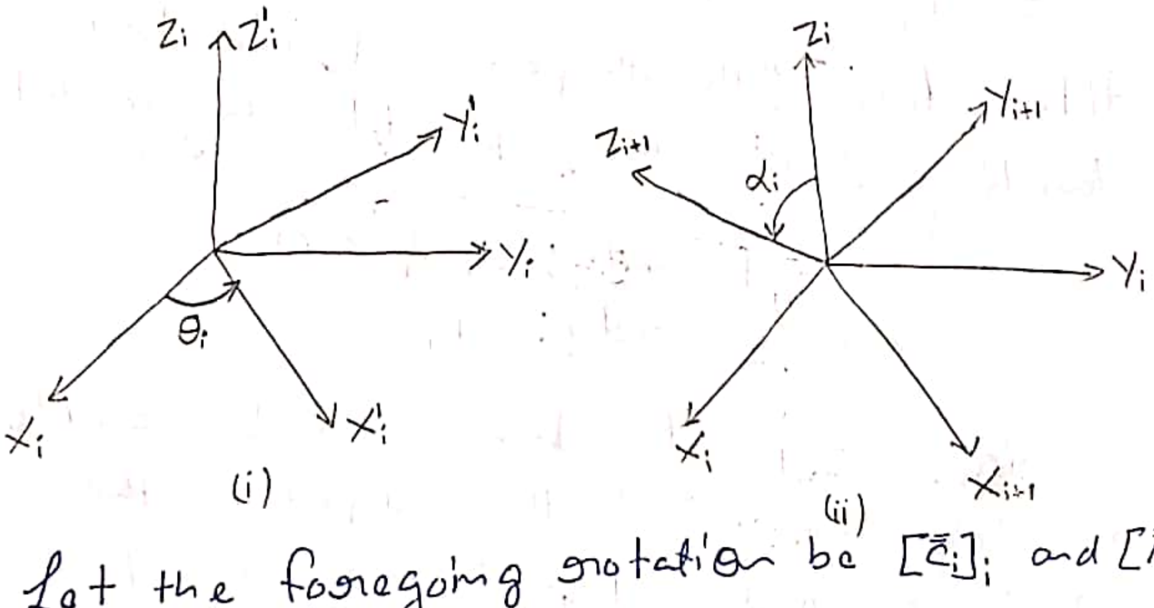
⇒ n-axis manipulator has n joints Variables
→ which are henceforth grouped in the n-dimensional Vector $\bar{\theta}$, regardless of whether joint variables are angular or translational.
→ and 3n Constant Parameter.
↳ Defines the architecture of Manipulator.

⇒ Manipulator architecture is fully defined by its 3n Denavit–Hartenberg (DH) parameters.

⇒ Its posture is fully defined by its n joint variables, also called its joint coordinates.

⇒ First, we obtain the matrix representation of the rotation $\bar{\bar{Q}}_i$ carrying F_i into an orientation coincident with that of F_{i+1}, assuming that the two origin are coincident

↳ Matrix is most easily derived if the rotation of interest is decomposed into two successive rotation.



⇒ Let the foregoing rotation be $[\bar{C}_i]_i$ and $[\bar{\Lambda}_i]_{i'}$ respectively.

$$F_i \xrightarrow{[\bar{C}_i]_i} F_i' \xrightarrow{[\bar{\Lambda}_i]_{i'}} F_{i+1}$$

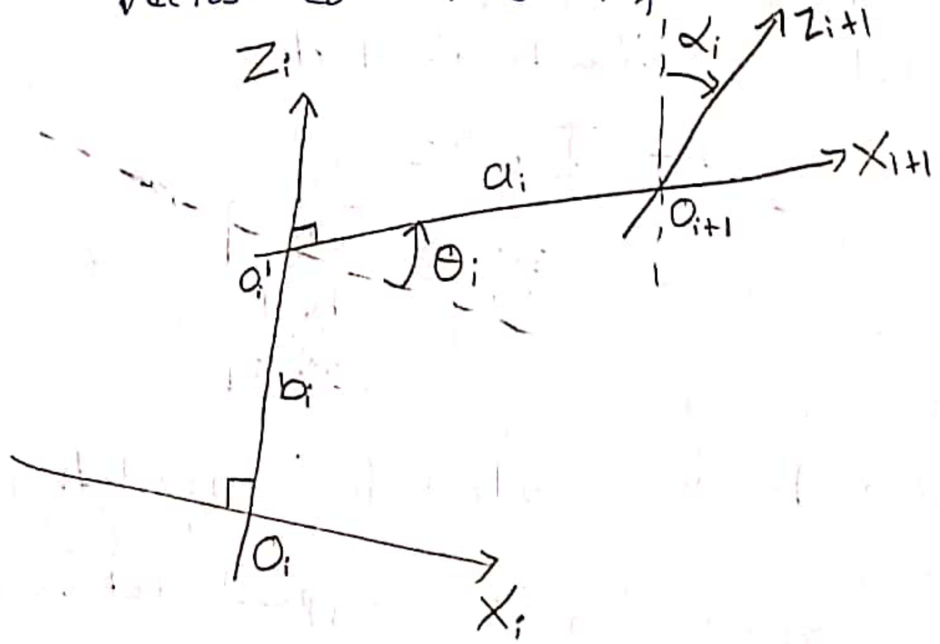
$$[\bar{C}_i]_i = \begin{bmatrix} \cos\theta & -\sin\theta_i & 0 \\ \sin\theta_i & \cos\theta_i & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[\Lambda_i]_{i'} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda_i & -\mu_i \\ 0 & \mu_i & \lambda_i \end{bmatrix} \quad \left\{ \text{where } \begin{array}{l} \lambda_i = \cos\alpha_i \\ \mu_i = \sin\alpha_i \end{array} \right\}$$

$$[\bar{\bar{Q}}_i]_i = [\bar{\bar{C}}_i]_i \, [A_i]_{i'}$$

$$[\bar{\bar{Q}}_i]_i = \begin{bmatrix} \cos\theta_i & -\lambda_i\sin\theta_i & \mu_i\sin\theta_i \\ \sin\theta_i & \lambda_i\cos\theta_i & -\mu_i\cos\theta_i \\ 0 & \mu_i & \lambda_i \end{bmatrix}$$

Le $\bar{a}_i$ be Vector Connecting O_i of F_i to O_{i+1} of F_{i+1}



$$\bar{a}_i = \overrightarrow{O_i O_{i+1}} = \overrightarrow{O_i O_i'} + \overrightarrow{O_i' O_{i+1}}$$

$$[\overrightarrow{O_i O_i'}]_i = \begin{bmatrix} 0 \\ 0 \\ b \end{bmatrix} \qquad [\overrightarrow{O_i' O_{i+1}}]_{i+1} = \begin{bmatrix} a_i \\ 0 \\ 0 \end{bmatrix}$$

$$[\overrightarrow{O_i' O_{i+1}}]_i = [\overline{\overline{Q}}_i]_i \, [\overrightarrow{O_i' O_{i+1}}]_{i+1} = \begin{bmatrix} a_i \cos\theta_i \\ a_i \sin\theta_i \\ 0 \end{bmatrix}$$

So $$[\overline{a}_i]_i = \begin{bmatrix} a_i \cos\theta_i \\ a_i \sin\theta_i \\ b_i \end{bmatrix}$$

$$\overline{\overline{Q}}_i \equiv [\overline{\overline{Q}}_i]_i \quad \forall \text{ Brevity}$$

& $$\overline{a}_i = [\overline{a}_i]_i$$

⇒ Similar to the foregoing factoring of $\overline{\overline{Q}}_i$, vector $\overline{a}_i$ admits the factoring.

$$\overline{a}_i = \overline{\overline{Q}}_i \, \overline{b}_i$$

$$\text{Where, } \overline{b}_i = \begin{bmatrix} a_i \\ b_i \mu_i \\ b_i \lambda_i \end{bmatrix}$$

Vector $\overline{b}_i$ is constant for revolute pair.

⇒ From geometry it is apparent that $\overline{b}_i$ is nothing but $\overline{a}_i$ in F_{i+1}

$$\overline{b}_i = [\overline{a}_i]_{i+1}$$

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⇒ Let $\bar{i}_i$ $\bar{j}_i$ and $\bar{K}_i$ be the unit vector ∥ to the X_i Y_i and Z_i axis respectively; directed to positive direction of these axes.

$$[\bar{i}_{i+1}]_i = \begin{bmatrix} \cos\theta_i \\ \sin\theta_i \\ 0 \end{bmatrix}, \quad [\bar{K}_{i+1}]_i = \begin{bmatrix} \mu_i \sin\theta_i \\ -\mu_i \cos\theta_i \\ \lambda_i \end{bmatrix}$$

$$[\bar{j}_{i+1}]_i = [\bar{K}_{i+1} \times \bar{i}_{i+1}]_i = \begin{bmatrix} -\lambda_i \sin\theta_i \\ \lambda_i \cos\theta_i \\ \mu_i \end{bmatrix}$$

⇒ Therefore, the components $\bar{i}_{i+1}$, $\bar{j}_{i+1}$ and $\bar{K}_{i+1}$ in F_i are nothing but the first, second & third columns of $\bar{\bar{Q}}_i$.

⇒ In general, any vector $\bar{V}$ in F_{i+1} is transformed into F_i in the form

$$[\bar{V}]_i = [\bar{\bar{Q}}_i]_i [\bar{V}]_{i+1}$$

⇒ Likewise, any matrix $\bar{M}$ in F_{i+1} is transformed into F_i by the corresponding similarity transformation:-

$$[\bar{\bar{M}}]_i = [\bar{\bar{Q}}_i]_i [\bar{\bar{M}}]_{i+1} [\bar{\bar{Q}}_i^T]_i$$

⇒ Moreover if we have a chain of i frame then the inward coordinate transformation from F_i to F_i is given by:-

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$$[\bar{V}]_1 = \bar{Q}_1 \bar{Q}_2 \cdots \bar{Q}_{i-1} [\bar{V}]_i$$

$$[\bar{M}]_1 = \bar{Q}_1 \bar{Q}_2 \cdots \bar{Q}_{i-1} [\bar{M}]_i (\bar{Q}_1 \bar{Q}_2 \cdots \bar{Q}_{i-1})^T$$

4.3) The geometric Model of Six-Revolute Manipulator

Joint Variable → For 6R Manipulator ⇒ $\theta_1\ \theta_2\ \theta_3\ \theta_4\ \theta_5\ \theta_6$

Cartesian Variable → Pose of End EFFECTOR (EE) (6 Independent Variable)

Joint Parameters → $\alpha_i\ a_i\ b_i\ \forall\, i = 1, 2 \cdots 6$

⇒ Besides geometry, the Kinematics of manipulator comprises the study of the relations between time-rates of change of the Joint Variables (joint rate) and the twist of the EE.

Joint Rate — Twist of EE

Joint acceleration — Time-rate of change of the twist of EE

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⇒ We distinguish two problem commonly referred to as :-

(i) Direct displacement problem (DDP)

(ii) Inverse displacement problem (IDP)

DDP → Six joint variables of a given six-axis manipulator are assumed to be known, the problem consists of finding the pose of the EE.

IDP → Pose of EE is given while the six joint variables that produce this pose are to be found.

⇒ In studying the DDP of six-axis manipulators, we need not limit ourselves to particular architecture.

⇒ Let us consider manipulator consists of seven rigid body or links (0 to 6) coupled by six revolute joints.

⇒ Correspondingly we have: F_1 F_2 F_3 --- F_7
7 frames

$\{F_i$ attached to $(i-1)^{th}$ link$\}$

⇒ Let line L_i is associated with the axis of the i^{th} revolute joint and a positive direction is defined arbitrarily through a unit vector $\bar{e}_i$.

⇒ Rotation of i^{th} link with respect to $(i-1)^{st}$ link or correspondingly of F_{i+1} with respect to F_i is totally defined by:-

- DH Parameters: a, b, & α_i & $\bar{e}_i$
- Joint Variables: θ_i

⇒ DH parameters & joint variables define uniquely the posture of the manipulator.

⇒ Relative position and orientation of F_{i+1} with respect to F_i is given by matrix $\bar{\bar{Q}}_i$ and vector $\bar{a}_i$.

$$[\bar{\bar{Q}}_i]_i = \begin{bmatrix} \cos\theta_i & -\lambda_i \sin\theta_i & \mu_i \sin\theta_i \\ \sin\theta_i & \lambda_i \cos\theta_i & -\mu \cos\theta_i \\ 0 & \mu_i & \lambda_i \end{bmatrix} \quad \left\{ \begin{array}{l} \lambda_i = \cos\alpha_i \\ \mu_i = \sin\alpha_i \end{array} \right\}$$

$$[\bar{a}_i]_i = \begin{bmatrix} a_i \cos\theta_i \\ a_i \sin\theta_i \\ b_i \end{bmatrix}$$

⇒ The equation leading to the geometric model under study are known as the displacement equation.

{Sometime also termed as Closure Equation}

⇒ Let $\overline{\overline{Q}}$ be orientation of F_7 and $\overline{P}$ be position vector of O_7.

⇒ Orientation $\overline{\overline{Q}}$ of EE is obtained as a result of the six individual rotations $\{Q_i\}_i^6$ about each revolute axis through an angle θ_i, in a sequential order, from 1 to 6.

$$[\overline{\overline{Q}}]_1 = [\overline{\overline{Q}}_6]_1 [\overline{\overline{Q}}_5]_1 [\overline{\overline{Q}}_4]_1 [\overline{\overline{Q}}_3]_1 [\overline{\overline{Q}}_2]_1 [\overline{\overline{Q}}_1]_1 \quad \text{—①}$$

$$[\overline{P}]_1 = [\overline{a}_6]_1 + [\overline{a}_5]_1 + [\overline{a}_4]_1 + [\overline{a}_3]_1 + [\overline{a}_2]_1 + [\overline{a}_1]_1 \quad \text{—②}$$

⇒ Notice that the above equations requires that all vectors and matrices involved be expressed in the same coordinate frame.

↳ However it is convenient to represent the foregoing relations in each individual frame.

↳ This can be readily done by means of similarity transformation.

⇒ If we apply similarity transformation to eq ①
it becomes:-

$$\left[\bar{\bar{Q}}\right]_1 = \left[\bar{\bar{Q}}_6\right]_6 \left[\bar{\bar{Q}}_5\right]_5 \left[\bar{\bar{Q}}_4\right]_4 \left[\bar{\bar{Q}}_3\right]_3 \left[\bar{\bar{Q}}_2\right]_2 \left[\bar{\bar{Q}}_1\right]_1$$

$$\Rightarrow \boxed{\bar{\bar{Q}} = \bar{\bar{Q}}_6 \bar{\bar{Q}}_5 \bar{\bar{Q}}_4 \bar{\bar{Q}}_3 \bar{\bar{Q}}_2 \bar{\bar{Q}}_1}$$

⇒ Like wise eq ② becomes:-

$$\boxed{\begin{aligned}\bar{P} = {} & \bar{a}_1 + \bar{\bar{Q}}_1 \bar{a}_2 + \bar{\bar{Q}}_1 \bar{\bar{Q}}_2 \bar{a}_3 + \bar{\bar{Q}}_1 \bar{\bar{Q}}_2 \bar{\bar{Q}}_3 \bar{a}_4 + \bar{\bar{Q}}_1 \bar{\bar{Q}}_2 \bar{\bar{Q}}_3 \bar{\bar{Q}}_4 \bar{a}_5 \\ & + \bar{\bar{Q}}_1 \bar{\bar{Q}}_2 \bar{\bar{Q}}_3 \bar{\bar{Q}}_4 \bar{\bar{Q}}_5 \bar{a}_5\end{aligned}}$$

⇒ Above equations will be more compact if homogeneous transformations are now introduced.

⇒ Let $\bar{\bar{T}}_i = \{\bar{\bar{T}}_i\}_i$ be 4×4 matrix transformin F_{i+1} coordinate into F_i coordinate, the foregoing equation can be written in 4×4 matrix form namely,

$$\boxed{\bar{\bar{T}} = \bar{\bar{T}}_6 \bar{\bar{T}}_5 \bar{\bar{T}}_4 \bar{\bar{T}}_3 \bar{\bar{T}}_2 \bar{\bar{T}}_1}$$

⇒ A Scalar, Vector or matrix expression is said to be <u>multilinear</u> in a set of Vectors $\{\bar{V}_i\}_1^N$ if each of those Vectors appears only linearly in the same expression.

⇒ Scalar, Vector or matrix expression is said to be <u>multiquadratic</u> in the same set of Vectors if those Vectors appear at most quadratically in the said expression.

⇒ Qualifiers like multicubic, multiquadratic etc. bear similar meanings.

⇒ Further, we partition matrix $\bar{Q}_i$ rowwise and columnwise, namely:

$$\bar{\bar{Q}}_i = \begin{bmatrix} \bar{m}_i^T \\ \bar{n}_i^T \\ \bar{o}_i^T \end{bmatrix} = \begin{bmatrix} \bar{p}_i & \bar{q}_i & \bar{u}_i \end{bmatrix}$$

$$[e_i]_i = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \bar{e} \qquad \bar{X}_i = \begin{bmatrix} \cos\theta_i \\ \sin\theta_i \end{bmatrix}$$

$$\bar{u}_i = \bar{\bar{Q}}_i \bar{e} \quad , \quad \bar{o}_i = \bar{\bar{Q}}^T \bar{e}$$

$$\bar{u}_i = [e_{i+1}]_i \qquad \bar{o}_i = [e_i]_{i+1}$$

4.4> The Inverse Displacement Analysis of Decoupled Manipulator

⇒ Industrial manipulators are frequently Supplied with a Special architecture that allows a decoupling of the Positioning Problem from Orientation Problem.

↳ Determinant design Criterion in this regard has been that the manipulator lend itself to a Closed-form Inverse displacement solution.

⇒ Decoupled manipulator are those whose last three joints have intersecting axes.

↳ These joints Constitute the wrist of the manipulator, which is said to be Spherical.

Because when the point of Intersection of the three wrist axes C is Kept fixed, all the Points of the wrist moves On Spheres. Centered at 'C'

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⇒ In terms of DH Parameters of the manipulator, In a decoupled manipulator $a_4 = a_5 = b_5 = 0$.

↳ So Origin of frames 5 & 6 are Coincident.

⇒ All other DH Parameter Can assume arbitrary Values.

4.4.1 > The Positioning Problem

⇒ Let C denotes the intersection of axes 4, 5 and 6 ie the Center of the Spherical wrist.

↳ Let $\bar{c}$ denote the position vector of this point

⇒ Apparently position Vector of C is indipendent of joint angles θ_4 θ_5 & θ_6.

$$[\bar{c}]_1 = [\bar{a}_1]_1 + [\bar{a}_2]_1 + [\bar{a}_3]_1 + [\bar{a}_4]_1$$

$$\cancel{\bar{c} = \bar{a}_1 + Q_1 Q_2 Q_3}$$

$$\boxed{\bar{c} = \bar{a}_1 + \bar{\bar{Q}}_1 \bar{a}_2 + \bar{\bar{Q}}_1 \bar{\bar{Q}}_2 \bar{a}_3 + \bar{\bar{Q}}_1 \bar{\bar{Q}}_2 \bar{\bar{Q}}_3 \bar{a}_4}$$

$$\Rightarrow \bar{\bar{Q}}_1^T (\bar{c} - \bar{a}_1) = \bar{a}_2 + \bar{\bar{Q}}_2 \bar{a}_3 + \bar{\bar{Q}}_2 \bar{\bar{Q}}_3 \bar{a}_4$$

$$= \bar{\bar{Q}}_2 (\bar{b}_2 + \bar{\bar{Q}}_3 \bar{b}_3 + \bar{\bar{Q}}_3 \bar{\bar{Q}}_4 \bar{b}_4)$$

$$\{ \text{as } \bar{a}_i = \bar{\bar{Q}}_i \bar{b}_i \}$$

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⇒ However since we are dealing with a decoupled manipulator, we have, from

$$\overline{a}_4 = \overline{\overline{Q}}_4 \overline{b}_4 = \begin{bmatrix} 0 \\ 0 \\ b_4 \end{bmatrix} = b_4 \overline{e}$$

So $\overline{\overline{Q}}_3 \overline{\overline{Q}}_4 \overline{b}_4 = b_4 \overline{\overline{Q}}_3 \overline{e} = b_4 \overline{u}_3$

$$\Rightarrow \overline{\overline{Q}}_1^T \overline{c} - \overline{b}_1 = \overline{\overline{Q}}_2 (\overline{b}_2 + \overline{\overline{Q}}_3 \overline{b}_3 + b_4 \overline{u}_3)$$

⇒ Further, an expression of $\overline{c}$ can be derived in terms of $\overline{p}$, the position vector of the operation point of the EE and $\overline{\overline{Q}}$, namely

$$\overline{c} = \overline{p} - \overline{\overline{Q}}_1 \overline{\overline{Q}}_2 \overline{\overline{Q}}_3 \overline{\overline{Q}}_4 \overline{a}_5 - \overline{\overline{Q}}_1 \overline{\overline{Q}}_2 \overline{\overline{Q}}_3 \overline{\overline{Q}}_4 \overline{\overline{Q}}_5 \overline{a}_6$$

$$\overline{c} = \overline{p} - \overline{\overline{Q}} \,\overline{\overline{Q}}_6^T \overline{a}_6 \quad \begin{cases} \text{as } \overline{a}_5 = \overline{0} \\ \text{\& } Q = \overline{\overline{Q}}_1 \overline{\overline{Q}}_2 \overline{\overline{Q}}_3 \overline{\overline{Q}}_4 \overline{\overline{Q}}_5 \overline{\overline{Q}}_6 \end{cases}$$

$$\boxed{\overline{c} = \overline{p} - \overline{\overline{Q}} \overline{b}_6} \quad \text{— (1)}$$

⇒ Moreover, the base coordinates of P and C in F1 components of their position vectors $\overline{p}$ and $\overline{c}$ are defined as

$$[\overline{p}]_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad [\overline{c}]_1 = \begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix}$$

$\Rightarrow$ Putting these things in eq ① we get:-

$$\begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = \begin{bmatrix} x - (a_{11} a_6 + a_{12} b_6 \mu_6 + a_{13} b_6 \lambda_6) \\ y - (a_{21} a_6 + a_{22} b_6 \mu_6 + a_{23} b_6 \lambda_6) \\ z - (a_{31} a_6 + a_{32} b_6 \mu_6 + a_{33} b_6 \lambda_6) \end{bmatrix} \quad \text{--②}$$

$\Rightarrow$ Now Positioning problem become one of Finding the first three joint angles necessary to position point C at a point of base coordinates x_c, y_c & z_c.

$$\bar{\bar{Q}}_2 (\bar{b}_2 + \bar{\bar{Q}}_3 \bar{b}_3 + b_4 \bar{u}_3) = \bar{\bar{Q}}_1^T \bar{c} - \bar{b}_1 \quad \text{--③}$$

(i) If the Euclidean norms of the two sides of that equation are equated, the resulting equation will not contain θ_2.

(ii) The third Scalar equation of the same equation is independent of θ_2, by Virtue of the Structure of $\bar{\bar{Q}}_i$ matrix.

$\Rightarrow$ Thus we have two equations free of θ_2 which allows us to Calculate the two remaining Unknowns θ_1 & θ_3.

⇒ Let the Euclidean norm of the left-hand side of Eq ③ be denoted by l, that of its right-hand side by r.

$$l^2 = a_2^2 + b_2^2 + a_3^2 + b_3^2 + b_4^2 + 2\bar{b}_2^T \bar{Q}_3 \bar{b}_3 + 2b_4 \bar{b}_2^T \bar{u}_3 + 2\lambda_3 b_3 b_4$$

$$r^2 = \|\bar{c}\|^2 + \|\bar{b}_1\|^2 - 2b_1^T Q_1^T c$$

⇒ From above it is apparent that l^2 is linear in $\bar{x}_3$ and r^2 is linear in $\bar{x}_1$.

⇒ Upon equating l^2 with r^2 we get:-

$$A\cos\theta_1 + B\sin\theta_1 + C\cos\theta_3 + D\sin\theta_3 + E = 0 \quad (4)$$

where,

$A = 2a_1 x_c$

$B = 2a_1 y_c$

$C = 2a_2 a_3 - 2b_2 b_4 \mu_2 \mu_3$

$D = 2a_3 b_2 \mu_2 + 2a_2 b_4 \mu_3$

$$E = a_2^2 + a_3^2 + b_2^2 + b_3^2 + b_4^2 - a_1^2 - x_c^2 - y_c^2 - (z_c - b_1)^2 + 2b_2 b_3 \lambda_2 + 2b_2 b_4 \lambda_2 \lambda_3 + 2b_3 b_4 \lambda_3$$

hand
of

$\Rightarrow$ Moreover, the third scalar equation of eq (3) takes the form:-

$$F\cos\theta_1 + G\sin\theta_1 + H\cos\theta_3 + I\sin\theta_3 + J = 0 \quad (5)$$

where,

$$F = y_c \mu_1$$
$$G = -x_c \mu_1$$
$$H = -b_4 \mu_2 \mu_3$$
$$I = a_3 \mu_2$$
$$J = b_2 + b_3\lambda_2 + b_4\lambda_2\lambda_3 - (z_c - b_1)\lambda_1$$

$\Rightarrow$ Thus we have derived two nonlinear equations in θ_1 and θ_3 that are linear in $\cos\theta_1$, $\sin\theta_1$, $\cos\theta_3$ & $\sin\theta_3$.
{Eq (4) & 5}

=0 $\Rightarrow$ Each of these equations (eq (4) & eq (5)) defines a contour in the $\theta_1 - \theta_3$ plane.

. (4) $\hookrightarrow$ Their intersection determine all real solutions to the problem at hand.

We know

$$\cos\theta_3 = \frac{1 - \tan^2(\theta_3/2)}{1 + \tan^2(\theta_3/2)}$$

)^2 {tan-half-angle Identities}

$$\sin\theta_3 = \frac{2\tan(\theta_3/2)}{1 + \tan^2(\theta_3/2)}$$

~~$\tan\theta_3 = \tan(\theta_3/2)$~~

⇒ If $\cos\theta_i$ & $\sin\theta_i$ are substituted for their equivalent in terms of $\tan(\theta_i/2)$, then two biquadratic polynomial equations in $\tan(\theta_1/2)$ and $\tan(\theta_3/2)$ are derived.

⇒ One can eliminate one of these variables from the foregoing equations, thereby reducing the two equations to a single quadratic polynomial equation in the other variable.

↳ Quadratic equation thus resulting is called the <u>characteristic equation</u>.

⇒ Eq (4) & (5) can be solved for say $\cos\theta_1$ & $\sin\theta_1$ in terms of $\cos\theta_3$ & $\sin\theta_3$.

$$\cos\theta_1 = \frac{-G(C\cos\theta_3 + D\sin\theta_3 + E) + B(H\cos\theta_3 + I\sin\theta_3 + J)}{\Delta_1} \quad (6)$$

$$\sin\theta_1 = \frac{F(C\cos\theta_3 + D\sin\theta_3 + E) - A(H\cos\theta_3 + I\sin\theta_3 + J)}{\Delta_1} \quad (7)$$

Where
Note,
the
⇒ C
d
⇒
⇒
⇒

where $\Delta_1 = AG - FB = -2a_1\mu_1(x_c^2 + y_c^2)$

Note, Δ_1 can be computed off-line, ie prior to setting the manipulator into operation.

⇒ Calculations are possible as long as Δ_1 does not vanish.

⇒ Δ_1 vanishes if and only if :-

(1) a_1 vanishes
(2) μ_1 vanishes } Architecture dependent }
(3) $x_c^2 + y_c^2$ vanishes. {Position dependent}

{ Means point C lies on the Z_1 axis }

{ This occurs frequently in industrial Manipulator }
{not both at same time}

⇒ Even if neither a_1 nor μ_1 vanishes, the manipulator can be postured in a configuration at which point C lies on the Z_1 axis.

↳ Such configuration is termed the $\boxed{\text{first Singularity.}}$

⇒ If both parameters a_1 & μ_1 vanished, then the arm would be useless to position arbitrarily a point in space.

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⇒ For the moment, it will be assumed that Δ_1 does not vanish, the particular case under which it does being studied presently.

⇒ Next, both sides of eq (6) & (7) are squared and added:-

$$K \cos^2\theta_3 + L \sin^2\theta_3 + M \cos\theta_3 \sin\theta_3 + N \cos\theta_3 + P \sin\theta_3 + Q = 0 \quad \text{—(8)}$$

whose coefficients, after simplification are given below:-

$$K = 4a_1^2 H^2 + \mu_1^2 C^2$$

$$L = 4a_1^2 I^2 + \mu_1^2 D^2$$

$$M = 2(4a_1^2 HI + \mu_1^2 CD)$$

$$N = 2(4a_1^2 HJ + \mu_1^2 CE)$$

$$P = 2(4a_1^2 IJ + \mu_1^2 DE)$$

$$Q = 4a_1^2 J^2 + \mu_1^2 E^2 - 4a_1^2 \mu_1^2 \rho^2$$

Where, $\rho^2 = x_c^2 + y_c^2$

⇒ Upon substitution of tan-half identities introduced, a quartic equation in $\tan\theta_3/2$ is obtained:-

$$R\tan^4(\theta_3/2) + S\tan^3(\theta_3/2) + T\tan^2(\theta_3/2) + U\tan(\theta_3/2) + V = 0 \quad \text{— } (9)$$

Where,

$$R = 4a_1^2(J-H)^2 + \mu_1^2(E-C)^2 - 4\rho^2 a_1^2 \mu_1^2$$

$$S = 4\left[4a_1^2 I(J-H) + \mu_1^2 D(E-C)\right]$$

$$T = 2\left[4a_1^2(J^2-H^2+2I^2) + \mu_1^2(E^2-C^2+2D^2) - 4\rho^2 a_1^2 \mu_1^2\right]$$

$$U = 4\left[4a_1^2 I(H+J) + \mu_1^2 D(C+E)\right]$$

$$V = 4a_1^2(J+H)^2 + \mu_1^2(E+C)^2 - 4\rho^2 a_1^2 \mu_1^2$$

⇒ Further, let $\{(\tan\theta_3/2)_i\}_1^4$ be the four roots of equation (9).

↳ Thus, up to four possible values of θ_3 can be obtained, namely.

$$(\theta_3)_i = 2\tan^{-1}\left[(\tan\theta_3/2)_i\right] \quad \forall\, i = 1, 2, 3, 4$$

⇒ Once the four values of θ_3 are available, each of these, is substituted into eq (6) & (7), which thus produce four different values of θ_1.

⇒ For each value of θ_1 & θ_3, then one value of θ_2 can be computed from the first two scalar equation of Eq ③, which is displayed below:-

$$A_{11}\cos\theta_2 + A_{12}\sin\theta_2 = x_c\cos\theta_1 + y_c\sin\theta_1 - a_1 \quad \text{—⑩}$$

$$-A_{12}\cos\theta_3 + A_{11}\sin\theta_2 = -x_c\lambda_1\sin\theta_1 + y_c\lambda_1\cos\theta_1 + (z_c - b_1)\mu_1 \quad \text{—⑪}$$

Where,

$$A_{11} = a_2 + a_3\cos\theta_3 + b_4\mu_3\sin\theta_3 \quad \text{— } 10^*$$

$$A_{12} = -a_3\lambda_2\sin\theta_3 + b_3\mu_2 + b_4\lambda_2\mu_3\cos\theta_3 + b_4\mu_2\lambda_3 \quad \text{— } 11^*$$

⇒ Thus, if A_{11} & A_{12} do not vanish simultaneously, angle θ_2 is readily computed in terms of θ_1 & θ_3 from eq ⑩ & ⑪:-

$$\cos\theta_2 = \frac{1}{\Delta_2}\left\{ A_{11}(x_c\cos\theta_1 + y_c\sin\theta_1 - a_1) - A_{12}\left[-x_c\lambda_1\sin\theta_1 + y_c\lambda_1\cos\theta_1 + (z_c - b_1)\mu_1\right]\right\} \quad \text{— ⑫}$$

$$\sin\theta_2 = \frac{1}{\Delta_2}\Big\{A_{12}\left(x_c\cos\theta_1 + y_c\sin\theta_1 - a_1\right) + A_{11}\left[-x_c\lambda_1\sin\theta_1 + y_c\lambda_1\cos\theta_1 + (z_c - b_1)\mu_1\right]\Big\} \quad \text{--- (13)}$$

Where Δ_2 is defined as:-

$$\Delta_2 \equiv A_{11}^2 + A_{12}^2$$

$$\begin{aligned}\equiv\ & a_2^2 + a_3^2(\cos^2\theta_3 + \lambda_2^2\sin^2\theta_3) + b_4^2\mu_3^2(\sin^2\theta_3 + \lambda_2^2\cos^2\theta_3) \\ & + 2a_2a_3\cos\theta_3 + 2a_2b_4\mu_3\sin\theta_3 \\ & + 2\lambda_2\mu_2(b_3 + b_4\lambda_3)(b_4\mu_3\cos\theta_3 - a_3\sin\theta_3) \\ & + 2a_3b_4\mu_2^2\mu_3\sin\theta_3\cos\theta_3 + (b_3 + \lambda_3 b_4)^2\mu_2^2\end{aligned}$$

$\Rightarrow$ The case in which $\Delta_2 = 0$, which leads to what is termed here the $\boxed{\text{Second Singularity.}}$

The Vanishing of Δ_1

$\Rightarrow$ In above derivations we have assumed that neither μ_1 nor a_1 vanishes.

$\Rightarrow$ If either of $\mu_1 = 0$ or $a_1 = 0$, then one can readily show that eq (9) reduces to a quadratic equation.

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⇒ Both $\mu_1 = 0$ & $a_1 = 0$ together never occur, because their simultaneous occurrence would render the axes of the first two revolutes coincident.

⇒ We thus have two cases:-

1. $\mu_1 = 0$, $a_1 \neq 0$. In this case

$$A, B \neq 0 \quad , F = G = 0$$

⇒ Under these conditions, Eq (5) & tan-half-angle identities give:-

$$(J - H)\tan^2(\theta_3/2) + 2I\tan(\theta_3/2) + (J + H) = 0$$

which thus produce two values of $\tan \theta_3/2$

$$(\tan \theta_3/2)_{1,2} = \frac{-I \pm \sqrt{I^2 - J^2 + H^2}}{J - H}$$

⇒ Once two values of θ_3 have been determined according to the above equation, θ_1 can be found using eq (4) & tan half identities:-

$$(E'-A)\tan^2(\theta_1/2) + 2B\tan(\theta_1/2) + (E'+A) = 0$$

where,

$$E' = \cancel{C_{C_3}+D}\ C\cos\theta_3 + D\sin\theta_3 + E.$$

whose roots are,

$$\left(\tan(\theta_1/2)\right)_{1,2} = \frac{-B \pm \sqrt{B^2 - E'^2 + A^2}}{E'-A}$$

⇒ Thus, two values of θ_1 are found for each of the two values of θ_3, which results in four positioning solutions.

↳ Value of θ_3 is obtained from eq (10) & (11).

2. $a_1 = 0$, $\mu_1 \neq 0$. We have now

$$A = B = 0, \quad F, G \neq 0$$

⇒ Under the present conditions Eq (4) is reduced to:-

$$(E-C)\tan^2(\theta_3/2) + 2D\tan(\theta_3/2) + (E+C) = 0$$

which produces two values of $\tan\theta_3/2$, namely,

$$\tan(\theta_3/2)_{1,2} = \frac{-D \pm \sqrt{D^2 - E^2 + C^2}}{E-C}$$

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⇒ With the two values of θ_3 obtained, θ_1 can be found using (5) and the tan-half-angle identities to produce.

$$(J' - F)\tan^2(\theta_1/2) + 2G\tan(\theta_1/2) + (J'+F) = 0$$

Where,

$$J' = H\cos\theta_3 + I\sin\theta_3 + J$$

Whose roots are,

$$\left(\tan(\theta_1/2)\right)_{1,2} = \frac{-G \pm \sqrt{G^2 - J'^2 + F^2}}{J' - F}$$

⇒ Once again the solution results in a cascade of two quadratic equations, one for θ_3 and one for θ_1, which yields four positioning solutions.

↳ As above θ_2 is determined using eq (10) & (11)

The Vanishing of Δ_2

⇒ Δ_2, may vanish at a certain posture thereby preventing the calculation of θ_2.

↳ This posture, termed the <u>Second Singularity</u>, occurs if both coefficients A_{11} & A_{12} vanishes.

⇒ From eq (10*) & (11*) it is evident that, these coefficients are not only position but also <u>architecture-dependent</u>.

↳ Thus, an arbitrary manipulator cannot take on this configuration unless its geometric dimensions allows it.

⇒ This type of singularity will be termed <u>architecture-dependent</u>.

⇒ First note that the right-hand side of Eq (3) is identical to $\bar{Q}_1^T(\bar{c}-\bar{a}_1)$.

↳ This means that this expression is nothing but the F_2-representation of the position vector of C.

⇒ That is, the Components of Vector $\overline{\overline{Q}}_1^T(\overline{c}-\overline{a}_1)$ are the F_2-Components of Vector $\overrightarrow{O_2C}$.

⇒ So sight had sides of eq (10) & (11) are respectively, the X_2- and Y_2-Components of Vector $\overrightarrow{O_2C}$.

⇒ If $A_{11} = A_{22} = 0$, then the two foregoing Components Vanish and hence, point C lies on the Z_2 axis.

Conclusion

(1) First Singularity occurs when poin C lies on the axis of the first revolute.

(2) Second Singular occurs when point C lies on the axis of Second revolute.

4.4.2> The Orientation Problem

⇒ This problem consists in determining the wrist angles that will produce a prescribed orientation of the end-effector.

⇒ The orientation, in turn is given in terms of the rotation matrix $\overline{\overline{Q}}$, taking the end-effector from its home attitude to its current one.

↳ Alternatively can be given by the natural invariants of the rotation matrix, vector $\overline{e}$ & angle ϕ.

⇒ In any event, all nine components of matrix $\overline{\overline{Q}}$ are known in F_1.

⇒ It is convenient to assume the columnwise partitioning of $[\overline{\overline{Q}}]_1$.

$$[\overline{\overline{Q}}]_1 = [\overline{p} \ \overline{q} \ \overline{u}] \quad \text{— (12)}$$

⇒ Without loss of generality it can be assumed that Z_7 is defined parallel to Z_6.

⇒ Since θ_1, θ_2 & θ_3 are available $\overline{\overline{Q}}_1$, $\overline{\overline{Q}}_2$ & $\overline{\overline{Q}}_3$ become data for this problem.

⇒ Let $[\bar{u}]_1 = [\bar{e}_7]_1 = [\bar{e}_6]_1$

⇒ Now since the orientation of the end-effector is given, the components of $[\bar{e}_6]_1$ are known, but they will be needed in frame 4.

$$[\bar{e}_6]_4 = (\bar{\bar{Q}}_1 \bar{\bar{Q}}_2 \bar{\bar{Q}}_3)^T [\bar{e}_6]_1 = (\bar{\bar{Q}}_1 \bar{\bar{Q}}_2 \bar{\bar{Q}}_3)^T [\bar{u}]_1$$

⇒ Let the components of $[\bar{e}_6]_4$, all of them known, be defined as:-

$$[\bar{e}_6]_4 = \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} \quad \text{—(13)}$$

⇒ Moreover, the components of vector $\bar{e}_5$ in F_4 are nothing but the entries of the third column of matrix: $\bar{\bar{Q}}_4$

$$[\bar{e}_5]_4 = \begin{bmatrix} \mu_4 \sin\theta_4 \\ -\mu_4 \cos\theta_4 \\ \lambda_4 \end{bmatrix} \quad \text{—(14)}$$

⇒ Furthermore, Vector $\bar{e}_5$ & $\bar{e}_6$ make an angle α_5, hence,

$$\bar{e}_6^T \bar{e}_5 = \lambda_5 \quad \text{or} \quad [e_6]_4^T [e_5]_4 = \lambda_5 \quad \text{—(15)}$$

⇒ Upon Substitution eq (13) & eq (14) in eq (15) we get:-

$$\xi \mu_4 \sin\theta_4 - \eta \mu_4 \cos\theta_4 + \zeta \lambda_4 = \lambda_5$$

⇒ which can be ~~exact~~ readily transformed, with the aid of the tan-half angle identities, into a quadratic equation in $\tan(\theta_4/2)$.

$$(\lambda_4 - \eta\mu_4 - \zeta\lambda_4)\tan^2(\theta_4/2) - 2\xi\mu_4 \tan(\theta_4/2) + (\lambda_5 + \eta\mu_4 - \zeta\lambda_4) = 0 \quad \text{—(16)}$$

its two roots being given by:-

$$\tan(\theta_4/2) = \frac{\xi\mu_4 \pm \sqrt{(\xi^2+\eta^2)\mu_4^2 - (\lambda_5 - \zeta\lambda_4)^2}}{\lambda_5 - \zeta\lambda_4 - \eta\mu_4}$$

⇒ Note that the two foregoing roots are real as long as the radical is positive, the two roots merging into a single one when the radical vanishes.

⇒ A negative radical means an attitude of the EE that is not feasible with the wrist.

⇒ Therefore the workspace W of the wrist is not unlimited, but rather defined by the set of values of ξ, η and ζ that satisfy the two relations shown below:-

$$\xi^2 + \eta^2 + \zeta^2 = 1$$

$$f(\xi, \eta, \zeta) \equiv (\xi^2 + \eta^2)\mu_4^2 - (\lambda_5 - \zeta\lambda_4)^2 \geq 0$$

So above can be simplified in ζ alone, namely,

$$F(\zeta) = \zeta^2 - 2\lambda_4\lambda_5\zeta - (\mu_4^2 - \lambda_5^2) \leq 0$$

As a Consequence

1. W is a region of the unit sphere S centered at the origin of the three-dimensional space.

2. W is bounded by the two parallels given by the roots of $F(\xi)=0$ on the Sphere.

3. The Wrist attains its Singular Configurations along the two foregoing parallels.

⇒ In order to gain more insight on the Shape of the workspace W, Let us look at the boundary defined by $F(\zeta)=0$.

$$\zeta_{1,2} = \lambda_4 \lambda_5 \pm |\mu_4 \mu_5|$$

Which thus defines two planes, Π_1 & Π_2 Parallel to the $\xi-\eta$ plane of the three dimentional Space, intersecting the ζ-axis at ζ_1 & ζ_2 respectively.

⇒ Thus, the workspace W of the Spherical wrist at hand is that region of the Surface of the Unit Sphere S Contained between the two Parallels defined by Π_1 & Π_2.

⇒ Once θ_4 is Calculated from the two foregoing Values of $\tan \theta_4/2$, if these are real, angle θ_5 is obtained uniquely for each value of θ_4 as explained below:-

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Let $\bar{\bar{R}} = \bar{\bar{Q}}_6 \bar{\bar{Q}}_5 \bar{\bar{Q}}_4$ — (17)

$\Rightarrow \quad \bar{\bar{Q}} = \bar{\bar{Q}}_6 \bar{\bar{Q}}_5 \bar{\bar{Q}}_4 \bar{\bar{Q}}_3 \bar{\bar{Q}}_2 \bar{\bar{Q}}_1$

$\Rightarrow \quad \bar{\bar{Q}} = \bar{\bar{R}} \bar{\bar{Q}}_3 \bar{\bar{Q}}_2 \bar{\bar{Q}}_1$

$$R = \bar{\bar{Q}}_3^T \bar{\bar{Q}}_2^T \bar{\bar{Q}}_1^T \bar{\bar{Q}} \quad — (18)$$

$\Rightarrow$ Let entries of $\bar{\bar{R}}$ in the fourth coordinate frame be given as :-

$$[\bar{\bar{R}}]_4 = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$\Rightarrow$ Expressions of θ_5 and θ_6 can be readily derived by solving first the $\bar{\bar{Q}}_5$ from eq (17) :-

$$\bar{\bar{Q}}_5 = \bar{\bar{Q}}_4^T \bar{\bar{R}} \bar{\bar{Q}}_6^T$$

$\Rightarrow$ Now, by virtue of the form of the $\bar{\bar{Q}}_i$ matrices, it is apparent that the third row of $\bar{\bar{Q}}_i$ does not contain θ_i

$\Rightarrow$ Thus, two equations of θ_5 are obtained by equating the first two components of the third column of that equation.

$$\mu_5 \sin\theta_5 = (\mu_6 r_{12} + \lambda_6 r_{13}) \cos\theta_4 + (\mu_6 r_{22} + \lambda_6 r_{23}) \sin\theta_4 \quad \text{— (19)}$$

$$-\mu_5 \sin\theta_5 = -\lambda_4 (\mu_6 r_{12} + \lambda_6 r_{13}) \sin\theta_4 + \lambda_4 (\mu_6 r_{22} + \lambda_6 r_{23}) \cos\theta_4 + \mu_4 (\mu_6 r_{32} + \lambda_6 r_{33}) \quad \text{— (20)}$$

⇒ Which thus yield a unique value of θ_5 for every value of θ_4.

⇒ Finally, with θ_4 and θ_5 known, it is a simple matter to calculate θ_6.

⇒ This is done upon solving for $\bar{Q}_6$ from eq (19) :-

$$\boxed{\bar{\bar{Q}}_6 = \bar{\bar{Q}}_5^T \bar{\bar{Q}}_4^T \bar{\bar{R}}} \quad \text{— (21)}$$

⇒ Using partitioning of $\bar{\bar{Q}}_i$ we get :-

$$\bar{P}_6 = \bar{\bar{Q}}_5^T \bar{\bar{Q}}_4^T \bar{r}_1 \qquad \left\{ \text{where } \bar{r}_1 \text{ is the first column of } \bar{\bar{R}} \right\}$$

Let $\bar{\omega} = \bar{Q}_4^T \bar{r}_1$

$$\bar{\omega} = \begin{bmatrix} r_{11} \cos\theta_4 + r_{21} \sin\theta_4 \\ -\lambda_4 (r_{11} \sin\theta_4 - r_{21} \cos\theta_4) + \mu_4 r_{31} \\ \mu_4 (r_{11} \sin\theta_4 - r_{21} \cos\theta_4) + \lambda_4 r_{31} \end{bmatrix}$$

Hence,

$$\bar{Q}_5^T \bar{Q}_4^T \bar{r}_1 = \begin{bmatrix} \omega_1 \cos\theta_5 + \omega_2 \sin\theta_5 \\ \lambda_4(-\omega_1 \sin\theta_5 + \omega_2 \cos\theta_5) + \omega_3 \mu_5 \\ \mu_5(\omega_1 \sin\theta_5 - \omega_2 \cos\theta_5) + \omega_3 \lambda_5 \end{bmatrix}$$

So,

$$\cos\theta_6 = \omega_1 \cos\theta_5 + \omega_2 \sin\theta_5$$

$$\sin\theta_6 = -\omega_1 \lambda_5 \sin\theta_5 + \omega_2 \lambda_5 \cos\theta_5 + \omega_3 \mu_5$$

$\Rightarrow$ Thereby deriving a unique valu. of θ_3 for every pair of values (θ_4, θ_5).

$\Rightarrow$ Therefore, there are two sets of solutions for the orientation problem under study, which leads to two corresponding wrist postures.

"Finaly a maximum of eight possible combinations of joint angles for a single pose of the end effector of a decoupled manipulator are found."

CHAPTER 5

Kinetostatics of Serial Robot

5

Kinetostatics of Serial Robot

5.1) Introduction

⇒ We derive first the relation between the twist of the robot EE and the set of joint rates.

↳ Which is given by a linear transformation induced by the robot Jacobian matrix.

⇒ Once the above relation is established for a general six-joint robot, the relation between the static wrench exerted by the environment on the EE and the balancing joint torques is derived by duality

⇒ Three-dimensional workspace is derived.

⇒ An algorithm is proposed for the display of this workspace as pertaining to general regional structures whose inverse displacement analysis leads to a quartic polynomial.

⇒ Chapter closes with Kinetostatic performance indices. Their purpose are:-

↳ Needed in robot design to help the designer best dimension the links of the robot in the early stage of design process, prior to the elastostatic & the electrodynamic design stage

→ Also needed in the Control of a given robot to ensure an acceptable Kinetostatic Performance under feedback Control.

→ Comparison of various Candidate robot when a robotic facility is being planned.

Elastostatic design pertains to the Structural design of robot to ensure that the links and the joint mechanical transmissions will be able to withstand the static load that arise when the robot is in operation.

Elastodynamic design Considers the inertia load of the structural elements while accounting for link flexibility, which gives rise to mechanical Vibration.

5.2) Velocity analysis of Serial Manipulator

⇒ First, a Serial n-axis manipulator Containing only revolute pairs is Considered.

⇒ Then, relations associated with prismatic Pair are introduced.

⇒ Finally, the joint rates of Six-axis manipulators are calculated in terms of the EE twist.

⇒ We Consider manipulator with joint Coordinate θ_i, joint rate $\dot{\theta}_i$, and a unit Vector $\bar{e}_i$ are associated with each revolute axis.

↳ The $\{X_i\ Y_i\ Z_i\}$ Coordinate frame, attached to the $(i-1)^{st}$ link, with Origin O_i

⇒ If the angular-velocity Vector of i^{th} link is denoted by $\bar{\omega}_i$, then we have,

$$\bar{\omega}_0 = \bar{0}$$
$$\bar{\omega}_1 = \dot{\theta}_1 \bar{e}_1$$
$$\bar{\omega}_2 = \dot{\theta}_1 \bar{e}_1 + \dot{\theta}_2 \bar{e}_2$$
$$\vdots$$
$$\bar{\omega}_n = \dot{\theta}_1 \bar{e}_1 + \dot{\theta}_2 \bar{e}_2 + \cdots + \dot{\theta}_n \bar{e}_n$$

⇒ and if the angular Velocity of the EE is denoted by $\bar{\omega}$ then,

$$\bar{\omega} = \bar{\omega}_n = \sum_{i=1}^{n} \dot{\theta}_i \bar{e}_i \quad \text{—— ①}$$

⇒ Position vector of point on EE is readily derived as:-

$$\bar{p} = \bar{a}_1 + \bar{a}_2 + \cdots + \bar{a}_n \quad \text{—— ②}$$

⇒ Upon differentiating both sid of eq ① we get

$$\dot{\bar{p}} = \dot{\bar{a}}_1 + \dot{\bar{a}}_2 + \cdots + \dot{\bar{a}}_n \quad \text{—— ③}$$

$\Rightarrow$ Since Vector a_i is fixed ~~vector~~ to the i^{th} link,

$$\dot{\bar{a}}_i = \bar{\omega}_i \times \bar{a}_i \qquad i = 1, 2 \ldots n \quad \text{— (4)}$$

$\Rightarrow$ Furthermore, substitution of Eq (1) & (4) into eq (3) yields:-

$$\dot{\bar{p}} = \dot{\theta}_1 \bar{e}_1 \times \bar{a}_1 + (\dot{\theta}_1 \bar{e}_1 + \dot{\theta}_2 \bar{e}_2) \times \bar{a}_2 + \cdots$$
$$+ (\dot{\theta}_1 \bar{e}_1 + \dot{\theta}_2 \bar{e}_2 + \cdots + \dot{\theta}_n \bar{e}_n) \times \bar{a}_n \quad \text{— (5)}$$

$\Rightarrow$ Above can be rearranged as:-

$$\dot{\bar{p}} = \dot{\theta}_1 \bar{e}_1 \times (\bar{a}_1 + \bar{a}_2 + \cdots + \bar{a}_n) + \dot{\theta}_2 \bar{e}_2 \times (\bar{a}_2 + \bar{a}_3 + \cdots \bar{a}_n)$$
$$+ \cdots + \dot{\theta}_n \bar{e}_n \times \bar{a}_n$$

$\Rightarrow$ Now Vector $\bar{r}_i$ is defined as that joining O_i with P, directed from the former to the latter.

$$\bar{r}_i = \bar{a}_i + \bar{a}_{i+1} + \cdots + \bar{a}_n \quad \text{— (6)}$$

$\Rightarrow$ hence $\dot{\bar{p}}$ can be rewritten as:-

$$\dot{\bar{p}} = \sum_{i=1}^{n} \dot{\theta}_i \bar{e}_i \times \bar{r}_i$$

$\Rightarrow$ Further, let A and B denote the $3 \times n$ matrices defined as:-

$$A = [\bar{e}_1 \; \bar{e}_2 \cdots \bar{e}_n] \quad \text{— (7)}$$

$$B = [\bar{e}_1 \times \bar{r}_1, \bar{e}_2 \times \bar{r}_2, \ldots, \bar{e}_n \times \bar{r}_n]$$

the n-dimensional joint-rate vector $\dot{\bar{\theta}}$ being defined as:-

$$\dot{\bar{\theta}} = [\dot{\theta}_1 \; \dot{\theta}_2 \cdots \dot{\theta}_n]^T$$

⇒ Thus $\bar{\omega}$ and $\dot{\bar{p}}$ can be expressed in a more compact form as:-

$$\bar{\omega} = \bar{\bar{A}} \dot{\theta}$$
$$\dot{\bar{p}} = \bar{\bar{B}} \dot{\theta}$$

⇒ the twist of the EE being defined, in turn as:-

$$\bar{t} = \begin{bmatrix} \bar{\omega} \\ \dot{\bar{p}} \end{bmatrix} \quad \text{——— (8)}$$

⇒ The EE twist is thus linearly related to the joint-rate vector $\dot{\bar{\theta}}$

$$\boxed{\bar{\bar{J}} \dot{\bar{\theta}} = \bar{t}} \quad \text{—— (9)}$$

⇒ where $\bar{\bar{J}}$ is the jacobian matrix.

→ First introduced by Whitney (1972)

→ It is $6 \times n$ matrix.

$$\bar{\bar{J}} = \begin{bmatrix} \bar{\bar{A}} \\ \bar{\bar{B}} \end{bmatrix}$$

$$\bar{\bar{J}} = \begin{bmatrix} \bar{e}_1 & \bar{e}_2 & \cdots & \bar{e}_n \\ \bar{e}_1 \times \bar{r}_1 & \bar{e}_2 \times \bar{r}_2 & \cdots & \bar{e}_n \times \bar{r}_n \end{bmatrix} \quad \text{—— (10)}$$

⇒ Apparently, an alternative definition of the foregoing Jacobian matrix can be given as:-

$$\bar{\bar{J}} = \frac{\delta \bar{t}}{\delta \dot{\bar{\theta}}} \qquad (11)$$

If $\bar{j}_i$ denotes i^{th} column of $\bar{\bar{J}}$ then,

$$\bar{j}_i = \begin{bmatrix} \bar{e}_i \\ \bar{e}_i \times \bar{r}_i \end{bmatrix} \qquad (12)$$

⇒ It is noteworthy that if the axis of the i^{th} revolute is denoted by R_i then $\bar{j}_i$ is nothing but the Plücker array of that line, with the moment of R_i being taken with respect to the operation point P of the EE.

⇒ If i^{th} pair is prismatic, then the $(i-1)^{st}$ and i^{th} links have the same angular velocity.

Hence,

$$\bar{\omega}_i = \bar{\omega}_{i-1} \qquad \dot{\bar{a}}_i = \bar{\omega}_{i-1} \times \bar{a}_i + \dot{b}_i \bar{e}_i$$

⇒ One can readily prove in this case that:-

$$\overline{\omega} = \dot{\theta}_1 \overline{e}_1 + \dot{\theta}_2 \overline{e}_2 + \cdots \dot{\theta}_{i-1} \overline{e}_{i-1} + \dot{\theta}_{i+1} \overline{e}_{i+1} + \cdots + \dot{\theta}_n \overline{e}_n$$

$$\dot{P} = \dot{\theta}_1 \overline{e}_1 \times \overline{r}_1 + \dot{\theta}_2 \overline{e}_2 \times \overline{r}_2 + \dot{\theta}_{i-1} \overline{e}_{i-1} \times \overline{r}_{i-1} + \dot{b}_i \overline{e}_i$$
$$+ \dot{\theta}_{i+1} \overline{e}_{i+1} \times \overline{r}_{i+1} + \cdots + \dot{\theta}_n \overline{e}_n \times \overline{r}_n$$

⇒ If Vector $\dot{\overline{\theta}}$ is now defined as

$$\dot{\overline{\theta}} = [\dot{\theta}_1 \; \dot{\theta}_2 \; \cdots \; \dot{\theta}_{i-1} \; \dot{b}_i \; \dot{\theta}_{i+1} \cdots \dot{\theta}_n]$$

the i^{th} Column of J then changing to

$$\overline{J}_i = \begin{bmatrix} \overline{0} \\ \overline{e}_i \end{bmatrix}$$

Plücker array of the axis of the i^{th} joint, is that of a line at infinity lying in a plane normal to the unit vector $\overline{e}_i$

⇒ In general, $\overline{\overline{J}}_A$ denotes the Jacobian defined for a point A of the EE and $\overline{\overline{J}}_B$ that defined for another point B, then the relation between $\overline{\overline{J}}_A$ and $\overline{\overline{J}}_B$ is

$$\boxed{\overline{\overline{J}}_B = \overline{\overline{U}}\,\overline{\overline{J}}_A} \quad \text{— (13)}$$

Where the 6×6 matrix $\overline{U}$ is defined as:-

$$\overline{U} = \begin{bmatrix} \overline{\overline{I}} & \overline{0} \\ \overline{\overline{A}}-\overline{B} & \overline{\overline{I}} \end{bmatrix}$$

Where, $\bar{\bar{A}}$ & $\bar{\bar{B}}$ are Cross-Product matrices of the position vectors $\bar{a}$ and $\bar{b}$ of points A and B respectively.

<u>Theorem 5.2.1</u>: The determinant of the Jacobian matrix of a six-axis manipulator is not affected under a change of operation point of the EE.

$$\det(\bar{\bar{J}}_B) = \det(\bar{\bar{J}}_A)$$

⇒ Equation (13) is called <u>Jacobian transfer Matrix</u>

⇒ In particular, for six-axis manipulators, $\bar{\bar{J}}$ is a 6×6 matrix. Whenever this matrix is non-singular eq (5) can be solved for $\dot{\theta}$.

$$\boxed{\dot{\theta} = \bar{\bar{J}}^{-1} E} \quad (14)$$

This is solved using Gauss-elimination algorithm or LU Decomposition.

⇒ Gaussian elimination produces the solution by recognizing that a system of linear equations is most easily solved when it is in either <u>upper or lower-triangular form</u>.

⇒ To exploit this fact, matrix $\bar{\bar{J}}$ is decomposed into the unique $\bar{L}$ & $\bar{U}$ factor in the form :-

$$\bar{\bar{J}} = \bar{L}\,\bar{U}$$

$\bar{L}$ → Lower-triangular

$\bar{U}$ → Upper-triangular

⇒ Thus, the unknown vectors of joint rates can now be computed from two triangular systems, namely,

$$\boxed{\bar{L}\,\bar{y} = \bar{t}} \quad \text{— (15)}$$

$$\boxed{\bar{U}\,\dot{\bar{\theta}} = \bar{y}} \quad \text{— (16)}$$

⇒ The solution of a system of n linear equation in n unknowns, using the LU-decomposition method, can be accomplished with M_n multiplications & A_n additions :-

$$\left.\begin{aligned} M_n &= \frac{n}{6}\left(2n^2+9n+1\right) \\ A_n &= \frac{n}{3}\left(n^2+3n-4\right) \end{aligned}\right\} \quad \left\{\frac{\text{Dahlquist \& Bjorck}}{1974}\right\}$$

$$\text{eg} \left\{\begin{aligned} M_6 &= 127 \\ A_6 &= 100 \end{aligned}\right\}$$

5.2.1) <u>Decoupled Manipulator</u>

⇒ For manipulator of this type of architecture, it is more convenient to deal with the velocity of the center C of the wrist.

↳ Than with that of the operation point P.

⇒ Thus,

$$\bar{t}_c = \bar{\bar{J}}\,\dot{\bar{\theta}}$$

where, $\bar{t}_c$ is defined as

$$\bar{t}_c = \begin{bmatrix} \bar{\omega} \\ \dot{\bar{c}} \end{bmatrix}$$

⇒ and can be obtained from $\bar{t}_p = [\bar{\omega}^T\ \dot{\bar{p}}^T]^T$ using the twist-transfer formula :-

$$\bar{t}_c = \begin{bmatrix} \bar{\bar{I}} & \bar{\bar{O}} \\ \bar{\bar{P}}-\bar{\bar{C}} & \bar{\bar{I}} \end{bmatrix} \bar{t}_p$$

with $\bar{C}$ & $\bar{P}$ defined as the cross-product matrices of the position vectors $\bar{c}$ & $\bar{p}$ respectively.

⇒ Since C is on the last three joint axes, its velocity is not affected by the motion of the last three joints, and hence, we can write :-

$$\bar{C} = \dot{\theta}_1 \bar{e}_1 \times \bar{r}_1 + \dot{\theta}_2 \bar{e}_2 \times \bar{r}_2 + \dot{\theta}_3 \bar{e}_3 \times \bar{r}_3$$

$\left\{ \bar{r}_i \text{ is defined as that directed from } O_i \text{ to } C \right\}$

$\Rightarrow$ On the

CHAPTER 6

Appendix: 1

Linear Transformation which Maps a Space On to itself is Called Linear Operator.

Reff 1

Not necessarily basis, Valid for any 3 Vector $\bar{p}, \bar{q}, \bar{r}$

Let $\bar{B} = \{\bar{p}\ \bar{q}\ \bar{r}\}$ be a basis of an $\mathcal{E}^3$ space.

$\Rightarrow$ After rotation let the basis be $\bar{B}' = \{\bar{p}'\ \bar{q}'\ \bar{r}'\}$

$$|\bar{B}| = \left|\begin{bmatrix}\bar{p} & \bar{q} & \bar{r}\end{bmatrix}\right| = \left|\begin{bmatrix}\bar{p}' & \bar{q}' & \bar{r}'\end{bmatrix}\right| = |\bar{B}'|$$

$$|\bar{p}\ \bar{q}\ \bar{r}| = \bar{r}\cdot(\bar{p}\times\bar{q})$$

$$|\bar{p}'\ \bar{q}'\ \bar{r}'| = \bar{r}'\cdot(\bar{p}'\times\bar{q}')$$

As relative orientation of $\bar{B} \longrightarrow \bar{B}'$ is not Changed, but only rotated, so Scalar tripple Product will not Change.

Hence,

$$\boxed{|\bar{B}| = |\bar{B}'|}$$

Orthogonal Matrix $\Rightarrow$ Square matrix whose Columns and rows are a Orthonormal Vector

$$\bar{\bar{Q}}^T\bar{\bar{Q}} = \bar{\bar{Q}}\bar{\bar{Q}}^T = \bar{\bar{I}}$$

Proper orthogonal Matrix $\Rightarrow$ Orthogonal matrix with determinant Unity.

$$\boxed{|\bar{\bar{Q}}| = 1}$$

Raff 2

$$\bar{a} \times (\bar{a} \times \bar{x}) = \bar{A}^2 \bar{x}$$

$$\Rightarrow \bar{A}^2 \bar{x} = \bar{a}(\bar{a}^T \bar{x}) - \bar{x}(\bar{a}^T \bar{a}) \quad \left\{ \begin{aligned} &\bar{a} \times (\bar{b} \times \bar{c}) \\ &= \bar{b}(\bar{a} \cdot \bar{c}) - \bar{c}(\bar{a} \cdot \bar{b}) \end{aligned} \right\}$$

$$\Rightarrow \bar{A}^2 \bar{x} = \bar{a}\bar{a}^T \bar{x} - \|\bar{a}\|^2 \bar{x}$$

$$\Rightarrow \bar{A}^2 \bar{x} = \left(-\|\bar{a}\|^2 \bar{I} + \bar{a}\bar{a}^T\right) \bar{x}$$

So $\boxed{\bar{A}^2 = -\|\bar{a}\|^2 \bar{I} + \bar{a}\bar{a}^T}$

$\Rightarrow$ For a fixed axis of rotation {fixed value of $\bar{e}$}

Raff 3

↳ Rotation matrix $\bar{\bar{Q}}$ is a function of the angle of rotation φ only.

$\Rightarrow$ Thus series expansion of $\bar{Q}$ in terms of φ is:-

$$\bar{\bar{Q}}(\varphi) = \bar{\bar{Q}}(0) + \bar{\bar{Q}}'(0)\varphi + \frac{1}{2!}\bar{\bar{Q}}''(0)\varphi^2 + \cdots + \frac{1}{k!}\bar{\bar{Q}}^{k}(0)\varphi^k + \cdots$$

$\Rightarrow$ from defination of $\bar{\bar{E}}$, one can readily prove the relations below:

$$\bar{\bar{E}}^{(2k+1)} = (-1)^k \bar{\bar{E}} \qquad \bar{\bar{E}}^{2k} = (-1)^k (\bar{\bar{I}} - \bar{e}\bar{e}^T) \quad \text{— (1)}$$

$\Rightarrow$ We know,

$$\boxed{\bar{\bar{Q}} = \bar{e}\bar{e}^T + \cos\varphi(\bar{\bar{I}} - \bar{e}\bar{e}^T) + \sin\varphi \bar{\bar{E}}} \quad \text{— (2)}$$

⇒ Using eq ① & ② one can readily show that :-

$$\bar{\bar{Q}}^{(K)}(0) = \bar{\bar{E}}^{K}$$

$$\frac{d\bar{\bar{Q}}}{d\phi} = -\sin\phi\,(\bar{\bar{I}} - \bar{e}\bar{e}^{T}) + \cos\phi\,\bar{\bar{E}}$$

$$\left.\frac{d\bar{\bar{Q}}}{d\phi}\right|_{\phi=0} = \bar{\bar{E}}$$

$$\frac{d^2\bar{\bar{Q}}}{d\phi^2} = -\cos\phi\,(\bar{\bar{I}} - \bar{e}\bar{e}^{T}) - \sin\phi\,\bar{\bar{E}}$$

$$\left.\frac{d^2\bar{\bar{Q}}}{d\phi^2}\right|_{\phi=0} = -(\bar{\bar{I}} - \bar{e}\bar{e}^{T}) = \bar{\bar{E}}^{2}$$

$$\frac{d^3\bar{\bar{Q}}}{d\phi^3} = \sin\phi\,(\bar{\bar{I}} - \bar{e}\bar{e}^{T}) - \cos\phi\,\bar{\bar{E}}$$

$$\left.\frac{d^3\bar{\bar{Q}}}{d\phi}\right|_{\phi=0} = -\bar{\bar{E}} = \bar{\bar{E}}^{3}$$

$$\frac{d^4\bar{\bar{Q}}}{d\phi^4} = \cos\phi\,(\bar{\bar{I}} - \bar{e}\bar{e}^{T}) + \sin\phi\,\bar{\bar{E}}$$

$$\left.\frac{d^4\bar{\bar{Q}}}{d\phi^4}\right|_{\phi=0} = (\bar{\bar{I}} - \bar{e}\bar{e}^{T}) = \bar{\bar{E}}^{4}$$

So

$$\overline{\overline{Q}}(\varphi) = \overline{\overline{I}} + \overline{\overline{E}}\varphi + \frac{1}{2!}\overline{\overline{E}}^2\varphi^2 + \cdots + \frac{1}{K!}\overline{\overline{E}}^K\varphi^K + \cdots$$

$$\boxed{\overline{\overline{Q}}(\varphi) = e^{\overline{\overline{E}}\varphi}}$$

$$\overline{\overline{Q}}(\varphi) = \overline{\overline{I}} + \left(\overline{\overline{E}}\varphi + \frac{1}{3!}\overline{\overline{E}}^3\varphi^3 + \frac{1}{5!}\overline{\overline{E}}^5\varphi^5 + \cdots\right)$$

$$+ \left(\frac{1}{2!}\overline{\overline{E}}^2\varphi^2 + \frac{1}{4!}\overline{\overline{E}}^4\varphi^4 + \cdots\right)$$

$$\Rightarrow \overline{\overline{I}} + \left(\varphi - \frac{1}{3!}\varphi^3 + \cdots + \frac{1}{(2K+1)!}(-1)^K\varphi^{2K+1} + \cdots\right)\overline{\overline{E}}$$

$$+ \left(-\frac{1}{2!}\varphi^2 + \frac{1}{4!}\varphi^4 - \cdots + \frac{1}{(2K)!}(-1)^K\varphi^{2K} + \cdots\right)(\overline{\overline{I}} - ee^T)$$

So $\boxed{\overline{\overline{Q}} = \overline{\overline{I}} + \sin\varphi\,\overline{\overline{E}} + (1 - \cos\varphi)\overline{\overline{E}}^2}$