Contents

5. State-Space Analysis

6. Pole placement and Observer Design.

7. Polynomial Equations Approach to Control System Design

8. Quadratic Optimal Control Systems.

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CHAPTER 1

State Space Analysis

5

State-Space Analysis

5.1) Introduction

⇒ The state-space method is based on the description of system equations in terms of n first order difference equations or differential equations.

⇓

⇒ This may be combined into a first order vector-matrix differential or difference equation.

⇓

⇒ The use of the vector-matrix notation greatly simplifies the mathematical representation of the systems of equations.

⇒ "Design in state space can be carried out for a class of input, instead of a specific input function such as the impulse function, step function or sinusoidal function.

↳ State-Space methods enables the engineer to include initial conditions in the design.

State ⇒ The state of a dynamic system is the smallest set of variables (called state variables) such that the knowledge of these variables at $t = t_0$, together with the knowledge of the input for $t \geq t_0$, completely determines the behavior of the system for any time $t \geq t_0$.

State Varibles ⇒ Variables making up the Smallest Set of Varibles that determine the state of the dynamic System.

⇒ State varibles need not be physically measurable or observable quantities.

State Vector ⇒ If n state Varibles are needed to completely describe the behavior of a given System, then these n State Varibles can be considered the n Components of a Vector $\overline{X}$. Such a vector is called State Vector.

State Space ⇒ The n-dimensional Space whose Coordinate axes consist of the x_1 axis, x_2 axis ---- x_n axis is Called a State-Space.

⇒ Any state can be represented by a point in the State Space.

State-Space Equation

⇒ There are three types of Varibles that are involved in the modeling of dynamic Systems :-

(1) Input varibles
(2) Output Varibles
(3) State Varibles

⇒ For time-Varying (linear or Nonlinear) descrete-time Control Systems, the State equation may be written as:-

$$\bar{x}(K+1) = f[\bar{x}(K), \bar{u}(K), K] \quad \text{--- (1)}$$

and the output equation as:-

$$\bar{y}(K) = g[\bar{x}(K), \bar{u}(K), K] \quad \text{--- (2)}$$

⇒ For linear time Varying discrete-time Systems, the state equation and output equation may be simplified to

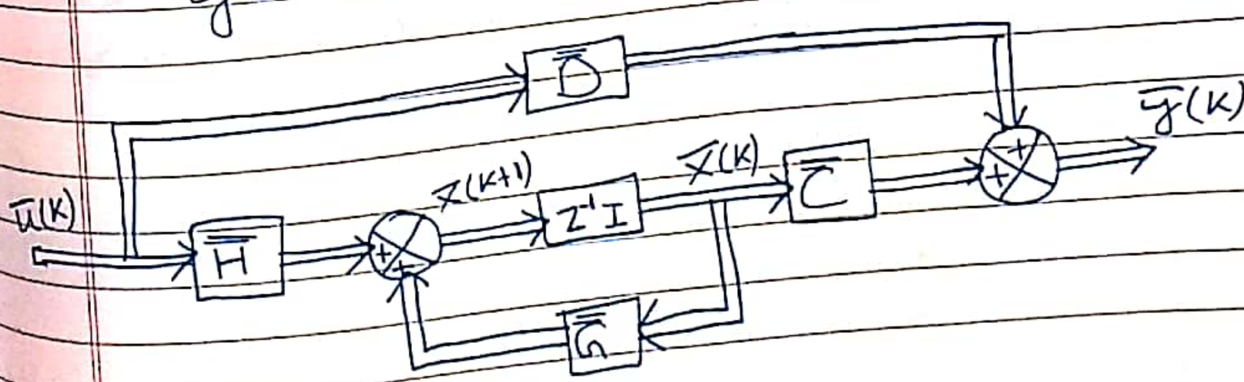
$$\bar{x}(K+1) = \bar{\bar{G}}(K)\bar{x}(K) + \bar{\bar{H}}(K)\bar{u}(K)$$

$$\bar{y}(K) = \bar{\bar{C}}(K)\bar{x}(K) + \bar{\bar{D}}(K)\bar{u}(K)$$

⇒ For linear time invariant descrete time control systems:-

$$\bar{x}(K+1) = \bar{\bar{G}}\bar{x}(K) + \bar{\bar{H}}u(t)$$

$$\bar{y}(K) = \bar{C}\bar{x}(K) + \bar{D}\bar{u}(K)$$



5.2) State Space Representations of Discrete-time System

Canonical forms of discrete-time State Space Equations

Consider the discrete time System described by:-

$$y(k) + a_1 y(k-1) + a_2 y(k-2) + \cdots + a_n y(k-1)$$
$$= b_0 u(k) + b_1 u(k-1) + \cdots + b_n u(k-n)$$

$u(k) \Rightarrow$ Input
$y(k) \Rightarrow$ Output
at k^{th} Sampling instat

⇒ The above equation may be written in the form of the pulse transfer function as:-

$$\frac{Y(z)}{U(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_n z^{-n}}{1 + a_1 z^{-1} + \cdots + a_n z^{-n}}$$

⇒ There are many ways to realize state-space representations for the discrete time System described above:-

1) Controllable Canonical form
2) Observable Canonical form
3) Diagonal Canonical form
4) Jordan Canonical form

① Controllable Canonical form

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_{n-1}(k+1) \\ x_n(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_{n-1}(k) \\ x_n(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \left[b_n - a_n b_0 \; \vdots \; b_{n-1} - a_{n-1} b_0 \; \vdots \; \cdots \; \vdots \; b_1 - a_1 b_0 \right] \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{bmatrix} + b_0 u(k)$$

② Observable Canonical form

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_{n-1}(k+1) \\ x_n(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & -a_n \\ 1 & 0 & \cdots & 0 & 0 & -a_{n-1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & -a_2 \\ 0 & 0 & \cdots & 0 & 1 & -a_1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_{n-1}(k) \\ x_n(k) \end{bmatrix} + \begin{bmatrix} b_n - a_n b_0 \\ b_{n-1} - a_{n-1} b_0 \\ \vdots \\ b_2 - a_2 b_0 \\ b_1 - a_1 b_0 \end{bmatrix} u(k)$$

$$y(k) = [0 \; 0 \; \cdots \; 0 \; 1] \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_{n-1}(k) \\ x_n(k) \end{bmatrix} + b_0 u(k)$$

③ Diagonal Canonical form

Let $P_1\ P_2 \cdots P_n$ be n poles of pulse transfer function and corresponding residue be $C_1\ C_2 \cdots C_n$.

⇒ If all poles are distinct, then the state-space representation may be put in the diagonal canonical form as follows:-

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_n(k+1) \end{bmatrix} = \begin{bmatrix} P_1 & 0 & \cdots & 0 \\ 0 & P_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & P_n \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} C_1 & C_2 & \cdots & C_n \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{bmatrix} + b_0 u(k)$$

④ Jordan Canonical Form

⇒ If the pulse transfer function given by involves a multiple pole of order m at $z = P_1$ and all other poles are distinct.

⇒ ~~PlA~~ $C_1\ C_2 \cdots C_m$ are residue at $\frac{1}{z-P_1}$ $\frac{1}{(z-P_1)^2} \cdots \frac{1}{(z-P_1)^m}$ respectively.

⇒ and $C_{m+1}\ C_{m+2} \cdots C_n$ are residue at poles $P_{m+1}\ P_{m+2} \cdots P_n$ respectively.

$$\begin{bmatrix} x_1(K+1) \\ x_2(K+1) \\ \vdots \\ x_m(K+1) \\ \hline x_{m+1}(K+1) \\ \vdots \\ x_n(K+1) \end{bmatrix} = \left[\begin{array}{ccccc|ccc} P_1 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & P_1 & 1 & \cdots & & 0 & \cdots & 0 \\ \vdots & & & & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & P_1 & 0 & \cdots & 0 \\ \hline 0 & 0 & 0 & \cdots & 0 & P_{m+1} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & P_n \end{array}\right] \begin{bmatrix} x_1(K) \\ x_2(K) \\ \vdots \\ x_m(K) \\ x_{m+1}(K) \\ \vdots \\ x_n(K) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} U(K),$$

$$y(K) = \begin{bmatrix} C_1 & C_2 & \cdots & C_n \end{bmatrix} \begin{bmatrix} x_1(K) \\ x_2(K) \\ \vdots \\ x_n(K) \end{bmatrix} + b_0 U(K)$$

★ Nonuniqueness of State Space Representation

⇒ For a given pulse-transfer function system the state-space representation is not unique.

⇒ The state equation is however related to each other by the similarity transformation.

⇒ Consider the system defined by:-

$$\boxed{\begin{aligned} \overline{X}(K+1) &= \overline{G}\,\overline{X}(K) + \overline{H}\,\overline{U}(K) \quad \text{— (1)} \\ \overline{Y}(K) &= \overline{C}\,\overline{X}(K) + \overline{D}\,\overline{U}(K) \quad \text{— (2)} \end{aligned}}$$

Let us define a new state vector as $\hat{\overline{X}}(K)$ by

$$\overline{X}(K) = \overline{P}\hat{X} \quad \{\overline{P} \text{ is non singular matrix}\}$$

Then,

$$\bar{\bar{P}}\,\hat{\bar{x}}(K+1) = \bar{\bar{G}}\,\bar{\bar{P}}\,\hat{\bar{x}}(K) + \bar{\bar{H}}\,\bar{u}(K)$$

$$\Rightarrow \hat{\bar{x}}(K+1) = \bar{\bar{P}}^{-1}\bar{\bar{G}}\,\bar{\bar{P}}\,\hat{\bar{x}}(K) + \bar{\bar{P}}^{-1}\bar{\bar{H}}\,u(K)$$

Let $\hat{\bar{\bar{G}}} = \bar{\bar{P}}^{-1}\bar{\bar{G}}\,\bar{\bar{P}}$ & $\bar{\bar{P}}^{-1}\bar{\bar{H}} = \hat{\bar{\bar{H}}}$

So $$\boxed{\hat{\bar{x}}(K+1) = \hat{\bar{\bar{G}}}\,\hat{\bar{x}}(K) + \hat{\bar{\bar{H}}}\,\bar{u}(K)} \quad \text{—③}$$

Similarly $$\boxed{\bar{y}(K) = \hat{\bar{\bar{C}}}\,\hat{\bar{x}}(K) + \hat{\bar{\bar{D}}}\,\bar{u}(K)} \quad \text{—④}$$

Where, $\hat{\bar{\bar{C}}} = \bar{\bar{C}}\,\bar{\bar{P}}$; $\bar{\bar{D}} = \hat{\bar{\bar{D}}}$

⇒ State-Space representation given by eq ①, ② is equivalent to the State Space representation by eq ③, ④.

5.3) Solving discrete-time State Space Equations

$$\bar{x}(K+1) = \bar{\bar{G}}\,\bar{x}(K) + \bar{\bar{H}}\,\bar{u}(K) \quad \text{—①}$$
$$\bar{y}(K) = \bar{\bar{C}}\,\bar{x}(K) + \bar{\bar{D}}\,u(K) \quad \text{—②}$$

⇒ In general, discrete-time equations are easier to solve than differential equations because the former can be solved easily by means of a recursion procedure.

↳ Recursion procedure is quite simple and convenient for digital computations.

$$\bar{x}(1) = \bar{\bar{G}}\bar{x}(0) + \bar{\bar{H}}\bar{u}(0)$$

$$\bar{x}(2) = \bar{\bar{G}}\bar{x}(1) + \bar{\bar{H}}\bar{u}(1)$$

$$= \bar{\bar{G}}^2\bar{x}(0) + \bar{\bar{G}}\bar{\bar{H}}u(0) + \bar{\bar{H}}\bar{u}(1)$$

$$\bar{x}(3) = \bar{\bar{G}}\bar{x}(2) + \bar{\bar{H}}\bar{u}(2)$$

$$= \bar{\bar{G}}^3\bar{x}(0) + \bar{\bar{G}}^2\bar{\bar{H}}\bar{u}(0) + \bar{\bar{G}}\bar{\bar{H}}\bar{u}(1) + \bar{\bar{H}}\bar{u}(2)$$

$$\vdots$$

$$\boxed{\bar{x}(k) = \bar{\bar{G}}^k x(0) + \sum_{j=0}^{k-1} \bar{\bar{G}}^{k-j-1}\bar{\bar{H}}\bar{u}(j)}$$

$$\bar{y}(k) = \bar{\bar{C}}\,\bar{\bar{G}}^k\bar{x}(0) + C\sum_{j=0}^{k-1} \bar{\bar{G}}^{k-j-1}\bar{\bar{H}}\bar{u}(j) + \bar{\bar{D}}\bar{u}(k)$$

★ State transition matrix

⇒ It is possible to write the solution of the homogeneous state equation

$$\bar{x}(k+1) = \bar{\bar{G}}\bar{x}(k)$$

as

$$\bar{x}(k) = \bar{\bar{\Psi}}(k)\,\bar{x}(0)$$

→ It is unique $n \times n$ matrix satisfying:

(1) $\bar{\bar{\Psi}}(k+1) = \bar{\bar{G}}\,\bar{\bar{\Psi}}(k)$

2) $\bar{\bar{\Psi}}(0) = I$

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So $\boxed{\bar{\bar{\Psi}}(k) = \bar{\bar{G}}^K}$

⇒ We see that solution is simply a transformation of the initial state. Therefore $\bar{\bar{\Psi}}(k)$ is called state transition matrix.

So,

$$\boxed{\bar{X}(k) = \bar{\bar{\Psi}}(k)\bar{X}(0) + \sum_{j=0}^{K-1} \bar{\bar{\Psi}}(k-j-1)\bar{\bar{H}}\bar{u}(j)}$$

$$\boxed{\bar{X}(k) = \bar{\bar{\Psi}}(k)\bar{X}(0) + \sum_{j=0}^{K-1} \bar{\bar{\Psi}}(j)\bar{\bar{H}}\bar{u}(k-j-1)}$$

So

$$\boxed{\bar{y}(k) = \bar{\bar{C}}\bar{\bar{\Psi}}(k)\bar{X}(0) + C\sum_{j=0}^{K-1} \bar{\bar{\Psi}}(j)\bar{\bar{H}}\bar{u}(k-j-1) + \bar{\bar{D}}\bar{u}(k)}$$

★ Z transform approach to the solution of Discrete-time State Equation

$$\bar{X}(k+1) = \bar{\bar{G}}\bar{X}(k) + \bar{\bar{H}}\bar{u}(k)$$

⇒ Taking Z-transform both side:-

$$z\bar{x}(z) - z\bar{x}(0) = \bar{\bar{G}}\bar{x}(z) + \bar{\bar{H}}\bar{u}(z)$$

$$\bar{x}(z) = Z\{\bar{X}(k)\} \;\&\; u(z) = Z\{u(k)\}$$

$$\Rightarrow (z\bar{\bar{I}} - \bar{\bar{G}})\bar{x}(z) = z\bar{X}(0) + \bar{\bar{H}}\bar{u}(z)$$

$$x(z) = (z\bar{\bar{I}} - \bar{\bar{G}})^{-1} z\bar{X}(0) + (z\bar{\bar{I}} - \bar{\bar{G}})^{-1}\bar{\bar{H}}\bar{u}(z)$$

$$\bar{X}(K) = Z^{-1}\left\{(z\bar{\bar{I}} - \bar{\bar{G}})^{-1} z\right\}\bar{X}(0) + Z^{-1}\left\{(z\bar{\bar{I}} - \bar{\bar{G}})^{-1}\bar{\bar{H}}\bar{U}(z)\right\}$$

So $G^{K} = Z^{-1}\left\{(z\bar{\bar{I}} - \bar{\bar{G}})^{-1} z\right\}$

& $\sum_{j=0}^{K-1} \bar{\bar{G}}^{K-j-1}\bar{\bar{H}}\bar{u}(j) = Z^{-1}\left\{(z\bar{\bar{I}} - \bar{\bar{G}})^{-1}\bar{\bar{H}}\bar{u}(z)\right\}$

★ <u>Computing $(zI-G)^{-1}$</u>

\# $(z\bar{\bar{I}} - \bar{\bar{G}})^{-1} = \dfrac{\text{adj}(z\bar{\bar{I}} - \bar{\bar{G}})}{|z\bar{\bar{I}} - \bar{\bar{G}}|}$

\# $|z\bar{\bar{I}} - \bar{\bar{G}}| = z^{n} + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n$

where, $a_1 = -\text{tr}\,\bar{\bar{G}}$

$a_2 = -\frac{1}{2}\text{tr}\,\bar{\bar{G}}\bar{\bar{H}}_1$ $\quad\quad \bar{\bar{H}}_1 = \bar{\bar{G}} + a_1\bar{\bar{I}}$

$a_3 = -\frac{1}{3}\text{tr}\,\bar{\bar{G}}\bar{\bar{H}}_2$ $\quad\quad \bar{\bar{H}}_2 = \bar{\bar{G}}\bar{\bar{H}}_1 + a_2\bar{\bar{I}}$

$\vdots \quad\quad \vdots$ $\quad\quad \bar{\bar{H}}_3 = \bar{\bar{G}}H_2 + a_3\bar{\bar{I}}$

$\vdots \quad\quad \vdots \quad\quad \vdots$

$a_n = -\frac{1}{n}\text{tr}\,\bar{\bar{G}}\bar{\bar{H}}_{n-1}$ $\quad\quad \bar{\bar{H}}_{n-1} = \bar{\bar{G}}\bar{\bar{H}}_{n-2} + a_{n-1}\bar{\bar{I}}$

$$\text{adj}(z\bar{I}-G) = \bar{I}z^{n-1} + \bar{H}_1 z^{n-2} + \bar{H}_2 z^{n-3} + \cdots + H_{n-1}$$

★ Solution of Linear Time-Varying Discrete Time State Equation

$$\bar{x}(k+1) + \bar{G}(k)\bar{x}(k) + \bar{H}(k)\bar{u}(k) \quad \text{--- (1)}$$

$$Y(k) = \bar{C}(k)\bar{x}(k) + \bar{D}(k)\bar{u}(k) \quad \text{--- (2)}$$

⇒ The Solution of Eq (1) can be easily found by recursion:-

$$\bar{x}(h+1) = \bar{G}(h)\bar{x}(h) + \bar{H}(h)\bar{u}(h)$$

$$\bar{x}(h+2) = \bar{G}(h+1)\bar{x}(h+1) + \bar{H}(h+1)\bar{u}(h+1)$$

$$\Rightarrow \bar{G}(h+1)\bar{G}(h)\bar{x}(h) + \bar{G}(h+1)\bar{H}(h)\bar{u}(h)$$

$$+ \bar{H}(h+1)\bar{u}(h+1)$$

⇒ Let us define the state transition matrix for the System defined by eq (1) as $\bar{\Psi}(k,h)$

⇒ It is a unique matrix satisfying the Conditions:-

$$\bar{\Psi}(k+1,h) = \bar{G}(k)\bar{\Psi}(k,h)$$

$$\bar{\Psi}(h,h) = \bar{I}$$

Where $k = h, h+1, h+2, \cdots$

$$\boxed{\bar{\Psi}(K,h) = \bar{\bar{G}}(K-1)\,\bar{\bar{G}}(K-2)\cdots\bar{\bar{G}}(h)} \quad K>h$$

So,

$$\boxed{\bar{X}(k) = \bar{\bar{\Psi}}(K,h)\,\bar{X}(h) + \sum_{j=h}^{K-1} \bar{\Psi}(K,j+1)\,\bar{H}(j)\,\bar{u}(j)}$$

⇒ Output Equation:-

$$\boxed{\bar{y}(K) = \bar{\bar{C}}(K)\,\bar{\Psi}(K,h)\,\bar{X}(h) + \sum_{j=h}^{K-1} \bar{\bar{C}}(K)\,\bar{\bar{\Psi}}(K,j+1)\,\bar{H}(j)\,\bar{u}(j) + \bar{\bar{D}}(K)\,\bar{u}(K)}$$

⇒ If $\bar{G}(K)$ is non Singular for all K values Considered, so that the inverse of $\bar{\Psi}(K,h)$ exists then the inverse of $\bar{\bar{\Psi}}(K,h)$ is denoted by $\bar{\Psi}(h,K)$

$$\bar{\Psi}^{-1}(K,h) = \bar{\Psi}(h,K)$$

$$= [G(K-1)\,G(K-2)\cdots G(h)]^{-1}$$

$$= G(h)^{-1}\,G^{-1}(h+1)\cdots G^{-1}(K-1)$$

* Properties of $\bar{\Psi}(K,h)$

1. $\bar{\Psi}(K,K) = \bar{\bar{I}}$
2. $\bar{\bar{\Psi}}(K,h) = \bar{\bar{G}}(K-1)\,\bar{\bar{G}}(K-2)\cdots\bar{\bar{G}}(h)$
3. If inverse of $\bar{\Psi}(K,h)$ exist then,

$$\bar{\Psi}^{-1}(K,h) = \bar{\bar{\Psi}}(h,K)$$

4. If $\bar{G}(K)$ is nonsingular for all K values considered, then

$$\bar{\Psi}(K,i) = \bar{\Psi}(K,j)\,\bar{\Psi}(j,i)$$

$$\forall\ i\ j\ \&\ K$$

If $\bar{G}(K)$ is singular for any value of K then

$$\bar{\Psi}(K,i) = \bar{\Psi}(K,j)\,\bar{\Psi}(j,i) \quad \forall\ K > j > i$$

5.4) Pulse-Transfer function Matrix

⇒ A single input - single output discrete time system may be modeled by a pulse transfer function.

↳ Extension of the pulse transfer function concept to a multiple input multiple output discrete time system gives us the pulse transfer function matrix.

★ Pulse transfer-function matrix

⇒ The state space representation of an nth order linear time-invariant discrete-time system with r input and m output can be given by:

$$\bar{x}(K+1) = \bar{G}\,\bar{x}(K) + \bar{H}\,\bar{u}(K)$$
$$\bar{y}(K) = \bar{C}\,\bar{x}(K) + \bar{D}\,\bar{u}(K)$$

⇒ Taking the Z transform we obtain:-

$$z\bar{X}(z) - z\bar{x}(0) = \bar{\bar{G}}\bar{X}(z) + \bar{\bar{H}}\bar{U}(z)$$
$$\bar{Y}(z) = C\bar{X}(z) + \bar{\bar{D}}\bar{U}(z)$$

⇒ Defination for pulse transfer function calls for the assumption of zero initial conditions.

$$\Rightarrow \quad \bar{X}(z) = (z\bar{\bar{I}} - \bar{\bar{G}})^{-1}\bar{\bar{H}}\bar{U}(z)$$

$$\bar{Y}(z) = \left[\bar{\bar{C}}(z\bar{\bar{I}} - \bar{\bar{G}})^{-1}\bar{\bar{H}} + \bar{\bar{D}}\right]\bar{U}(z) = F(z)\,U(z)$$

Where, $F(z) = \bar{\bar{C}}(z\bar{\bar{I}} - \bar{\bar{G}})^{-1}\bar{\bar{H}} + \bar{\bar{D}}$

⇒ $F(z)$ is called the pulse transfer function matrix

Since $(z\bar{\bar{I}} - \bar{\bar{G}})^{-1} = \dfrac{\mathrm{adj}\,(z\bar{\bar{I}} - \bar{\bar{G}})}{|z\bar{\bar{I}} - \bar{\bar{G}}|}$

$$\bar{\bar{F}}(z) = \frac{\bar{\bar{C}}\,\mathrm{adj}(z\bar{\bar{I}} - \bar{\bar{G}})\,\bar{\bar{H}}}{|z\bar{\bar{I}} - \bar{\bar{G}}|} + \bar{\bar{D}}$$

⇒ Clearly poles of $F(z)$ are the zeros of $|z\bar{\bar{I}} - \bar{\bar{G}}|$

⇒ This means characteristic equation of the discrete-time system is given by

$$|z\bar{\bar{I}} - \bar{\bar{G}}| = 0$$

$$\Rightarrow \boxed{z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0}$$

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Date
Page

★ Similarity Transformation

Let $\bar{x}(k) = \bar{P}\,\hat{\bar{x}}(k)$

So, $\hat{\bar{x}}(k+1) = \hat{\bar{G}}\hat{\bar{x}}(k) + \hat{\bar{H}}\bar{u}(k)$

$\bar{y}(k) = \hat{\bar{C}}\hat{\bar{x}}(k) + \hat{\bar{D}}u(k)$

$$\hat{\bar{G}} = \bar{P}^{-1}\bar{G}\bar{P}$$
$$\hat{\bar{H}} = \bar{P}^{-1}\bar{H}$$
$$\hat{\bar{C}} = \bar{C}\bar{P}$$
$$\hat{\bar{D}} = \bar{D}$$

$$\boxed{\hat{\bar{F}}(z) = \hat{\bar{C}}(z\bar{I}-\hat{\bar{G}})^{-1}\hat{\bar{H}} + \hat{\bar{D}} = \bar{F}(z)}$$

⇒ Thus pulse transfer function is invariant under Similarity transformation.

5.5) Discretization of Continuous Time State-Space Equations

⇒ We assume that the input vector $\bar{u}(t)$ changes only at equally spaced Sampling instants.

⇒ Consider the Continuous-time state equation and output equation as:-

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}\bar{u} \quad \text{—} (1)$$
$$\bar{y} = \bar{C}\bar{x} + \bar{D}\bar{u} \quad \text{—} (2)$$

⇒ The discrete-time representation of Equation ① will take the form :-

$$\bar{x}((k+1)T) = \bar{\bar{G}}(T)\,\bar{x}(kT) + \bar{\bar{H}}(T)\,\bar{u}(kT) \quad \text{—②}$$

→ $\bar{\bar{G}}$ & $\bar{\bar{H}}$ depends on Sampling period T. Once the Sampling Period T is fixed, G and H are Constant matrix.

We Know,

$$\bar{x}(t) = e^{\bar{A}t}\bar{x}(0) + \int_0^t e^{\bar{A}(t-\tau)}\bar{B}u(\tau)\,d\tau$$

⇒ We assume that the input $u(t)$ is sampled and fed to a Zero Order hold so that all the Components of $u(t)$ are Constant over the interval between any two Consecutive Sampling instants.

$$u(t) = u(kT) \quad \forall \; kT \leq t \leq kT+T \quad \text{—③*}$$

$$\bar{x}((k+1)T) = e^{\bar{A}(k+1)T}\bar{x}(0) + e^{\bar{A}(k+1)T}\int_0^{(k+1)T} e^{-\bar{A}\tau}\bar{\bar{B}}\bar{u}(\tau)\,d\tau \quad \text{—④}$$

&

$$\bar{x}(kT) = e^{\bar{A}kT}\bar{x}(0) + e^{\bar{A}kT}\int_0^{kT} e^{-\bar{A}\tau}\bar{\bar{B}}u(\tau)\,d\tau \quad \text{—⑤}$$

⇒ Multiplying eq (5) with $e^{\bar{A}T}$ and Subtracting it from eq (4) we get:-

$$\bar{x}((K+1)T) = e^{\bar{A}T}\bar{x}(KT) + e^{\bar{A}(K+1)T}\int_{KT}^{(K+1)T} e^{-\bar{A}\tau}\bar{B}\bar{u}(\tau)\,d\tau \quad \text{— (6)}$$

Sol:

As Using eq (3*) and (6) and Comparing it with eq (2) we get.

$$\boxed{G(T) = e^{\bar{A}T}} \quad \text{— (7)}$$

$$\boxed{H(T) = \left(\int_0^T e^{\bar{A}\lambda}\,d\lambda\right)\bar{B}} \quad \text{— (8)}$$

⇒ The output equation becomes :-

$$\bar{y}(KT) = \bar{C}\,\bar{x}(KT) + \bar{D}\,\bar{u}(KT)$$

→ Matrix C and D are constant matrices and do not depend on the Sampling period T.

⇒ If matrix $\bar{A}$ is non singular, then $\bar{H}(T)$ Can be Simplified to :-

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$$\bar{H}(T) = \left(\int_0^T e^{\bar{A}\lambda} d\lambda\right)\bar{B} = \bar{A}^{-1}(e^{\bar{A}T} - I)\bar{B} = (e^{\bar{A}T} - I)A^{-1}B$$

★ Time response between two consecutive Sampling Instants

⇒ In a Sampled Continuous time System the output is Continuous in time.

↳ But Z transform solution of the discrete time System equation gives the output only between two consecutive sampling instants.

⇒ Consider the time invariant Continuous-time System defined by

$$\dot{\bar{X}} = \bar{A}\bar{X} + \bar{B}\bar{u}$$
$$\bar{y} = \bar{C}\bar{x} + \bar{D}\bar{u}$$

Let us assume that the input $\bar{u}$ is Sampled and fed to a Zero Order hold.

Then $u(\tau) = u(KT) \quad \forall \; KT \leq \tau < KT+T$

⇒ Solution of the State equation starting with the initial state $\bar{x}(t_0)$ is

$$\bar{X}(t) = e^{\bar{A}(t-t_0)}\bar{x}(t_0) + \int_0^{t} e^{\bar{A}(t-\tau)}\bar{B}\bar{u}(\tau)\,d\tau$$

⇒ To obtain the response of the Sampled

System at $t = KT + \Delta T$ when $0 < \Delta T < T$

We put $t = KT + \Delta T$, $t_0 = KT$ & $u(\tau) = u(KT)$

$$\Rightarrow \bar{x}(KT+\Delta T) = e^{\bar{A}\Delta T} x(KT) + \int_{KT}^{KT+\Delta T} e^{A(KT+\Delta T-\tau)} B u(KT)\, d\tau$$

$$\Rightarrow \bar{x}(KT+\Delta T) = e^{\bar{A}\Delta T} \bar{x}(KT) + \int_{0}^{\Delta T} e^{\bar{A}\lambda} \bar{B}\bar{u}(KT)\, d\lambda$$

Let us define,

$$\bar{G}(\Delta T) = e^{\bar{A}\Delta T}$$

$$\bar{H}(\Delta T) = \left(\int_0^{\Delta T} e^{\bar{A}\lambda} d\lambda\right) \bar{B}$$

We obtain,

$$\bar{x}(KT+\Delta T) = G(\Delta T)\, x(KT) + H(\Delta T)\, u(KT)$$

So

$$\bar{y}(KT+\Delta T) = \bar{C}\bar{x}(KT+\Delta T) + \bar{D}\bar{u}(KT)$$

$$\boxed{\bar{y}(KT+\Delta T) = \bar{C}\bar{G}(\Delta T)\bar{x}(KT) + \left[\bar{C}\bar{H}(\Delta T) + \bar{D}\right]\bar{u}(KT)}$$

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5.6) Liapunov Stability Analysis

→ Plays an important role in the stability analysis of control system described by State-Space equations.

Method 1

Consists entirely of procedures in which the explicit form of the solution is used

Method 2

Dose not require the solutions of differential or difference equations

(Also called Direct Method)

(Due to this second method is so useful in practice)

⇒ Liapunov Stability Criteria is applicable to both linear and nonlinear, time invariant or time varying System.

★ Second Method of Liapunov

⇒ From the classical theory of mechanics, we know that a vibratory system is stable if its total energy is continually decreasing until a equilibrium state is reached.

→ Second method of Liapunov is generalization of this fact.

⇒ For purely mathematical System, however, there is no simple way of defining an "energy function".

↳ To avoid this difficulty, Liapunov introduced the Liapunov function, a fictitious energy function.

Positive definiteness of Scalar function

A Scalar function $V(\bar{x})$ is said to be positive definite in a region Ω if $V(\bar{x}) > 0$ for all nonzero State $\bar{x}$ in the region Ω and if $V(\bar{0}) = 0$.

(It includes origin of the State space)

A time-Varying function $V(\bar{x}, t)$ is said to be positive definite in a region Ω if it is bounded from below by a time invariant positive definite function, that is if there exists a positive definite function $V(\bar{x})$ Such that

$$V(\bar{x}, t) > V(\bar{x}) \quad \forall \; t \geq t_0$$

$$\cancel{V(\bar{0}, t) = 0 \quad \forall \; t \geq t_0}$$

Negative definiteness of Scalar function

A Scalar function $V(\bar{x})$ is said to be negative definite if $-V(\bar{x})$ is positive definite.

Positive Semidefinitness of Scalar function

A Scalar function $V(\bar{x})$ is said to be positive Semidefinite if it is possible at all states in the region Ω except at the origin and at certain other states, where it is zero.

Negative Semidefiniteness of Scalar function

A Scalar function $V(\bar{x})$ is said to be negative Semidefinite if $-V(\bar{x})$ is positive semidefinite.

Indefiniteness of Scalar function

A Scalar function $V(\bar{x})$ is said to be indefinite if in the region Ω it assumes both positive and negative values, no matter how small the region Ω is.

★ Liapunov Function

⇒ It is a Scalar function, is a positive definite function, and it is Continuous together with its first partial derivatives in the region Ω about the origin and has a time derivative that, when taken along the trajectory is negative definite (or negative semidefinite).

⇒ $\dot{V}(\bar{x},t)$ is total derivative of $V(\bar{x},t)$ with respect to 't' along a solution of the System.

↳ Hence $\dot{V}(\bar{x},t) < 0$ means $V(\bar{x}t)$ is a decreasing function of time.

⇒ A Liapunov function is not unique for a given System.

⇒ Later in this Section we will show that in the Second method of Liapunov the Sign behaviour of $V(\bar{x},t)$ and that of its time derivative $\dot{V}(\bar{x},t)$ gives information about the Stability of an equilibrium state without having the Solution.

⇒ Simplest positive definite function is of a quadratic form:-

$$V(\bar{x}) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\, x_i x_j \quad \forall\, i,j = 1,2 \dots n$$

⇒ In general, Liapunov functions may not be of a Simple quadratic form.

Let $\frac{x_1}{x_n} = \hat{x}_1 \quad \frac{x_2}{x_n} = \hat{x}_2 \quad \dots \quad \frac{x_{n-1}}{x_n} = \hat{x}_{n-1}$

⇒ then in the neighbourhood of the origin the lowest-degree terms alone will become dominant and we can write $V(\bar{x})$ as:-

$$V(\bar{x}) = x_n^p \, V(\hat{x}_1, \hat{x}_2 \cdots \hat{x}_{n-1}, 1)$$

⇒ For p odd, x_n^p can assume both positive and negative values near the origin, which means that $V(\bar{x})$ is not positive definite. Hence p must be even.

→ Lowest degree term in V must be even

★ <u>System</u>: The System we consider here is defined by:-

$$\boxed{\dot{\bar{x}} = f(\bar{x}, t)} \quad \text{——— (1)}$$

where $\bar{x}$ is a state vector (an n-vector) and $\bar{f}(\bar{x},t)$ is an n-vector whose elements are functions of $x_1, x_2 \cdots x_n$ and t.

⇒ We assume that the system of eq (1) has a unique solution starting at a given initial condition.

⇒ Let Solution be $\bar{\phi}(t; \bar{x}_0, t_0)$.

So $\bar{x}_0 = \bar{\phi}(t_0; \bar{x}_0, t_0)$

★ **Equilibrium State:** In the System of eq ①
a State $\bar{X}_e$, where,

$$f(\bar{x}_e, t) = 0 \quad \forall t$$

is called an equilibrium state of the System.

If the system is linear and time invariant that is, if $\bar{f}(\bar{x}, t) = \bar{A}\bar{x}$, then,

→ There exist only one equilibrium state if A is non singular.

→ There exist infinitely many equilibrium States if $\bar{A}$ is singular.

For nonlinear System, there may be one or more equilibrium states.

⇒ Any isolation equilibrium state can be shifted to the origin of the coordinates, or $\bar{f}(\bar{0}, t) = \bar{0}$, by a translation of coordinates.

★ **Stability in the Sense of Liapunov**

⇒ We denote a Spherical region of radius r about an equilibrium state $\bar{x}_e$ as

$$\|\bar{x} - \bar{x}_e\| \leq r$$

→ Euclidean norm

~~Consider~~

⇒ Let $S(\delta)$ consists of all points such that

$$\|\bar{x}_0 - \bar{x}_e\| \leq \delta$$

and let $S(\epsilon)$ consist of all points such that

$$\|\bar{\phi}(t; \bar{x}_0, t_0) - \bar{x}_e\| \leq \epsilon \quad \forall\ t \geq t_0$$

⇒ An equilibrium state $\bar{x}_e$ of the System of eq. ① is said to be <u>Stable</u> in the Sense of Liapunov if, Corresponding to each $S(\epsilon)$ ~~....~~, there is an $S(\delta)$ such that trajectories starting in $S(\delta)$ do no leave $S(\epsilon)$ as t increases indefenitely.

⇒ The real number δ depends on ϵ and in general, also depends on t_0.

↳ If δ does not depend on t_0, the equilibrium state is said to be Uniformly stable.

★ <u>Asymptootic Stability</u>

"An equilibrium state $\bar{x}_e$ of the System of eq. ① is said to be asymptotically stable if it is Stable in the Sense of Liapunove and if every solution Starting within $S(\delta)$ Converges, without leaving $S(\epsilon)$, to $\bar{x}_e$ at time increases indefinitely."

⇒ ~~Region~~ of largest region of asymptotic stability is called domain of attraction.

{ That part of State Space in which asymptotically stable trajectories originate }

★ Asymptotic stability in the Large

If asymptotic stability holds for all states from which trajectories originate, the equilibrium state is said to be asymptotically stable in the large.

{ If every solution converges to $\bar{X}_e$ as t increases indefinitely }

⇒ Obviously, a necessary condition for asymptotic stability in the large is that there be only one equilibrium state in the whole state space.

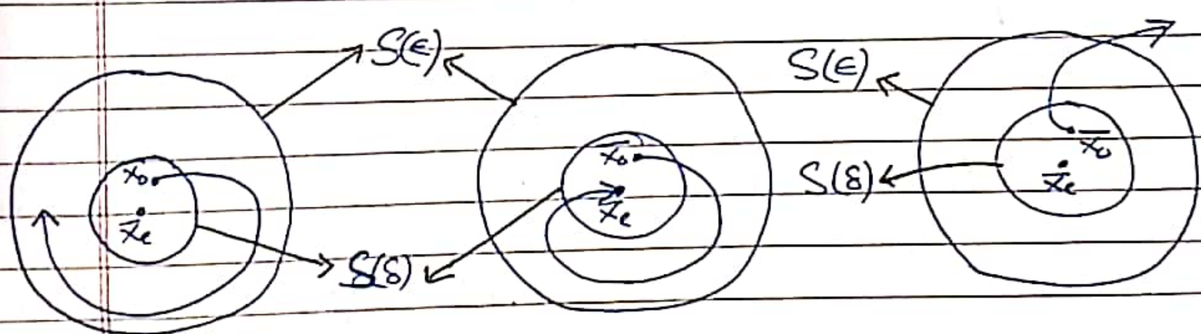
⇒ In control engineering problems, asymptotic stability in the large is a desirable feature.

↳ If the equilibrium state is not asymptotically stable in the large, then the problem becomes one of determining the largest region of asymptotic stability.

↳ This is usually very difficult.

★ <u>Instability</u>: An equilibrium state $\overline{x}_e$ is said to be unstable if for some real number $\epsilon > 0$ and any real number $\delta > 0$, no matter how small, there is always a state $\overline{x}_0$ in $S(\delta)$ such that the trajectory starting at this state leaves $S(\epsilon)$.

★ <u>Graphical representation of Stability, asymptotic stability, and Instability</u>



(a) <u>Stable Equilibrium State</u>

(b) <u>Asymptotically stable equilibrium</u>

(c) <u>Unstable Equilibrium State</u>

\# $S(\delta) \rightarrow$ Bounds the initial state.

\# $S(\epsilon) \rightarrow$ Boundary for trajectory starting from initial state $\overline{x}_0$ in the region $S(\delta)$

★ Liapunov Theorem on Asymptotic Stability

If a scalar function $V(\bar{x})$, where $\bar{x}$ is a n-vector, is positive definite then the states $\bar{x}$ that satisfy

$$V(\bar{x}) = C$$

where C is a positive constant, lie on a closed hypersurface in the n-dimensional state space, at least in the neighborhood of the origin.

⇒ If $V(\bar{x}) \to \infty$ as $\|\bar{x}\| \to \infty$, then such closed surface extend over the entire state space.

⇒ The hypersurface $V(\bar{x}) = C_1$ lies entirely inside the hypersurface $V(\bar{x}) = C_2$ if $C_1 < C_2$

⇒ For a given system, if a positive definite scalar function $V(\bar{x})$ can be found such that its time derivative taken along a trajectory is always negative, then as time increases, $V(\bar{x})$ takes smaller & smaller values of C.

↳ As time increases, $V(\bar{x})$ finally shrink to zero, and therefore $\bar{x}$ also shrink to zero.

Theorem 5.1: Suppose a system is described by

$$\dot{\bar{X}} = \bar{f}(\bar{x}, t)$$

Where,
$$\bar{f}(\bar{0}, t) = \bar{0} \quad \forall\, t$$

If there exists a scalar function $V(\bar{x}, t)$ having continuous first partial derivative & satisfying the conditions:-

1. $V(\bar{x}, t)$ is positive definite
2. $\dot{V}(\bar{x}, t)$ is negative definite.

then the equilibrium state at the origin is uniformly asymptotically stable.

⇒ If, in addition, $V(\bar{x}, t) \to \infty$ as $\|\bar{x}\| \to \infty$, then the equilibrium state at the origin is uniformly asymptotically stable in the large.

⇒ The conditions of this theorem may be modified

i) $V(\bar{x}, t)$ is positive definite.
ii) $\dot{V}(\bar{x}, t)$ is negative semidefinite.
iii) $\dot{V}(\bar{\phi}(t; x_0, t_0), t)$ does not vanish identically in $t \geq t_0$ for any t_0 and any $\bar{X}_0 \neq 0$ where $\bar{\phi}(t; \bar{x}_0, t_0)$ denotes the solution starting from $\bar{X}_0$ at $t = t_0$.

⇒ Then the origin of the system is uniformly asymptotically stable in the large.

Theorem 5-2: Suppose a system is described by:-

$$\dot{\bar{x}} = \bar{f}(\bar{x}, t)$$

where $\bar{f}(\bar{0}, t) = \bar{0} \;\forall t$.

⇒ If there exist a scalar function $V(\bar{x}, t)$ having continuous first partial derivatives and satisfying the conditions

1. $V(\bar{x}, t)$ is positive definite.
2. $\dot{V}(\bar{x}, t)$ is negative semidefinite

then the equilibrium state at the origin is uniformly stable.

{Not necessarily Uniformly asymptotically stable}

★ Instability

Theorem 5-3: Suppose a system is ~~described~~ described by:-

$$\dot{\bar{x}} = \bar{f}(\bar{x}, t)$$

where,

$$\bar{f}(\bar{0}, t) = \bar{0} \quad \forall \; t \geq t_0$$

If there exist a scalar function $W(\bar{x}, t)$ having continuous first partial derivatives and satisfying the conditions

1. $W(\bar{x},t)$ is positive definite in some region about the origin.

2. $\dot{W}(\bar{x},t)$ is positive definite in the same region.

then equilibrium state at the origin is unstable.

★ **Stability Analysis of linear time-Invariant System**

⇒ There are many approaches to investigate asymptotic stability of linear time-invariant system.

↳ Example: For a continuous-time system described by the equation

$$\dot{\bar{x}} = \bar{A}\bar{x}$$

it can be stated that a necessary and sufficient condition for the asymptotic stability of the origin of the system is that all eigenvalues of A have negative real part.

↳ Or the zeros of the characteristic polynomial

$$|s\bar{I}-\bar{A}| = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$$

have negative real parts.

⇒ Similarly, for a discrete-time system represented by the equation:-

$$\bar{x}(k+1) = \bar{G}\,\bar{x}(k)$$

a necessary and sufficient condition that can be stated for the asymptotic stability of the origin is that all eigenvalues of $\bar{G}$ be less than unity in their magnitude.

↳ Or that the zeros of the characteristic polynomial.

$$|z\bar{\bar{I}} - \bar{\bar{G}}| = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$$

lies within the unit circle centered at the origin of the z-plane.

⇒ For linear time invariant systems the second method of Liapunov gives not just sufficient conditions, but the necessary and sufficient conditions for stability or asymptotic stability.

⇒ The conjugate transpose of A is denoted be $\bar{\bar{A}}^*$.

→ Conjugate

$$\boxed{\bar{\bar{A}}^* = \bar{\bar{A}}^T}$$

→ Transpose

★ Liapunov stability analysis of Linear time Invariant Continuous time System

⇒ Consider the following linear time-invariant system:

$$\dot{\bar{x}} = \bar{A}\bar{x} \quad \text{—} ①$$

⇒ We assume that A is non singular.

- Then the only equilibrium state is the origin, $\bar{x} = 0$.
- The stability of the equilibrium state can easily be investigated by second method of Liapunov.

⇒ For the system defined by eq ①, let us choose as a possible Liapunov function.

$$V(\bar{x}) = \bar{x}^* \bar{P} \bar{x}$$

Where P is positive definite Hermitian matrix.

⇒ The time derivative of $V(\bar{x})$ along any trajectory is

$$\begin{aligned}
\dot{V}(\bar{x}) &= \dot{\bar{x}}^* \bar{P} \bar{x} + \bar{x}^* \bar{P} \dot{\bar{x}} \\
&= (\bar{A}\bar{x})^* \bar{P} \bar{x} + \bar{x}^* \bar{P} \bar{A} \bar{x} \\
&= \bar{x}^* \bar{A}^* \bar{P} \bar{x} + \bar{x}^* \bar{P} \bar{A} \bar{x} \\
&= \bar{x}^* (\bar{A}^* \bar{P} + \bar{P} \bar{A}) \bar{x}
\end{aligned}$$

⇒ Since $V(\bar{x})$ was chosen to be positive definite, we require, for asymptotic stability, that $\dot{V}(\bar{x})$ be negative definite.

→ Therefor, we require that,

$$\dot{V}(\bar{x}) = -\bar{x}^* \bar{Q} \bar{x}$$

Where,

$$\bar{\bar{Q}} = -(\bar{A}^* \bar{P} + \bar{P}\bar{A}) = \text{Positive definite}$$

Hence, for the asymptotic stability of the System of eq (1), it is Sufficient that $\bar{Q}$ be positive definite.

Theorem 5-4: Consider the System described by:-

$$\dot{\bar{x}} = \bar{\bar{A}}\bar{x}$$

A necessary and sufficient condition for the equilibrium state $\bar{x} = \bar{0}$ to be asymptotically stable in the large is that, given any positive definite Hermitian matrix $\bar{\bar{Q}}$, there exists a positive definite Hermitian matrix $\bar{\bar{P}}$ such that

$$\bar{A}^* \bar{P} + \bar{\bar{P}}\bar{A} = -Q$$

The scalar function $\bar{x}^* \bar{\bar{P}} \bar{x}$ is a Liapunov function for this System.

★ Liapunov stability analysis of discrete-time Systems

\# **Theorem 5-5:** Consider the discrete-time system

$$x((K+1)T) = f(x(KT))$$

Where, $\bar{x}$ = n-Vector
$\bar{f}(\bar{x})$ = n-Vector with property that $\bar{f}(\bar{0})=\bar{0}$
T = Sampling period

Suppose there exists a scalar function $V(\bar{x})$ Continuous in $\bar{x}$ such that

1. $V(\bar{x}) > 0 \quad \forall \bar{x} \neq 0$
2. $\Delta V(\bar{x}) < 0 \quad \forall \bar{x} \neq 0$ where,

$$\Delta V(\bar{x}(KT)) = V(x(K+1)T) - V(x(K)T)$$
$$= V(\bar{f}(\bar{x}(KT))) - V(x(KT))$$

3. $V(\bar{0}) = 0$
4. $V(\bar{x}) \to \infty$ as $\|\bar{x}\| \to \infty$

then the equilibrium state $\bar{x}=0$ is asymptotically Stable in the large and $V(\bar{x})$ is a Liapunov function.

★ **Liapunov Stability analysis of Linear, Time-Invariant Discrete-Time System**

⇒ Consider the discrete-time system described by:-

$$\bar{x}(K+1) = \bar{\bar{G}}\,\bar{x}(K)$$

⇒ The origin $\bar{x} = \bar{0}$ is the equilibrium state.

⇒ Let us choose as a possible Liapunov function

$$V(\bar{x}(K)) = \bar{x}^*(K)\,\bar{\bar{P}}\,\bar{x}(K)$$

Where P is a positive definit Hermitian matrix

Then,

$$\Delta V(\bar{x}(K)) = V(\bar{x}(K+1)) - V(\bar{x}(K))$$
$$= \bar{x}^*(K+1)\,\bar{\bar{P}}\,\bar{x}(K+1) - \bar{x}^*(K)\,\bar{\bar{P}}\,\bar{x}(K)$$
$$= [\bar{\bar{G}}\,\bar{x}(K)]^*\,\bar{\bar{P}}\,[\bar{\bar{G}}\,\bar{x}(K)] - \bar{x}^*(K)\,\bar{\bar{P}}\,\bar{x}(K)$$
$$\Rightarrow \bar{x}^*(K)\left(\bar{\bar{G}}^*\bar{\bar{P}}\bar{\bar{G}} - \bar{\bar{P}}\right)\bar{x}(K)$$

⇒ Since $V(\bar{x}(K))$ is chosen to be positive definit, we require, for asymptotic stability that $\Delta V(\bar{x}(K))$ be negative definite.

Therefore,

$$\Delta V(\bar{x}(K)) = -\bar{x}^*(K)\,\bar{\bar{Q}}\,\bar{x}(K)$$

Where, $\bar{\bar{Q}} = -\left(\bar{\bar{G}}^* \bar{\bar{P}} \bar{\bar{G}} - \bar{\bar{P}}\right)$ = Positive definite

≠ <u>Theorem 5-6</u>: Consider the discrete-time system

$$\bar{x}(k+1) = \bar{\bar{G}}\bar{x}(k)$$

A necessary and sufficient condition for the equilibrium state $\bar{x}=0$ to be asymptotically stable in the large is that, given any positive-definit Hermitian matrix $\bar{\bar{Q}}$, there exist a positive definit Hermitian matrix $\bar{\bar{P}}$ Such that

$$\bar{\bar{G}}^* \bar{\bar{P}} \bar{\bar{G}} - \bar{\bar{P}} = -\bar{\bar{Q}}$$

The scalar function $\bar{x}^* \bar{\bar{P}} \bar{x}$ is a Liapunov function for this system.

★ <u>Stability of discrete-time system Obtained by discretizing a Continuous-time System</u>:

⇒ The asymptotic stability of an equilibrium state of a descrete-time system obtained by discretizing a Continuous-time system is equivalent to that of the original Continuous-time system.

⇒ Consider a continuous-time system:-

$$\dot{\bar{x}} = \bar{\bar{A}}\bar{x} \quad \text{— (1)}$$

and the Corresponding discrete-time system

$$\bar{x}((k+1)T) = \bar{\bar{G}}\,\bar{x}(kT) \quad \text{---} (2)$$

Where, $\bar{\bar{G}} = e^{\bar{\bar{A}}T}$

⇒ If the Continuous-time System is asymptotically Stable, that is, if all the eigenvalues of the matrix $\bar{A}$ have negative real parts, then

$$\|\bar{\bar{G}}^n\| \to 0 \quad \text{as } n \to \infty$$

and the discretized system is also asymptotically Stable. This is because if the λ_i's are the eigenvalues of $\bar{A}$ then the $e^{\lambda_i T}$'s are the eigenvalues of $\bar{\bar{G}}$. (Note that $|e^{\lambda_i T}| < 1$ if $\lambda_i T$ is negative).

⇒ It should be noted here that, if a Continuous-time system having complex poles is discretized, then in certain exceptional cases hidden instability may occur, depending on the choice of the Sampling period T.

★ **Contraction:** A function $\bar{f}(\bar{x})$ is said to be a Contraction if $\bar{f}(\bar{0}) = \bar{0}$ &

$$\|\bar{f}(\bar{x})\| < \|\bar{x}\|$$

for some set of values of $\bar{x} \neq 0$.

CHAPTER 2

Pole Placement and Observer Design

6

Pole Placement and Observer Design

6.1 Introduction

Controllability

→ Concerned with the problem of whether it is possible to steer a System from a given initial state to an arbitrary state.

Observability

→ Concerned with the problem of determining the State of a dynamic System from observations of the output and Control vectors in a finite number of Sampling period.

⇒ The Concept of Controllability and Observability were introduced by R.E. Kalman.

⇒ The design approach of placing the closed loop poles in the desired locations in the Z plane is called the pole placement design technique.

→ In this technique we feed back all state Variables so that all poles of the closed-loop System are placed at desired location.

⇒ The pole placement design process of Control systems may be separated into two phases:-

First Phase: We design the system assuming that all state variables are available for feed back.

Second Phase: We design the state observer that estimates all state variables (or only those that are not directly measurable) that are required for feedback to complete design.

⇒ Assumptions in this chapter:

- Disturbance are impulses that take place randomly.
- Effect of such impulses is to change the system state. So disturbance mabe be represented as an initial state.
- Spacing between adjacent disturbances is sufficiently wide that any response to such a disturbance settels down before the next disturbance takes place.

Regulator Problem

Control Problem

⇒ Both Control & Regulator problem boils down to the determination of the desired state feedback matrix.

* Regulator ⇒ We desire to transfer nonzero error vector to the origin

* Servo ⇒ We require output to follow the command input.

→ Servo System must follow the command input and at the same time must solve any regulator problem.

6.2) Controllability

"A Control System is said to be completely state controllable if it is possible to transfer the system from any arbitrary initial state to any desired state in a finite time period"

⇒ If any state variable is independent of the control signal, then it is impossible to control this state variable and therefore the system is uncontrollable.

★ Complete State Controllability for a linear time-invariant Discrete time System

⇒ Consider the discrete-time control system defined by:-

$$\overline{X}((K+1)T) = \overline{\overline{G}}\,\overline{X}(KT) + \overline{H}\,\overline{u}(KT) \quad \text{--- (1)}$$

State Vector

Control Signal

$\Rightarrow$ We assume that $u(KT)$ is constant for $KT \leq t \leq (K+1)T$

$\Rightarrow$ The discrete-time control system given by eq ① is said to be state controllable if

→ There exist a piecewise constant control signal $u(KT)$ defined over a finite number of sampling period that,

→ Starting from any initial state, the state $\overline{X}(KT)$ can be transferred to the desired state $\overline{X}_f$ in at most n sampling periods.

{In this discussion $\overline{X}_f = \overline{0}$}

$\Rightarrow$ Solution of eq ① is given by :-

$$\overline{X}(nT) = \overline{\overline{G}}^{n}\overline{X}(0) + \sum_{j=0}^{n-1} \overline{\overline{G}}^{n-j-1}\overline{H}u(jT)$$

$$\Rightarrow \overline{X}(nT) - \overline{\overline{G}}^{n}\overline{X}(0) = \left[\overline{H} \,\vdots\, \overline{\overline{G}}\overline{H} \,\vdots\, \cdots \,\vdots\, \overline{\overline{G}}^{n-1}\overline{H}\right]\begin{bmatrix} u((n-1)T) \\ u((n-1)T) \\ \vdots \\ u(0) \end{bmatrix} \quad ②$$

$\Rightarrow$ If $\text{Rank}\left[\overline{H} \,\vdots\, \overline{\overline{G}}\overline{H} \,\vdots\, \cdots \,\vdots\, \overline{\overline{G}}^{n-1}\overline{H}\right] = n$ ③ then

the n vectors $\overline{H}, \overline{\overline{G}}\overline{H} \cdots \overline{\overline{G}}^{n-1}\overline{H}$ can span the n dimensional space.

⇒ The matrix $[\overline{H} \vdots \overline{G}\overline{H} \vdots \cdots \vdots \overline{G}^{n-1}\overline{H}]$ is commonly called the Controllability matrix.

⇒ If Rank of Controllability matrix is n, then for an arbitrary state $\overline{x}(nT) = \overline{x}_f$ there exists a sequence of unbounded control signals $u(0), u(T) \cdots u((n-1)T)$ that satisfies equation ②.

↳ Hence rank of the controllability matrix be n gives a sufficient condition for complete state controllability.

⇒ To prove that eq ③ it is also a necessary condition for complete state controllability, let us assume that,

$$\text{Rank}\,[\overline{H} \vdots \overline{G}\overline{H} \vdots \cdots \vdots \overline{G}^{n-1}\overline{H}] < n$$

⇒ By use of Cayley-Hamilton theorem, it can be shown that, for an arbitrary i $\overline{G}^{i}\overline{H}$ can be expressed as a linear combination of $\overline{H}, \overline{G}\overline{H} \cdots \overline{G}^{n-1}\overline{H}$.

↳ Consequently for any i

$$\text{Rank}\,[\overline{H} \vdots \overline{G}\overline{H} \vdots \cdots \vdots \overline{G}^{i-1}\overline{H}] < n$$

So vectors $\overline{H}, \overline{G}\overline{H}, \cdots \overline{G}^{i-1}\overline{H}$ cannot span the n-dimensional space.

⇒ Therefore, for some $\overline{X}_f$, it is not possible to have $\overline{X}(iT) = \overline{X}_f \ \forall\ i$.

↳ Thus the condition given by eq ③ is necessary.

★ **Complete State Controbality in the Case where $\overline{u}(KT)$ is a Vector**

⇒ If the system is defined by :-

$$\overline{X}((K+1)T) = \overline{\overline{G}}\,\overline{X}(KT) + \overline{\overline{H}}\,\overline{u}(KT)$$

$\overline{\overline{G}}$: $n \times n$; $\overline{\overline{H}}$: $n \times r$; $\overline{X}(KT)$: n-State Vector; $\overline{u}(KT)$: r-Control Vector

⇒ It can be proved that the condition for Complete State Controllability is :-

$$\text{Rank}\underbrace{\left[\overline{H} \ \vdots\ \overline{G}\,\overline{H} \ \vdots\ \cdots \ \vdots\ \overline{G}^{\,n-1}\overline{H}\right]}_{n \times nr} = n$$

★ **Determining Control Sequence to Bring the initial State to a Desired State**

⇒ If Controbality matrix is of rank n and $u(KT)$ is a Scalar, then, it is possible to find n linearly independent scalar equations

↳ From which a sequence of unbounded control signal $u(KT)\ \forall\ K = 0, 1, \ldots, n-1$ can be uniquely determined.

↳ Such that any initial state $\overline{X}(0)$ is transferred to the desired state in n sampling periods. (eq ②)

⇒ If the control signal is not a scalar but a vector, then the sequence of $\overline{u}(KT)$ is not unique.

★ Alternative form of the Condition for Complete State Controllability

⇒ Consider the system defined by

$$\overline{X}((K+1)T) = \overline{G}\,\overline{X}(KT) + \overline{H}\,\overline{u}(KT)$$

⇒ If the eigenvectors of $\overline{G}$ are distinct, then it is possible to find a transformation matrix $\overline{P}$ such that:

$$\overline{P}^{-1}\overline{G}\,\overline{P} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

⇒ **Note:** i^{th} column of the $\overline{P}$ matrix is an eigenvector of $\overline{G}$ associated with i^{th} eigenvalue λ_i.

Let us define,

$$\overline{X}(KT) = \overline{\overline{P}}\hat{X}(KT) \quad — (5)$$

So,

$$\hat{X}((K+1)T) = \overline{\overline{P}}^{-1}\overline{\overline{G}}\,\overline{\overline{P}}\hat{X}(KT) + \overline{\overline{P}}^{-1}\overline{\overline{H}}\overline{u}(KT) \quad — (6)$$

Let $\overline{\overline{F}} = \overline{\overline{P}}^{-1}\overline{\overline{H}}$

⇒ Then eq (6) may be written as:-

$$\hat{X}_1((K+1)T) = \lambda_1\hat{X}_1(KT) + f_{11}u_1(KT) + f_{12}u_2(KT) + \cdots + f_{1r}u_r(KT)$$

$$\hat{X}_2((K+1)T) = \lambda_2\hat{X}_2(KT) + f_{21}u_1(KT) + f_{22}u_2(KT) + \cdots f_{2r}u_r(KT)$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\hat{X}_n((K+1)T) = \lambda_n\hat{X}_n(KT) + f_{n1}u_1(KT) + f_{n2}u_2(KT) + \cdots + f_{nr}u_r(KT)$$

⇒ If the element of any one row of the $n \times r$ matrix $\overline{\overline{F}}$ are all zero, then the corresponding state variables cannot be controlled by any of the $u_i(KT)$.

↳ Hence, the condition for complete state controllability is that, if the eigenvectors of $\overline{\overline{G}}$ are distinct, then the system is completely state controllable if and only if no row of $\overline{\overline{P}}^{-1}\overline{\overline{H}}$ has all zero elements.

⇒ In case $\bar{\bar{G}}$ is not diagonalizable then we may transform $\bar{\bar{G}}$ into a Jordan Canonical form.

⇒ Suppose it is possible to find a transformation matrix $\bar{\bar{S}}$ such that

$$\bar{\bar{S}}^{-1}\bar{\bar{G}}\bar{\bar{S}} = \bar{\bar{J}}$$

⇒ If we define a new state vector $\hat{x}$ by

$$\bar{x}(KT) = \bar{\bar{S}}\hat{x}(KT) \quad — ⑦$$

So, $\hat{x}((K+1)T) = \bar{\bar{S}}^{-1}\bar{\bar{G}}\bar{\bar{S}}\hat{x}(KT) + \bar{\bar{S}}^{-1}\bar{\bar{H}}\bar{u}(KT)$

$$= \bar{\bar{J}}\hat{x}(KT) + \bar{\bar{S}}^{-1}\bar{\bar{H}}\bar{u}(KT) \quad — ⑧$$

⇒ The System given by eq. ⑧ is Completely state Controllable if and only if :-

(1) No two Jordan blocks in $\bar{\bar{J}}$ are associated with the same eigen value.

(2) Elements of any row of $\bar{\bar{S}}^{-1}\bar{\bar{H}}$ that Corresponds to the last row of each Jordan block are not all zeros.

(3) Elements of each row of $\bar{\bar{S}}^{-1}\bar{\bar{H}}$ that Corresponds to distinct eigenvalues are not zero.

★ **Condition for Complete State Controllability in the Z-plane**

"A necessary and sufficient condition for complete state controllability is that no cancellation occurs in the pulse transfer function.

↳ If cancellation occurs, the system cannot be controlled in the direction of the cancelled mode.

★ **Complete Output Controllability**

⇒ If practical design of a control system we may want to control the output rather than the state of the system.

↳ Complete state controllability is neither necessary nor sufficient for controlling the output of the system.

⇒ Consider the system defined by the equations

$$\bar{X}((K+1)T) = \bar{G}\bar{X}(KT) + \bar{H}u(KT) \quad \text{— (9)}$$

$$\bar{Y}(KT) = \bar{C}\bar{X}(KT) \quad \text{— (10)}$$

⇒ The system defined by eq (9) & (10) is said to be completely output controllable, if it is possible to construct an unconstrained control signal $u(KT)$ defined over a finite number of sampling periods $0 \leq KT \leq nT$ such that,

↳ Starting from any initial output $\bar{y}(0)$, the output $\bar{y}(KT)$ can be transferred to the desired point $\bar{y}_f$ in the output space in at most n sampling periods.

⇒ Solution of eq (9) is:-

$$\bar{x}(nT) = \bar{\bar{G}}^{n}\bar{x}(0) + \sum_{j=0}^{n-1} \bar{\bar{G}}^{n-j-1}\bar{H}u(jT)$$

We have $\bar{y}(nT) = \bar{C}\,\bar{x}(nT)$

$$\bar{y}(nT) = \bar{\bar{C}}\,\bar{\bar{G}}^{n}\bar{x}(0) + \sum_{j=0}^{n-1} \bar{\bar{C}}\,\bar{\bar{G}}^{n-j-1}\bar{H}u(jT)$$

$$\bar{y}(nT) - \bar{C}\bar{\bar{G}}^{n}\bar{x}(0) = \left[\bar{C}\bar{H} \;\vdots\; \bar{C}\bar{G}\bar{H} \;\vdots\; \cdots \;\vdots\; \bar{C}\bar{G}^{n-1}\bar{H}\right]\begin{bmatrix} u((n-1)T) \\ u((n-2)T) \\ \vdots \\ u(0) \end{bmatrix}$$

⇒ A necessary and sufficient condition for the system to be completely output controllable is that vectors $\bar{C}\bar{H}$, $\bar{C}\bar{G}\bar{H}$ ... $\bar{C}\bar{G}^{n-1}\bar{H}$ spans m-D output space.

$$\text{Rank}\left[\bar{\bar{C}}\bar{H} \;\vdots\; \bar{\bar{C}}\bar{\bar{G}}\bar{H} \;\vdots\; \cdots \;\vdots\; \bar{C}\bar{\bar{G}}^{n-1}\bar{H}\right] = m \quad \text{—— (11)}$$

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⇒ Next, Consider the system defined by the equations:-

$$\bar{x}((k+1)T) = \bar{G}\bar{x}(kT) + \bar{H}\bar{u}(kT) \quad \text{--- (12)}$$
$$\bar{y}(kT) = \bar{C}\bar{x}(kT) + \bar{D}\bar{u}(kT) \quad \text{--- (13)}$$

A necessary & Sufficient Condition for the System defined by eq (12) & (13) to be Completely output Controllable if:-

$$\text{Rank}\left[\bar{D} \; \vdots \; \bar{C}\bar{H} \; \vdots \; \bar{C}\bar{G}\bar{H} \; \vdots \; \cdots \; \vdots \; \bar{C}\bar{G}^{n-1}\bar{H}\right] = m$$

6-3) Observability