Lecture 8

Date
Page

Frequency Response

⇒ The steady-state response of a linear, time-invariant system to a complex exponential input of the form $u(t) = e^{st}$ is

$$y_{ss}(t) = G(s)\,e^{st}$$

⇒ If we chose $s = j\omega$ (real part of input is $\cos(\omega t)$), and we assume that the system is stable, then the steady-state output will be given by

$$y_{ss}(t) = G(s)\,e^{j\omega t} = |G(j\omega)|\,e^{j\omega t + \angle G(j\omega)}$$

(real part of output)

$$Re\,[y_{ss}(t)] = |G(j\omega)|\cos(\omega t + \angle G(j\omega))$$

⇒ The steady-state response to a sinusoidal input of frequency ω is a sinusoidal output of the same frequency such that:

- The amplitude of the output is $|G(j\omega)|$ times the amplitude of the input.
- The phase of the output lags the phase of the input by $\angle G(j\omega)$.

Date
Page

★ Frequency response plots

⇒ The frequency response $G(j\omega) \in \mathbb{C}$ is a complex function of a single real argument $\omega \in \mathbb{R}$.

⇒ We basically have two options to plot the frequency response:

① A parametric curve showing $G(j\omega)$ in the complex plane, in which ω is implicit.
↳ This leads to the polar plot and eventually to the Nyquist plot.

② Two separate plots for e.g. real and imaginary part of $G(j\omega)$ or better the magnitude and phase of $G(j\omega)$ as a function of ω.
↳ The latter choice leads to the Bode plot.

★ The Bode Plot

⇒ The Bode plot is actually composed of two plot:
- → Magnitude plot
- → Phase plot

⇒ On the horizontal axis of both plots, we report the frequency ω on a logarithmic scale (base 10).

⇒ On the vertical axis we report:

↳ $|G(j\omega)|$ in dB (deciBels)

$$|G(j\omega)|\,[dB] = 20\log_{10}|G(j\omega)|$$

Note: One decade = 20dB

↳ The phase $\angle G(j\omega)$ in degree or radians.

⇒ Since magnitudes multiply (i.e. there log add) and phase add.

↳ These choices of vertical coordinates makes it possible to just add Bode plots of serial connections.

⇒ Inverting the transfer function is equivalent to reflection about the horizontal axis, in both plots.

★ Drawing approximate bode plot

$$G(s) = \cancel{K(1+\tau s)} \; K \frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$

⇒ It can be broken down into product of following terms:

① K (constant)

② $\cancel{(1+\tau s)}$ s , $1/s$

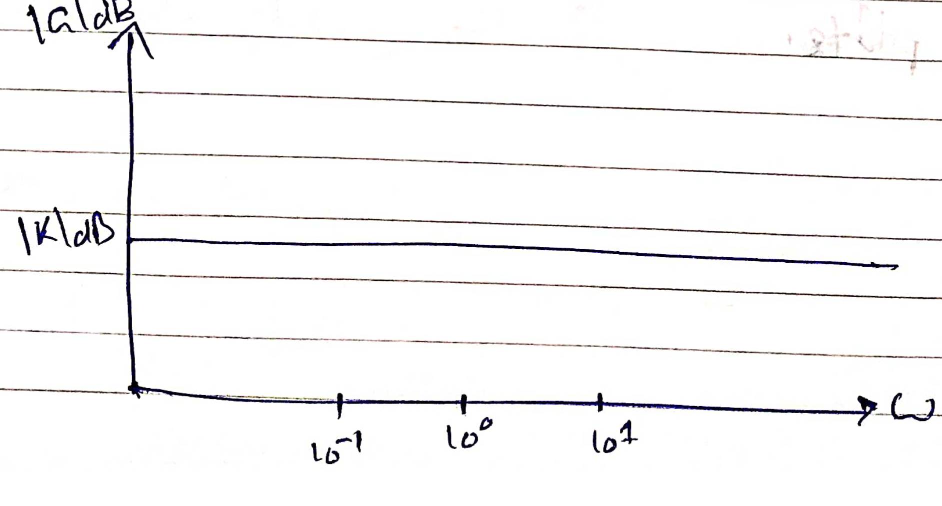
③ $(1+\tau s)$, $1/(1+\tau s)$

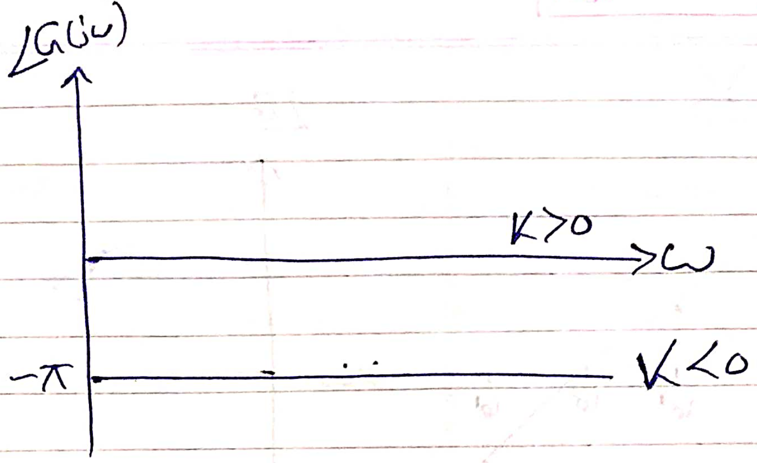
④ $\frac{s^2}{\omega_n^2} + 2\xi \frac{s}{\omega_n} + 1$, $\frac{1}{\frac{s^2}{\omega_n^2} + 2\xi \frac{s}{\omega_n} + 1}$

① K (constant)

$$|K|_{dB} = 20\log_{10}|K|$$

$$\angle K = \tan^{-1}\left(\frac{0}{K}\right) = \begin{cases} 0 & \forall\ K > 0 \\ -\pi & \forall\ K < 0 \end{cases}$$





② S , $1/S$

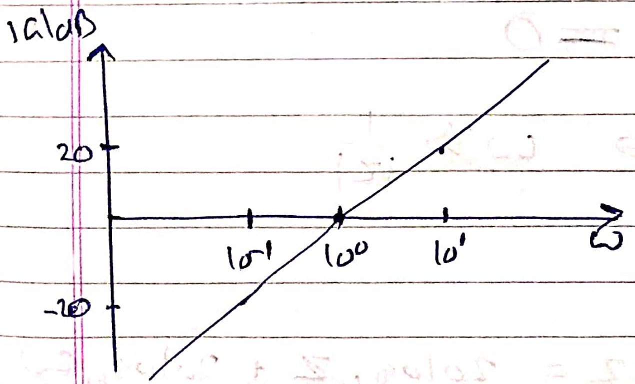
⇒ If we have bode plot of S, so bode plot of $1/S$ is negative of bode plot of S.

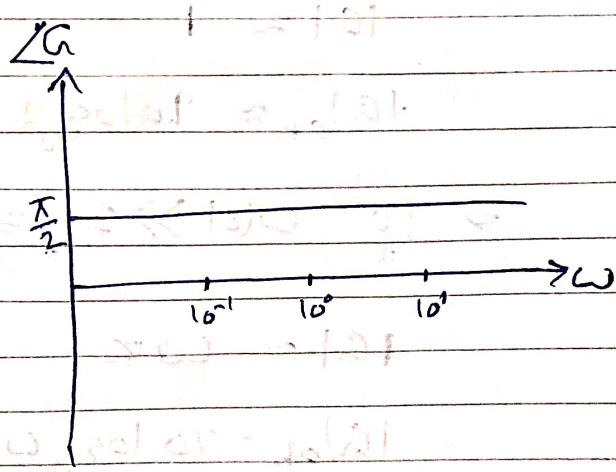
⇒ Or more generally if we have bode plot of $f(s)$, bode plot of $1/f(s)$ is negative of bode plot of $f(s)$.

$G(s) = S \qquad G(j\omega) = j\omega$

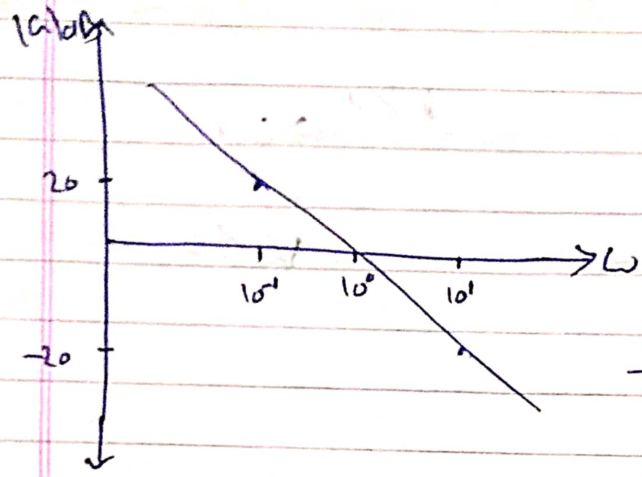
$|G(j\omega)| = \omega \qquad \angle G(j\omega) = \tan^{-1}\left(\frac{\omega}{0}\right) = \pi/2$

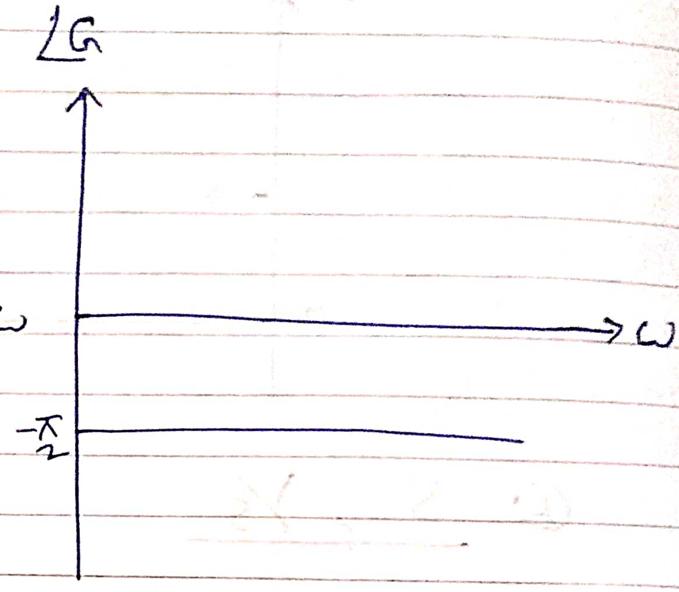
$|G|_{dB} = 20\log_{10}\omega$





⇒ For 1/s





③ $(1+\tau s)$, $\frac{1}{(1+\tau s)}$

$G(s) = 1+\tau s$

$G(j\omega) = 1+j\omega\tau$

$|G| = \sqrt{1^2+\omega^2\tau^2} = \sqrt{1+\omega^2\tau^2}$

⇒ If $1 \gg |\omega\tau| \Rightarrow \omega \ll \frac{1}{|\tau|}$

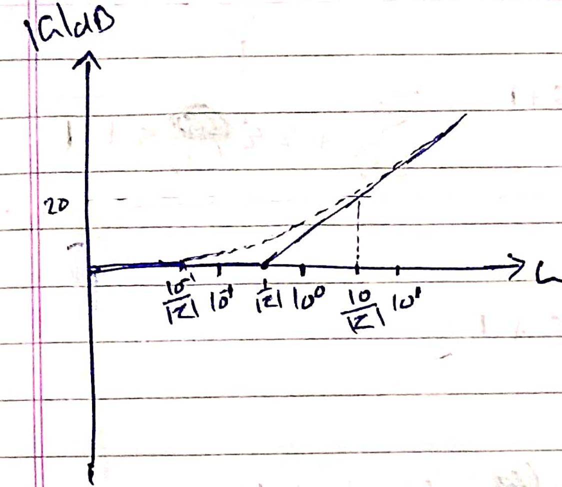
$|G| \approx 1$

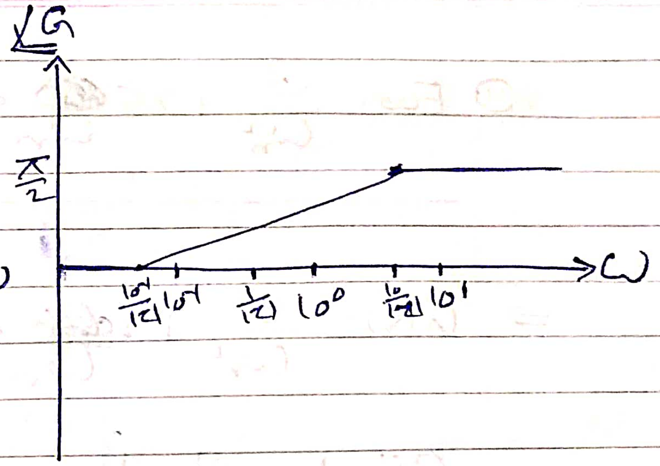
$|G|_{dB} \approx 20\log_{10} 1 \approx 0$

⇒ If $\omega|\tau| \gg 1 \Rightarrow \omega \gg \frac{1}{|\tau|}$

$|G| \approx \omega\tau$

$|G|_{db} = 20\log_{10}\omega\tau = 20\log_{10}\tau + 20\log_{10}\omega$



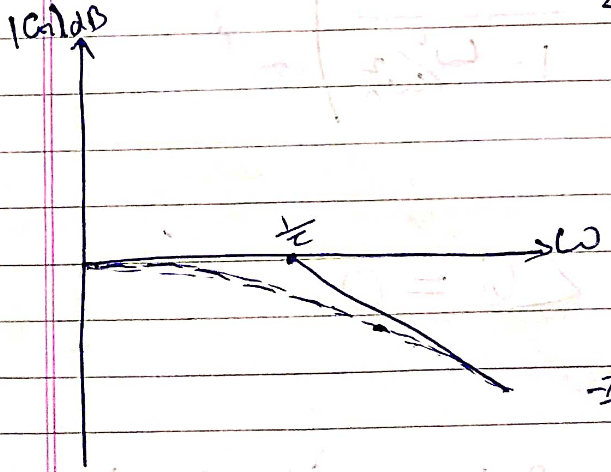


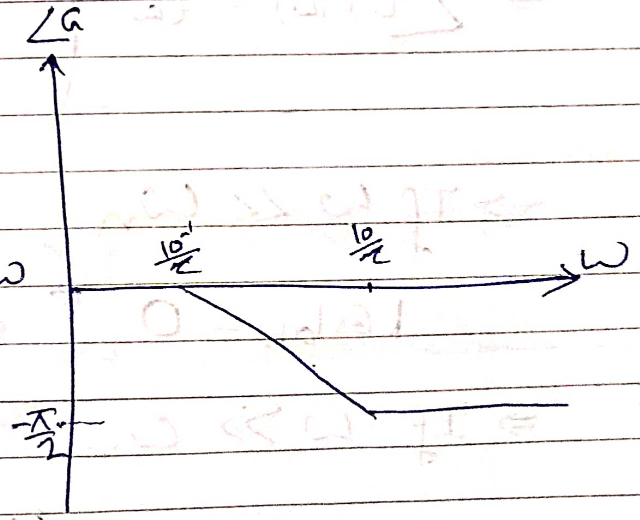
$$\angle G(j\omega) = \tan^{-1}\left(\frac{\omega\tau}{1}\right)$$

$\Rightarrow$ If $\omega\tau << 1$: If $\omega\tau >> 1$

$\angle G(j\omega) \approx 0$ $\quad$ $\angle G(j\omega) \approx \pi/2$

$\Rightarrow$ For $\dfrac{1}{(1+\tau s)}$





④ For $\frac{s^2}{\omega_n^2} + \frac{2\xi}{\omega_n} s + 1$, $\frac{1}{\frac{s^2}{\omega_n^2} + \frac{2\xi}{\omega_n} s + 1}$

$$\Rightarrow G(s) = \frac{s^2}{\omega_n^2} + \frac{2\xi}{\omega_n} s + 1$$

$$\Rightarrow G(j\omega) = \frac{(j\omega)^2}{\omega_n^2} + \frac{2\xi}{\omega_n}(j\omega) + 1$$

$$= \left(1 - \frac{\omega^2}{\omega_n^2}\right) + \frac{2\xi\omega}{\omega_n} j$$

$$\Rightarrow |G(j\omega)|_{db} = 20\log_{10}\left(\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2 + \frac{4\xi^2\omega^2}{\omega_n^2}}\right)$$

$$\Rightarrow \angle G(j\omega) = \tan^{-1}\left(\frac{2\xi\omega/\omega_n}{1-\omega^2/\omega_n^2}\right)$$

$\Rightarrow$ If $\omega << \omega_n$

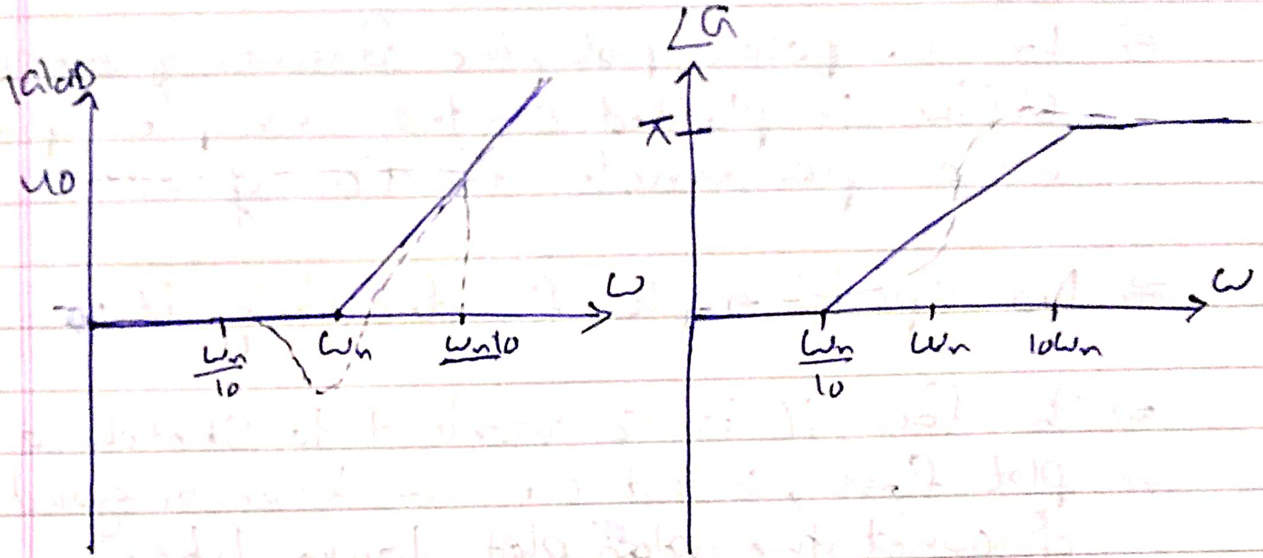
$$|G|_{db} = 0 \qquad \angle G = 0$$

$\Rightarrow$ If $\omega >> \omega_n$

$$|G|_{db} = 20\log_{10}\left(\frac{\omega^2}{\omega_n^2}\right) = 40\log\left(\frac{\omega}{\omega_n}\right)$$

$$= 40\log(\omega) - 40\log(\omega_n)$$

$\angle G \approx \pi$



$\Rightarrow$ For $G(s) = \dfrac{1}{\dfrac{s^2}{\omega_n^2} + \dfrac{2\xi}{\omega_n}s + 1}$