2

Mathematical Background

2.1> Preamble

⇒ Study of motions undergone by robotic mechanical systems, requires a suitable motion representation.

⇒ The most general kind of rigid body motion consists of both translation and rotation.

Translation can be studied simply with the aid of 3-D Vector Calculus

Rigid body rotation requires introduction of Tensor

Entities mapping Vector space into Vector space

Invariant → dose not depends on the choice of frame.

Coordinate frame is required to compute but the final result will be independent of the choice of frame

⇒ Right and Left hand side quantities are not the same.

Vector	1-D array
Tensor	2-D array

2.2) Linear Transformation

⇒ 3D Space is particular case of a vector space.

Vector Space → A vector space is a set of object called vectors, that follows certain algebraic rules.

→ Every linear combination of vectors in the vector space lies in the same vector space.

* Vector ⇒ will be denoted by lower case letters with a bar on its top.

  eg ⇒ $\bar{a}$, $\bar{b}$

Tensor ⇒ Will be denoted by Upper case letters with two bars on its top

  eg ⇒ $\bar{A}$, $\bar{B}$

⇒ In elementary mechanics, the dimension of the vector space needed is usually three. but when studying multibody system, an arbitrary finite dimension will be required.

⇒ Linear transformation

→ Represented by operator $\bar{L}$ of a vector space U into a vector space V, is a rule that assigns to every vector $\bar{u}$ of U at least one vector $\bar{v}$ of V, represented as $\bar{v} = \bar{L}\bar{u}$.

Example of Linear Transformation ⇒
* Projection
* Reflections
* Rotations

Example of Non linear Transformation → affine transformation

Kernel

Kernel of $\bar{L}$ is the set of vectors $\bar{u}_N$ of $\mathcal{U}$ that are mapped by $\bar{L}$ into the zero vector $\bar{0} \in \mathcal{V}$

Range

The range of a linear transformation $\bar{L}$ of $\mathcal{U}$ into $\mathcal{V}$ is the set of vectors $\bar{v}$ of $\mathcal{V}$ into which some vector $\bar{u}$ of $\mathcal{U}$ is mapped.

⇒ It can be proved that Kernel & Range, they are themselves a vector space.

⇒ Kernel of a linear transformation is often called the null space.

⇒ 3-d Euclidean space is denoted by $\mathcal{E}^3$.

⇒ A matrix $\bar{P}$ is said to be idempotent of degree l then.

$$\bar{P}^{l} = \bar{P}$$

Projection of a position vector $\bar{P}$ denoted by $\bar{P'}$, onto a plane π of unit normal $\bar{n}$ is

$$\bar{P'} = \bar{P} - \bar{n}(\bar{n}^T\bar{P})$$

dot product

$$\left\{ \begin{array}{c} \bar{a}\cdot\bar{b} = \bar{a}^T\bar{b} = \bar{b}^T\bar{a} \\ \parallel \\ \bar{b}\cdot\bar{a} \end{array} \right\}$$

$\Rightarrow \quad \bar{p}' = (\bar{\bar{I}} - \bar{n}\bar{n}^T)\bar{p}$

So $\quad \bar{P} = (\bar{\bar{I}} - \bar{n}\bar{n}^T)$

$\rightarrow$ Orthogonal Projection

A reflection $\bar{R}$ of $\mathcal{E}^3$ on to plane Π passing through the origin and having a unit normal $\bar{n}$ is a linear transformation of the said space into itself such that a position vector $\bar{p}$ is mapped by $\bar{R}$ in a vector $\bar{p}'$ given by:-

$$\bar{p}' = (\bar{p} - \bar{n}(\bar{n}^T\bar{p})) - \bar{n}(\bar{n}^T\bar{p})$$

$$\bar{p}' = (\bar{p} - 2\bar{n}(\bar{n}^T\bar{p}))$$

$$\bar{p}' = (\bar{\bar{I}} - 2\bar{n}\bar{n}^T)\bar{p}$$

$$\boxed{\bar{\bar{R}} = \bar{\bar{I}} - 2\bar{n}\bar{n}^T}$$

$\rightarrow$ Reflection

(i) Symmetric
$\bar{\bar{R}}^T = \bar{\bar{R}}$

(ii) Has idempotent of degree 3
$\bar{\bar{R}}^3 = \bar{\bar{R}}$
or
$\bar{\bar{R}}^2 = \bar{\bar{I}}$

$\bar{\bar{R}}^{-1} = \bar{\bar{R}}$
"Self-inverse"

Basis of a vector space V is a set of linearly independent vectors of V, $\{\bar{v}_i\}_1^n$, in terms of which any vector $\bar{v}$ of V can be expressed as:-

$$\bar{v} = \alpha_1\bar{v}_1 + \alpha_2\bar{v}_2 + \cdots \alpha_n\bar{v}_n$$

$\Rightarrow$ The number n is called dimension of the space.

Scanned by CamScanner

Note: Any set of n linearly independent vectors of V can play the role of a basis of this space.

⇒ Let U and V be two vector space of dimensions m and n respectively and $\bar{L}$ a linear transformation of U into V, and define B_U and B_V basis for U and V.

$$B_U = \{\bar{U}_j\}_1^m \quad \& \quad B_V = \{\bar{V}_i\}_1^n$$

$$\bar{L}\bar{U}_j = l_{1j}\bar{V}_1 + l_{2j}\bar{V}_2 + \cdots l_{nj}\bar{V}_n$$

$$[L]_{B_U}^{B_V} = \begin{bmatrix} l_{11} & l_{12} & \cdots & l_{1m} \\ l_{21} & l_{22} & \cdots & l_{2m} \\ \vdots & \vdots & & \vdots \\ l_{n1} & l_{n2} & \cdots & l_{nm} \end{bmatrix}$$

⇒ The foregiven array is thus called the matrix representation of $\bar{L}$ with respect to B_U and B_V.

⇒ When mapping $\bar{L}$ is a mapping U onto itself, then single basis suffices to represent L in matrix form.

\# A mapping $\bar{L}$ of an n-dimensional vector space V, a U into the n-dimensional vector space that is mapped by $\bar{L}$ into non zero vector $\bar{e}$ a multiple of itself, $\lambda\bar{e}$ is called an <u>eigen vector</u>

of $\bar{\bar{L}}$, the Scalar λ being called <u>eigen values</u> of $\bar{\bar{L}}$.

⇒ Eigen Values λ can be determined by

$$|\lambda \bar{\bar{I}} - \bar{\bar{L}}| = 0$$

→ This is called Characteristics Polynomial of $\bar{\bar{L}}$.

2.3> <u>Rigid body rotations</u>

⇒ A linear isomorphism (one to one Linear transformation mapping a Space V onto itself) is called isometry if it preserves distance between any two points of V.

⇒ Let $\bar{P}$ be the position vector of any point of $\mathcal{E}^3$ its image under a rotation $\bar{\bar{Q}}$ be $\bar{P}'$.

→ Distance preservation requires that

$$\bar{P}^T \bar{P} = \bar{P}'^T \bar{P}' \quad \text{— (1)}$$

where, $$\bar{P}' = \bar{\bar{Q}} \bar{P} \quad \text{— (2)}$$

Putting (2) in (1) we get

$$\bar{P}^T \bar{P} = \bar{P}^T \bar{\bar{Q}}^T \bar{\bar{Q}} \bar{P}$$

So, $$\boxed{\bar{\bar{Q}}^T \bar{\bar{Q}} = \bar{\bar{I}}}$$

⇒ So $\bar{\bar{Q}}$ is orthogonal matrix.

Let $T = [\bar{u}\ \bar{v}\ \bar{w}]$ $\quad$ $T' = [\bar{u}'\ \bar{v}'\ \bar{w}']$

⇒ From above it is clear that:-

$$\boxed{\bar{\bar{T}}' = \bar{\bar{Q}}\,\bar{\bar{T}}}$$

⇒ For isometry to represent rotation

$$|\bar{\bar{T}}| = |\bar{\bar{T}}'|$$

\# Reff1
else it represent reflection

So $\boxed{|\bar{\bar{Q}}| = 1}$

⇒ Therefore $\bar{\bar{Q}}$ is a proper orthogonal matrix.

\# Theorem 1: The eigen values of a proper orthogonal matrix $\bar{\bar{Q}}$ lie on the unit circle centered at the origin of the complex plane.

Proof: Let λ be one of the eigen values of $\bar{\bar{Q}}$ and $\bar{e}$ the corresponding eigenvector, so that

$$\bar{\bar{Q}}\,\bar{e} = \lambda \bar{e} \quad \text{——} \; (1)$$

⇒ The transpose conjugate of the above equation takes form

$$\bar{e}^* \bar{\bar{Q}}^* = \lambda^* \bar{e}^* \quad \text{——} \; (2)$$

⇒ Multiplying eqn (1) & (2) we get:-

$$\bar{e}^* \bar{\bar{Q}}^* \bar{\bar{Q}}\,\bar{e} = \lambda\lambda^* \bar{e}^* \bar{e}$$

$$\Rightarrow \bar{e}^* \bar{\bar{Q}}^T \bar{\bar{Q}}\,\bar{e} = \lambda\lambda^* \bar{e}^* \bar{e}$$ {$\bar{\bar{Q}}$ is assumed to be real.}

$\Rightarrow \bar{e}^* \bar{e} = |\lambda|^2 \bar{e}^* \bar{e}$ {as Q is orthogonal}

$\Rightarrow \boxed{|\lambda|^2 = 1}$

Theorem 3: (Cayley-Hamilton) Let $P(\lambda)$ be the characteristic polynomial of a $n \times n$ matrix A

$$P(\lambda) = |\lambda \bar{I} - \bar{A}| = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots a_1 \lambda + a_0$$

Then $\bar{A}$ satisfies its characteristic equation:-

$$\bar{A}^n + a_{n-1}\bar{A}^{n-1} + \cdots + a\bar{A} + a_0 \bar{I} = \bar{O}$$

{$n \times n$ zero matrix}

Theorem 2: (Euler 1776) A rigid-body motion about a point O leaves fixed set of point lying on a line $\mathcal{L}$ that passes through O and is parallel to the eigenvector $\bar{e}$ of $\bar{Q}$ associated with the eigen value $+1$.

@Analytic function $\Rightarrow$ function that is locally given by a convergent power series.

$\Rightarrow$ An important consequence of Cayley-Hamilton is that any analytic matrix function of A can be expressed not as an infinite series, but as a sum, namely, a linear combination of first n powers of $\bar{A}$.

example $\Rightarrow$ Exponential function.

2.3.1) The Cross-Product Matrix

⇒ Let $\bar{u}$ and $\bar{v}$ be vectors of space U and V, of dimensions m and n respectively.

⇒ Let t be a real variable and f be real-valued functions of t, with $f = f(\bar{u}, \bar{v})$

⇒ The partial derivative of f with respect to $\bar{u}$ is a ~~m~~-dimensional vector whose i^{th} component is partial derivative of f with respect to u_i.

↳ Similarly for partial derivative of f with respect to $\bar{V}$.

$$\frac{\delta f}{\delta \bar{u}} = \begin{bmatrix} \delta f/\delta u_1 \\ \delta f/\delta u_2 \\ \vdots \\ \delta f/\delta u_m \end{bmatrix} \qquad \frac{\delta f}{\delta \bar{V}} = \begin{bmatrix} \delta f/\delta v_1 \\ \delta f/\delta v_2 \\ \vdots \\ \delta f/\delta v_n \end{bmatrix}$$

⇒ Furthermore, the partial derivative of $\bar{V}$ with respect to $\bar{u}$ is an $n \times m$ array whose (i,j) entry is defined as $\delta V_i/\delta V_j$.

$$\frac{\delta \bar{V}}{\delta \bar{u}} = \begin{bmatrix} \delta u_1/\delta u_1 & \delta V_1/\delta u_2 & \cdots & \delta V_1/\delta u_m \\ \delta u_2/\delta u_1 & \delta V_2/\delta u_2 & \cdots & \delta V_2/\delta u_m \\ \vdots & \vdots & & \vdots \\ \delta V_m/\delta u_1 & \delta V_n/\delta u_2 & \cdots & \delta V_n/\delta u_m \end{bmatrix}$$

⇒ Hence, the total derivative of f with respect to $\bar{u}$ can be written as:-

$$\boxed{\frac{df}{d\bar{u}} = \frac{\delta f}{\delta \bar{u}} + \left(\frac{\delta \bar{v}}{\delta \bar{u}}\right)^T \frac{\delta f}{\delta \bar{v}}}$$

& $\bar{v} = \bar{v}(\bar{u}, t)$

⇒ If f is an explicit function (ie $f = f(\bar{u}, \bar{v}, t)$) then one can write total derivative of f w.r.t t as:-

$$\boxed{\frac{df}{dt} = \frac{\delta f}{\delta t} + \left(\frac{\delta f}{\delta \bar{u}}\right)^T \frac{d\bar{u}}{dt} + \left(\frac{\delta f}{\delta \bar{v}}\right)^T \frac{\delta \bar{v}}{\delta t} + \left(\frac{\delta f}{\delta \bar{v}}\right)^T \frac{\delta \bar{v}}{\delta \bar{u}} \frac{d\bar{u}}{dt}}$$

⇒ The total derivative of $\bar{V}$ with respect to t can be written, likewise as:-

$$\boxed{\frac{d\bar{V}}{dt} = \frac{\delta \bar{V}}{\delta t} + \frac{\delta \bar{V}}{\delta \bar{u}} \frac{d\bar{u}}{dt}}$$

Let the Components of $\bar{V}$ and $\bar{X}$ in a certain reference frame F is given by

$$[\bar{V}]_F = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} \quad ; \quad [\bar{X}]_F = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

$$[\bar{V} \times \bar{X}]_F = \begin{bmatrix} V_2 x_3 - V_3 x_2 \\ V_3 x_1 - V_1 x_3 \\ V_1 x_2 - V_2 x_1 \end{bmatrix}$$

Hence,

$$\left[\frac{d\,[\bar{V} \times \bar{X}]}{d\bar{X}}\right]_F = \begin{bmatrix} 0 & -V_3 & V_2 \\ V_3 & 0 & -V_1 \\ -V_2 & V_1 & 0 \end{bmatrix}$$

Scanned by CamScanner

⇒ The partial derivative of the cross product of any 3-dimensional vector $\bar{v}$ and $\bar{x}$ will be denoted by the 3×3 matrix $\bar{\bar{V}}$.

↳ $\bar{\bar{V}}$ is termed cross product matrix of $\bar{v}$.

⇒ The cross product can be alternative represented as:-

$$\bar{v}\times\bar{x} = \bar{\bar{V}}\bar{x}$$

Theorem 4: The cross product matrix $\bar{A}$ of any 3D vector $\bar{a}$ is skew symmetric :-

$$A^T = -A$$

and as a consequence

$$\bar{a}\times(\bar{a}\times\bar{b}) = \bar{A}^2\bar{b}$$

$\bar{A}^2$ can be readily proven to be

$$\bar{A}^2 = -\|a\|^2\bar{I} + \bar{a}\bar{a}^T$$

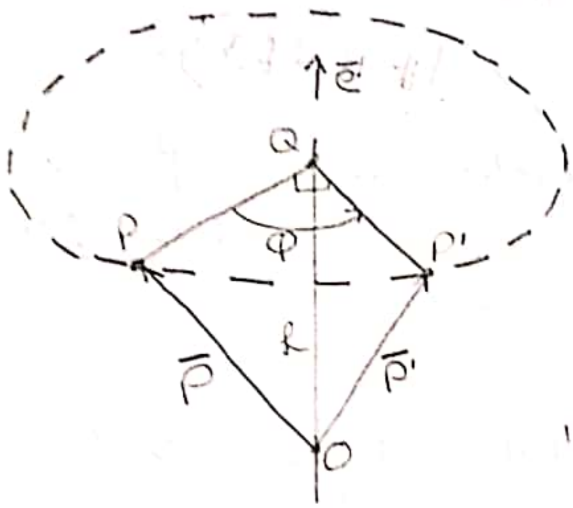
Reff 2 {using vector triple product expansion}

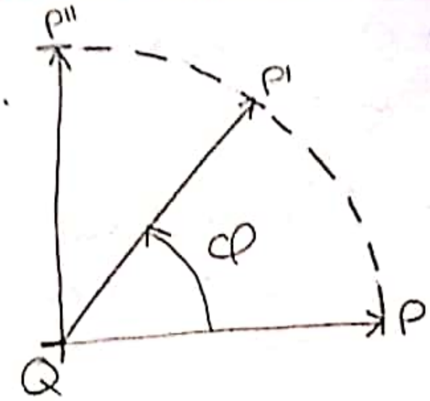
→ Euclidean norm

2.3.2) The rotation matrix

⇒ Line $\mathcal{L}$ mentioned in Euler's Theorem, is termed the axis of rotation of the motion of interest.

⇒ Consider the rotation depicted in figure bellow of angle φ about line $\mathcal{L}$.





$$\overline{P'} = \overline{OQ} + \overline{QP'} \quad \text{— ①}$$

$\overline{OQ} \rightarrow \bar e \bar e^T \bar P$

$$\overline{QP'} = \cos\varphi\, \overline{QP} + \sin\varphi\, \overline{QP''}$$

Normal component of $\bar P$ with respect to $\bar e$

$$\overline{QP} = (1 - \bar e \bar e^T)\bar P$$

$$\overline{QP''} = \bar e \times \bar P = \bar{\bar E}\, \bar P$$

So $\overline{P'} = \bar e \bar e^T \bar P + \cos\varphi (1 - \bar e \bar e^T)\bar P + \sin\varphi\, \bar{\bar E}\, \bar P$

$$\overline{P'} = \left[\bar e \bar e^T + \cos\varphi (1 - \bar e \bar e^T) + \sin\varphi\, \bar{\bar E}\right] \bar P$$

Rotation matrix $\bar{\bar Q}$ is given by

$$\boxed{\bar{\bar Q} = \bar e \bar e^T + \cos\varphi (1 - \bar e \bar e^T) + \sin\varphi\, \bar{\bar E}} \quad \text{— ①}$$

One more representation of $\overline{\overline{Q}}$ in terms of $\bar{e}$ and φ is given:

{# Ref 3}

⇒ For a fixed axis rotation (ie for a fixed value of $\bar{e}$), the rotation matrix $\overline{\overline{Q}}$ is a function of the angle of rotation φ only.

So $$\overline{\overline{Q}}(\varphi) = \overline{\overline{Q}}(0) + \overline{\overline{Q}}'(0)\varphi + \frac{1}{2!}\overline{\overline{Q}}''(0)\varphi^2 + \cdots + \frac{1}{K!}\overline{\overline{Q}}^K\varphi^K + \cdots$$

⇒ Now, from the definition of $\overline{\overline{E}}$ one can readily prove the relations below:-

$$\overline{\overline{E}}^{2K+1} = (-1)^K\overline{\overline{E}} \quad ; \quad \overline{\overline{E}}^{2K} = (-1)^K(\overline{\overline{I}} - \bar{e}\bar{e}^T) \quad \text{—— (2)}$$

⇒ Using eq (1) and (2) one can readily show that

$$\overline{\overline{Q}}^K(0) = \overline{\overline{E}}^K$$

So, $$\overline{\overline{Q}}(\varphi) = \overline{\overline{I}} + \overline{\overline{E}}\varphi + \frac{1}{2!}\overline{\overline{E}}^2\varphi^2 + \cdots \frac{1}{K!}\overline{\overline{E}}^K\varphi^K + \cdots$$

$$\Rightarrow \boxed{\overline{\overline{Q}}(\varphi) = e^{\overline{\overline{E}}\varphi}}$$

Exponential representation of the rotation matrix in terms of its natural invariants $\bar{e}$ and φ

⇒ The foregoing parameters are termed invariant because they are clearly independent of the coordinate axis chosen to represent the rotation under study.

⇒ Alternative representation of $\bar{\bar{Q}}$:-

$$\boxed{\bar{\bar{Q}} = \bar{\bar{I}} + \sin\phi\, \bar{\bar{E}} + (1-\cos\phi)\, \bar{\bar{E}}^2}$$

→ Expected result in view of Cayley-Hamilton theorem.

Canonical forms of the Rotation Matrix

⇒ The rotation matrix takes on especially simple form if the axis of rotation coincides with one of the coordinate axis.

eg ⇒ x-axis ∥ to axis of rotation :-

$$[\bar{e}]_x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad [E]_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \quad [E^2]_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\text{So } [\bar{\bar{Q}}]_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \quad \text{— a}$$

Similarly,

$$[\bar{\bar{Q}}]_y = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix} \quad \text{— b} \qquad [\bar{\bar{Q}}]_x = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{— c}$$

⇒ The representation of eq ⓐ ⓑ & ⓒ can be called the X- Y- and Z- Canonical forms of the rotation matrix.

⇒ An application of Canonical forms lies in the parameterization of rotations by means of Euler angle consisting of three successive rotations φ, θ and ψ about the axis of coordinate frame.

2.3.3> The Linear Invariants of a 3×3 Matrix

⇒ Any 3×3 matrix $\bar{\bar{A}}$ consists of the sum of its symmetric part $\bar{\bar{A}}_s$, and its skew-symmetric part $\bar{\bar{A}}_{ss}$.

$$\bar{\bar{A}} = \bar{\bar{A}}_s + \bar{\bar{A}}_{ss} \qquad \begin{cases} \bar{\bar{A}}_s = \frac{1}{2}(\bar{\bar{A}} + \bar{\bar{A}}^T) \\ \bar{\bar{A}}_{ss} = \frac{1}{2}(\bar{\bar{A}} - \bar{\bar{A}}^T) \end{cases}$$

⇒ Let $\bar{\bar{A}}_{ss}$ has vector $\bar{a}$ so,

$$\bar{a} \times \bar{v} = \bar{\bar{A}}_{ss}\bar{v}$$

for any 3D vector $\bar{v}$.

⇒ Let us assume that the entries of matrix $\bar{\bar{A}}$ in a certain coordinate frame are given by the array of real numbers a_{ij} $\forall$ $i,j = 1,2,3$.

Let $\bar{a}$ has components a_i $\forall$ $i = 1,2,3$. In the same frame.

$$\text{Vec}(\bar{A}_{ss}) = \bar{a} = \frac{1}{2}\begin{bmatrix} a_{32} - a_{23} \\ a_{13} - a_{31} \\ a_{21} - a_{12} \end{bmatrix}$$

$$\text{tr}(\bar{A}) = a_{11} + a_{22} + a_{33}$$

Let us call Vector of $\bar{A}$ as $\bar{a}$. when $\bar{a}$ is Vector of $\bar{\bar{A}}_{ss}$.

$$\text{So} \quad \text{Vec}(\bar{A}) = \bar{a} = \frac{1}{2}\begin{bmatrix} a_{32} - a_{23} \\ a_{13} - a_{31} \\ a_{21} - a_{12} \end{bmatrix}$$

$$\text{tr}(\bar{A}) = a_{11} + a_{22} + a_{33}.$$

Theorem 3: The Vector of a 3×3 matrix Vanishes if and Only if it is Symmetric, whereas as the trace of an n×n matrix Vanishes if the matrix is skew Symmetric.

⇒ Other useful relations are given bellow. For any 3-D Vector $\bar{a}$ and $\bar{b}$.

$$\text{Vec}(\bar{a}\bar{b}^T) = -\frac{1}{2}(\bar{a} \times \bar{b})$$

$$\& \quad \text{tr}(\bar{a}\bar{b}^T) = \bar{a}^T\bar{b} \qquad \{\text{Straight forward}\}$$

Proof

Let W denote $\text{Vec}(\bar{a}\bar{b}^T)$, for and 3 D Vector $\bar{V}$

$$\bar{w} \times \bar{V} = \bar{\bar{W}}\bar{V}$$

Scanned by CamScanner

where, $\overline{\overline{W}}$ is the skew symmetric component of $\bar{a}\bar{b}^T$.

$$\overline{\overline{W}} = \frac{1}{2}\left(\bar{a}\bar{b}^T - \bar{b}\bar{a}^T\right)$$

$$\overline{\overline{W}}\bar{v} = \bar{\omega}\times\bar{v} = \frac{1}{2}\left[\bar{a}(\bar{b}^T\bar{v}) - \bar{b}(\bar{a}^T\bar{v})\right]$$

$$= \frac{1}{2}\,\bar{b}\times\bar{a}\times\bar{v}$$

So $\boxed{\bar{\omega} = \frac{1}{2}\bar{b}\times\bar{a}}$

2.3.4> The Linear Invariants of a Rotation

$$\overline{\overline{Q}} = \bar{e}\bar{e}^T + \cos\phi\,(\overline{\overline{I}} - \bar{e}\bar{e}^T) + \sin\phi\,\overline{\overline{E}}$$

⇒ From above it is clear that first two terms of $\overline{\overline{Q}}$, $\bar{e}\bar{e}^T$ and $\cos\phi(\overline{\overline{I}} - \bar{e}\bar{e}^T)$ are symmetric and third one, $\sin\phi\,\overline{\overline{E}}$ is skew-symmetric.

$$tr(\overline{\overline{Q}}) = tr\left[\bar{e}\bar{e}^T + \cos\phi\,(\overline{\overline{I}} - \bar{e}\bar{e}^T)\right]$$

$$= \bar{e}^T\bar{e} + \cos\phi\,(3 - \bar{e}^T\bar{e}) = 1 + 2\cos\phi$$

So $\boxed{\cos\phi = \frac{tr(\overline{\overline{Q}}) - 1}{2}}$

⇒ Henceforth, the vector of $\overline{\overline{Q}}$ will be denoted by $\bar{q}$ and its components in a given coordinate frame by $q_1\, q_2\, q_3$.

⇒ Rather than $tr(\bar{Q})$ as the Other linear invariant $q_0 = \cos\phi$ will be linear invariants of the rotation matrix

⇒ The rotation matrix is fully defined by four Scalar parameters, namely $\{q_i\}_0^3$, which will be conveniently stored in the 4D array $\bar{\lambda}$ defined as :-

$$\bar{\lambda} = [q_1\; q_2\; q_3\; q_0]^T$$

⇒ However, that the four components of $\bar{\lambda}$ are not independent as:

$$\bar{q} = \sin\phi\, \bar{e} \Rightarrow |\bar{q}|^2 = \sin^2\phi$$
$$q_0 = \cos\phi \Rightarrow q_0^2 = \cos^2\phi$$

$$\Rightarrow \|\bar{q}\|^2 + q_0^2 = 1$$

$$\Rightarrow q_1^2 + q_2^2 + q_3^2 + q_0^2 = 1$$

$$\Rightarrow \boxed{\|\bar{\lambda}\|^2 = 1}$$

Geometric Interpretation of above

"As a body rotates about a fixed point, its motion Can be described in a 4D Space by the motion of a point of Vector $\bar{\lambda}$ that moves on the Surface of the Unit Sphere Centered at the Origin"

⇒ Given the dependence of the four Components of Vector $\bar{\lambda}$, One might be tempted to solve for q_0.

$\Rightarrow q_0 = \pm\sqrt{1-(q_1^2+q_2^2+q_3^2)}$

Sign ambiguity leaves angle ϕ Undefined

$\Rightarrow$ Moreover, the three Components of vector $\bar{q}$ alone, do not Suffice to define the rotation represented by $\bar{Q}$.

$$\bar{e} = \frac{\bar{q}}{\sin\phi} \; ; \; \sin\phi = \pm\|\bar{q}\|$$

angle ϕ undefined.

In terms of $\bar{q}$ an q_0 rotation matrix can be written as :-

$$\bar{\bar{Q}} = \frac{\bar{q}\bar{q}^T}{\|\bar{q}\|^2} + q_0\left(\bar{\bar{I}} - \frac{\bar{q}\bar{q}^T}{\|\bar{q}\|^2}\right) + \bar{\bar{Q}}'$$

where $\bar{\bar{Q}}' = \dfrac{\delta(\bar{q}\times\bar{x})}{\delta x}$

$\forall$ any vector $\bar{x}$

$$\bar{\bar{Q}} = q_0\bar{\bar{I}} + \bar{\bar{Q}}' + \frac{\bar{q}\bar{q}^T}{1+q_0}$$

$\Rightarrow$ From above equation it is clear that linear invariants are <u>not Suitable to represent a rotation</u> when the associated angle is either <u>π or close to it.</u>

⇒ Rotation through an angle α about an axis given by vector $\bar{e}$ is identical to a rotation through an angle $-\alpha$ about an axis given by vector $-\bar{e}$.

↳ Henceforth we will choose the sign of the components of $\bar{e}$ so that $\sin\alpha \geq 0$
↓
equivalent to $0 \leq \alpha \leq \pi$

Thus $\sin\alpha = \|a\|$ → $\{\bar{e}$ is simply normalized $\bar{a}\}$

$\cos\alpha = a_0$
↓
$\dfrac{\operatorname{tr}(\bar{\bar{Q}})-1}{2}$

2.3.6) The Euler-Rodrigues Parameters

Natural Invariant ⇒ $\operatorname{tr}(\bar{\bar{A}})$, $\operatorname{Vec}(\bar{\bar{A}})$

Linear Invariant ⇒ a_0 $\quad a_1\ a_2\ a_3$

⇒ Natural and the linear invariants of a rotation matrix, are not the only one that are used in kinematics.

↳ Additionally one has Euler-Rodrigues Parameter, represented here as $\bar{r}$ and r_0.

$$\bar{r} = \sin\left(\frac{\alpha}{2}\right)\bar{e} \qquad r_0 = \cos\left(\frac{\alpha}{2}\right)$$

⇒ One can readily show that $\bar{\bar{Q}}$ takes on a quite simple form in terms of Euler-Rodrigues Parameters.

$$\overline{\overline{Q}} = (r_0^2 - \overline{r}\cdot\overline{r})\overline{\overline{I}} + 2\overline{r}\,\overline{r}^T + 2r_0\overline{\overline{R}}$$

where $\overline{\overline{R}} = \dfrac{\delta(\overline{r}\times\overline{x})}{\delta\overline{x}}$ $\forall$ arbitrary $\overline{x}$

⇒ The relationship between the linear invariants and the Euler-Rodrigues parameters can be readily derived.

$$r_0 = \pm\sqrt{\frac{1+q_0}{2}} \qquad \overline{r} = \frac{\overline{q}}{2r_0} \qquad \varphi \neq \pi$$

$$\overline{r} = \overline{e} \quad \& \quad r_0 = 0 \quad \forall\ \varphi = \pi$$

⇒ We now derive invariant relations between the rotation matrix and the Euler-Rodrigues Parameters.

↳ To do this we resort to the concept of Matrix Square root.

⇒ From the geometric meaning of a rotation through the angle φ about an axis ∥ to the unit vector $\overline{e}$,

↳ It is apparent that the square of the matrix representing the foregoing rotation it itself a rotation about the same axis, but through the angle 2φ.

↳ By the Same token, the Square root of ~~the~~ the Same matrix is again a rotation matrix about the Same axis but through an angle $\phi/2$.

⇒ While the Square of a matrix is Unique, its Square root is not.

↳ Of these Square roots, nevertheless, there is one that is proper orthogonal, the one representing a rotation of $\phi/2$

⇒ We will denote the particular Squa. root of $\bar{\bar{Q}}$ by $\sqrt{\bar{\bar{Q}}}$.

⇒ The Euler-Rodrigues parameters of $\bar{\bar{Q}}$ can thus be expressed as the linear invariants of $\sqrt{\bar{\bar{Q}}}$ namely:-

$$\bar{r} = \text{Vect}\left(\sqrt{\bar{\bar{Q}}}\right) \qquad r_0 = \frac{\text{tr}\left(\sqrt{\bar{\bar{Q}}}\right) - 1}{2}$$

⇒ It is important to recognize the basic difference between

linear Invariants

↳ Easily derived from matrix representation of rotation.

↳ fails to provide axis of rotation when angle is π.

Euler-Rodrigues Parameters

↳ Require Square root & involves ~~[illegible]~~ Sign ambiguities.

↳ Gives axis of rotation for any value of the angle of rotation.

⇒ The Euler-Rodrigues Parameters are nothing but the quaternions invented by Sir William Rowan Hamilton (1844).

2.4) Composition of Reflections and Rotation

Let $\bar{\bar{R}}$ ⇒ Pure reflection $\left\{ \begin{array}{l} \to \text{Symmetric} \\ \to \text{Self inverse} \end{array} \right\}$

$\bar{\bar{Q}}$ ⇒ arbitrary rotation {Proper Orthogonal}

⇒ The product of these two transformations $\bar{\bar{Q}}\,\bar{\bar{R}}$, denoted by $\bar{\bar{T}}$ is apparently neither symmetric nor self-inverse.

↳ Same true for reverse order

⇒ As a consequence, ~~of~~ an improper orthogonal transformation that is not symmetric can always be decomposed into the product of a rotation and a pure reflection.

⇒ If we want to decompose $\bar{\bar{T}}$ in the above paragraph into the product $\bar{\bar{Q}}\,\bar{\bar{R}}$,

↳ then we can freely close the unit normal vector $\bar{n}$ of the plane of reflection and write

$$\bar{\bar{R}} = \bar{\bar{I}} - 2\bar{n}\bar{n}^T$$

⇒ Hence the factor $\bar{\bar{Q}}$ is obtained as:-

$$\bar{\bar{Q}} = \bar{\bar{T}}\bar{\bar{R}}^{-1} = \bar{\bar{T}}\bar{\bar{R}} = \bar{\bar{T}} - 2(\bar{\bar{T}}\bar{n})\bar{n}^T$$

2.5) Coordinate Transformations and Homogeneous Coordinates

⇒ Crucial to robotics is the unambiguous description of the geometrical relation among the various bodies in the environment surrounding a robot.

↳ These relations are established by means of coordinate frame
↓
{attached to each rigid body in the scene, including the robot links.}

⇒ The origin of these frames, moreover, are set at landmark points and orientations defined by key geometric entities like lines and planes.

↳ Z-axis of each frame is defined, according to the Denavit-Hartenberg notation.

2.5.1) Coordinate transformation between frames With a Common Origin

Let two Coordinate frame be $A = \{X\ Y\ Z\}$ & $B = \{x\ y\ z\}$.

Let $\bar{Q}$ be the rotation carrying A into B.

$$Q : A \rightarrow B$$

Let us denote position vector of a point P in A and B, denoted by $[\bar{P}]_A$ and $[\bar{P}]_B$ respectively.

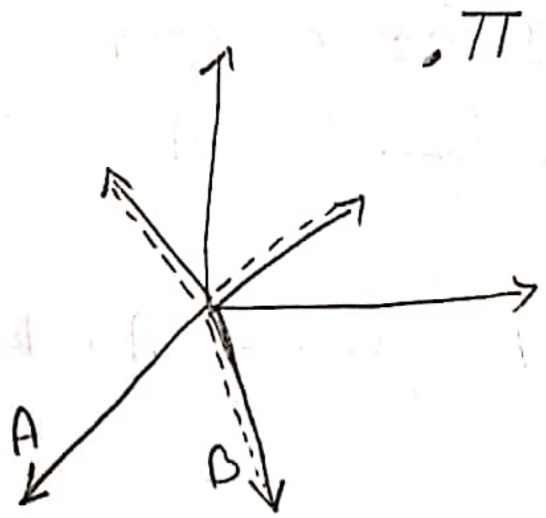
Let

$$[P]_A = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

⇒ We want to find $[P]_B$ in terms of $[P]_A$ and $\bar{Q}$.

⇒ Suppose P is attached to frame A.

⇒ Now frame A undergo rotation $\bar{Q}$ about its origin, that carries it to new configuration, that of frame B.



⇒ Point P in its rotated position is labeled Π, of position vector $\bar{\pi}$.

$\bar{r} = \bar{Q}\bar{P}$

& $[\bar{r}]_B = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Let $[\bar{r}]_A = \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix}$

Theorem 2.5.1: The representations of the position vector $\bar{r}$ of any point in two frame A & B, denoted by $[\bar{r}]_A$ and $[\bar{r}]_B$, respectively are related by

$$[\bar{r}]_A = [\bar{Q}]_A [\bar{r}]_B$$

Proof: $[\bar{r}]_A = [\bar{Q}]_A [\bar{P}]_A$ —— ①

& $[\bar{r}]_B = [\bar{P}]_A$ —— ②

So using ① & ②

$$\boxed{[\bar{r}]_A = [\bar{Q}]_A [\bar{r}]_B}$$

Theorem 2.5.2: The representations of $\bar{Q}$ carrying A and B in these two frames are identical.

$$[\bar{Q}]_A = [\bar{Q}]_B$$

Proof

$$[\bar{\pi}]_A = [\bar{\bar{Q}}]_A [\bar{\pi}]_B \quad \text{—} ①$$

$$\bar{\pi} = \bar{\bar{Q}}\bar{P} \quad \text{—} ②$$

$\Rightarrow$ Using ① & ② we get :-

$$[\bar{\bar{Q}}\bar{P}]_A = [\bar{\bar{Q}}]_A [\bar{\bar{Q}}\bar{P}]_B$$

$$\Rightarrow [\bar{\bar{Q}}]_A [\bar{P}]_A = [\bar{\bar{Q}}]_A [\bar{\bar{Q}}]_B [\bar{P}]_B$$

$$\Rightarrow [\bar{P}]_A = [\bar{\bar{Q}}]_B [\bar{P}]_B$$

~~We know $[\bar{P}]_A = \bar{P}$~~

$\Rightarrow$ By Virtue of Theorem 2.5.1, the two representations of P observe the relation

$$[P]_A = [Q]_A [P]_B$$

Hence $[Q]_A = [Q_B]$

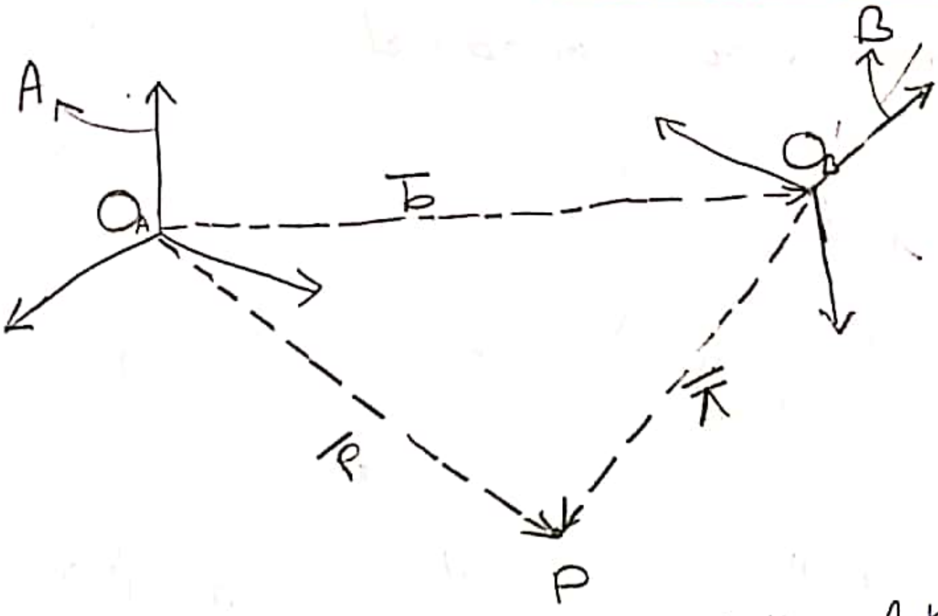
2.5.2) Coordinate transformation with Origin Shift

Fram A : Origin O_A

Fram B : Origin O_B

Let $\overline{O_A O_B} = \bar{b}$

& $\bar{\bar{Q}} : A \rightarrow B$



Theorem 2.5.4 : The representations of the position vector $\bar{p}$ of point P of Euclidean 3D space in two frames A and B are related by

$$[\bar{p}]_A = [b]_A + [\bar{Q}]_A [\bar{\pi}]_B$$

Proof

$$\bar{p} = \bar{b} + \bar{\pi}$$

$$\Rightarrow [\bar{p}]_A = [\bar{b}]_A + [\bar{\pi}]_A \quad \text{—} (1)$$

$$[\bar{\pi}]_A = [\bar{Q}]_A [\bar{\pi}]_B \quad \text{—} (2)$$

Using eq (1) & (2) we get :-

$$\boxed{[\bar{p}]_A = [\bar{b}]_A + [\bar{Q}]_A [\bar{\pi}]_B}$$

2.5.3> Homogeneous Coordinates

⇒ The general coordinate transformation, involving a shift of the origin, is not linear, in general, as can be readily realized by virtue of the nonhomogeneous term involved.

⇒ Let $[\bar{P}]_M$ be the coordinate array of a finite point P in reference frame M.

{finite points are one whose coordinates are all finite}

⇒ The homogeneous coordinates of P are those in the 4-dimensional array $\{P\}_M$ defined as:-

$$\{\bar{P}\}_M = \begin{bmatrix} [\bar{P}]_M \\ 1 \end{bmatrix}$$

We know

$$[\bar{P}]_A = [\bar{b}]_A + [\bar{\bar{Q}}]_A [\bar{A}]_B$$

⇒ The affine transformation of above equation can be rewritten in homogeneous-coordinate form as:

$$\{\bar{P}\}_A = \{\bar{\bar{T}}\}_A \langle \bar{\pi} \}_B$$

where,

$$\{\bar{\bar{T}}\}_A = \begin{bmatrix} [\bar{\bar{Q}}]_A & [b]_A \\ [O^T]_A & 1 \end{bmatrix}$$

$$\{\bar{\bar{T}}^{-1}\}_B = \begin{bmatrix} [\bar{\bar{Q}}]_B & [-b]_B \\ [O^T]_B & 1 \end{bmatrix}$$

⇒ Furthermore, homogeneous transformation can be Concatenated
{Link together in chain}

⇒ Let $F_K \ \forall \ K = i-1, i, i+1$, denote three coordinate frames with origin at O_K.

⇒ Let $\bar{\bar{Q}}_{i-1}$ be the rotation carrying F_{i-1} into orientation coinciding with that of F_i.

Similarly, $\bar{\bar{Q}}_i : F_i \rightarrow F_{i+1}$

⇒ First the case in which all the three origins coincide is considered.

$$[\bar{P}]_i = [\bar{\bar{Q}}^T_{i-1}]_{i-1} [\bar{P}]_{i-1}$$

$$[\bar{P}]_{i+1} = [\bar{\bar{Q}}^T_i]_i [\bar{P}]_i \Rightarrow [\bar{\bar{Q}}^T_i]_i [\bar{\bar{Q}}^T_{i-1}]_{i-1} [\bar{P}]_{i-1}$$

⇒ If now the origins do not coincide, let $\bar{a}_{i-1}$ and $\bar{a}_i$ denote the vectors $\overline{O_{i-1}O_i}$ and $\overline{O_i O_{i+1}}$ respectively.

⇒ The homogeneous-coordinate transformation $\{\bar{\bar{T}}_{i-1}\}_{i-1}$ and $\{\bar{\bar{T}}_i\}_i$, thus arising are obviously

$$\{\bar{\bar{T}}_{i-1}\}_{i-1} = \begin{bmatrix} [\bar{\bar{Q}}_{i-1}]_{i-1} & [a_{i-1}]_{i-1} \\ [O^T]_{i-1} & 1 \end{bmatrix}$$

$$\{\bar{\bar{T}}_i\}_i = \begin{bmatrix} [\bar{\bar{Q}}_i]_i & [\bar{a}_i]_i \\ [O^T]_i & 1 \end{bmatrix}$$

$$\{T_{i-1}^{-1}\}_i = \begin{bmatrix} [\bar{\bar{Q}}_{i-1}^T]_i & [\bar{\bar{Q}}_{i-1}^T]_i [-\bar{a}_{i-1}]_{i-1} \\ [O^T]_i & 1 \end{bmatrix}$$

$$\{T_i^{-1}\}_{i+1} = \begin{bmatrix} [\bar{\bar{Q}}_i^T]_{i+1} & [\bar{\bar{Q}}_i^T]_{i+1} [-\bar{a}_i]_i \\ [O^T]_{i+1} & 1 \end{bmatrix}$$

⇒ Hence the coordinate transformations involved are:

$$\{\bar{P}\}_{i-1} = \{\bar{\bar{T}}_{i-1}\}_{i-1} \{\bar{P}\}_i$$

$$\{\bar{P}\}_{i-1} = \{\bar{\bar{T}}_{i-1}\}_{i-1} \{\bar{\bar{T}}_i\}_i \{\bar{P}\}_{i+1}$$

$$\{\bar{P}\}_i = \{\bar{\bar{T}}_{i-1}^{-1}\}_i \{\bar{P}\}_{i-1}$$

$$\{\bar{P}\}_{i+1} = \{\bar{\bar{T}}_i^{-1}\}_{i+1} \{\bar{\bar{T}}_{i-1}^{-1}\}_i \{\bar{P}\}_{i-1}$$

⇒ Now, if P lies at infinity, we can express its homogeneous coordinates in a simple form

$$\{\bar{P}\}_M = \begin{bmatrix} [P]_M \\ 1 \end{bmatrix}$$

{unit vector of $[P]_M$}

$$\Rightarrow \{\bar{P}\}_M = \|\bar{P}\| \begin{bmatrix} [e]_M \\ 1/\|P\| \end{bmatrix}$$

{Length of $\bar{P}$}

$$\lim_{\|P\|\to\infty} \{\bar{P}\}_M = \left(\lim_{\|P\|\to\infty} \|P\|\right) \begin{bmatrix} [e]_M \\ 0 \end{bmatrix}$$

⇒ We now define the homogeneous coordinates of a point P lying at infinity as the 4-D array:

$$\{P_\infty\}_M = \begin{bmatrix} [e]_M \\ 0 \end{bmatrix}$$

{Point at infinity in homogeneous coordinates, has only direction given by unit vector $\bar{e}$ but an undefined length}

If $[\overline{\overline{Q}}]_A = [\bar{e}_1\ \bar{e}_2\ \bar{e}_3]$

$$\{\overline{\overline{T}}\}_A = \begin{bmatrix} \bar{e}_1 & \bar{e}_2 & \bar{e}_3 & \bar{b} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2.6) Similarity Transformation

⇒ Transformations of the position vector of points under a change of coordinate frame involving both a translation of the origin and a rotation of the coordinate axis. was in Section 2.5.

⇒ In this section, we study the transformations of components of vector other than the position vectors, while extending the concept to the transformation of matrix entries.

⇒ Let $A = \{\bar{a}_i\}_1^n$ and $B = \{\bar{b}_i\}_1^n$ be two different bases of the same space V.

⇒ Hence any vector $\bar{v}$ of V can be expressed in either two ways,

$$\bar{V} = \alpha_1 \bar{a}_1 + \alpha_2 \bar{a}_2 + \cdots \alpha_n \bar{a}_n$$

$$\bar{V} = \beta_1 \bar{b}_1 + \beta_2 \bar{b}_2 + \cdots \beta_n \bar{b}_n$$

$$[\bar{V}]_A = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} \qquad [\bar{V}]_B = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix}$$

⇒ Let the two foregoing bases be related by

$$b_j = a_{1j} \bar{a}_1 + a_{2j} \bar{a}_2 + \cdots a_{nj} \bar{a}_n \qquad j = 1 \cdots n$$

⇒ In order to find the relationship between the two representation:

$$\bar{V} = \beta_1 (a_{11} \bar{a}_1 + a_{21} \bar{a}_2 + \cdots a_{n1} \bar{a}_n)$$
$$+ \beta_2 (a_{12} \bar{a}_1 + a_{22} \bar{a}_2 + \cdots + a_{n2} \bar{a}_n)$$
$$\vdots$$
$$+ \beta_n (a_{1n} \bar{a}_1 + a_{2n} \bar{a}_2 + \cdots + a_{nn} \bar{a}_n)$$

This can be written as:

$$\boxed{[\bar{V}]_A = [\bar{A}]_A [\bar{V}]_B} \quad \text{— ①}$$

$$[\bar{A}]_A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Scanned by CamScanner

So

$$\boxed{[\bar{V}]_B = [\bar{A}^{-1}]_A \, [\bar{V}]_A} \quad \text{--- (2)}$$

⇒ Next, let $\bar{L}$ have the representation in A given below:

$$[\bar{L}]_A = \begin{bmatrix} l_{11} & l_{12} & \cdots & l_{1n} \\ l_{21} & l_{22} & \cdots & l_{2n} \\ \vdots & \vdots & & \vdots \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{bmatrix}$$

⇒ Now we aim at finding the relationship between $[\bar{L}]_A$ and $[\bar{L}]_B$.

⇒ Let $\bar{\omega}$ be the image of $\bar{V}$ under $\bar{L}$

$$\bar{L}\bar{V} = \bar{\omega}$$

Which can be expressed in terms of either A or B as:-

$$[\bar{L}]_A [\bar{V}]_A = [\bar{\omega}]_A$$

$$[\bar{L}]_B [\bar{V}]_B = [\bar{\omega}]_B$$

We Know, $[\bar{\omega}]_A = [\bar{A}]_A [\bar{\omega}]_B$ & $[\bar{V}_A] = [\bar{A}]_A [\bar{V}]_B$

$$\Rightarrow [\bar{A}]_A [\bar{\omega}]_B = [\bar{L}]_A [\bar{V}]_A = [\bar{L}]_A [\bar{A}]_A [\bar{V}]_B$$

$$[\bar{\omega}]_B = [\bar{A}^{-1}]_A [\bar{L}]_A [\bar{A}]_A [\bar{V}]_B$$

So $$\boxed{[\bar{L}]_B = [A^{-1}]_A [L]_A [\bar{A}]_A} \quad \text{— ③}$$

$\&$ $$\boxed{[\bar{L}]_A = [A]_A [\bar{L}]_B [A^{-1}]_A} \quad \text{— ④}$$

⇒ Relations ①, ②, ③ & ④ Constitute what are called Similarity Transformation.

These are important because they preserve invariant qualities.
{eg ⇒ eigen values, eigen vectors, magnitude, angle between vectors etc....}

Theorem 7.6.1: The characteristic polynomial of a given $n \times n$ matrix remains unchanged under a similarity transformation.

Theorem 7.6.2: If $[L]_A$ and $[\bar{L}]_B$ are related by the similarity transformation then,

$$[\bar{L}^K]_B = [\bar{A}^{-1}]_A [\bar{L}^K]_A [\bar{A}]_A$$

for any integer K.

Theorem 7.6.3: The trace of an $n \times n$ matrix does not change under a similarity transformation.

2.7) Invariance Concept

Let a Scalar, Vector, and matrix function of the position Vector $\bar{P}$ be denoted by $f(\bar{P})$, $\bar{f}(\bar{P})$ and $\bar{\bar{F}}(\bar{P})$ respectively.

⇒ Representations of $\bar{f}(\bar{P})$ in two different Coordinate frames, labelled A & B, will be indicated as $[\bar{f}(\bar{P})]_A$ and $[\bar{f}(\bar{P})]_B$ respectively.

↳ Similar for $f(\bar{P})$ and $\bar{\bar{F}}(\bar{P})$

Let $[\bar{\bar{Q}}]_A$ denote rotation of Coordinate frame A into B.

⇒ The Scalar function $f(\bar{P})$ is said to be frame invariant if:-

$$\boxed{f([\bar{P}]_B) = f([\bar{P}]_A)}$$

⇒ Vector quantity $\bar{f}$ is said to be invariant if:

$$\boxed{[\bar{f}]_A = [\bar{\bar{Q}}]_A [\bar{f}]_B}$$

⇒ Matrix quantity $\bar{\bar{F}}$ is said to be invariant if:-

$$\boxed{[\bar{\bar{F}}]_A = [\bar{\bar{Q}}]_A [\bar{\bar{F}}]_B [\bar{\bar{Q}}^T]_A}$$

⇒ The k^{th} moment of an $n \times n$ matrix $\bar{\bar{T}}$, denoted by I_K is defined as (Leigh 1968)

$$\boxed{I_K = tr(\bar{\bar{T}}^K)}$$

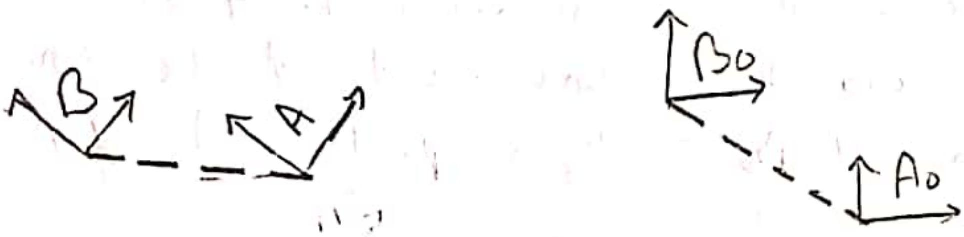
$$\{\text{where } I_0 = tr(\bar{\bar{I}}) = n\}$$

2.7.1) Applications to Redundant Sensing

⇒ A Sensor, such as a Camera or a range finder, is often mounted on a robotic end-effector to determine the Pose. (Position and Orientation)

⇒ If redundant Sensors are introduced, and we attach frames A and B to each of these, then each can be used to determine the each Sensor Can be used to determine the Orientation of the end-effector with respect to a reference Configuration.

⇒ This is a Simple task, for all that is needed is to measure the rotation $\bar{R}$ that each of the foregoing frames Undergo from the reference Configuration.



⇒ With this information we would like to determine the relative orientation $\overline{\overline{Q}}$ of frame B with respect to frame.

↓

This problem is called Instrument Calibration

Let $\overline{\overline{A}} \equiv [\overline{\overline{R}}]_A$ $\overline{\overline{B}} \equiv [\overline{\overline{R}}]_B$

$$\Rightarrow \overline{\overline{A}} = [\overline{\overline{Q}}]_A \overline{\overline{B}} [\overline{\overline{Q}}^T]_A$$

$$\Rightarrow \overline{\overline{A}} [\overline{\overline{Q}}]_A = [\overline{\overline{Q}}]_A \overline{\overline{B}}$$

⇒ This problem can be solved if we have three invariant vectors associated with each of the two matrices $\overline{\overline{A}}$ and $\overline{\overline{B}}$.

⇒ However, since $\overline{\overline{A}}$ and $\overline{\overline{B}}$ are orthogonal matrices, they admit only one real invariant vector, namely, their axial vector.

↳ we are short of two vector equation.

⇒ One way ~~to~~ of obtaining one additional vector in each frame is to take not one, but two measurements of the orientation of A_0 and B_0 with respect to A and B respectively.

⇒ Let the matrices representing these orientation be given, in each of the two coordinate frames, by $\bar{A}_i$ and $\bar{B}_i$ $\forall i = 1,2$.

↳ Let $\bar{a}_i$ and $\bar{b}_i$ $\forall i = 1,2$ be the axial vectors of the matrices $\bar{A}_i$ and $\bar{B}_i$ respectively.

⇒ Now we have two possibilities :-

(i) Neither $\bar{a}_1$ and $\bar{a}_2$ and, consequently neither of b_1 and b_2 is zero.

⇒ Here nothing prevents us from computing a third vector of each set, namely

$$\bar{a}_3 = \bar{a}_1 \times \bar{a}_2 \qquad \bar{b}_3 = \bar{b}_1 \times \bar{b}_2$$

(ii) At least one of $\bar{a}_1$ and $\bar{a}_2$ and consequently the corresponding vector of the $\{\bar{b}_1, \bar{b}_2\}$ pair vanishes.

ⓐ The angle of rotation of the orthogonal matrix $\bar{A}_1$ or $\bar{A}_2$, whose axial vector vanishes is either 0 or π.

⇒ A underwent a pure translation from A_0 ; the same holding of course for B and B_0.

⇒ This means new measurements are needed involving rotation.

ⓑ If angle is π, then the associated rotation is symmetric and the unit vector $\bar{e}$ ∥ to its axis can be determined.

$$\bar{a}_i = [\bar{Q}]_A \bar{b}_i \quad \text{for } i = 1,2,3$$

$$\Rightarrow \quad \bar{\bar{E}} = [\bar{Q}]_A \bar{\bar{F}}$$

$$\left\{ \bar{\bar{E}} = [\bar{a}_1 \ \bar{a}_2 \ \bar{a}_3] \ , \ \bar{\bar{F}} = [\bar{b}_1 \ \bar{b}_2 \ \bar{b}_3] \right\}$$

So $[\bar{Q}]_A = \bar{\bar{E}} \bar{\bar{F}}^{-1}$

$$\bar{\bar{F}}^{-1} = \frac{1}{\Delta} \begin{bmatrix} (\bar{b}_2 \times \bar{b}_3)^T \\ (\bar{b}_3 \times \bar{b}_1)^T \\ (\bar{b}_1 \times \bar{b}_2)^T \end{bmatrix} \qquad \Delta \equiv \bar{b}_1 \times \bar{b}_2 \cdot \bar{b}_3$$

Therefore,

$$\boxed{[\bar{\bar{Q}}]_A = \frac{1}{\Delta} \left[a_1 (b_2 \times b_3)^T + a_2 (b_3 \times b_1)^T + a_3 (b_1 \times b_2)^T \right]}$$