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Robust Control Theory

⇒ The first step in the design of a control system is to obtain a mathematical model of the control object based on the physical law.

⇒ Quite often the modle may be nonlinear and possibly with distributed parameters.

↳ Such a model may be difficult to analyze.

⇒ It is desirable to approximate it by a linear constat-coefficient system that will approximate the actual object fairly well.

⇒ In frequency-response approach to control system design we may use phase & gain margins to take care of the modeling errors.

↳ However, in state-space approach, which is based on the differential equation of the plant dynamics, no such margins are involved in the design process.

⇒ The actual plant differs from the model used in the design, a question arises whether the controller desined using a model will work satisfactory with the actual plat.

⇒ To ensure that it will do so, robust control theory has been developed since around 1980.

⇒ Robust Control theory assumes that there is an Uncertainity or error between the actual plant and its mathematical model.

⇒ Systems designed based on the robust Control theory will possess the following Properties:-

(i) <u>Robust stability</u> ⇒ The Control System designed is stable in the Presence of Perturbation.

(2) <u>Robust performance</u> ⇒ The Control System exhibits predetermined response characteristic in the presence of Perturbation.

1. <u>Uncertain Elements in Plant Dynamics</u>

⇒ The term Uncertainity reffers to the difference or error between the model of the plant and the actual plant.

Structured Uncertainity

eg ⇒ Parameter variation in Plant dynamics.

Unstructured Uncertainity

Eg ⇒ due to linearizatio of a nonlinear Plant

→ In the robust Control theory, we define Unstructured Uncertainty as $\Delta(s)$.

↳ Since exact description of $\Delta(s)$ is Unknown, we use an estimate of $\Delta(s)$ (as the Magnitude & phase Characteristics) and use this estimate in the design of the Controller that Stablizes the Control System.

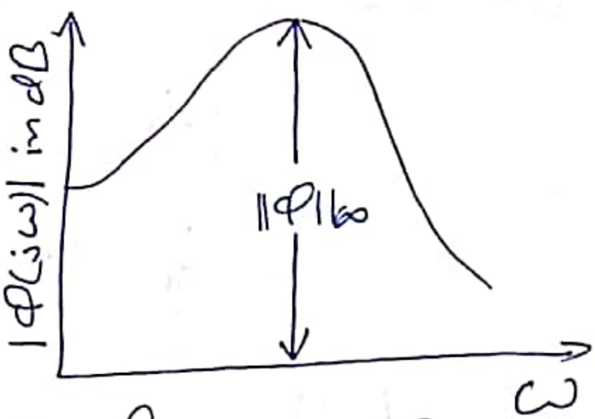
2. <u>H_∞ Norm</u>

The H_∞ Norm of a stable Single-input-single output System is the largest possible amplification factor of the steady-State response to Sinusoidal excitation.

⇒ For a Scalar $\Phi(s)$ $\|\Phi\|_\infty$ gives the maximum Value of $|\Phi(j\omega)|$.

↳ It is called the H_∞ norm.

$$\omega \longrightarrow \boxed{\Phi(s)} \longrightarrow z$$



⇒ Assume the transfer function $\Phi(s)$ is Proper and Stable.

The H_∞ norm of $\Phi(s)$ is defined by

$$\|\Phi\|_\infty = \bar{\sigma}\left[\Phi(j\omega)\right]$$

$\bar{\sigma}[\Phi(j\omega)]$ means the maximum Singular Value of $[\Phi(j\omega)]$.

↳ ($\bar{\sigma}$ means σ_{max})

⇒ Singular Value of a transfer function Φ is defined by

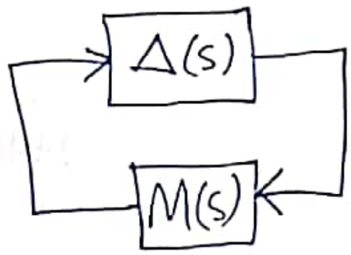
$$\sigma_i(\Phi) = \sqrt{\lambda_i(\Phi * \Phi)}$$

Where $\lambda_i(\Phi * \Phi)$ is the i^{th} longest eigenvalue of $\Phi * \Phi$ and is always a non-negative real value.

⇒ By making $\|\Phi\|_\infty$ smaller, we make the effect of input ω on the output z smaller.

3. Small-Gain theorem

⇒ Consider the Closed-loop System shown:-



$\Delta(s)$ and $M(s)$ are stable and Proper transfer function.

⇒ The Small gain theorem States that if

$$\|\Delta(s) M(s)\|_\infty < 1$$

then this closed-loop System is Stable.

⇒ This theorem is an extension of the Nyquist stability Criteria.

4. System with Unstructured Uncertainty

⇒ In some cases an unstructured uncertainty error may be considered multiplicative such that

$$\underbrace{\tilde{G}}_{\text{True plant dynamics}} = \underbrace{G}_{\text{Model plant dynamics}}(1+\Delta_m)$$

⇒ In other cases an unstructured ~~unstructured~~ additive uncertainity may be considered such that:-

$$\tilde{G} = G + \Delta_a$$

⇒ In either case we assume that the norm of Δ_m or Δ_a is bounded such that

$$\|\Delta_m\| < \gamma_m \qquad \|\Delta_a\| < \gamma_a$$

where γ_m and γ_a are positive constants.

5. Robust Stability

Let us define

$\tilde{G}$ = true plant dynamics

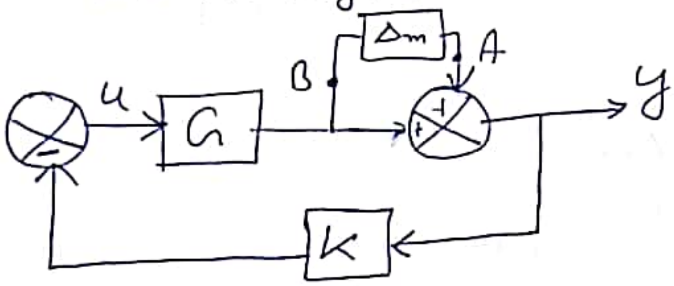
G = model of plant dynamics

Δ_m = Unstructured multiplicative uncertanity.

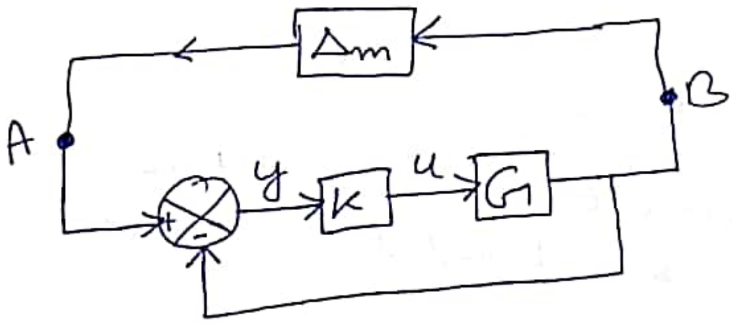
⇒ We assume that Δ_m is stable & its upper bound is known.

$$\tilde{G} = G(I+\Delta_m)$$

⇒ Consider the system shown in figure bellow:-

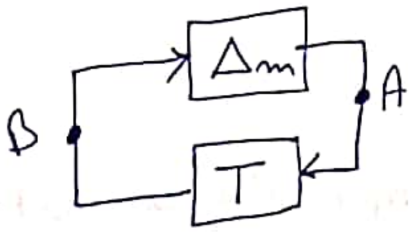


⇒ To obtain transfer function between point A and point B, the above can be re drawn as:-



$$\frac{KG}{1+KG} = (1+KG)^{-1}KG$$

Let $T = (1+KG)^{-1}KG$



upper

bellow:-

Point A
n coi:-

⇒ Applying the Small gain theorem to the System Consisting of Δ_m and T, we obtain the Condition for stability to be:-

$$\|\Delta_m T\|_\infty < 1 \quad \text{—— (1)}$$

⇒ At general, it is impossible to precisely model Δ_m.

↳ Therefore let us use a Scalar transfer function $W_m(j\omega)$ Such that

$$\bar{\sigma}\{\Delta_m(j\omega)\} < |W_m(j\omega)|$$

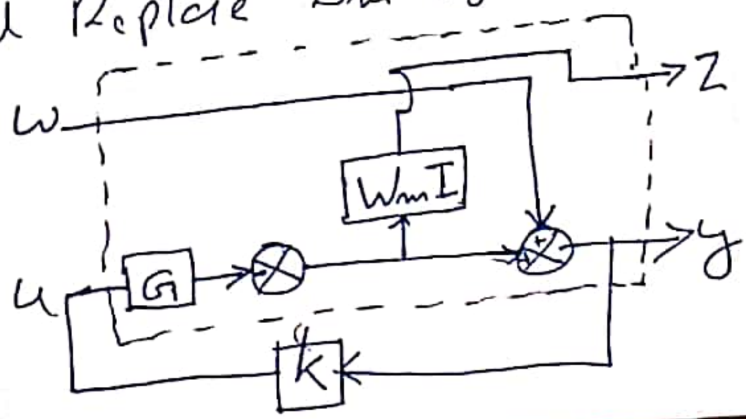
⇒ Consider, instead of above inequality, the following inequality:

$$\|W_m T\|_\infty < 1 \quad \text{—— (2)}$$

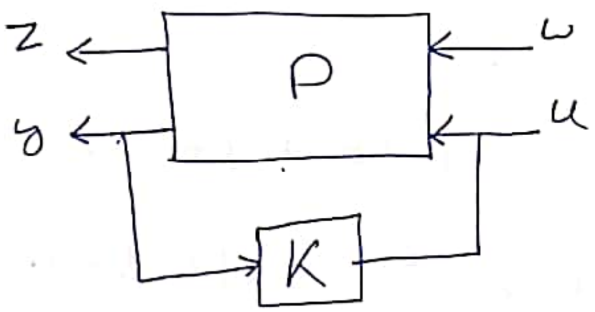
⇒ If inequality (2) holds true, Inequality (1) will always be Satisfied.

↳ By making the H_∞ norm of $W_m T$ to be less than 1, we obtain the controller K that will make the System stable.

⇒ Suppose that we Cut the line at point A and Replace Δ_m by $W_m I$ we obtain:-



⇒ Redrawing above we obtain what is called generalized plant diagram.



$$\Rightarrow \quad \| W_m T \|_\infty < 1$$

$$\Rightarrow \quad \left\| \frac{W_m K(s) G(s)}{1 + K(s) G(s)} \right\|_\infty < 1$$

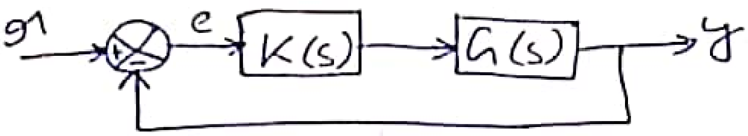
⇒ For a stable plant model $G(s)$, $K(s) = 0$ will satisfy above inequality. However it is not desirable.

⇒ To find an acceptable transfer function for $K(s)$, we may ~~find~~ add another condition, that the resulting system will have robust performance.

↳ Such that the system output follows the input with minimum error or another reasonable condition.

6. Robust Performance

⇒ Consider the System shown bellow.



⇒ Suppose that we want the output $y(t)$ to follow the input $r(t)$ as closely as possible

$$\lim_{t\to\infty}[r(t)-y(t)] = \lim_{t\to\infty} e(t) \to 0$$

$$\frac{Y(s)}{R(s)} = \frac{KG}{1+KG}$$

we have $\dfrac{E(s)}{R(s)} = \dfrac{1}{1+KG} = S$

$S \Rightarrow$ Sensitivity function

$T \Rightarrow$ Complementary Sensitivity function.

⇒ In robust performance Problem we want to make the H_∞ norm of S smaller than the desired transfer function W_s^{-1}.

$$\|W_s S\|_\infty < 1$$

⇒ Combining inequalities we get

$$\left\| \begin{matrix} W_m T \\ W_s S \end{matrix} \right\| < 1 \qquad \text{where } T+S=1$$

$$\left\| \begin{matrix} W_m(s)\,\dfrac{K(s)\,G(s)}{1+K(s)\,G(s)} \\[2ex] W_s(s)\,\dfrac{1}{1+K(s)\,G(s)} \end{matrix} \right\| < 1 \quad \text{—} \; (1)$$

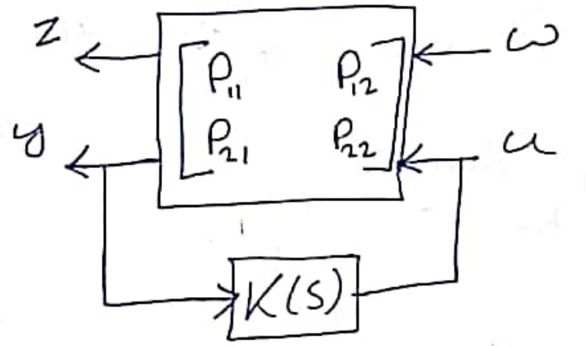
⇒ Our Problem then becomes to find $K(s)$ that will satisfy Inequality above.

⇒ Depending on the chosen $W_m(s)$ and $W_s(s)$ there may be many $K(s)$ that satisfy Inequality or may be no $K(s)$ that satisfies Inequality.

⇒ Such a robust Control problem using Inequality is called a mixed-Sensitivity Problem.

7. <u>Finding Transfer function $Z(s)/W(s)$ from a Generalized Plant diagram</u>

Consider the generalized plant diagram shown below:-



$W(s)$ ⇒ Exogenous disturbance

$u(s)$ ⇒ Manipulated Variable

$Z(s)$ ⇒ Controlled Variables

$y(s)$ ⇒ Observed Variable

$$\begin{bmatrix} Z(s) \\ y(s) \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} \omega(s) \\ u(s) \end{bmatrix}$$

$\Rightarrow$ The equation that relates $u(s)$ and $y(s)$ is given by:-

$$u(s) = K(s) y(s)$$

$\Rightarrow$ Let us define the transfer function that relates the controlled variables $Z(s)$ to the exogenous disturbance $\omega(s)$ as $\Phi(s)$.

$$Z(s) = \Phi(s)\,\omega(s)$$

$\Rightarrow$ $\Phi(s)$ can be determined as follows:-

$$Z(s) = P_{11}\,\omega(s) + P_{12}\,u(s)$$

$$y(s) = P_{21}\,\omega(s) + P_{22}\,u(s)$$

$$u(s) = K(s)\,y(s)$$

$$\Rightarrow \quad y(s) = P_{21}\,\omega(s) + P_{22} K(s) y(s)$$

$$y(s) = [I - P_{22} K(s)]^{-1} P_{21}\,\omega(s)$$

$$\Rightarrow \quad Z(s) = P_{11}\,\omega(s) + P_{12} K(s) [I - P_{22} K(s)]^{-1} P_{21}\,\omega(s)$$

$$Z(s) = \{P_{11} + P_{12} K(s) [I - P_{22} K(s)]^{-1} P_{21}\}\,\omega(s)$$

Hence,

$$\boxed{\Phi(s) = P_{11} + P_{12} K(s) [I - P_{22} K(s)]^{-1} P_{21}}$$

8. H Infinity Control Problem

⇒ To design a Controller K of a Control System to Satisfy Various Stability and performance Specifications, we utilize the Concept of the generalized plant.

⇒ The reason to use generalized plants, rather than individual block diagrams of Control Systems, is that a number of Control Systems with uncertain elements have been designed using generalized plant and Consequently, established design approaches using such plants are available.

⇒ Controller that is the Solution to the H infinity Control problems is commonly Called the H infinity Controller.

9. Solving Robust Control Problem

⇒ There are three established approaches to Solve robust Control problems. They are

1. Solve robust Control problems by deriving Riccati equation and Solve them.
2. Solve robust Control Problems by using the linear matrix inequality approach.

3. Solve rob
structure
analysis

"Solving
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math

3. Solve robust Control problems that involves structured uncertainty by using the μ analysis and μ Synthesis approach.

"Solving robust Control Problems by use of any of the above method requires a broad mathematical background..."

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