4

Geometry of Decoupled Serial Robot

4.1) Introduction

Kinematic State ⇒ Position & Orientation

Dynamic State ⇒ Velocities

Statics ⇒ Bodies are at Rest

4.2) The Denavit-Hartenberg Notation

⇒ The first task of a robotics engineer is the Kinematic modeling of a robotic manipulator.

⇒ The Simplest way to Kinematically modeling a robotic manipulator is by means of the Concept of Kinematic chain.

↓

{ Set of rigid body (links), Coupled by Kinematic Pairs. }

↓

{ Coupling of two rigid bodies so as to Constrain their relative motion }

Kinematic Pair

Upper Kinematic Pair

"Contact takes place along a line or at a point"

eg = Cam & follower ; gear trains, roller bearing.

Lower Kinematic Pair

"Contact takes place along a Common Surface"

Rotating Pair (Revolute)

R

- Common surface of Contact is Circular Cylinder.
- The two rigid body can rotate relative to each other about the axis of Common Cylinder. (Axis of Revolute)

Sliding Pair (Prismatic)

P

- Common Surface is Prism of arbitrary Cross-section.
- Can move only in a pure translation motion along a direction ∥ to the Axis of Prism

Simple Kinematic Chain

Kinematic Chain with each link connected to at most two other links.

Simple Kinematic chain

Closed

'every link is coupled to two other links!

⇒ This chain is called <u>Linkage</u>.

Open

"Contains exactly two links, coupled to only one other link"

⇒ In Simple Open Kinematic chain the first link is called <u>Manipulator base</u>, whereas last link is termed the <u>End-effector (EE)</u>.

Manipulator base — Binary links — End Effector

⇒ In order to uniquely describe the architecture of a kinematic chain (Relative location and orientation of its neighbouring ~~links~~ links), the ~~Daven~~ Denavit-Hartenberg notation (1955) is introduced.

⇒ links are numbered 0, 1, ... n.
⇒ The i^{th} pair being defined as that coupling the $(i-1)^{st}$ link to the i^{th} link.

Hence, the manipulator is assumed to be Composed of n+1 links and n Pairs.

⇒ link 0 is fixed Base.
⇒ each pair can be either R or P.
⇒ Link n is the End-Effector.

⇒ Next, a Coordinate frame F_i is defined with Origin O_i and axes X_i Y_i Z_i · is attached to (i-1)st link $\forall$ i = 1, 2, ... n+1.

⇒ For the first n frames, this is done following the rules given below:-

1) Z_i is the axis of the i^{th} pair.
↳ There are two possibilities of defining the positive direction of this axis, Since each pair axis is only a line, not a directed Segment.

2) X_i is defined as the Common perpendicular to Z_{i-1} and Z_i, directed from the former to the latter.

If these two axis intersect the positive direction of X_i is undefined and hence can be freely assigned.

↳ Henceforth we will follow the right-hand rule in this case.

⇒ If Unit Vector $\bar{i}_i$, $\bar{K}_{i-1}$ & $\bar{K}_i$ are attached to X_i, Z_{i-1} & Z_i then $\bar{i}_i = \bar{K}_{i-1} \times \bar{K}_i$.

⇒ Point of intersection is defined the Origin.

If Z_{i-1} & Z_i are $\parallel$, the location of X_i is Undefined.

↳ In order to define it uniquely we will specify X_i, as passing through the Origin of the $(i+1)$st frame.

3> Distance between Z_i and Z_{i+1} is defined as a_i, which is thus non negative.

4> The Z_i-Coordinate of the intersection O_i' of Z_i with X_{i+1} is denoted by b_i

→ It can be either positive or Negative.

→ It is also called <u>Offset</u> between common perpendiculars.

5> The angle between Z_i and Z_{i+1} is defined as α_i and is measured about the positive direction of X_{i+1}.

↳ <u>Twist angle</u> between Successive Pair axes.

6) The angle between X_i and X_{i+1} is defined as θ_i and is measured about the positive direction of Z_i.

7) The $(n+1)$st Coordinate frame is attached to the far end of the nth link.

↳ We have the freedom to define this frame as it best suits the task in hand.

⇒ All quantities involved in these definitions are constant, except θ_i, which is variable and is thus termed the joint variable of the ith pair.

↳ The other quantities ie a_i, b_i & α_i are the joint parameters of the said pair.

⇒ Alternatively, the ith pair is P, then b_i is variable and the other quantities are constant.

↳ Joint Parameter: a_i α_i & θ_i

↳ Joint Variable: b_i

⇒ n-axis manipulator has n joints variables

→ which are henceforth grouped in the n-dimensional vector $\bar{\theta}$, regardless of whether joint variables are angular or translational.

→ and 3n Constant Parameter.

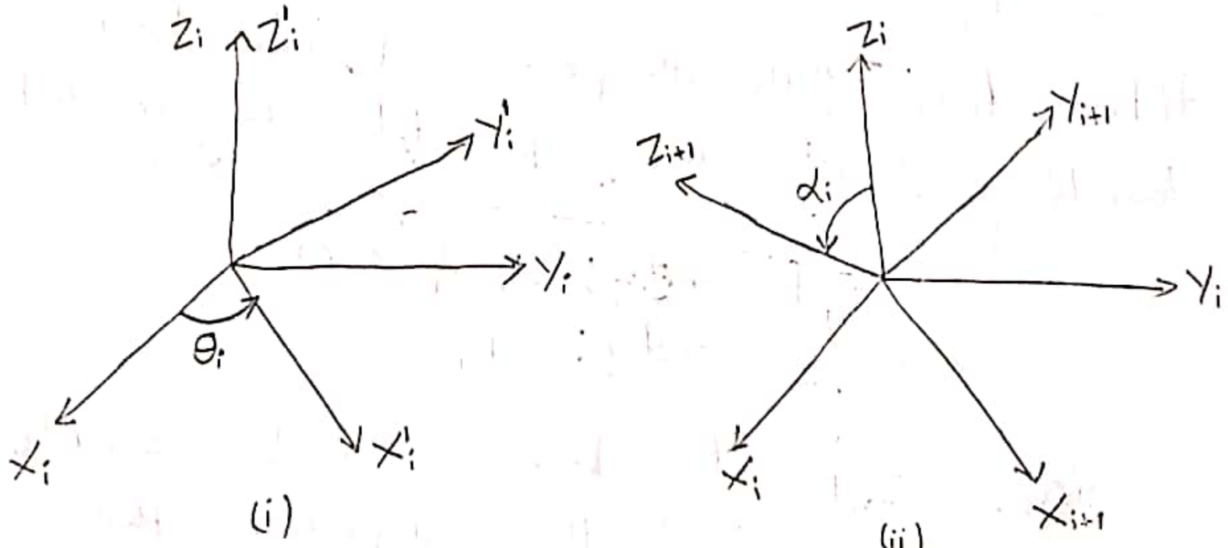
↳ Defines the architecture of Manipulator.

⇒ Manipulator architecture is fully defined by its 3n Denavit–Hartenberg (DH) parameter.

⇒ Its posture is fully defined by its n joint variables, also called its joint coordinates.

⇒ First, we obtain the matrix representation of the rotation $\overline{\overline{Q}}_i$ carrying F_i into an orientation coincident with that of F_{i+1}, assuming that the two origin are coincident

↳ Matrix is most easily derived if the rotation of interest is decomposed into two successive rotation.



⇒ Let the foregoing rotation be $[\check{C}_i]_i$ and $[\check{\Lambda}_i]_{i'}$ respectively.

$$F_i \xrightarrow{[\check{C}_i]_i} F_i' \xrightarrow{[\check{\Lambda}_i]_{i'}} F_{i+1}$$

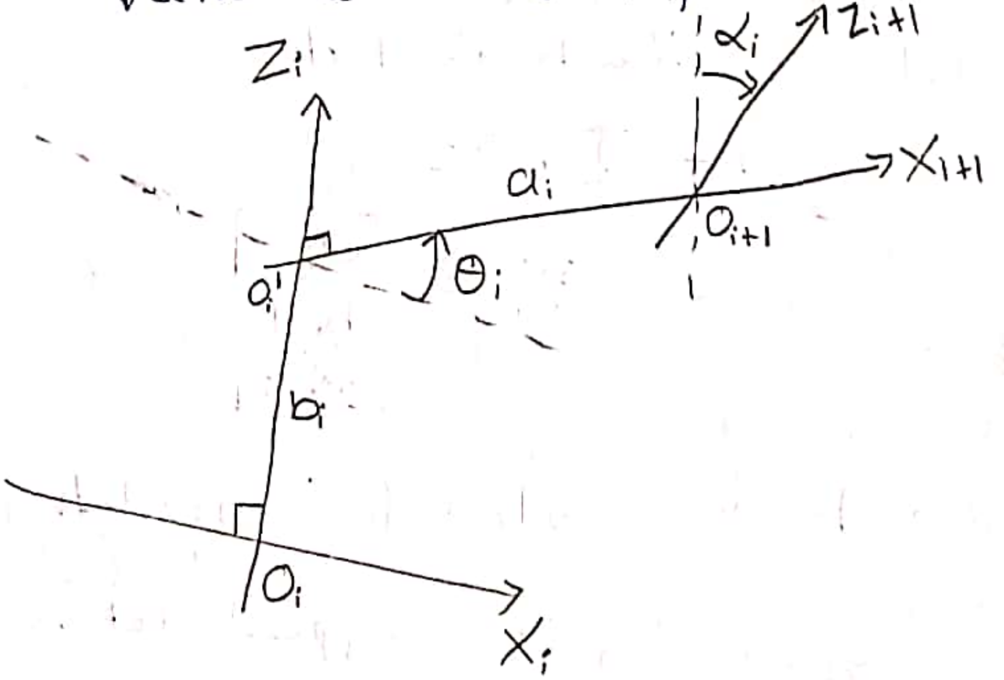
$$[\overline{C}_i]_i = \begin{bmatrix} \cos\theta & -\sin\theta_i & 0 \\ \sin\theta_i & \cos\theta_i & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[\overline{\Lambda}_i]_{i'} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda_i & -\mu_i \\ 0 & \mu_i & \lambda_i \end{bmatrix} \quad \left\{ \text{where } \begin{array}{l} \lambda_i = \cos\alpha_i \\ \mu_i = \sin\alpha_i \end{array} \right\}$$

$$[\overline{\overline{Q}}_i]_i = [\overline{\overline{C}}_i]_i \, [A_i]_{i'}$$

$$[\overline{\overline{Q}}_i]_i = \begin{bmatrix} \cos\theta_i & -\lambda_i\sin\theta_i & \mu_i\sin\theta_i \\ \sin\theta_i & \lambda_i\cos\theta_i & -\mu_i\cos\theta_i \\ 0 & \mu_i & \lambda_i \end{bmatrix}$$

Le $\overline{a}_i$ be Vector Connecting O_i of F_i to O_{i+1} of F_{i+1}



$$\overline{a}_i = \overrightarrow{O_i O_{i+1}} = \overrightarrow{O_i O_i'} + \overrightarrow{O_i' O_{i+1}}$$

$$[\overrightarrow{O_i O_i'}]_i = \begin{bmatrix} 0 \\ 0 \\ b \end{bmatrix} \qquad [\overrightarrow{O_i' O_{i+1}}]_{i+1} = \begin{bmatrix} a_i \\ 0 \\ 0 \end{bmatrix}$$

$$[\overrightarrow{O_i O_{i+1}}]_i = [\bar{\bar{Q}}_i]_i [\overrightarrow{O_i' O_{i+1}}]_{i+1} = \begin{bmatrix} a_i \cos\theta_i \\ a_i \sin\theta_i \\ 0 \end{bmatrix}$$

So
$$[\bar{a}_i]_i = \begin{bmatrix} a_i \cos\theta_i \\ a_i \sin\theta_i \\ b_i \end{bmatrix}$$

$$\bar{\bar{Q}}_i \equiv [\bar{\bar{Q}}_i]_i \qquad \forall \text{ Brevity}$$
$$\& \quad \bar{a}_i = [\bar{a}_i]_i$$

⇒ Similar to the foregoing factoring of $\bar{\bar{Q}}_i$, Vector $\bar{a}_i$ admits the factoring.

$$\bar{a}_i = \bar{\bar{Q}}_i \bar{b}_i$$

$$\text{Where, } \bar{b}_i = \begin{bmatrix} a_i \\ b_i \mu_i \\ b_i \lambda_i \end{bmatrix}$$

Vector $\bar{b}_i$ is constant for revolute pair.

⇒ From geometry it is apparent that $\bar{b}_i$ is nothing but $\bar{a}_i$ in F_{i+1}

$$\bar{b}_i = [\bar{a}_i]_{i+1}$$

⇒ Let $\bar{i}_i$, $\bar{j}_i$ and $\bar{k}_i$ be the unit vector ∥ to the X_i, Y_i and Z_i axis respectively; directed to positive direction of these axes.

$$[\bar{i}_{i+1}]_i = \begin{bmatrix} \cos\theta_i \\ \sin\theta_i \\ 0 \end{bmatrix}, \quad [\bar{K}_{i+1}]_i = \begin{bmatrix} \mu_i \sin\theta_i \\ -\mu_i \cos\theta_i \\ \lambda_i \end{bmatrix}$$

$$[\bar{j}_{i+1}]_i = [\bar{K}_{i+1} \times \bar{i}_{i+1}]_i = \begin{bmatrix} -\lambda_i \sin\theta_i \\ \lambda_i \cos\theta_i \\ \mu_i \end{bmatrix}$$

⇒ Therefore, the Components $\bar{i}_{i+1}$, $\bar{j}_{i+1}$ and $\bar{K}_{i+1}$ in F_i are nothing but the first, second & third Columns of $\bar{\bar{Q}}_i$.

⇒ In general, any vector $\bar{V}$ in F_{i+1} is transformed into F_i in the form

$$[\bar{V}]_i = [\bar{\bar{Q}}_i]_i [\bar{V}]_{i+1}$$

⇒ Likewise, any matrix $\bar{M}$ in F_{i+1} is transformed into F_i by the Corresponding Similarity transformation:-

$$[\bar{\bar{M}}]_i = [\bar{\bar{Q}}_i]_i [\bar{\bar{M}}]_{i+1} [\bar{\bar{Q}}_i^T]_i$$

⇒ Moreover if we have a chain of i frame then the inward coordinate transformation from F_i to F_i is given by:-

$$[\bar{V}]_1 = \bar{Q}_1 \bar{Q}_2 \cdots \bar{Q}_{i-1} [\bar{V}]_i$$

$$[\bar{M}]_1 = \bar{Q}_1 \bar{Q}_2 \cdots \bar{Q}_{i-1} [\bar{M}]_i (\bar{Q}_1 \bar{Q}_2 \cdots \bar{Q}_{i-1})^T$$

4.3> The geometric Model of Six-Revolute Manipulator

Joint Variable

For 6R Manipulator ⇒ $\theta_1\ \theta_2\ \theta_3$
$\theta_4\ \theta_5\ \theta_6$

Cartesian Variable

Pose of End EFFECTOR (EE)
(6 Independent Variable)

Joint Parameters

$\alpha_i\ a_i\ b_i\ \forall\ i = 1, 2 \cdots 6$

⇒ Besides geometry, the kinematics of manipulator comprises the study of the relations between time-rates of change of the Joint Variables (joint rate) and the twist of the EE.

Joint Rate

Twist of EE

Joint acceleration

Time-rate of change of the twist of EE

⇒ We distinguish two problem commonly referred to as :-

(i) Direct displacement Problem (DDP)

(ii) Inverse displacement Problem (IDP)

DDP → Six joint variables of a given six-axis manipulator are assumed to be known , the problem consists of finding the POSE of the EE.

IDP → Pose of EE is given while the six joint variables that produce this pose are to be found.

⇒ In studying the DDP of six-axis manipulators , we need not limit ourselves to particular architecture.

⇒ Let us consider manipulator consists of seven rigid body or links (0 to 6) coupled by six revolute joints.

⇒ Correspondingly we have : F_1 F_2 F_3 ... F_7 7 frames

{ F_i attached to $(i-1)^{th}$ link }

⇒ Let line L_i is associated with the axis of the i^{th} revolute joint and a positive direction is defined arbitrarily through a unit vector $\bar{e}_i$.

⇒ Rotation of i^{th} link with respect to $(i-1)^{st}$ link or correspondingly of F_{i+1} with respect to F_i is totally defined by:-

- → DH Parameters: a_i, b_i & α_i & $\bar{e}_i$
- → Joint Variables: θ_i

⇒ DH parameters & joint variables define uniquely the posture of the manipulator.

⇒ Relative position and orientation of F_{i+1} with respect to F_i is given by matrix $\bar{\bar{Q}}_i$ and vector $\bar{a}_i$.

$$[\bar{\bar{Q}}_i]_i = \begin{bmatrix} \cos\theta_i & -\lambda_i\sin\theta_i & \mu_i\sin\theta_i \\ \sin\theta_i & \lambda_i\cos\theta_i & -\mu\cos\theta_i \\ 0 & \mu_i & \lambda_i \end{bmatrix} \quad \begin{Bmatrix} \lambda_i = \cos\alpha_i \\ \mu_i = \sin\alpha_i \end{Bmatrix}$$

$$[\bar{a}_i]_i = \begin{bmatrix} a_i\cos\theta_i \\ a_i\sin\theta_i \\ b_i \end{bmatrix}$$

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⇒ The equation leading to the geometric model under study are known as the <u>displacement equation</u>.

{ Sometime also termed as Closure Equation }

⇒ Let $\overline{\overline{Q}}$ be orientation of F_7 and P be position vector of O_7.

⇒ Orientation $\overline{\overline{Q}}$ of EE is obtained as a result of the six individual rotations $\{Q_i\}_1^6$ about each revolute axis through an angle θ_i, in a sequential order, from 1 to 6.

$$[\overline{\overline{Q}}]_1 = [\overline{\overline{Q}}_6]_1 [\overline{\overline{Q}}_5]_1 [\overline{\overline{Q}}_4]_1 [\overline{\overline{Q}}_3]_1 [\overline{\overline{Q}}_2]_1 [\overline{\overline{Q}}_1]_1 \quad \text{—} \; ①$$

$$[\overline{P}]_1 = [\overline{a}_6]_1 + [\overline{a}_5]_1 + [\overline{a}_4]_1 + [\overline{a}_3]_1 + [\overline{a}_2]_1 + [\overline{a}_1]_1 \quad \text{—} \; ②$$

⇒ Notice that the above equations requires that all vectors and matrices involved be expressed in the same coordinate frame.

↳ However it is convenient to represent the foregoing relations in each individual frame.

↳ This can be readily done by means of similarity transformation.

⇒ If we apply similarity transformation to eq ①
it becomes:-

$$[\bar{\bar{Q}}]_1 = [\bar{\bar{Q}}_6]_6 [\bar{\bar{Q}}_5]_5 [\bar{\bar{Q}}_4]_4 [\bar{\bar{Q}}_3]_3 [\bar{\bar{Q}}_2]_2 [\bar{\bar{Q}}_1]_1$$

$$\Rightarrow \boxed{\bar{\bar{Q}} = \bar{\bar{Q}}_6 \bar{\bar{Q}}_5 \bar{\bar{Q}}_4 \bar{\bar{Q}}_3 \bar{\bar{Q}}_2 \bar{\bar{Q}}_1}$$

⇒ Likewise eq ② becomes:-

$$\boxed{\bar{P} = \bar{a}_1 + \bar{\bar{Q}}_1 a_2 + \bar{\bar{Q}}_1 \bar{\bar{Q}}_2 \bar{a}_3 + \bar{\bar{Q}}_1 \bar{\bar{Q}}_2 \bar{\bar{Q}}_3 \bar{a}_4 + \bar{\bar{Q}}_1 \bar{\bar{Q}}_2 \bar{\bar{Q}}_3 \bar{\bar{Q}}_4 \bar{a}_5 + \bar{\bar{Q}}_1 \bar{\bar{Q}}_2 \bar{\bar{Q}}_3 \bar{\bar{Q}}_4 \bar{\bar{Q}}_5 \bar{a}_5}$$

⇒ Above equations will be more compact if homogeneous transformations are now introduced.

⇒ Let $\bar{\bar{T}}_i = \{\bar{\bar{T}}_i\}_i$ be 4×4 matrix transformation F_{i+1} coordinate into F_i coordinate, the foregoing equation can be written in 4×4 matrix form namely,

$$\boxed{\bar{\bar{T}} = \bar{\bar{T}}_6 \bar{\bar{T}}_5 \bar{\bar{T}}_4 \bar{\bar{T}}_3 \bar{\bar{T}}_2 \bar{\bar{T}}_1}$$

⇒ A Scalar, Vector or matrix expression is said to be multilinear in a set of Vectors $\{\bar{V}_i\}_1^N$ if each of those vectors appears only linearly in the same expression.

⇒ Scalar, Vector or matrix expression is said to be multiquadratic in the same set of Vectors if those Vectors appear at most quadratically in the said expression.

⇒ Qualifiers like multicubic, multiquadratic etc. bear similar meanings.

⇒ Further, we partition matrix $\bar{Q}_i$ rowwise and columnwise, namely:

$$\bar{\bar{Q}}_i = \begin{bmatrix} \bar{m}_i^T \\ \bar{n}_i^T \\ \bar{O}_i^T \end{bmatrix} = \begin{bmatrix} \bar{p}_i & \bar{q}_i & \bar{u}_i \end{bmatrix}$$

$$[\bar{e}_i]_i = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \bar{e} \qquad \bar{X}_i = \begin{bmatrix} \cos\theta_i \\ \sin\theta_i \end{bmatrix}$$

$$\bar{u}_i = \bar{\bar{Q}}_i \bar{e} \quad , \quad \bar{O}_i = \bar{\bar{Q}}^T \bar{e}$$

$$\bar{u}_i = [\bar{e}_{i+1}]_i \qquad \bar{O}_i = [\bar{e}_i]_{i+1}$$

4.4> The Inverse Displacement Analysis of Decoupled Manipulator

⇒ Industrial manipulators are frequently supplied with a special architecture that allows a decoupling of the Positioning Problem from Orientation Problem.

↳ Determinant design criterion in this regard has been that the manipulator lend itself to a Closed-form Inverse displacement solution.

⇒ Decoupled manipulator are those whose last three joints have intersecting axes.

↳ These joints Constitute the wrist of the manipulator, which is said to be Spherical.

Because when the point of Intersection of the three wrist axes C is Kept fixed, all the Points of the wrist moves on Spheres. Centered at 'C'

⇒ In terms of DH Parameters of the manipulator, In a decoupled manipulator $a_4 = a_5 = b_5 = 0$.

↳ So Origin of frames 5 & 6 are Coincident.

⇒ All other DH Parameter Can assume arbitrary Values.

4.4.1) The Positioning Problem

⇒ Let C denotes the intersection of axes 4,5 and 6 ie the Center of the Spherical wrist.

↳ Let $\bar{c}$ denote the position vector of this point

⇒ Apparently position Vector of C is indipendent of joint angles θ_4 θ_5 & θ_6.

$$[\bar{c}]_1 = [\bar{a}_1]_1 + [\bar{a}_2]_1 + [\bar{a}_3]_1 + [\bar{a}_4]_1$$

$$\boxed{\bar{c} = \bar{a}_1 + \bar{\bar{Q}}_1 \bar{a}_2 + \bar{\bar{Q}}_1 \bar{\bar{Q}}_2 \bar{a}_3 + \bar{\bar{Q}}_1 \bar{\bar{Q}}_2 \bar{\bar{Q}}_3 \bar{a}_4}$$

$$\Rightarrow \bar{\bar{Q}}_1^T (\bar{c} - \bar{a}_1) = \bar{a}_2 + \bar{\bar{Q}}_2 \bar{a}_3 + \bar{\bar{Q}}_2 \bar{\bar{Q}}_3 \bar{a}_4$$

$$= \bar{\bar{Q}}_2 (\bar{b}_2 + \bar{\bar{Q}}_3 \bar{b}_3 + \bar{\bar{Q}}_3 \bar{\bar{Q}}_4 \bar{b}_4)$$

$$\{\text{as } \bar{a}_i = \bar{\bar{Q}}_i \bar{b}_i\}$$

⇒ However since we are dealing with a decoupled manipulator, we have, from

$$\bar{a}_4 = \bar{\bar{Q}}_4 \bar{b}_4 = \begin{bmatrix} 0 \\ 0 \\ b_4 \end{bmatrix} = b_4 \bar{e}$$

So $\bar{\bar{Q}}_3 \bar{\bar{Q}}_4 \bar{b}_4 = b_4 \bar{\bar{Q}}_3 \bar{e} = b_4 \bar{u}_3$

$$\Rightarrow \bar{\bar{Q}}_1^T \bar{c} - \bar{b}_1 = \bar{\bar{Q}}_2 (\bar{b}_2 + \bar{\bar{Q}}_3 \bar{b}_3 + b_4 \bar{u}_3)$$

⇒ Further, an expression of $\bar{c}$ can be derived in terms of $\bar{P}$, the position vector of the operation point of the EE and $\bar{\bar{Q}}$ i.e. manalyst

$$\bar{c} = \bar{P} - \bar{Q}_1 \bar{Q}_2 \bar{Q}_3 \bar{Q}_4 \bar{a}_5 - \bar{Q}_1 \bar{Q}_2 \bar{Q}_3 \bar{Q}_4 \bar{Q}_5 \bar{a}_6$$

$$\bar{c} = \bar{P} - \bar{\bar{Q}} \bar{\bar{Q}}_6^T \bar{a}_6 \quad \begin{cases} \text{as } \bar{a}_5 = \bar{0} \\ \text{kk } Q = \bar{Q}_1 \bar{Q}_2 \bar{Q}_3 \bar{Q}_4 \bar{Q}_5 \bar{Q}_6 \end{cases}$$

$$\boxed{\bar{c} = \bar{P} - \bar{\bar{Q}} \bar{b}_6} \quad \text{—— (1)}$$

⇒ Moreover, the base coordinates of P and C in F1 components of their position vectors $\bar{P}$ and $\bar{c}$ are defined as

$$[\bar{P}]_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad [\bar{C}]_1 = \begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix}$$

⇒ Putting these things in eq ① we get:-

$$\begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = \begin{bmatrix} x - (a_{11} a_6 + a_{12} b_6 \mu_6 + a_{13} b_6 \lambda_6) \\ y - (a_{21} a_6 + a_{22} b_6 \mu_6 + a_{23} b_6 \lambda_6) \\ z - (a_{31} a_6 + a_{32} b_6 \mu_6 + a_{33} b_6 \lambda_6) \end{bmatrix} \quad \text{— ②}$$

⇒ Now Positioning problem become one of Finding the first three joint angles necessary to position point C at a point of base coordinates x_c, y_c & z_c.

$$\bar{\bar{Q}}_2 (\bar{b}_2 + \bar{\bar{Q}}_3 \bar{b}_3 + b_4 \bar{u}_3) = \bar{\bar{Q}}_1^T \bar{c} - \bar{b}_1 \quad \text{— ③}$$

(i) If the Euclidean norms of the two side of that equation are equated, the resulting equation will not contain θ_2.

(ii) The third Scalar equation of the same equation is independent of θ_2, by virtue of the Structure of $\bar{\bar{Q}}_i$ matrix.

⇒ Thus we have two equations free of θ_2 which allows us to calculate the two remaining unknowns θ_1 & θ_3.

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⇒ Let the Euclidean norm of the left-hand Side of Eq ③ be denoted by l, that of its right-hand side by r.

$$l^2 = a_2^2 + b_2^2 + a_3^2 + b_3^2 + b_4^2 + 2\bar{b}_2^T \bar{Q}_3 \bar{b}_3 + 2b_4 \bar{b}_2^T \bar{u}_3 + 2\lambda_3 b_3 b_4$$

$$r^2 = \|\bar{c}\|^2 + \|\bar{b}_1\|^2 - 2b_1^T Q_1^T c$$

⇒ From above it is apparent that l^2 is linear in $\bar{x}_3$ and r^2 is linear in $\bar{x}_1$.

⇒ Upon equating l^2 with r^2 we get:-

$$A\cos\theta_1 + B\sin\theta_1 + C\cos\theta_3 + D\sin\theta_3 + E = 0 \quad (4)$$

Where,

$A = 2a_1 x_c$

$B = 2a_1 y_c$

$C = 2a_2 a_3 - 2b_2 b_4 \mu_2 \mu_3$

$D = 2a_3 b_2 \mu_2 + 2a_2 b_4 \mu_3$

$$E = a_2^2 + a_3^2 + b_2^2 + b_3^2 + b_4^2 - a_1^2 - x_c^2 - y_c^2 - (z_c - b_1)^2 + 2b_2 b_3 \lambda_2 + 2b_2 b_4 \lambda_2 \lambda_3 + 2b_3 b_4 \lambda_3$$

had

of

=0

(4)

$)^2$

⇒ Moreover, the third scalar equation of eq (3) takes the form:-

$$F\cos\theta_1 + G\sin\theta_1 + H\cos\theta_3 + I\sin\theta_3 + J = 0$$ (5)

where,

$$F = y_c\mu_1$$
$$G = -x_c\mu_1$$
$$H = -b_4\mu_2\mu_3$$
$$I = a_3\mu_2$$
$$J = b_2 + b_3\lambda_2 + b_4\lambda_2\lambda_3 - (z_c - b_1)\lambda_1$$

⇒ Thus we have derived two nonlinear equations in θ_1 and θ_3 that are linear in $\cos\theta_1$, $\sin\theta_1$, $\cos\theta_3$ & $\sin\theta_3$. {Eq (4) & 5}

⇒ Each of these equations (eq (4) & eq (5)) defines a contour in the $\theta_1 - \theta_3$ plane.

↳ Their intersection determine all real solutions to the problem at hand.

We know

$$\cos\theta_3 = \frac{1 - \tan^2(\theta_3/2)}{1 + \tan^2(\theta_3/2)}$$

$$\sin\theta_3 = \frac{2\tan(\theta_3/2)}{1 + \tan^2(\theta_3/2)}$$

{tan-half-angle Identities}

~~$\tan\theta_3 = \tan\left(\frac{\theta_3}{2}\right)$~~

⇒ If $\cos\theta_i$ & $\sin\theta_i$ are substituted for their equivalent in terms of $\tan(\theta_i/2)$, then two biquadratic polynomial equations in $\tan(\theta_1/2)$ and $\tan(\theta_3/2)$ are derived.

⇒ One can eliminate one of these variables from the foregoing equations, thereby reducing the two equations to a single quadratic polynomial equation in the other variable.

↳ Quadratic equation thus resulting is called the characteristic equation.

⇒ Eq ④ & ⑤ can be solved for say $\cos\theta_1$ & $\sin\theta_1$ in terms of $\cos\theta_3$ & $\sin\theta_3$.

$$\cos\theta_1 = \frac{-G\left(C\cos\theta_3 + D\sin\theta_3 + E\right) + B\left(H\cos\theta_3 + I\sin\theta_3 + J\right)}{\Delta_1} \quad \text{—⑥}$$

$$\sin\theta_1 = \frac{F\left(C\cos\theta_3 + D\sin\theta_3 + E\right) - A\left(H\cos\theta_3 + I\sin\theta_3 + J\right)}{\Delta_1} \quad \text{—⑦}$$

where $\Delta_1 = AG - FB = -2a_1\mu_1(x_c^2 + y_c^2)$

Note, Δ_1 can be computed off-line, ie prior to setting the manipulator into operation.

⇒ Calculations are possible as long as Δ_1 does not vanish.

⇒ Δ_1 vanishes if and only if :-

(1) a_1 vanishes
(2) μ_1 vanishes } Architecture dependent}
(3) $x_c^2 + y_c^2$ vanishes. {Position dependent}

{Means point C lies on the Z_1 axis}

{This occurs frequently in industrial Manipulator}
{not both at same time}

⇒ Even if neither a_1 nor μ_1 vanishes, the manipulator can be positioned in a configuration at which point C lies on the Z_1 axis.

↳ Such configuration is termed the $\boxed{\text{first Singularity.}}$

⇒ If both parameters a_1 & μ_1 vanished, then the arm would be useless to position arbitrarily a point in space.

⇒ For the moment, it will be assumed that Δ_1 does not vanish, the particular case under which it does being studied presently.

⇒ Next, both sides of eq (6) & (7) are squared and added:-

$$K \cos^2\theta_3 + L \sin^2\theta_3 + M \cos\theta_3 \sin\theta_3 + N \cos\theta_3 + P \sin\theta_3 + Q = 0 \quad \text{--- (8)}$$

whose coefficients, after simplification are given below:-

$$K = 4a_1^2 H^2 + \mu_1^2 C^2$$
$$L = 4a_1^2 I^2 + \mu_1^2 D^2$$
$$M = 2(4a_1^2 HI + \mu_1^2 CD)$$
$$N = 2(4a_1^2 HJ + \mu_1^2 CE)$$
$$P = 2(4a_1^2 IJ + \mu_1^2 DE)$$
$$Q = 4a_1^2 J^2 + \mu_1^2 E^2 - 4a_1^2 \mu_1^2 \rho^2$$

Where, $\rho^2 = x_c^2 + y_c^2$

⇒ Upon substitution of tan-half identities introduced, a quartic equation in $\tan\theta_3/2$ is obtained:-

$$R \tan^4(\theta_3/2) + S\tan^3(\theta_3/2) + T\tan^2(\theta_3/2) + U\tan(\theta_3/2) + V = 0 \quad — (9)$$

Where,

$$R = 4a_1^2(J-H)^2 + \mu_1^2(E-C)^2 - 4\rho^2 a_1^2 \mu_1^2$$

$$S = 4\left[4a_1^2 I(J-H) + \mu_1^2 D(E-C)\right]$$

$$T = 2\left[4a_1^2(J^2-H^2+2I^2) + \mu_1^2(E^2-C^2+2D^2) - 4\rho^2 a_1^2 \mu_1^2\right]$$

$$U = 4\left[4a_1^2 I(H+J) + \mu_1^2 D(C+E)\right]$$

$$V = 4a_1^2(J+H)^2 + \mu_1^2(E+C)^2 - 4\rho^2 a_1^2 \mu_1^2$$

⇒ Further, let $\{(\tan\theta_3/2)_i\}_1^4$ be the four roots of equation 9.

↳ Thus, up to four possible values of θ_3 can be obtained, namely.

$$(\theta_3)_i = 2\tan^{-1}\left[(\tan\theta_3/2)_i\right] \quad \forall\, i = 1,2,3,4$$

⇒ Once the four values of θ_3 are available, each of these, is substituted into eq (6) & (7), which thus produce four different values of θ_1.

⇒ For each value of θ_1 & θ_3, then one value of θ_2 can be computed from the first two scalar equation of Eq (3), which is displayed below:-

$$A_{11}\cos\theta_2 + A_{12}\sin\theta_2 = x_c\cos\theta_1 + y_c\sin\theta_1 - a_1 \quad (10)$$

$$-A_{12}\cos\theta_3 + A_{11}\sin\theta_2 = -x_c\lambda_1\sin\theta_1 + y_c\lambda_1\cos\theta_1 + (z_c - b_1)\mu_1 \quad (11)$$

Where,

$$A_{11} = a_2 + a_3\cos\theta_3 + b_4\mu_3\sin\theta_3 \quad (10^*)$$

$$A_{12} = -a_3\lambda_2\sin\theta_3 + b_3\mu_2 + b_4\lambda_2\mu_3\cos\theta_3 + b_4\mu_2\lambda_3 \quad (11^*)$$

⇒ Thus, if A_{11} & A_{12} do not vanish simultaneously, angle θ_2 is readily computed in terms of θ_1 & θ_3 from eq (10) & (11):-

$$\cos\theta_2 = \frac{1}{\Delta_2}\Big\{A_{11}(x_c\cos\theta_1 + y_c\sin\theta_1 - a_1) - A_{12}\left[-x_c\lambda_1\sin\theta_1 + y_c\lambda_1\cos\theta_1 + (z_c - b_1)\mu_1\right]\Big\} \quad (12)$$

$$\sin\theta_2 = \frac{1}{\Delta_2}\Big\{A_{12}(x_c\cos\theta_1 + y_c\sin\theta_1 - a_1) + A_{11}[-x_c\lambda_1\sin\theta_1 + y_c\lambda_1\cos\theta_1 + (z_c - b_1)\mu_1]\Big\} \quad \text{—(13)}$$

Where Δ_2 is defined as:-

$$\Delta_2 \equiv A_{11}^2 + A_{12}^2$$

$$\equiv a_2^2 + a_3^2(\cos^2\theta_3 + \lambda_2^2\sin^2\theta_3) + b_4^2\mu_3^2(\sin^2\theta_3 + \lambda_2^2\cos^2\theta_3)$$
$$+ 2a_2a_3\cos\theta_3 + 2a_2b_4\mu_3\sin\theta_3$$
$$+ 2\lambda_2\mu_2(b_3 + b_4\lambda_3)(b_4\mu_3\cos\theta_3 - a_3\sin\theta_3)$$
$$+ 2a_3b_4\mu_2^2\mu_3\sin\theta_3\cos\theta_3 + (b_3 + \lambda_3b_4)^2\mu_2^2$$

$\Rightarrow$ The Case in which $\Delta_2 = 0$, which leads to what is termed here the $\boxed{\text{Second Singularity.}}$

The Vanishing of Δ_1

$\Rightarrow$ In above derivations we have assumed that neither μ_1 nor a_1 vanishes.

$\Rightarrow$ If either of $\mu_1 = 0$ or $a_1 = 0$, then one can readily show that eq (8) reduces to a quadratic equation.

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⇒ Both $\mu_1 = 0$ & $a_1 = 0$ together never occurs, because their simultaneous occurrence would render the axes of the first two revolutes coincident.

⇒ We thus have two cases:-

1. $\mu_1 = 0$, $a_1 \neq 0$. In this case

$$A, B \neq 0 \quad , F = G = 0$$

⇒ Under these conditions, Eq ⑤ & tan-half-angle identities give:-

$$(J-H)\tan^2(\theta_3/2) + 2I\tan(\theta_3/2) + (J+H) = 0$$

which thus produce two values of $\tan \theta_3/2$

$$(\tan \theta_3/2)_{1,2} = \frac{-I \pm \sqrt{I^2 - J^2 + H^2}}{J-H}$$

⇒ Once two values of θ_3 have been determined according to the above equation, θ_1 can be found using eq ④ & tan half identities:-

$$(E'-A)\tan^2(\theta_1/2) + 2B\tan(\theta_1/2) + (E'+A) = 0$$

where,

$$E' = \cancel{C c_3 + D}\ C\cos\theta_3 + D\sin\theta_3 + E.$$

whose roots are,

$$\left(\tan(\theta_1/2)\right)_{1,2} = \frac{-B \pm \sqrt{B^2 - E'^2 + A^2}}{E' - A}$$

⇒ Thus, two values of θ_1 are found for each of the two values of θ_3, which results in four positioning solutions.

↳ Value of θ_3 is obtained from eq (10) & (11).

2. $a_1 = 0$, $\mu_1 \neq 0$. We have now

$$A = B = 0, \quad F, G \neq 0$$

⇒ Under the present conditions Eq (4) is reduced to:-

$$(E-C)\tan^2(\theta_3/2) + 2D\tan(\theta_3/2) + (E+C) = 0$$

Which produces two values of $\tan\theta_3/2$, namely,

$$\tan(\theta_3/2)_{1,2} = \frac{-D \pm \sqrt{D^2 - E^2 + C^2}}{E - C}$$

⇒ With the two values of θ_3 obtained, θ_1 can be found using (5) and the tan-half-angle identities to produce.

$$(J' - F)\tan^2(\theta_1/2) + 2G\tan(\theta_1/2) + (J'+F) = 0$$

Where,

$$J' = H\cos\theta_3 + I\sin\theta_3 + J$$

whose roots are,

$$\left(\tan(\theta_1/2)\right)_{1,2} = \frac{-G \pm \sqrt{G^2 - J'^2 + F^2}}{J' - F}$$

⇒ Once again the solution results in a cascade of two quadratic equations, one for θ_3 and one for θ_1, which yields four positioning solutions.

↳ As above θ_2 is determined using eq (10) & (11)

The Vanishing of Δ_2

⇒ Δ_2, may vanish at a certain posture thereby preventing the calculation of θ_2.

↳ This posture, termed the <u>Second Singularity</u>, occurs if both coefficients A_{11} & A_{12} vanishes.

⇒ From eq (10*) & (11*) it is evident that, these coefficients are not only position-but also <u>architecture-dependent</u>.

↳ Thus, an arbitrary manipulator cannot take on this configuration unless its geometric dimensions allows it.

⇒ This type of singularity will be termed <u>architecture-dependent</u>.

⇒ First note that the right-hand side of Eq (3) is identical to $\overline{Q}_1^T(\bar{c}-\bar{a}_1)$.

↳ This means that this expression is nothing but the F_2-representation of the position vector of C.

⇒ That is, the Components of Vector $\overline{\overline{Q}}_1^T(\overline{c}-\overline{a}_1)$ are the F_2-Components of Vector $\overrightarrow{O_2C}$.

⇒ So right had sides of eq (10) & (11) are respectively, the X_2- and Y_2-Components of Vector $\overrightarrow{O_2C}$.

⇒ If $A_{11} = A_{22} = 0$, then the two foregoing Components vanish and hence, point C lies on the Z_2 axis.

Conclusion

(1) First Singularity occurs when poin C lies on the axis of the first revolute.

(2) Second Singular occurs when point C lies on the axis of Second revolute.

4.4.2> The Orientation Problem

⇒ This problem consists in determining the wrist angles that will produce a prescribed orientation of the end-effector.

⇒ The orientation, in turn is given in terms of the rotation matrix $\bar{\bar{Q}}$, taking the end-effector from its home attitude to its current one.

↳ Alternatively can be given by the natural invariants of the rotation matrix, vector $\bar{e}$ & angle ϕ.

⇒ In any event, all nine components of matrix $\bar{\bar{Q}}$ are known in F_1.

⇒ It is convenient to assume the columnwise partitioning of $[\bar{\bar{Q}}]_1$.

$$[\bar{\bar{Q}}]_1 = [\bar{p} \ \bar{q} \ \bar{u}] \quad \text{—— (12)}$$

⇒ Without loss of generality it can be assumed that Z_7 is defined parallel to Z_6.

⇒ Since θ_1, θ_2 & θ_3 are available $\bar{\bar{Q}}_1$, $\bar{\bar{Q}}_2$ & $\bar{\bar{Q}}_3$ become data for this problem.

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⇒ Let $[\bar{u}]_1 = [\bar{e}_7]_1 = [\bar{e}_6]_1$

⇒ Now since the orientation of the end-effector is given, the components of $[\bar{e}_6]_1$ are known, but they will be needed in frame 4.

$$[\bar{e}_6]_4 = (\bar{\bar{Q}}_1 \bar{\bar{Q}}_2 \bar{\bar{Q}}_3)^T [\bar{e}_6]_1 = (\bar{\bar{Q}}_1 \bar{\bar{Q}}_2 \bar{\bar{Q}}_3)^T [\bar{u}]_1$$

⇒ Let the components of $[\bar{e}_6]_4$, all of them known, be defined as:-

$$[\bar{e}_6]_4 = \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} \quad \text{—(13)}$$

⇒ Moreover, the components of vector $\bar{e}_5$ in F_4 are nothing but the entries of the third column of matrix $\bar{\bar{Q}}_4$

$$[\bar{e}_5]_4 = \begin{bmatrix} \mu_4 \sin\theta_4 \\ -\mu_4 \cos\theta_4 \\ \lambda_4 \end{bmatrix} \quad \text{—(14)}$$

⇒ Furthermore, Vector $\overline{e}_5$ & $\overline{e}_6$ make an angle α_5, hence,

$$\overline{e}_6^T\,\overline{e}_5 = \lambda_5 \quad \text{or} \quad [e_6]_4^T [e_5]_4 = \lambda_5 \quad \text{--- (15)}$$

⇒ Upon Substitution eq (13) & eq (14) in eq (15) we get:-

$$\xi \mu_4 \sin\theta_4 - \eta \mu_4 \cos\theta_4 + \zeta \lambda_4 = \lambda_5$$

⇒ Which can be ~~used~~ readily transformed, with the aid of the tan-half angle identities, into a quadratic equation in $\tan(\theta_4/2)$.

$$(\lambda_4 - \eta\mu_4 - \zeta\lambda_4)\tan^2(\theta_4/2) - 2\xi\mu_4 \tan(\theta_4/2) + (\lambda_5 + \eta\mu_4 - \zeta\lambda_4) = 0 \quad \text{--- (16)}$$

its two roots being given by:-

$$\tan(\theta_4/2) = \frac{\xi\mu_4 \pm \sqrt{(\xi^2+\eta^2)\mu_4^2 - (\lambda_5 - \zeta\lambda_4)^2}}{\lambda_5 - \zeta\lambda_4 - \eta\mu_4}$$

⇒ Note that the two foregoing roots are real as long as the radical is positive, the two roots merging into a single one when the radical vanishes.

⇒ A negative radical means an attitude of the EE that is not feasible with the wrist.

⇒ Therefore the Workspace W of the Wrist is not unlimited, but rather defined by the set of values of ξ, η and ζ that satisfy the two relations shown below:-

$$\xi^2 + \eta^2 + \zeta^2 = 1$$

$$f(\xi, \eta, \zeta) \equiv (\xi^2 + \eta^2)\mu_u^2 - (\lambda_s - \zeta\lambda_u)^2 \geq 0$$

So above can be simplified in ζ alone, namely,

$$F(\zeta) = \zeta^2 - 2\lambda_u\lambda_s\zeta - (\mu_u^2 - \lambda_s^2) \leq 0$$

As a Consequence

1. W is a region of the Unit Sphere S Centered at the origin of the three-dimensional Space.

2. W is bounded by the two parallels given by the roots of $F(\zeta)=0$ on the Sphere.

3. The Wrist attains its Singular Configurations along the two foregoing parallels.

⇒ In order to gain more insight on the Shape of the workspace W, Let us look at the boundary defined by $F(\zeta)=0$.

$$\zeta_{1,2} = \lambda_4\lambda_5 \pm |\mu_4\mu_5|$$

Which thus defines two planes, Π_1 & Π_2 Parallel to the ξ–η plane of the three dimensional Space, intersecting the ζ-axis at ζ_1 & ζ_2 respectively.

⇒ Thus, the workspace W of the Spherical wrist at hand is that region of the Surface of the Unit Sphere S Contained between the two Parallels defined by Π_1 & Π_2.

⇒ Once θ_4 is Calculated from the two foregoing Values of $\tan\theta_4/2$, if these are real, angle θ_5 is obtained uniquely for each value of θ_4 as explained below:-

Let $\bar{\bar{R}} = \bar{\bar{Q}}_6 \bar{\bar{Q}}_5 \bar{\bar{Q}}_4$ — (17)

$\Rightarrow$ $\bar{\bar{Q}} = \bar{\bar{Q}}_6 \bar{\bar{Q}}_5 \bar{\bar{Q}}_4 \bar{\bar{Q}}_3 \bar{\bar{Q}}_2 \bar{\bar{Q}}_1$

$\Rightarrow$ $\bar{\bar{Q}} = \bar{\bar{R}} \bar{\bar{Q}}_3 \bar{\bar{Q}}_2 \bar{\bar{Q}}_1$

$R = \bar{\bar{Q}}_3^T \bar{\bar{Q}}_2^T \bar{\bar{Q}}_1^T \bar{\bar{Q}}$ — (18)

$\Rightarrow$ Let entries of $\bar{\bar{R}}$ in the fourth coordinate frame be given as :-

$$[\bar{\bar{R}}]_4 = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$\Rightarrow$ Expressions of θ_5 and θ_6 can be readily derived by solving first the $\bar{\bar{Q}}_5$ from eq (17) :-

$$\bar{\bar{Q}}_5 = \bar{\bar{Q}}_4^T \bar{\bar{R}} \bar{\bar{Q}}_6^T$$

$\Rightarrow$ Now, by virtue of the form of the $\bar{\bar{Q}}_i$ matrices, it is apparent that the third row of $\bar{\bar{Q}}_i$ does not contain θ_i

$\Rightarrow$ Thus, two equations of θ_5 are obtained by equating the first two components of the third column of the equation.

$$\mu_5 \sin\theta_5 = (\mu_6 r_{12} + \lambda_6 r_{13})\cos\theta_4 + (\mu_6 r_{22} + \lambda_6 r_{23})\sin\theta_4 \quad \text{--- (19)}$$

$$-\mu_5 \sin\theta_5 = -\lambda_4(\mu_6 r_{12} + \lambda_6 r_{13})\sin\theta_4 + \lambda_4(\mu_6 r_{22} + \lambda_6 r_{23})\cos\theta_4 + \mu_4(\mu_6 r_{32} + \lambda_6 r_{33}) \quad \text{--- (20)}$$

⇒ which thus yield a unique value of θ_5 for every value of θ_4.

⇒ Finally, with θ_4 and θ_5 known, it is a simple matter to calculate θ_6.

⇒ This is done upon solving for $\bar{Q}_6$ from eq (18) :-

$$\boxed{\bar{\bar{Q}}_6 = \bar{\bar{Q}}_5^T \bar{\bar{Q}}_4^T \bar{\bar{R}}} \quad \text{--- (21)}$$

⇒ using Partitioning of $\bar{\bar{Q}}_i$ we get :-

$$\bar{P}_6 = \bar{\bar{Q}}_5^T \bar{\bar{Q}}_4^T \bar{r}_1 \quad \left\{ \text{where } \bar{r}_1 \text{ is the first column of } \bar{\bar{R}} \right\}$$

Let $\bar{\omega} = \bar{Q}_4^T \bar{r}_1$

$$\bar{\omega} = \begin{bmatrix} r_{11}\cos\theta_4 + r_{21}\sin\theta_4 \\ -\lambda_4(r_{11}\sin\theta_4 - r_{21}\cos\theta_4) + \mu_4 r_{31} \\ \mu_4(r_{11}\sin\theta_4 - r_{21}\cos\theta_4) + \lambda_4 r_{31} \end{bmatrix}$$

Hence,

$$\bar{Q}_5^T \bar{Q}_4^T \bar{r}_1 = \begin{bmatrix} \omega_1 \cos\theta_5 + \omega_2 \sin\theta_5 \\ \lambda_4(-\omega_1 \sin\theta_5 + \omega_2 \cos\theta_5) + \omega_3 \mu_5 \\ \mu_5(\omega_1 \sin\theta_5 - \omega_2 (\cos\theta_5) + \omega_3 \lambda_5 \end{bmatrix}$$

So,

$$\cos\theta_6 = \omega_1 \cos\theta_5 + \omega_2 \sin\theta_5$$

$$\sin\theta_6 = -\omega_1 \lambda_5 \sin\theta_5 + \omega_2 \lambda_5 \cos\theta_5 + \omega_3 \mu_5$$

⇒ Thereby deriving a unique valu. of θ_3 for every pair of values (θ_4, θ_5).

⇒ Therefor, there are two sets of solutions for the orientation problem under study, which leads to two corresponding wrist postures.

" Finaly a maximum of eight possible combinations of joint angles for a single pose of the end effector of a decoupled manipulator are found."