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CHAPTER 1

Modeling in State Space

1

Modeling in State Space

Modern Control Theory

- Multiple-input Multiple-output System
- Linear/Nonlinear
- time invariant/ time Varying
- Time domain

Classical Control Theory

- Single Input Single Output System.
- Linear
- Time invariant
- Laplace domain or frequency domain

<u>State</u>: The State of a dynamic System is the Smallest set of variable (Called $\boxed{\text{State Variable}}$) Such that Knowledge of these variable at $t = t_0$, together with knowledge of the input for $t > t_0$, Completely determines the behavior of the System for any time $t \geq t_0$.

<u>State Vector</u>: If n state variables are needed to Completely describe the behavior of a given System, then these n State Variables Can be Considered the n Components of a Vector $\vec{X}$. Such a vector is called a state vector.

State Space: The n-dimentional space whose coordinate axes consist of the x_1 axis, x_2 axis —— x_n axis, where $x_1, x_2, \cdots x_n$ are state variables, is called a state space. Any state can be represented by a point in the state space.

State-Space Equation

⇒ In state-space analysis we are concerned with three types of variables that are involved in the modeling of dynamic systems:-

⇒ Input Variables
⇒ Output Variables
⇒ State Variables

⇒ The dynamic system must involve elements that memorize the values of the input for $t > t_1$.

⇒ Since integrator serves as memory devices in a continuous-time control system, the output of such integrators can be considered as the variable that define the internal state of the dynamic system.

↳ Thus the output of integrators serve as state variable.

⇒ Assume that a multiple-input, multiple-output system involve n integrators. {Let us define n output of the integrators as state variable $x_1(t)$, $x_2(t)$ --- $x_n(t)$} Assume also there are r inputs $u_1(t), u_2(t) \cdots u_r(t)$ and m outputs $y_1(t), y_2(t) --- y_m(t)$

Then the System may be described by:

$$\dot{x}_1(t) = f_1(x_1, x_2, \cdots x_n; u_1, u_2 \cdots u_r; t)$$

$$\dot{x}_2(t) = f_2(x_1, x_2, \cdots x_n; u_1, u_2 \cdots u_r; t)$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$\dot{x}_n(t) = f_n(x_1, x_2 \cdots x_n; u_1, u_2 \cdots u_r; t)$$

The Output $y_1(t)\, y_2(t) \cdots y_m(t)$ of the System may be given by

$$y_1(t) = g_1(x_1, x_2 \cdots x_n; u_1, u_2 \cdots u_r; t)$$

$$y_2(t) = g_2(x_1, x_2 \cdots x_n; u_1, u_2 \cdots u_r; t)$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$y_m = g_m(x_1, x_2 \cdots x_n; u_1, u_2 \cdots u_r; t)$$

So $\boxed{\dot{\vec{x}}(t) = \vec{f}(\vec{x}, \vec{u}, t)}$ → State Equation

$\boxed{\dot{\vec{y}}(t) = \vec{g}(\vec{x}, \bar{u}, t)}$ → Output Equation

⇒ If the above are linearized about the operating state, then we have the following linearized <u>State equation</u> and <u>Output Equation</u>.

$$\dot{\vec{X}}(t) = \vec{A}(t)\,\vec{X}(t) + \vec{B}(t)\,\vec{u}(t)$$

$$\vec{Y}(t) = \vec{C}(t)\,\vec{X}(t) + \vec{D}(t)\,\vec{u}(t)$$

{ From now we can ignore vector Sign }

$$\boxed{\dot{X}(t) = A(t)X(t) + B(t)U(t)} \rightarrow \text{State Equation}$$

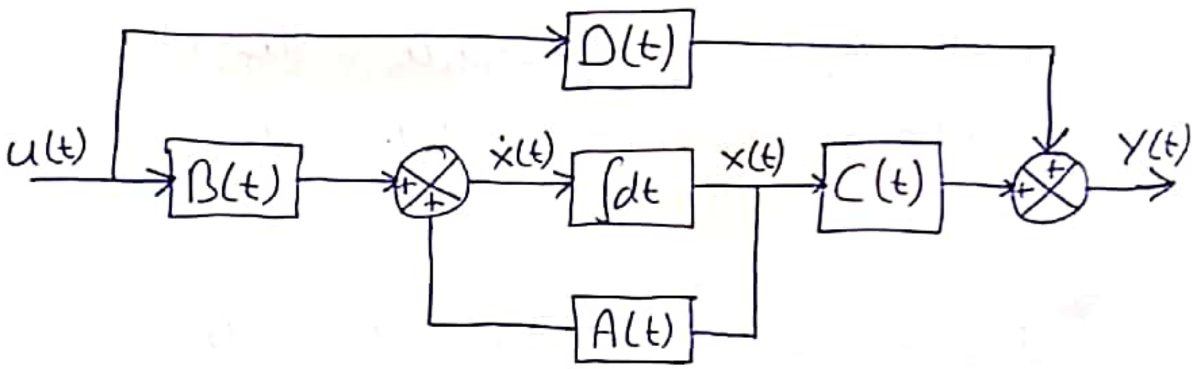
$$\boxed{Y(t) = C(t)X(t) + D(t)U(t)} \rightarrow \text{Output Equation}$$

$A(t)$ = State matrix

$B(t)$ = Input matrix

$C(t)$ = Output matrix

$D(t)$ = Direct transmission matrix



⇒ If vector functions f and g do not involve time t explicitly then the System is called a time-invariant system.

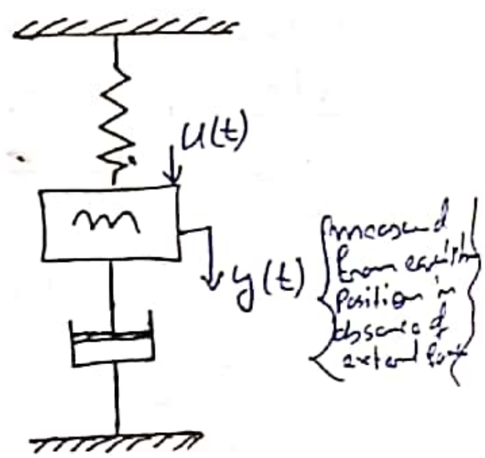
$$\left.\begin{array}{l}\boxed{\dot{X} = AX(t) + BU(t)} \\ \boxed{Y = CX(t) + DU(t)}\end{array}\right\} \text{For } \underline{\text{time-invariant linear}}$$

<u>Example 1.1:</u>

<u>Input</u> : $u(t)$

<u>Output</u> : $y(t)$

<u>State Variable</u> : $y(t)$ $\dot{y}(t)$



$m\ddot{y}(t) + b\dot{y}(t) + Ky(t) = u(t)$ {from mechanics}

$$X(t) = \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix} \quad \dot{X}(t) = \begin{bmatrix} \dot{y}(t) \\ \ddot{y}(t) \end{bmatrix}$$

$$Y(t) = [y(t)] \quad u(t) = [u(t)]$$

$$\dot{X}(t) = AX(t) + BU(t)$$
$$Y(t) = CX(t) + DU(t)$$

$$\begin{bmatrix} \dot{y}(t) \\ \ddot{y}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{-K}{m} & \frac{-b}{m} \end{bmatrix} \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u(t)$$

$$Y(t) = [1 \ 0] \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix}$$

So $A = \begin{bmatrix} 0 & 1 \\ \frac{-K}{m} & \frac{-b}{m} \end{bmatrix}$ $B = \begin{bmatrix} 0 \\ 1/m \end{bmatrix}$ $C = [1, 0]$, $D = 0$

★ State-Space Representation of Scalar Differential Equation System

⟹ By use of Vector-matrix notation, an n^{th} order differential equation may be expressed by a first order vector-matrix differential equation.

$$\boxed{\dot{X}(t) = AX(t) + BU(t)}$$

State-Space Representation of n^{th}-Order System of Linear Differential Equation in which the Forcing function Does not involve Derivative Term

$$\overset{n}{y} + a_1 \overset{n-1}{y} + \cdots + a_{n-1}\dot{y} + a_n y = u$$

Output: y Input: u

⇒ We may take $y(t), \dot{y}(t) \cdots \overset{n-1}{y}(t)$ as a set of n state variables.

So
$$x_1 = y \qquad \dot{x}_1 = x_2$$
$$x_2 = \dot{y} \qquad \dot{x}_2 = x_3$$
$$\vdots \qquad \Rightarrow \qquad \vdots$$
$$x_n = \overset{n-1}{y} \qquad \dot{x}_{n-1} = x_n$$
$$\dot{x}_n = u - (a_1 x_n + \cdots a_n x_1)$$

$$\dot{X} = AX + BU$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & \cdots & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u$$

$$Y = CX + DU$$

$$y = \begin{bmatrix} 1, 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + [0]\,u$$

State-Space Representation of n^{th} Order Systems of Linear differential Equations in which the Forcing function Involves Derivative terms

$$\overset{n}{y} + a_1 \overset{n-1}{y} + \cdots\cdots a_{n-1}\dot{y} + a_n y = b_0 \overset{n}{u} + b_1 \overset{n-1}{u} + \cdots + b_{n-1}\dot{u} + b_n u$$

⇒ The main problem in defining the state variables for this case lies in the derivative terms of the input u.

↳ The state variables must be such that they will eliminate the derivatives of u in the state equation.

⇒ One way to obtain a state equation and output equation for this case is to define the following n variables as a set of n state variables.

$$x_1 = y - \beta_0 u$$

$$x_2 = \dot{y} - \beta_0 \dot{u} - \beta_1 u = \dot{x}_1 - \beta_1 u$$

$$x_3 = \ddot{y} - \beta_0 \ddot{u} - \beta_1 \dot{u} - \beta_2 u = \dot{x}_2 - \beta_2 u$$

$$\vdots$$

$$x_n = \overset{n-1}{y} - \beta_0 \overset{n-1}{u} - \beta_1 \overset{n-2}{u} - \cdots \beta_{n-1} u = \dot{x}_{n-1} - \beta_{n-1} u$$

Where, $\beta_0, \beta_1, \beta_2 \cdots \beta_{n-1}$ are determined from:

$$\beta_0 = b_0$$

$$\beta_1 = b_1 - a_1\beta_0$$

$$\beta_2 = b_2 - a_1\beta_1 - a_2\beta_2$$

$$\beta_3 = b_3 - a_1\beta_2 - a_2\beta_1 - a_3\beta_0$$

$$\vdots$$

⇒ With the present choice of state variables, we obtain:

$$\dot{x}_1 = x_2 + B_1 u$$
$$\dot{x}_2 = x_3 + B_2 u$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$\dot{x}_{n-1} = x_n + B_{n-1} u$$
$$\dot{x}_n = -a_1 x_1 - a_{n-1} x_2 - \cdots - a_1 x_n + B_n u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_{n-1} \\ B_n \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + B_0 u$$

CHAPTER 2

Control System Analysis in State Space

2

Control System Analysis in State Space

★ State-Space Representations of Transfer-Function System

(1) State-Space Representations in Canonical Forms

⇒ Consider a system defined by :-

$$\overset{n}{y} + a_1 \overset{n-1}{y} + \cdots a_{n-1}\dot{y} + a_n y = b_0 \overset{n}{u} + b_1 \overset{n-1}{u} + \cdots + b_{n-1}\dot{u} + b_n u$$

where u is the input and y is the output.

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}$$

Controllable Canonical Form (a)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \cdots & 1 \\ -a_n & -a_{n-1} & \cdots & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \left[b_n - a_n b_0 \;\vdots\; b_{n-1} - a_{n-1} b_0 \;\vdots\; \cdots \;\vdots\; b_1 - a_1 b_0 \right] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0 u$$

⇒ The Controllable Canonical form is important in discussing the pole-placement approach to Control system design.

Observable Canonical form

(b)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_n - a_n b_0 \\ b_{n-1} - a_{n-1} b_0 \\ \vdots \\ b_1 - a_1 b_0 \end{bmatrix} u$$

$$y = [0 \; 0 \cdots 0 \; 1] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0 u$$

Diagonal Canonical Form

(c)

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1} s + b_n}{(s+P_1)(s+P_2)\cdots(s+P_n)}$$

All distinct roots

$$= b_0 + \frac{C_1}{s+P_1} + \frac{C_2}{s+P_2} + \cdots \frac{C_n}{s+P_n}$$

⇒ The diagonal canonical form of the state space representation of this system is given by :-

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -P_1 & & & 0 \\ & -P_2 & & \\ & & \ddots & \\ 0 & & & -P_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u$$

$$y = [C_1\ C_2 \cdots C_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0 u$$

Jordan Canonical Form

u

⇒ Case where denominator polynomial Contains multiple roots.

Suppose,

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1} s + b_n}{(s+P_1)^3 + (s+P_4) + (s+P_5) \cdots (s+P_n)}$$

$$= b_0 + \frac{C_1}{(s+P_1)^3} + \frac{C_2}{(s+P_1)^2} + \frac{C_3}{s+P_1} + \frac{C_4}{s+P_4} + \cdots \frac{C_n}{s+P_n}$$

⇒ State-space representation in the Jordan Canonical form is given by.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \left[\begin{array}{ccc|ccc} -P_1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & -P_1 & 1 & & & \\ 0 & 0 & -P_1 & 0 & \cdots & 0 \\ \hline 0 & \cdots & 0 & -P_4 & & 0 \\ \vdots & & \vdots & & \ddots & \\ 0 & \cdots & 0 & 0 & & -P_n \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u$$

$$y = [C_1\ C_2 \cdots C_n] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + b_0 u$$

★ Eigenvalues of an $n \times n$ Matrix A

⇒ The eigenvalues of an $n \times n$ matrix A are the roots of the characteristic equation

$$|\lambda I - A| = 0$$

★ Diagonalization of $n \times n$ Matrix

⇒ If $n \times n$ matrix A with distinct eigenvalues is given by

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & \cdots & \cdots & -a_1 \end{bmatrix}$$

the transformation $X = PZ$, where

$$P = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \cdots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \cdots & \lambda_n^2 \\ \vdots & \vdots & \vdots & & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \cdots & \lambda_n^{n-1} \end{bmatrix}$$

{where $\lambda_1, \lambda_2 \cdots \lambda_n$ = n distinct Eigenvalues of A}

will transform $P^{-1}AP$ into the diagonal matrix.

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

⇒ If the matrix A involves multiple eigenvalues, then diagonalization is impossible.

⇒ Eigenvalues of A and those of $P^{-1}AP$ are identical.
We shall prove this for a general case in what follows.

1

★ **Invariance of Eigen Values**

To prove the invariance of the eigenvalues under a linear transformation, we must show that the characteristic polynomials $|\lambda I - A|$ and $|\lambda I - P^{-1}AP|$ are identical.

$$
\begin{aligned}
|\lambda I - P^{-1}AP| &= |\lambda P^{-1}P - P^{-1}AP| \\
&= |P^{-1}(\lambda I - A)P| \\
&= |P^{-1}||\lambda I - A||P| \\
&= |P^{-1}||P||\lambda I - A| \\
&= |P^{-1}P||\lambda I - A| \\
&= |\lambda I - A|
\end{aligned}
$$

★ **Nonuniqueness of Set of State Variables**

Let $x_1, x_2 \cdots x_n$ are a set of state variables.
Let $x_1, x_2 \cdots x_n$ are a set of state variables
⇒ Then we may take as another set of state variables any set of functions.

$$
\begin{aligned}
\hat{x}_1 &= X_1(x_1, x_2 \cdots x_n) \\
\hat{x}_2 &= X_2(x_1, x_2 \cdots x_n) \\
&\vdots \\
\hat{x}_n &= X_n(x_1, x_2 \cdots x_n)
\end{aligned}
$$

Provided that, for every set of values $\hat{x}_1, \hat{x}_2 \cdots \hat{x}_n$, there corresponds to a unique set of values $x_1, x_2 \cdots x_n$ and vice-versa

⇒ Thus, if x is a state vector, then $\hat{x}$, where

$$\boxed{\hat{x} = Px}$$

is also a state vector, provided the matrix P is nonsingular.

↳ Different state vectors convey the same information about the system behavior.

★ Solving the time-invariant State Equation

Solving Homogeneous State Equations

⇒ Before we solve vector-matrix differential equations, let us review the solution of the scalar differential equation.

$$\dot{x} = ax$$

⇒ In solving this equation, we may assume a solution $x(t)$ of the form.

$$x(t) = b_0 + b_1 t + b_2 t^2 + \cdots b_k t^k + \cdots$$

⇒ By substituting this assumed solution we obtain:-

$$b_1 + 2b_2 t + 3b_3 t^2 + \cdots + kb_k t^{k-1} + \cdots$$
$$= a(b_0 + b_1 t + b_2 t^2 + \cdots b_k t^k + \cdots)$$

⇒ If the assumed solution is to be true solution, the above equation must hold for any t.

$$b_1 = ab_0$$
$$b_2 = \frac{1}{2}ab_1 = \frac{1}{2}a^2 b_0$$

$$b_3 = \frac{1}{3} a b_2 = \frac{1}{6} a^3 b_0$$

$$\vdots \quad \vdots \quad \vdots$$

$$b_k = \frac{1}{k!} a^k b_0$$

$\Rightarrow$ The value of b_0 is determined by substituting $t=0$ in above equation.

$$b_0 = x(0)$$

$$\Rightarrow x(t) = \left(1 + at + \frac{1}{2!} a^2 t^2 + \cdots + \frac{1}{k!} a^k t^k + \cdots \right) x(0)$$

$$\Rightarrow \boxed{x(t) = e^{at} x(0)}$$

※ We shall now solve the vector-matrix differential equation.

$$\boxed{\dot{x} = Ax} \qquad \left\{ \begin{array}{l} x = n \text{ Vector} \\ A = n \times n \text{ Vector} \end{array} \right\}$$

$\Rightarrow$ By analogy with the scalar case, we assume that the solution is in the form of a vector power series in t.

(Vector's)

$$x(t) = b_0 + b_1 t + b_2 t^2 + \cdots + b_k t^k + \cdots$$

$\Rightarrow$ By substituting this assumed solution into the vector differential equation we get:-

$$b_1 + 2b_2 t + 3b_3 t^2 + \cdots + k b_k t^{k-1} + \cdots$$
$$= A(b_0 + b_1 t + b_2 t^2 + \cdots + b_k t^k + \cdots)$$

$\Rightarrow$ By equating the like powers of t on both side, we obtain,

$$b_1 = A b_0$$
$$b_2 = \frac{1}{2} A b_1 = \frac{1}{2} A^2 b_0$$
$$\vdots \quad \vdots \quad \vdots$$

$$b_K = \frac{1}{K!} A^K b_0$$

So $X(t) = \underbrace{\left(I + At + \frac{1}{2!}A^2t^2 + \cdots + \frac{1}{K!}A^K t^K + \cdots\right)}_{e^{At}\ \{\text{Matrix Exponential}\}} X(0)$

$$\Rightarrow \boxed{X(t) = e^{At} X(0)}$$

Matrix Exponential

2

$$\boxed{e^{At} = \sum_{K=0}^{\infty} \frac{A^K t^K}{K!}}$$

{At converges for all finite t.}

$$\frac{de^{At}}{dt} = A + A^2 t + \frac{A^3 t^2}{2!} + \cdots \frac{A^K t^{K-1}}{(K-1)!} + \cdots$$

$$= A\left[I + At + \frac{A^2t^2}{2!} + \cdots \frac{A^{K-1}t^{K-1}}{(K-1)!} + \cdots\right]$$

$$\boxed{\frac{de^{At}}{dt} = Ae^{At} = e^{At}A}$$

$$\boxed{e^{A(t+s)} = e^{At}e^{As}}$$

Proof

$$e^{At}e^{As} = \left(\sum_{K=0}^{\infty} \frac{A^K t^K}{K!}\right)\left(\sum_{K=0}^{\infty} \frac{A^K s^K}{K!}\right)$$

$$= \sum_{K=0}^{\infty} A^K \left(\sum_{i=0}^{\infty} \frac{t^i s^{K-i}}{i!\,(K-i)!}\right) = \sum_{K=0}^{\infty} A^K \frac{(t+s)^K}{K!}$$

$$= e^{A(t+s)}$$

x(0)

omentially

⇒ In particular if $s=-t$ then

$$e^{At}e^{-At} = e^{-At}e^{At} = e^{A(t-t)} = I$$

⇒ Thus the inverse of e^{At} is e^{-At}. $\boxed{(e^{At})^{-1} = e^{-At}}$

⇒ Since the inverse of e^{At} always exists, e^{At} is nonsingular.

$$\boxed{e^{(A+B)t} = e^{At}e^{Bt}} \quad \{\text{if } AB = BA\}$$

$$\boxed{e^{(A+B)t} \neq e^{At}e^{Bt}} \quad \{\text{if } AB \neq BA\}$$

Laplace Transform approach to the Solution of Homogeneous State Equation

$$\dot{x}(t) = Ax(t)$$

$$sX(s) - x(0) = AX(s) \quad \{X(s) = \mathcal{L}[x(t)]\}$$

$$X(s) = (sI-A)^{-1}x(0)$$

$$x(t) = \mathcal{L}^{-1}\{(sI-A)^{-1}x(0)\}$$

$$\Rightarrow \boxed{(sI-A)^{-1} = \frac{I}{s} + \frac{A}{s^2} + \frac{A^2}{s^3} + \cdots}$$

$$\Rightarrow \mathcal{L}^{-1}\{(sI-A)^{-1}\} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots$$

$$= e^{At} \Rightarrow \boxed{x(t) = e^{At}x(0)}$$

★ State-Transition matrix

We can write solution of the homogeneous state equation

$$\dot{x} = Ax$$

as $\boxed{x(t) = \phi(t)x(0)}$ {State transition matrix}

K

Where $\phi(t)$ is an $n \times n$ matrix and is the unique solution of

$$\boxed{\dot{\phi}(t) = A\,\phi(t)} \qquad \phi(0) = I$$

$$x(0) = \phi(0)\,x(0) = x(0)$$

$$\phi(t) = e^{At} = \mathcal{L}^{-1}\{(sI - A)^{-1}\}$$

$$\boxed{\phi^{-1}(t) = e^{-At} = \phi(-t)}$$

⇒ $\phi(t)$ is called the state

⇒ If the eigenvalues $\lambda_1, \lambda_2, \ldots \lambda_n$ of the matrix A are distinct, then $\phi(t)$ will contain the n exponentials.

$$e^{\lambda_1 t}, e^{\lambda_2 t} \ldots e^{\lambda_n t}$$

⇒ In particular, if the matrix A is diagonal then,

$$\phi(t) = e^{At} = \begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix}$$

⇒ If there is a multiplicity in the eigenvalues then $\phi(t)$ will contain, in addition to the exponentials $e^{\lambda_1 t}, e^{\lambda_2 t} \ldots e^{\lambda_n t}$, terms like $te^{\lambda_1 t}$ and $t^2 e^{\lambda_1 t}$.

* Properties of State-Transition Matrices

1. $\phi(0) = e^{A0} = I$
2. $\phi(t) = e^{At} = (e^{-At})^{-1} = [\phi(-t)]^{-1}$ or $\boxed{\phi^{-1}(t) = \phi(-t)}$
3. $\phi(t_1 + t_2) = \phi(t_1)\phi(t_2) = \phi(t_2)\phi(t_1)$
4. $[\phi(t)]^n = \phi(nt)$
5. $\phi(t_2 - t_1)\phi(t_1 - t_0) = \phi(t_2 - t_0) = \phi(t_1 - t_0)\phi(t_2 - t_1)$

Example 9-5: Obtain the state-transition matrix $\phi(t)$ of the following System:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\Rightarrow$ $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

$$\boxed{\phi(t) = e^{At} = \mathcal{L}^{-1}[(SI-A)^{-1}]}$$

$$SI - A = \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} S & -1 \\ 2 & S+3 \end{bmatrix}$$

$$(SI-A)^{-1} = \cancel{\frac{A}{(S+2)(S+1)}} \frac{1}{|SI-A|} Adj(SI-A)$$

$$= \frac{1}{S(S+3)+2} \begin{bmatrix} S+3 & -2 \\ +1 & S \end{bmatrix}^T$$

$$= \frac{1}{(S+1)(S+2)} \begin{bmatrix} S+3 & 1 \\ -2 & S \end{bmatrix}$$

$$= \begin{bmatrix} \frac{(S+3)}{(S+1)(S+2)} & \frac{1}{(S+1)(S+2)} \\ \frac{-2}{(S+1)(S+2)} & \frac{S}{(S+1)(S+2)} \end{bmatrix}$$

$$\Phi(t) = \mathcal{L}^{-1}\left[(sI-A)^{-1}\right]$$

$$= \begin{bmatrix} 2e^{-t}-e^{-2t} & e^{-t}-e^{-2t} \\ -2e^{-t}+2e^{-2t} & -e^{-t}+2e^{-2t} \end{bmatrix}$$

★ **Solving Nonhomogeneous State Equation**

We shall begin by considering the scalar case :-

$$\dot{x} = ax + bu$$

$$\dot{x} - ax = bu$$

⇒ Multiplying both side by e^{-at}, we obtain

$$e^{-at}\left[\dot{x}(t) - ax(t)\right] = \frac{d}{dt}\left[e^{-at}x(t)\right] = e^{-at}bu(t)$$

⇒ Integrating this equation between 0 and t gives

$$e^{-at}x(t) - x(0) = \int_0^t e^{-at}bu(t)\,dt$$

$$\boxed{x(t) = e^{at}x(0) + e^{at}\int_0^t e^{-a\tau}bu(\tau)\,d\tau}$$

Response to the initial condition

Response to the input $u(t)$

⇒ Let us consider the nonhomogeneous state equation described by

$$\boxed{\dot{X} = AX + BU}$$

Where, X = n-Vector
U = r-Vector
A = n×n constant matrix
B = n×r constant matrix

$$\dot{x}(t) - Ax(t) = Bu(t)$$

$$\Rightarrow \quad e^{-At}\left[\dot{x}(t) - Ax(t)\right] = \frac{d}{dt}\left[e^{-At}x(t)\right] = e^{-At}Bu(t)$$

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau$$

$$\Rightarrow \quad \boxed{x(t) = \phi(t)x(0) + \int_0^t \phi(t-\tau)Bu(\tau)\,d\tau}$$

★ Laplace Transform Approach to the Solution of Nonhomogeneous State Equation

$$\dot{x} = Ax + BU$$

$$\Rightarrow \quad sX(s) - x(0) = AX(s) + BU(s)$$

$$\Rightarrow \quad (sI - A)X(s) = x(0) + BU(s)$$

$$X(s) = (sI-A)^{-1}x(0) + (sI-A)^{-1}BU(s)$$

$$X(s) = \mathcal{L}[e^{At}]x(0) + \mathcal{L}[e^{At}]BU(s)$$

$$x(t) = e^{At}x(0) + \underbrace{\int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau}_{\text{Convolution Integral}}$$

Solution in Terms of $x(t_0)$

⇒ Thus far we have assumed the initial time to be zero. If however, the initial time is given by t_0 instead of 0 then:

$$\boxed{x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^t e^{A(t-\tau)}Bu(\tau)\,d\tau}$$

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★ Some useful Results in Vector matrix Analysis

Cayley - Hamilton Theorem

Consider an $n \times n$ matrix A and its characteristic equation:

$$|\lambda I - A| = \lambda^n + a_1 \lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0$$

The Cayley-Hamilton theorem states that the matrix A satisfies its own characteristic equation

$$A^n + a_1 A^{n-1} + \cdots + a_{n-1} A + a_n I = 0$$

⇒ To prove this theorem note that $\text{adj}(\lambda I - A)$ is a Polynomial in λ of degree $n-1$

$$\text{adj}(\lambda I - A) = B_1 \lambda^{n-1} + B_2 \lambda^{n-2} + \cdots + B_{n-1}\lambda + B_n$$

Where $B_1 = I$ since,

$$(\lambda I - A)\,\text{adj}(\lambda I - A) = [\text{adj}(\lambda I - A)](\lambda I - A) = |\lambda I - A| I$$

We obtain

$$|\lambda I - A| I = I\lambda^n + a_1 I \lambda^{n-1} + \cdots + a_{n-1} I \lambda + a_n I$$

$$= (\lambda I - A)(B_1 \lambda^{n-1} + B_2 \lambda^{n-2} + \cdots B_{n-1}\lambda + B_n)$$

$$= (B_1 \lambda^{n-1} + B_2 \lambda^{n-2} + \cdots + B_{n-1}\lambda + B_n)(\lambda I - A)$$

⇒ If A is substituted for λ in this last equation, then clearly $\lambda I - A$ becomes zero.

We obtain

$$\boxed{A^n + a_1 A^{n-1} + \cdots + a_{n-1} A + a_n I = 0}$$

Minimal polynomial

3

"The least degree polynomial having A as a root is called the minimal polynomial"

⇒ Minimal polynomial of an $n\times n$ matrix A is defined as the polynomial $\phi(\lambda)$ of least degree

$$\phi(\lambda) = \lambda^m + a_1\lambda^{m-1} - \cdots + a_{m-1}\lambda + a_m \quad \{m \leq n\}$$

Such that $\phi(A) = 0$

⇒ The minimal polynomial plays an important role in the Computation of polynomials in an $n\times n$ matrix.

⇒ Let us suppose that $d(\lambda)$, a polynomial in λ, is the greatest Common divisor of all the elements of $adj(\lambda I - A)$.

↳ We can show that if the Coefficient of the highest-degree term in λ of $d(\lambda)$ is chosen as 1, then the minimal polynomial $\phi(\lambda)$ is given by

$$\phi(\lambda) = \frac{|\lambda I - A|}{d(\lambda)}$$

Minimal polynomial $\phi(\lambda)$ of an $n\times n$ matrix A can be determined by the following procedure:-

1. Form $adj(\lambda I - A)$ and write the elements of $adj(\lambda I - A)$ as factored polynomial in λ.
2. Determine $d(\lambda)$ as the greatest Common divisor of all the elements of $adj(\lambda I - A)$. Choose the coefficient of the highest degree term in λ of $d(\lambda)$ to be 1. If there is no common divisor, $d(\lambda) = 1$.
3. The minimal polynomial:-

$$\boxed{\phi(\lambda) = \frac{|\lambda I - A|}{d(\lambda)}}$$

★ Matrix Exponential e^{At}

Matlab provides a simple way to compute e^{AT}, where T is a constant.

Computation of e^{At}: Method 1

If matrix A can be transformed into a diagonal form, then e^{At} can be given by

$$e^{At} = Pe^{Dt}P^{-1} = P\begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix}P^{-1}$$

where P is a diagonalizing matrix for A.

⇒ If matrix A can be transformed into a Jordan canonical form, then e^{At} can be given by

$$e^{At} = Se^{Jt}S^{-1}$$

{S = transformation matrix that transforms matrix A into a Jordan Canonical form J}

Computation of e^{At}: Method 2

⇒ The second method of computing e^{At} uses the Laplace transform approach.

$$e^{At} = \mathcal{L}^{-1}\left[(sI - A)^{-1}\right]$$

Computation of e^{At}: Method 3

⇒ This method is based on Sylvester's Interpolation method.

Case 1: Minimal polynomial of A Involves Only Distinct Roots

4

⇒ We shall assume that the degree of the minimal Polynomial of A is m.

⇒ By using Sylvester's interpolation formula, it can be shown that e^{At} can be obtained by solving the following determinant equation.

$$\begin{vmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{m-1} & e^{\lambda_1 t} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{m-1} & e^{\lambda_2 t} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & \lambda_m & \lambda_m^2 & \cdots & \lambda_m^{m-1} & e^{\lambda_m t} \\ I & A & A^2 & \cdots & A^{m-1} & e^{At} \end{vmatrix} = 0$$

$\lambda_1, \lambda_2 \cdots \lambda_m$ are roots of minimal Polynomial

⇒ Solving above equation for e^{At} is the same as:

$$\boxed{e^{At} = \alpha_0(t) I + \alpha_1(t) A + \alpha_2(t) A^2 + \cdots + \alpha_{m-1} A^{m-1}}$$

where $\alpha_K(t)$ can be determined by solving the following set of m equations.

$$\alpha_0(t) + \alpha_1(t)\lambda_1 + \alpha_2(t)\lambda_1^2 + \cdots + \alpha_{m-1}(t)\lambda_1^{m-1} = e^{\lambda_1 t}$$

$$\alpha_0(t) + \alpha_1(t)\lambda_2 + \alpha_2(t)\lambda_2^2 + \cdots + \alpha_{m-1}(t)\lambda_2^{m-1} = e^{\lambda_2 t}$$

$$\vdots$$

$$\alpha_0(t) + \alpha_1(t)\lambda_m + \alpha_2(t)\lambda_m^2 + \cdots + \alpha_{m-1}(t)\lambda_m^{m-1} = e^{\lambda_m t}$$

Case 2: Minimal Polynomial of A Involves Multiple Roots

⇒ As an example, consider the case where the minimal polynomial of A involves three equal roots ($\lambda_1=\lambda_2=\lambda_3$) and has other roots ($\lambda_4, \lambda_5 \cdots \lambda_m$) that are all distinct.

⇒ By applying Sylvester's interpolation formula it can be shown that e^{At} can be obtained from the following determinant equation.

$$\begin{vmatrix} 0 & 0 & 1 & 3\lambda_1 & \cdots & \frac{(m-1)(m-2)}{2}\lambda_1^{m-3} & \frac{t^2}{2}e^{\lambda_1 t} \\ 0 & 1 & 2\lambda_1 & 3\lambda_1^2 & \cdots & (m-1)\lambda_1^{m-2} & te^{\lambda_1 t} \\ 1 & \lambda_1 & \lambda_1^2 & \lambda_1^3 & \cdots & \lambda_1^{m-1} & e^{\lambda_1 t} \\ 1 & \lambda_4 & \lambda_4^2 & \lambda_4^3 & \cdots & \lambda_4^{m-1} & e^{\lambda_4 t} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & \lambda_m & \lambda_m^2 & \lambda_m^3 & \cdots & \lambda_m^{m-1} & e^{\lambda_m t} \\ I & A & A^2 & A^3 & \cdots & A^{m-1} & e^{At} \end{vmatrix} = 0$$

⇒ Note that if the minimal polynomial of A is not found, it is possible to substitute the characteristic polynomial for the minimal polynomial.

★ Linear Independence of Vector

⇒ The vectors $X_1, X_2 \cdots X_m$ are said to be linearly independent if

$$C_1X_1 + C_2X_2 + C_3X_3 + \cdots C_mX_m = 0$$

where $C_1, C_2 \cdots C_m$ are constant, implise that

$$C_1 = C_2 = C_3 = \cdots = C_m = 0$$

⇒ The vectors $x_1, x_2 \cdots x_n$ are said to be linearly dependent if and only if x_i can be expressed as a linear combination of x_j $(j = 1, 2 \cdots n, j \neq i)$

$$x_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} c_j x_j \quad \left\{ \text{for some set of constants } c_j \right\}$$

★ Controllability

Controllable ⇒ A system is said to be controllable at time t_0 if it is possible by means of an unconstrained control vector to transfer the system from any initial state $x(t_0)$ to any other state in a finite interval of time.

Observable ⇒ A system is said to be observable at time t_0 if, with the system in state $x(t_0)$, it is possible to determine this state from the observation of the output over a finite time interval.

⇒ The concept of controllability and observability were introduced by Kalman.

Complete State Controllability of Continuous-Time System

⇒ Consider the continuous-time system

$$\boxed{\dot{x} = Ax + Bu}$$

where, x = State vector (n-vector)

u = Control signal (scalar)

A = $n \times n$ matrix

B = $n \times 1$ matrix

⇒ The system described above is said to be state controllable at $t = t_0$ if it is possible to construct an unconstrained control signal that will transfer an initial state to any final state in a finite time interval $t_0 \le t \le t_1$.

↳ If every state is controllable, then the system is said to be completely state controllable.

⇒ Without any loss of generality, we can assume that the final state is the origin of the state space and that the initial time is zero.

⇒ The solution of above equation is :-

$$X(t) = e^{At} x(0) + \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau$$

$$X(0) = -\int_0^{t_1} e^{-A\tau} Bu(\tau) d\tau$$

$$e^{-A\tau} = \sum_{K=0}^{n-1} \alpha_K(\tau) A^K$$

So
$$X(0) = -\sum_{K=0}^{n-1} A^K B \int_0^{t_1} \alpha_K(\tau) u(\tau) d\tau$$

Let us put
$$\beta_K = \int_0^{t_1} \alpha_K(\tau) u(\tau) d\tau$$

So
$$X(0) = -\sum_{K=0}^{n-1} A^K B \beta_K$$

$$= -\left[B \,\vdots\, AB \,\vdots\, \cdots \,\vdots\, A^{n-1}B\right] \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{n-1} \end{bmatrix}$$

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⇒ If the System is Completely State Controllable, then given any initial State $x(0)$ above equation must be Satisfied.

↳ This requires that the rank of the $n \times n$ matrix

$$[B : AB : \cdots : A^{n-1}B]$$ be n.

// Condition for State Controllability ÷

The System given by

$$\dot{x} = Ax + Bu$$

is Completely State Controllable if and only if the vectors $B, AB \cdots A^{n-1}B$ are linearly independent, or the $n \times n$ matrix

$$[B : AB : \cdots : A^{n-1}B]$$ is of rank n.

// The result just obtained can be extended to the case where the Control Vector u is r-dimensional.

If the System is described by

$$\dot{x} = Ax + BU$$

then it can be proved that the Condition for Complete State Controllability is that the $n \times nr$ matrix

$$[B : AB : \cdots : A^{n-1}B]$$

be of rank n, or Contain n linear independent Column Vectors.

// The matrix

$$[B : AB : \cdots : A^{n-1}B]$$

is Commonly called the Controllability matrix.

Alternative Form of the Condition for Complete State Controllability

Consider a System defined by

$$\dot{X} = AX + BU$$

Where, X = State Vector (n-Vector)
U = Control Vector (r-Vector)
A = $n \times n$ Matrix
B = $n \times r$ matrix

$\Rightarrow$ If the ~~eigenvalue~~ eigenvector of A are distinct, then it is Possible to find a transformed matrix P Such that

$$P^{-1}AP = D = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

Note: If the eigenvalues of A are distinct, then the eigenvectors of A are distinct; however Converse is not true.

Let us define

$$X = PZ$$

$$\Rightarrow (\dot{PZ}) = A(PZ) + BU$$

$$\Rightarrow \dot{Z} = P^{-1}APZ + P^{-1}BU \quad \text{—} (1)$$

Let $P^{-1}B = F = (f_{ij})$

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We can rewrite Eq ① as:

$$\dot{z}_1 = \lambda_1 z_1 + f_{11}u_1 + f_{12}u_2 + \cdots + f_{1r}u_r$$
$$\dot{z}_2 = \lambda_2 z_2 + f_{21}u_1 + f_{22}u_2 + \cdots + f_{2r}u_r$$
$$\vdots$$
$$\dot{z}_n = \lambda_n z_n + f_{n1}u_1 + f_{n2}u_2 + \cdots f_{nr}u_r$$

⇒ If the elements of any one row of the $n \times r$ matrix F are all zero, then the corresponding state variable cannot be controlled by any of the u_i.

"The condition of complete state controllability is that if the eigenvectors of A are distinct, then the system is completely state controllable if and only if no row of $P^{-1}B$ has all zero element"

⇒ If the A matrix does not possess distinct eigenvectors, then diagonalization is impossible.

↳ In such case we may transform A into a Jordan canonical form.

⇒ Suppose that we can find a transformation matrix S such that

$$S^{-1}AS = J$$

If we define new state vector z by

$$x = Sz$$

So, $\dot{Z} = S^{-1}ASZ + S^{-1}BU$

$\Rightarrow \dot{Z} = JZ + S^{-1}BU$

"The System is Completely State Controllable if and only if :-

1) No two Jordan blocks in J are associated with the same eigen values.

2) The elements of any row of $S^{-1}B$ that Correspond to the last row of each Jordan block are not all zero.

3) The elements of row of $S^{-1}B$ that Corresponds to distinct eigenvalues are not all zero.

Condition for Complete State Controllability in the S plane

"Necessary and Sufficient Condition for Complete State Controllability is that no Cancellation occur in the transfer function or transfer matrix. If Cancellation occure, the System Cannot be Controlled in the direction of the Canceled mode"

Output Controllability

5

$\Rightarrow$ In the practical design of a Control System, we may want to Control the output rather than the State of the System.

⇒ For this reason, it is desirable to define separately complete output controllability.

⇒ Consider the system defined by

$$\dot{X} = AX + BU$$
$$Y = CX + DU$$

Where, X = State Vector (n-Vector)
U = Control Vector (r-Vector)
Y = Output Vector (m-Vector)
A = $n \times n$ matrix
B = $n \times r$ matrix
C = $m \times n$ matrix
D = $m \times r$ matrix

"The system described above is said to be completely output controllable if it is possible to construct an unconstrained control vector $u(t)$ that will transfer any given initial output $Y(t_0)$ to any final output $y(t_1)$ in a finite time interval $t_0 \le t \le t_1$"

" The system described above is completely output controllable if and only if the $m \times (n+1)r$ matrix.

$$\left[CB \,\vdots\, CAB \,\vdots\, CA^2B \,\vdots\, ---- \,\vdots\, CA^{n-1}B \,\vdots\, D \right]$$

is of rank m.

Un Controllable System ⇒ System which has a subsystem that is physically disconnected from the input

Stabilizability ⇒ For a partially Controllable System, if the UnControllable modes are stable and the ~~Stable~~ unstable modes are Controllable, the System is said to be Stabilizable.

★ Observability

⇒ Consider an unforced System described by the following equations:-

$$\dot{x} = Ax \quad \text{—①} \qquad \left\{ \begin{array}{l} x \Rightarrow \text{State Vector (n-Vector)} \\ y \Rightarrow \text{Output Vector (m-Vector)} \\ A \Rightarrow n\times n \text{ matrix} \\ C \Rightarrow m\times n \text{ matrix} \end{array} \right\}$$

$$y = Cx \quad \text{—②}$$

⇒ System is <u>Completely observable</u> if:-

"Every State $x(t_0)$ Can be determined from the observation of $y(t)$ over a finite time interval, $t_0 \leq t \leq t_1$"

⇒ In this Section we treat only linear, time-invariant System. Therefore, without loss of generality, we can assume that $t_0 = 0$.

Importance of Concept of Observability

↳ In practice, the difficulty encountered with State feedback Control is that some of the State Variables are not accessible for direct measurement, with the result it become necessary to estimate the unmeasurable State Variables in order to Construct the Control signals.

If we describe System by:-

$$\boxed{\dot{x} = Ax + Bu} \Rightarrow x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)} Bu(\tau)d\tau$$

$$\boxed{Y = Cx + Du} \Rightarrow y(t) = Ce^{At}x(0) + C\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau + Du$$

Since the matrix A, B, C, & D are Known and u(t) is also Known, the last two terms on the right hand side of this last equation are Known quantity.

⇒ Hence, for investigating a necessary & Sufficient Condition for Complete observability, it suffices to consider the System described by Equation ① & ②

Complete Observability of Continuous-time System

⇒ The output Vector y(t) is:-

$$y(t) = Ce^{At}x(0)$$

{degree of Characteristic Polynomial}

$$\text{We Know, } e^{At} = \sum_{K=0}^{n-1} \alpha_K(t) A^K$$

$$\text{So, } y(t) = \sum_{K=0}^{n-1} \alpha_K(t)\, CA^K x(0)$$

$$y(t) = \alpha_0(t) Cx(0) + \alpha_1(t) CAx(0) + \dots + \alpha_{n-1}(t)\, CA^{n-1}x(0)$$

⇒ It Can be Shown that for Complete observability requires the rank of this matrix of $mn \times n$ to be n.

$$\begin{bmatrix} C \\ \hline CA \\ \hline \vdots \\ \hline CA^{n-1} \end{bmatrix}$$

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Condition for Complete Observability

The System described by eq ① & ② is Completely Observable if & only if the $n \times nm$ matrix

$$\left[C^T \mid A^T C^T \mid \cdots \mid (A^T)^{n-1} C^T\right]$$

is of rank n or has n linearly independent Column vectors. This matrix is called the <u>Observability matrix</u>.

Conditions for Complete Observability in the S plane

"The necessary & sufficient Conditions for Complete observability is that no Cancellation occur in the transfer function or transfer matrix"

Alternate Form of Condition for Complete Observability

⇒ Consider the System described by equation ① & ②.

Let $P^{-1}AP = D$ $\left\{ \begin{array}{l} D \Rightarrow \text{Diagonal matrix} \\ P \Rightarrow \text{Transformation matrix} \end{array} \right\}$

Let $x = PZ$

$$\Rightarrow \quad \dot{Z} = P^{-1}APZ = DZ$$
$$y = CPZ$$

Hence, $y(t) = CPe^{Dt}Z(0)$

$$y(t) = CP \begin{bmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{bmatrix} Z(0) = CP \begin{bmatrix} e^{\lambda_1 t} Z_1(0) \\ e^{\lambda_2 t} Z_2(0) \\ \vdots \\ e^{\lambda_n t} Z_n(0) \end{bmatrix}$$

⇒ The System is Completely observable if none of the columns of the $m \times n$ matrix CP consists of all zero element.

⇒ If the matrix A cannot be transformed into a diagonal matrix, then by use of a suitable transformation matrix S, we can transform A into a Jordan canonical form.

★ Principle of Duality

⇒ Consider the System S_1 described by

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

⇒ And the dual System S_2 defined by

$$\dot{z} = A^T z + C^T v$$
$$n = B^T z$$

⇒ The principle of duality states that the system S_1 is completely state controllable (observable) if and only if system S_2 is completely observable (state controllable).

For System S1

State Controllability

$[B \,;\, AB \,;\, --- \,;\, A^{n-1}B]$ → rank n

State observability

$[C^T \,;\, A^T C^T \,;\, --- \,;\, (A^T)^{n-1} C^T]$ → Rank n

For Dual System S2

State Controllability

$[C^T \,;\, A^T C^T \,|\, --- \,|\, (A^T)^{n-1} C^T]$ → Rank n

State observability

$[B \,;\, AB \,|\, \cdots \,|\, A^n B]$ → rank n

2.1 Appendix 1: Controllable Canonical Form

(A) Controllable Canonical Form

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n} \quad \text{--- (1)}$$

(1)
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u \quad \text{--- (2)}$$

(2)
$$y = \begin{bmatrix} b_n - a_n b_0 & \vdots & b_{n-1} - a_{n-1} b_0 & \vdots & \cdots & \vdots & b_1 - a_1 b_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_1 \\ \vdots \\ x_n \end{bmatrix} + b_0 u \quad \text{--- (3)}$$

Solution

Equation (1) Can be written as:-

$$\frac{Y(s)}{U(s)} = \frac{b_0\left(s^n + a_1 s^{n-1} + \cdots a_{n-1} s + a_n\right) + \left\{ (b_1 - b_0 a_1) s^{n-1} + (b_2 - b_0 a_2) s^{n-2} + \cdots + (b_{n-1} - a_{n-1} b_0) s + (b_n - a_n b_0) \right\}}{\left(s^n + a_1 s^{n-1} + \cdots a_{n-1} s + a_n\right)}$$

$$\Rightarrow \frac{Y(s)}{U(s)} = b_0 + \left(\frac{(b_1 - a_1 b_0) s^{n-1} + \cdots + (b_{n-1} - a_{n-1} b_0) s + (b_n - a_n b_0)}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n} \right)$$

$\otimes\, U(s)$

So $\boxed{Y(s) = b_0 U(s) + \hat{Y}(s)}$

$$\hat{Y}(s) = \frac{(b_1 - a_1 b_0)s^{n-1} + \cdots + (b_{n-1} - a_{n-1} b_0)s + (b_n - a_n b_0)}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

$\Rightarrow$ Let
$$Q(s) = \frac{\hat{Y}(s)}{(b_1 - a_1 b_0)s^{n-1} + \cdots + (b_{n-1} - a_{n-1} b_0)s + (b_n - a_n b_0)}$$

$$= \frac{U(s)}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

$$s^n Q(s) = U(s) - a_1 s^{n-1} Q(s) \cdots\cdots - a_{n-1} s Q(s) - a_n Q(s) \quad \text{--- (4)}$$

$$\hat{Y}(s) = (b_1 - a_1 b_0)s^{n-1} Q(s) + \cdots + (b_{n-1} - a_{n-1} b_0) s Q(s) + (b_n - a_n b_0) Q(s) \quad \text{--- (5)}$$

$\Rightarrow$ Now let us define state variable as follows :-

$$X_1(s) = Q(s)$$
$$X_2(s) = sQ(s)$$
$$\vdots \qquad \vdots$$
$$X_{n-1}(s) = s^{n-2} Q(s)$$
$$X_n(s) = s^{n-1} Q(s)$$

So clearly,

$$X_2 = sX_1$$
$$X_3 = sX_2$$
$$\vdots \quad \vdots \quad \vdots$$
$$X_n = sX_{n-1}$$

(6)
$$\boxed{\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_{n-1} &= x_n \end{aligned}}$$

{Time domain}

Space D. ... System of

$\Rightarrow$ Rewriting eq 4 $\{ s^n Q(s) = sX_n \}$

$$sX_n = -a_1 X_n(s) - \cdots - a_{n-1} X_2(s) - a_n X_1(s) + U(s)$$

$\downarrow$ {Time domain}

$\cancel{\dot{x}_n = -a_1 x_1 \cdots - a_{n-1} x_1 - x}$

$$\boxed{\dot{x}_n = -a_n x_1 - a_{n-1} x_2 - \cdots - a_1 x_n + u} \quad \text{— (b)}$$

Using a & b we got eq 2 {Part 1 of solution}

$\Rightarrow$ We know that $Y(s) = b_0 U(s) + \hat{Y}(s)$

$$Y(s) = b_0 U(s) + (b_1 - a_1 b_0) s^{n-1} Q(s) + \cdots + (b_{n-1} - a_{n-1} b_0) s Q(s) + (b_n - a_n b_0) Q(s)$$

$$\Rightarrow Y(s) = b_0 U(s) + (b_1 - a_1 b_0) X_n(s) + \cdots + (b_{n-1} - a_{n-1} b_0) X_2(s) + (b_n - a_n b_0) X_1(s)$$

$\downarrow$ {Time domain}

$$\boxed{y = b_0 u + (b_n - a_n b_0) x_1 + (b_{n-1} - a_{n-1} b_0) x_2 + \cdots + (b_1 - a_1 b_0) x_n} \quad \text{— (c)}$$

From eq (c) we get eq 3 {Part 2 of the solution}

2.2 Appendix 2: Observable Canonical Form

(B) Observable Canonical Form

Standard form

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n} \quad \text{—(1)}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_n - a_n b_0 \\ b_{n-1} - a_{n-1} b_0 \\ \vdots \\ b_1 - a_1 b_0 \end{bmatrix} u \quad \text{—(2)}$$

$$y = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0 u \quad \text{—(3)}$$

Solution

⇒ Rearranging eq (1) we got:-

$$s^n [Y(s) - b_0 U(s)] + s^{n-1} [a_1 Y(s) - b_1 U_s] + \cdots$$

$$+ s [a_{n-1} Y(s) - b_{n-1} U(s)] + [a_n Y(s) - b_n U(s)] = 0$$

⇒ Dividing by s^n and rearranging:-

$$\boxed{Y(s) = b_0 U(s) + \frac{1}{s}[b_1 U(s) - a_1 Y(s)] + \cdots + \frac{1}{s^{n-1}}[b_{n-1} U(s) - a_{n-1} Y(s)] + \frac{1}{s^n}[b_n U(s) - a_n Y(s)]} \quad \text{—(4)}$$

Now define state variables as follows:-

$$X_n = \frac{1}{s}\left[b_1 U(s) - a_1 Y(s) + X_{n-1}(s)\right]$$

$$X_{n-1}(s) = \frac{1}{s}\left[b_2 U(s) - a_2 Y(s) + X_{n-2}(s)\right]$$

$$\vdots$$

$$X_2(s) = \frac{1}{s}\left[b_{n-1} U(s) - a_{n-1} Y(s) + X_1(s)\right]$$

$$X_1(s) = \frac{1}{s}\left[b_n U(s) - a_n Y(s)\right]$$

⇒ Using above definition of state variable eq ④ can be written as:-

$$Y(s) = b_0 U(s) + X_n(s) \quad \text{— ⑤}$$

⇒ Putting $Y(s)$ from eq ⑤ in state variables we get:-

$$sX_n = X_{n-1} - a_1 X_n + (b_1 - a_1 b_0) U(s)$$

$$sX_{n-1} = X_{n-1} - a_2 X_n + (b_2 - a_2 b_0) U(s)$$

$$\vdots$$

$$sX_2(s) = X_1(s) - a_{n-1} X_n + (b_{n-1} - a_{n-1} b_0) U(s)$$

$$sX_1 = -a_n X_n + (b_n - a_n b_0) U(s)$$

⇒ Taking Inverse Laplace transform of above set of equation we get:

$$\dot{x}_1 = -a_n x_n + (b_n - a_n b_0) u$$

$$\dot{x}_2 = x_1 - a_{n-1} x_n + (b_{n-1} - a_{n-1} b_0) u$$

$$\vdots$$

$$\dot{x}_{n-1} = x_{n-2} - a_2 x_n + (b_2 - a_2 b_0)u$$

$$\dot{x}_n = x_{n-1} - a_1 x_n + (b_1 - a_1 b_0)u$$

⇒ The above set of equation gives eq ①
{Part 1 of solution}

⇒ Taking inverse Laplace transform of $Y(s) = b_0 U(s) + X_n(s)$
{eq ⑤}

$$\Rightarrow \boxed{y = x_n + b_0 u} \quad \text{—⑥}$$

⇒ Equation ⑥ gives eq ③ {Part 2 of solution}

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2.3 Appendix 3: Diagonal Canonical Form

(c) Diagonal Canonical form

Consider the transfer-function of a system defined by :-

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1} s + b_n}{(s+P_1)(s+P_2)\cdots(s+P_n)} = b_0 + \frac{C_1}{s+P_1} + \frac{C_2}{s+P_2} + \cdots \frac{C_n}{s+P_n} \quad (1)$$

where $P_i \neq P_j$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -P_1 & & & 0 \\ & -P_2 & & \\ & & \ddots & \\ 0 & & & -P_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u \quad (2)$$

$$y = \begin{bmatrix} C_1 & C_2 & \cdots & C_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0 u \quad (3)$$

Solution

$$Y(s) = b_0 U(s) + \frac{C_1}{s+P_1} U(s) + \frac{C_2}{s+P_2} U(s) + \cdots \frac{C_n}{s+P_n} U(s)$$

Lets define state variable as follows:

$$X_1(s) = \frac{U(s)}{s+P_1}$$

$$X_2(s) = \frac{1}{s+P_2} U(s)$$

$$\vdots$$

$$X_n(s) = \frac{1}{s+P_n} U(s)$$

⇒ The above equations may be re-written as :-

$$sX_1(s) = -P_1 X_1(s) + U(s)$$
$$sX_2(s) = -P_2 X_2(s) + U(s)$$
$$\vdots$$
$$sX_n(s) = -P_n X_n(s) + U(s)$$

$\Longrightarrow$ {Time domain}

$$\dot{x}_1 = -P_1 x_1 + u$$
$$\dot{x}_2 = -P_2 x_2 + u$$
$$\vdots$$
$$\dot{x}_n = -P_n x_n + u$$

{from this we can get eq 2 (first part of the solution)}

$$\Rightarrow Y(s) = b_0 U(s) + C_1 X_1(s) + C_2 X_2(s) + \cdots C_n X_n(s)$$

$\Downarrow$ {Time domain}

$$\boxed{y = b_0 u + C_1 x_1 + C_2 x_2 + \cdots C_n x_n}$$

{from this we can get eq 3 (Second part of solution)}

2.4 Appendix 4: Jordan Canonical Form

(D) Jordan Canonical Form

Consider the System defined by :-

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots b_{n-1} s + b_n}{(s+P_1)^3 (s+P_4)(s+P_5)\cdots(s+P_n)} \quad \text{—} \; 0$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \left[\begin{array}{ccc|ccc} -P_1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & -P_1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & -P_1 & 0 & \cdots & 0 \\ \hline 0 & \cdots & 0 & -P_4 & & 0 \\ \vdots & & \vdots & & \ddots & \\ 0 & \cdots & 0 & 0 & & -P_n \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u \quad \text{—} \; 1$$

$$y = \begin{bmatrix} C_1 & C_2 & \cdots & C_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0 u \quad \text{—} \; 2$$

Solution

⇒ Partial-fraction expansion of Eq (0) :-

$$\frac{Y(s)}{U(s)} = b_0 + \frac{C_1}{(s+P_1)^3} + \frac{C_2}{(s+P_1)^2} + \frac{C_3}{(s+P_1)} + \frac{C_4}{s+P_4} + \cdots + \frac{C_n}{s+P_n}$$

$$\Rightarrow Y(s) = b_0 U(s) + \frac{C_1}{(s+P_1)^3} U(s) + \frac{C_2}{(s+P_1)^2} U(s) + \frac{C_3}{s+P_1} U(s) + \frac{C_4}{s+P_4} U(s) + \cdots + \frac{C_n}{s+P_n}$$

Let us define State Variable as :-

$$X_1(s) = \frac{1}{(s+P_1)^3} U(s)$$

$$X_2(s) = \frac{1}{(s+P_1)^2} U(s)$$

$$X_3(s) = \frac{1}{s+P_1} U(s) \Longrightarrow (a)\ sX_3(s) = -P_1 X_3(s) + U(s)$$

$$X_4(s) = \frac{1}{s+P_4} (U(s) \longrightarrow (b)\ sX_4(s) = -P_4 X_4(s) + U(s)$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$X_n(s) = \frac{1}{s+P_n} U(s) \Longrightarrow (f)\ sX_n(s) = -P_4 X_n(s) + U(s)$$

⇒ Notice the following relationships between $X_1(s)$, $X_2(s)$ & $X_3(s)$:

$$\frac{X_1(s)}{X_2(s)} = \frac{1}{s+P_1} \Rightarrow sX_1 = -P_1 X_1(s) + X_2(s) \quad (g)$$

$$\frac{X_2(s)}{X_3(s)} = \frac{1}{s+P_1} \Rightarrow sX_2 = -P_1 X_2(s) + X_3(s) \quad (h)$$

⇒ Converting eq (a), (b) - - - (f) (g) & (h) in time domain we get:-

$$\dot{x}_1 = -P_1 x_1 + x_2$$
$$\dot{x}_2 = -P_1 x_2 + x_3$$
$$\dot{x}_3 = -P_1 x_3 + u$$
$$\dot{x}_4 = -P_4 x_4 + u$$
$$\vdots$$
$$\dot{x}_n = -P_n x_n + u$$

From this we get eq (1) {first part of solution}

$$Y(s) = b_0 U(s) + C_1 X_1(s) + C_2 X_2(s) + \cdots + C_n X_n(s)$$

$\Downarrow$ {Time domain}

$$\boxed{y = C_1 x_1 + C_2 x_2 + \cdots C_n x_n + b_0 u}$$

{ from this we get eq ① {Second part of the solution} }

2.5 Appendix 5: Correlation Between Transfer Function and State Space Equation

★ Correlation Between Transfer functions & State-Space Equation

Let us Consider the System whose TF is given by:-

$$G(s) = \frac{Y(s)}{U(s)}$$

⇒ The System may be represented in state Space by the following equations:-

$$\dot{x} = Ax + Bu \quad \text{—— (1)}$$

$$y = Cx + Du \quad \text{—— (2)}$$

⇒ Taking Laplace transform of eq (1) & (2) ÷

$$sX(s) = AX(s) + BU(s)$$

$$Y(s) = CX(s) + DU(s)$$

$$(sI - A)X(s) = BU(s)$$

$$X(s) = (sI-A)^{-1} BU(s)$$

$$Y(s) = C(sI-A)^{-1} BU(s) + DU(s)$$

$$\boxed{G(s) = \frac{Y(s)}{U(s)} = C(sI-A)^{-1}B + D}$$

$$\boxed{G(s) = \frac{Q(s)}{|sI-A|}}$$

Same polynomial in s.

Characteristic Polynomial of G(s)

⇒ Eigen Values of A are identical to the Poles of G(s).

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Transfer Matrix

⇒ Consider a multiple input - Multiple Output system.

⇒ Assume that there are r inputs & m outputs.

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} \qquad u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_r \end{bmatrix}$$

$$\boxed{G(s) = \frac{Y(s)}{U(s)} = C(sI-A)^{-1}B + D}$$

→ Transfer matrix

2.6 Appendix 6: (1)

Consider a System is represented in Standard State Space form as:-

$$\boxed{\dot{X} = AX + BU} \quad \text{—— (1)}$$

$$\boxed{y = CX + DU} \quad \text{—— (2)}$$

Let us define a new set of state variables by the linear transformation

$$\boxed{X = PZ} \quad \text{—— (3)}$$

{This mean x_i is linear Combination of Z_i.}

~~Putting~~ {where P is eigen Vector matrix of A}

⇒ Putting eq (3) in (1) we get:-

$$P\dot{Z} = APZ + BU$$

$$\Rightarrow \dot{Z} = P^{-1}APZ + P^{-1}BU$$

$$\Rightarrow \boxed{\dot{Z} = \Lambda Z + P^{-1}BU}$$

{Λ is diagonal matrix}

So finally System Can be represented in diagonal form as:-

$$\boxed{\begin{aligned} \dot{Z} &= \Lambda Z + B'U \\ Y &= C'Z + DU \end{aligned}}$$

where,

$$\Lambda = P^{-1}AP$$

$$B' = P^{-1}B$$

$$C' = CP$$

2.7 Appendix 7: (2)

② Properties of Matrix Exponential

① $e^{At} = \sum_{K=0}^{\infty} \frac{A^K t^K}{K!}$ {definition}

② $\frac{d}{dt} e^{At} = A e^{At}$

③ $e^{A(t+s)} = e^{At} e^{As}$

④ $e^{(A+B)t} \neq e^{At} e^{Bt}$ {If $AB \neq BA$}

$e^{(A+B)t} = e^{At} e^{Bt}$ {If $AB = BA$}

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2.8 Appendix 8: (3)

Minimal polynomial of an $n \times n$ matrix A is defined as the polynomial $\phi(\lambda)$ of least degree,

(3)

$$\phi(\lambda) = \lambda^m + a_1 \lambda^{m-1} + \cdots + a_{m-1}\lambda + a_m$$

$$m \leq n$$

Such that

$$\phi(\bar{A}) = \bar{O}$$

Let $d(\lambda)$, a polynomial in λ, is the greatest common divisor of all the elements of $adj(\lambda I - A)$.

If the coefficient of the highest-degree term in λ of $d(\lambda)$ is chosen as 1, then the minimum polynomial $\phi(\lambda)$ is given by

$$\phi(\lambda) = \left| \frac{\lambda\bar{I} - \bar{A}}{d(\lambda)} \right|$$

$$\Rightarrow adj(\lambda\bar{I} - \bar{A}) = d(\lambda)\bar{B}(\lambda) \quad \left\{ \begin{array}{l} \text{as } d(\lambda) \text{ is greatest common} \\ \text{divisor of the matrix } adj(\lambda I - A) \end{array} \right\}$$

$$\Rightarrow (\lambda\bar{I} - \bar{A})\, adj(\lambda\bar{I} - \bar{A}) = |\lambda\bar{I} - \bar{A}|\,\bar{I}$$

$$\Rightarrow d(\lambda)(\lambda\bar{I} - \bar{A})\bar{B}(\lambda) = |\lambda\bar{I} - \bar{A}|\,\bar{I}$$

From above we find that $|\lambda\bar{I} - \bar{A}|$ is divisible by $d(\lambda)$.

$$\Rightarrow |\lambda\bar{I} - \bar{A}| = d(\lambda)\psi(\lambda)$$

Because the coefficient of the higher degree term in λ of $d(\lambda)$ has been chosen as 1, the coefficient of the higher-degree term in λ of $\psi(\lambda)$ is also 1.

$$\Rightarrow (\lambda\bar{I} - \bar{A})\bar{B}(\lambda) = \psi(\lambda)\bar{I}$$

hence, $\psi(A) = 0$

So $\boxed{\psi(\lambda) = \frac{|\lambda I - A|}{d(\lambda)}}$

Minimal Polynomial

2.9 Appendix 9: (4)

(4) Computation of e^{At} {Method 3}

→ Using Sylvester's interpolation formula

Consider the following polynomial in λ of degree $m-1$, where we assume $\lambda_1, \lambda_2 \cdots \lambda_m$ to be distinct.

$$P_K(\lambda) = \frac{(\lambda-\lambda_1)(\lambda-\lambda_2)\cdots\cancel{(\lambda-\lambda_{K})}(\lambda-\lambda_{K+1})\cdots(\lambda-\lambda_m)}{(\lambda_K-\lambda_1)(\lambda_K-\lambda_2)\cdots(\lambda_K-\lambda_{K-1})(\lambda_K-\lambda_{K+1})\cdots(\lambda_K-\lambda_m)}$$

Where $K = 1, 2, \cdots m$.

$$P_K(\lambda_i) = \begin{cases} 1 & \text{if } i = K \\ 0 & \text{if } i \neq K \end{cases}$$

⇒ Then the polynomial $f(\lambda)$ of degree $m-1$

$$f(\lambda) = \sum_{K=1}^{m} f(\lambda_K) P_K(\lambda)$$

$$f(\lambda) = \sum_{K=1}^{m} f(\lambda_K) \frac{(\lambda-\lambda_1)(\lambda-\lambda_2)\cdots(\lambda-\lambda_{K-1})(\lambda-\lambda_{K+1})\cdots(\lambda-\lambda_m)}{(\lambda_K-\lambda_1)\cdots(\lambda_K-\lambda_{K-1})(\lambda_K-\lambda_{K+1})\cdots(\lambda_K-\lambda_m)}$$

The above takes value $f(\lambda_K)$ at the points λ_K.

⇒ The above equation is called <u>Lagrange's interpolation formula</u>.

⇒ Assuming that the eigenvalues of an $n \times n$ matrix A are distinct, substitute A for λ in the polynomial $P_K(\lambda)$, we get:-

$$\bar{\bar{P}}_K(\bar{\bar{A}}) = \frac{(\bar{\bar{A}}-\lambda_1\bar{\bar{I}})\cdots(\bar{\bar{A}}-\lambda_{K-1}\bar{\bar{I}})(\bar{\bar{A}}-\lambda_{K+1}\bar{\bar{I}})\cdots(\bar{\bar{A}}-\lambda_m\bar{\bar{I}})}{(\lambda_K-\lambda_1)\cdots(\lambda_K-\lambda_{K-1})(\lambda_K-\lambda_{K+1})\cdots(\lambda_K-\lambda_m)}$$

So $\bar{\bar{P}}_K(\lambda_i I) = \begin{cases} \bar{\bar{I}} & \text{if } i=K \\ \bar{\bar{O}} & \text{if } i\neq K \end{cases}$

Now define

$$\bar{\bar{f}}(\bar{\bar{A}}) = \sum_{K=1}^{m} f(\lambda_K)\,\bar{\bar{P}}_K(\bar{\bar{A}})$$

$$= \sum_{K=1}^{m} f(\lambda_K)\,\frac{(A-\lambda_1 I)\cdots(A-\lambda_{K-1}I)(A-\lambda_{K+1}I)\cdots(A-\lambda_m I)}{(\lambda_K-\lambda_1)\cdots(\lambda_K-\lambda_{K-1})(\lambda_K-\lambda_{K+1})\cdots(\lambda_K-\lambda_m)}$$

The above equation is known as <u>Sylvester's interpolation formula</u>.

$\Rightarrow$ The above equation can be equivalently written as:-

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & \bar{\bar{I}} \\ \lambda_1 & \lambda_2 & \lambda_3 & \cdots & \lambda_m & \bar{\bar{A}} \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \cdots & \lambda_m^2 & \bar{\bar{A}}^2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \lambda_1^{m-1} & \lambda_2^{m-2} & \lambda_3^{m-3} & \cdots & \lambda_m^{m-1} & \bar{\bar{A}}^{m-1} \\ f(\lambda_1) & f(\lambda_2) & f(\lambda_3) & \cdots & f(\lambda_m) & \bar{\bar{f}}(\bar{\bar{A}}) \end{vmatrix} = 0$$

$\Rightarrow$ Sylvester's interpolation formula is frequently used in evaluating $\bar{\bar{f}}(\bar{\bar{A}})$.

2.10 Appendix 10: (5)

/ 5

Output Controllability

Let the System be described by:-

$$\dot{\bar{x}} = \bar{\bar{A}}\bar{x} + \bar{\bar{B}}\bar{u} \qquad \begin{cases} \bar{x} \Rightarrow n\text{-Vector} \\ \bar{u} \Rightarrow r\text{-Vector} \\ \bar{y} \Rightarrow m\text{-Vector} \end{cases}$$

$$\bar{y} = \bar{\bar{C}}\bar{x}$$

It is completely output Controllable if and only if the Composite $m \times nr$ matrix $\bar{\bar{P}}$, where

$$\bar{\bar{P}} = \left[\bar{\bar{C}}\bar{\bar{B}} \,\vdots\, \bar{\bar{C}}\bar{\bar{A}}\bar{\bar{B}} \,\vdots\, \cdots \,\vdots\, \bar{\bar{C}}\bar{\bar{A}}^{n-1}\bar{\bar{B}}\right]$$

is of rank m.

Notice: Complete state Controllability is neither necessary nor sufficient for complete output Controllability

⇒ Suppose the System is output Controllable and the Output $y(t)$ Starting from any $y(0)$ the initial output, can be transferred to the origin of the output Space in finite ~~in~~ time interval $0 \leq t \leq T$.

$$\bar{y}(T) = \bar{\bar{C}}\bar{x}(T) = 0$$

$$\bar{x}(t) = e^{\bar{\bar{A}}t}\left[\bar{x}(0) + \int_0^t e^{-\bar{\bar{A}}\tau}\bar{\bar{B}}\bar{u}(\tau)\,d\tau\right]$$

at $t = T$

$$\bar{x}(T) = e^{\bar{\bar{A}}T}\left[\bar{x}(0) + \int_0^T e^{-\bar{\bar{A}}\tau}\bar{\bar{B}}\bar{u}(\tau)\,d\tau\right]$$

So, $\bar{y}(T) = \bar{C}\bar{x}(T) = \bar{C}e^{\bar{A}T}\left[\bar{x}(0) + \int_0^T e^{-\bar{A}\tau}\bar{B}\bar{u}(\tau)\,d\tau\right] = 0$ Let

$$\Rightarrow \bar{C}e^{\bar{A}T}\bar{x}(0) = -\bar{C}e^{\bar{A}T}\int_0^T e^{-\bar{A}\tau}\bar{B}\bar{u}(\tau)\,d\tau$$

$$= -\bar{C}\int_0^T e^{\bar{A}(T-\tau)}\bar{B}\bar{u}(\tau)\,d\tau$$

$$= -\bar{C}\int_0^T e^{\bar{A}\tau}\bar{B}\bar{u}(T-\tau)\,d\tau$$

$$e^{\bar{A}\tau} = \sum_{i=0}^{P-1}\alpha_i(\tau)A^i \quad \left\{P \Rightarrow \text{degree of the minimal polynomial of } A\right\}$$

$$\bar{C}e^{\bar{A}T}\bar{x}(0) = -\bar{C}\int_0^T \sum_{i=0}^{P-1}\left(\alpha_i(\tau)\bar{A}^i\right)\bar{B}\bar{u}(T-\tau)\,d\tau$$

$$= -\bar{C}\sum_{i=0}^{P-1}\bar{A}^i\bar{B}\int_0^T \alpha_i(\tau)\bar{u}(T-\tau)\,d\tau$$

$$= -\sum_{i=0}^{P-1}\bar{C}\bar{A}^i\bar{B}\int_0^T \alpha_i(\tau)\bar{u}(T-\tau)\,d\tau$$

Let $Y_{ij} = \int_0^T \alpha_i(\tau)\underbrace{u_j}_{j^{th} \text{ component of vector } u}(T-\tau)\,d\tau$

Let $\bar{B}_j$ be j^{th} column of $\bar{B}$

So $\bar{C}e^{\bar{A}T}\bar{x}(0) = -\sum_{i=0}^{P-1}\bar{C}\bar{A}^i\sum_{j=1}^{r}Y_{ij}\bar{B}_j$

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Let $\bar{Y}_i = \int_0^T \alpha_i(\tau)\,\bar{u}(T-\tau)\,d\tau$

$$\Rightarrow \bar{C} e^{\bar{A}T} \bar{x}(0) = -\sum_{i=0}^{p-1} \bar{C} \bar{A}^i \bar{B} \bar{Y}_i$$

$$= -\left[\bar{C}\bar{B} \;\vdots\; \bar{C}\bar{A}\bar{B} \;\vdots\; \bar{C}\bar{A}^2\bar{B} \;\vdots\; \cdots \;\vdots\; \bar{C}\bar{A}^{p-1}\bar{B}\right] \cdot \begin{bmatrix} \bar{Y}_1 \\ \bar{Y}_2 \\ \vdots \\ \bar{Y}_{p-1} \end{bmatrix}$$

$$\bar{C} e^{\bar{A}T} \bar{x}(0) = -\bar{\bar{Q}}\,\bar{Y}$$

If $\bar{\bar{Q}}$ is of rank m then $\bar{C} e^{\bar{A}T} \bar{x}(0)$ spans the m-dimensional output space.

↓

This means that if the rank of Q is m the $\bar{C}\bar{x}(0)$ also spans the m-dimensional output space and the system is completely output controllable.

CHAPTER 3

Control System Design in State Space

3

Control System design in State Space

10.1) Introduction

→ Pole-placement method
→ Observers
→ Quadratic optimal regulator.
→ Robust Control {Introduction}

{Somewhat similar to the root-locus method}

{The basic difference is that in the (root-locus) design we place only the dominant closed-loop poles at the desired locations while in the (pole placement) design we place all closed-loop poles at desired location.}

10.2) Pole Placement (or Pole-assignment technique)

⇒ We assume all state variables are measurable and are available for feedback.

⇒ If the System Considered is completely state Controllable, then poles of the Closed-loop System may be placed at any desired locations by means of state feedback through an appropriate State feedback gain matrix.

⇒ The present design technique begins with a determination of the desired closed-loop poles based on the transient response & frequency-response requirements.
eg → Speed, damping ratio, bandwidth, as well as steady-state requirement.

⇒ Let us assume that we decided that the desired Closed-loop poles are to be at $s = \mu_1,\ s = \mu_2 \cdots s = \mu_n$.

↳ By choosing an appropriate gain matrix for state feedback it is possible to force the System to have Closed-loop poles at the desired locations, Provided the original System is Completely State Controllable.

> In this Chapter we limit our discussions to Single-input, Single-output Systems.

→ When Control Signal is a Vector quantity, the state feedback gain matrix is not Unique.

★ Design by pole Placement

⇒ There is a Cost associated with placing all closed-loop poles, however, because placing all closed loop poles requires successful measurements of all state Variables or else requires the inclusions of a state observer in the System.

⇒ There is also requirement that the System is Completely State Controllable.

⇒ Consider a Control System be:

$$\dot{X} = \bar{A}X + \bar{B}u$$

$$y = \bar{C}X + Du \qquad (1)$$

- X = State Vector (n-Vector)
- y = output Signal (scalar)
- u = Control Signal (scalar)
- A = $n \times n$ Constant matrix
- B = $n \times 1$ Constant matrix
- C = $1 \times n$ Constant matrix
- D = Constant (scalar)

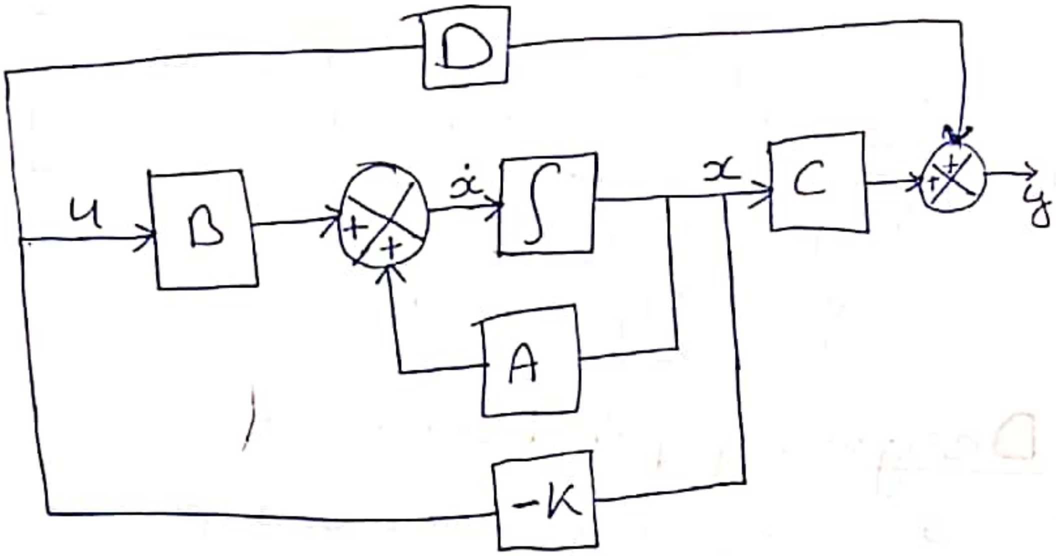
⇒ We choose the Control Signal to be:

$$\boxed{u = -\vec{K}\vec{x}} \quad \text{——} ②$$

$(1 \times n)$
State feedback gain matrix

$(n \times 1)$
State vector

⇒ We assume { (1) all state variable are available for feedback. (2) u is unconstrained }



⇒ This closed-loop System has no input. Its objective is to maintain the zero output.

⇒ Such System where the reference input is always zero is called a regulator system.

(or non zero constant)

⇒ Substituting ② in ① we get:-

$$\dot{x} = Ax + B(-Kx)$$

$$\Rightarrow \dot{x} = (A - KB)x$$

(Solution

$$\boxed{x(t) = e^{(A-BK)t}x(0)} \quad \text{——} ③$$

{ Initial state caused by external disturbances }

⇒ The stability and transient response characteristics are determined by the eigenvalues of the matrix $(A-BK)$.

↳ If matrix K is chosen properly, the matrix $A-BK$ can be made an asymptotically stable matrix for all $x(0)$.

↳ (i.e. $x(t)$ approaches 0 as $t \to \infty$)

⇒ The eigen-values of matrix $A-BK$ are called the regulator poles.

\# If these regulator poles are placed in the left half s-plane, then $x(t) \to 0$ at $t \to \infty$.

* The problem of placing the regulator poles at the desired location is called a pole-placement problem.

★ Necessary and Sufficient Condition for Arbitrary Pole Placement

⇒ Suppose the system of Equation ① is not completely state controllable.

⇒ Then the rank of the controllability matrix is less than n

$$\text{rank}\,[B \,|\, AB \,|\, \cdots \,|\, A^{n-1}B] = q < n$$

⇒ This means that there are q linearly independent column vectors in the controllability matrix.

↳ Let q linearly independent column vectors be $f_1\ f_2 \cdots f_q$

↳ Let the remaining column vectors be $V_{q+1}, V_{q+2} \cdots V_n$

Let, $P = [f_1 \mid f_2 \mid \cdots \mid f_q \mid V_{q+1} \mid V_{q+2} \mid \cdots \mid V_n]$

$\Rightarrow$ By using matrix P as the transformation matrix define

$$\hat{A} = P^{-1}AP \quad , \quad \hat{B} = P^{-1}B$$

$\Rightarrow$ It can be proved that :-

$$\underset{n\times n}{\hat{A}} = \begin{bmatrix} \underset{q\times q}{A_{11}} & \underset{q\times(n-q)}{A_{12}} \\ \underset{(n-q)\times q}{0} & \underset{(n-q)\times(n-q)}{A_{22}} \end{bmatrix} \qquad \underset{n\times 1}{\hat{B}} = \begin{bmatrix} \underset{q\times 1}{B_{11}} \\ \underset{(n-q)\times 1}{0} \end{bmatrix}$$

Let $\hat{K} = KP = [\underset{1\times q}{K_1} \mid \underset{1\times(n-q)}{K_2}]$

Then we have

$$|SI - A + BK| = |P^{-1}(SI - A + BK)P|$$

$$\Rightarrow |SP^{-1}P - P^{-1}AP + P^{-1}BKP|$$

$$\Rightarrow \left| SI - \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} + \begin{bmatrix} B_{11} \\ 0 \end{bmatrix} [K_1 \mid K_2] \right|$$

$$\Rightarrow \begin{vmatrix} SI_q - A_{11} + B_{11}K_1 & -A_{12} + B_{11}K_2 \\ 0 & SI_{n-q} - A_{22} \end{vmatrix}$$

$$\Rightarrow |SI_q - A_{11} + B_{11}K_1|\,|SI_{n-q} - A_{22}| = 0$$

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⇒ Notice that the eigenvalues of A_{22} do not depend on K.

⇒ Thus if the System is not Completely State Controllable, then there are eigenvalues of matrix A that can not be arbitrarily placed.

⇒ To place the eigenvalues of matrix $A-BK$ arbitrarily, the System must be Completely State Controllable. (Necessary Condition)

Sufficient Condition

→ {If the System is completely state controllable, then all eigenvalues of matrix A can be arbitrarily placed.}

⇒ In proving a Sufficient Condition, it is Convenient to transform the state equation given by eq ① into Controllable Canonical form.

⇒ Define a transformation matrix T by

$$T = MW$$

M → Controllability Matrix

$$[B \,\vdots\, AB \,\vdots\, \cdots \,|\, A^{n-1}B]$$

$$W = \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & 1 \\ a_{n-2} & a_{n-3} & \cdots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ a_1 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

{a_i are Coefficient of the characteristic polynomial}

$$|SI-A| = S^n + a_1 S^{n-1} + \cdots + S\, a_{n-1} + a_n$$

Let us define new state vector $\hat{x}$ by

$$x = T\hat{x}$$

⇒ If the rank of the Controllability matrix M is n, then the inverse of matrix T exists.

$$\Rightarrow \dot{\hat{x}} = T^{-1}AT\hat{x} + T^{-1}Bu \quad \text{—} \quad (9)$$

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & & 1 \\ -a_n & -a_{n-1} & \cdots & & -a_1 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

⇒ Thus given a state equation, it can be transformed into the Controllable Canonical form if the system is completely state controllable.

⇒ Let us choose a set of the desired eigenvalues as $\mu_1 \mu_2 \cdots \mu_n$.

⇒ Then the desired characteristic equation becomes:-

$$(s-\mu_1)(s-\mu_2)\cdots(s-\mu_n) = s^n + \alpha_1 s^{n-1} + \cdots \alpha_{n-1}s + \alpha_n = 0$$

⇒ Let us write,

$$KT = [\delta_n \ \delta_{n-1} \ \cdots \ \delta_1]$$

⇒ When $u = -KT\hat{x}$ is used ~~to~~ control the System, eq (9) becomes:-

$$\dot{\hat{x}} = T^{-1}AT\hat{x} - T^{-1}BKT\hat{x}$$

⇒ The Characteristic equation is

$$|SI - T^{-1}AT + T^{-1}BKT| = 0$$

$\Rightarrow$ Now let us simplify the characteristic equation of the system in the controllable canonical form.

$$\left| SI - \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & 1 \\ -a_n & -a_{n-1} & & -a_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} \delta_n & \delta_{n-1} & \cdots & \delta_1 \end{bmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} S & -1 & \cdots & 0 \\ 0 & S & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ a_n + \delta_n & a_{n-1} + \delta_{n-1} & \cdots & S + a_1 + \delta_1 \end{vmatrix} = 0$$

$$\Rightarrow S^n + (a_1 + \delta_1) S^{n-1} + \cdots + (a_{n-1} + \delta_{n-1}) S + (a_n + \delta_n) = 0$$

So $\alpha_1 = (a_1 + \delta_1)$

$\alpha_2 = (a_2 + \delta_2)$

$\vdots$

$\alpha_n = (a_n + \delta_n)$

$$\Rightarrow K = \left[\alpha_n - a_n \,\vdots\, \alpha_{n-1} - a_{n-1} \,\vdots\, \cdots \,\vdots\, \alpha_1 - a_1 \right] T^{-1}$$

$\Rightarrow$ Thus, if the system is completely state controllable, all eigenvalues can be arbitrarily placed by choosing suitable matrix K. {Sufficient Condition}

★ Determination of Matrix K Using Transforming Matrix T

Let the System be defined by:-

$$\dot{x} = Ax + Bu$$

and the Control Signal is given by:-

$$u = -Kx$$

⇒ The feedback gain matrix K forces the eigen value of A−BK to be $\mu_1 . \mu_2 - \mu_n$ (desired values)

{may be real on Complex}

<u>Step 1:</u> Check the controllability condition for the System. If the System is Completely state Controllable, then use the following steps.

<u>Step 2:</u> Form the Characteristic polynomial of matrix A, that is

$$|sI-A| = s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n$$

determine the values of $a_1 . a_2 \cdots a_n$.

<u>Step 3:</u> Determine the transformation matrix T that transforms the system state equation into the Controllable Canonical form.

$$T = MW$$ {M & W are as shown as Previous}

<u>Step 4:</u> Using the desired eigen value, write the desired Characteristic polynomial.

$$(s-\mu_1)(s-\mu_2)\cdots(s-\mu_n) = s^n + \alpha_1 s^{n-1} + \cdots + \alpha_{n-1} s + \alpha_n$$

and determine the values of $\alpha_1 \alpha_2 \cdots \alpha_n$.

Step 5: The required state feedback gain matrix K can be determined as:-

$$K = [\alpha_n - a_n \,\vdots\, \alpha_{n-1} - a_{n-1} \,\vdots\, \cdots \,\vdots\, \alpha_1 - a_1] T^{-1}$$

★ Determination of Matrix K Using Direct Substitution Method

⇒ If the system is of low order ($n \leq 3$), direct substitution of matrix K into the desired characteristic polynomial may be simpler.

⇒ For example if $n = 3$

$$K = [K_1 \; K_2 \; K_3]$$

⇒ Substitute this K matrix into the desired characteristic polynomial and equate it to $(s-\mu_1)(s-\mu_2) \cdots (s-\mu_3)$.

$$|sI - A + BK| = (s-\mu_1)(s-\mu_2)(s-\mu_3)$$

⇒ By equating the coefficients of the like powers of s on both sides, it is possible to determine the values of K_1, K_2 & K_3.

↳ {If system is not completely controllable}, matrix K cannot be determined.

↳ {no solution exists}

★ Determination of Matrix K Using Ackermann's Formula

⇒ Consider the System:-

$$\dot{x} = Ax + Bu \quad — ①$$

⇒ where we use the state feedback control $u = -Kx$. ——②

⇒ We assume that the System is completely State Controllable.

⇒ We also assume that the desired Closed-loop Poles are at $S = \mu_1$, $S = \mu_2$ - - - $S = \mu_n$.

⇒ Using ② in ① we get:-

$$\dot{x} = (A - BK)x$$

⇒ Let us define,

$$\tilde{A} = A - BK$$

⇒ The desired Characteristic equation is

$$|SI - A + BK| = |SI - \tilde{A}| = (s-\mu_1)(s-\mu_1)' \cdots (s-\mu_n)$$

$$= s^n + \alpha_1 s^{n-1} + \cdots \alpha_{n-1} s + \alpha_n = 0$$

⇒ Using Cayley-Hamilton theorem we get.

$$\phi(\tilde{A}) = \tilde{A}^n + \alpha_1 \tilde{A}^{n-1} + \cdots \alpha_{n-1}\tilde{A} + \alpha_n I = 0 \quad ③$$

⇒ To Simplify the derivation, we consider the Case when $n = 3$. The following derivation can easily be extended for any value of n.

⇒ Consider the following:-

- $I = I$
- $\tilde{A} = A - BK$
- $\tilde{A}^2 = (A-BK)^2 = A^2 - 2ABK + (BK)^2$
  $\Rightarrow A^2 - ABK - BK(A-BK)$
  $\Rightarrow A^2 - ABK - BK\tilde{A}$
- $\tilde{A}^3 = (A-BK)^3 = A^3 - A^2BK - ABK\tilde{A} - BK\tilde{A}^2$

⇒ Multiplying the preceding equations in order by α_3 α_2 α_1 & α_0 $(\alpha_0 = 1)$ and adding the result.

$$\alpha_3 I + \alpha_2 \tilde{A} + \alpha_1 \tilde{A}^2 + \tilde{A}^3.$$

$$= \alpha_3 I + \alpha_2 (A-BK) + \alpha_1 (A^2 - ABK - BK\tilde{A}) + A^3 - A^2BK - ABK\tilde{A} - BK\tilde{A}^2$$

$$= \alpha_3 I + \alpha_2 A + \alpha_1 A^2 + A^3 - \alpha_2 BK - \alpha_1 ABK - \alpha_1 BK\tilde{A} - A^2BK - ABK\tilde{A} - BK\tilde{A}^2$$

we know, $\phi(A) \neq 0$ & $\phi(\tilde{A}) = 0$

$$\phi(\tilde{A}) = \phi(A) - \alpha_2 BK - \alpha_1 BK\tilde{A} - BK\tilde{A} - \alpha_1 ABK - ABK\tilde{A} - A^2BK$$

(3)

$$\Rightarrow \phi(A) = B(\alpha_2 K + \alpha_1 K\tilde{A} + K\tilde{A}^2) + AB(\alpha_1 K + K\tilde{A}) + A^2BK$$

$$= [B \,\vdots\, AB \,\vdots\, A^2B] \begin{bmatrix} \alpha_2 K + \alpha_1 K\tilde{A} + K\tilde{A}^2 \\ \alpha_1 K + K\tilde{A} \\ K \end{bmatrix}$$

⇒ Since the system is completely state controllable, the inverse of the controllability matrix exists.

$$[B \,\vdots\, AB \,\vdots\, A^2B]^{-1}\phi(A) = \begin{bmatrix} \alpha_2 K + \alpha_1 K\hat{A} + K\hat{A}^2 \\ \alpha_1 K + K\hat{A} \\ K \end{bmatrix}$$

⇒ Premultiplying both side of this last equation ~~of~~ by [0 0 1] we obtain:-

$$K = [0\ 0\ 1]\,[B \,\vdots\, AB \,|\, A^2B]\,\phi(A)$$

⇒ For any arbitrary positive integer n we have:-

$$\boxed{K = [0\ 0 \cdots 1]\,[B \,\vdots\, AB \,\vdots \cdots |\, A^{n-1}B]^{-1}\phi(A)}$$

→ {Ackermann's formula for the determination of the state feedback matrix K}

Controllers

- Regulator system {Reference input is constant including zero}
- Control system {when reference input is time varying}

★ Choosing the Locations of Desired Closed-loop Poles

The most frequently used approach is to choose such poles based on experience in the root-locus design.

→ **Note**: If we place the dominant closed loop poles far from the $j\omega$ axis, so that the

System response become very fast, the Signal in the System become very large, with the result that the System may become non-linear. This should be avoided.

Another approach is based on the <u>quadratic optimal Control approach</u>.

→ This approaches will determine the desired closed loop poles such that it balances between the acceptable response and the amount of Control energy required.

10.4> Design of Servo System

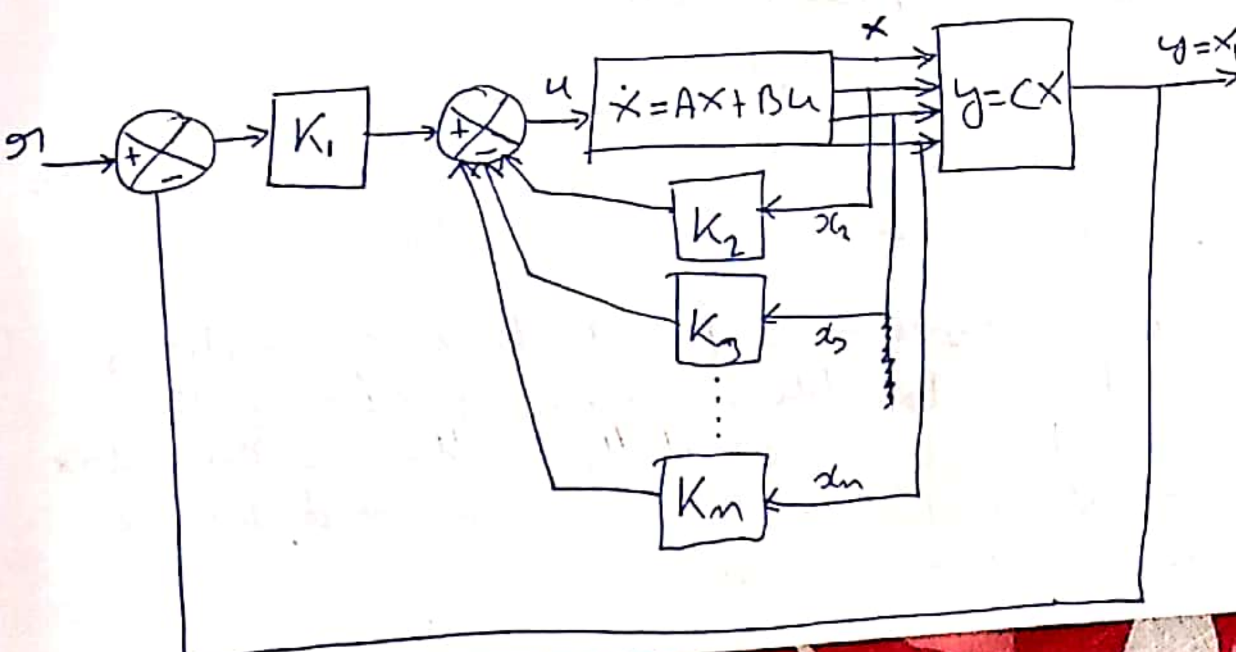
★ <u>Design of Type 1 Servo System when the Plant has an Integrator</u>

Let the plant be defined by:-

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

⇒ By proper choice of a set of state variables, it is possible to choose the output to be equal to one of the state variable.

↳ Let us assume $y = x_1$



⇒ In present analysis we assume that the reference input r is a step function.

$$u = -\begin{bmatrix} 0 & K_2 & K_3 & \cdots & K_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + K_1 (r - x_1)$$

$$u = -Kx + K_1 r \quad \longrightarrow \quad [K_1 \; K_2 \; \cdots \; K_n]$$

So, ⇒ $\dot{x} = Ax + Bu = \cancel{Ax}\,(A-BK)x + BK_1 r$

$$\boxed{\dot{x} = (A-BK)x + BK_1 r}$$

⇒ $y(\infty) \to r$ ~~ds~~ $u(\infty) \to 0$ {r = step input}

<u>at Steady State:</u> $\dot{x}(\infty) = (A-BK)x(\infty) + BK_1 r(\infty)$

$r(\infty) \to r$ (const)

$$\dot{x}(t) - \dot{x}(\infty) = (A-BK)[x(t) - x(\infty)] \qquad r(t) \; \forall \, t > 0$$

Let $e(t) = x(t) - x(\infty)$

So, above condition becomes:

$$\boxed{\dot{e} = (A-BK)e} \quad \text{— (1)}$$

eq (1) describes the error dynamics.

⇒ If the System defined above is completely State Controllable then by specifying the desired eigenvalues $\mu_1 \mu_2 \cdots \mu_n$ for the matrix A-BK matrix K can be determined by the Pole placement technique.

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⇒ The steady state values of $x(t)$ & $u(t)$ can be found out as follows:-

$$\dot{x}(\infty) = 0 = (A-BK)x(\infty) + BK_1 r$$

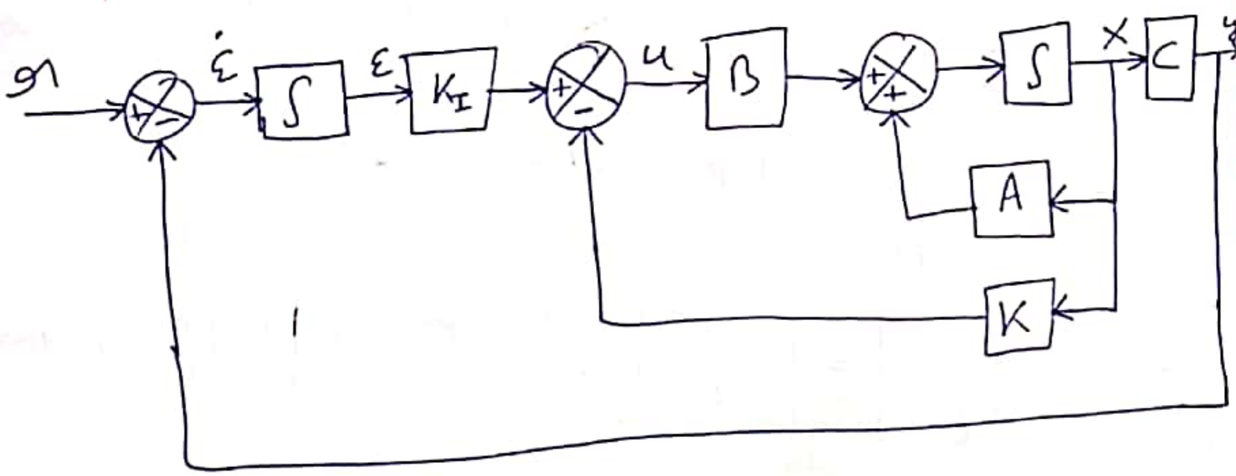
$$\Rightarrow \boxed{x(\infty) = -(A-BK)^{-1} BK_1 r}$$

&

$$\boxed{u(\infty) = -Kx(\infty) + K_1 r}$$

★ Design of Type 1 Servo System when the plant has no Integrator.

⇒ If the plant has no integrator (type 0 plant), the basic principle of the design of a type 1 servo system is to insert an integrator in the feed forward path between the error comparator and the plant as shown below.



⇒ From diagram we obtain:

$$\boxed{\dot{x} = Ax + Bu} \quad \text{—— (1)}$$

$$y = Cx$$

$$u = -Kx + K_I\varepsilon$$

$$\dot{\varepsilon} = r - y = r - Cx \quad \text{—— (2)}$$

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⇒ Assume that the reference input (step function) is applied at $t=0$. Then for $t>0$, the system dynamics can be determined by an equation that is combination of eq ① & ②

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\varepsilon}(t) \end{bmatrix} = \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ \varepsilon(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r(t) \quad ③$$

⇒ We shall design an asymptotically stable system such that $x(\infty)$, $\varepsilon(\infty)$ & $u(\infty)$ approach constant values, respectively

→ At steady state $\dot{\varepsilon}(t)=0$ & $y(\infty) = r$

⇒ At steady state we have:-

$$\begin{bmatrix} \dot{x}(\infty) \\ \dot{\varepsilon}(\infty) \end{bmatrix} = \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix} \begin{bmatrix} x(\infty) \\ \varepsilon(\infty) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(\infty) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r(\infty) \quad ④$$

⇒ as $r(t)$ ⇒ Step Input ⇒ $r(\infty) = r(t) = r$ (const.)

⇒ Subtracting eq ④ from eq ③ we get

$$\begin{bmatrix} \dot{x}(t) - \dot{x}(\infty) \\ \dot{\varepsilon}(t) - \dot{\varepsilon}(\infty) \end{bmatrix} = \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix} \begin{bmatrix} x(t) - x(\infty) \\ \varepsilon(t) - \varepsilon(\infty) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} [u(t) - u(\infty)]$$

Let us define; $x(t) - x(\infty) = x_e(t)$

$\varepsilon(t) - \varepsilon(\infty) = \varepsilon_e(t)$

$u(t) - u(\infty) = u_e(t)$

$$\Rightarrow \begin{bmatrix} \dot{x}_e(t) \\ \dot{\varepsilon}_e(t) \end{bmatrix} = \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix} \begin{bmatrix} x_e(t) \\ \varepsilon_e(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u_e(t)$$

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Where, $u_e(t) = -K x_e(t) + K_I \varepsilon_e(t)$

Let us define :-

(3) $e(t) = \begin{bmatrix} x_e(t) \\ \varepsilon_e(t) \end{bmatrix}$ $\hat{A} = \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix}$ $\hat{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}$

⇒ So previous equation become:-

$$\dot{e} = \hat{A} e + \hat{B} u_e \quad \text{— (5)}$$

$$\left\{ u_e = -\hat{K} e \text{ — (6)}, \; \hat{K} = [K \; \vdots \; -K_I] \right\}$$

⇒ The State error equation can be obtained by substituting eq (6) in (5)

$$\dot{e} = (\hat{A} - \hat{B}\hat{K}) e$$

⇒ If the desired eigenvalues of matrix $\hat{A} - \hat{B}\hat{K}$ are specified as $\mu_1, \mu_2 \cdots \mu_{n+1}$, then the state feedback gain matrix K and the integral gain constant K_I can easily be determined by the pole-placement technique.

⇒ Not all State variables can be measured directly. If this is the case, we need to use a <u>State observer</u>.

10.5) State Observers

⇒ In the pole-placement approach to the design of control system, we assumed that all state variables are feedback. ~~Exponents~~

⇒ In practice however, not all state variables are available for feedback.

⇒ Estimation of unmeasurable state variable is commonly called observation.

↳ A device (or computer program) that estimates or observes the state variables is called a state observer.

Full order state observer → If the state observer observes all state variables of the system.

Reduced order state observer → If the state observer observes not all state variables of the system.

Minimum order state observer → If the order of the reduced order state observer is the minimum possible it is called minimum order state observer.

★ State Observer

↳ State observer estimates the state variables based on the measurements of the output and control variables.

⇒ We shall use $\tilde{x}$ to designate the observed state vector.

⇒ Consider the plant defined by

$$\dot{x} = Ax + Bu \quad \text{—①}$$
$$y = Cx \quad \text{—②}$$

⇒ The mathematical model of the observer is basically the same as that of the plant, except that we include additional term that includes the estimation error to compensate for inaccuracy in Matrix A & B & the lack of the initial error.

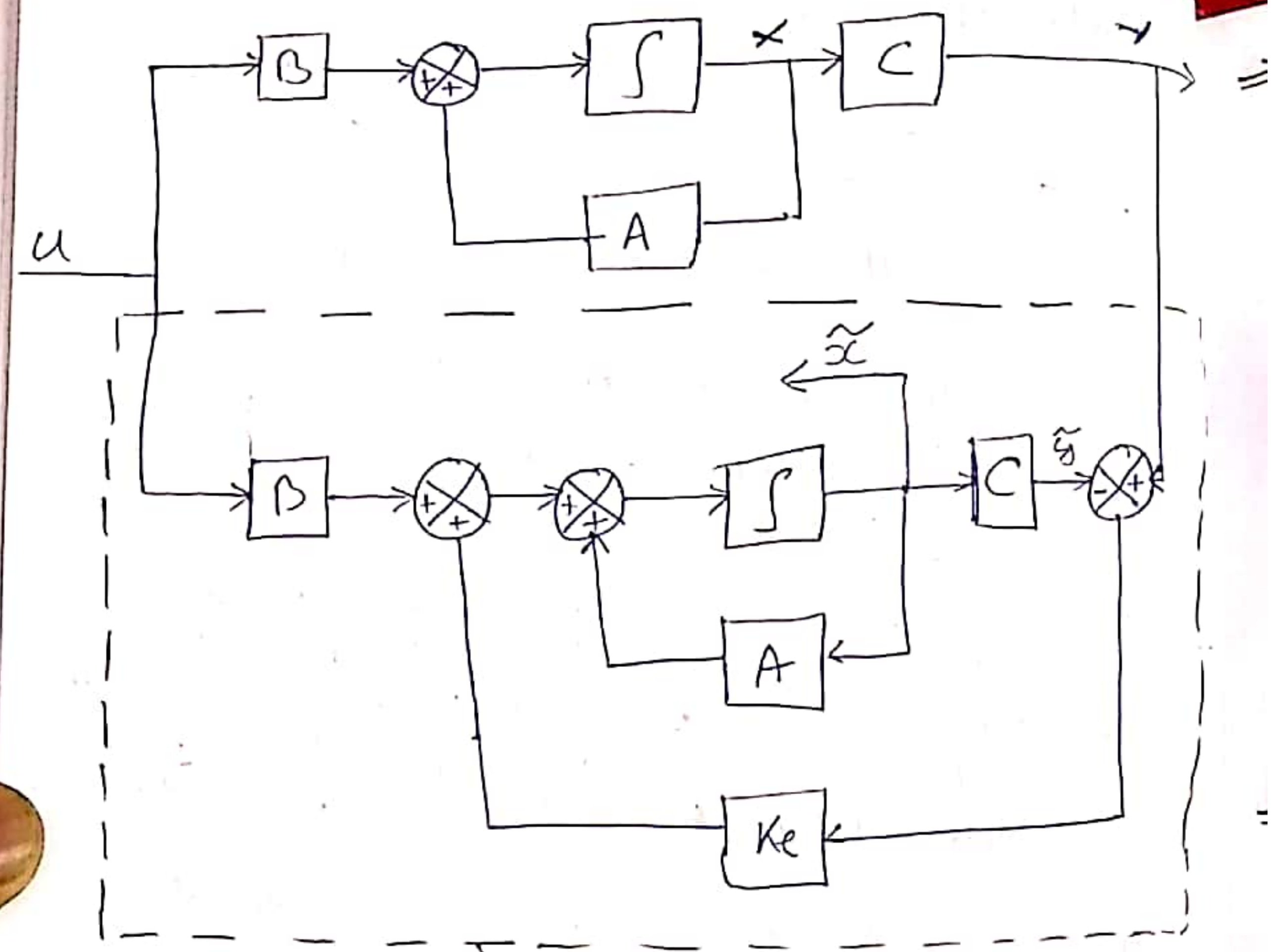
⇒ Thus we define the mathematical model of the observer to be:

$$\dot{\tilde{x}} = A\tilde{x} + Bu + K_e(y - C\tilde{x}) \quad \text{—③}$$
$$= (A - K_eC)\tilde{x} + Bu + K_e y$$

⇒ Matrix K_e is called observer gain matrix.

$\tilde{x} \rightarrow$ estimated state
$C\tilde{x} \rightarrow$ estimated output
$y \rightarrow$ actual output
$u \rightarrow$ actual input.

↳ This term continuously corrects the model output & improves the performance of observer.



{Full-order state observer}

★ <u>Full-order State observer</u>

⇒ Consider plant is described by eq ① & ② & Observer by eq ③.

⇒ Substracting eq ① & ③ we get:-

$$\dot{x} - \dot{\tilde{x}} = Ax - A\tilde{x} - K_e(Cx - C\tilde{x})$$

$$= (A - K_eC)(x - \tilde{x})$$

Let $e = x - \tilde{x}$

⇒ $\dot{e} = (A - K_eC)e$ — ④

⇒ From eq (5) we see that dynamic behavior of the error vector is determined by the eigen-values of matrix $(A-K_eC)$.

↳ If matrix $(A-K_eC)$ is a stable matrix the error vector will converge to zero for any initial error vector $e(0)$.

↳ If the eigenvalues of matrix $A-K_eC$ are chosen in such a way that the dynamic behaviour of the error vector is asymptotically stable and is adequately fast, then any error vector will tend to zero with an adequate speed.

⇒ If the plant is completely observable, then it can be proved that it is possible to choose matrix Ke such that $A-K_eC$ has arbitrarily desired eigen value.

★ Dual Problem

⇒ The problem of designing a full-order observer becomes that of determining the observer gain matrix Ke such that the error dynamics are asymptotically stable with sufficient speed of response.

↳ (ie determining an appropriate Ke such that $A-K_eC$ has desired eigen value)

⇒ In designing the full-order state observer, we may solve the dual problem, that is, solve the pole-placement problem for the dual system.

$$\dot{Z} = A^T Z + C^T V$$
$$\eta = B^T Z$$

assume the Control Signal V to be
$$V = -KZ$$

⇒ If the dual system is completely state Controllable, then the State feedback gain matrix K can be determined such that matrix $A^T - C^T K$ will yield a set of desired eigen values.

⇒ If $\mu_1\ \mu_2 \cdots \mu_n$ are the desired eigen-values of the State observer matrix, then by taking the same μ_i's as the desired eigen Values of the State - feedback matrix of the dual System, we obtain:-

$$|SI - (A^T - C^T K)| = (S-\mu_1)(S-\mu_2)\cdots(S-\mu_n)$$

⇒ eigen Value of $A^T - C^T K$ are same as eigen Value of $A - K^T C$.

We find $\boxed{K_e = K^T}$

⇒ Thus, using the matrix K determined by the pole-placement approach in the dual System, the observer gain matrix Ke for the Original System can be determined by using the relationship $K_e = K^T$.

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* Transformation Approach to Obtain State Observer Gain Matrix K_e

By following the same approach as we used in deriving the equation for the state feedback gain matrix K, we can obtain

$$K_e = Q \begin{bmatrix} \alpha_n - a_n \\ \alpha_{n-1} - a_{n-1} \\ \vdots \\ \alpha_1 - a_1 \end{bmatrix} = (WN^T)^{-1} \begin{bmatrix} \alpha_n - a_n \\ \alpha_{n-1} - a_{n-1} \\ \vdots \\ \alpha_1 - a_1 \end{bmatrix}$$

$$Q = (WN^T)^{-1}$$

$$W = \begin{bmatrix} a_{n-1} & a_{n-1} & \cdots & a_1 & 1 \\ a_{n-2} & a_{n-1} & & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ a_1 & 1 & & 0 & 0 \\ 1 & 0 & & 0 & 0 \end{bmatrix}$$

$$N = \left[C^T \,\vdots\, A^TC^T \,\vdots\, \cdots \,(A^T)^{n-1}C^T \right]$$

* Direct-Sustitution Approach to obtain State Observer gain Matrix K_e

If the system is of low order, then direct substitution of matrix K_e into the desired characteristic polynomial may be simpler.

ex ⇒ If x is a 3-vector

$$K_e = \begin{bmatrix} K_{e1} \\ K_{e2} \\ K_{e3} \end{bmatrix}$$

$$|SI - (A - K_eC)| = (S-\mu_1)(S-\mu_1)(S-\mu_3)$$

⇒ By equating the coefficients of the like power of s on both side, we get K_{e1}, K_{e2} & K_{e3}.

★ Ackermann's Formula

⇒ Consider the System defined by

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

$$K = [0\ 0 \cdots 0\ 1]\left[B \vdots AB \vdots \cdots \vdots A^{n-1}B\right]^{-1} \phi(A)$$

{from pole placement}

⇒ For the dual of the System:

$$\dot{z} = A^T z + C^T v$$
$$n = B^T z$$

⇒ The preceding Ackermann's formula for Pole placement can be modified to

$$K = [0\ 0 \cdots 1]\left[C^T \vdots A^T C^T \vdots \cdots \vdots (A^T)^{n-1} C^T\right]^{-1} \phi(A^T)$$

⇒ As stated earlier $K_e = K^T$

$$K_e = K^T = \phi(A^T)^T \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-2} \\ CA^{n-1} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$\boxed{K_e = \phi(A) \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-2} \\ CA^{n-1} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}$$

where $\phi(s)$ is the desired characteristic polynomial for the state observer.

$$\phi(s) = (s-\mu_1)(s-\mu_2)\cdots(s-\mu_n)$$

★ Comment on Selecting best K_e

⇒ The choice of a set of $\mu_1, \mu_2 \cdots \mu_n$ is in many instances, not unique.

↳ As a general rule however, the observer poles must be two to five times faster than the controller poles to make sure the observation error converges to zero quickly.

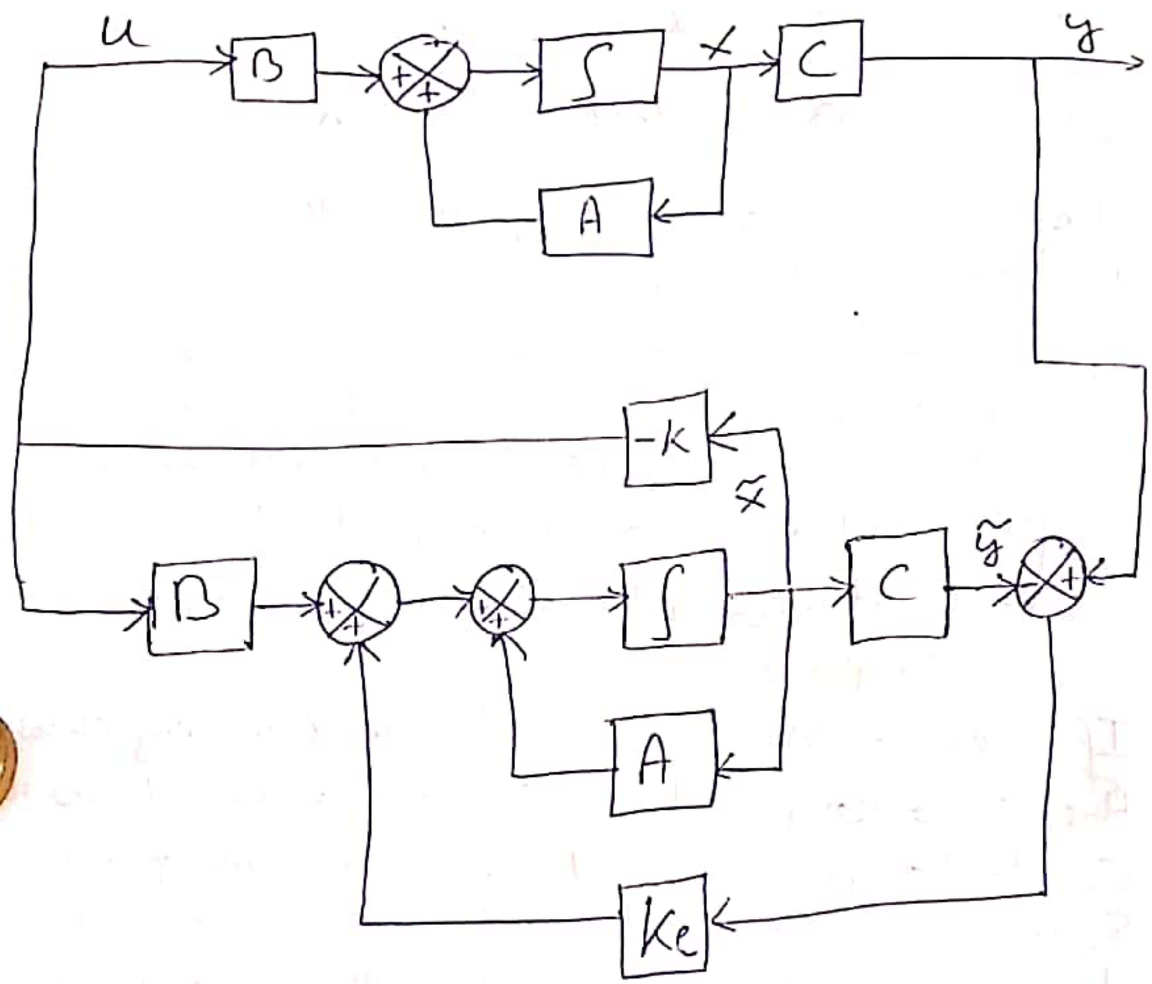
↳ Controller poles dominate the system response.

⇒ If sensor noise is considerable, we may choose the observer poles to be slower than two times the controller poles, so that the bandwidth of the system will become lower & smooth the noise.

↳ Response will be strongly influenced by the observer poles.

⇒ In many practical cases, the selection of the best matrix K_e boils down to a compromise between speedy response & sensitivity to disturbance & noises.

★ Effects of the Addition of the observer on a Closed-Loop System



Observed-state feedback Control System

⇒ The design process, therefore, becomes a two stage process, i.e.

1) The first stage being the determination of the feedback gain matrix K to yield the desired characteristic equation.

2) Second stage being the determination of the observer gain matrix K_e to yield the desired observer characteristic equation.

⇒ Let us now investigate the effects of the use of the observed state $\tilde{x}(t)$, rather than the actual state $x(t)$, on the characteristic equation of a closed-loop control system.

⇒ Consider the completely state controllable & completely observable system defined by equations:

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

⇒ For the state-feedback control based on the observed state $\tilde{x}$.

$$u = -K\tilde{x}$$

⇒ With the control, the state equation become

$$\dot{x} = Ax - BK\tilde{x} = (A-BK)x + BK(x-\tilde{x})$$

Let $e(t) = x(t) - \tilde{x}(t)$

$$\Rightarrow \dot{x} = (A-BK)x + BKe \quad — ①$$

⇒ Observer error is given by equation as:

$$\dot{e} = (A-K_eC)e \quad — ②$$

Combining eq ① & ② we get:-

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A-BK & BK \\ 0 & A-K_eC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} \quad — ③$$

⇒ Equation ③ describes the dynamics of the observed-state feedback control system.

⇒ The characteristic equation for the system is:-

$$\begin{vmatrix} SI - A + BK & -BK \\ 0 & SI - A + K_e C \end{vmatrix} = 0$$

$$\Rightarrow |SI - A + BK| \, |SI - A + K_e C| = 0$$

⇒ <u>Note</u>: Closed-loop poles of the observed state feedback control system consists of the poles due to the:-

→ Pole-placement design done

→ Pole - due to the observed design done.

⇒ This means that the pole-placement design and the observer design are independent of each other.

↳ They can be designed separately & combined.

★ <u>Transfer Function of the Observer-Based Controller</u>

⇒ Consider the plant defined by:-

$$\dot{x} = Ax + Bu \quad \text{——} \; (1)$$

$$y = Cx \quad \text{——} \; (2)$$

⇒ Use observed-state feedback control

$$u = -K\tilde{x}. \quad \text{——} \; (3)$$

⇒ Then equations for the Observer are given by:-

$$\dot{\tilde{x}} = (A - K_e C - BK)\tilde{x} + K_e y \quad \text{—④}$$

$$u = -K\tilde{x} \quad \text{—⑤}$$

⇒ By taking laplace transform of eq ④ with zero initial Condition we get:-

$$\tilde{X}(s) = (sI - A + K_e C + BK)^{-1} K_e Y(s)$$

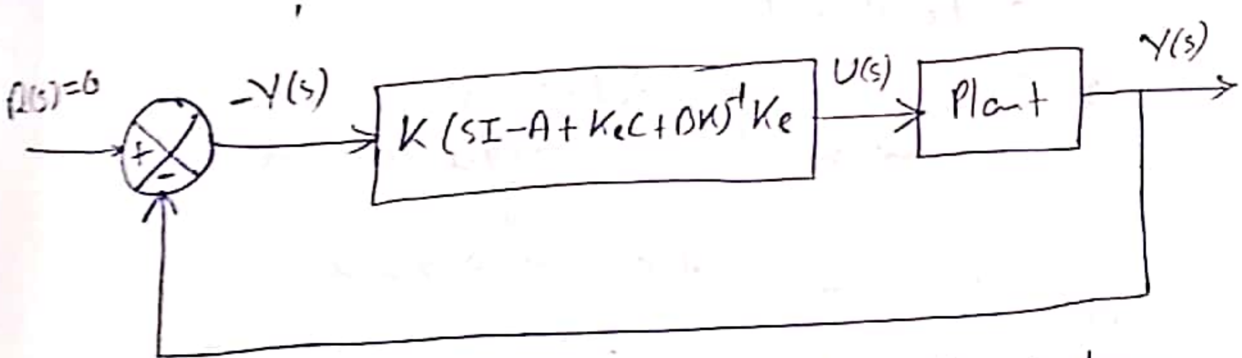
⇒ By substituting this $\tilde{X}(s)$ from Laplace transform of Eq ⑤ we obtain:-

$$U(s) = -K(sI - A + K_e C + BK)^{-1} K_e Y(s)$$

⇒ Then the TF $U(s)/Y(s)$ can be obtained as:-

$$\frac{U(s)}{Y(s)} = -K(sI - A + K_e C + BK)^{-1} K_e$$

⇒ Block diagram representation of the System.



⇒ The TF as a Controller for the System.

★ Minimum-Order Observer

⇒ The Observers discussed So far are designed to reconstruct all the State variables.

- ↳ In practice, Some of the State Variables may be accurately measured.
- ↳ Such accurately measured State variable need not be estimated.

⇒ Suppose

- → State vector x is an n-vector
- → Output Vector y is m-Vector

→ Since m output variables are linear combination of the state variable

↓

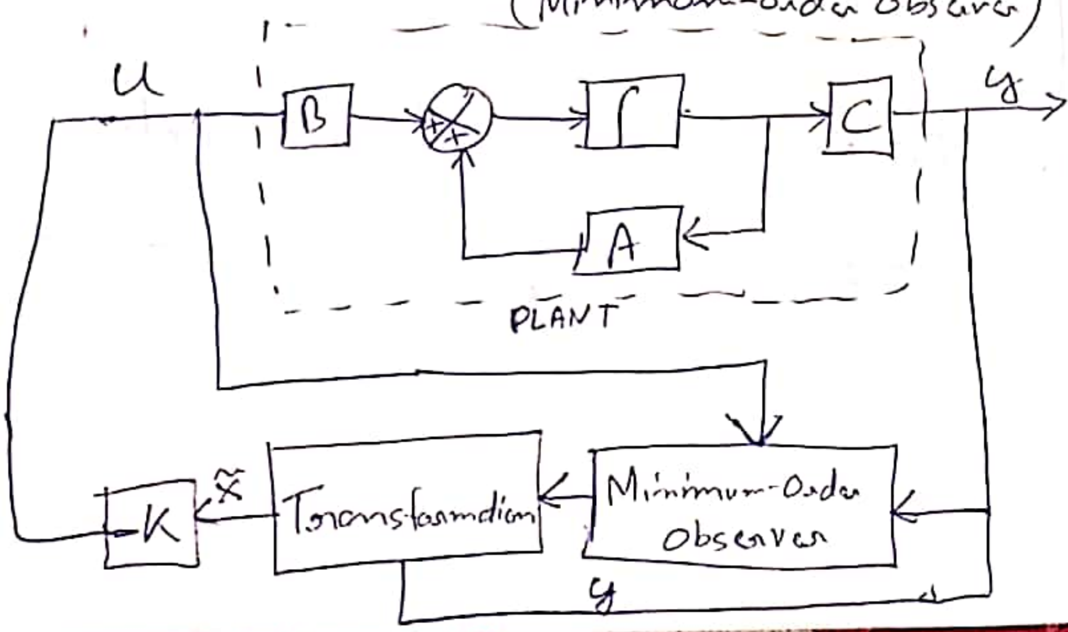
(m State variable need not be estimated)

→ We need to estimate only n−m State variables.

→ Then the reduced-order observer becomes an $(n-m)^{th}$ order observer.

↓

(Minimum-order observer)



⇒ To present the basic idea of the minimum-order observer, without ~~unduly~~ mathematical complications, we shall present the case where the output is scalar.

↳ Consider the System

$$\dot{X} = AX + Bu$$
$$y = CX$$

⇒ Let state vector X be partitioned into two part as X_a (scalar) & X_b [an $(n-1)$ vector].

↳ Hence state variable x_a is equal to the output y and thus can be directly measured.

{ Unmeasurable Portion of State Vector }

⇒ Then the partitioned state & output equation become:-

$$\begin{bmatrix} \dot{x}_a \\ \dot{x}_b \end{bmatrix} = \left[\begin{array}{c|c} A_{aa} & A_{ab} \\ \hline A_{ba} & A_{bb} \end{array}\right] \left[\begin{array}{c} X_a \\ \hline X_b \end{array}\right] + \left[\begin{array}{c} B_a \\ \hline B_b \end{array}\right] u$$

$$y = \left[\begin{array}{c|c} 1 & 0 \end{array}\right] \left[\begin{array}{c} X_a \\ \hline X_b \end{array}\right]$$

$$\dot{x}_a - A_{aa}x_a - B_a u = A_{ab} X_b \quad \text{(a)}$$

{ Measured Portion }

$$\dot{x}_b = A_{ba}x_a + A_{bb}x_b + B_b u \quad \text{(b)}$$

{ Unmeasured Portion }

⇒ Let us compare the state equation for the full-order observer with that for the minimum-order observer

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⇒ The state equation for the full-order observer is

$$\dot{x} = Ax + Bu \quad \text{—— (1)}$$

⇒ and the state equation for the minimum order observer is

$$\dot{x}_b = A_{bb}x_b + A_{ba}x_a + B_b u \quad \text{—— (2)}$$

⇒ The output equation for the full-order observer is

$$y = Cx \quad \text{—— (3)}$$

⇒ and the "output equation" for the minimum-order observer is

$$\dot{x}_a - A_{aa}x_a - B_a u = A_{ab}x_b \quad \text{—— (4)}$$

Full-order State Observer	Minimum-Order State Observer
$\tilde{x}$	$\tilde{x}_b$
A	A_{bb}
Bu	$A_{ba}x_a + B_b u$
y	$\dot{x}_a - A_{aa}x_a - B_a u$
C	A_{ab}
K_e [$n \times 1$ matrix]	K_e [$(n-1) \times 1$ matrix]

⇒ Observer equation for full order observer:

$$\dot{\tilde{x}} = (A - K_e C)\tilde{x} + Bu + K_e y \quad — (5)$$

⇒ By making substitutions from table we get:-

$$\dot{\tilde{x}}_b = (A_{bb} - K_e A_{ab})\tilde{x}_b + A_{ba} x_a + B_b u + K_e(\dot{x}_a - A_{aa}x_a - B_a u) \quad — (6)$$

⇒ If x_a is noisy the use of $\dot{x}_a$ is unacceptable.

⇒ To avoide this difficulty, we eliminate $\dot{x}_a$ in the following way:-

$$\dot{\tilde{x}}_b - K_e \dot{x}_a = (A_{bb} - K_e A_{ab})\tilde{x}_b + (A_{ba} - K_e A_{aa})y + (B_b - K_e B_a)u$$

$$\Rightarrow (A_{bb} - K_e A_{ab})(\tilde{x}_b - K_e y) + \left[(A_{bb} - K_e A_{ab})K_e + A_{ba} - K_e A_{aa}\right]y + (B_b - K_e B_a)u$$

Let us define $\quad x_b - K_e y = x_b - K_e x_a = \eta$

$$\tilde{x}_b - K_e y = \tilde{x}_b - K_e x_a = \hat{\eta}$$

$$\Rightarrow \dot{\hat{\eta}} = (A_{bb} - K_e A_{ab})\hat{\eta} + \left[(A_{bb} - K_e A_{ab})K_e + A_{ba} - K_e A_{aa}\right]y + (B_b - K_e B_a)u$$

Let us define $\quad \hat{A} = A_{bb} - K_e A_{ab}$

$$\hat{B} = \hat{A}K_e + A_{ba} - K_e A_{aa}$$

$$\hat{F} = B_b - K_e B_a$$

$$\boxed{\dot{\tilde{\eta}} = \hat{A}\tilde{\eta} + \hat{B}y + \hat{F}u} \quad \text{—} (8)$$

$$\tilde{x} = \begin{bmatrix} x_a \\ \tilde{x}_b \end{bmatrix} = \begin{bmatrix} y \\ \tilde{x}_b \end{bmatrix} = \begin{bmatrix} 0 \\ I_{n-1} \end{bmatrix} [\tilde{x}_b - K_e y] + \begin{bmatrix} 1 \\ K_e \end{bmatrix} y$$

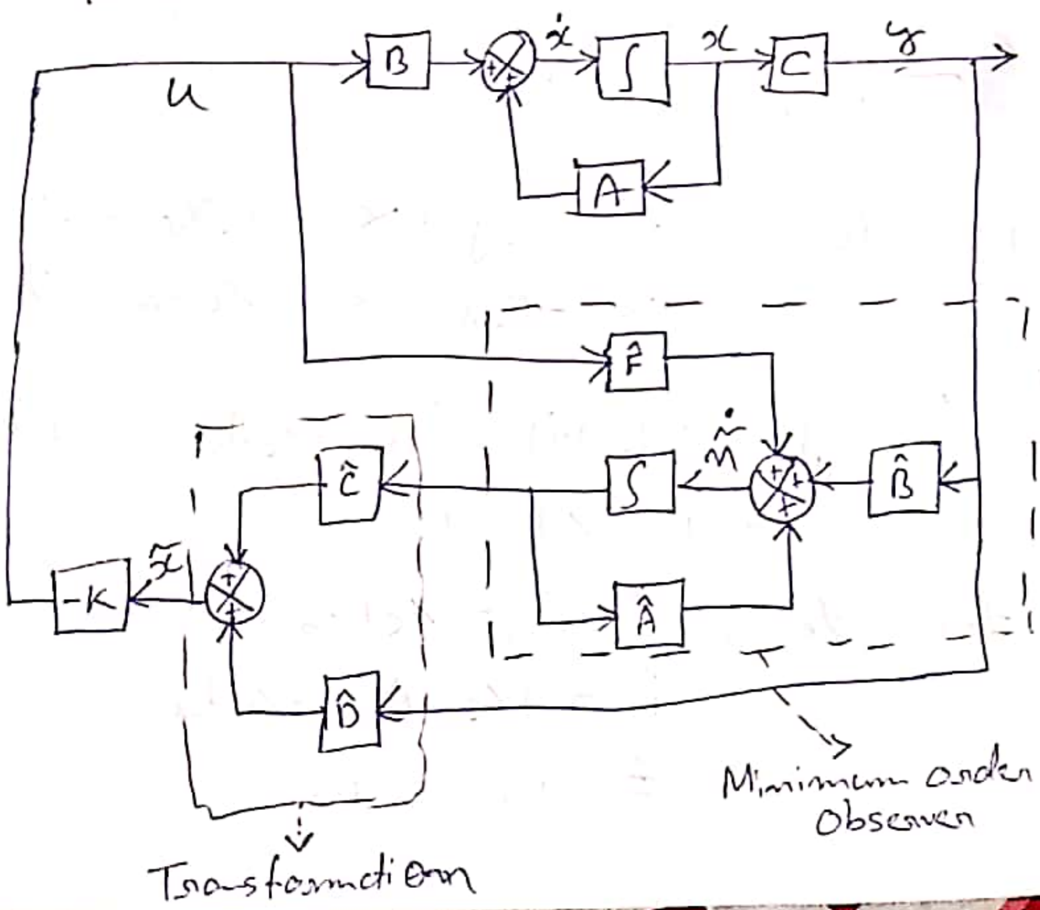
0 is row vector consisting of $(n-1)$ zeros.

Let $\hat{C} = \begin{bmatrix} 0 \\ I_{n-1} \end{bmatrix}$ $\quad \hat{D} = \begin{bmatrix} 1 \\ K_e \end{bmatrix}$

⇒ then $\tilde{x}$ in terms of $\tilde{\eta}$ & y:-

$$\boxed{\tilde{x} = \hat{C}\tilde{\eta} + \hat{D}y}$$

⇒ This equation gives the transformation from $\tilde{\eta}$ to $\tilde{x}$.



⇒ Next we shall derive the observer error equation.
{using eq 8 eq 6 can be modified as}

$$\dot{\tilde{x}}_b = (A_{bb} - K_e A_{ab})\tilde{x}_b + A_{ba}x_a + B_b u + K_e A_{ab} x_b \quad \text{--- (9)}$$

⇒ By subtracting eq 9 & 6 we obtain:-

$$\dot{x}_b - \dot{\tilde{x}}_b = (A_{bb} - K_e A_{ab})(x_b - \tilde{x}_b)$$

Let $e = x_b - \tilde{x}_b = n - \hat{n}$

$$\Rightarrow \boxed{\dot{e} = (A_{bb} - K_e A_{ab})e} \quad \text{--- (10)}$$

⇒ Provided the rank of matrix

$$\begin{bmatrix} A_{ab} \\ A_{ab}A_{bb} \\ \vdots \\ A_{ab}A_{bb}^{n-2} \end{bmatrix}$$

is $n-1$. → {This is the complete observability condition applicable to the minimum order observer}

⇒ The characteristic equation for the minimum order observer is obtained from eq 10 as:-

$$|SI - A_{bb} + K_e A_{ab}| = (s-\mu_1)(s-\mu_2)\cdots(s-\mu_{n-1})$$
$$= s^{n-1} + \hat{\alpha}_1 s^{n-2} + \cdots \hat{\alpha}_{n-2} s + \hat{\alpha}_{n-1} = 0$$

where $\mu_1, \mu_2 \cdots \mu_{n-1}$ are desired eigen value for the minimum-order observer.

$$K_e = \hat{Q} \begin{bmatrix} \hat{\alpha}_{n-1} - \hat{a}_{n-1} \\ \hat{\alpha}_{n-2} - \hat{a}_{n-2} \\ \vdots \\ \hat{\alpha}_1 - \hat{a}_1 \end{bmatrix} = (\hat{W}\hat{N}^T)^{-1} \begin{bmatrix} \hat{\alpha}_{n-1} - \hat{a}_{n-1} \\ \hat{\alpha}_{n-2} - \hat{a}_{n-2} \\ \vdots \\ \hat{\alpha}_1 - \hat{a}_1 \end{bmatrix}$$

$$\hat{N} = \left[A_{ab}^T \,\vdots\, A_{bb}^T A_{ab}^T \,\vdots\, \cdots \,\vdots\, A_{bb}^{T\,n-2} A_{ab}^T \right]$$

$$\hat{W} = \begin{bmatrix} \hat{a}_{n-2} & \hat{a}_{n-3} & \cdots & \hat{a}_1 & 1 \\ \hat{a}_{n-1} & \hat{a}_{n-4} & \cdots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ \hat{a}_1 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

⇒ $\hat{a}_1, \hat{a}_2 \cdots \hat{a}_{n-2}$ are coefficients in the characteristics equation for the state eqution

$$|IS - A_{bb}| = S^{n-1} + \hat{\alpha}_1 S^{n-2} + \cdots + \hat{a}_{n-2} S + \hat{a}_{n-1} = 0$$

⇒ Ackermann's formula can be modified to :-

$$K_e = \phi(A) \begin{bmatrix} A_{ab} \\ A_{ab} A_{bb} \\ \vdots \\ A_{ab} A_{bb}^{n-3} \\ A_{ab} A_{bb}^{n-2} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Where $\phi(A_{bb}) = A_{bb}^{n-1} + \hat{\alpha}_1 A_{bb}^{n-2} + \cdots + \hat{\alpha}_{n-2} A_{bb} + \hat{\alpha}_{n-1} I$

* Observed-State feedback Control System with Minimum-Order Observer

⇒ The closed-loop poles of the observed-state feedback control system with a minimum order ~~poles of~~ observer comprise of:-

(1) Closed loop poles due to pole placement

(2) Closed loop poles due to the minimum order observer

⇒ The system characteristic equation can be derived as:-

$$|SI - A + BK|\,|SI - A_{bb} + K_e A_{ab}| = 0$$

⇒ Therefore, the pole-placement design and the minimum order observer are independent of each other.

* Transfer function of Minimum-order observer based Controller

⇒ minimum-order observer equation is given by

$$\dot{\tilde{\eta}} = (A_{bb} - K_e A_{ab})\tilde{\eta} + [(A_{bb} - K_e A_{ab})K_e + A_{ba} - K_e A_{aa}]y + (B_b - K_e B_a)u$$

⇒ Let us define
$$\hat{A} = A_{bb} - K_e A_{ab}$$
$$\hat{B} = \hat{A} K_e + A_{ba} - K_e A_{aa}$$
$$\hat{F} = B_b - K_e B_a$$

⇒ Then the following three equations define the minimum-order observer:

$$\dot{\tilde{\eta}} = \hat{A}\tilde{\eta} + \hat{B}y + \hat{F}u \quad \text{—— (1)}$$

$$\tilde{\eta} = \tilde{x}_b - K_e y \quad \text{—— (2)}$$

$$u = -K\tilde{x} \quad \text{—— (3)}$$

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⇒ Eq ③ Can be rewritten as:-

$$u = -K\tilde{x} = -[K_a \; K_b]\begin{bmatrix} y \\ \tilde{x}_b \end{bmatrix} = -K_a y - K_b \tilde{x}_b$$

$$u = -K_b \tilde{\eta} - (K_a + K_b K_e) y \quad \text{—— ④}$$

⇒ Substituting eq ④ in ① we get

$$\dot{\tilde{\eta}} = \hat{A}\tilde{\eta} + \hat{B}y + \hat{F}\left[-K_b\tilde{\eta} - (K_a + K_b K_e)y\right]$$

$$\dot{\tilde{\eta}} = (\hat{A} - \hat{F}K_b)\tilde{\eta} + [\hat{B} - \hat{F}(K_a + K_b K_e)]y \quad \text{—— ⑤}$$

Let us define, $\tilde{A} = \hat{A} - \hat{F}K_b$

$\tilde{B} = \hat{B} - \hat{F}(K_a + K_b K_e)$

$\tilde{C} = -K_b$

$\tilde{D} = -(K_a + K_b K_e)$

Thus eq ④ & ⑤ become:-

$$\dot{\tilde{\eta}} = \tilde{A}\tilde{\eta} + \tilde{B}y$$

$$u = \tilde{C}\tilde{\eta} + \tilde{D}y$$

$$\Rightarrow \boxed{\frac{U(s)}{-Y(s)} = -\left[\tilde{C}(sI - \tilde{A})^{-1}\tilde{B} + \tilde{D}\right]}$$

{Since input to the observer controller is $-Y(s)$ rather than $Y(s)$}

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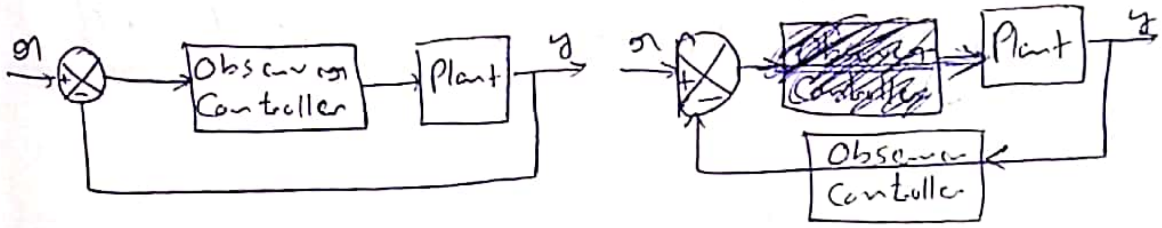
10.6) Design of Regulator System with Observer

1. Derive a state-space model of the plant.
2. Choose the desired closed-loop poles for pole placement. Choose the desired observer poles.
3. Determine the state feedback gain matrix K and the observer gain matrix.
4. Using the gain matrices K and Ke obtained in Step 3, derive the transfer function of the observer controller. If it is a stable controller, check the response to the given initial condition. If the response is not acceptable, adjust the closed-loop pole location and/or observer pole location until an acceptable response is obtained.

10.7) Design of Control System with Observer

⇒ In this Section we consider the design of Control System with observer when the Systems have reference inputs.

↳ Output of Control System must follow the input that is time varying.



Configuration 1 Configuration 2

10.8 > Quadratic Optimal Regulator System

An advantage of the quadratic optimal control method over the pole-placement method is that the former provides a systematic way of computing the state feedback control gain matrix

★ Quadratic Optimal Regulator Problem

Let system be given by :-

$$\dot{x} = Ax + Bu \quad \text{—— (1)}$$

We have to determine matrix K of the optimal control vector

$$u(t) = -K\,x(t) \quad \text{—— (2)}$$

⇒ So as to minimize the performance index

$$\boxed{J = \int_0^\infty (x^T Q x + u^T R u)\,dt} \quad \text{—— (3)}$$

(accounts for error)   accounts for energy expenditure of control signal

Where Q is a positive-definite Hermitian matrix

& R is a positive-definite Hermitian matrix

⇒ The matrix Q & R determines the relative importance of the error & the expenditure of this energy.

⇒ Therefore, the unknown elements of the matrix K are determined so as to minimize the performance index.

⇒ Substituting eq ② in ① we get

$$\dot{x} = Ax - BKx = (A-BK)x \quad \text{—— ④}$$

{We assume matrix (A−BK) is stable or eigenvalues of A−BK have negative real part}

⇒ Substituting eq ② in ③ we get:-

$$J = \int_0^\infty \left(x^T Q x + x^T K^T R K x\right) dt$$

$$J = \int_0^\infty x^T \left(Q + K^T R K\right) x \, dt$$

Let us set:

$$x^T (Q + K^T R K) x = -\frac{d}{dt}(x^T P x)$$

{where P is a positive-definite Hermitian matrix}

$$x^T (Q + K^T R K) x = -\dot{x}^T P x - x^T P \dot{x}$$

$$= -x^T \left[(A-BK)^T P + P(A-BK)\right] x$$

{using eq ④}

⇒ Comparing both side of last eqn di@

$$-\left(Q + K^T R K\right) = (A-BK)^T P + P(A-BK) \quad \text{—— ⑤}$$

⇒ It Can be proved that if (A − BK) is stable, there exists a positive definit matrix P that satifies eq ⑤.

⇒ Hence our procedure is to determine the elements P from the eq (8) & see if it is positive definite.

→ More than one matrix P may satisfy this equation.

→ If the system is stable, there always exist one positive definite matrix P to satisfy this equation.

→ This means that, if we solve this equation & find one positive definite matrix P, the system is stable.

⇒ The performance index J can be evaluated as:-

$$J = \int_0^\infty x^T(Q+K^TRK)x\,dt = -x^TPx\Big|_0^\infty$$

$$J = -x^T(\infty)Px(\infty) + x^T(0)Px(0)$$

⇒ Since all eigen value of A-BK are assumed to have negative real parts we have $x(\infty)\to 0$

So $$\boxed{J = x^T(0)Px(0)} \quad -(6)$$

⇒ To obtain the solution to the quadratic optimal control problem, we proceed as follows:

⇒ Since R has been assumed to be a positive definite Hermitian matrix

$R = T^TT$ {where T is not singular matrix}

$\Rightarrow$ Then equation ⑥ can be ~~~~ rewritten as

$$(A^T - K^T B^T)P + P(A - BK) + Q + K^T T^T T K = 0$$

$$\Rightarrow A^T P + PA + [TK - (T^T)^{-1} B^T P]^T [TK - (T^T)^{-1} B^T P] - PBR^{-1}B^T P + Q = 0$$

$\Rightarrow$ Minimization of J with respect to K requires minimization of

$$x^T [TK - (T^T)^{-1} B^T P]^T [TK - (T^T)^{-1} B^T P] x$$

with respect to K.

$\Rightarrow$ Since this last expression is nonnegative, the minimum occurs when it is zero.

$$TK = (T^T)^{-1} B^T P$$

$$\Rightarrow K = T^{-1}(T^T)^{-1} B^T P = R^{-1} B^T P$$

$$\boxed{K = R^{-1} B^T P} \quad \{\text{optimal } K\} \quad \text{— ⑦}$$

$\Rightarrow$ Thus the optimal control law to the quadratic optimal control problem is given by:

$$u(t) = -Kx(t) = -R^{-1} B^T P x(t)$$

$\Rightarrow$ The matrix P must satisfy the following reduced equation:-

$$\boxed{A^T P + PA - PBR^{-1} B^T P + Q = 0} \quad \text{— ⑧}$$

{Reduced-matrix Riccati equation}

⇒ The design step :-

1. Solve , the reduced-matrix Riccati equation , for the matrix P

$$A^T P + PA - PBR^{-1}B^T P + Q = 0$$

2. Substitute this matrix P into Equation

$$K = R^{-1}B^T P$$

The resulting matrix K is the optimal matrix.

CHAPTER 4

Robust Control Theory

4

Robust Control Theory

⇒ The first step in the design of a control system is to obtain a mathematical model of the control object based on the physical law.

⇒ Quite often the modle may be nonlinear and possibly with distributed parameters.

↳ Such a model may be difficult to analyze.

⇒ It is desirable to approximate it by a linear constat-coefficient system that will approximate the actual object fairly well.

⇒ In frequency-response approach to control system design we may use phase & gain margins to take care of the modeling errors.

↳ However, in state-space approach, which is based on the differential equation of the plant dynamics, no such margins are involved in the design process.

⇒ The actual plant differs from the model used in the design, a question arises whether the controller desined using a model will work satisfactory with the actual plat.

⇒ To ensure that it will do so, robust control theory has been developed since around 1980.

⇒ Robust Control theory assumes that there is an uncertainity or error between the actual plant and its mathematical model.

⇒ Systems designed based on the robust Control theory will possess the following Properties:-

(i) <u>Robust stability</u> ⇒ The Control System designed is stable in the Presence of Perturbation.

(2) <u>Robust performance</u> ⇒ The Control System exhibits predetermined response characteristic in the presence of Perturbation.

1. Uncertain Elements in Plant Dynamics

⇒ The term Uncertainity reffers to the difference or error between the model of the plant and the actual plant.

Structured Uncertainity

eg ⇒ Parameter variation in Plant dynamics.

Unstructured Uncertainity

Eg ⇒ due to linearization of a nonlinear Plant

⇒ In the robust Control theory, we define Unstructured Uncertainity as $\Delta(s)$.

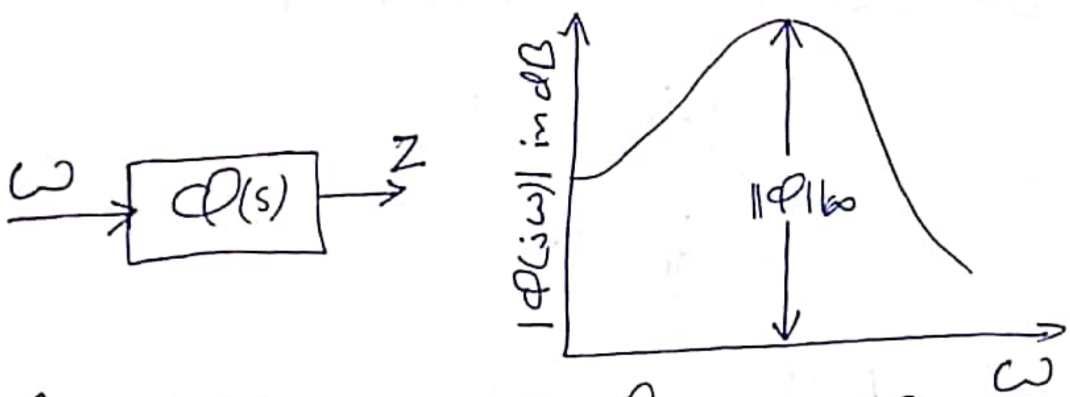
↳ Since exact description of $\Delta(s)$ is Unknown, we use an estimate of $\Delta(s)$ (as the Magnitude & phase Characteristics) and use this estimate in the design of the Controller that Stablizes the Control System.

2. <u>H∞ Norm</u>

The H∞ Norm of a stable Single-input-single output System is the largest possible amplification factor of the steady-state response to Sinusoidal excitation.

⇒ For a Scalar $\Phi(s)$ $\|\Phi\|_\infty$ gives the maximum value of $|\Phi(j\omega)|$.

↳ It is called the H∞ norm.



⇒ Assume the transfer function $\Phi(s)$ is Proper and Stable.

The H∞ norm of $\Phi(s)$ is defined by

$$\|\Phi\|_\infty = \bar{\sigma}[\Phi(j\omega)]$$

$\bar{\sigma}[\Phi(j\omega)]$ means the maximum Singular Value of $[\Phi(j\omega)]$.

↳ ($\bar{\sigma}$ means σ_{max})

⇒ Singular Value of a transfer function Φ is defined by

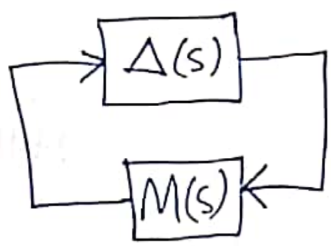
$$\sigma_i(\Phi) = \sqrt{\lambda_i(\Phi^* \Phi)}$$

Where $\lambda_i(\Phi^*\Phi)$ is the i^{th} largest eigenvalue of $\Phi^*\Phi$ and is always a non-negative real value.

⇒ By making $\|\Phi\|_\infty$ Smaller, we make the effect of input ω on the output Z Smaller.

3. Small-Gain theorem

⇒ Consider the Closed-loop System shown:-



$\Delta(s)$ and $M(s)$ are stable and Proper transfer function.

⇒ The Small gain theorem States that if

$$\|\Delta(s)M(s)\|_\infty < 1$$

then this closed-loop System is Stable.

⇒ This theorem is an extension of the Nyquist Stability Criteria.

4. System with Unstructured Uncertainity

⇒ In some cases an unstructured uncertainty error may be considered multiplicative such that

$$\tilde{G} = G(1+\Delta m)$$

$\tilde{G}$ → True plant dynamics

G → Model plant dynamics

⇒ In other cases an unstructured unstructured uncertainity may be considered additive such that:-

$$\tilde{G} = G + \Delta a$$

⇒ In either case we assume that the norm of Δm or Δa is bounded such that

$$\|\Delta m\| < \gamma m \qquad \|\Delta a\| < \gamma a$$

where γm and γa are positive constants.

5. Robust Stability

Let us define

$\tilde{G}$ = true plant dynamics

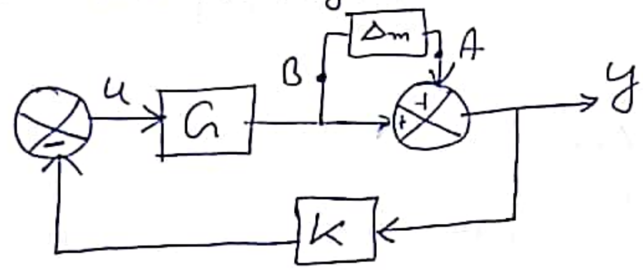
G = model of plant dynamics

Δm = Unstructured multiplicative uncertainity.

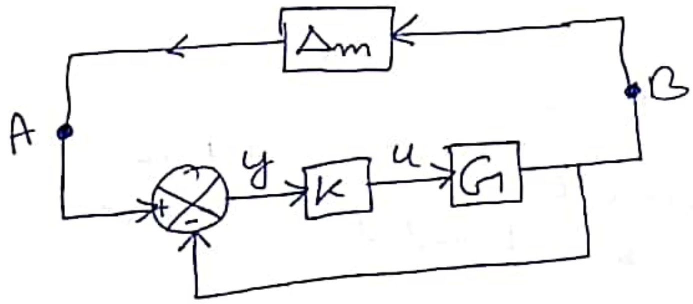
⇒ We assume that Δ_m is stable & its upper bound is known.

$$\tilde{G} = G(I + \Delta_m)$$

⇒ Consider the system shown in figure bellow:-

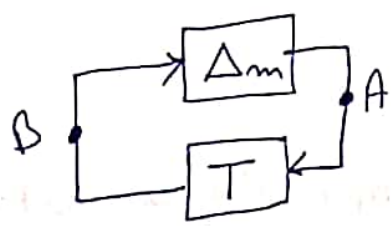


⇒ To obtain transfer function between Point A and point B, the above can be re drawn as:-



$$\frac{KG}{1+KG} = (1+KG)^{-1}KG$$

Let $T = (1+KG)^{-1}KG$



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upper

bellow:-

Point A
n co:-

⇒ Applying the Small gain theorem to the System consisting of Δm and T, we obtain the Condition for stability to be :-

$$\|\Delta_m T\|_\infty < 1 \quad \text{—— (1)}$$

⇒ In general, it is impossible to precisely model Δm.

↳ Therefore let us use a Scalar transfer function $W_m(j\omega)$ Such that

$$\bar{\sigma}\{\Delta_m(j\omega)\} < |W_m(j\omega)|$$

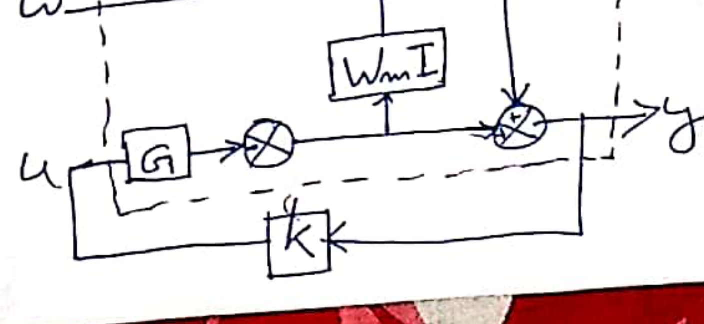
⇒ Consider, instead of above inequality, the following inequality:

$$\|W_m T\|_\infty < 1 \quad \text{—— (2)}$$

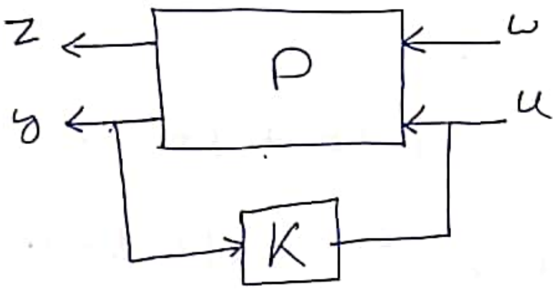
⇒ If inequality (2) holds true, Inequality (1) will always be Satisfied.

↳ By making the H∞ norm of $W_m T$ to be less than 1, we obtain the controller K that will make the System Stable.

⇒ Suppose that we Cut the line at point A and Replace Δm by $W_m I$ we obtain :-



⇒ Redrawing above we obtain what is Called generalized Plant diagram.



$$\Rightarrow \|W_m T\|_\infty < 1$$

$$\Rightarrow \left\| \frac{W_m K(s) G(s)}{1 + K(s) G(s)} \right\|_\infty < 1$$

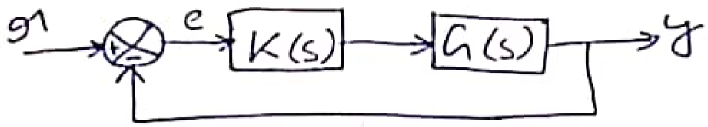
⇒ For a Stable plant model $G(s)$, $K(s)=0$ will Satisfy above inequality. However It is not desirable.

⇒ To find an acceptable transfer function ~~find~~ add another Condition for $K(s)$, we may that the resulting System will have robust Performance.

↳ Such that the System output follows the input with minimum error or another reasonable Condition.

6. Robust Performance

⇒ Consider the System shown bellow.



⇒ Suppose that we want the output $y(t)$ to follow the input $r(t)$ as closely as possible

$$\lim_{t\to\infty}[r(t)-y(t)] = \lim_{t\to\infty} e(t) \to 0$$

$$\frac{Y(s)}{R(s)} = \frac{KG}{1+KG}$$

we have $\dfrac{E(s)}{R(s)} = \dfrac{1}{1+KG} = S$

$S \Rightarrow$ Sensitivity function

$T \Rightarrow$ Complementary Sensitivity function.

⇒ In robust performance Problem we want to make the H_∞ norm of S smaller than the desired transfer function W_s^{-1}.

$$\|W_s S\|_\infty < 1$$

⇒ Combining Inequalities we get

$$\left\| \begin{matrix} W_m T \\ W_s S \end{matrix} \right\| < 1 \qquad \text{where } T+S=1$$

$$\left\| \begin{matrix} W_m(s)\dfrac{K(s)\,G(s)}{1+K(s)\,G(s)} \\[2ex] W_s(s)\dfrac{1}{1+K(s)\,G(s)} \end{matrix} \right\| < 1 \quad \text{—} \; (1)$$

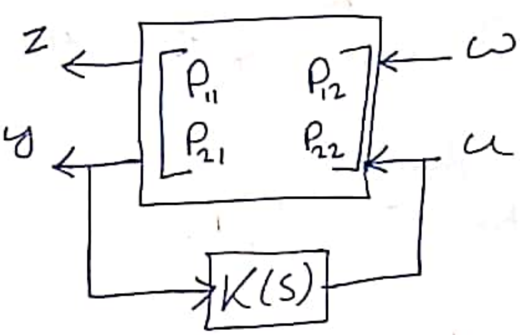
⇒ Our Problem then becomes to find $K(s)$ that will satisfy Inequality above.

⇒ Depending on the Chosen $W_m(s)$ and $W_s(s)$ there may be many $K(s)$ that Satisfy Inequality or may be no $K(s)$ that satisfies Inequality.

⇒ Such a robus Control problem using Inequality is Called a mixed-Sensitivity Problem.

7. Finding Transfer function $Z(s)/\omega(s)$ from a Generalized Plat diagram

Consider the generalized Plat diagram shown bellow:-



$\omega(s)$ ⇒ Exogenous disturbance

$u(s)$ ⇒ Manipulated Variable

$Z(s)$ ⇒ Controlled Variables

$y(s)$ ⇒ Observed Variable

$$\begin{bmatrix} Z(s) \\ y(s) \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} \omega(s) \\ u(s) \end{bmatrix}$$

$\Rightarrow$ The equation that relates $u(s)$ and $y(s)$ is given by:-

$$u(s) = K(s)\, y(s)$$

$\Rightarrow$ Let us define the transfer function that relates the controlled variables $Z(s)$ to the exogenous disturbance $\omega(s)$ as $\Phi(s)$.

$$Z(s) = \Phi(s)\, \omega(s)$$

$\Rightarrow$ $\Phi(s)$ can be determined as follows:-

$$Z(s) = P_{11}\, \omega(s) + P_{12}\, u(s)$$

$$y(s) = P_{21}\, \omega(s) + P_{22}\, u(s)$$

$$u(s) = K(s)\, y(s)$$

$$\Rightarrow \quad y(s) = P_{21}\, \omega(s) + P_{22}\, K(s)\, y(s)$$

$$y(s) = [I - P_{22} K(s)]^{-1} P_{21}\, \omega(s)$$

$$\Rightarrow Z(s) = P_{11}\, \omega(s) + P_{12}\, K(s) \, [I - P_{22} K(s)]^{-1} P_{21}\, \omega(s)$$

$$Z(s) = \{P_{11} + P_{12} K(s)\} [I - P_{22} K(s)]^{-1} P_{21} \} \, \omega(s)$$

Hence,

$$\boxed{\Phi(s) = P_{11} + P_{12} K(s) [I - P_{22} K(s)]^{-1} P_{21}}$$

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8. H Infinity Control Problem

⇒ To design a Controller K of a Control System to Satisfy Various Stability and performance Specifications, we utilize the Concept of the generalized plant.

⇒ The reason to use generalized plants, rather than individual block diagrams of Control Systems, is that a number of Control Systems with uncertain elements have been designed using generalized plant and consequently, established design approaches using such plants are available.

⇒ Controller that is the Solution to the H infinity Control problems is commonly Called the H infinity Controller.

9. Solving Robust Control Problem

⇒ There are three established approaches to Solve robust Control problems. They are

1. Solve robust Control problems by deriving Riccati equation and Solve them.
2. Solve robust Control Problems by using the linear matrix inequality approach.

3. Solve rob
structure
analysis

"Solving
of the
mathe

3. Solve robust Control Problems that involves structured uncertainity by using the μ analysis and μ Synthesis approach.

"Solving robust Control Problems by use of any of the above method requires a broad mathamatical background..."