Linear Transformation which Maps a Space on to itself is Called Linear Operator.

Reff 1

$\Rightarrow$ {Not necessarily basis, Valid for any 3 Vector B.}

Let $\bar{B} = \{\bar{p}\ \bar{q}\ \bar{r}\}$ be a basis of an $\mathcal{E}^3$ space.

$\Rightarrow$ After rotation let the basis be $\bar{B}' = \{\bar{p}'\ \bar{q}'\ \bar{r}'\}$

$$|\bar{B}| = \left|\begin{bmatrix}\bar{p} & \bar{q} & \bar{r}\end{bmatrix}\right| = \left|\begin{bmatrix}\bar{p}' & \bar{q}' & \bar{r}'\end{bmatrix}\right| = |\bar{B}'|$$

$$|\bar{p}\ \bar{q}\ \bar{r}| = \bar{r}\cdot(\bar{p}\times\bar{q})$$

$$|\bar{p}'\ \bar{q}'\ \bar{r}'| = \bar{r}'\cdot(\bar{p}'\times\bar{q}')$$

As relative orientation of $\bar{B} \longrightarrow \bar{B}'$ is not Changed, but only rotated, so Scalar tripple Product will not change.

Hence,

$$\boxed{|\bar{\bar{B}}| = |\bar{\bar{B}}'|}$$

Orthogonal Matrix $\Rightarrow$ Square matrix whose Columns and rows are o Orthonormal Vector

$$\bar{\bar{Q}}^T\bar{\bar{Q}} = \bar{\bar{Q}}\bar{\bar{Q}}^T = \bar{\bar{I}}$$

Proper orthogal Matrix $\Rightarrow$ Orthogonal matrix with determinant unity.

$$\boxed{|\bar{\bar{Q}}| = 1}$$

Raff 2

$\bar{a} \times (\bar{a} \times \bar{x}) = \bar{\bar{A}}^2 \bar{x}$

$\Rightarrow \bar{\bar{A}}^2 \bar{x} = \bar{a}(\bar{a}^T \bar{x}) - \bar{x}(\bar{a}^T \bar{a})$ $\left\{ \bar{a} \times (\bar{b} \times \bar{c}) = \bar{b}(\bar{a} \cdot \bar{c}) - \bar{c}(\bar{a} \cdot \bar{b}) \right\}$

$\Rightarrow \bar{\bar{A}}^2 \bar{x} = \bar{a}\bar{a}^T \bar{x} - \|\bar{a}\|^2 \bar{x}$

$\Rightarrow \bar{\bar{A}}^2 \bar{x} = \left(-\|\bar{a}\|^2 \bar{\bar{I}} + \bar{a}\bar{a}^T\right) \bar{x}$

So $\boxed{\bar{\bar{A}}^2 = -\|\bar{a}\|^2 \bar{\bar{I}} + \bar{a}\bar{a}^T}$

Raff 3

$\Rightarrow$ For a fixed axis of rotation {fixed value of $\bar{e}$}

$\hookrightarrow$ Rotation matrix $\bar{\bar{Q}}$ is a function of the angle of rotation φ only.

$\Rightarrow$ Thus series expansion of $\bar{\bar{Q}}$ in terms of φ is:-

$$\bar{\bar{Q}}(\varphi) = \bar{\bar{Q}}(0) + \bar{\bar{Q}}'(0)\varphi + \frac{1}{2!}\bar{\bar{Q}}''(0)\varphi^2 + \cdots + \frac{1}{k!}\bar{\bar{Q}}^{k}(0)\varphi^k + \cdots$$

$\Rightarrow$ from defination of $\bar{\bar{E}}$, one can readily prove the relations below:

$$\bar{\bar{E}}^{(2k+1)} = (-1)^k \bar{\bar{E}} \qquad \bar{\bar{E}}^{2k} = (-1)^k (\bar{\bar{I}} - \bar{e}\bar{e}^T) \quad \text{--- (1)}$$

$\Rightarrow$ We know,

$$\boxed{\bar{\bar{Q}} = \bar{e}\bar{e}^T + \cos\varphi(\bar{\bar{I}} - \bar{e}\bar{e}^T) + \sin\varphi \bar{\bar{E}}} \quad \text{--- (2)}$$

⇒ Using eq ① & ② one can readily show that :-

$$\bar{\bar{Q}}^{(K)}(0) = \bar{\bar{E}}^{K}$$

$$\frac{d\bar{\bar{Q}}}{d\phi} = -\sin\phi\,(\bar{\bar{I}} - \bar{e}\bar{e}^{T}) + \cos\phi\,\bar{\bar{E}}$$

$$\left.\frac{d\bar{\bar{Q}}}{d\phi}\right|_{\phi=0} = \bar{\bar{E}}$$

$$\frac{d^2\bar{\bar{Q}}}{d\phi^2} = -\cos\phi\,(\bar{\bar{I}} - \bar{e}\bar{e}^{T}) - \sin\phi\,\bar{\bar{E}}$$

$$\left.\frac{d^2\bar{\bar{Q}}}{d\phi^2}\right|_{\phi=0} = -(\bar{\bar{I}} - \bar{e}\bar{e}^{T}) = \bar{\bar{E}}^{2}$$

$$\frac{d^3\bar{\bar{Q}}}{d\phi^3} = \sin\phi\,(\bar{\bar{I}} - \bar{e}\bar{e}^{T}) - \cos\phi\,\bar{\bar{E}}$$

$$\left.\frac{d^3\bar{\bar{Q}}}{d\phi}\right|_{\phi=0} = -\bar{\bar{E}} = \bar{\bar{E}}^{3}$$

$$\frac{d^4\bar{\bar{Q}}}{d\phi^4} = \cos\phi\,(\bar{\bar{I}} - \bar{e}\bar{e}^{T}) + \sin\phi\,\bar{\bar{E}}$$

$$\left.\frac{d^4\bar{\bar{Q}}}{d\phi^4}\right|_{\phi=0} = (\bar{\bar{I}} - \bar{e}\bar{e}^{T}) = \bar{\bar{E}}^{4}$$

So

$$\bar{\bar{Q}}(\varphi) = \bar{\bar{I}} + \bar{\bar{E}}\varphi + \frac{1}{2!}\bar{\bar{E}}^2\varphi^2 + \cdots + \frac{1}{K!}\bar{\bar{E}}^K\varphi^K + \cdots$$

$$\boxed{\bar{\bar{Q}}(\varphi) = e^{\bar{\bar{E}}\varphi}}$$

$$\bar{\bar{Q}}(\varphi) = \bar{\bar{I}} + \left(\bar{\bar{E}}\varphi + \frac{1}{3!}\bar{\bar{E}}^3\varphi^3 + \frac{1}{5!}\bar{\bar{E}}^5\varphi^5 + \cdots\right)$$

$$+ \left(\frac{1}{2!}\bar{\bar{E}}^2\varphi^2 + \frac{1}{4!}\bar{\bar{E}}^4\varphi^4 + \cdots\right)$$

$$\Rightarrow \bar{\bar{I}} + \left(\varphi - \frac{1}{3!}\varphi^3 + \cdots + \frac{1}{(2K+1)!}(-1)^K\varphi^{2K+1} + \cdots\right)\bar{\bar{E}}$$

$$+ \left(-\frac{1}{2!}\varphi^2 + \frac{1}{4!}\varphi^4 - \cdots + \frac{1}{(2K)!}(-1)^K\varphi^{2K} + \cdots\right)(\bar{\bar{I}} - ee^T)$$

So $$\boxed{\bar{\bar{Q}} = \bar{\bar{I}} + \sin\varphi\,\bar{\bar{E}} + (1-\cos\varphi)\bar{\bar{E}}^2}$$