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Appendix: Chapter-6
Manipulator dynamics

★ Introduction

⇒ Here, we consider the equations of motion for a manipulator:

↳ The way in which motion of the manipulator arises from torque applied by the actuators or from external forces applied to the manipulator.

★ Acceleration of a Rigid body

⇒ At any instant the linear and angular velocity vectors have derivatives that are called the Linear & Angular acceleration respectively.

$$^{B}\dot{V}_{Q} = \frac{d}{dt}\,{}^{B}V_{Q} = \lim_{\Delta t \to 0} \frac{{}^{B}V_{Q}(t+\Delta t) - {}^{B}V_{Q}(t)}{\Delta t}$$

& $$^{A}\dot{\Omega}_{B} = \frac{d}{dt}\,{}^{A}\Omega_{B} = \lim_{\Delta t \to 0} \frac{{}^{A}\Omega_{B}(t+\Delta t) - {}^{A}\Omega_{B}(t)}{\Delta t}$$

⇒ As with velocities, when the reference frame of the differentiation is understood to be some universal reference frame {U}, we will use the notation.

$$\dot{V}_{A} = {}^{U}\dot{V}_{AORG}$$

$$\dot{\omega}_{A} = {}^{U}\dot{\Omega}_{A}$$

(ii) Linear acceleration

$$ {}^{A}V_{Q} = {}^{A}R_{B}\,{}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}^{A}R_{B}\,{}^{B}Q $$

$$ \frac{d}{dt}\left({}^{A}R_{B}\,{}^{B}Q\right) $$

$\Rightarrow$ On differentiating :-

$$ {}^{A}\dot{V}_{Q} = \frac{d}{dt}\left({}^{A}R_{B}\,{}^{B}V_{Q}\right) + {}^{A}\dot{\Omega}_{B} \times {}^{A}R_{B}\,{}^{B}Q + {}^{A}\Omega_{B} \times \frac{d}{dt}\left({}^{A}R_{B}\,{}^{B}Q\right) $$

$$ \Rightarrow {}^{A}\dot{V}_{Q} = {}^{A}R_{B}\,{}^{B}\dot{V}_{Q} + {}^{A}\Omega_{B} \times {}^{A}R_{B}\,{}^{B}V_{Q} + {}^{A}\dot{\Omega}_{B} \times {}^{A}R_{B}\,{}^{B}Q + {}^{A}\Omega_{B} \times \left({}^{A}R_{B}\,{}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}^{A}R_{B}\,{}^{B}Q\right) $$

$$ \Rightarrow {}^{A}\dot{V}_{Q} = {}^{A}R_{B}\,{}^{B}\dot{V}_{Q} + 2\,{}^{A}\Omega_{B} \times {}^{A}R_{B}\,{}^{B}V_{Q} + {}^{A}\dot{\Omega}_{B} \times {}^{A}R_{B}\,{}^{B}Q + {}^{A}\Omega_{B} \times \left({}^{A}\Omega_{B} \times {}^{A}R_{B}\,{}^{B}Q\right) $$

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⇒ Finally, to generalize to the case in which the origins are not coincident, we add one term which gives the linear acceleration of the origin of {B}, resulting in the final general formula:

$$\boxed{\begin{aligned} {}^{A}\dot{V}_{Q} &= {}^{A}\dot{V}_{BORG} + {}^{A}R_{B}\,{}^{B}\dot{V}_{Q} + 2\,{}^{A}\Omega_{B} \times {}^{A}R_{B}\,{}^{B}V_{Q} \\ &\quad + {}^{A}\dot{\Omega}_{B} \times {}^{A}R_{B}\,{}^{B}Q + {}^{A}\Omega_{B}\,({}^{A}\Omega_{B} \times {}^{A}R_{B}\,{}^{B}Q) \end{aligned}}$$

⇒ If ${}^{B}Q$ is Constant ⇒ ${}^{B}V_{Q} = {}^{B}\dot{V}_{Q} = 0$

$$\boxed{{}^{A}\dot{V}_{Q} = {}^{A}\dot{V}_{BORG} + {}^{A}\Omega_{B} \times ({}^{A}\Omega_{B} \times {}^{A}R_{B}\,{}^{B}Q) + {}^{A}\dot{\Omega}_{B} \times {}^{A}R_{B}\,{}^{B}Q}$$

(#) Angular acceleration

⇒ Consider the case in which {B} is rotating relative to {A} with ${}^{A}\Omega_{B}$ and {C} is rotating relative to {B} with ${}^{B}\Omega_{C}$.

$$ {}^{A}\Omega_{C} = {}^{A}\Omega_{B} + {}^{A}R_{B}\,{}^{B}\Omega_{C} $$

$$ \Rightarrow {}^{A}\dot{\Omega}_{C} = {}^{A}\dot{\Omega}_{B} + \frac{d}{dt}\left({}^{A}R_{B}\,{}^{B}\Omega_{C}\right) $$

$$ \Rightarrow \boxed{{}^{A}\dot{\Omega}_{C} = {}^{A}\dot{\Omega}_{B} + {}^{A}R_{B}\,{}^{B}\dot{\Omega}_{C} + {}^{A}\Omega_{B} \times {}^{A}R_{B}\,{}^{B}\Omega_{C}}$$

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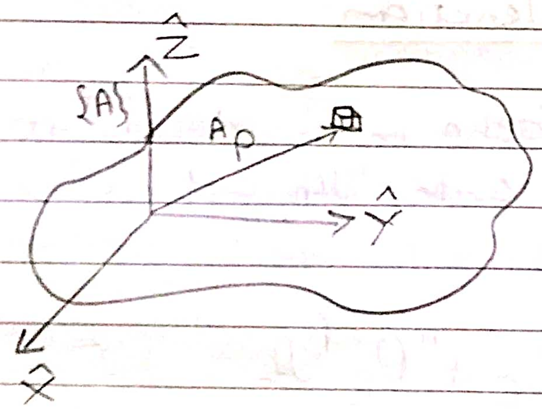
★ Mass distribution

⇒ In system with a single degree of freedom, we often talk about the mass of a rigid body.

⇒ In the case of rotational motion about a single axis, the notion of the moment of inertia is a familiar one.

⇒ For a rigid body that is force to move in three dimensions there are infinitely many possible rotation axes.

↳ Here, we introduce the inertia tensor.



$$^{A}P = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

⇒ The inertia tensor relative to frame {A} is expressed in the matrix form as the 3×3 matrix:

$$^{A}I = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix}$$

Where the Scalar element are given by:

$$I_{xx} = \iiint_V (y^2 + z^2) \rho \, dv$$

$$I_{yy} = \iiint_V (x^2 + z^2) \rho \, dv$$

$$I_{zz} = \iiint_V (x^2 + y^2) \rho \, dv$$

$$I_{xy} = \iiint_V xy \rho \, dv$$

$$I_{xz} = \iiint_V xz \rho \, dv$$

$$I_{yz} = \iiint_V yz \rho \, dv$$

⇒ The elements I_{xx} I_{yy} & I_{zz} are called the 'mass moment of inertia'

⇒ The element with mixed indices are called the 'mass product of Inertia'

⇒ The set of six independent quantities will for a given body, depend on the position and orientation of the frame in which they are defined.

⇒ If we are free to choose the orientation of the reference frame, it is possible to cause the product of inertia to be zero.

- → The axis of the reference frame which are so aligned are called the principal axes.
- → The corresponding mass moments are the principal moments of inertia.

⇒ A well-known result, the parallel-axis theorem is one way of computing how the inertia tensor changes under translation of the reference frame.

- → The parallel-axis theorem relates the inertia tensor in a frame with origin at the center of mass to the inertia tensor with respect to another reference frame.

⇒ When {C} is located at the center of mass of the body, and {A} is an arbitrarily translated frame, the theorem can be stated as

$$^{A}I_{zz} = {}^{C}I_{zz} + m(x_c^2 + y_c^2)$$

$$^{A}I_{xy} = {}^{C}I_{xy} - m x_c y_c$$

where $P_c = [x_c\ y_c,\ z_c]^T$ locates the center of mass relative to {A}.

⇒ The theorem can be stated in vector matrix form as

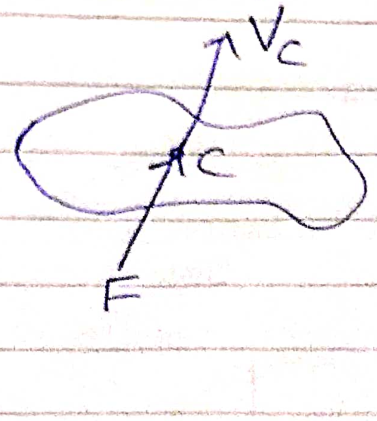
$$\boxed{^{A}I = {}^{C}I + m\left[P_c^T P_c I_3 - P_c P_c^T\right]}$$

★ **Newton's Equation, Euler Equation**

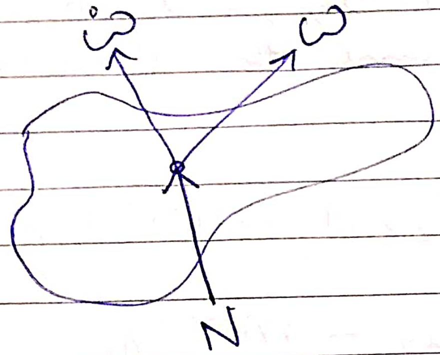
⇒ If we know the location of the center of mass and the inertia tensor of the link, then its mass distribution is completely characterized.

⊕ **Newton's Equation**

$$\boxed{F = m\dot{v}_c}$$



Euler's equation



$$\boxed{N = {}^{c}I\dot{\omega} + \omega \times {}^{c}I\omega}$$