

5 State-Space Analysis

classmate

Date _____
Page _____

5.1) Introduction

- ⇒ The State-Space method is based on the description of System equations in terms of ~~n~~ first order difference equations or differential equations.
 - ↓
- ⇒ This may be combined into a first order Vector-matrix differential or difference equation.
 - ↓
- ⇒ The use of the Vector-matrix notation greatly simplifies the mathematical representation of the Systems of equations.
- ⇒ "Design in State Space can be carried out for a class of input, instead of a specific input function such as the impulse function, Step function or Sinusoidal function."
 - ↳ State-Space methods enables the engineer to include initial conditions in the design.

State ⇒ The state of a dynamic system is the smallest set of variables (called state variables) such that the knowledge of these variables at $t=t_0$, together with the knowledge of the input for $t \geq t_0$, completely determines the behavior of the system for any time $t > t_0$.

State Variables \Rightarrow Variables making up the smallest set of variables that determine the state of the dynamic system.

\Rightarrow State variables need not be physically measurable or observable quantities.

State Vector \Rightarrow If n state variables are needed to completely describe the behavior of a given system, then these n state variables can be considered the n components of a vector \bar{x} . Such a vector is called State Vector:-

State Space \Rightarrow The n -dimensional space whose coordinate axes consist of the x_1 axis, x_2 axis --- x_n axis is called a State-Space.

\Rightarrow Any state can be represented by a point in the State Space.

State-Space Equation

\Rightarrow There are three types of variables that are involved in the modeling of dynamic systems:-

- (1) Input Variables
- (2) Output Variables
- (3) State Variables

⇒ For time-varying (linear or nonlinear) discrete-time Control Systems, the state equation may be written as:-

$$\bar{x}(k+1) = f[\bar{x}(k), \bar{u}(k), k] \quad \text{--- (1)}$$

and the output equation as:-

$$\bar{y}(k) = g[\bar{x}(k), \bar{u}(k), k] \quad \text{--- (2)}$$

⇒ For linear time varying discrete-time Systems, the state equation and output equation may be simplified to

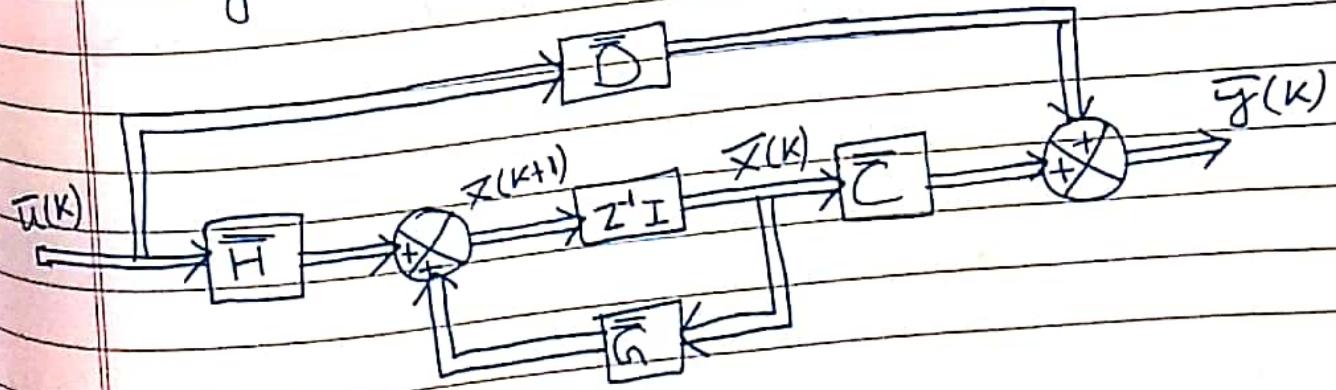
$$\bar{x}(k+1) = \bar{G}(k) \bar{x}(k) + \bar{H}(k) \bar{u}(k)$$

$$\bar{y}(k) = \bar{C}(k) \bar{x}(k) + \bar{D}(k) \bar{u}(k)$$

⇒ For linear time invariant discrete time control systems:-

$$\bar{x}(k+1) = \bar{G} \bar{x}(k) + \bar{H} u(t)$$

$$\bar{y}(k) = \bar{C} \bar{x}(k) + \bar{D} \bar{u}(k)$$



5.2 State Space Representations of Discrete-time System

Canonical forms of discrete-time State Space Equations

Consider the discrete time System described by:-

$$y(k) + a_1 y(k-1) + a_2 y(k-2) + \dots + a_n y(k-n) \\ = b_0 u(k) + b_1 u(k-1) + \dots + b_m u(k-m)$$

$u(k) \Rightarrow$ Input at k^{th} Sampling instant
 $y(k) \Rightarrow$ Output

\Rightarrow The above equation may be written in the form of the pulse transfer function as:-

$$\frac{Y(z)}{U(z)} = \frac{b_0 + b_1 z^{-1} + \dots + b_m z^{-m}}{1 + a_1 z^{-1} + \dots + a_n z^{-n}}$$

\Rightarrow There are many ways to realize state-space representations for the discrete time system described above:-

- 1) Controllable canonical form
- 2) Observable Canonical form
- 3) Diagonal Canonical form
- 4) Jordan Canonical form

① Controllable Canonical form

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_{n-1}(k+1) \\ x_n(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_{n-1}(k) \\ x_n(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = [b_n - a_n b_0; b_{n-1} - a_{n-1} b_0; \dots; b_1 - a_1 b_0] \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{bmatrix} + b_0 u(k)$$

② Observable Canonical form

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_{n-1}(k+1) \\ x_n(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & -a_n \\ 1 & 0 & \cdots & 0 & 0 & -a_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & -a_2 \\ 0 & 0 & \cdots & 0 & 1 & -a_1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_{n-1}(k) \\ x_n(k) \end{bmatrix} + \begin{bmatrix} b_n - a_n b_0 \\ b_{n-1} - a_{n-1} b_0 \\ \vdots \\ b_2 - a_2 b_0 \\ b_1 - a_1 b_0 \end{bmatrix}$$

$$y(k) = [0 \ 0 \ \cdots \ 0 \ 1] \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_{n-1}(k) \\ x_n(k) \end{bmatrix} + b_0 u(k)$$

③ Diagonal Canonical form

Let P_1, P_2, \dots, P_n be n poles of pulse transfer function and corresponding residues be C_1, C_2, \dots, C_n .

⇒ If all poles are distinct, then the state-space representation may be put in the diagonal canonical form as follows:-

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_n(k+1) \end{bmatrix} = \begin{bmatrix} P_1 & 0 & \cdots & 0 \\ 0 & P_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & P_n \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u(k)$$

$$y(k) = [C_1 \ C_2 \ \cdots \ C_n] \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{bmatrix} + b_0 u(k)$$

④ Jordan Canonical Form

⇒ If the pulse transfer function given by involves a multiple pole of order m at $Z = P_1$ and all other poles are distinct.

⇒ ~~Residues~~ C_1, C_2, \dots, C_m are residues at $\frac{1}{Z-P_1}$

$\frac{1}{(Z-P_1)^2}, \dots, \frac{1}{(Z-P_1)^m}$ respectively.

⇒ And $C_{m+1}, C_{m+2}, \dots, C_n$ are residues at Poles $P_{m+1}, P_{m+2}, \dots, P_n$ respectively.

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_m(k+1) \\ x_{m+1}(k+1) \\ \vdots \\ x_n(k+1) \end{bmatrix} = \begin{bmatrix} P_1 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & P_1 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & P_1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & P_{m+1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & P_n \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_m(k) \\ x_{m+1}(k) \\ \vdots \\ x_n(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u(k)$$

$$y(k) = [C_1 \ C_2 \ \cdots \ C_n] \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{bmatrix} + b_0 u(k)$$

* Nonuniqueness of State Space Representation

⇒ For a given pulse-transfer function System, the state-space representation is not unique.

⇒ The state equation is however related to each other by the similarity transformation.

⇒ Consider the system defined by:-

$$\begin{aligned} \bar{x}(k+1) &= \bar{G}\bar{x}(k) + \bar{H}\bar{u}(k) & \text{--- (1)} \\ \bar{y}(k) &= \bar{C}\bar{x}(k) + \bar{D}\bar{u}(k) & \text{--- (2)} \end{aligned}$$

Let us define a new state vector as $\hat{x}(k)$ by

$$\bar{x}(k) = \bar{P}\hat{x} \quad \left\{ \bar{P} \text{ is non singular matrix} \right\}$$

Then,

$$\bar{P} \bar{x}(k+1) = \bar{G} \bar{P} \bar{x}(k) + \bar{H} \bar{u}(k)$$

$$\Rightarrow \bar{x}(k+1) = \bar{P}^{-1} \bar{G} \bar{P} \bar{x}(k) + \bar{P}^{-1} \bar{H} \bar{u}(k)$$

Let $\bar{G} = \bar{P}^{-1} \bar{G} \bar{P}$ & $\bar{P}^{-1} \bar{H} = \bar{H}$

So $\boxed{\bar{x}(k+1) = \bar{G} \bar{x}(k) + \bar{H} \bar{u}(k)} - (3)$

Similarly $\boxed{\bar{y}(k) = \bar{C} \bar{x}(k) + \bar{D} \bar{u}(k)} - (4)$

where, $\bar{C} = \bar{C} \bar{P}$; $\bar{D} = \bar{D}$

\Rightarrow State-Space representation given by eq (1), (2) is equivalent to the State Space representation by eq (3), (4).

5.3) Solving discrete-time State Space Equations

$$\bar{x}(k+1) = \bar{G} \bar{x}(k) + \bar{H} \bar{u}(k) - (1)$$

$$\bar{y}(k) = \bar{C} \bar{x}(k) + \bar{D} \bar{u}(k) - (2)$$

\Rightarrow In general, discrete-time equations are easier to solve than differential equations because the former can be solved easily by means of a recursion procedure.

→ Recursion procedure is quite simple and convenient for digital computations.

$$\bar{x}(1) = \bar{G}\bar{x}(0) + \bar{H}\bar{u}(0)$$

$$\begin{aligned} \bar{x}(2) &= \bar{\bar{G}}\bar{x}(1) + \bar{\bar{H}}\bar{u}(1) \\ &= \bar{\bar{G}}^2\bar{x}(0) + \bar{\bar{G}}\bar{H}\bar{u}(0) + \bar{\bar{H}}\bar{u}(1) \end{aligned}$$

$$\bar{x}(3) = \bar{\bar{\bar{G}}}\bar{x}(2) + \bar{\bar{H}}\bar{u}(2)$$

$$= \bar{\bar{\bar{G}}}^3\bar{x}(0) + \bar{\bar{\bar{G}}}^2\bar{H}\bar{u}(0) + \bar{\bar{G}}\bar{H}\bar{u}(1) + \bar{\bar{H}}\bar{u}(2)$$

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$$\boxed{\bar{x}(k) = \bar{\bar{\bar{G}}}^k\bar{x}(0) + \sum_{j=0}^{k-1} \bar{\bar{\bar{G}}}^{k-j-1} \bar{H}\bar{u}(j)}$$

$$\bar{y}(k) = \bar{\bar{C}}\bar{\bar{G}}^k\bar{x}(0) + C \sum_{j=0}^{k-1} \bar{\bar{G}}^{k-j-1} \bar{H}\bar{u}(j) + \bar{D}\bar{u}(k)$$

* State transition matrix

\Rightarrow It is possible to write the solution of the homogeneous state equation

$$\bar{x}(k+1) = \bar{\bar{G}}\bar{x}(k)$$

as

$$\bar{x}(k) = \bar{\varphi}(k)\bar{x}(0)$$

\Rightarrow It is unique $n \times n$ matrix
Satisfies

$$(1) \quad \bar{\varphi}(k+1) = \bar{\bar{G}}\bar{\varphi}(k)$$

$$(2) \quad \bar{\varphi}(0) = I$$

$$\text{So } \boxed{\bar{\Psi}(k) = \bar{G}^k}$$

\Rightarrow We see that Solution is simply a transformation of the initial state. Therefore $\bar{\Psi}(k)$ is called state transition matrix.

So,

$$\boxed{\bar{x}(k) = \bar{\Psi}(k) \bar{x}(0) + \sum_{j=0}^{k-1} \bar{\Psi}(k-j-1) \bar{H} \bar{u}(j)}$$

$$\boxed{\bar{x}(k) = \bar{\Psi}(k) \bar{x}(0) + \sum_{j=0}^{k-1} \bar{\Psi}(j) \bar{H} \bar{u}(k-j-1)}$$

Sp

$$\boxed{\bar{y}(k) = \bar{C} \bar{\Psi}(k) \bar{x}(0) + C \sum_{j=0}^{k-1} \bar{\Psi}(j) \bar{H} \bar{u}(k-j-1) + \bar{D} \bar{u}(k)}$$

* Z transform approach to the Solution of Discrete-time State Equation

$$\bar{x}(k+1) = \bar{G} \bar{x}(k) + \bar{H} \bar{u}(k)$$

\Rightarrow Taking Z-transform both side:-

~~$$z \bar{x}(z) = \bar{G} \bar{x}(z) + \bar{H} \bar{u}(z)$$~~

$$= z \bar{x}(0)$$

$$\bar{x}(z) = Z\{\bar{x}(k)\} \quad \& \quad u(z) = Z\{u(k)\}$$

$$\Rightarrow (z\bar{I} - \bar{G}) \bar{x}(z) = z\bar{X}(0) + \bar{H}\bar{U}(z)$$

$$\bar{x}(z) = (z\bar{I} - \bar{G})^{-1} z \bar{X}(0) + (z\bar{I} - \bar{G})^{-1} \bar{H}\bar{U}(z)$$

$$\bar{X}(k) = z^{-1} \left\{ (z\bar{I} - \bar{G})^{-1} z \right\} \bar{X}(0)$$

$$+ z^{-1} \left\{ (z\bar{I} - \bar{G})^{-1} \bar{H}\bar{U}(z) \right\}$$

$$\text{So } G^k = z^{-1} \left\{ (z\bar{I} - \bar{G})^{-1} z \right\}$$

$$\& \sum_{j=0}^{K-1} \bar{G}^{K-j-1} \bar{H}\bar{U}(j) = z^{-1} \left\{ (z\bar{I} - \bar{G})^{-1} \bar{H}\bar{U}(z) \right\}$$

* Computing $(z\bar{I} - \bar{G})^{-1}$

$$\# (z\bar{I} - \bar{G})^{-1} = \frac{\text{adj}(z\bar{I} - \bar{G})}{|z\bar{I} - \bar{G}|}$$

$$\# |z\bar{I} - \bar{G}| = z^n + a_1 z^{n-1} + a_2 z^{n-2} + \dots + a_n$$

$$\text{where, } a_1 = -\text{tr } \bar{G}$$

$$a_2 = -\frac{1}{2} \text{tr } \bar{G} \bar{H},$$

$$a_3 = -\frac{1}{3} \text{tr } \bar{G} \bar{H}_1 \bar{H}_2$$

$$a_m = -\frac{1}{m} \text{tr } \bar{G} \bar{H}_{m-1}$$

$$\bar{H}_1 = \bar{G} + a_1 \bar{I}$$

$$\bar{H}_2 = \bar{G} \bar{H}_1 + a_2 \bar{I}$$

$$\bar{H}_3 = \bar{G} \bar{H}_2 + a_3 \bar{I}$$

$$\bar{H}_{m-1} = \bar{G} \bar{H}_{m-2} + a_{m-1} \bar{I}$$

$$\Rightarrow \text{adj}(z\bar{I} - \bar{G}) = \bar{I} z^{n-1} + \bar{H}_1 z^{n-2} + \bar{H}_2 z^{n-3} + \dots + \bar{H}_{n-1}$$

* Solution of Linear Time-Varying Discrete Time State Equation

$$\bar{x}(k+1) + \bar{G}(k)\bar{x}(k) + \bar{H}(k)\bar{u}(k) \quad \text{--- (1)}$$

$$y(k) = \bar{C}(k)\bar{x}(k) + \bar{D}(k)\bar{u}(k) \quad \text{--- (2)}$$

\Rightarrow The solution of Eq. (1) can be easily found by recursion :-

$$\bar{x}(h+1) = \bar{G}(h)\bar{x}(h) + \bar{H}(h)\bar{u}(h)$$

$$\bar{x}(h+2) = \bar{G}(h+1)\bar{x}(h+1) + \bar{H}(h+1)\bar{u}(h+1)$$

$$\Rightarrow \bar{G}(h+1)\bar{G}(h)\bar{x}(h) + \bar{G}(h+1)\bar{H}(h)\bar{u}(h)$$

$$+ \bar{H}(h+1)\bar{u}(h+1)$$

\Rightarrow Let us define the state transition matrix for the system defined by eq. (1) as $\bar{\Psi}(k, h)$

\Rightarrow It is a unique matrix satisfying the conditions:-

$$\bar{\Psi}(k+1, h) = \bar{G}(k)\bar{\Psi}(k, h)$$

$$\bar{\Psi}(h, h) = \bar{I}$$

where $k = h, h+1, h+2, \dots$

$$\bar{\Psi}(k, h) = \bar{G}(k-1) \bar{G}(k-2) \cdots \bar{G}(h) \quad k > h$$

So,

$$\bar{X}(k) = \bar{\Psi}(k, h) \bar{X}(h) + \sum_{j=h}^{k-1} \bar{\Psi}(k, j+1) \bar{H}(j) \bar{U}(j)$$

⇒ Output Equation:-

$$\bar{y}(k) = \bar{C}(k) \bar{\Psi}(k, h) \bar{X}(h) + \sum_{j=h}^{k-1} \bar{C}(k) \bar{\Psi}(k, j+1) \bar{H}(j) \bar{U}(j) + \bar{D}(k) \bar{U}(k)$$

⇒ If $\bar{G}(k)$ is non singular for all k value
Considered, so that the inverse of $\bar{\Psi}(k, h)$ exists
then the inverse of $\bar{\Psi}(k, h)$ is denoted by $\bar{\Psi}(h, k)$

$$\begin{aligned}\bar{\Psi}^{-1}(k, h) &= \bar{\Psi}(h, k) \\ &= [G(k-1) G(k-2) \cdots G(h)]^T \\ &= G(h)^{-1} G(h+1)^{-1} \cdots G(k-1)^{-1}\end{aligned}$$

* Properties of $\bar{\Psi}(k, h)$

$$1. \bar{\Psi}(k, k) = \bar{I}$$

$$2. \bar{\Psi}(k, h) = \bar{G}(k-1) \bar{G}(k-2) \cdots \bar{G}(h)$$

3. If inverse of $\bar{\Psi}(k, h)$ exist then,

$$\bar{\Psi}^{-1}(k, h) = \bar{\Psi}(h, k)$$

4. If $\bar{G}(k)$ is nonsingular for all k values
 Consider then

$$\bar{\Psi}(k,i) = \bar{\Psi}(k,j) \bar{\Psi}(j,i)$$

$\forall i, j < k$

If $\bar{G}(k)$ is singular for any value of k then

$$\bar{\Psi}(k,i) = \bar{\Psi}(k,j) \bar{\Psi}(j,i) \quad \forall k > j > i$$

5.4) Pulse - Transfer function Matrix

\Rightarrow A single input - Single output discrete time system may be modeled by a pulse transfer function.

\hookrightarrow Extension of the pulse transfer function concept to a multiple input multiple output discrete time system gives us the pulse transfer function matrix.

* Pulse transfer - function matrix

\Rightarrow The state space representation of an n^{th} order linear time-invariant discrete-time system with m input and m output can be given by:

$$\bar{x}(k+1) = \bar{G} \bar{x}(k) + \bar{H} \bar{u}(k)$$

$$\bar{y}(k) = \bar{C} \bar{x}(k) + \bar{D} \bar{u}(k)$$

⇒ Taking the Z transform we obtain:-

$$Z \bar{X}(z) - z \bar{x}(0) = \bar{G} \bar{X}(z) + \bar{H} \bar{U}(z)$$

$$\bar{Y}(z) = C \bar{X}(z) + D \bar{U}(z)$$

⇒ Definition for pulse transfer function calls for the assumption of zero initial conditions.

$$\Rightarrow \bar{X}(z) = (z\bar{I} - \bar{G})^{-1} \bar{H} \bar{U}(z)$$

$$\bar{Y}(z) = [\bar{C}(z\bar{I} - \bar{G})^{-1} \bar{H} + \bar{D}] \bar{U}(z) = F(z) \bar{U}(z)$$

$$\text{where, } F(z) = \bar{C}(z\bar{I} - \bar{G})^{-1} \bar{H} + \bar{D}$$

⇒ F(z) is called the pulse transfer function matrix.

$$\text{Since } (z\bar{I} - \bar{G})^{-1} = \frac{\text{adj}(z\bar{I} - \bar{G})}{|z\bar{I} - \bar{G}|}$$

$$\bar{F}(z) = \frac{\bar{C} \text{adj}(z\bar{I} - \bar{G}) \bar{H}}{|z\bar{I} - \bar{G}|} + \bar{D}$$

⇒ Clearly poles of F(z) are the zeros of $|z\bar{I} - \bar{G}|$

⇒ This means characteristic equation of the discrete-time system is given by

$$|z\bar{I} - \bar{G}| = 0$$

$$\Rightarrow z^n + a_1 z^{n-1} + \dots + a_{n-1} z + a_n = 0$$

* Similarity Transformation

$$\text{Let } \bar{x}(k) = \bar{P} \hat{x}(k)$$

$$\text{So, } \bar{x}(k+1) = \bar{A} \bar{x}(k) + \bar{B} u(k)$$

$$\bar{y}(k) = \bar{C} \bar{x}(k) + \bar{D} u(k)$$

$$\bar{\bar{A}} = \bar{P}^{-1} \bar{A} \bar{P}$$

$$\bar{\bar{B}} = \bar{P}^{-1} \bar{B}$$

$$\bar{\bar{C}} = \bar{C} \bar{P}$$

$$\bar{\bar{D}} = \bar{D}$$

$$\bar{\bar{F}}(z) = \bar{\bar{C}}(z\bar{\bar{I}} - \bar{\bar{A}})^{-1} \bar{\bar{B}} + \bar{\bar{D}} = \bar{\bar{F}}(z)$$

\Rightarrow This pulse transfer function is invariant under similarity transformation.

5.5) Discretization of Continuous-time State-Space Equations

\Rightarrow We assume that the input vector $\bar{u}(t)$ changes only at equally spaced sampling instants.

\Rightarrow Consider the Continuous-time state and a and output equation as:-

$$\dot{\bar{x}} = \bar{A} \bar{x} + \bar{B} \bar{u} \quad \text{--- (1)}$$

$$\bar{y} = \bar{C} \bar{x} + \bar{D} \bar{u} \quad \text{--- (2)}$$

⇒ The discrete-time representation of Equation ① will take the form :-

$$\bar{x}((k+1)T) = \bar{G}(T) \bar{x}(kT) + \bar{H}(T) \bar{u}(kT) \quad \textcircled{0}$$

→ \bar{G} & \bar{H} depends on Sampling period T. Once the Sampling period T is fixed, G and H are constant matrix.

We know,

$$\bar{x}(t) = e^{\bar{A}t} \bar{x}(0) + \int_0^t e^{\bar{A}(t-\tau)} \bar{B} u(\tau) d\tau$$

⇒ We assume that the input $u(t)$ is sampled and fed to a zero Order hold so that all the components of $u(t)$ are constant over the interval between any two consecutive Sampling instants.

$$u(t) = u(kT) \quad \forall \quad kT \leq t \leq KT + T \quad \textcircled{3*}$$

$$\bar{x}((k+1)T) = e^{\bar{A}(k+1)T} \bar{x}(0) + e^{\bar{A}(k+1)T} \int_0^{(k+1)T} e^{-\bar{A}\tau} \bar{B} u(\tau) d\tau \quad \textcircled{4}$$

&

$$\bar{x}(KT) = e^{\bar{A}KT} \bar{x}(0) + e^{\bar{A}KT} \int_0^{KT} e^{-\bar{A}\tau} \bar{B} u(\tau) d\tau \quad \textcircled{5}$$

\Rightarrow Multiplying eqn (5) with $e^{\bar{A}T}$ and subtracting it from eqn (4) we get:-

$$\bar{x}(k+1)T = e^{\bar{A}T} \bar{x}(kT) + e^{\bar{A}(k+1)T} \int_{kT}^{(k+1)T} e^{\bar{A}x} \bar{B} \bar{U}(x) dx \quad (6)$$

Sol:

Using eqn (3*) and (6) and Comparing it with eqn (2) we got.

$$G(T) = e^{\bar{A}T} \quad (7)$$

$$H(T) = \left(\int_0^T e^{\bar{A}x} dx \right) \bar{B} \quad (8)$$

\Rightarrow The output equation becomes :-

$$\bar{y}(kT) = \bar{C} \bar{x}(kT) + \bar{D} \bar{u}(kT)$$

\Rightarrow Matrix C and D are constant matrices and do not depend on the Sampling period T.

\Rightarrow If matrix \bar{A} is non singular, then $H(T)$ can be simplified to :-

$$\bar{H}(T) = \left(\int_0^T e^{\bar{A}\lambda} d\lambda \right) \bar{B} = \bar{A}^{-1} (e^{\bar{A}T} - I) \bar{B} = (e^{\bar{A}T} - I) A^{-1} B$$

* Time response between two consecutive Sampling Instants

⇒ In a Sampled Continuous time System the output is continuous in time.

But Z transform solution of the discrete time System equation gives the output only between two consecutive sampling instants.

⇒ Consider the time invariant Continuous-time System defined by

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u$$

$$\bar{y} = \bar{C}\bar{x} + \bar{D}u$$

Let us assume that the input u is Sampled and fed to a Zero Order hold.

Then $u(\gamma) = u(KT) \quad \forall KT \leq \gamma < KT + T$

⇒ Solution of the state equation starting with the initial state $\bar{x}(t_0)$ is

$$\bar{x}(t) = e^{\bar{A}(t-t_0)} \bar{x}(t_0) + \int_{t_0}^t e^{\bar{A}(t-\tau)} \bar{B} \bar{u}(\tau) d\tau$$

⇒ To obtain the response of the Sampled

System at $t = KT + \Delta T$ when $0 < \Delta T < T$

Let us put $t = KT + \Delta T$, $t_0 = KT$ & $u(\tau) = u(K\tau)$

$$\Rightarrow \bar{x}(KT + \Delta T) = e^{\bar{A}\Delta T} \bar{x}(KT) + \int_{KT}^{KT + \Delta T} e^{A(KT + \Delta T - \tau)} B u(KT) d\tau$$

$$\Rightarrow \bar{x}(KT + \Delta T) = e^{\bar{A}\Delta T} \bar{x}(KT) + \int_0^{\Delta T} e^{\bar{A}\lambda} \bar{B} \bar{u}(KT) d\lambda$$

Let us define,

$$\bar{G}(\Delta T) = e^{\bar{A}\Delta T}$$

$$\bar{H}(\Delta T) = \left(\int_0^{\Delta T} e^{\bar{A}\lambda} d\lambda \right) \bar{B}$$

We obtain,

$$\bar{x}(KT + \Delta T) = G(\Delta T) \bar{x}(KT) + H(\Delta T) u(KT)$$

So

$$\bar{y}(KT + \Delta T) = \bar{C} \bar{x}(KT + \Delta T) + \bar{D} \bar{u}(KT)$$

$$\boxed{\bar{y}(KT + \Delta T) = \bar{C} \bar{G}(\Delta T) \bar{x}(KT) + [\bar{C} \bar{H}(\Delta T) + \bar{D}] \bar{u}(KT)}$$

5.6) Liapunov Stability Analysis

→ Plays an important role in the stability analysis of control system described by State-Space equations.

Method 1

Method 2

Consists entirely of procedures
in which the explicit form
of the Solution is used

Dose not give the
Solutions of differential
or difference equations

(Also called
Direct Method)

(Due to this Second
method is so useful)
in practice

⇒ Liapunov Stability Criterion is applicable to both linear and nonlinear, time invariant or time varying system.

* Second Method of Liapunov

⇒ From the classical theory of mechanics, we know that a Vibratory System is stable if its total energy is continually decreasing until a equilibrium state is reached.

→ Second method of Liapunov is generalization of this fact.

⇒ For purely mathematical System, however, there is no simple way of defining an "energy function".

↳ To avoid this difficulty, Liapunov introduced the Liapunov functions, a fictitious energy function.

Positive definiteness of Scalar Function

A scalar function $V(\bar{x})$ is said to be positive definite in a region S_1 if $V(\bar{x}) > 0$ for all nonzero states \bar{x} in the region S_1 and if $V(\bar{0}) = 0$.

} It includes origin
of the state space

A time-varying function $V(\bar{x}, t)$ is said to be positive definite in a region S_2 if it is bounded from below by a time invariant positive definite function, that is if there exists a positive definite function $V(\bar{x})$ such that

$$V(\bar{x}, t) > V(\bar{x}) \quad \forall t \geq t_0$$

$$V(\bar{0}, t) = 0 \quad \forall t \geq t_0$$

Negative definiteness of Scalar function

A Scalar function $V(\bar{x})$ is said to be negative definite if $-V(\bar{x})$ is positive definite.

Positive Semidefiniteness of Scalar function

A Scalar function $V(\bar{x})$ is said to be positive semidefinite if it is possible at all states in the region S_2 except at the origin and at certain other states, where it is zero.

Negative Semidefiniteness of Scalar function

A Scalar function $V(\bar{x})$ is said to be negative semidefinite if $-V(\bar{x})$ is positive semidefinite.

Indefiniteness of Scalar function

A Scalar function $V(\bar{x})$ is said to be indefinite if in the region S_2 it assumes both positive and negative values, no matter how small the region S_2 is.

★ Liapunov Function

⇒ It is a Scalar function, is a positive definite function, and it is continuous together with its first partial derivatives in the region S_2 about the origin and has a time derivative that, when taken along the trajectory is negative definite (or negative semidefinite).

⇒ $\dot{V}(\bar{x}, t)$ is total derivative of $V(\bar{x}, t)$ with respect to 't' along a solution of the system.

↳ Hence $\dot{V}(\bar{x}, t) < 0$ means $V(\bar{x}, t)$ is a decreasing function of time.

⇒ A Liapunov function is not unique for a given system.

⇒ Later in this section we will show that in the second method of Liapunov the sign behaviour of $V(\bar{x}, t)$ and that of its time derivative $\dot{V}(\bar{x}, t)$ gives information about the stability of an equilibrium state without having the solution.

⇒ Simplest positive definite function is of a quadratic form:-

$$V(\bar{x}) = \sum_{i=1}^m \sum_{j=1}^n a_{ij} x_i x_j \quad \forall i, j = 1, 2, \dots, n$$

⇒ In general, Liapunov functions may not be of a simple quadratic form.

$$\text{Let } \frac{x_1}{x_m} = \hat{x}_1, \quad \frac{x_2}{x_m} = \hat{x}_2, \quad \dots, \quad \frac{x_{m-1}}{x_m} = \hat{x}_{m-1}$$

⇒ Then in the neighbourhood of the origin the lowest-degree terms alone will become dominant and we can write $V(\bar{x})$ as:-

$$V(\bar{x}) = x_n^p V(\hat{x}_1, \hat{x}_2, \dots, \hat{x}_{n-1}, 1)$$

\Rightarrow For p odd, x_n^p can assume both positive and negative values near the origin, which means that $V(\bar{x})$ is not positive definite.
• Hence p must be even.

↓
 Lowest degree terms in V must
 be even

★ System: The system we consider here is defined by:

$$\dot{\bar{x}} = f(\bar{x}, t) \quad \text{--- (1)}$$

where \bar{x} is a state vector (an n -vector) and $f(\bar{x}, t)$ is an n -vector whose elements are functions of x_1, x_2, \dots, x_n and t .

\Rightarrow We assume that the system of eq(1) has a unique solution starting at a given initial condition.

\Rightarrow Let solution be $\bar{\varphi}(t; \bar{x}_0, t_0)$.

$$\text{So } \bar{x}_0 = \bar{\varphi}(t_0; \bar{x}_0, t_0)$$

* Equilibrium State: In the system of eq. ① a state \bar{x}_e , where,

$$f(\bar{x}_e, t) = 0 \quad \forall t$$

is called an equilibrium state of the system.

If the system is linear and time invariant that is, if $f(\bar{x}, t) = \bar{A}\bar{x}$, then,

→ There exist only one equilibrium state if \bar{A} is non singular.

→ There exist infinitely many equilibrium states if \bar{A} is singular.

For nonlinear system, there may be one or more equilibrium states.

⇒ Any isolated equilibrium state can be shifted to the origin of the coordinates, on $f(\bar{o}, t) = \bar{0}$, by a translation of coordinates.

* Stability in the Sense of Liapunov

⇒ We denote a spherical region of radius r_1 about an equilibrium state \bar{x}_e as

$$\|\bar{x} - \bar{x}_e\| \leq r_1$$

→ Euclidean norm

~~Correspondence~~

⇒ Let $S(\delta)$ consists of all points such that

$$\|\bar{x}_0 - \bar{x}_e\| \leq \delta$$

and let $S(\epsilon)$ consist of all points such that

$$\|\varphi(t; \bar{x}_0, t_0) - \bar{x}_e\| \leq \epsilon \quad \forall t \geq t_0$$

⇒ An equilibrium state \bar{x}_e of the System of eq. ① is said to be stable in the sense of Liapunov if, corresponding to each $S(\epsilon)$ ~~containing~~, there is an $S(\delta)$ such that trajectories starting in $S(\delta)$ do not leave $S(\epsilon)$ as t increases indefinitely.

⇒ The real number δ depends on ϵ and in general, also depends on t_0 .

↳ If S does not depend on t_0 , the equilibrium state is said to be uniformly stable.

* Asymptotic Stability

• An equilibrium state \bar{x}_e of the System of eq. ① is said to be asymptotically stable if it is stable in the sense of Liapunov and if every solution starting within $S(\delta)$ converges without leaving $S(\epsilon)$, to \bar{x}_e at time increases indefinitely.

\Rightarrow ~~Region of~~ Largest region of asymptotic stability is called domain of attraction.

) That part of State Space in {
which asymptotically
stable trajectories originate}

* Asymptotic stability in the Large

If asymptotic stability holds for all states from which trajectories originate, the equilibrium state is said to be asymptotically stable in the large.

) If every solution converges to \bar{x}_e as t increases indefinitely

\Rightarrow Obviously, a necessary condition for asymptotic stability in the large is that there be only one equilibrium state in the whole state space.

\Rightarrow In control engineering problems, asymptotic stability in the large is a desirable feature.

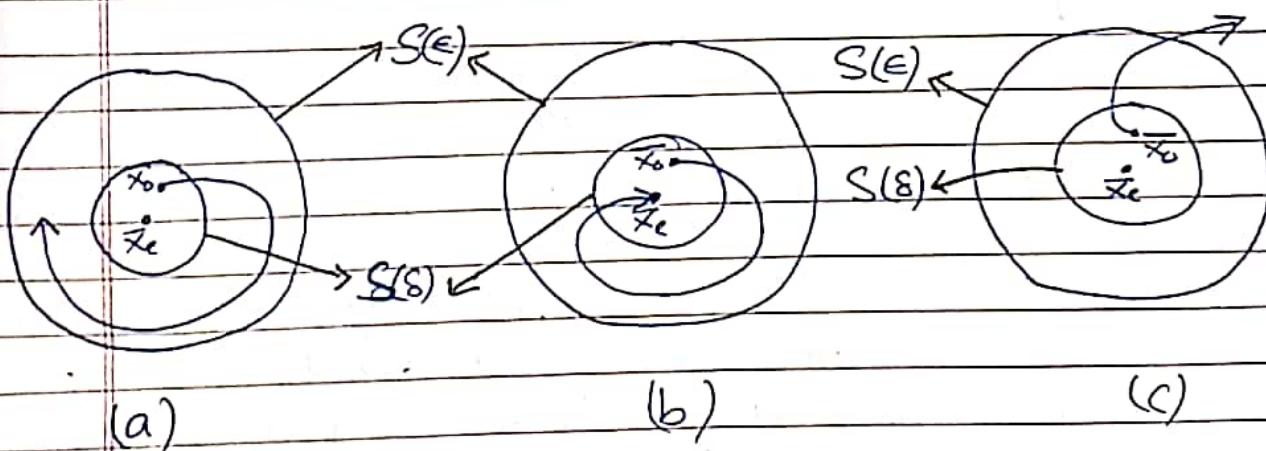
\hookrightarrow If the equilibrium state is not asymptotically

stable in the large, then the problem becomes one of determining the largest region of asymptotic stability.

→ This is usually very difficult.

* Instability: An equilibrium state \bar{x}_e is said to be unstable if for some real number $\epsilon > 0$ and any real number $\delta > 0$, no matter how small, there is always a state \bar{x}_0 in $S(\delta)$ such that the ~~tex~~ trajectory starting at this state leaves $S(\epsilon)$.

* Graphical representation of Stability, Asymptotic stability, and Instability



Stable Equilibrium State

Asymptotically Stable equilibrium

Unstable Equilibrium State

$S(\delta) \rightarrow$ Bounds the initial state.

$S(\epsilon) \rightarrow$ Boundary for trajectory starting from initial state \bar{x}_0 in the region $S(\delta)$

* Liapunov Theorem on Asymptotic Stability

If a scalar function $V(\bar{x})$, where \bar{x} is an n -vector, is positive definite then the states \bar{x} that satisfy

$$V(\bar{x}) = C$$

where C is a positive constant, lie on a closed hypersurface in the n -dimensional state space, at least in the neighbourhood of the origin.

\Rightarrow If $V(\bar{x}) \rightarrow \infty$ as $\|\bar{x}\| \rightarrow \infty$, then such closed surface extend over the entire state space.

\Rightarrow The hypersurface $V(\bar{x}) = C_1$ lies entirely inside the hypersurface $V(\bar{x}) = C_2$ if $C_1 < C_2$.

\Rightarrow For a given system, if a positive definite scalar function $V(\bar{x})$ can be found such that its time derivative taken along a trajectory is always negative, then as time increases, $V(\bar{x})$ takes smaller & smaller values of C .

\hookrightarrow As time increases, $V(\bar{x})$ finally shrink to zero, and therefore \bar{x} also shrink to zero.

Theorem 5.1: Suppose a system is described by

$$\dot{\bar{x}} = \bar{f}(\bar{x}, t)$$

where,

$$\bar{f}(\bar{0}, t) = \bar{0} \quad \forall t$$

If there exists a scalar function $V(\bar{x}, t)$ having continuous first partial derivative & satisfying the conditions:-

1. $V(\bar{x}, t)$ is positive definite.
2. $\dot{V}(\bar{x}, t)$ is negative definite.

then the equilibrium state at the origin is uniformly asymptotically stable.

\Rightarrow If, in addition, $V(\bar{x}, t) \rightarrow \infty$ as $\|\bar{x}\| \rightarrow \infty$, then the equilibrium state at the origin is uniformly asymptotically stable in the large.

\Rightarrow The conditions of this theorem may be modified

- i) $V(\bar{x}, t)$ is positive definite.
- ii) $\dot{V}(\bar{x}, t)$ is negative semidefinite.
- iii) $\dot{V}(\bar{Q}(t; \bar{x}_0, t_0), t)$ does not vanish identically in $t \geq t_0$ for any t_0 and any $\bar{x}_0 \neq 0$ where $\bar{Q}(t; \bar{x}_0, t_0)$ denotes the solution starting from \bar{x}_0 at $t = t_0$.

\Rightarrow Then the origin of the system is uniformly asymptotically stable in the large.

Theorem 5-2: Suppose a system is described by:-

$$\dot{\bar{x}} = \bar{f}(\bar{x}, t)$$

where $\bar{f}(\bar{0}, t) = \bar{0} \forall t$.

\Rightarrow If there exist a scalar function $V(\bar{x}, t)$ having continuous first partial derivatives and satisfying the conditions

1. $V(\bar{x}, t)$ is positive definite.

2. $\dot{V}(\bar{x}, t)$ is negative Semidefinite

then the equilibrium state at the origin is uniformly stable.

\rightarrow { Not necessarily Uniformly
Asymptotically Stable }

* Instability

Theorem 5-3: Suppose a system is ~~described~~ described by:-

$$\dot{\bar{x}} = \bar{f}(\bar{x}, t)$$

where,

$\bar{f}(\bar{0}, t) = \bar{0} \forall t \geq t_0$

If there exist a scalar function $W(\bar{x}, t)$ having continuous first partial derivatives

and satisfying the conditions

1. $W(\bar{x}, t)$ is positive definite in some region about the origin.

2. $\dot{W}(\bar{x}, t)$ is positive definite in the same region then equilibrium state at the origin is unstable.

★ Stability Analysis of Linear time-Invariant System

⇒ There are many approaches to investigate asymptotic stability of linear time-invariant system.

↳ Example: For a continuous-time system described by the equation

$$\dot{\bar{x}} = \bar{A}\bar{x}$$

it can be stated that a necessary and sufficient condition for the asymptotic stability of the origin of the system is that all eigenvalues of \bar{A} have negative real part.

↳ On the zeros of the characteristic polynomial

$$|S\bar{I} - \bar{A}| = S^n + a_1 S^{n-1} + \dots + a_{n-1} S + a_n$$

have negative real parts.

\Rightarrow Similarly, for a discrete-time system represented by the equation:-

$$\bar{x}(k+1) = \bar{G} \bar{x}(k)$$

A necessary and sufficient condition that can be stated for the asymptotic stability of the origin is that all eigenvalues of \bar{G} be less than unity in their magnitude.

\hookrightarrow Or that the zeros of the characteristic polynomial:

$$|z\bar{I} - \bar{G}| = z^n + a_1 z^{n-1} + \dots + a_{n-1} z + a_n$$

lies within the unit circle centered at the origin of the Z -plane.

\Rightarrow For linear time invariant systems the second method of Liapunov gives not just sufficient conditions, but the necessary and sufficient conditions for stability or asymptotic stability.

\Rightarrow The Conjugate transpose of A is denoted by \bar{A}^* .

$$\boxed{\bar{A}^* = \overline{A}^T}$$

\nearrow Conjugate; \searrow Transpose

* Liapunov stability analysis of linear time invariant
Continuous time System

⇒ Consider the following linear time invariant system:

$$\dot{\bar{X}} = \bar{A}\bar{X} \quad \text{--- (1)}$$

⇒ We assume that \bar{A} is non singular.

→ Then the only equilibrium state is the origin, $\bar{X}=0$.

→ The stability of the equilibrium state can easily be investigated by second method of Liapunov.

⇒ For the system defined by eq (1), let us choose a possible Liapunov function.

$$V(\bar{X}) = \bar{X}^* \bar{P} \bar{X}$$

where P is positive definite Hermitian matrix.

⇒ The time derivative of $V(\bar{X})$ along any trajectory is

$$\begin{aligned}\dot{V}(\bar{X}) &= \dot{\bar{X}}^* \bar{P} \bar{X} + \bar{X}^* \bar{P} \dot{\bar{X}} \\ &= (\bar{A}\bar{X})^* \bar{P} \bar{X} + \bar{X}^* \bar{P} \bar{A}\bar{X} \\ &= \bar{X}^* \bar{A}^* \bar{P} \bar{X} + \bar{X}^* \bar{P} \bar{A}\bar{X} \\ &= \bar{X}^* (\bar{A}^* \bar{P} + \bar{P} \bar{A})\bar{X}\end{aligned}$$

\Rightarrow Since $V(\bar{x})$ was chosen to be positive definite, we require, for asymptotic stability, that $\dot{V}(\bar{x})$ be negative definite.



\rightarrow Therefore, we require that,

$$V(\bar{x}) = -\bar{x}^* \bar{Q} \bar{x}$$

where,

$$\bar{Q} = -(\bar{A}^* \bar{P} + \bar{P} \bar{A}) = \text{Positive definite}$$

Hence, for the asymptotic stability of the system of eq (i), it is sufficient that \bar{Q} be positive definite.

Theorem 5-4: Consider the system described by:-

$$\dot{\bar{x}} = \bar{A} \bar{x}$$

A necessary and sufficient condition for the equilibrium state $\bar{x} = \bar{0}$ to be asymptotically stable in the large is that, given any positive definite Hermitian matrix \bar{Q} , there exists a positive definite Hermitian matrix \bar{P} such that

$$\bar{A}^* \bar{P} + \bar{P} \bar{A} = -\bar{Q}$$

The scalar function $\bar{x}^* \bar{P} \bar{x}$ is a Liapunov function for this system.

★ Liapunov stability analysis of discrete-time systems

Theorem 5-5: Consider the discrete-time system

$$\bar{x}((k+1)T) = \bar{f}(\bar{x}(kT))$$

Where, \bar{x} = n-Vector

$\bar{f}(\bar{x})$ = n-Vector with property
that $\bar{f}(\bar{0}) = \bar{0}$

T = Sampling period

Suppose there exists a scalar function $V(\bar{x})$
Continuous in \bar{x} such that

$$1. V(\bar{x}) > 0 \quad \forall \bar{x} \neq \bar{0}$$

$$2. \Delta V(\bar{x}) < 0 \quad \forall \bar{x} \neq \bar{0} \text{ where,}$$

$$\Delta V(\bar{x}(kT)) = V(\bar{x}(k+1)T) - V(\bar{x}(kT))$$

$$= V(\bar{f}(\bar{x}(kT))) - V(\bar{x}(kT))$$

$$3. V(\bar{0}) = 0$$

$$4. V(\bar{x}) \rightarrow \infty \text{ as } \|\bar{x}\| \rightarrow \infty$$

then the equilibrium state $\bar{x} = \bar{0}$ is asymptotically stable in the large and $V(\bar{x})$ is a Liapunov function.

classmate
Date _____
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* Liapunov Stability analysis of Linear Time-Invariant Discrete-Time System

Consider the discrete-time system described by:-

$$\bar{x}(k+1) = \bar{G} \bar{x}(k)$$

The origin $\bar{x} = \bar{0}$ is the equilibrium state.

Let us choose as a possible Liapunov function

$$V(\bar{x}(k)) = \bar{x}^*(k) \bar{P} \bar{x}(k)$$

where P is a positive definite Hermitian matrix.

Then,

$$\begin{aligned}\Delta V(\bar{x}(k)) &= V(\bar{x}(k+1)) - V(\bar{x}(k)) \\ &= \bar{x}^*(k+1) \bar{P} \bar{x}(k+1) - \bar{x}^*(k) \bar{P} \bar{x}(k) \\ &= [\bar{G} \bar{x}(k)]^* \bar{P} [\bar{G} \bar{x}(k)] - \bar{x}^*(k) \bar{P} \bar{x}(k) \\ &\Rightarrow \bar{x}^*(k) (\bar{G}^* \bar{P} \bar{G} - \bar{P}) \bar{x}(k)\end{aligned}$$

Since $V(\bar{x}(k))$ is chosen to be positive definite, we require, for asymptotic stability that $\Delta V(\bar{x}(k))$ be negative definite.

Therefore,

$$\Delta V(\bar{x}(k)) = -\bar{x}^*(k) \bar{Q} \bar{x}(k)$$

∴ Where,

$$\bar{Q} = -(\bar{G}^* \bar{P} \bar{G} - \bar{P}) = \text{Positive definit.}$$

Theorem 5-6: Consider the discrete-time system

$$\bar{x}(k+1) = \bar{G}\bar{x}(k)$$

A necessary and sufficient condition for the equilibrium state $\bar{x}=0$ to be asymptotically stable in the large is that, given any positive-definite Hermitian matrix \bar{Q} , there exist a positive definite Hermitian matrix \bar{P} such that

$$\bar{G}^* \bar{P} \bar{G} - \bar{P} = -\bar{Q}$$

The scalar function $\bar{x}^* \bar{P} \bar{x}$ is a Lyapunov function for this system.

* Stability of discrete-time System obtained by discretizing a Continuous-time System.

⇒ The asymptotic stability of an equilibrium state of a discrete-time system obtained by discretizing a continuous-time system is equivalent to that of the original continuous-time system.

⇒ Consider a continuous-time system:-

$$\dot{x} = \bar{A}x \quad \text{--- (1)}$$

and the corresponding discrete-time system

$$\bar{x}((k+1)T) = \bar{G}\bar{x}(kT) \quad \text{--- (2)}$$

where, $\bar{G} = e^{\bar{A}T}$

\Rightarrow If the Continuous-time System is asymptotically stable, that is, if all the eigenvalues of the matrix \bar{A} have negative real parts, then

$$\|\bar{G}^n\| \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

and the discretized system is also asymptotically stable. This is because if the λ_i 's are the eigenvalues of \bar{A} then the $e^{\lambda_i T}$'s are the eigenvalues of \bar{G} . (Note that $|e^{\lambda_i T}| < 1$ if $\lambda_i T$ is negative).

\Rightarrow It should be noted here that, if a continuous-time system having complex poles is discretized, then in certain exceptional cases hidden instability may occur, depending on the choice of the Sampling period T .

* Contraction: A function $\bar{f}(\bar{x})$ is said to be a contraction if $\bar{f}(0) = 0$ &

$$\|\bar{f}(\bar{x})\| < \|\bar{x}\|$$

for some set of values of $\bar{x} \neq 0$.