Lecture 4

Diagonalization, Modal Analysis, Intro to Feedback

★ Similarity Transform

⇒ Let T be an invertible matrix, and consider a coordinate transformation $x = T\tilde{x}$.

↳ This is called similarity transform.

⇒ The standard state-space model can be written as:

$$\Rightarrow \boxed{\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}} \Longrightarrow \begin{aligned} T\dot{\tilde{x}} &= AT\tilde{x} + Bu \\ y &= CT\tilde{x} + Du \end{aligned}$$

$$\Rightarrow \begin{aligned} \dot{\tilde{x}} &= (T^{-1}AT)\tilde{x} + (T^{-1}B)u \\ y &= (CT)\tilde{x} + Du \end{aligned}$$

$$\Rightarrow \boxed{\begin{aligned} \dot{\tilde{x}} &= \tilde{A}\tilde{x} + \tilde{B}u \\ y &= \tilde{C}\tilde{x} + \tilde{D}u \end{aligned}} \quad \text{where, } \left\{ \begin{aligned} \tilde{A} &= T^{-1}AT \\ \tilde{B} &= T^{-1}B \\ \tilde{C} &= CT \\ \tilde{D} &= D \end{aligned} \right\}$$

⇒ The choice of a state-space model for a given system is not unique.

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★ Controllability and Observability

⇒ A LTI system of the form $\dot{x} = Ax + Bu$ is said to be **controllable** if for any given initial state $x(0) = x_c$ there exists a control signal that takes the state to the origin $x(t) = 0$ for some finite time t.

⇒ A LTI system of the form $\dot{x} = Ax + Bu$, $y = Cx + Du$ is said to be **observable** if any given initial condition $x(0) = x_0$ can be reconstructed based on the knowledge of the output and input signal only over a finite time interval $[0, t]$.

⇒ An LTI system is **stabilizable** if all unstable modes are controllable.

⇒ An LTI system is **detectable** if all unstable modes are observable.

★ Diagonalization

⇒ Lets assume we have n independent eigenvectors; then we can assemble the eigenvectors into an invertible matrix V whose columns are the eigenvectors v_i.

$$V = [v_1 \; v_2 \; \dots \; v_n] \qquad \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \; \lambda_n \end{bmatrix}$$

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So $\boxed{AV = V\Lambda}$

★ Modal Coordinates

⇒ Eigen value and eigen vectors of A define the modes of the system.

⇒ The transformed coordinates $X(t) = V\tilde{X}$ are called modal coordinates.

⇒ The eigen vector V_i defines the shape of the ith mode.

⇒ The eigen value λ_i defines how the amplitude of the mode evolves over time.

⇒ Let $\boxed{\dot{X}(t) = AX(t)}$ be the system.

⇒ Let $V = [V_1\ V_2 \cdots V_n]$ be the eigen vector matrix.

⇒ Let $\Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$ be eigen value matrix.

⇒ Let $X(t) = V\tilde{X}(t)$

$$\Rightarrow \dot{\tilde{X}} = (V^{-1}AV)\tilde{X} \quad \{\Lambda = V^{-1}AV\}$$

$$\Rightarrow \dot{\tilde{X}}(t) = \Lambda \tilde{X}(t)$$

$$\Rightarrow \tilde{X}(t) = e^{\Lambda t}\tilde{X}(0)$$

$$\Rightarrow V^{-1}x(t) = \cancel{e^{\Lambda t}V^{-1}X(0)}\ e^{\Lambda t}\tilde{X}(0)$$

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$$X(t) = V e^{\Lambda t} \tilde{X}(0)$$

$$\Rightarrow [V_1 \; V_2 \; \cdots \; V_n] \begin{bmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} \tilde{X}_1(0) \\ \tilde{X}_2(0) \\ \vdots \\ \tilde{X}_n(0) \end{bmatrix}$$

$$\Rightarrow \boxed{\sum_{i=1}^{n} e^{\lambda_i t} \tilde{X}_i(0) V_i = X(t)}$$

$\Rightarrow$ Any trajectory can be expressed as a linear combination of modes.

★ <u>Pole placement – First order systems</u>

$\Rightarrow$ Consider a first order control system with dynamics $\dot{x} = ax + bu$, and assume that we are not happy about its behaviour

$\Rightarrow$ <u>Feedback control</u>: Choose $u = -Kx$, then the dynamics would be

$$\dot{x} = (a - bK)x$$

$\Rightarrow$ As long as $b \neq 0$ (i.e. the system is controllable)

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. by choosing $K = (a - a^*)/b$, we can place the "closed-loop" eigenvalue at a desired value a^*, or anywhere we want on the real axis!

⇒ This is the simplest example of a general technique called "pole placement".

★ Effect on feedback for closed-loop dynamics

⇒ If we have an open-loop LTI system

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$Y(t) = CX(t)$$

⇒ By choosing a linear feedback $u = -KY = -KCX$, we can transform it into another closed-loop LTI system:

$$\boxed{\begin{aligned} \dot{X}(t) &= (A - BKC)X(t) \\ Y(t) &= CX(t) \end{aligned}}$$

⇒ In general "negative feedback" has stabilizing effects and the bigger K, the faster the closed-loop system is, and the smaller the errors are.

⇒ However this is not generally the case.