(1) Controllable Canonical Form

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n} \quad \text{--- (1)}$$

$$(1) \quad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u \quad \text{--- (2)}$$

$$(2) \quad y = \left[b_n - a_n b_0 \,\vdots\, b_{n-1} - a_{n-1} b_0 \,\vdots\, \cdots \,\vdots\, b_1 - a_1 b_0 \right] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0 u \quad \text{--- (3)}$$

Solution

Equation (1) can be written as:-

$$\frac{Y(s)}{U(s)} = \frac{b_0 \left(s^n + a_1 s^{n-1} + \cdots a_{n-1} s + a_n \right) + \left\{ (b_1 - b_0 a_1) s^{n-1} + (b_2 - b_0 a_2) s^{n-2} + \cdots + (b_{n-1} - a_{n-1} b_0) s + (b_n - a_n b_0) \right\}}{\left(s^n + a_1 s^{n-1} + \cdots a_{n-1} s + a_n \right)}$$

$$\Rightarrow \frac{Y(s)}{U(s)} = b_0 + \frac{(b_1 - a_1 b_0) s^{n-1} + \cdots + (b_{n-1} - a_{n-1} b_0) s + (b_n - a_n b_0)}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}$$

$\otimes U(s)$

So $$\boxed{Y(s) = b_0 U(s) + \hat{Y}(s)}$$

$$\hat{Y}(s) = \frac{(b_1 - a_1 b_0)s^{n-1} + \cdots + (b_{n-1} - a_{n-1} b_0)s + (b_n - a_n b_0)}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

$\Rightarrow$ Let $$Q(s) = \frac{\hat{Y}(s)}{(b_1 - a_1 b_0)s^{n-1} + \cdots + (b_{n-1} - a_{n-1} b_0)s + (b_n - a_n b_0)}$$

$$= \frac{U(s)}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

$$s^n Q(s) = U(s) - a_1 s^{n-1} Q(s) \cdots\cdots - a_{n-1} s Q(s) - a_n Q(s) \quad \text{— (4)}$$

$$\hat{Y}(s) = (b_1 - a_1 b_0)s^{n-1}Q(s) + \cdots + (b_{n-1} - a_{n-1} b_0) s Q(s) + (b_n - a_n b_0) Q(s) \quad \text{— (5)}$$

$\Rightarrow$ Now let us define state variable as follows :-

$$X_1(s) = Q(s)$$
$$X_2(s) = sQ(s)$$
$$\vdots \qquad \vdots$$
$$X_{n-1}(s) = s^{n-2}Q(s)$$
$$X_n(s) = s^{n-1}Q(s)$$

So Clearly,

$$X_2 = sX_1$$
$$X_3 = sX_2$$
$$\vdots \quad \vdots \quad \vdots$$
$$X_n = sX_{n-1}$$

(6)

$$\boxed{\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_{n-1} &= x_n \end{aligned}}$$

{Time domain}

--> Space D. ... System of

$\Rightarrow$ Rewriting eq ④ $\{ s^n Q(s) = sX_n \}$

$$sX_n = -a_1 X_n(s) - \cdots - a_{n-1} X_2(s) - a_n X_1(s) + U(s)$$

$\downarrow$ {Time domain}

$$\cancel{\dot{x}_n = -a_1 x_1 \cdots - a_{n-1} x_1 - a_n}$$

$$\boxed{\dot{x}_n = -a_n x_1 - a_{n-1} x_2 - \cdots - a_1 x_n + u} \quad \text{— ⓑ}$$

Using a & b we got eq ② {Part 1 of solution}

$\Rightarrow$ We know that $Y(s) = b_0 U(s) + \hat{Y}(s)$

$$Y(s) = b_0 U(s) + (b_1 - a_1 b_0) s^{n-1} Q(s) + \cdots + (b_{n-1} - a_{n-1} b_0) s Q(s) + (b_n - a_n b_0) Q(s)$$

$$\Rightarrow Y(s) = b_0 U(s) + (b_1 - a_1 b_0) X_n(s) + \cdots + (b_{n-1} - a_{n-1} b_0) X_2(s) + (b_n - a_n b_0) X_1(s)$$

$\downarrow$ {Time domain}

$$\boxed{y = b_0 u + (b_n - a_n b_0) x_1 + (b_{n-1} - a_{n-1} b_0) x_2 + \cdots + (b_1 - a_1 b_0) x_n} \quad \text{— ⓒ}$$

From eq ⓒ we got eq ③ {Part 2 of the solution}