Design of Linear feedback Control System

⇒ Feedback is the most important idea in the design of control system.

⇒ We will study what can be accomplished when the state feedback or output feedback is applied to system described by:

→ $\dot{x}(t) = Ax(t) + Bu(t)$
$y(t) = Cx(t) + Du(t)$ { CT-LTI System }

→ $x(k+1) = Ax(k) + Bu(k)$
$y(k) = Cx(k) + Du(k)$ { DT-LTI System }

$x \in R^{n\times 1}$

$u \in R^{r\times 1}$

$y \in R^{m\times 1}$

★ State Feedback and Eigenvalue Assignment

⇒ Consider the state feedback control law:

$u(t) = -Kx(t) + Fv(t)$; $u(k) = -Kx(k) + Fv(k)$

where F is nonsingular.

⇒ The closed loop system is described by

$$\boxed{\begin{aligned}\dot{x}(t) &= [A-BK]x(t) + BFv(t)\\ y(t) &= [C-DK]x(t) + DFv(t)\end{aligned}}$$

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$$\boxed{\begin{aligned} x(k+1) &= [A-BK]x(k) + BFV(k) \\ y(k) &= [C-DK]x(k) + DFV(k) \end{aligned}}$$

★ **Dynamic properties of Closed loop system**

⇒ The presence of a feedback loop drastically changes the dynamic properties of a system.

⇒ The eigenvalues of the closed loop system matrix $(A-BK)$ can be arbitrarily assigned if the (open loop) system is controllable.

Theorem F1: The state feedback dynamical system is controllable for any feedback gain matrix K and any nonsingular matrix F if & only if the dynamical system is controllable.

⇒ For observability, there is no analogous result.

→ The observable system may loose observability after the loop is closed.

→ And unobservable system may gain observability after the loop is closed.

★ Eigenvalue assignment for single-input system

⇒ Let a controllable single-input system is represented in the controllable canonical form.

$$\dot{X} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & & \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} X + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

⇒ The state feedback control law is

$$u(t) = -K_1 x_1(t) - K_2 x_2(t) - \cdots - K_n x_n(t) + V(t)$$

$$= -\sum_{i=1}^{n} K_i x_i(t) + V(t)$$

⇒ The closed loop system is described by

$$\dot{X} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -(K_1+a_0) & -(K_2+a_1) & -(K_3+a_2) & \cdots & -(K_n+a_{n-1}) \end{bmatrix} X + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} V$$

$\Rightarrow$ The characteristic equation for this system is

$$\boxed{\lambda^n + (K_n + a_{n-1})\lambda^{n-1} + \cdots (K_2 + a_1)\lambda + (K_1 + a_0) = 0}$$

$\Rightarrow$ Every coefficient of this characteristic equation can take any real value by proper selection of the state feedback gain.

$\hookrightarrow$ This implies that the closed loop eigenvalue can be placed anywhere in the complex plane (symmetric to real axis)

$\Rightarrow$ Specifying λ_{ci} $i = 1, \ldots n$ as the desired closed loop eigenvalues, the system of characteristic equation can be written as:

$$\Rightarrow (\lambda - \lambda_{c1})(\lambda - \lambda_{c2}) \cdots (\lambda - \lambda_{cn}) = 0$$

$$\Rightarrow \lambda^n + a_{c(n-1)}\lambda^{n-1} + \cdots + a_{c1}\lambda + a_{c0} = 0$$

So $\boxed{K_i = a_{c(i-1)} - a_{i-1}}$ $\forall\ i = 1, 2 \cdots n$

Controllable SI System not in Controllable Canonical form

Theorem F2: If the n-dimensional time-invariant single input system $\dot{x} = Ax + bu$ is controllable, then it can be transformed, by a equivalence transformation $x = Qx_c$ so that $\dot{x}_c = Q^{-1}AQx_c + Q^{-1}bu$ is in the controllable canonical form.

★ Stabilization

⇒ It has been shown that a single-input uncontrollable system can be decomposed to

$$\begin{bmatrix} \dot{\bar{x}}_c \\ \dot{\bar{x}}_{uc} \end{bmatrix} = \begin{bmatrix} \bar{A}_c & \bar{A}_{12} \\ 0 & \bar{A}_{uc} \end{bmatrix} \begin{bmatrix} \bar{x}_c \\ \bar{x}_{uc} \end{bmatrix} + \begin{bmatrix} \bar{b}_c \\ 0 \end{bmatrix} u$$

{Kalman decomposition}

⇒ Applying the control law:

$$u = -[\bar{K}_c \ \bar{K}_{uc}] \begin{bmatrix} \bar{x}_c \\ \bar{x}_{uc} \end{bmatrix} + v$$

⇒ The closed loop system is

$$\begin{bmatrix} \dot{\bar{x}}_c \\ \dot{\bar{x}}_{uc} \end{bmatrix} = \begin{bmatrix} \bar{A}_c - \bar{b}_c \bar{K}_c & \bar{A}_{12} - \bar{b}_c \bar{K}_{uc} \\ 0 & \bar{A}_{uc} \end{bmatrix} \begin{bmatrix} \bar{x}_c \\ \bar{x}_{uc} \end{bmatrix} + \begin{bmatrix} \bar{b}_c \\ 0 \end{bmatrix} v$$

⇒ The closed loop characteristic polynomial is written as:

$$\det[\lambda I - \bar{A}] = \det[\lambda I_{n_1} - (\bar{A}_c - \bar{b}_c \bar{K}_c)]$$
$$\det[\lambda I_{n-n_1} - \bar{A}_{uc}]$$

n_1 is the dimn of A_c

⇒ The equation implies that:

→ Eigenvalues of the controllable subsystem can be arbitrarily assigned.

→ While those of the uncontrollable subsystem remain fixed under the state feedback.

⇒ Therefore, the single-input system is stabilizable if & only if the uncontrollable portion of the system does not have any unstable eigenvalue.

⇒ The eigenvalues of the controllable portion can be obtained by applying the eigenvalue assignment algorithm algorithm to A_c and b_c.

★ Output Feedback

⇒ If the state vector is not directly measurable, the state feedback control cannot be implemented.

$$\boxed{\begin{aligned}\dot{X} &= Ax + Bu \\ Y &= CX\end{aligned}} \xrightarrow{u = -KY = -KCX} \begin{aligned}&\dot{X} = AX - BKCX \\ &\boxed{\dot{X} = (A - BKC)X}\end{aligned}$$

⇒ The closed loop eigenvalues are the roots of

$$\det[\lambda I - (A - BKC)] = 0$$

⇒ In this case, it is not possible to assign arbitrary closed loop eigenvalues.