Lecture 6:
Poles and Zeros

⇒ The transfer function $g(s)$ can be written as:

$$g(s) = \frac{r_1}{s-p_1} + \frac{r_2}{s-p_2} + \dots + \frac{r_n}{s-p_n} + r_0$$

→ $p_1, \dots p_n$ are the poles i.e the roots of the characteristic polynomial $\det(sI-A)$ i.e. the eigenvalues of A.

→ The numbers $r_1, \dots r_n$ are called the residues.

$$\boxed{r_i = \lim_{s \to p_i} (s-p_i)\, g(s)}$$

⇒ $g(s)$ can also be written as: (root locus form)

$$g(s) = K_{rl} \frac{(s-z_1)(s-z_2)\dots(s-z_m)}{(s-p_1)(s-p_2)\dots(s-p_n)}$$

→ $z_1, z_2 \dots z_m$ are called the zeros of $g(s)$.

⇒ $g(s)$ can also be written as: (Bode form)

$$g(s) = K_{Bode} \frac{\left(\frac{s}{-z_1}+1\right)\left(\frac{s}{-z_2}+1\right)\dots\left(\frac{s}{-z_m}+1\right)}{\left(\frac{s}{-p_1}+1\right)\left(\frac{s}{-p_2}+1\right)\dots\left(\frac{s}{-p_n}+1\right)}$$

* **Example**

Let $g(s) = 2 \dfrac{s+1}{s^3+4s^2+6s+4}$

$\Rightarrow$ If $u(t) = \sin t$

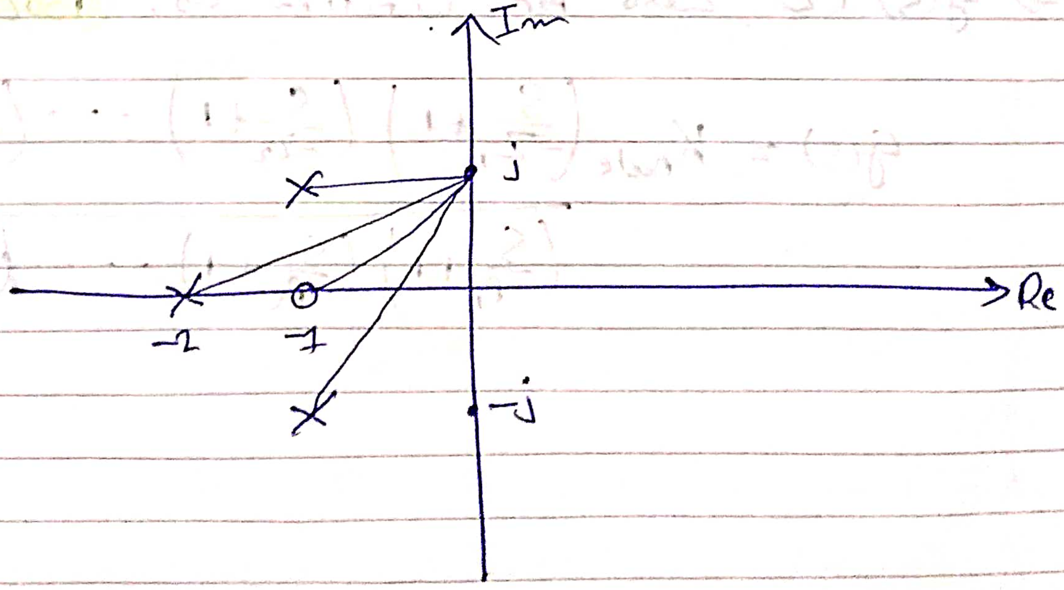
$$y_{ss}(t) = |g(j)| \sin(t + \angle g(j))$$

$$g(s) = 2 \frac{(s+1)}{(s+2)(s+1+j)(s+1-j)}$$ {After factorising}

$$|g(s)| = |2| \frac{|s+1|}{|s+2|\,|s+1+j|\,|s+1-j|}$$

o Zeros : -1

× Poles : -2, $-1 \pm j$



$$|g(j)| = 2 \frac{\sqrt{2}}{\sqrt{5}\cdot\sqrt{5}\cdot 1} = \frac{2\sqrt{2}}{5} \approx 0.5657$$

$$\angle g(s) = \angle(2) + \angle(s+1) - \angle(s+2) - \angle(s+1+j) - \angle(s+1-j)$$

$$\Rightarrow 0 + 45^\circ - \tan^{-1}\left(\tfrac{1}{2}\right) - \tan^{-1}(2) + 0$$

$$= -45^\circ$$

$\Rightarrow$ $y_{ss}(t) = 0.5657 \sin(t - 45^\circ)$

★ <u>Transient response to special inputs</u>

1. <u>Unit impulse $u(t) = \delta(t)$</u>

$\Rightarrow$ This is not really a function, but a mathematical construct such that

$$\int_0^{\epsilon} \delta(t) = 1 \quad \text{for any } \epsilon > 0$$

$$\int_0^{t} f(\tau)\delta(\tau)\,d\tau = f(0) \quad \forall\ t > 0$$

2. <u>Unit step input $u(t) = 1 \ \forall\ t > 0$</u>

$\Rightarrow$ Same as $u(t) = e^{0t}$

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★ Impulse response

⇒ Assume $D=0$, $X(0)=0$ & $u(t)=\delta(t)$. Then:

$$y_{imp}(t) = \int_0^t C e^{A(t-\tau)} B \delta(\tau)\, d\tau$$

$$= C e^{At} B$$

⇒ The impulse response is the same as the response to an initial condition $X(0)=B$.

★ Unit Step response

⇒ Assume $D=0$, $X(0)=0$, $u(t)=1=e^{0t}$ $\forall\ t>0$ and A is invertible then:

$$y_{step}(t) = \int_0^t C e^{A(t-\tau)} B\, d\tau$$

$$= C \int_0^t e^{A(t-\tau)} d\tau\, B$$

$$= -CA^{-1}\left(I - e^{At}\right) B$$

$$\Rightarrow -CA^{-1}B + CA^{-1}e^{At}B$$

⇒ The steady state response is given by

$$y_{ss} = G(0) = -CA^{-1}B$$

★ **First-order system (impulse response)**

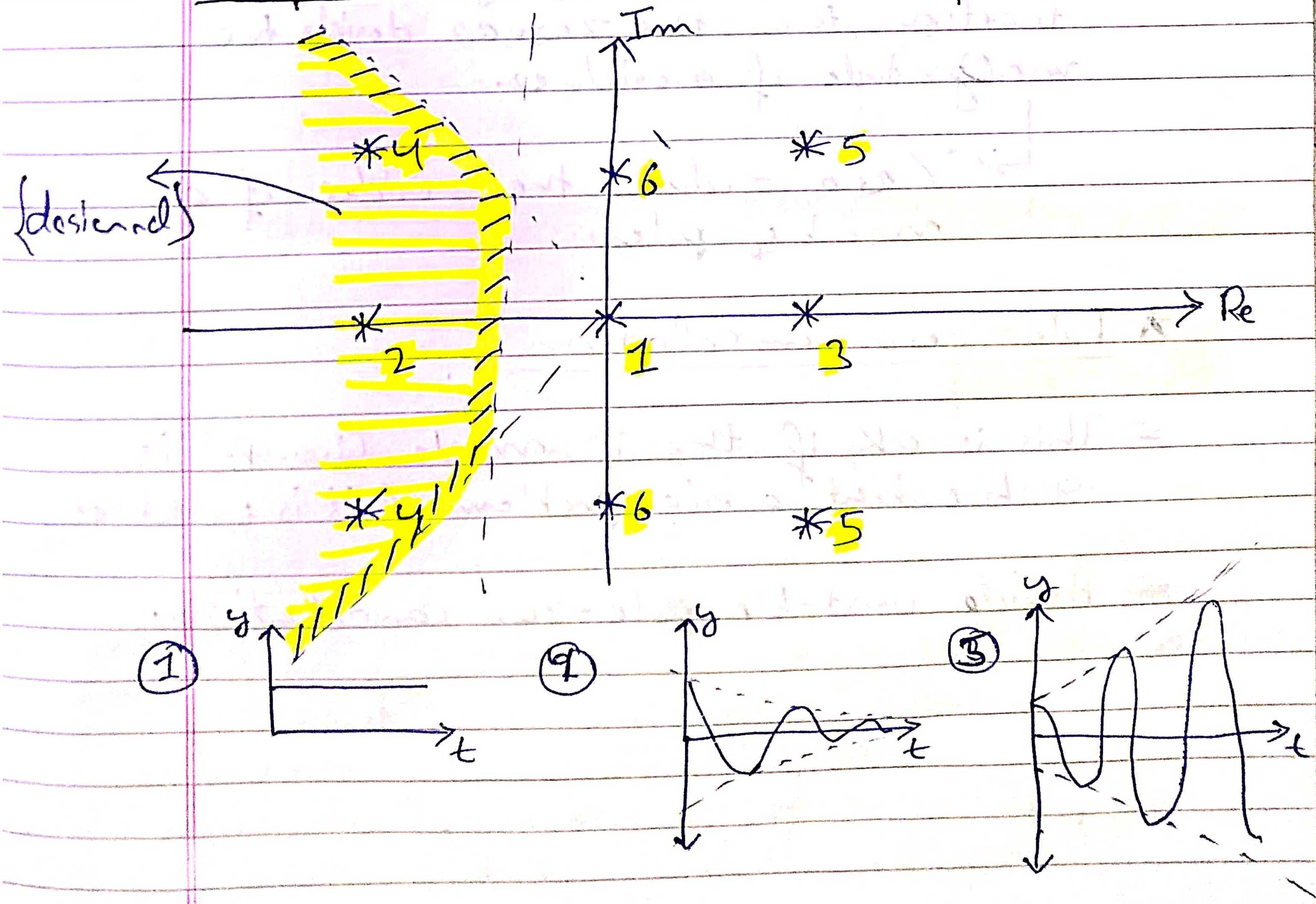
$$g(s) = \frac{r}{s-a} \Rightarrow y(t) = r e^{at}$$

★ **Higher-order system (impulse response)**

$$g(s) = \frac{r_1}{s-p_1} + \frac{r_2}{s-p_2} + \cdots + \frac{r_m}{s-p_m}$$

$$y(t) = r_1 e^{p_1 t} + r_2 e^{p_2 t} + \cdots + r_m e^{p_m t}$$

★ **Response shape as function of pole location**



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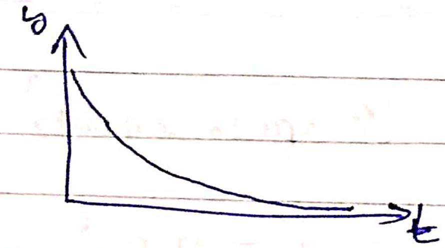

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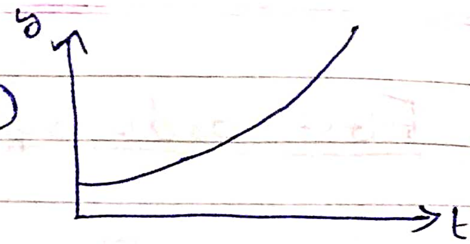

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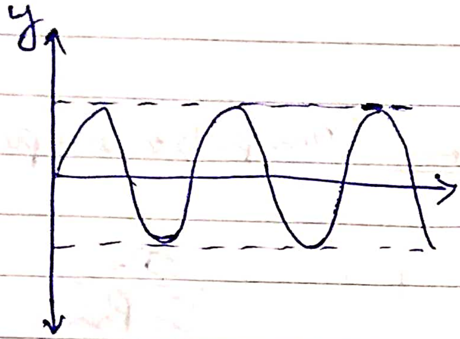

★ Effects of zeros on the response

⇒ If we are decomposing $g(s)$ in partial fraction, then the zeroes decide the magnitude of residues.

↳ Zero reduces the residue of a nearby poles.

★ Pole-Zero cancellation

⇒ This is ok if the i^{th} mode (i.e. p_i) is stable, but a big problem if it is unstable.

⇒ Avoid unstable pole-zero cancellation.

★ Zero with ~~positive~~ negative real part

⇒ If we have a transfer function $g(s) = (s+z)\tilde{g}(s)$, we can decompose it into

$$g(s) = z\tilde{g}(s) + s\tilde{g}(s)$$

⇒ If the impulse response of $\tilde{g}(s)$ is given by $\tilde{y}(s)$, and the impulse response of $g(s)$ is $y(t)$, then remembering that s is the transfer function of a differentiator, we can write

$$y(t) = z\tilde{y}(t) + \dot{\tilde{y}}(t)$$

⇒ In other words, the zero is effectively adding a derivative term to the output.

↳ This typically has an "anticipatory effect"

★ Zero with positive real part (Non-minimum phase zeros)

⇒ Stability of the system is preserved as growth/decay of the terms in the response is not affected by the zeros.

⇒ However, a zero in the right half plane effectively means a "negative" derivative action.

↳ This is the opposite of anticipatory

↳ The output will tend to move in wrong direction initially.