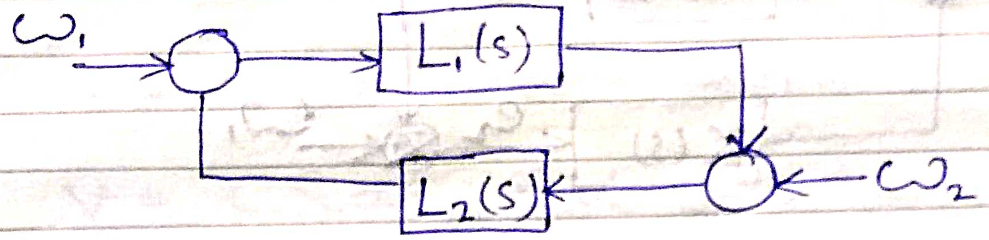
MIMO stability, performance & robustness

① MIMO Stability

⇒ Necessary and Sufficient condition for BIBO stability:

↳ Closed loop transfer function $P(s) = C(sI-A)^{-1} + D$ has poles all in the open left half plane.

★ Small gain theorem



⇒ Consider $L_1(s) \cdot L_2(s)$ stable, rational and proper and let $\gamma_1, \gamma_2 \in \mathbb{R}$ such that

$$\|L_1(s)\| = \gamma_1 \qquad \|L_2(s)\| = \gamma_2$$

where the norm can be any matrix norm.

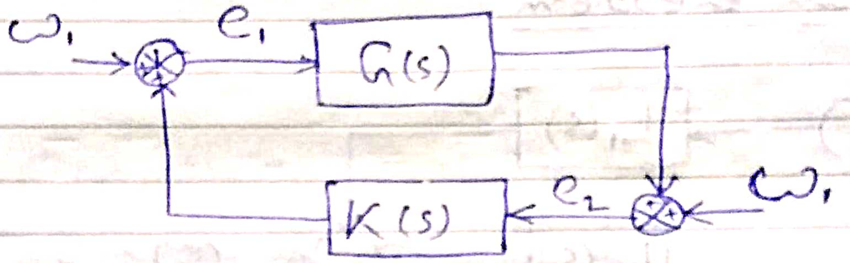
⇒ Then the closed loop interconnection is stable if

$$\boxed{\gamma_1 \gamma_2 < 1}$$

⇒ Provides a sufficient condition (not necessary) for closed loop stability

★ Internal stability

⇒ A System is Internally stable if for all Initial Conditions, and all bounded signals Injected at any place in the system, all states remain bounded for all future time.



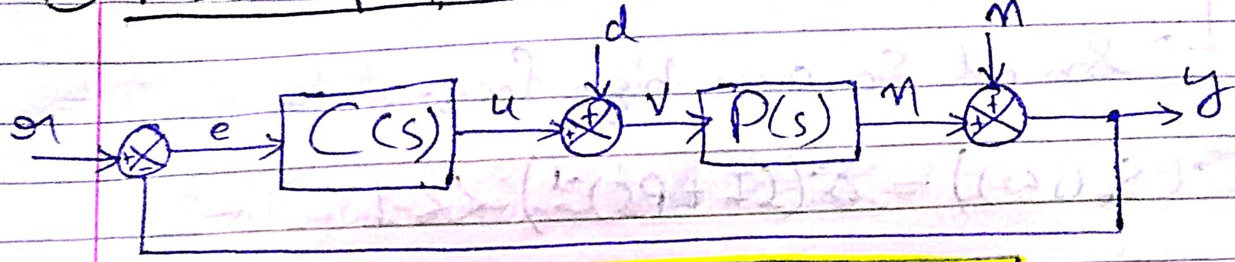
$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} (I-KG)^{-1} & (I-KG)^{-1}K \\ (I-GK)^{-1}G & (I-GK)^{-1} \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}$$

Necessary & sufficient condition (Internal Stability)

↳ Each of the four transfer functions in the above relation are stable.

★ ~~Controllability~~

② MIMO performance



$$\boxed{y = S_o L_o r + S_o P d + S_o n}$$

$$e = S_o (r - n) - S_o P d$$

$$u = C S_o r - T_I d - C S_o n$$

$$v = C S_o r + S_I d - C S_o n$$

⇒ A good performance means:
- → Disturbance attenuation
- → Noise attenuation
- ↳ Good reference tracking

at input and output.

⇒ Output disturbance attenuation:

↳ Small $S_o P$ over low frequencies.

$$\bar{\sigma}(S_o(j\omega)P(j\omega)) = \bar{\sigma}((I + PC)^{-1}P)$$
$$= \bar{\sigma}(P S_I) << 1$$

⇒ Output noise attenuation:

↳ Small S_o over high frequencies

$$\bar{\sigma}(S_o(j\omega)) = \bar{\sigma}((I+PC)^{-1}) \ll 1$$

⇒ Input disturbance attenuation:

↳ Small $S_I(j\omega)$ over low frequencies

$$\bar{\sigma}(S_I(j\omega)) = \bar{\sigma}((I+CP)^{-1}) \ll 1$$

⇒ Input noise attenuation:

↳ Small $C(j\omega)S_o(j\omega)$ over high frequencies

★ Useful properties of singular values

$$\bar{\sigma}(A^{-1}) = \frac{1}{\underline{\sigma}(A)}$$

★ Disturbance attenuation

① Output

$$\underline{\sigma}(P(j\omega)C(j\omega)) \gg 1$$

$$\Rightarrow \bar{\sigma}\left((I+PC)^{-1}P\right) \approx \bar{\sigma}(C^{-1}) = \frac{1}{\underline{\sigma}(C)}$$

$$\approx \frac{1}{\underline{\sigma}(C)} \ll 1$$

$$\Rightarrow \boxed{\underline{\sigma}(C) \gg 1 \quad \forall \omega \in (0, \omega_{low})}$$

② Input

$$\text{Let } \frac{1}{\underline{\sigma}(CP)+1} \leq \bar{\sigma}(S_I)$$

$$\Rightarrow \bar{\sigma}(S_I) = \bar{\sigma}\left((I+CP)^{-1}\right) = \frac{1}{\underline{\sigma}(I+CP)} \ll 1$$

$$\Rightarrow \underline{\sigma}(I+CP) \gg 1$$

$$\Rightarrow \boxed{\underline{\sigma}(CP) \gg 1 \quad \forall \omega \in (0, \omega_{low})}$$

* **Noise attenuation**

Output

$$\boxed{\underline{\sigma}(PC) \gg 1 \quad \forall \omega \in (\omega_{high}, \infty)}$$

Input

$$\boxed{\underline{\sigma}(P(j\omega)) \gg 1 \quad \forall \omega \in (\omega_{high}, \infty)}$$

* **Trade offs**

① Robust stability

⇒ Let Δ be a stable uncertainty matrix such that

$$P_{real} = (I + \Delta) P_{nominal}$$

⇒ The perturbed closed loop transfer function is then characterized by

$$\det(I + PC) \rightarrow \det(I + (I + \Delta) PC)$$

$$= \det(I + PC) \det(I + \Delta T_o)$$

$$\det(I + \Delta T_o) \approx 1 \Rightarrow \|\Delta T_o\| \text{ small}$$

$$\bar{\sigma}(T_o(j\omega)) \ll 1$$

$$\Rightarrow \boxed{\bar{\sigma}(L_o(j\omega)) \ll 1 \quad \forall \omega \in (\omega_{high}, \infty)}$$

② Actuator saturation

$$u = CS_0 r - T_I d - CS_0 n \approx C(r-n)$$

$\Rightarrow$ Controller gain "not too big"

$$\boxed{\bar{\sigma}(C(j\omega)) \le M \quad \forall \omega \in (\omega_{high}, \infty)}$$