Modeling in State Space

Modern Control Theory

- Multiple-input Multiple-output System
- Linear/Nonlinear
- time invariant/ time varying
- Time domain

Classical Control Theory

- Single Input Single output System.
- Linear
- Time invariant
- Laplace domain or frequency domain

State: The State of a dynamic System is the Smallest set of variable (Called $\boxed{\text{State Variable}}$) Such that Knowledge of these variable at $t = t_0$, together with knowledge of the input for $t > t_0$, Completely determines the behavior of the System for any time $t \geq t_0$.

State Vector: If n State Variables are needed to Completely describe the behavior of a given System, then these n State Variables can be Considered the n Components of a Vector $\vec{X}$. Such a vector is called a state vector.

State Space: The n-dimentional space whose coordinate axes consist of the x_1 axis, x_2 axis —— x_n axis, where $x_1, x_2, \cdots x_n$ are state variables, is called a state space. Any state can be represented by a point in the state space.

State-Space Equation

⇒ In state-space analysis we are concerned with three types of variables that are involved in the modeling of dynamic systems:-

- ⇒ Input Variables
- ⇒ Output Variables
- ⇒ State Variables

⇒ The dynamic system must involve elements ~~tat~~ that memorize the values of the input for $t > t_1$.

⇒ Since integrator serves as memory devices in a continuous-time control system, the output of such integrators can be considered as the variable that define the internal state of the dynamic system.

↳ Thus the output of integrators serve as state variable.

⇒ Assume that a multiple-input, multiple-output system involve n integrators. {Let us define n output of the integrators as state variable $x_1(t)$, $x_2(t)$ --- $x_n(t)$} Assume also there are r inputs $u_1(t), u_2(t) \cdots u_r(t)$ and m outputs $y_1(t), y_2(t) --- y_m(t)$

Then the System may be described by:-

$$\dot{x}_1(t) = f_1(x_1, x_2, \cdots x_n; u_1, u_2 \cdots u_r; t)$$
$$\dot{x}_2(t) = f_2(x_1, x_2, \cdots x_n; u_1, u_2 \cdots u_r; t)$$
$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$
$$\dot{x}_n(t) = f_n(x_1, x_2 \cdots x_n; u_1, u_2 \cdots u_r; t)$$

The output $y_1(t)\ y_2(t) \cdots y_m(t)$ of the system may be given by

$$y_1(t) = g_1(x_1, x_2 \cdots x_n; u_1, u_2 \cdots u_r; t)$$
$$y_2(t) = g_2(x_1, x_2 \cdots x_n; u_1, u_2 \cdots u_r; t)$$
$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$
$$y_m = g_m(x_1, x_2 \cdots x_n; u_1, u_2 \cdots u_r; t)$$

So
$$\boxed{\dot{\vec{x}}(t) = \vec{f}(\vec{x}, \vec{u}, t)} \longrightarrow \text{State Equation}$$
$$\boxed{\vec{y}(t) = \vec{g}(\vec{x}, \vec{u}, t)} \longrightarrow \text{Output Equation}$$

⇒ If the above are linearized about the operating state, then we have the following linearized <u>state equation</u> and <u>Output Equation</u>.

$$\dot{\vec{X}}(t) = \vec{A}(t)\vec{X}(t) + \vec{B}(t)\vec{u}(t)$$
$$\vec{Y}(t) = \vec{C}(t)\vec{X}(t) + \vec{D}(t)\vec{u}(t)$$

{ From now we can ignore vector sign }

$$\boxed{\dot{X}(t) = A(t)X(t) + B(t)U(t)} \rightarrow \text{State Equation}$$

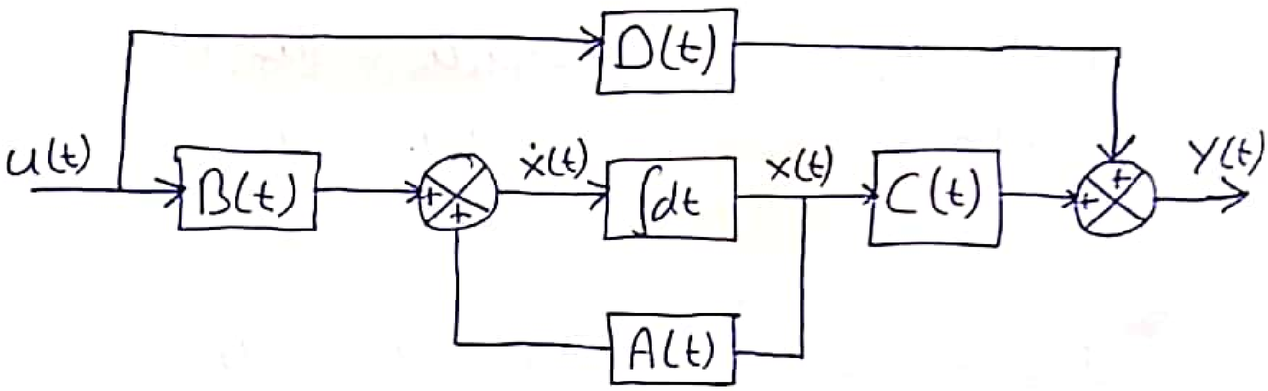
$$\boxed{Y(t) = C(t)X(t) + D(t)U(t)} \rightarrow \text{Output Equation}$$

$A(t)$ = State matrix

$B(t)$ = Input matrix

$C(t)$ = Output matrix

$D(t)$ = Direct transmission matrix



⇒ If vector functions f and g do not involve time t explicitly then the System is called a time-invariant System.

$$\boxed{\dot{X} = AX(t) + BU(t)}$$
$$\boxed{Y = CX(t) + DU(t)}$$

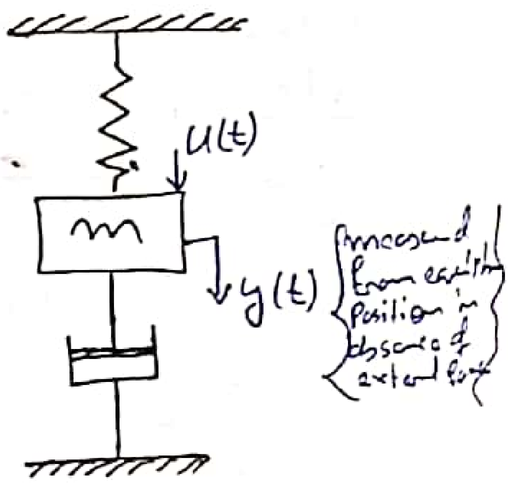
For time-invariant linear

Example 1.1:

Input : $u(t)$

Output : $y(t)$

State Variable : $y(t)$ $\dot{y}(t)$



$m\ddot{y}(t) + b\dot{y}(t) + Ky(t) = u(t)$ {from mechanics}

$$X(t) = \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix} \quad \dot{X}(t) = \begin{bmatrix} \dot{y}(t) \\ \ddot{y}(t) \end{bmatrix}$$

$$Y(t) = [y(t)] \quad u(t) = [u(t)]$$

$$\dot{X}(t) = AX(t) + BU(t)$$
$$Y(t) = CX(t) + DU(t)$$

$$\begin{bmatrix} \dot{y}(t) \\ \ddot{y}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{K}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u(t)$$

$$Y(t) = [1 \; 0] \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix}$$

So $A = \begin{bmatrix} 0 & 1 \\ -\frac{K}{m} & -\frac{b}{m} \end{bmatrix}$ $B = \begin{bmatrix} 0 \\ 1/m \end{bmatrix}$ $C = [1, 0]$, $D = 0$

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★ State-Space Representation of Scalar Differential Equation System

⇒ By use of Vector-matrix notation, an n^{th} order differential equation may be expressed by a first order vector-matrix differential equation.

$$\boxed{\dot{X}(t) = AX(t) + BU(t)}$$

State-Space Representation of n^{th}-Order System of Linear Differential Equation in which the Forcing function Dose not involve Derivative Term

Output

Input

$$\overset{n}{y} + a_1 \overset{n-1}{y} + \cdots + a_{n-1}\dot{y} + a_n y = u$$

⇒ We may take $y(t), \dot{y}(t) \cdots \overset{n-1}{y}(t)$ as a set of n state variables.

So

$$\begin{aligned} x_1 &= y \\ x_2 &= \dot{y} \\ &\vdots \\ x_n &= \overset{n-1}{y} \end{aligned} \quad \Rightarrow \quad \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= u - (a_1 x_n + \cdots a_n x_1) \end{aligned}$$

$$\dot{X} = AX + BU$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & \cdots & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u$$

$$Y = CX + DU$$

$$y = \begin{bmatrix} 1, 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + [0] u$$

State-Space Representation of n^{th} Order Systems of Linear differential Equations in which the Forcing function Involves Derivative terms

$$\overset{n}{y} + a_1 \overset{n-1}{y} + \cdots a_{n-1}\dot{y} + a_n y = b_0 \overset{n}{u} + b_1 \overset{n-1}{u} + \cdots + b_{n-1}\dot{u} + b_n u$$

⇒ The main problem in defining the state variables for this case lies in the derivative terms of the input u.

↳ The state variables must be such that they will eliminate the derivatives of u in the state equation.

⇒ One way to obtain a state equation and output equation for this case is to define the following n variables as a set of n state variables.

$$x_1 = y - B_0 u$$

$$x_2 = \dot{y} - B_0 \dot{u} - B_1 u = \dot{x}_1 - B_1 u$$

$$x_3 = \ddot{y} - B_0 \ddot{u} - B_1 \dot{u} - B_2 u = \dot{x}_2 - B_2 u$$

$$\vdots$$

$$x_n = \overset{n-1}{y} - B_0 \overset{n-1}{u} - B_1 \overset{n-2}{u} - \cdots - B_{n-1} u = \dot{x}_{n-1} - B_{n-1} u$$

Where, $B_0, B_1, B_2 \cdots B_{n-1}$ are determined from :-

$$B_0 = b_0$$

$$B_1 = b_1 - a_1 B_0$$

$$B_2 = b_2 - a_1 B_1 - a_2 B_2$$

$$B_3 = b_3 - a_1 B_2 - a_2 B_1 - a_3 B_0$$

$$\vdots$$

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⇒ With the present choice of state variables, we obtain:

$$\dot{x}_1 = x_2 + B_1 u$$

$$\dot{x}_2 = x_3 + B_2 u$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\dot{x}_{n-1} = x_n + B_{n-1} u$$

$$\dot{x}_n = -a_n x_1 - a_{n-1} x_2 - \cdots - a_1 x_n + B_n u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_{n-1} \\ B_n \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + B_0 u$$