(4) Computation of e^{At} {Method 3}

→ Using Sylvester's interpolation formula

Consider the following polynomial in λ of degree $m-1$, where we assume $\lambda_1, \lambda_2 \cdots \lambda_m$ to be distinct.

$$P_K(\lambda) = \frac{(\lambda-\lambda_1)(\lambda-\lambda_2)\cdots(\lambda-\lambda_{K-1})(\lambda-\lambda_{K+1})\cdots(\lambda-\lambda_m)}{(\lambda_K-\lambda_1)(\lambda_K-\lambda_2)\cdots(\lambda_K-\lambda_{K-1})(\lambda_K-\lambda_{K+1})\cdots(\lambda_K-\lambda_m)}$$

Where $K = 1, 2, \cdots m$.

$$P_K(\lambda_i) = \begin{cases} 1 & \text{if } i = K \\ 0 & \text{if } i \neq K \end{cases}$$

⇒ Then the polynomial $f(\lambda)$ of degree $m-1$

$$f(\lambda) = \sum_{K=1}^{m} f(\lambda_K)\, P_K(\lambda)$$

$$f(\lambda) = \sum_{K=1}^{m} f(\lambda_K) \frac{(\lambda-\lambda_1)(\lambda-\lambda_2)\cdots(\lambda-\lambda_{K-1})(\lambda-\lambda_{K+1})\cdots(\lambda-\lambda_m)}{(\lambda_K-\lambda_1)\cdots(\lambda_K-\lambda_{K-1})(\lambda_K-\lambda_{K+1})\cdots(\lambda_K-\lambda_m)}$$

The above takes value $f(\lambda_K)$ at the points λ_K.

⇒ The above equation is called <u>Lagrange's interpolation formula</u>.

⇒ Assuming that the eigenvalues of an $n\times n$ matrix A are distinct, substitute A for λ in the polynomial $P_K(\lambda)$, we get:-

$$\bar{P}_K(\bar{A}) = \frac{(\bar{A}-\lambda_1\bar{\bar{I}})\cdots(\bar{A}-\lambda_{K-1}\bar{\bar{I}})(\bar{A}-\lambda_{K+1}\bar{\bar{I}})\cdots(\bar{A}-\lambda_m\bar{\bar{I}})}{(\lambda_K-\lambda_1)\cdots(\lambda_K-\lambda_{K-1})(\lambda_K-\lambda_{K+1})\cdots(\lambda_K-\lambda_m)}$$

So $\bar{\bar{P}}_K(\lambda_i I) = \begin{cases} \bar{\bar{I}} & \text{if } i=K \\ \bar{\bar{O}} & \text{if } i\neq K \end{cases}$

Now define

$$\bar{f}(\bar{A}) = \sum_{K=1}^{m} f(\lambda_K)\,\bar{P}_K(\bar{A})$$

$$= \sum_{K=1}^{m} f(\lambda_K)\,\frac{(A-\lambda_1 I)\cdots(A-\lambda_{K-1}I)(A-\lambda_{K+1}I)\cdots(A-\lambda_m I)}{(\lambda_K-\lambda_1)\cdots(\lambda_K-\lambda_{K-1})(\lambda_K-\lambda_{K+1})\cdots(\lambda_K-\lambda_m)}$$

The above equation is known as <u>Sylvester's interpolation formula</u>.

⇒ The above equation can be equivalently written as:-

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & \bar{\bar{I}} \\ \lambda_1 & \lambda_2 & \lambda_3 & \cdots & \lambda_m & \bar{A} \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \cdots & \lambda_m^2 & \bar{A}^2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \lambda_1^{m-1} & \lambda_2^{m-2} & \lambda_3^{m-3} & \cdots & \lambda_m^{m-1} & \bar{A}^{m-1} \\ f(\lambda_1) & f(\lambda_2) & f(\lambda_3) & \cdots & f(\lambda_m) & \bar{f}(\bar{A}) \end{vmatrix} = 0$$

⇒ Sylvester's interpolation formula is frequently used in evaluating $\bar{f}(\bar{A})$.