3

Fundamentals of Rigid-Body Mechanics

3.1) Introduction

⇒ This chapter lay down the foundations of the kinetostatics and dynamics of rigid bodys.

3.2) General Rigid-Body Motion and its associated Screw

Let A and P be two points of the same rigid body B.

(Particular reference Point) (Arbitrary Point)

⇒ Let the position vector of point A in the original configuration is $\bar{a}$ and the position vector of the same point in the ~~displacement~~ displaced configuration, denoted by A', is $\bar{a'}$.

⇒ Similarly $P \rightarrow \bar{P}$
$P' \rightarrow \bar{P'}$

⇒ Furthermore $\bar{P'}$ is to be determined, while $\bar{a}$, $\bar{a'}$ and $\bar{P}$ are given along with the rotation matrix $\bar{\bar{Q}}$.

⇒ Vector $\bar{p}-\bar{a}$ can be considered to undergo a rotation $\bar{\bar{Q}}$ about point A.

⇒ Since vector $\bar{p}-\bar{a}$ is mapped into $\bar{p}'-\bar{a}'$ under the rotation one can write

$$\bar{p}'-\bar{a}' = \bar{\bar{Q}}(\bar{p}-\bar{a})$$

$$\Rightarrow \bar{p}' = \bar{a}' + \bar{\bar{Q}}(\bar{p}-\bar{a}) \quad \text{--- (1)}$$

⇒ Moreover, let $\bar{d}_A$ and $\bar{d}_P$ denote the displacements of A and P respectively

$$\bar{d}_A = \bar{a}'-\bar{a} \quad \& \quad \bar{d}_P = \bar{p}'-\bar{p} \quad \text{--- (2)}$$

⇒ from eq (1),

$$\bar{p}'-\bar{p} = \bar{a}'-\bar{p} + \bar{\bar{Q}}(\bar{p}-\bar{a})$$

$$= \bar{a}'-\bar{p}-\bar{a}+\bar{a}+\bar{\bar{Q}}(\bar{p}-\bar{a})$$

$$= \bar{d}_A + (\bar{\bar{Q}}-\bar{\bar{I}})(\bar{p}-\bar{a})$$

$$\Rightarrow \boxed{\bar{d}_P = \bar{d}_A + (\bar{\bar{Q}}-\bar{\bar{I}})(\bar{p}-\bar{a})} \quad \text{--- (3)}$$

Displacement of an arbitrary Point P of a rigid body ~~whose position vector~~ is determined by displacement of one certain point A and the concomitant rotation $\bar{\bar{Q}}$

Theorem 3.2.1: The component of the displacements of all the points of a rigid body undergoing a general motion along the axis of the underlying rotation is a constant.

Proof: Multiply both side of eq ⊙ by $\bar{e}^T$.

{where $\bar{e}$ is unit vector parallel to the axis of rotation represented by $\bar{Q}$}

$$\Rightarrow \bar{e}^T d_p = \bar{e}^T d_A + e^T(\bar{\bar{Q}} - \bar{\bar{I}})(\bar{P} - \bar{a})$$

$\Rightarrow$ Second term of the right-hand side of the above equation vanishes because $\bar{Q}\bar{e} = \bar{e}$ and hence $\bar{Q}^T\bar{e} = \bar{e}$.

$$\longrightarrow \bar{e}^T = \bar{e}^T \bar{\bar{Q}}$$

$$\bar{e}^T \bar{\bar{Q}} - \bar{e}^T = \bar{e}^T - \bar{e}^T = \bar{0}^T$$

$$\text{So } \bar{e}^T d_p = \bar{e}^T d_A \equiv d_0$$

Theorem 3.2.2: (Mozzi 1763; Chasles 1830)

Given a rigid body undergoing a general motion, a set of its points located on a line $\mathcal{L}$ undergo indentical displacements of minimum magnitude. Moreover, line $\mathcal{L}$ and the minimum-magnitude displacement are parallel to the axis of the rotation involved.

Proof: We can express the displacement of an arbitrary point P as the sum of two orthogonal components.

- $\rightarrow d_{\parallel} \Rightarrow \parallel$ to axis of rotation
- $\rightarrow d_{\perp} \Rightarrow \perp$ to axis of rotation

$$\bar{d}_p = \bar{d}_{\parallel} + \bar{d}_{\perp}$$

$$\bar{e}\bar{e}^T\bar{d}_p = d_0\bar{e} \qquad (\bar{\bar{I}} - \bar{e}\bar{e}^T)\bar{d}_p$$

$$\|\bar{d}_p\|^2 = \|\bar{d}_{\parallel}\|^2 + \|\bar{d}_{\perp}\|^2 = d_0^2 + \|d_{\perp}\|^2$$

$\Rightarrow$ In order to minimize $\|\bar{d}_p\|$ we have to make $\|\bar{d}_{\perp}\|$ equal to zero.

$\hookrightarrow$ So $\bar{d}_p$ must be $\parallel$ to $\bar{e}$

$$\bar{d}_p = \alpha\bar{e} \quad \{\forall \text{ Certain scalar } \alpha\}$$

$\Rightarrow$ If P^* is a point of minimum magnitude of position vector $\overline{P^*}$, its component $\perp$ to the axis of rotation must vanish.

$$\bar{d}_{\perp}^* = (\bar{\bar{I}} - \bar{e}\bar{e}^T)\bar{d}_p^*$$

$$\Rightarrow (\bar{\bar{I}} - \bar{e}\bar{e}^T)\bar{d}_A + (\bar{\bar{I}} - \bar{e}\bar{e}^T)(\bar{\bar{Q}} - \bar{\bar{I}})(\overline{P^*} - \bar{a}) = 0$$

$$\Rightarrow (\bar{\bar{I}} - \bar{e}\bar{e}^T)\bar{d}_A + (\bar{\bar{Q}} - \bar{\bar{I}})(\overline{P^*} - \bar{a}) = 0$$

$$\left\{ \begin{aligned} &(\bar{\bar{I}} - \bar{e}\bar{e}^T)(\bar{\bar{Q}} - I) \\ &= \bar{\bar{Q}} - \bar{e}\bar{e}^T\bar{\bar{Q}} - \bar{\bar{I}} + \bar{e}\bar{e}^T \\ &= (\bar{\bar{Q}} - \bar{\bar{I}}) \quad \{\text{as } \bar{e}\bar{e}^T = \bar{e}\bar{e}^T\bar{\bar{Q}}\} \end{aligned} \right.$$

$$\left\{ \begin{aligned} &\bar{\bar{Q}}\bar{e} = \bar{e} \\ \Rightarrow\ &\bar{e} = \bar{\bar{Q}}^T\bar{e} \\ \Rightarrow\ &\bar{e}^T = e^T\bar{\bar{Q}} \\ \Rightarrow\ &\boxed{\bar{e}\bar{e}^T = \bar{e}\bar{e}^T\bar{\bar{Q}}} \end{aligned} \right.$$

⇒ Now, if we define a line $\mathcal{L}$ passing through p^* and $\parallel$ to $\bar{e}$, then the position vector $p^* + \lambda\bar{e}$ of any points P satisfies the foregoing equation.

↳ As a consequence, all points of minimum magnitude lie in a line $\parallel$ to the axis of rotation of $\bar{\bar{Q}}$.

⇒ A rigid body can attain an arbitrary configuration from a given original one, following a screw-like motion of axis $\mathcal{L}$ and pitch P.

↳ It seems appropriate to call $\mathcal{L}$ the screw axis of the rigid body motion.

$$\text{Pitch, } P = \frac{d_0}{\phi} = \frac{\bar{d}_p^T \bar{e}}{\phi} \quad \text{or} \quad P = \frac{2\pi d_0}{\phi}$$

Rotation involved

m/rad

m/turn

⇒ The angle ϕ is called amplitude associated with the said motion.

3.2.1) The Screw of a rigid body Motion

⇒ The Screw axis $\mathcal{L}$ is totally specified by a given point P_0 of $\mathcal{L}$ and a unit vector $\bar{e}$ defining its direction.

(Can be chosen that lying closest to origin)

⇒ Expression for the position vector P_0, $\bar{P_0}$, in terms of $\bar{a}$ $\bar{a}'$ and $\bar{\bar{Q}}$ are derived below:-

⇒ If P_0 is defined as the point of $\mathcal{L}$ lying closest to the origin, then obviously $\bar{P_0}$ is $\perp$ to $\bar{e}$.

$$\bar{e}^T \bar{P_0} = 0 \quad — ①$$

⇒ Displacement $\bar{d_0}$ of P_0 (assuming P_0 is attached to the Body) is $\parallel$ to $\bar{e}$ and hence is identical to $d_\parallel$ given by:-

$$\bar{d_\parallel} = \bar{e}\,\bar{e}^T \bar{d_p}$$

{∀ arbitrary Point P on the body
→ $\bar{d_p}$ displacement vector of point P}

Where $\bar{d_0}$ is given by

$$\bar{d_0} = \bar{d_A} + (\bar{\bar{Q}} - \bar{\bar{I}})(\bar{P_0} - \bar{a})$$

$\Rightarrow$ Now Since $\bar{d_0}$ is identical to $\bar{d_{11}}$ we have

$$\bar{d_A} + (\bar{\bar{Q}} - \bar{\bar{I}})(\bar{P_0} - \bar{a}) = \bar{d_{11}} = \bar{e}\bar{e}^T\bar{d_0}$$

$\Rightarrow$ But from Theorem 3.1.1 we have

$$e^T d_0 = e^T d_A$$

So also

$$\bar{d_A} + (\bar{\bar{Q}} - \bar{\bar{I}})(\bar{P_0} - \bar{a}) = \bar{e}\bar{e}^T \bar{d_A}$$

$$\Rightarrow \boxed{(\bar{\bar{Q}} - \bar{\bar{I}})\bar{P_0} = (\bar{\bar{Q}} - \bar{\bar{I}})\bar{a} - (1 - \bar{e}\bar{e}^T)\bar{d_A}} \text{ — } ②$$

$\Rightarrow$ In order to find an expression for $\bar{P_0}$ eq ① is adjoined to eq ②, then by obtaining :-

$$\boxed{\bar{\bar{A}}\bar{P_0} = \bar{b}} \text{ — } ③$$

where,

$$\bar{\bar{A}} = \begin{bmatrix} \bar{\bar{Q}} - \bar{\bar{I}} \\ \bar{e}^T \end{bmatrix}_{4\times 3} \qquad \bar{b} = \begin{bmatrix} (\bar{\bar{Q}} - \bar{\bar{I}})\bar{a} - (\bar{\bar{I}} - \bar{e}\bar{e}^T)\bar{d_A} \\ 0 \end{bmatrix}$$

$\Rightarrow$ Equation ③ Cannot be Solved for $\bar{P_0}$ directly, because $\bar{\bar{A}}$ is not a Square matrix.

↳ Equation represents overdeterminant System of four equation and three unknown.

⇒ In fact if both sides of eq ① are multiplied from the left by A^T, we have:

$$\boxed{\bar{A}^T\bar{A}\bar{P}_o = \bar{A}^T\bar{b}} \text{ — } ④$$

⇒ If the product $\bar{A}^T\bar{A}$ which is 3×3 matrix, is invertible, then $\bar{P}_o$ can be computed from eq ④.

⇒ In fact, the said product is not only invertible, but also admits an inverse that is rather simple to derive.

⇒ Now the rotation matrix $\bar{\bar{Q}}$ is recalled in terms of its natural invariants.

$$\boxed{\bar{\bar{Q}} = \bar{e}\bar{e}^T + \cos\phi(\bar{\bar{I}} - \bar{e}\bar{e}^T) + \sin\phi\,\bar{\bar{E}}} \text{ — } ⑤$$

(cross product Matrix of $\bar{e}$)

⇒ further eq ⑤ can be substituted in $\bar{A}$ from eq ③ we get:—

$$\boxed{\bar{\bar{A}}^T\bar{\bar{A}} = 2(1 - \cos\phi)\bar{\bar{I}} - (1 - 2\cos\phi)\bar{e}\bar{e}^T} \text{ — } ⑥$$

<u>Proof</u>

$$\bar{\bar{A}} = \begin{bmatrix} \bar{\bar{Q}} - \bar{\bar{I}} \\ \bar{e}^T \end{bmatrix} \qquad \bar{\bar{A}}^T = \begin{bmatrix} \bar{\bar{Q}}^T - \bar{\bar{I}}^T, & \bar{e} \end{bmatrix}$$

$$\bar{\bar{A}}^T\bar{\bar{A}} = (\bar{\bar{Q}}^T - \bar{\bar{I}}^T)(\bar{\bar{Q}} - \bar{\bar{I}}) + \bar{e}\bar{e}^T$$

$$\Rightarrow \left(\bar{\bar{Q}}^T\bar{\bar{Q}} - \bar{\bar{I}}^T\bar{\bar{Q}} - \bar{\bar{Q}}^T\bar{\bar{I}} + \bar{\bar{I}}^T\bar{\bar{I}}\right) + \bar{e}\bar{e}^T$$

$$\Rightarrow \left(\bar{\bar{I}} - \bar{\bar{Q}} - \bar{\bar{Q}}^T + \bar{\bar{I}}\right) + \bar{e}\bar{e}^T$$

$\{\text{as } \bar{\bar{Q}}^{-1} = \bar{\bar{Q}}^T\} \ \{\bar{\bar{I}}^T = \bar{\bar{I}}\}$

$$\Rightarrow 2\bar{\bar{I}} + \bar{e}\bar{e}^T - (\bar{\bar{Q}} + \bar{\bar{Q}}^T)$$

$$\Rightarrow 2\bar{\bar{I}} + \bar{e}\bar{e}^T - \left(2\bar{e}\bar{e}^T + 2\cos\phi\,(\bar{\bar{I}} - \bar{e}\bar{e}^T)\right)$$

$$\Rightarrow \boxed{\bar{\bar{A}}^T\bar{\bar{A}} = 2(1-\cos\phi)\bar{\bar{I}} - (1-2\cos\phi)\,\bar{e}\bar{e}^T}$$

$\Rightarrow \bar{\bar{A}}^T\bar{\bar{A}}$ is linear Combination of $\bar{e}\bar{e}^T$ and $\bar{\bar{I}}$.

↳ This suggest that its inverse is very likely to be a linear Combination of these two matrices as well

$\Rightarrow$ If this is in fact true, then we can write

$$(\bar{\bar{A}}^T\bar{\bar{A}})^{-1} = \alpha\bar{\bar{I}} + \beta\bar{e}\bar{e}^T$$

$\Rightarrow$ The Coefficients α and β being determined from the Condition that the product of $\bar{\bar{A}}^T\bar{\bar{A}}$ by its inverse should be $\bar{\bar{I}}$, which yields

$$\alpha = \frac{1}{2(1-\cos\phi)} \qquad \beta = \frac{1-2\cos\phi}{2(1-\cos\phi)}$$

Hence,

$$(\bar{\bar{A}}^T\bar{\bar{A}})^{-1} = \frac{\bar{\bar{I}}}{2(1-\cos\phi)} + \frac{1-2\cos\phi}{2(1-\cos\phi)}\,\bar{e}\bar{e}^T$$

$\Rightarrow$ On the other hand,

$$\bar{\bar{A}}^T \bar{b} = (\bar{\bar{Q}} - \bar{\bar{I}})^T [(\bar{\bar{Q}} - \bar{\bar{I}})\bar{a} - \bar{d}_A]$$

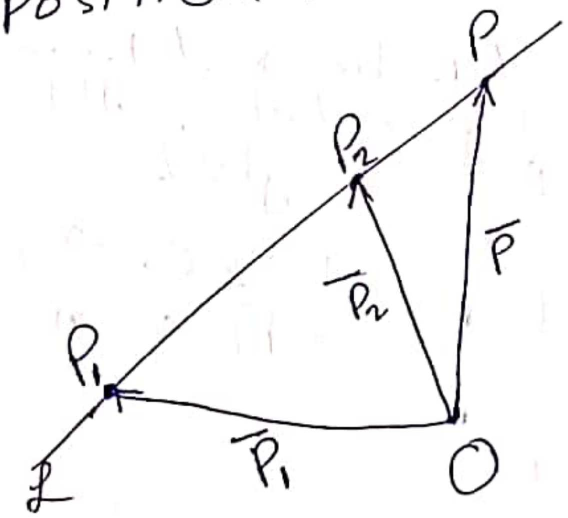
So finaly we get :-

$$\bar{P}_0 = \frac{(\bar{\bar{Q}} - \bar{\bar{I}})^T (\bar{\bar{Q}}\bar{a} - \bar{a}')}{2(1 - \cos\varphi)} \quad \forall\ \varphi \neq 0$$

3.2.2> The Plücker Coordindes of a Line

$\Rightarrow$ Alternatively, the Screw axis, and any line for that matter, can be defined more Conveniently by its <u>Plucker Coordinates</u>.

$\Rightarrow$ Lets recall the equation of a line $\mathcal{L}$ passing through two points P_1 and P_2 of position vectors $\bar{P}_1$ and $\bar{P}_2$.



$$(\bar{P}_2 - \bar{P}_1) \times (\bar{P} - \bar{P}_1) = 0$$

$$\Rightarrow (\bar{P}_2 - \bar{P}_1) \times \bar{P} + \bar{P}_1 \times (\bar{P}_2 - \bar{P}_1) = 0$$

⇒ If we introduce the Cross-Product matrix $\bar{\bar{P}}_1$ and $\bar{\bar{P}}_2$ of Vector $\bar{P}_1$ & $\bar{P}_2$ then,

$$(\bar{\bar{P}}_2 - \bar{\bar{P}}_1)\bar{P} + \bar{P}_1 \times (\bar{P}_2 - \bar{P}_1) = 0$$

$$\Rightarrow \left[\bar{\bar{P}}_2 - \bar{\bar{P}}_1 \,,\; \bar{P}_1 \times (\bar{P}_2 - \bar{P}_1)\right] \begin{bmatrix} \bar{P} \\ 1 \end{bmatrix} = 0 \quad \text{--- (1)}$$

⇒ It is apparent that the line is defined Completly by two Vectors, the $\bar{P}_2 - \bar{P}_1$ and the Cross product $\bar{P}_1 \times (\bar{P}_2 - \bar{P}_1)$.

↳ We will thus define a 6-dimensional array $\bar{\gamma}_L$ Containing these two Vectors;

$$\bar{\gamma}_L = \begin{bmatrix} \bar{P}_2 - \bar{P}_1 \\ \bar{P}_1 \times (\bar{P}_2 - \bar{P}_1) \end{bmatrix} \quad \text{--- (2)}$$

whose Six Scalar entries are the Plucker Coordinates of $\mathcal{L}$.

if $\bar{e} = \dfrac{\bar{P}_2 - \bar{P}_1}{\|\bar{P}_2 - \bar{P}_1\|}$ & $\bar{m} = \bar{P}_1 \times \bar{e}$

then, $\bar{\gamma}_L = \|P_2 - P_1\| \begin{bmatrix} \bar{e} \\ \bar{m} \end{bmatrix} \quad \text{--- (3)}$

⇒ The Six scalar entries of the above array are the <u>normalized Plücker Coordinates</u>.

$\bar{e}$ determines direction of $\mathcal{L}$
$\bar{n}$ determine its location
→ also called moment of $\mathcal{L}$

⇒ The Plücker coordinate thus defined will be thus stored in a plücker array $\bar{K}_L$ in the form

$$\bar{K}_L = \begin{bmatrix} \bar{e} \\ \bar{n} \end{bmatrix} \quad —④$$

⇒ However, six components of plücker array are not independent.

$$\bar{e} \cdot \bar{e} = 1 \;, \quad \bar{n} \cdot \bar{e} = 0 \quad —⑤$$

⇒ Hence, any line $\mathcal{L}$ has only four independent Plucker coordinates.

⇒ The set of Plucker array is array of real number not constituting a vector space.

⇒ Let $\bar{K}_A$ and $\bar{K}_B$ denote the Plücker array of the same line $\mathcal{L}$ when its moment is measured at points A and B, respectively.

⇒ This line passes through a point P of position vector $\bar{P}$ for a particular origin O.

$$n_A \equiv (\bar{P} - \bar{a}) \times \bar{e} \quad n_B = (\bar{P} - \bar{b}) \times \bar{e} \quad —⑥$$

Hence,
$$\overline{K_A} = \begin{bmatrix} \bar{e} \\ \eta_A \end{bmatrix} \qquad \overline{K_B} = \begin{bmatrix} \bar{e} \\ \eta_B \end{bmatrix}$$

Obviously, $\eta_B - \eta_A = (\bar{a} - \bar{b}) \times \bar{e}$ — 6*

$$\Rightarrow \overline{K_B} = \begin{bmatrix} \bar{e} \\ \bar{\eta}_A + (\bar{a} - \bar{b}) \times \bar{e} \end{bmatrix} \quad \text{— 7}$$

Which can be re-written as

$$\boxed{\overline{K_B} = \overline{\overline{U}}\, \overline{K_A}} \quad \text{— 8}$$

$$\overline{\overline{U}} = \begin{bmatrix} 1 & 0 \\ \bar{\bar{A}} - \bar{\bar{B}} & 1 \end{bmatrix}$$

$\bar{\bar{A}}$ = Cross product Matrix of $\bar{a}$
$\bar{\bar{B}}$ = " " " $\bar{b}$

⇒ Given the lower-triangular structure of the matrix $\overline{\overline{U}}$, its determinant is simply the product of its diagonal entries, which are all unity; Hence

$$\det(\overline{\overline{U}}) = 1 \quad \text{— 9}$$

⇒ $\overline{\overline{U}}$ thus belongs to the Unimodular group of 6×6 matrices. These matrices are rather simple to invert.

$$\overline{\overline{U}}^{-1} = \begin{bmatrix} 1 & 0 \\ \bar{\bar{B}} - \bar{\bar{A}} & 1 \end{bmatrix} \quad \text{— 10}$$

⇒ Equation (8) can then be called the Plücker-coordinate transform formula.

⇒ Multiplying both side of eq (6A) by $(\bar{a}-\bar{b})^T$

$$(\bar{a}-\bar{b})^T \bar{n}_0 = (\bar{a}-\bar{b})^T \bar{n}_A$$

⇒ Hence, the moments of the same line $\mathcal{L}$ with respect to two points are not independent.

↳ They have same component along the line joining the two points.

⇒ A special case of a line of interest in Kinematics is a line at infinity.

$$\bar{K} = \|\bar{n}\| \begin{bmatrix} e/\|n\| \\ n/\|n\| \end{bmatrix}$$

$$\lim_{\|n\| \to \infty} \bar{K} = \left(\lim_{\|n\| \to \infty} \|n\| \right) \begin{bmatrix} \bar{0} \\ \bar{f} \end{bmatrix}$$

$\to \lim_{\|n\| \to \infty} \frac{n}{\|n\|}$

$$\bar{K}_\infty = \begin{bmatrix} \bar{0} \\ \bar{f} \end{bmatrix}$$

⇒ Line at infinity of unit moment $\bar{f}$ can be thought of as being a line lying in a plane ⊥ to the unit vector $\bar{f}$, but otherwise with an indefinite location in the plane, except that it is an infinitely large distance from the origin.

3.2

3.2.3> The Pose of a Rigid Body

⇒ A possible form of describing a general rigid body motion, then is through a Set of 8 real numbers :-

(i) Six Plücker coordinates of Screw axis

(ii) Its pitch

(iii) Its amplitude (ie angle)

⇒ Rigid-body motion is fully described by six independent parameters.

⇒ Alternatively, a rigid body motion can be described by seven dependent parameters as follows.

(i) four invariants of the Concomitant Rotation

→ Natural Invariants

→ Linear Invariants

→ Euler – Rodrigues parameters.

(ii) Three Components of the displacement of a arbitrary point.

⇒ Let a rigid body undergo a general motion of rotation $\overline{\overline{Q}}$ and displacement $\overline{d}$ from a reference Configuration C_0.

⇒ If in the new Configuration C a landmark Point A of the body has a position vector $\overline{a}$.

Scanned by CamScanner

⇒ Then the pose array or simply the Pose $\bar{S}$ of the body, is defined as a 7D array namely

$$\bar{S} \equiv \begin{bmatrix} \bar{q} \\ q_0 \\ d_A \end{bmatrix}$$

$\bar{q}$, q_0 are four invariants of $\bar{\bar{Q}}$

d_A displacem of point A

⇒ An important problem in Kinematics is the Computation of the Screw parameters

{Components of $\bar{S}$}

→ From coordinate measurement over a certain finite set of points.

⇒ This problem is known as Pose estimation.

{Position and Orientation}

3.3 Rotation of a Rigid body about a fixed Point

⇒ This motion is fully described by a rotation matrix $\bar{\bar{Q}}$ that is proper orthogonal.

⇒ The position vector of point P in an original configuration, denoted here by $\bar{P}_0$, is mapped smoothly into a new vector $\bar{P}(t)$, namely

$$\bar{P}(t) = \bar{\bar{Q}}(t)\,\bar{P}_0 \quad \text{—— (1)}$$

⇒ The velocity of P is computed by differentiating both side of eq (1)

$$\dot{\bar{P}}(t) = \dot{\bar{\bar{Q}}}(t)\,\bar{P}_0 \quad \text{—— (2)}$$

→ Not very useful expression, because it requires knowledge of the original position of P.

$$\Rightarrow \quad \boxed{\dot{\bar{P}} = \dot{\bar{\bar{Q}}}\,\bar{\bar{Q}}^T\,\bar{P}} \quad \text{—— (3)}$$

⇒ The product $\dot{\bar{\bar{Q}}}\bar{\bar{Q}}^T$ is known as the angular-velocity matrix of the rigid-body motion and is denoted by $\bar{\bar{\Omega}}$.

$$\bar{\bar{\Omega}} = \dot{\bar{\bar{Q}}}\,\bar{\bar{Q}}^T$$

Theorem 3.3.1: The angular-velocity matrix is a skew symmetric matrix.

angular velocity vector $\quad \bar{\omega} = \text{vec}(\bar{\bar{\Omega}})$

⇒ Hence eq ⑤ Can be written as:-

$$\dot{\bar{P}} = \bar{\Omega}\bar{P} = \bar{\omega} \times \bar{P}$$

⇒ So, Velocity of any point P of a body moving with a point O fixed is perpendicular to line OP.

3.4) General Instantaneous Motion of a Rigid body

⇒ If a rigid body now undergoes the most general motion, none of its points remains fixed, and the position vector of any of these P in a displaced Configuration is given by:-

$$\bar{P}(t) = \bar{a}(t) + \bar{\bar{Q}}(t)\,(\bar{P}_0 - \bar{a}_0) \quad \text{—— ①}$$

⇒ $\bar{a}_0$, $\bar{P}_0$ are Position vectors of point A and P in reference Configuration C_0.

⇒ $\bar{a}(t)$ and $\bar{P}(t)$ being the position vector of the same point in displaced Configuration C.

⇒ $\bar{\bar{Q}}(t)$ denotes rotation matrix

⇒ Now the Velocity of P is Computed by differentiating both side of eq ①

$$\dot{\bar{P}}(t) = \dot{\bar{a}}(t) + \dot{\bar{\bar{Q}}}(t)\,(\bar{P}_0 - \bar{a}_0) \quad \text{—— ②}$$

$$\Rightarrow \boxed{\dot{\bar{P}}(t) = \dot{\bar{a}}(t) + \bar{\Omega}(\bar{P}(t) - \bar{a}(t))} \quad \text{--- (3)}$$

or

$$\boxed{\dot{\bar{P}} = \dot{\bar{a}} + \bar{\omega} \times (\bar{P} - \bar{a})} \quad \text{--- (4)}$$

Theorem 3.4.1: The relative velocity of two Points of the Same rigid body is ⊥ to the line joining them.

Corollary 3.4.1: The Projections of all velocities of all the points of a rigid body onto the angular-velocity Vector are identical.

Theorem 3.4.2: Given a rigid body under general motion, a set of its points located on a line $\mathcal{L}'$ undergoes the identical minimum-magnitude velocity $\bar{V}_0$ parallel to the angular velocity.

Defination 3.4.1: The line Containing the points of a rigid body undergoing minimum-magnitude velocities is Called the instant screw axis (ISA) of the body under the given motion.

3.4.1> The Instat Screw of a Rigid Body Motion

⇒ The instantaneous motion of a body is equivalent to that of the bolt of a screw of axis $\mathcal{L}'$, the ISA.

⇒ As the body moves the ISA changes, and the motion of the body is called instantaneous Screw.

⇒ Since $\overline{V}_o$ is ∥ to $\overline{\omega}$, it can be written in the form

$$\overline{V}_o = v_o \frac{\overline{\omega}}{\|\overline{\omega}\|}$$

{where V_o is scalar denoting the signed magnitude of $\overline{V}_o$ and bears sign of $\overline{V}_o \cdot \overline{\omega}$}

⇒ Furthermore the pitch of the instantaneous Screw p' is defined as

$$p' \equiv \frac{v_o}{\|\overline{\omega}\|} \equiv \frac{\dot{\overline{p}} \cdot \overline{\omega}}{\|\overline{\omega}\|^2} \quad \text{or} \quad p' \equiv \frac{2\pi v_o}{\|\overline{\omega}\|}$$

(m/rad) (m/turn)

⇒ Again ISA $\mathcal{L}'$ can be specified uniquely through its plücker coordinates, stored in $\overline{P}_{\mathcal{L}'}$ array:-

$$\overline{P}_{\mathcal{L}'} = \begin{bmatrix} \overline{e}' \\ \overline{n}' \end{bmatrix} \begin{matrix} \rightarrow \text{direction of } \mathcal{L}' \\ \rightarrow \text{its moment about origin.} \end{matrix}$$

$$\overline{e}' = \frac{\overline{\omega}}{\|\overline{\omega}\|} \qquad n' = \overline{p} \times \overline{e}'.$$

{Position vector of any point of the ISA}

⇒ $\overline{e}'$ is defined uniquely but become trivial when the rigid body instantaneously undergoes a Pure translation

Scanned by CamScanner

↳ In this case e' is defined as the unit vector $\parallel$ to the associated displacement field.

⇒ Instantaneous rigid body motion is defined by a line $\mathcal{L}'$, a pitch p' and an amplitude $\|\bar{\omega}\|$

↳ Such a motion is, then fully determined by six independent parameters:-
- ↳ 4 Independent Plücker Coordinate of $\mathcal{L}'$
- ↳ Its pitch
- ↳ Its amplitude.

⇒ A line supplied with a pitch ⇒ <u>Screw</u>

⇒ Screw supplied with amplitude ⇒ <u>Twist</u>.

⇒ The ISA can be <u>alternatively</u> described in terms of the position vector $\bar{P}_0'$. → {Its point lying closest to the origin}

⇒ Expression for $\bar{P}_0'$ in terms of the position and the velocity of a arbitrary body-point and the angular velocity are derived below.

⇒ We decompose $\dot{\bar{P}}$ into two orthogonal components, $\dot{\bar{P}}_{\parallel}$ & $\dot{\bar{P}}_{\perp}$ along and transverse respectively to the angular-velocity vectors.

⇒ $\dot{\bar{a}}$ is first decomposed into two such orthogonal components $\dot{\bar{a}}_{\parallel}$ and $\dot{\bar{a}}_{\perp}$

$$\dot{\bar{a}} = \dot{\bar{a}}_{\parallel} + \dot{\bar{a}}_{\perp}$$

$$\dot{\bar{a}}_{\parallel} = \left(\dot{\bar{a}} \cdot \frac{\bar{\omega}}{\|\bar{\omega}\|}\right)\frac{\bar{\omega}}{\|\bar{\omega}\|} = (\dot{\bar{a}} \cdot \bar{\omega})\frac{\bar{\omega}}{\|\bar{\omega}\|^2} = \frac{\bar{\omega}\,\bar{\omega}^T \dot{\bar{a}}}{\|\bar{\omega}\|^2}$$

$$\dot{\bar{a}}_{\perp} = \left(\bar{\bar{I}} - \frac{\bar{\omega}\bar{\omega}^T}{\|\bar{\omega}\|}\right)\dot{\bar{a}} = -\frac{1}{\|\bar{\omega}\|^2}\bar{\bar{\Omega}}^2 \dot{\bar{a}}$$

$$\left\{ \text{as } \Omega^2 = \bar{\omega}\bar{\omega}^T - \|\bar{\omega}\|^2 \bar{\bar{I}} \right\}$$

$$\dot{\bar{P}} = \underbrace{\frac{\bar{\omega}\bar{\omega}^T}{\|\bar{\omega}\|^2}\dot{\bar{a}}}_{\dot{\bar{P}}_{\parallel}} \underbrace{- \frac{1}{\|\bar{\omega}\|^2}\bar{\bar{\Omega}}^2 \dot{\bar{a}} + \bar{\bar{\Omega}}(\bar{P} - \bar{a})}_{\dot{\bar{P}}_{\perp}} \quad \text{—— (1)}$$

$\Rightarrow$ Vanishing of the normal component obviously implises the minimization of the magnitude of $\dot{\bar{P}}$.

$$\dot{\bar{P}}_{\perp} = 0$$

$$\Rightarrow \bar{\bar{\Omega}}(\bar{P} - \bar{a}) - \frac{1}{\|\bar{\omega}\|}\bar{\bar{\Omega}}^2 \dot{\bar{a}} = 0$$

$$\Rightarrow \bar{\bar{\Omega}}\bar{P} = \bar{\bar{\Omega}}\left(a + \frac{1}{\|\omega\|^2}\bar{\bar{\Omega}}\dot{\bar{a}}\right)$$

$$\Rightarrow \bar{\bar{\Omega}}(\bar{P} - \bar{\sigma}) = 0 \quad \text{—— (2)}$$

$$\left\{ \text{where, } \bar{\sigma} = \bar{a} + \frac{1}{\|\bar{\omega}\|^2}\Omega\dot{\bar{a}} \right\}$$

⇒ A possible solution of the foregoing problem is

$$\bar{P} = \bar{\pi} = \bar{a} + \frac{1}{\|\bar{\omega}\|^2}\bar{\bar{\Omega}}\dot{\bar{a}} \quad — (3)$$

⇒ This solution is not unique.

↳ eq (2) does not require that $\bar{P} - \bar{\pi}$ be zero, only that the difference lie in the null space of $\bar{\bar{\Omega}}$.

↓

$\bar{P} - \bar{\pi}$ is linearly dependent with $\bar{\omega}$.

⇒ In other words if a vector $\alpha\bar{\omega}$ is added to $\bar{P}$ then the sum also satisfy the equation.

(1)

⇒ Let $\bar{P}_0'$ be position vector of point of ISA lying closest to origin.

↳ This vector is obviously perpendicular to $\bar{\omega}$

$$\bar{\omega}^T \bar{P}_0' = 0 \quad — (4)$$

⇒ eq (2) in conjunction with (4) can be written as:-

$$\boxed{\bar{\bar{A}}\,\bar{P}_0' = \bar{b}} \quad — (5)$$

where,

$$A = \begin{bmatrix} \bar{\bar{\Omega}} \\ \bar{\omega}^T \end{bmatrix}_{4\times 3} \qquad \bar{b} = \begin{bmatrix} \bar{\bar{\Omega}}\bar{a} + \frac{1}{\|\bar{\omega}\|^2}\bar{\bar{\Omega}}^2\dot{\bar{a}} \\ 0 \end{bmatrix}_{4\times 1}$$

$\Rightarrow$ Multiplying both side by A^T

$$\bar{\bar{A}}^T \bar{\bar{A}}\, \bar{P}_0' = \bar{\bar{A}}^T \bar{b}$$

Where $A^T A = \Omega^T \Omega + \omega\omega^T = -\Omega^2 + \omega\omega^T$

$\Rightarrow$ $\bar{\bar{A}}^T \bar{\bar{A}} = \|\bar{\omega}\|^2 \bar{\bar{I}}$ $\qquad \bar{\bar{A}}^T \bar{b} = \bar{\bar{\Omega}}(\dot{\bar{a}} - \bar{\bar{\Omega}}\bar{a})$

So $\boxed{\bar{P}_0' = \dfrac{\bar{\bar{\Omega}}(\dot{\bar{a}} - \bar{\bar{\Omega}}\bar{a})}{\|\omega\|^2} = \dfrac{\bar{\omega} \times (\dot{\bar{a}} - \bar{\omega} \times \bar{a})}{\|\omega\|^2}}$

3.4.2> The twist of a rigid body

$\Rightarrow$ A line, as we saw earlier, is fully defined by its 6-dimensional Plücker array, which contains only four independent components.

$\Rightarrow$ Now if a pitch is added as a fifth feature to the line or correspondingle to its plücker array we obtain a screw $\bar{\bar{S}}$.

$$\bar{S} = \begin{bmatrix} \bar{e} \\ \bar{P} \times \bar{e} + p\bar{e} \end{bmatrix} \quad \text{—(1)}$$

⇒ An amplitude is any scalar A multiplying the foregoing screw.

↳ The amplitude produces a twist on a wrench.

⇒ The twist on the wrench thus defined can be regarded as an eight-parameter array.

⇒ Twist can be regarded as a 6-dimensional array defining completely the velocity field of a rigid body, and it comprises :-

(i) Three components of angular velocity

(ii) Three components of velocity

of any of the points of the body.

$$\text{twist, } \bar{t} = \begin{bmatrix} \bar{A}\bar{e} \\ \bar{p} \times (A\bar{e}) + p(A\bar{e}) \end{bmatrix} \quad \text{—— (2)}$$

⇒ $A\bar{e}$ can be readily identified as the angular velocity $\bar{\omega}$ parallel to vector $\bar{e}$, of magnitude A.

⇒ If we regard the line $\mathcal{L}$ and point O as set of ~~point P moves~~ a rigid body B moving with an angular velocity $\bar{\omega}$ and such that point P moves with velocity $p\bar{\omega}$ ∥ to the angular velocity, then the lower vector of $\bar{t}$ denoted by $\bar{v}$, represents velocity of Point O, ie

$$\bar{V} = -\bar{\omega}\times\bar{P} + p\bar{\omega}$$

We can thus express the twist $\bar{t}$ as

$$\bar{t} = \begin{bmatrix}\bar{\omega}\\ \bar{V}\end{bmatrix} \quad — ③$$

⇒ A Special case of great interest in Kinematics is the Screw of infinitely large pitch.

$$\lim_{p\to\infty}\begin{bmatrix}\bar{e}\\ \bar{P}\times\bar{e} + p\bar{e}\end{bmatrix} = \lim_{p\to\infty} p\begin{bmatrix}e/p\\ (\bar{P}\times\bar{e})/p + \bar{e}\end{bmatrix}$$

$$= \left(\lim_{p\to\infty} p\right)\underbrace{\begin{bmatrix}\bar{0}\\ \bar{e}\end{bmatrix}}_{\bar{S}_\infty}$$

{Screw of infinite Pitch}

⇒ The twist array, as defined in eq ③, with $\bar{\omega}$ on top, represents the ray coordinates of the twist.

↳ An exchange of the order of the two Cartesian vectors of this array, in turns, gives rise to the <u>axis Coordinates of the twist</u>.

Scanned by CamScanner

⇒ The relationships between the angular-velocity vector and the time derivatives of the invariants of the associated rotation are linear.

⇒ Let the three sets of four invariants of rotation, namely, the natural invariants, the linear invariants and the Euler-Rodrigues parameters be grouped in the 4 dimensional array $\bar{V}$ $\bar{\lambda}$ $\bar{\eta}$ respectively.

$$\bar{V} = \begin{bmatrix} \bar{e} \\ \phi \end{bmatrix} \qquad \bar{\lambda} = \begin{bmatrix} (\sin\phi)\,\bar{e} \\ \cos\phi \end{bmatrix} \qquad \bar{\eta} = \begin{bmatrix} (\sin \phi/2)\,\bar{e} \\ \cos \phi/2 \end{bmatrix}$$

$$\dot{\bar{V}} = \bar{\bar{N}}\,\bar{\omega} \quad ; \quad \dot{\bar{\lambda}} = \bar{\bar{L}}\,\bar{\omega} \qquad \dot{\bar{\eta}} = \bar{\bar{H}}\,\bar{\omega}$$

With

$$\bar{\bar{N}} = \begin{bmatrix} [\sin\phi/(2(1-\cos\phi))]\,(1 - \bar{e}\bar{e}^T) - \frac{1}{2}\bar{\bar{E}} \\ \bar{e}^T \end{bmatrix}$$

$$\bar{\bar{L}} = \begin{bmatrix} \frac{1}{2}[\operatorname{tr}(\bar{\bar{Q}})\bar{\bar{I}} - \bar{\bar{Q}}] \\ -(\sin\phi)\,\bar{e}^T \end{bmatrix}$$

$$\bar{\bar{H}} = \frac{1}{2}\begin{bmatrix} \cos(\phi/2)\bar{\bar{I}} - \sin(\phi/2)\bar{\bar{E}} \\ -\sin(\phi/2)\,\bar{e}^T \end{bmatrix}$$

⇒ The inverse relations:-

$$\cancel{\bar{\omega} = \bar{\bar{N}}\dot{\bar{V}} = \bar{\bar{L}}\dot{\bar{\lambda}} = \bar{\bar{H}}\dot{\bar{\eta}}}$$

$$\boxed{\bar{\omega} = \tilde{\bar{\bar{N}}}\,\dot{\bar{V}} = \tilde{\bar{\bar{L}}}\,\dot{\bar{\lambda}} = \tilde{\bar{\bar{H}}}\,\dot{\bar{\eta}}}$$

Where,

$$\tilde{\bar{\bar{N}}} = \left[\sin\varphi\,\bar{\bar{I}} + (1-\cos\varphi)\bar{\bar{E}} \quad \bar{e}\right]$$

$$\tilde{\bar{\bar{L}}} = \left[\bar{\bar{I}} + \frac{\sin}{1+\cos\varphi}\bar{\bar{E}} \quad - \frac{\sin\varphi}{1+\cos\varphi}\bar{e}\right]$$

$$\tilde{\bar{\bar{H}}} = 2\left[\cos\frac{\varphi}{2}\bar{\bar{I}} + \sin\frac{\varphi}{2}\bar{\bar{E}} \quad -\sin\varphi/_2\,\bar{e}\right]$$

⇒ Now we can write the relationship between the twist and the time-rate of change of the 7-dimensional pose array namely

$$\boxed{\dot{\bar{s}} = \bar{\bar{T}}\,\bar{t}_i}$$

Where, $\bar{\bar{T}} \equiv \begin{bmatrix} \bar{\bar{F}} & \bar{\bar{O}}_{43} \\ \bar{\bar{O}} & \bar{\bar{I}} \end{bmatrix}$

$\bar{\bar{O}}$ = 3×3 ~~Identity~~ zero matrix

$\bar{\bar{O}}_{43}$ = 4×3 Zero matrix

$\bar{\bar{I}}$ = 3×3 Identity matrix

$\bar{\bar{F}} = \bar{\bar{N}}, \bar{\bar{L}}$ or $\bar{\bar{H}}$ depending upon the invariant representation chosen for the rotation

⇒ The inverse relationship takes form:

$$\bar{t} = \bar{\bar{S}}\,\dot{\bar{s}}$$

where $\bar{\bar{S}} = \begin{bmatrix} \bar{\bar{E}} & \bar{\bar{O}} \\ \bar{\bar{O}}_{34} & \bar{\bar{I}} \end{bmatrix}$

⇒ A formula that relates the twist of the same rigid body at two different points is now derived.

⇒ Let A and P be two arbitrary points of a rigid body.
↳ The twist at each of these points is defined as:-

$$\bar{t}_A = \begin{bmatrix} \bar{\omega} \\ \bar{V}_A \end{bmatrix}, \quad \bar{t}_P = \begin{bmatrix} \bar{\omega} \\ \bar{V}_P \end{bmatrix}$$

$$\bar{V}_P = \bar{V}_A + (\bar{a} - \bar{p}) \times \bar{\omega}$$

⇒ Combining above eqution we get

$$\bar{t}_P = \bar{\bar{U}}\,\bar{t}_A \quad \left\{ \text{where } \bar{\bar{U}} = \begin{bmatrix} \bar{\bar{I}} & \bar{\bar{O}} \\ \bar{\bar{A}} - \bar{\bar{P}} & \bar{\bar{I}} \end{bmatrix} \right\}$$

{Cross product matrix of vector $\bar{a}$ and $\bar{P}$}

tion This is called the <u>Twist transfer formula.</u>

3.5> Acceleration analysis of Rigid body Motions

$$\dot{\bar{P}} = \dot{\bar{a}} + \bar{\bar{\Omega}}(\bar{P}-\bar{a})$$

$\Rightarrow$ upon differentiating both side with time:-

$$\Rightarrow \ddot{\bar{P}} = \ddot{\bar{a}} + \dot{\bar{\bar{\Omega}}}(\bar{P}-\bar{a}) + \bar{\bar{\Omega}}(\dot{\bar{P}}-\dot{\bar{a}})$$

$$\Rightarrow \ddot{\bar{P}} = \ddot{\bar{a}} + (\dot{\bar{\bar{\Omega}}} + \bar{\bar{\Omega}}^2)(\bar{P}-\bar{a})$$

$$\ddot{\bar{P}} = \ddot{\bar{a}} + \bar{\bar{W}}(\bar{P}-\bar{a}) \quad \text{—— (1)}$$

where, $\bar{\bar{W}} = \dot{\bar{\bar{\Omega}}} + \Omega^2$ is termed as angular acceleration matrix of the rigid body ~~both~~ motion.

$\Rightarrow$ Clearly first term is Skew-Symmetric where as Second one is Symmetric.

So $Vect(\bar{\bar{W}}) = Vect(\dot{\bar{\bar{\Omega}}}) = \dot{\bar{\omega}}$

{Termed as angular acceleration vector}

$\Rightarrow$ Eq (1) Can be rewritten as:-

$$\ddot{\bar{P}} = \ddot{\bar{a}} + \dot{\bar{\omega}} \times (\bar{P}-a) + \bar{\omega} \times [\bar{\omega} \times (\bar{P}-\bar{a})] \quad \text{—— (2)}$$

⇒ On the other hand, the time derivative of $\bar{t}$, referred as twist rate is :-

$$\dot{\bar{t}} = \begin{bmatrix} \dot{\bar{\omega}} \\ \dot{\bar{v}} \end{bmatrix}$$

③ acceleration of a point of the body.

⇒ The relationship between the twist rate and the second time derivative of the screw is derived by differentiating

$$\dot{\bar{S}} = \bar{\bar{T}}\,\bar{t}$$

$$\ddot{\bar{S}} = \bar{\bar{T}}\,\dot{\bar{t}} + \dot{\bar{\bar{T}}}\,\bar{t}$$

$$\text{where, } \dot{\bar{\bar{T}}} = \begin{bmatrix} \dot{\bar{\bar{F}}} & \bar{\bar{O}}_{4\times3} \\ \bar{\bar{O}} & \bar{\bar{O}} \end{bmatrix}$$

Similarly, Inverse relationship :-

$$\dot{\bar{t}} = \bar{\bar{S}}\,\ddot{\bar{s}} + \dot{\bar{\bar{S}}}\,\dot{\bar{s}}$$

$$\text{where } \dot{\bar{\bar{S}}} = \begin{bmatrix} \dot{\bar{\bar{F}}} & \bar{\bar{O}} \\ \bar{\bar{O}}_{3\times4} & \bar{\bar{O}} \end{bmatrix}$$

3.6) Rigid-Body Motion referred to Moving Coordinate Axis

⇒ Although in Kinematics no "preferred" coordinate system exists.

↳ In dynamics the governing equation of rigid-body motion are valid only in inertial frames.

Scanned by CamScanner

⇒ Consider the fixed coordinate frame X Y Z which we lable F, and the moving coordinate frame x y and z which will be labeled M.

⇒ Let $\bar{\bar{Q}}$ be the rotation matrix taking from F into the orientation M.

⇒ Let $\bar{O}$ be the position vector of the origin of M from the origin of F.

⇒ Further let $\bar{P}$ be the position vector of point P from the origin of F and $\bar{\rho}$ be position vector of the same point from the origin of M.

$$[\bar{P}]_F = [\bar{O}]_F + [\bar{\rho}]_F$$

We assume $\bar{\rho}$ is not available in frame F but in M

$$\text{So} \quad [\bar{\rho}]_F = [\bar{\bar{Q}}]_F [\bar{\rho}]_M$$

$$\text{So} \quad [\bar{P}]_F = [\bar{O}]_F + [\bar{\bar{Q}}]_F [\bar{\rho}]_M$$

⇒ Now in order to compute the velocity of $\bar{P}$, both side are differentiated.

$$[\dot{\bar{P}}]_F = [\dot{\bar{O}}]_F + [\dot{\bar{\bar{Q}}}]_F [\bar{\rho}]_M + [\bar{\bar{Q}}]_F [\dot{\bar{\rho}}]_M$$

$$[\dot{\bar{Q}}]_F = [\Omega]_F [\bar{Q}]_F$$

So
$$\boxed{[\dot{\bar{P}}]_F = [\dot{\bar{O}}]_F + [\bar{\Omega}]_F [\bar{Q}]_F [\bar{\rho}]_M + [\bar{Q}]_F [\dot{\bar{\rho}}]_M}$$

⇒ The above expression gives Velocity of P in F in terms of Velocity of P in M and the twist of M with respect to F.

⇒ Similarly for acceleration :-

$$\boxed{\begin{aligned}[\ddot{\bar{P}}]_F &= [\ddot{\bar{O}}]_F + \left([\dot{\bar{\Omega}}]_F + [\bar{\Omega}^2]_F\right)[\bar{Q}]_F [\bar{\rho}]_M \\ &+ 2[\bar{\Omega}]_F [\bar{Q}]_F [\dot{\bar{\rho}}]_M + [\bar{Q}]_F [\ddot{\bar{\rho}}]_M\end{aligned}}$$

(Acceleration of P as point wrt M)

(Coriolis acceleration)

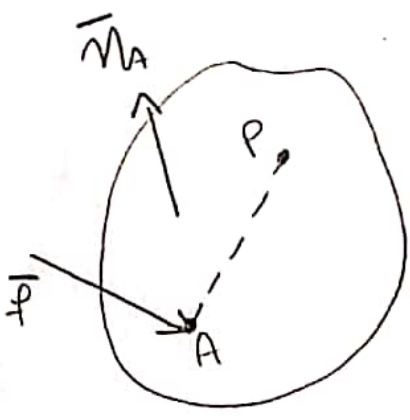
(Acceleration of P Measured from M)

3.7) Static Analysis of Rigid Body

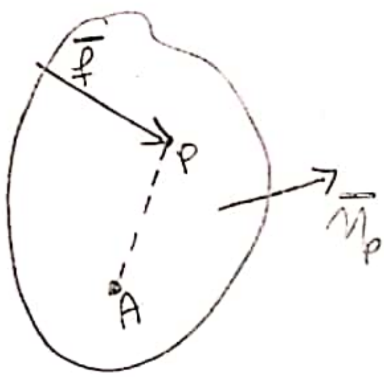
⇒ From elementary statics it is known that resultant of all external forces & moments, exerted on a body can be reduced to a force $\bar{f}$ acting at a point, say A and a moment $\bar{M}_A$.

↳ Alternatively, the aforementioned force f can be defined as acting at an arbitrary point P of the body but then the resultant moment $\bar{M}_P$ changes correspondingly.

⇒ In order to establish a relationship between $\bar{M}_A$ and $\bar{M}_P$, the moment of the first system of forces and moment with respect to the point P is equated to the moment about the same point of the second system thus obtaining:-



(a)



(b)

$$\bar{M}_P = \bar{M}_A + (\bar{a}-\bar{p})\times\bar{f}$$

$$\Rightarrow \bar{M}_P = \bar{M}_A + \bar{f}\times(\bar{p}-\bar{a})$$

Theorem 3.7.1: For a given system of forces & moment acting on a rigid body, if the resultant force is applied at any point of a particular line $\mathcal{L}''$, then the resultant moment is of minimum magnitude. Moreover that minimum magnitude moment is parallel to the resultant force.

$\Rightarrow$ Let $\bar{M}_0$ be the minimum magnitude moment.

$$\bar{M}_0 = M_0 \frac{\bar{f}}{\|\bar{f}\|}, \quad M_0 = \frac{\bar{M}_P\cdot \bar{f}}{\|\bar{f}\|}$$

Moreover, the pitch of the wrench p'' is defined as:-

$$p'' = \frac{M_0}{\|\bar{f}\|} = \frac{\bar{M}_P\cdot \bar{f}}{\|\bar{f}\|^2} \quad \text{or} \quad p'' = \frac{2\pi \bar{M}_P\cdot \bar{f}}{\|\bar{f}\|^2}$$

$\Rightarrow$ Of course the wrench axis can be defined by its Plücker array $\bar{P}_{L''}$

$$\bar{P}_{L''} = \begin{bmatrix}\bar{e}''\\ \bar{M}''\end{bmatrix}, \quad \bar{e}'' = \frac{\bar{f}}{\|\bar{f}\|}, \quad \bar{M}'' = \bar{p}\times \bar{e}''$$

{Unit vector $\parallel$ to $\mathcal{L}''$}

{Position vector of any point of $\mathcal{L}''$}

⇒ Let P_0'' be point on L'' lying closest to origin with position vector $\overline{P_0''}$.

$$\boxed{\overline{P_0''} = \frac{1}{\|\bar{f}\|^2}\,\bar{f}\times(\overline{n}_A - \bar{f}\times\bar{a})}$$

Theorem 3.7.2: The projection of the resultant moment of a system of moments and forces acting on a rigid body that arises when the resultant force is applied to an arbitrary point of the body onto the wrench axis is constant.

⇒ Upon the multiplication of the screw by an amplitude A with the unit of force, what we will obtain would be a wrench $\overline{W}$.

- → first 3 component unit of force
- → last 3 component unit of Moment

⇒ We would like to be able to obtain the power developed by the wrench on the body moving with the twist $\bar{t}$ by a simple inner product of two array.

⇒ This can be done ~~as~~ if we redefine wrench not simply as the product of a screw by an amplitude, but as a linear transformation of that screw involving the 6×6 array $\bar{\bar{\Gamma}}$ defined as:-

$$\bar{\bar{\Gamma}} = \begin{bmatrix} \bar{\bar{O}} & \bar{\bar{I}} \\ \bar{\bar{I}} & \bar{\bar{O}} \end{bmatrix}$$

→ 3×3 Identity Matrix

→ 3×3 Zero Matrix

⇒ Now we define the wrench as a linear transformation of the screw S defined as:-

$$\bar{W} = \bar{A}\bar{\bar{\Gamma}}\bar{S} = \begin{bmatrix} \bar{P}\times(A\bar{e}) + P(A\bar{e}) \\ A\bar{e} \end{bmatrix}$$

A ⇒ Unit of force

First 3 components of the foregoing array can be readily identified as the moment of force of magnitude A acting along a line of action, with respect to a point P, to which moment ∥ to that line and of magnitude PA is added.

Last 3 components of that array is a force of magnitude A and ∥ to the same line.

Scanned by CamScanner

⇒ So, Wrench has been defined so that the inner product $\bar{t}^T \bar{\omega}$ will product the power Π.

$$\boxed{\Pi = \bar{t}^T \bar{\omega}}$$

⇒ Let the wrench and the twist be given in terms of their respective screws $\bar{S}_\omega$ and $\bar{S}_t$ as

$$\bar{W} = W \bar{\bar{\Gamma}} \bar{S}_\omega \qquad \bar{t} = T \bar{S}_t$$

{Where W and T are the amplitudes of the wrench and the twist respectively}

⇒ We say wrench and the twist are reciprocal to each other if zero power is developed.

$$(\bar{\bar{\Gamma}} \bar{S}_\omega)^T \bar{S}_t = \bar{S}_\omega^T \bar{\bar{\Gamma}}^T \bar{S}_t = 0$$

$$\Rightarrow \bar{S}_\omega^T \bar{\bar{\Gamma}} \bar{S}_t = 0 \quad \text{or} \quad \bar{S}_t^T \bar{\bar{\Gamma}} \bar{S}_\omega = 0$$

{as $\bar{\bar{\Gamma}}$ is symmetric}

⇒ Now if A and P are arbitrary points of a rigid body, we define the wrench at these points as

$$\overline{W}_A = \begin{bmatrix} \overline{n}_A \\ \overline{f} \end{bmatrix} \qquad \overline{W}_P = \begin{bmatrix} \overline{n}_P \\ \overline{f} \end{bmatrix}$$

So $\boxed{\overline{W}_P = \overline{\overline{V}}\, \overline{W}_A}$ {wrench-transfer formula}

where $\overline{\overline{V}} = \begin{bmatrix} \overline{\overline{I}} & \overline{\overline{A}} - \overline{\overline{P}} \\ \overline{\overline{0}} & \overline{\overline{I}} \end{bmatrix}$

{$\overline{\overline{A}}$ & $\overline{\overline{P}}$ are cross product matrix of $\overline{a}$ & $\overline{p}$}

3.8) Dynamics of Rigid Body

⇒ If a rigid body has a mass density ρ then its mass m is defined as:-

$$m = \int_B \rho \, dB \quad \text{——— (1)}$$

where B denotes the region of the 3D space occupied by the body.

⇒ If $\overline{p}$ denotes position vector of an arbitrary point of the body, from a previously defined origin O, the mass first moment of the body with respect to O, $\overline{q}_0$ is defined as:-

$$\overline{q}_0 = \int_B \rho \, \overline{p} \, dB \quad \text{——— (2)}$$

⇒ Furthermore, the mass Second moment of the body with respect to O is defined as:-

$$\bar{\bar{I}}_O = \int_B \rho \left[(\bar{p}\cdot\bar{p})\bar{\bar{I}} - \bar{p}\bar{p}^T \right] dB \quad \text{—} (3)$$

$\downarrow$

It is clearly a Symmetric matrix and is called the moment of Inertia matrix of the body under study with respect to O

Theorem 3.8.1: The moment of inertia of a rigid body with respect to a point O is positive definite.

Proof: for any vector $\bar{\omega}$, the quadratic form

$$\bar{\omega}^T \bar{\bar{I}}_O \bar{\omega} > 0$$

$$\bar{\omega}^T \bar{\bar{I}}_O \bar{\omega} = \int_B \rho \left[\|\bar{p}\|^2 \|\bar{\omega}\|^2 - (\bar{p}\cdot\bar{\omega})^2 \right] dB$$

Now $\bar{p}\cdot\bar{\omega} = \|\bar{p}\| \|\bar{\omega}\| \cos(\bar{p},\bar{\omega})$

So,

$$\bar{\omega}^T \bar{\bar{I}}_O \bar{\omega} = \int_B \rho \left[\|\bar{p}\|^2 \|\bar{\omega}\|^2 - \|\bar{p}\|^2 \|\bar{\omega}\|^2 \cos^2(\bar{p},\bar{\omega}) \right] dB$$

$$\Rightarrow \int_B \rho \|\bar{P}\|^2 \|\bar{\omega}\|^2 \sin^2(\bar{P}, \bar{\omega})\, dB$$

⇒ Which is positive quatity, that vanishes only in the ideal case of <u>slender body</u> having all its mass concentrated along a line passing through O and ∥ to $\bar{\omega}$.

⇒ If vector $\bar{\omega}$ of the previous discussion is the angular velocity of the rigid body, then the quadratic form turns out to be twice the kinetic energy of the body.

$$T = \int_B \frac{1}{2} \rho \|\dot{\bar{P}}\|^2 dB$$

→ for this purpose it is assumed that point O is instantaneously at rest.

$$\dot{\bar{P}} = \bar{\omega} \times \bar{P} = -\bar{\bar{P}}\bar{\omega}$$

{Cross product matrix of $\bar{P}$}

$$\|\dot{\bar{P}}\|^2 = (\bar{\bar{P}}\bar{\omega})^T(\bar{\bar{P}}\bar{\omega}) = \bar{\omega}^T\bar{\bar{P}}^T\bar{\bar{P}}\bar{\omega} = -\bar{\omega}^T\bar{\bar{P}}^2\bar{\omega}$$

we know $\bar{\bar{P}}^2 = PP^T - \|\bar{P}\|^2 I$

so $\|\dot{\bar{P}}\|^2 = \bar{\omega}^T \left(\|\bar{P}\|^2 \bar{\bar{I}} - \bar{P}\bar{P}^T\right) \bar{\omega}$

$$T = \frac{1}{2} \int_B \rho\, \bar{\omega}^T \left(\|\bar{P}\|^2 \bar{\bar{I}} - \bar{P}\bar{P}^T\right) \bar{\omega}\, dB$$

$$\boxed{T = \frac{1}{2}\bar{\omega}^T \bar{\bar{I}}_0 \bar{\omega}} \quad \text{--- (4)}$$

⇒ The mass Center of a rigid body, measured from O, is defined as a point C, not necessarily within the body, of Position vector $\bar{c}$ given by:

$$\boxed{\bar{c} = \frac{\bar{q}_0}{m}} \quad \text{--- (5)}$$

⇒ Mass moment of Inertia of a body with respect to its Centroid is defined as:-

$$\boxed{\bar{\bar{I}}_c = \int_B \rho \left[\|\bar{r}\|^2 \bar{\bar{I}} - \bar{r}\bar{r}^T \right] dB} \quad \text{--- (6)}$$

where r is defined as $\boxed{\bar{r} = \bar{p} - \bar{c}}$

{Called Centroidal mass moment of Inertia}

⇒ The three eigenvalues are positive and are referred to as the principal moments of Inertia of the body.

⇒ The eigen Vector of the inertia matrix are furthermore mutually orthogonal and defined the principal axis of Inertia of the body.

⇒ Let $\bar{\bar{I}}_o$ & $\bar{\bar{I}}_c$ be defined as in ⑤ & ⑥.
It is possible to show that:

$$\boxed{\bar{\bar{I}}_o = \bar{\bar{I}}_c + m\left(\|\bar{c}\|^2 \bar{\bar{I}} - \bar{c}\bar{c}^T\right)} \quad (7)$$

→ Theorem of ∥ axis.

⇒ Smallest principal moment of inertia of a rigid body attains its minimum value at the mass center of the body.

⇒ **Newton - Euler Equation governing the Motion of a rigid body**

Let the body at hand be acted upon by a wrench of force $\bar{f}$ applied at its center, and a moment $\bar{n}_c$.

\# Newtons Equation: $\boxed{\bar{f} = m\ddot{\bar{c}}}$ — (8)

\# Euler Equation: $\boxed{\bar{n}_c = \bar{\bar{I}}_c \dot{\bar{\omega}} + \bar{\omega} \times \bar{\bar{I}}_c \bar{\omega}}$ — (9)

⇒ The momentum $\bar{m}$

$$\boxed{\bar{m} = m\dot{\bar{c}}} \quad (10)$$

⇒ Angular momentum $\bar{h}_c$ of rigid body with respect to mass center..

$$\boxed{\bar{h}_c = \bar{\bar{I}}_c \bar{\omega}} \quad (11)$$

So $\dot{\bar{m}} = m\ddot{c}$ & $\dot{h}_c = \bar{\bar{I}}_c\dot{\bar{\omega}} + \bar{\omega} \times \bar{\bar{I}}_c\bar{\omega}$

Hence Law of Motion take form:-

$$\boxed{\bar{f} = \dot{\bar{m}}} \quad \text{—— } (8^*)$$

$$\boxed{M_c = \dot{h}_c} \quad \text{—— } (9^*)$$

⇒ Above equation can be written in a more Compact form as:-

⇒ Let us introduce a 6×6 matrix $\bar{\bar{M}}$.

$$\bar{\bar{M}} = \begin{bmatrix} \bar{\bar{I}}_c & \bar{O} \\ \bar{O} & m\bar{\bar{I}} \end{bmatrix}$$

⇒ Now Newton-Euler equations can be written as

$$\boxed{\bar{\bar{M}}\dot{\bar{E}} + \bar{\bar{W}}\bar{\bar{M}}\bar{E} = \bar{w}}$$

$$\bar{\bar{W}} = \begin{bmatrix} \bar{\Omega} & \bar{O} \\ \bar{O} & \bar{O} \end{bmatrix}$$

⇒ The momentum Screw of the rigid body about the mass Center is the 6-dimensional vector $\bar{\mu}$ defined as

$$\bar{\mu} = \begin{bmatrix} \bar{\bar{I}}_c\bar{\omega} \\ m\dot{c} \end{bmatrix} = \bar{\bar{M}}E$$

$$\dot{\mu} = \bar{\bar{M}}\dot{E} + \bar{\bar{W}}\mu = \bar{\bar{M}}\dot{E} + \bar{\bar{W}}\bar{\bar{M}}E$$

⇒ Kinetic energy can be written in Compact form as:

$$\boxed{T = \frac{1}{2} E^T \bar{\bar{M}} E}$$

⇒ Finaly Newton-Euler equations can be written in an even more Compact form as:-

$$\boxed{\dot{\mu} = \bar{\bar{W}}}$$
