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Transient and Steady-State Responce Analysis

⇒ In analysing and designing control systems, we must have a basis of comparison of performance of various control systems.

⇒ This basis may be set up by specifying particular test input signals and by comparing the response ~~of such test signals~~ of various systems to these input signals.

★ **Typical test Signals:**

⇒ Common test signals are:-
- (i) Step function
- (ii) Ramp function
- (iii) acceleration function
- (iv) Impulse function
- (v) Sinusoidal function
- (vi) White noise.

⇒ Which of these test input signals to use for analysing system characteristics may be determined by the form of the input that the system will be subjected to most frequently under normal condition.

★ Transient Responce and Steady-state Responce

 # Transient Responce ⇒ By transient responce, we mean that which goes from the initial state to the final state.

 # Steady-State Responce ⇒ By steady state responce, we mean the manner in which the System output behave as t approaches infinity.

★ Absolute Stability, Relative Stability and Steady-State Error

 # Absolute Stability ⇒ Whether the System is Stable or Unstable.

 # Equilibrium ⇒ A Control System is in equilibrium if, in the absence of any disturbance or input the Output Stays in the Same State.

 # Stable ⇒ A System is stable if the output eventually Comes back to its equilibrium state when the System is Subjected to initial Condition.

 # Critically Stable ⇒ If oscillation of output Continue forever.

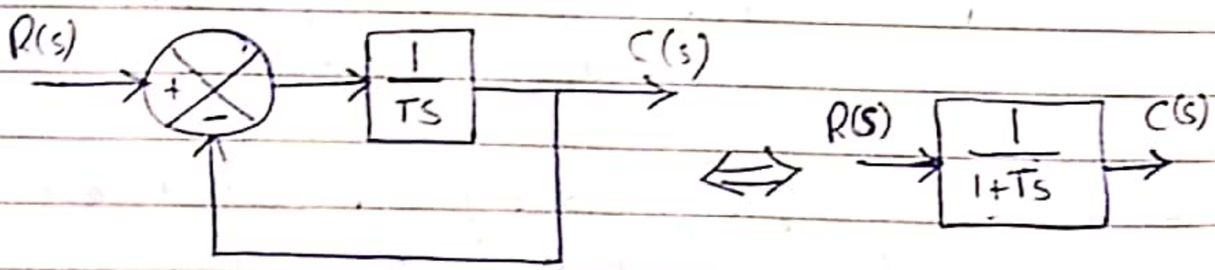
Unstable ⇒ The output diverges without bound from its equilibrium state when the system is subjected to initial condition.

Steady state Error ⇒ If output of a system at steady state does not exactly agree with the input, the system is said to have steady state error.

Relative Stability ⇒ It gives the degree of stability or how close it is to instability.

★ First-Order System {Systems whose closed loop Transfer function has one pole}

$$\boxed{\frac{C(s)}{R(s)} = \frac{1}{1+Ts}}$$



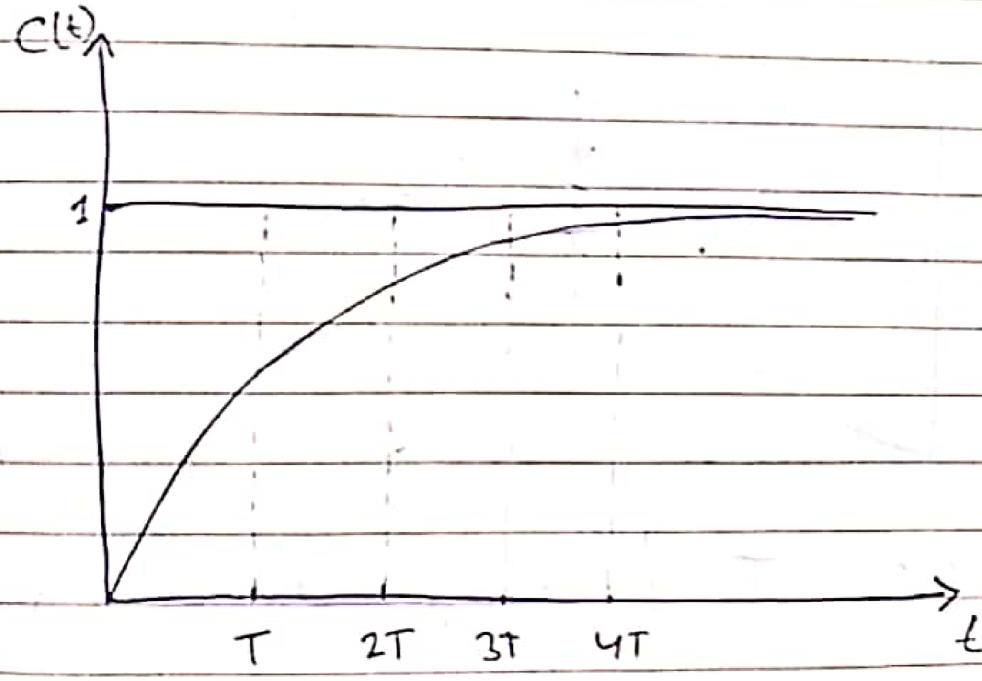
★ Unit-Step Responce of First Order System

$$\Rightarrow \quad r(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$

$$R(s) = \frac{1}{s}$$

$$C(s) = \frac{1}{s(Ts+1)} = \frac{1}{s} - \frac{1}{s+\frac{1}{T}}$$

$$\boxed{c(t) = 1 - e^{-\frac{t}{T}}} \quad \forall\, t \geq 0$$



★ Unit-Ramp Responce of First Order System

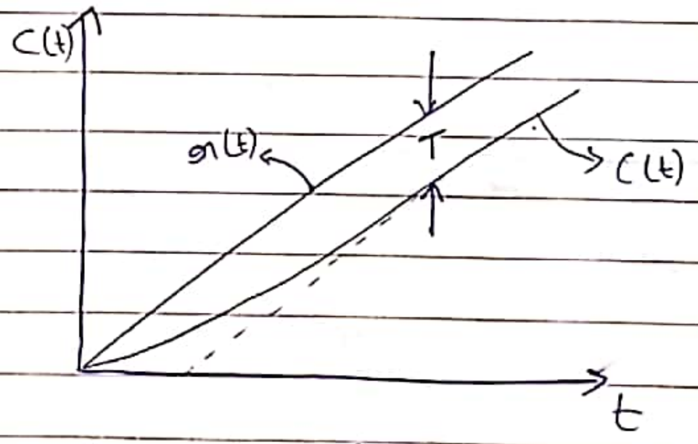
$$r(t) = \begin{cases} 0 & \forall t < 0 \\ t & \forall t > 0 \end{cases}$$

$$R(s) = \frac{1}{s^2}$$

$$C(s) = \frac{1}{s^2(Ts+1)} = \frac{1}{s^2} - \frac{T}{s} + \frac{T^2}{Ts+1}$$

$$\boxed{c(t) = t - T + Te^{-\frac{t}{T}}}$$

$$e(t) = r(t) - c(t) = T\left(1 - e^{-\frac{t}{T}}\right)$$



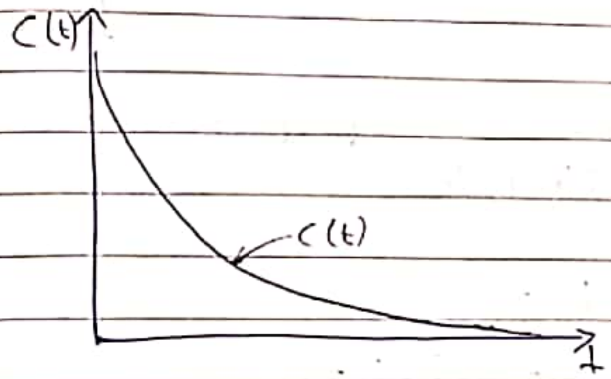
$e(\infty) = T$ {Steady State error}

★ **Unit-Impulse Response of First Order System:**

$r(t) = \delta(t)$ {dirac delta function}

$R(s) = 1$

$$C(s) = \frac{1}{Ts+1} \Rightarrow C(t) = \frac{1}{T} e^{-t/T} \quad \forall\ t \geq 0$$



★ **Summary of Response of First Order System**

Test Input	C(t)
1. Unit Step Input	$C(t) = 1 - e^{-\frac{t}{T}}$
2. Unit Ramp Input	$C(t) = t - T + Te^{-t/T}$ $e(t) = T(1 - e^{-t/T})$
3. Unit-Impulse	$C(t) = \frac{1}{T} e^{-\frac{t}{T}}$

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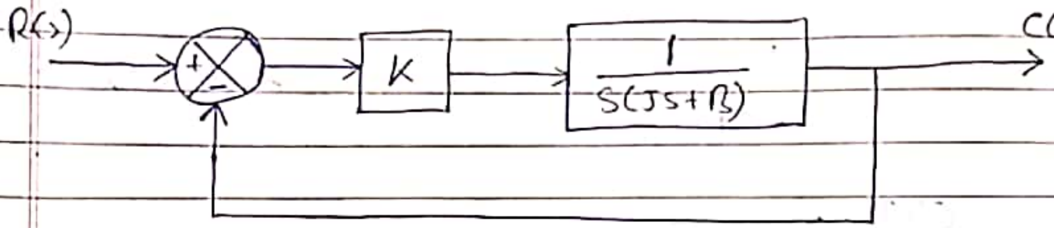
★ Second Order System

{System whose closed loop Transfer function have two Poles}

Servo System



Servo System with Proportional Controller

$$\frac{C(s)}{R(s)} = \frac{K/J}{s^2 + (B/J)s + (K/J)}$$

★ Step Responce of Second-Order System

$$\frac{C(s)}{R(s)} = \frac{K/J}{\left[S + \frac{B}{2J} + \sqrt{\left(\frac{B}{2J}\right)^2 - \frac{K}{J}}\right]\left[S + \frac{B}{2J} - \sqrt{\left(\frac{B}{2J}\right)^2 - \frac{K}{J}}\right]}$$

In the transient-response analysis, it is Convenient to write.

$$\frac{K}{J} = \omega_n^2 \qquad \frac{B}{J} = 2\xi\omega_n = 2\sigma$$

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where, σ = attenuation

ω_n = Undamped natural frequency.

ξ = damping Ratio

So $$\boxed{\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}}$$

→ {Standard form of Second order System}

⇒ The dynamic behavior of the second order system can then be described in terms of two parameters ξ and ω_n.

Case 1: $0 \leq \xi < 1$ {Closed-loop poles are complex conjugates}

Case 2: $\xi = 1$ {Closed loop poles are Real and equal}

Case 3: $\xi > 1$ {Closed loop poles are Real and different}

Case 1: $0 \leq \xi < 1$ {Underdamped Case}

$$\boxed{\frac{C(s)}{R(s)} = \frac{\omega_n^2}{\left(s+\xi\omega_n+\omega_n\sqrt{\xi^2-1}\right)\left(s+\xi\omega_n-\omega_n\sqrt{\xi^2-1}\right)}}$$

Let $\omega_d = \omega_n\sqrt{1-\xi^2}$ {damped natural frequency}

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s+\xi\omega_n+j\omega_d)(s+\xi\omega_n-j\omega_d)}$$

For unit step input $R(s) = \frac{1}{s}$

$$C(s) = \frac{1}{s} - \frac{s+\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2} - \frac{\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2}$$

$$c(t) = 1 - e^{-\xi\omega_n t}\cos\omega_d t - \frac{\xi\omega_n}{\omega_d}e^{-\xi\omega_n t}\sin\omega_d t$$

$$\Rightarrow 1 - e^{-\xi\omega_n t}\left(\cos\omega_d t + \frac{\xi}{\sqrt{1-\xi^2}}\sin\omega_d t\right)$$

$$\Rightarrow 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin\left(\omega_d t + \tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right)\right)$$

$e(t) = r(t) - c(t)$

$$c(t) = \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin\left(\omega_d t + \tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right)\right)$$

→ damped Sinusoidal Oscillation

→ No Steady state error

⇒ If $\xi = 0$, the response becomes Undamped and oscillation continue indefinitely.

Case 2: {Critically damped Case}

$(\xi = 1)$

$$C(s) = \frac{\omega_n^2}{s(s+\omega_n)^2}$$

$$C(t) = 1 - e^{-\omega_n t}(1+\omega_n t) \quad \forall\ t \geq 0$$

Case 3: {Overdamped Case}

$\{\xi > 1\}$

$$C(s) = \frac{\omega_n^2}{s\left(s+\xi\omega_n+\omega_n\sqrt{\xi^2-1}\right)\left(s+\xi\omega_n-\omega_n\sqrt{\xi^2-1}\right)}$$

$$\Rightarrow C(t) = 1 + \frac{\omega_n}{2\sqrt{\xi^2-1}}\left(\frac{e^{-s_1 t_1}}{s_1} - \frac{e^{-s_2 t_2}}{s_2}\right)$$

$$\left\{s_1 = \left(-\xi + \sqrt{\xi-1}\right)\omega_n \ \ \& \ \ s_2 = \left(\xi - \sqrt{\xi-1}\right)\omega_n\right\}$$

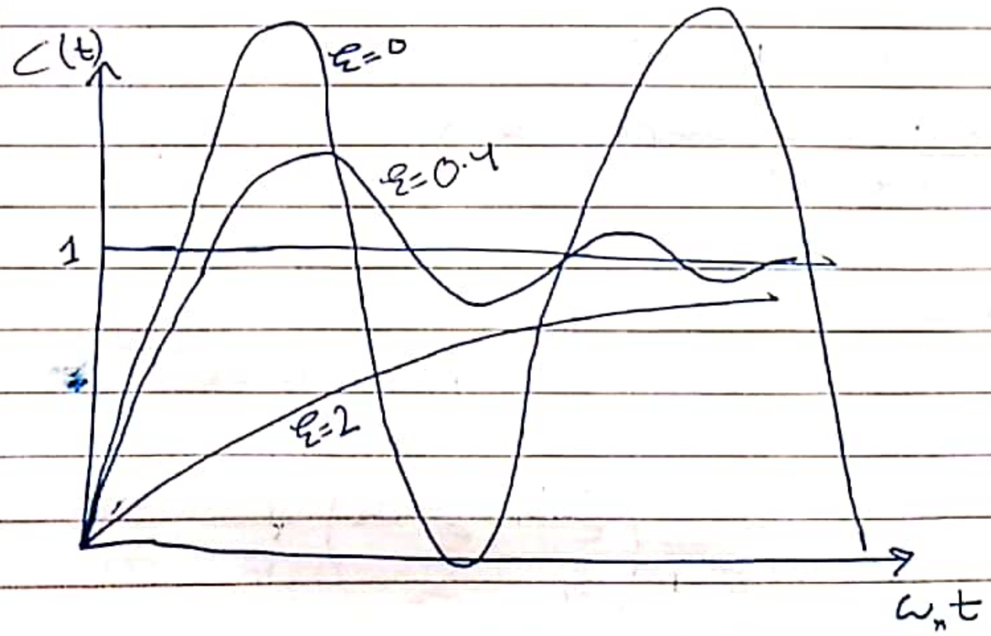
⇒ as $\xi > 1$, One of the two decaying exponentials decrease much faster than the other, so the faster-decaying exponentials term may be neglected.

⇒ So $\frac{C(s)}{R(s)}$ may be approximated as :-

$$\frac{C(s)}{R(s)} = \frac{(\xi - \sqrt{\xi^2-1})\omega_n}{s + (\xi - \sqrt{\xi^2-1})\omega_n} = \frac{S_2}{s+S_2}$$

⇒ The response is similar to that of first order system.

$$C(t) = 1 - e^{-S_2 t} \quad \forall \; t \geq 0$$



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Note

* Two 2^{nd} order system having same ξ but different ω_n will exhibit the same overshoot and the same oscillatory pattern. Such system is said to have same relative stability.

* Definitions of Transient Response Specification

It is common practice to use the standard initial condition that the system is at rest initially with the output and all time derivatives thereof zero.

1. **Delay time (t_d)**: Time required for the response to reach half the final value the very first time.

2. **Rise time (t_r)**: time required for the response to rise from (10% to 90%), (5% to 95%) or (0% to 100%) of its final value.

3. **Peak time (t_p)**: Time required for the response to reach the first peak of the overshoot.

4. **Maximum (%) overshoot (M_p)**: $\dfrac{C(t_p) - C(\infty)}{C(\infty)} \times 100$

5. Settling time: Time required for the response curve to reach and stay within a range about the final value. (Usually 2% to 5%)

★ Second Order System and Transient-Response Specification

1. Delay time (t_d)

$$\boxed{e^{-\xi\omega_n t_d}\sin\left(\omega_d t_d + \tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right)\right) = \frac{1}{2}\sqrt{1-\xi^2}}$$

2. Rise time (t_r)

$$\boxed{t_r = \frac{1}{\omega_d}\tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{-\xi}\right)}$$

3. Peak time (t_p)

$$\sin \omega_d t_p = 0$$

or

$$\boxed{t_p = \frac{\pi}{\omega_d}}$$

4) Maximum overshoot (M_p)

$$\boxed{M_p = e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}}}$$

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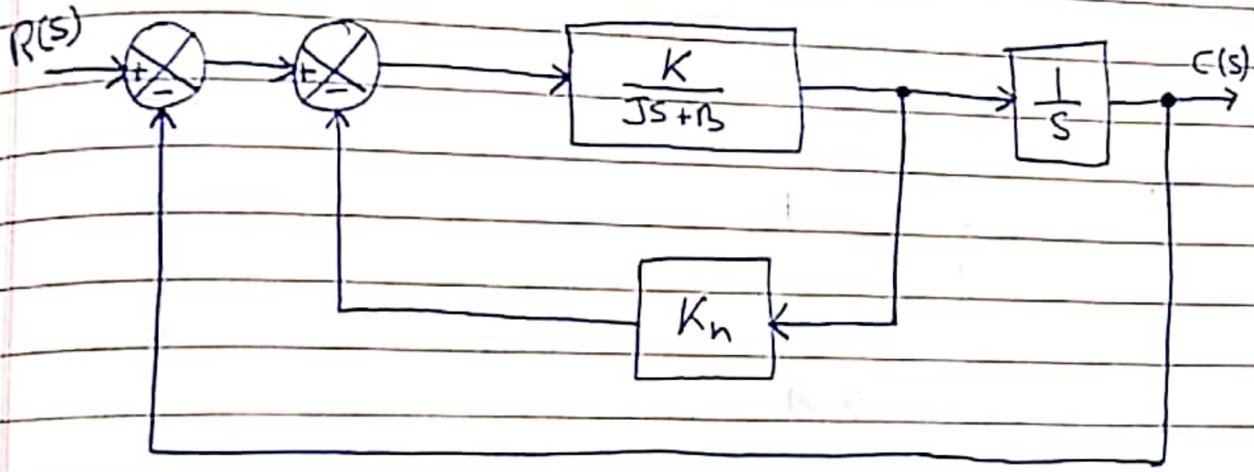
5> Setting time

$$\boxed{t_s = \frac{4}{\xi \omega_n}} \quad \{2\% \text{ Criterion}\}$$

★ Servo System with Velocity Feedback



$$\frac{C(s)}{R(s)} = \frac{K}{J(s^2) + (B+KK_n)s + K}$$

$$\xi = \frac{B + KK_n}{2\sqrt{KJ}}$$

⇒ Velocity feedback has the effect of increasing damping.

→ We can adjust K_n so that ξ is between 0.4 to 0.7.

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★ Impulse Response of Second-Order System

$$C(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

Case 1: $0 \le \xi < 1$

$$\boxed{C(t) = \frac{\omega_n}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t} \sin\sqrt{1-\xi^2}\, t}$$

Case 2: $\xi = 1$

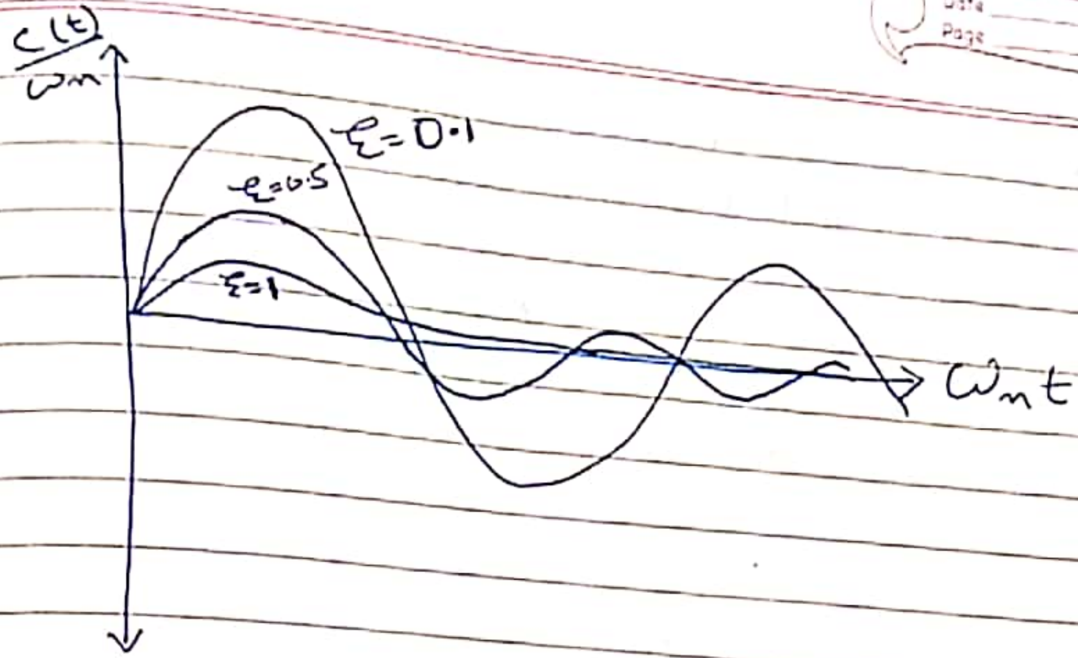
$$\boxed{C(t) = \omega_n^2\, t\, e^{-\omega_n t}}$$

Case 3: $\xi > 1$

$$C(t) = \frac{\omega_n}{2\sqrt{\xi^2-1}} e^{-C_1 t} - \frac{\omega_n}{2\sqrt{\xi^2-1}} e^{-C_2 t}$$

$$\Rightarrow \boxed{C(t) = \frac{\omega_n}{2\sqrt{\xi^2-1}} \left(e^{-C_1 t} - e^{-C_2 t}\right)}$$

$$\left\{ \begin{aligned} C_1 &= \left(-\xi - \sqrt{\xi^2-1}\right)\omega_n \\ C_2 &= \left(-\xi + \sqrt{\xi^2-1}\right)\omega_n \end{aligned} \right\}$$

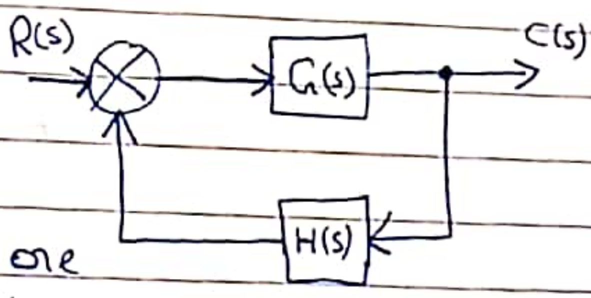


★ Higher Order Systems

Response of a higher-order system is the sum of the responses of first-order system & second-order systems.

★ Transient Response of Higher Order System

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$



In general G(s) and H(s) are given as ratios of polynomials in s

$$G(s) = \frac{P(s)}{q(s)} \quad \& \quad H(s) = \frac{n(s)}{d(s)}$$

$$\Rightarrow \frac{C(s)}{R(s)} = \frac{P(s)\,d(s)}{q(s)\,d(s) + P(s)\,n(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_n}$$

⇒ The transient response of this System to any given input can be obtained by a Computer Simulation.

⇒ If an analytical expression of the ~~transfer~~ transient response is desired, then it is necessary to factor the denominator polynomial.

$$\frac{C(s)}{R(s)} = \frac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$

⇒ Let us examine the response behavior of this System to a unit Step input.

Case 1: Closed loop poles are real & distinct

$$C(s) = \frac{a}{s} + \sum_{i=1}^{n} \frac{a_i}{s+p_i}$$

$\left\{ \text{where } a_i \text{ is the residue of the pole at } s = -p_i \right\}$

$$\boxed{C(t) = a + \sum_{i=1}^{n} a_i e^{-p_i t}}$$

⇒ If all closed-loop poles lie in the left half S plane, then $C(\infty) = a$.

Case 2: Closed loop poles of $C(s)$ consist of real poles and pairs of complex-conjugate poles.

$$C(s) = \frac{a}{s} + \sum_{j=1}^{q} \frac{a_j}{s+p_j} + \sum_{K=1}^{r} \frac{b_K(s+\xi_K\omega_K) + c_K\omega_K\sqrt{1-\xi_K^2}}{s^2+2\xi_K\omega_K s+\omega_K^2}$$

$$\boxed{C(t) = a + \sum_{j=1}^{q} a_j e^{-p_j t} + \sum_{K=1}^{r} b_K e^{-\xi_K\omega_K t} \cos\omega_K\sqrt{1-\xi_K^2}\,t + \sum_{K=1}^{r} c_K e^{-\xi_K\omega_K t} \sin\omega_K\sqrt{1-\xi_K^2}\,t}$$

⇒ Thus the response curve of a stable higher order system is the sum of a number of exponential curves and damped sinusoidal curves.

⇒ If all closed-loop poles lie in the left half s-plane, then the exponential terms and the damped exponential term will approach zero as time t increases.

↳ The steady state output is then $C(\infty) = a$.

* Dominant Closed loop poles

⇒ The relative dominance of closed-loop poles is determined by the ratio of the real part of the Closed-loop poles, as well as by the relative magnitudes of the residues evaluated at the Closed loop poles.

→ {depends on both Closed loop Poles & Zeros}

⇒ If the ratio of the real parts of the Closed-loop poles exceed 5 and there are no zeros nearby, then the Closed-loop pole nearest the $j\omega$ axis will dominate in the transient response behavior.

→ Those Closed loop poles that have dominant effects on the transient response behavior are called dominant closed-loop poles.

* Stability Analysis in Complex plane

⇒ The Stability of a linear closed-loop System can be determined from the location of the Closed loop poles in the s plane.

Closed loop poles on Left half of S-plane ⇒ Stable

Closed loop poles on Right half of S-plane ⇒ Unstable

⇒ Whether a linear System is stable or Unstable is a property of the System itself and does not depend on the input or driving function of the System

↳ The poles of the input or driving function, do not affect the property of stability of the system, but they Contribute only to Steady-State response terms in the Solution.

⇒ If dominant complex-Conjugate closed loop poles lies close to the jω axis, the transient response may exhibit excessive oscillations or may be very slow

↳ To guarantee fast yet well-damped, transient response characteristics, it is necessary that the closed-loop poles of the System lie in a particular region on the complex plane.