Recursive State Estimation

2.1 Introduction

⇒ At the core of probabilistic robotics is the idea of estimating state from sensor data.

"estimating quantities from sensor data that are not directly observable, but can be inferred"

⇒ The goal of this chapter is to introduce the basic vocabulary & mathematical tools for estimating state from sensor data.

2.2 Basic Concepts in Probability.

⇒ Let X denote a random variable and x denote a specific event that X might take on.

⇒ $p(X=x)$, denotes probability that the random variable has value x.

⇒ Discrete probabilities sums to one, that is,

$$\boxed{\sum_x p(X=x) = 1}$$

⇒ Probabilities are always non-negative.

⇒ We usually omit explicit mention

$$p(X=x) \iff p(x)$$

⇒ Continuous spaces are characterized by random variables that can take on a continuum of values.

⇒ We assume that all continuous random variables possess probability density function (PDFs).

⇒ A common density function is that of the one-dimensional <u>normal distribution</u> with <u>mean μ</u> & <u>variance σ^2</u>.

↳ This distribution is given by the following <u>Gaussian function</u>:

$$p(x) = (2\pi\sigma^2)^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right\}$$

$$\mu = \frac{\int_{x_{min}}^{x_{max}} p(x)\,dx}{\int_{x_{min}}^{x_{max}} dx}$$

{Average value}

$$\sigma^2 = \int_{x_{min}}^{x_{max}} (x-\mu)^2 p(x)\,dx$$

{It measures how far a set of random numbers are spread out from their average value}

⇒ Normal distribution will be abbreviated as:

$$\boxed{N(x : \mu, \sigma^2)}$$

random Variable, mean, Variance

⇒ x can be a vector. Normal distributions over vectors are called multivariate.

⇒ Multivariate normal distributions are characterized by density functions of the following form:

$$p(x) = \det(2\pi \Sigma)^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}(x-\mu)^T \Sigma^{-1} (x-\mu)\right\}$$

{Positive Semidefinite Symmetric matrix called Covariance matrix}

⇒ The joint distribution of two random variables X and Y is given by:

$$p(x,y) = p(X=x, \& \ Y=y)$$

⇒ If X and Y are independent, we have

$$p(x,y) = p(x)\,p(y)$$

⇒ If we already know that Y's value is y and we would like to know probability that X's value is x conditioned on that fact.

↳ Such probability is called Conditional probability and is denoted as:

$$\boxed{p(x|y)}$$

⇒ If $P(y) > 0$,

$$\boxed{P(x|y) = \frac{P(x,y)}{P(y)}}$$

⇒ <u>Theorem of total probability</u>

$$\boxed{P(x) = \int P(x|y)\, P(y)\, dy} \text{ or } \boxed{P(x) = \sum_y P(x|y)\, P(y)}$$

⇒ <u>Bayes rule</u>: It states that if $P(y) > 0$

$$\boxed{P(x|y) = \frac{P(y|x)\, P(x)}{P(y)}}$$

{ Predominant role in 'Probabilistic robotics }

Let x ⇒ quantity that we would like to know
y ⇒ It is the data that we have. (Sensor measurement)

$P(x)$ ⇒ It is called prior probability distribution.
$P(x|y)$ ⇒ It is called posterior probability distribution

⇒ In robotics, this inverse probability is often coined "<u>generative model</u>".

⇒ $P(y)$, does not depend on x.

↳ Thus the factor $P(y)^{-1}$ will be same for any value x in the posterior $p(x|y)$.

↳ For this reason, $P(y)^{-1}$ is often written as a normalizer variable, and generically denoted η:

$$P(x|y) = \eta \, P(y|x)\, p(x)$$

⇒ If X is discrete, equation of this type can be computed as follows:

$$\forall x: \; aux_{x|y} = P(y|x)\, P(x)$$

$$aux_y = \sum_x aux_{x|y}$$

$$\forall x: \; p(x|y) = \frac{aux_{x|y}}{aux_y}$$

⇒ The expectation of a random variable X is given by

$$\boxed{E[X] = \int x \, p(x)\, dx} \qquad E[X] = \sum_x x\, P(x)$$

⇒ For arbitrary numerical value a and b,

$$E[aX + b] = a\,E[X] + b$$

⇒ The covariance of X is obtained as follows:

$$Cov[X] = E[X - E[X]]^2 = E[X^2] - E[X]^2$$

The covariance measures the squared expected deviation from the mean

⇒ Another important characteristic of a random variable is its entropy.

⇒ For discrete random variables, the entropy is given by the following expression.

$$H(P) = E[-\log_2 P(x)] = -\sum_x P(x) \log_2 P(x)$$

→ The concept of entropy originates in information theory.

→ Entropy is the expected information that the value of x carries.

→ $-\log_2 P(x)$ is the number of bits required to encode x using an optimal encoding

2.3) Robot Environment Interaction

2.3.1) State

→ Environments are characterized by state.

State → { Collection of all aspect of the robot and its environment that can impact the future }

⇒ State that changes ⇒ dynamic state
which not ⟶ static state

⇒ State will be denoted as x.
⇒ State at time t will be denoted as x_t.

Examples of State

robot pose (location and Orientation)
↳ Also referred to as Kinematic state.

Configuration of robot's actuators
↳ Also part of Kinematic state of the robot.

robot velocity

The location and features of surrounding objects in the environment.

The location and velocity of moving objects and people.

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⇒ A state x_t is called Complete if Knowledge of past carry no additional information that would help us to predict the future more accurately.

⇒ Process where no variable prior to x_t influence the evolution of future states is commonly Known as <u>Markov chains</u>.

2.3.2> Environment Interaction

→ Robot can influence the state of its environment through its actuators.

→ Gather information about the state through its sensors.

{ Typically, Sensor measurement arrive with some delay. Hence they provide information about the state a few moments ago. }

⇒ In accordance with the two types of environment interaction, the robot has access to two different data streams.

① Measurement data

⇒ Provides information about a momentary state of the environment.

⇒ The measurement data at time t will be denoted.

$$z_t$$

⇒ The notation,

$$z_{t_1:t_2} = z_{t_1}, z_{t_1+1} \cdots z_{t_2}$$

{denotes the Set of all measurements acquired from time t_1 to t_2}

② Control data

↳ Carry information about the Change of State

Example: Velocity, odometer etc.

⇒ Control data will be denoted u_t.

⇒ The variable u_t will always correspond to the change of state in the time interval $(t-1; t]$

2.3.3> Probabilistic Generative Laws

⇒ Probabilistic law characterizing the evolution of state might be given by a probability distribution of the following form:

$$p(x_t \mid x_{0:t-1}, z_{1:t-1}, u_{1:t}) = p(x_t \mid x_{t-1}, u_t)$$

⇓

{If state x is Complete}

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⇒ If x_t is complete, we have an important conditional independence.

$$p(z_t \mid x_{0:t}, z_{1:t-1}, u_{1:t}) = p(z_t \mid x_t)$$

$p(x_t \mid x_{t-1}, u_t)$ ⇒ State transition probability

→ It specifies how environment state evolves over time as a function of robot controls u_t.

$p(z_t \mid x_t)$ ⇒ Measurement probability

⇒ The state transition probability and the measurement probability together describe the dynamical stochastic of the robot and its environment.

⇒ This generative model is known as hidden Markov model (HMM) or dynamic Bayes network (DBN).

2.3.4> Belief Distributions

⇒ A belief reflects the robot's internal knowledge about the state of the environment.

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⇒ We will denote belief over a state variable x_t by $bel(x_t)$, which is an abbreviation for the posterior

$$bel(x_t) = p(x_t \mid z_{1:t}, u_{1:t})$$

⇒ Posterior before incorporating z_t just after executing the control u_t

$$\overline{bel}(x_t) = p(x_t \mid z_{1:t-1}, u_{1:t})$$

This probability distribution is often referred to as prediction

⇒ Calculating $bel(x_t)$ from $\overline{bel}(x_t)$ is called correction or measurement update

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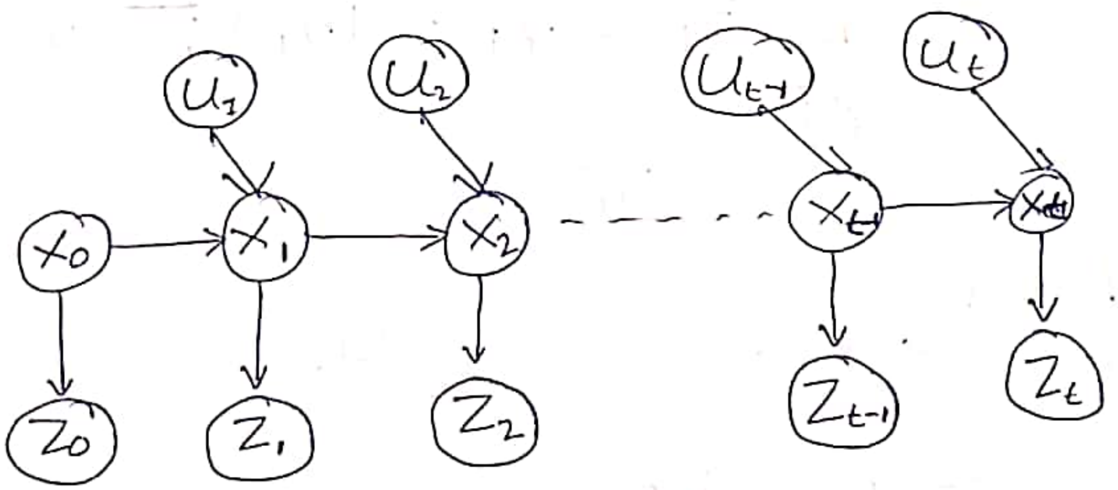

2.4) Bayes Filters

2.4.1) The Bayes Filter Algorithm

⇒ The most general algorithm for calculating beliefs is given by the Bayes filter algorithm.

↳ Calculates belief distribution bel from measurement and control data.

Algorithm Bayes filter $(bel(x_{t-1}), u_t, z_t)$:

for all x_t do

$\overline{bel}(x_t) = \int p(x_t \mid u_t, x_{t-1})\, bel(x_{t-1})\, dx_{t-1}$

$bel(x_t) = \eta\, p(z_t \mid x_t)\, \overline{bel}(x_t)$

endfor

return $bel(x_t)$

→ Measurement update.

→ Prediction.

⇒ To compute the posterior belief recursively, the algorithm requires an initial belief $bel(x_0)$ at time $t=0$ as boundary condition.

→ Sensor reliability data and actuator reliability data is needed to perform bayes filter algorithm.

2.4.4> The Markov Assumption

"Markov assumption postulates that past and future data are independent if one knows the current state x_t"

2.5> Representation and Computation

⇒ When choosing an approximation, one has to trade off a range of properties:

1) Computational Efficiency

2) Accuracy of the approximation

3) Ease of implementation.