3

Z-plane Analysis of Discrete time Control Systems

⇒ The z-transform method is particularly useful for analyzing and designing single input single output linear time invariant discrete time control systems.

⇒ The main advantage of the z transform method is that it enables the engineer to apply conventional continuous-time design methods to discrete-time systems that may be partly discrete time and partly continuous time.

3.2) Impulse Sampling and Data Hold

★ Impulse Sampling

⇒ We shall consider a fictious sampler commonly called an impulse sampler.

⇒ The output of this sampler is considered to be a train of impulse that begin with $t=0$, with the sampling period equal to T and the strength of each impulse equal to the sampled value of the continuous-time signal at the corresponding sampling instant.

⇒ So impulse-sampled output is a sequence of impulses, with the strength of each impulse equal to the magnitude of $x(t)$ at the corresponding instant of time.

Sampled Signal

$$\delta(t) = \begin{cases} 1 & t=0 \\ 0 & t \neq 0 \end{cases}$$

$$x^*(t) = \sum_{K=0}^{\infty} x(KT)\,\delta(t-KT) \quad \text{——} \; ①$$

⇒ We shall define a train of unit impulse as :-

$$\delta_T(t) = \sum_{K=0}^{\infty} \delta(t-KT)$$

⇒ The Sampler output is equal to the product of the Continuous-time input $x(t)$ and the train of unit impulses $\delta_T(t)$.

⇒ Next Consider the Laplace transform of Eq ①.

$$X^*(s) = \mathcal{L}[x^*(t)] = x(0)\,\mathcal{L}[\delta(t)] + x(T)\,\mathcal{L}[\delta(t-T)] + x(2T)\,\mathcal{L}[\delta(t-2T)] + \cdots$$

$$= x(0) + x(T)e^{-TS} + x(2T)e^{-2TS} + \cdots$$

$$X^*(s) = \sum_{K=0}^{\infty} x(KT)e^{-KTS}$$

If we define e^{TS} as z

then above equation become

$$X^*(s)\Big|_{s=\frac{\ln z}{T}} = \sum_{K=0}^{\infty} x(KT)z^{-K}$$

⇒ It is the z transform of the Sequence $x(0), x(T), x(2T)\cdots$ generated from $x(t)$ at $t=KT$ where $K=0,1,2\cdots$

⇒ Hence we may write

$$X^*(s)\Big|_{s=\frac{\ln z}{T}} = X(z)$$

Summary: If the Continuous-time Signal $x(t)$ is impulse sampled in a periodic manner, mathematically the sampled signal may be represented by :-

$$x^*(t) = \sum_{K=0}^{\infty} x(t)\,\delta(t-KT)$$

⇒ The Laplace transform of the impulse-sampled Signal $x^*(t)$ has been shown to be the same as the z transform of signal $x(t)$ if e^{Ts} is defined as z.

★ Data-Hold Circuits

⇒ Data-hold is a process of generating a continuous-time Signal $h(t)$ from a discrete-time Sequence $x(KT)$.

~~⇒ A hold circuit Converts the Sampled signal into a Continuous-time signal, which approximately~~

⇒ The Signal $h(t)$ during the time interval $KT \leq t \leq (K+1)T$ may be approximated by a polynomial in τ as follows :-

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$$h(KT+\tau) = a_n\tau^n + a_{n-1}\tau^{n-1} + \cdots + a_1\tau + a_0$$

$\{$ where $0 \leq \tau \leq T$

$$\Rightarrow h(KT+\tau) = a_n\tau^n + a_{n-1}\tau^{n-1} + \cdots + a_1\tau + x(KT)$$

$\{$ as $h(KT) = x(KT)$ $\}$

⇒ If the data hold circuit is an n^{th} order polynomial extrapolator, it is called an n^{th} order hold.

$\{$ The n^{th} order hold uses the past $n+1$ discrete data $x((K-n)T), x((K-n+1)T) \cdots x(KT)$ the generate a signal $h(KT+\tau)$. $\}$

⇒ It will be seen later that the TF G_h of the Zero-order hold may be given by

$$G_h = \frac{1-e^{-TS}}{S}$$

Zero Order Hold

$$\boxed{h(KT+\tau) = x(KT) \quad \forall\ 0 \leq \tau < T}$$

Let $H'(t)$ be Heaviside Unit step function

$$H'(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

$$h(t) = x(0)\left[H'(t) - H'(t-T)\right] + x(T)\left[H'(t-T) - H'(t-2T)\right] + x(2T)\left[H'(t-2T) - H'(t-3T)\right] + \cdots$$

$$h(t) = \sum_{K=0}^{\infty} x(KT)\left[H'(t-KT) - H'(t-(K+1)T)\right]$$

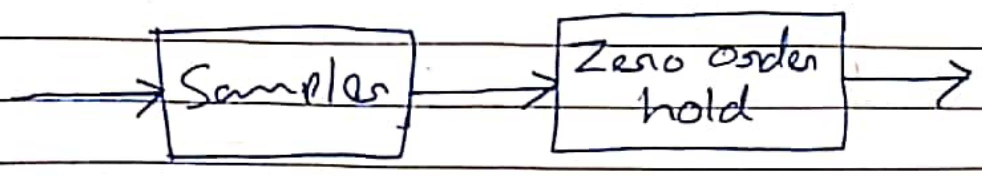
$$\mathcal{L}\left[h(t)\right] = H(s) = \sum_{K=0}^{\infty} x(KT)\left(\frac{e^{-KTs} - e^{-(K+1)Ts}}{s}\right)$$

$$\Rightarrow \frac{1-e^{-Ts}}{s} \sum_{K=0}^{\infty} x(KT)e^{-KTs}$$

$$X^*(s) = \sum_{K=0}^{\infty} x(KT)e^{-KTs}$$

So $\Rightarrow$ $\boxed{G_{ho}(s) = \frac{1-e^{-Ts}}{s}}$

{Transfer function of zero order hold}



First Order Hold

Although we do not use first order holds in control systems, it is worthwhile to see what the TF of first order holds may look like.

$$h(KT+\tau) = a_1\tau + x(KT)$$

$$\{0 \leq \tau \leq T \quad \& \quad K = 0, 1, 2 \ldots\}$$

$$h((K-1)T) = x((K-1)T)$$

$$\Rightarrow h((K-1)T) = -a_1 T + x(KT) = x((K-1)T)$$

$$a_1 = \frac{1}{T}\left(x(KT) - x((K-1)T)\right)$$

So $$\boxed{h(KT+\tau) = x(KT) + \frac{x(KT) - x((K-1)T)}{T}\tau}$$

$$h(t) = \left(x(0) + \frac{x(0) - x(-T)}{T}t\right)\left(H'(t) - H'(t-T)\right)$$

$$+ \left(x(1) + \frac{\{x(1) - x(0)}{T}t\right)\left(H'(t-T) - h'(t-2T)\right)$$

$$+ \ldots\ldots$$

$$\cancel{\mathcal{L}[h(t)] = H(s) = \left(\frac{x(0)}{s} + \frac{x(0) - x(-T)}{Ts^2}\right)e^{-0Ts} - \left(\frac{x(0)e^{-Ts}}{s} + \frac{(x(0)-x(-T))e^{-Ts}}{Ts^2}\right)}$$

$$\cancel{- \quad + \ldots\ldots}$$

$$H(s) = \frac{1-e^{-TS}}{TS^2}\left(x(0)-x(-T)\right) + \frac{1-e^{-TS}}{S}x(0)$$

$$+ \text{———}$$

$$H(s) = \sum_{K=0}^{\infty} \frac{1-e^{-TS}}{TS^2}\left(x(K)-x(K-1)\right) + \frac{1-e^{-TS}}{S}x(K)$$

$$H(s) = \sum_{K=0}^{\infty}\left[\frac{x(KT)}{S} + \frac{x(KT)-x((K-1)T)}{TS^2}\right]e^{-KTS}$$

$$- \left[\frac{x(KT)}{S} + \frac{x(KT)-x((K-1)T)}{TS^2}\right]e^{-(K-1)TS}$$

$$\Rightarrow \sum_{K=0}^{\infty}\left[\frac{e^{-KTS} - e^{-(K+1)TS}}{S} x(KT)\right.$$

$$\left. + \frac{e^{-KTS} - e^{-(K+1)TS}}{TS^2}\left(x(KT)-x((K-1)T)\right)\right]$$

$$\Rightarrow \cancel{\sum_{K=0}^{\infty}} \frac{1-e^{-TS}}{S}\sum_{K=0}^{\infty} x(KT)e^{-KTS}$$

$$+ \frac{1-e^{-TS}}{TS^2}\left(\sum_{K=0}^{\infty} x(KT)e^{-KTS} - \sum_{K=0}^{\infty} x((K-1)T)e^{-KTS}\right)$$

$$\Rightarrow \frac{1-e^{-TS}}{S}X(Z) + \frac{1-e^{-TS}}{TS^2}\left(X(Z) - \frac{1}{Z}X(Z)\right)$$

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$$\cancel{\left\{ z = e^{TS} \right\}}$$

$$\cancel{H(s) = X(z) * \left[\frac{1-e^{-TS}}{s} + \frac{1-e^{-TS}}{TS^2}\left(1-e^{-TS}\right) \right]}$$

$$\cancel{\left(\frac{1-e^{-TS}}{s} \right)^2 \left[\frac{s}{1-e^{-TS}} + \frac{1}{T} \right]}$$

$$\Rightarrow h(t) = \sum_{K=0}^{\infty} \left\{ \left(x(KT) + \left(\frac{x(KT) - x((K-1)T)}{T} \right) \gamma \right) \left(H(t-KT) - H(t-(K+1)T) \right) \right\}$$

A, B

$$\Rightarrow \mathcal{L}[h(t)] = \sum_{K=0}^{\infty} \left\{ x(KT)\, \mathcal{L}[H(t-KT)] + \frac{x(KT) - x((K-1)T)}{T} \mathcal{L}[t H(t-KT)] \right.$$

$$\left. - \left(x(KT)\, \mathcal{L}[H(t-(K+1)T)] + \frac{x(KT) - x((K-1)\gamma)}{T} \mathcal{L}[t H(t-(K+1)T)] \right) \right\}$$

$$\Rightarrow H(s) = \sum_{K=0}^{\infty} \left[\left(\frac{A e^{-KTS}}{s} + \frac{B}{s^2} e^{-KTS} \right) - \left(\frac{A}{s} e^{-(K+1)TS} + \frac{B}{s^2} e^{-(K+1)TS} \right) \right]$$

$$= \frac{1-e^{-TS}}{s} \sum A e^{-KTS} + \frac{1-e^{-TS}}{s^2} \sum B e^{-KTS}$$

$$\Rightarrow \frac{1-e^{-TS}}{s} \sum_{K=0}^{\infty} x(KT) e^{-KTS} + \frac{1-e^{-TS}}{Ts^2} \left(\sum x(KT) e^{-KTS} - \sum x((K-1)T) e^{-KTS} \right)$$

$$X^*(s) \left\{ \frac{1-e^{-TS}}{s} + \frac{1-e^{-TS}}{Ts^2} \left(1 - e^{-TS}\right) \right\}$$

$$G_{H1}(s) = \left(\frac{1-e^{-Ts}}{s}\right)^2\left(\frac{Ts+1}{T}\right)$$

$$\boxed{G_{h1}(s) = \frac{1-e^{-Ts}}{s^2}\left(\frac{Ts+1-e^{-Ts}}{T}\right)}$$??

3.3) Obtaining the z transform by the Convolution Integral method

$$x(t) \xrightarrow{\delta_T} x^*(t) \quad \text{Impulse Sampler}$$

$$x^*(t) = \sum_{K=0}^{\infty} x(t)\,\delta(t-KT) = x(t)\sum_{K=0}^{\infty}\delta(t-KT) \quad —(1)$$

$$\mathcal{L}\left\{\sum_{K=0}^{\infty}\delta(t-KT)\right\} = \sum_{K=0}^{\infty} e^{-KTs} = \frac{1}{1-e^{-Ts}} \quad —(2)$$

$$X^*(s) = \mathcal{L}[x^*(t)] = \mathcal{L}\left[x(t)\sum_{K=0}^{\infty}\delta(t-KT)\right] \quad —(3)$$

$$\mathcal{L}[f(t)g(t)] = \int_0^{\infty} f(t)g(t)e^{-st}\,dt$$

$$= \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(p)G(s-p)\,dp$$

Where $F(s)$ & $G(s)$ are Laplace transform of $f(t)$ & $g(t)$ respectively

Let $f(t) = x(t)$

& $g(t) = \sum_{K=0}^{\infty} \delta(t-KT)$

$$\boxed{X^*(s) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} X(P)\,\frac{1}{1-e^{-T(s-P)}}\,dP}$$

Where the integration is along the line $c-j\infty$ to $c+j\infty$ and this line is parallel to the imaginary axis in the P plane and Separates the poles of $X(P)$ from those of $\frac{1}{1-e^{-T(s-P)}}$.

⇒ Such an integral can be evaluated in terms of Residues by forming a closed contour consisting of a line from $c-j\infty$ to $c+j\infty$ and a Semicircle of infinite radius in the left or right half plane, Provided that the integral along the added Semicircle is a constant.

$$X^*(s) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} X(P)\frac{1}{1-e^{-T(s-P)}}\,dP$$

$$\Rightarrow \frac{1}{2\pi j}\oint \frac{X(P)}{1-e^{-T(s-P)}}\,dP - \frac{1}{2\pi j}\int_{\Gamma} \frac{X(P)}{1-e^{-T(s-P)}}\,dP$$

{where Γ is a Semicircle of infinite radius in the left or right half of P plane}

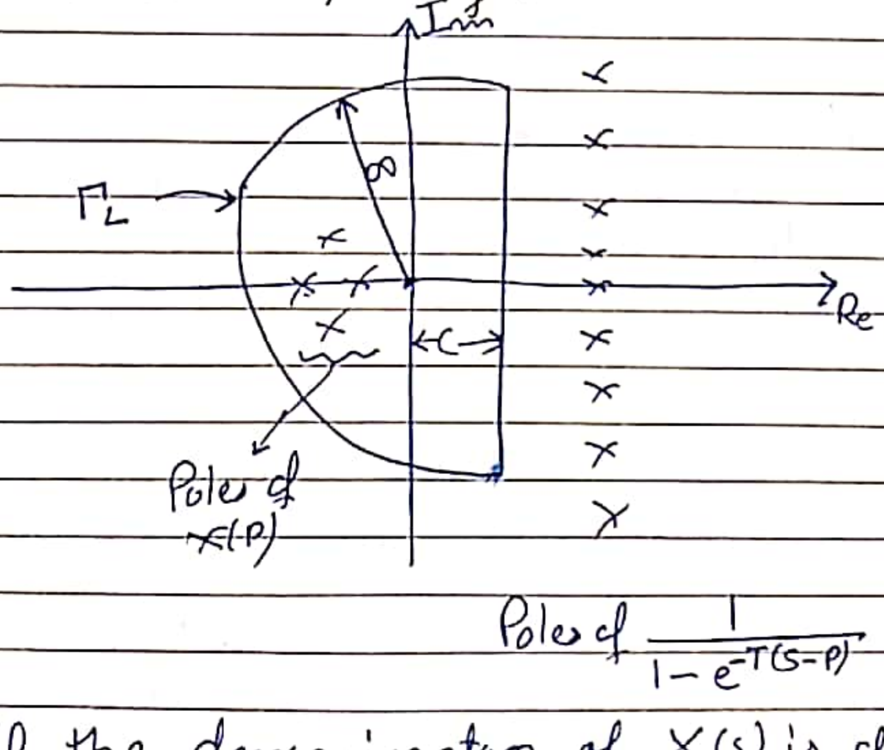
⇒ There are two way to evaluate this integral: One using an Infinite Semicircle in the left-half plane and the other an infinite Semicircle in the right half plane.

⇒ In our analysis here, we assume that the poles of $X(s)$ lie in the left half-plane and $X(s)$ can be expressed as a ratio of polynomials in s

$$X(s) = \frac{Q(s)}{P(s)} \quad \left\{ P(s) \;\&\; Q(s) \text{ are polynomial in } s \right\}$$

⇒ We assume $P(s)$ is of higher degree in s than $Q(s)$. $\boxed{\lim_{s\to\infty} X(s) = 0}$

Evaluation of Convolution Integral in the Left half plane



⇒ If the denominator of $X(s)$ is of higher degree in s than the numerator, the integral along Γ_L vanishes.

$$X^*(s) = \frac{1}{2\pi j} \oint \frac{X(P)}{1-e^{-T(s-P)}}\, dP$$

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⇒ This integral is equal to the sum of the residues of $X(P)$ in the closed contour.

$$X^*(s) = \sum \left[\text{residue of } \frac{X(P)}{1-e^{-T(s-P)}} \text{ at poles of } X(P) \right]$$

⇒ By substituting z for e^{TS} and changing complex variable notation from P to S, we obtain.

$$X^*(z) = \sum \left[\text{residue of } \frac{X(s)z}{z-e^{TS}} \text{ at poles of } X(s) \right]$$

⇒ Assuming that $X(s)$ has poles $S_1, S_2 \dots S_m$.

→ If a pole at $S=S_i$ is a simple pole, then the corresponding residue K_i is

$$K_i = \lim_{S \to S_i} \left[(S-S_i) \frac{X(s)z}{z-e^{TS}} \right]$$

→ If a pole at $S=S_i$ is a multiple pole of order n_i, then the residue K_i is

$$K_i = \frac{1}{(n_i-1)!} \lim_{S \to S_i} \frac{d^{n_i-1}}{ds^{n_i-1}} \left[(S-S_i)^{n_i} \frac{X(s)z}{z-e^{TS}} \right]$$

Example 3-1: $X(s) = \frac{1}{s^2(s+1)}$

Obtain $X(z)$ by use of the Convolution integral in the left half plane.

$\Rightarrow$ $X(s)$ has double pole at $s=0$ & simple pole at $s=-1$

$$X(z) = \sum \left[\text{residue of } \frac{X(s)}{z-e^{Ts}} \text{ at pole of } X(s) \right]$$

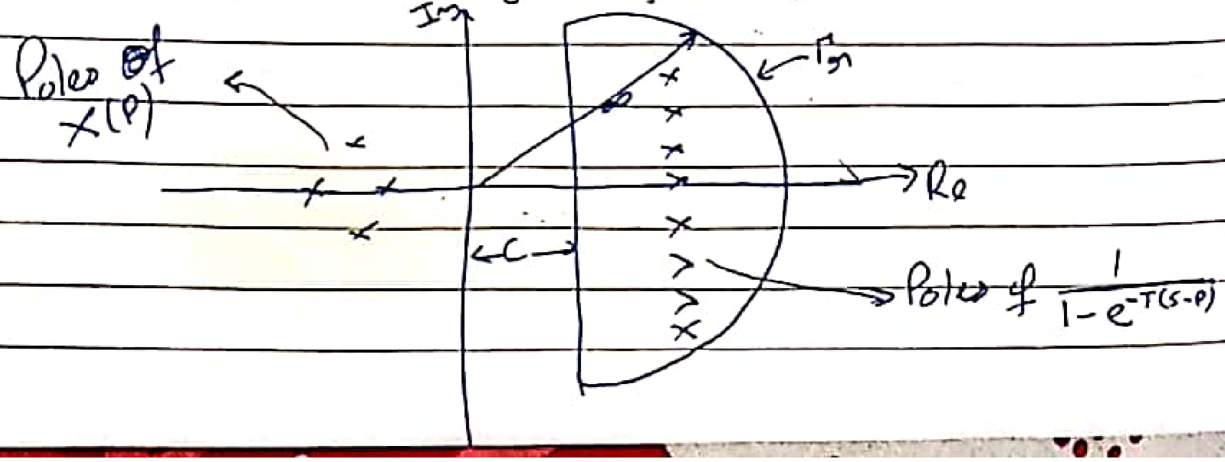
$$K_0 = \frac{1}{(2-1)!} \lim_{s\to 0} \frac{d}{ds} \left[\frac{s^2}{s^2(s+1)} \frac{z}{z-e^{Ts}} \right]$$

$$= \lim_{s\to 0} \frac{-z\left[z-e^{Ts} + (s+1)(-T)e^{Ts}\right]}{(s+1)^2 (z-e^{Ts})^2} = \frac{-z(z-1-T)}{(z-1)^2}$$

$$K_{-1} = \lim_{s\to -1} \left[\cancel{(s+1)} \frac{1}{s^2 \cancel{(s+1)}} \frac{z}{z-e^{Ts}} \right] = \frac{z}{z-e^{-T}}$$

$$X(z) = \frac{z}{z-e^{-T}} - \frac{z(z-1-T)}{(z-1)^2}$$

Evaluation of Convolution Integral in the Right Half-Plane



⇒ The closed contour encloses all poles of $\frac{1}{1-e^{-T(s-p)}}$, but it does not enclose any poles of $X(p)$.

$$X^*(s) = \frac{1}{2\pi j}\oint \frac{X(p)}{1-e^{-T(s-p)}}dp - \frac{1}{2\pi j}\int_{\Gamma_1} \frac{X(p)}{1-e^{-T(s-p)}}dp$$

⇒ In evaluating the integrals on the right-half side of above equation we need to consider two cases separately:-

- One case where denominator of $X(s)$ is two or more degree higher in s than the numerator;
- And another case where the denominator of $X(s)$ is only one degree higher in s than the numerator.

Case 1: $X(s)$ has a denominator two or more degree higher in s than the numerator.

$$\lim_{s\to\infty} sX(s) = x(0+) = 0$$

⇒ Then the integral along Γ_1 is zero. Thus in present case

$$\frac{1}{2\pi j}\int_{\Gamma_1} \frac{X(p)}{1-e^{-T(s-p)}}dP = 0$$

Thus, ~~XXXX~~

$$X^*(s) = \frac{1}{T}\sum_{K=-\infty}^{\infty} X(s + j\omega_s K)$$

$$\Rightarrow X(z) = \frac{1}{T}\sum_{K=-\infty}^{\infty} X(s + j\omega_s k)\Big|_{s=\frac{\ln z}{T}}$$

⇒ It is very tedious to obtain z transform expression of commonly encountered functions by this method.

Case 2: $X(s)$ has a denominator one degree higher in s than the numerator.

⇒ For this case $\lim_{s\to\infty} sX(s) = x(0+) \neq 0 < \infty$ and the integral along Γ_R is not zero.

⇒ It can be shown that the contribution of the integral along Γ_R is $-\frac{1}{2}x(0+)$.

$$\frac{1}{2\pi j}\int_{\Gamma_R} \frac{x(p)}{1 - e^{-T(s-p)}}\,dp = -\frac{1}{2}x(0+)$$

So $X^*(s) = \frac{1}{T}\sum_{K=-\infty}^{\infty} X(s + j\omega_s k) + \frac{1}{2}x(0+)$

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Example 3.2: So that $X^*(s)$ is periodic with period $2\pi/\omega_s$.

$$X^*(s) = \frac{1}{T}\sum_{h=-\infty}^{\infty} X(s+j\omega_s h) + \frac{1}{2}x(0+)$$

$$\Rightarrow X^*(s+j\omega_s K) = \frac{1}{T}\sum_{h=-\infty}^{\infty} X(s+j\omega_s K+j\omega_s h) + \frac{1}{2}x(0+)$$

Let $K+h=m$

$$\Rightarrow X^*(s+j\omega_s K) = \frac{1}{T}\sum_{m=-\infty}^{\infty} X(s+j\omega_s m) + \frac{1}{2}x(0+)$$

$$= X^*(s)$$

Therefore,

$$X^*(s) = X^*(s \pm j\omega K) \quad K = 0, 1, 2 \ldots$$

⇒ Thus $X^*(s)$ is periodic, with period $2\pi/\omega_s$. This means that, if a function $X(s)$ has a pole at $s=s_1$ in the s plane, then $X^*(s)$ has poles at $s=s_1 \pm j\omega_s K$ $(K = 0, 1, 2 \ldots)$

★ Obtaining Z Transforms of Function Involving the term $(1-e^{-Ts})/s$.

⇒ Suppose the transfer function $G(s)$ follows the zero-order hold.

$$X(s) = \frac{1-e^{-Ts}}{s} G(s)$$

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$$\Rightarrow X(s) = (1-e^{-Ts})\frac{G(s)}{s} = (1-e^{-Ts})\,G_1(s)$$

$$\left\{ \text{where } G_1(s) = \frac{G(s)}{s} \right\}$$

Consider the function $\cancel{X}_1(s) = e^{-Ts}G_1(s)$

$$\Rightarrow x_1(t) = \int_0^t g_0(t-\tau)\,g_1(\tau)\,d\tau$$

$$\left\{ \begin{array}{l} g_0(t) = \mathcal{L}^{-1}[e^{-Ts}] = \delta(t-T) \\ g_1(t) = \mathcal{L}^{-1}[G_1(s)] \end{array} \right.$$

$$x_1(t) = \int_0^t \delta(t-T-\tau)\,g_1(\tau)\,d\tau$$

$$= g_1(t-T)$$

$$\Rightarrow Z[g_1(t)] = G_1(z)$$

$$Z[x_1(t)] = Z[g_1(t-T)] = z^{-1}G_1(z)$$

$$\Rightarrow X(z) = Z[G_1(s) - e^{-Ts}G_1(s)]$$

$$= Z[g_1(t)] - Z[x_1(t)]$$

$$= G_1(z) - z^{-1}G_1(z)$$

$$= (1-z^{-1})\,G_1(z)$$

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$$\boxed{X(z) = Z[X(s)] = (1 - z^{-1})\, Z\left[\frac{G(s)}{s}\right]}$$

Example 3.3 $X(s) = \frac{1-e^{-Ts}}{s} \frac{1}{s+1}$

$$X(z) = Z\left[\frac{1-e^{-Ts}}{s} \times \frac{1}{s+1}\right]$$

$$= (1 - z^{-1})\, Z\left[\frac{1}{s(s+1)}\right]$$

$$= (1 - z^{-1})\, Z\left[\frac{1}{s} - \frac{1}{s+1}\right]$$

$$\frac{1}{s} \rightarrow \frac{1}{1-z^{-1}} \qquad \frac{1}{s+1} \rightarrow \frac{1}{1-e^{-T}z^{-1}}$$

$$\Rightarrow \boxed{X(z) = \frac{(1-e^{-T})z^{-1}}{1-e^{-T}z^{-1}}}$$

3.4) Reconstructing Original Signal From Sampled Signals

Sampling Theorem: If ω_s, defined as $2\pi/T$, where T is the sampling period, is greater than $2\omega_1$

$$\omega_s > 2\omega_1$$

where ω_1 is the highest-frequency component present in the continuous-time signal $x(t)$ then the signal $x(t)$ can be reconstructed completly from the sampled signal $x^*(t)$.

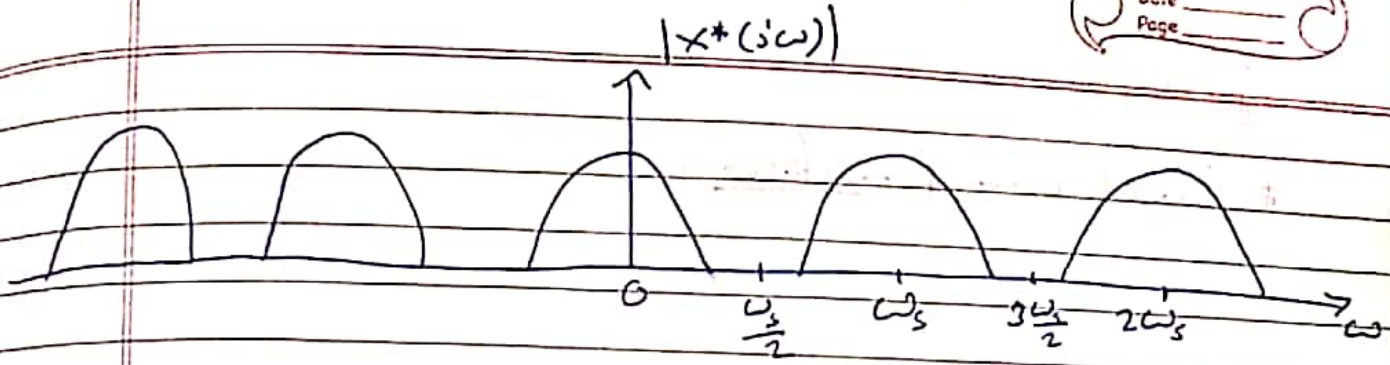
⇒ To show the validity of sampling theorem, we need to find the frequency spectrum of the sampled signal $x^*(t)$

⇒ To obtain the frequency spectrum, we substitute $j\omega$ for s in $X^*(s)$.

$$X^*(j\omega) = \frac{1}{T}\sum_{K=-\infty}^{\infty} X(j\omega + j\omega_s K)$$

⇒ Thus, the process of impulse modulation of the continuous-time signal produces a series of side bands since $X^*(s)$ is periodic with period $2\pi/\omega_s$.

→ If a function $X(s)$ has a pole at $s=s_1$ then $X^*(s)$ has poles at $s = s_1 \pm j\omega_s K$.



⇒ Each plot of $|X^*(j\omega)|$ versus ω consists of $|X(j\omega)|/T$ repeated every $\omega_s = 2\pi/T$ rad/sec.

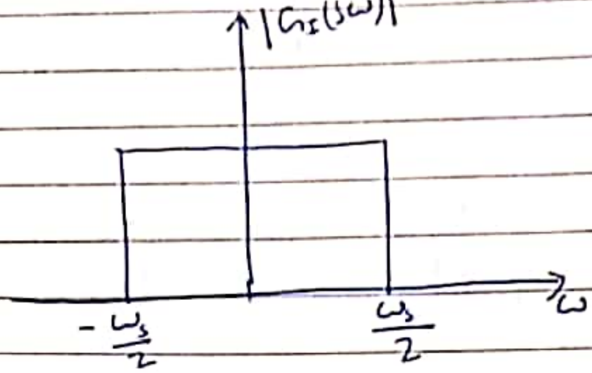
⇒ In the frequency spectrum of $|X^*(j\omega)|$ the component $|X(j\omega)|/T$ is called the primary component, and the other components, $|X(j\omega \pm \omega_s k)|/T$ are called complementary component.

→ If $\omega_s > 2\omega_1$ no two components of $X^*(j\omega)$ will overlap, and the sampled frequency spectrum will be repeated every ω_s rad/sec.

→ If $\omega_s < 2\omega_1$, the original signal of $|X(j\omega)|$ no longer appears in the plot of $|X^*(j\omega)|$ versus ω because of the superposition of the spectra.

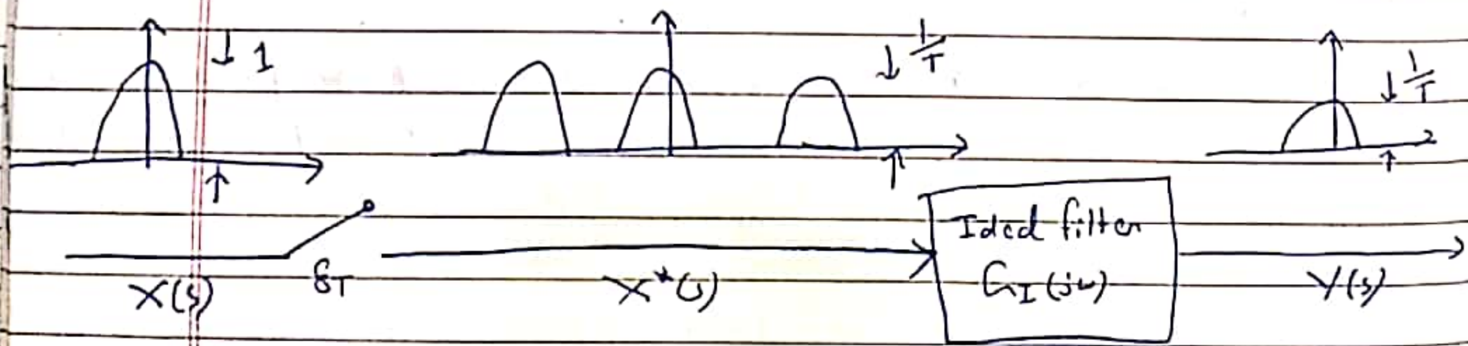
→ Therefore, we see that the continuous-time signal $x(t)$ can be reconstructed from the impulse-sampled signal $x^*(t)$ by filtering if and only if $\omega_s > 2\omega_1$.

Ideal Low-pass filter



⇒ The magnitude of the ideal filter is unity over the frequency range $-\frac{1}{2}\omega_s \le \omega \le \frac{1}{2}\omega_s$ and is zero outside this frequency range.

⇒ Such an ideal filter reconstructs the continuous-time signals before and after ideal filtering.

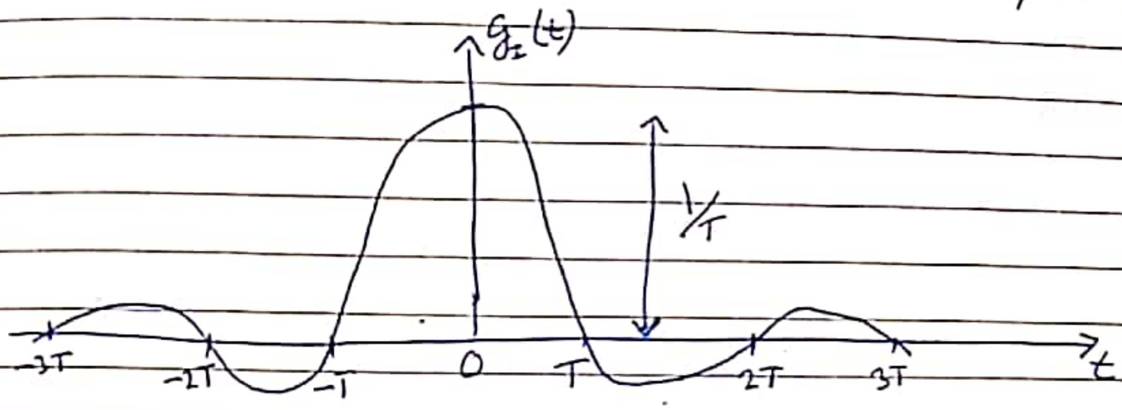


Ideal Low pass filter is not physically Realizable

$$G_I(j\omega) = \begin{cases} 1 & -\frac{1}{2}\omega_s \le \omega \le \frac{1}{2}\omega_s \\ 0 & \text{elsewhere} \end{cases}$$

$$g_I(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G_I(j\omega)\, e^{j\omega t}\, d\omega$$

$$g_I(t) = \frac{1}{2\pi}\int_{-\omega_s/2}^{\omega_s/2} e^{j\omega t}\,d\omega = \frac{1}{T}\frac{\sin(\omega_s t/2)}{(\omega_s t/2)}$$



⇒ Notice that the response extends from $t = -\infty$ to $t = \infty$. This implies that there is a response $\forall\, t < 0$ to a unit impulse at $t = 0$. This can not be true in real physical world. Hence, such an ideal filter is physically unrealizable.

⇒ So it is no possible, in practice, to exactly reconstruct a continuous time signal from the sampled signal, no matter what sampling frequency is chosen.

Frequency-Response Characteristics of the Zero-Order Hold

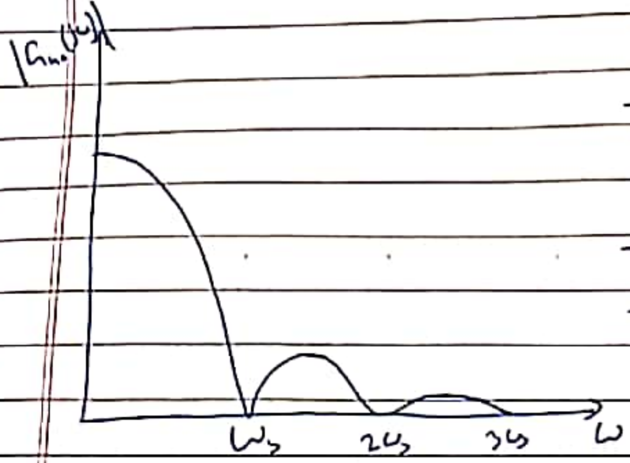
$$G_{ho}(s) = \frac{1-e^{-Ts}}{s}$$

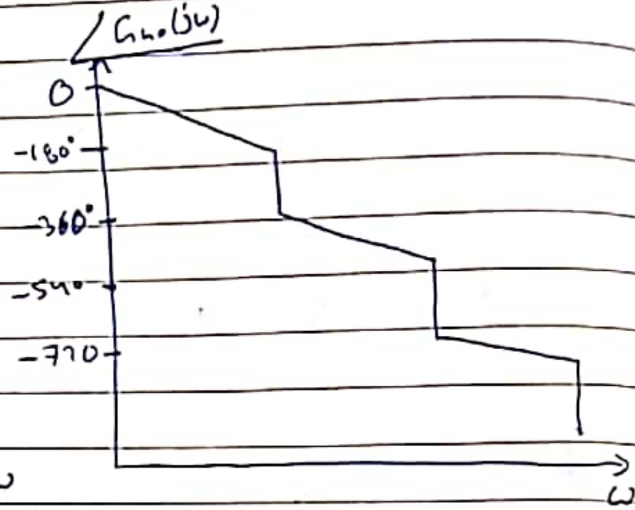
$$G_{ho}(j\omega) = \frac{1-e^{-Tj\omega}}{j\omega} = T\frac{\sin(\omega T/2)}{\omega T/2}\, e^{-\frac{Tj\omega}{2}}$$

$$\left|G_{ho}(j\omega)\right| = T\left|\frac{\sin(\omega T/2)}{\omega T/2}\right|$$

$$\angle G_{ho}(j\omega) = \angle \sin\frac{\omega T}{2} - \frac{\omega T}{2}$$

$\angle \sin\frac{\omega T}{2} \rightarrow 0°$ or $\pm 180°$





⇒ The frequency spectrum of the output of the zero order hold includes complementary components, since the magnitude characteristics shows that the magnitude of $G_{ho}(j\omega)$ is not zero for $|\omega| > \frac{1}{2}\omega_s$, except at points where $\omega = \pm\omega_s$, $\omega = \pm 2\omega_s$ - - .

⇒ In the phase curve there are phase discontinuities of $\pm 180°$ at frequency points that are multiples of ω_s.

Folding: The phenomenon of the overlap in the frequency spectrum is known as folding.

⇒ The frequency $\frac{1}{2}\omega_s$ is called folding frequency or Nyquist frequency ω_N.

Aliasing: The phenomenon that the frequency component $n\omega_s - \omega_2$, where n is an integer shows up at frequency ω_2 when the signal $x(t)$ is sampled is called ~~at~~ aliasing.

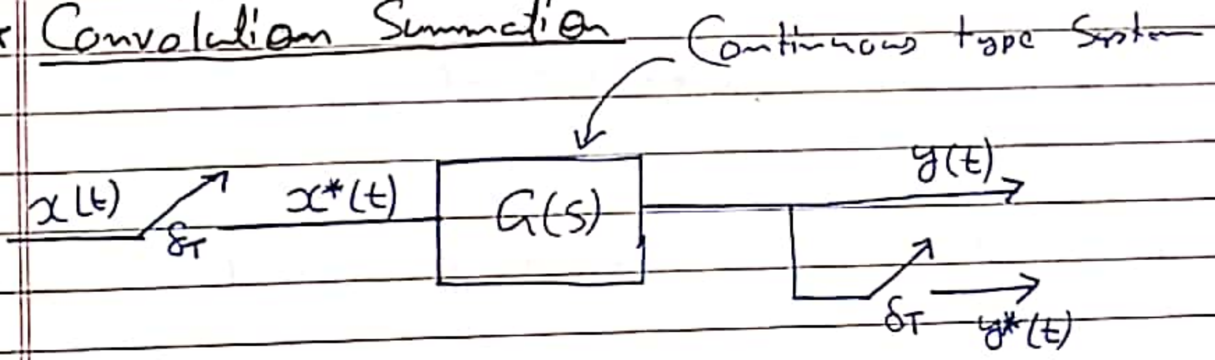
↳ The frequency $n\omega_s \pm \omega_2$ is called an alias of ω_2

3.5) The Pulse Transfer function

⇒ The transfer function for the continuous-time system relates the Laplace transform of the continuous-time output to that of the continuous time input.

⇒ Pulse transfer function relates the Z transform of the output at the sampling instants to that of the sampled ~~output~~ input.

★ **Convolution Summation**



Let $x(t) = 0 \ \forall \ t < 0$

$$Z[y(t)] = Y(z) = \sum_{k=0}^{\infty} y(kT) z^{-k}$$

⇒ For a continuous-time system, it is a well-known fact that the output $y(t)$ of the system is related to the input $x(t)$ by the convolution integral

$$y(t) = \int_0^t g(t-\tau)\,x(\tau)\,d\tau = \int_0^t x(t-\tau)\,g(\tau)\,d\tau$$

$\left\{ g(t) = \right.$ Impulse response function of the system or weighting function of the system. $\left. \right\}$

⇒ For discrete-time system we have a convolution summation, which is similar to the convolution integral.

$$x^*(t) = \sum_{K=0}^{\infty} x(KT)\,\delta(t-KT)$$

$$y(t) = \sum_{h=0}^{K} g(t-hT)\,x(hT) \quad \{0 \leq t \leq KT\}$$

⇒ The value of the output $y(t)$ at the s

$$\boxed{y(KT) = \sum_{h=0}^{K} x(KT-hT)\,g(hT)}$$

⇒ This summation is called Convolution Summation.

⇒ Simplified notation of Convolution Summation.

$$\boxed{y(KT) = x(KT) * g(KT)}$$

⇒ Above equation can be taken from 0 to K without changing the value of the Summation.

$$y(KT) = \sum_{h=0}^{\infty} g(KT-hT)x(hT) = \sum_{h=0}^{\infty} x(KT-hT)g(hT)$$

→ If $G(s)$ is a ratio of polynomials in s and if the degree of the denominator polynomial exceeds the degree of the numerator polynomial only by 1 the output $y(t)$ is discontinuous.

→ If the degree of the denominator polynomial exceeds that of the numerator polynomial by 2 or more, however output is Continuous.

★ Pulse transfer function

$$y(KT) = \sum_{h=0}^{\infty} g(KT-hT)x(hT) \quad \forall K = 0,1,2\ldots$$

$$\left\{ \cancel{g(KT-hT) = 0 \ \forall \ h > K} \right\}$$

⇒ The Z transform of $y(KT)$ become

$$y(KT) = \sum_{h=0}^{\infty} g(KT-hT)\,x(hT)$$

$$Y(z) = \sum_{K=0}^{\infty} y(KT)\,z^{-K}$$

$$\Rightarrow \sum_{K=0}^{\infty}\sum_{h=0}^{\infty} g(KT-hT)\,x(hT)\,z^{-K}$$

$\{m = K-h\}$

$$\Rightarrow \sum_{m=0}^{\infty}\sum_{h=0}^{\infty} g(mT)\,x(hT)\,z^{-(m+h)}$$

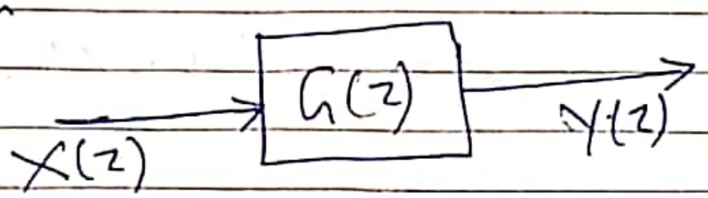
$$\Rightarrow \sum_{m=0}^{\infty} g(mT)\,z^{-m} \sum_{h=0}^{\infty} x(hT)\,z^{-h}$$

$$\Rightarrow G(z)\,X(z)$$

Z transform of g(t)   Z transform of x(t)

$$\Rightarrow \boxed{G(z) = \frac{Y(z)}{X(z)}}$$

Pulse transfer function



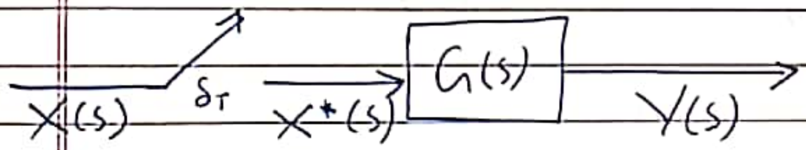
⇒ $G(z)$ is also the z transform of the system's response to the Kronecker delta input:

$$x(KT) = \delta_0(KT) \begin{cases} 1 & \forall K = 0 \\ 0 & \forall K \neq 0 \end{cases}$$

Z transform of the Kronecker delta input is

$$X(z) = \sum_{K=0}^{\infty} x(KT) z^{-K} = 1$$

★ <u>Starred Laplace transform of the Signal Involving Both Ordinary and Starred Laplace Transform</u>



$$Y(s) = G(s) X^*(s)$$

$$Y^*(s) = [G(s) X^*(s)]^* = G^*(s) X^*(s)$$

$$y(t) = \mathcal{L}^{-1}[G(s) X^*(s)]$$

$$= \int_0^t g(t-\tau) x^*(\tau) d\tau$$

$$= \int_0^t g(t-\tau) \sum_{K=0}^{\infty} x(\tau) \delta(\tau - KT) d\tau$$

$$\Rightarrow \sum_{K=0}^{\infty} \int_0^t g(t-\tau) x(\tau) \delta(\tau - KT)$$

$$\Rightarrow \sum_{K=0}^{\infty} g(t-KT)x(KT)$$

$$Y(z) = Z[y(t)]$$

$$= \sum_{n=0}^{\infty}\left[\sum_{K=0}^{\infty} g(nT-KT)x(KT)\right] z^{-n}$$

$\{m=n-K\}$ $$= \sum_{m=0}^{\infty} g(mT)z^{-m} \sum_{K=0}^{\infty} x(KT)z^{-K}$$

$$\Rightarrow G(z)X(z)$$

⇒ Since the Z transform can be understood as the Starred Laplace transform with e^{TS} replaced by Z, the Z transform may be considered as to be a shorthand notation for the Starred Laplace transform

⇒ It is noted that systems become periodic under Starred Laplace transform operations. Such periodic system are generally more complicated to analyze than the original non periodic ones but the former may be analysed without difficulty if carried out in the Z plane.

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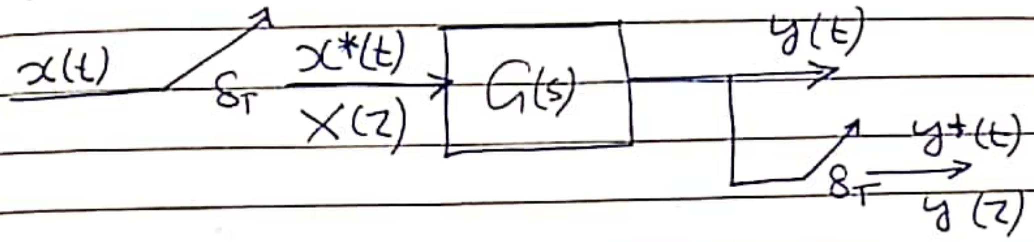
★ General procedure for obtaining pulse transfer function

Example 3-4



$$G(s) = \frac{1}{s+a}$$

$$Z\left[\frac{1}{s+a}\right] = \frac{1}{1-e^{-aT}z^{-1}} = G(z)$$

Example 3-5

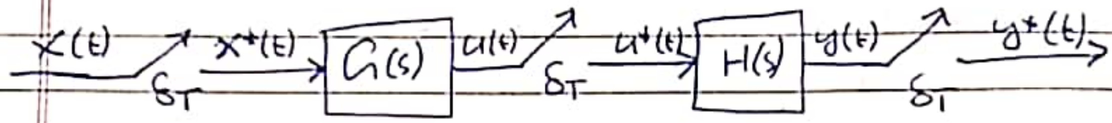
$$G(s) = \frac{1-e^{-Ts}}{s} \frac{1}{s(s+1)}$$

$$G(z) = Z\left[\frac{1-e^{-Ts}}{s} \frac{1}{s(s+1)}\right]$$

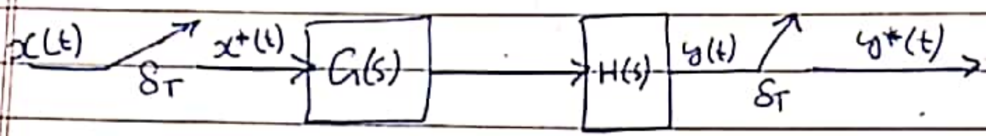
$$= (1-z^{-1})\, Z\left[\frac{1}{s^2(s+1)}\right]$$

$$G(z) = (1-z^{-1})\left[\frac{Tz^{-1}}{(1-z^{-1})^2} - \frac{1}{1-z^{-1}} + \frac{1}{1-e^{-T}z^{-1}}\right]$$

* Pulse transfer function of Cascaded Element



(a)



(b)

(a) $U(s) = G(s)\, X^*(s)$

$Y(s) = H(s)\, U^*(s)$

$\Rightarrow U^*(s) = G^*(s)\, X^*(s)$

$Y^*(s) = H^*(s)\, U^*(s)$

$\Rightarrow Y^*(s) = G^*(s)\, H^*(s)\, X^*(s)$

$\Rightarrow Y(z) = G(z)\, H(z)\, X(z)$

$$\frac{Y(z)}{X(z)} = G(z)\, H(z)$$

(b) $Y(s) = G(s)\, H(s)\, X^*(s)$

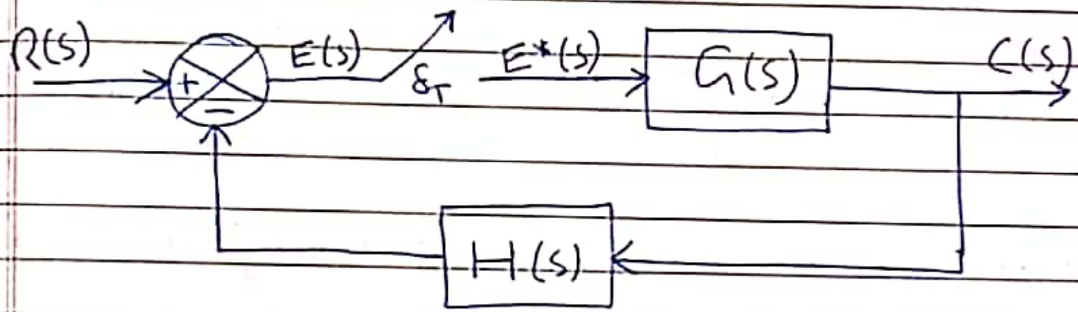
$Y^*(s) = \boxed{GH(s)}^*\, X^*(s)$

$\Rightarrow Y(z) = GH(z)\, X(z)$

$$GH(z) = Z[G(s)H(s)] = \frac{Y(z)}{X(z)}$$

$$Z[G(s)H(s)] \neq G(z)H(z)$$

★ Pulse transfer function of Closed-loop System



$$E(s) = R(s) - H(s)C(s)$$
$$C(s) = G(s)E^*(s)$$

$$\Rightarrow E(s) = R(s) - H(s)G(s)E^*(s)$$

$$\Rightarrow E^*(s) = R^*(s) - GH^*(s)E^*(s)$$

$$\Rightarrow E^*(s) = \frac{R^*(s)}{1 + GH^*(s)}$$

$$C^*(s) = G^*(s)E^*(s)$$

$$\Rightarrow C^*(s) = \frac{G^*(s)R^*(s)}{1 + GH^*(s)}$$

$$\Rightarrow C(z) = \frac{G(z)R(z)}{1 + GH(z)}$$

classmate
Date
Page

⇒ The pulse transfer function for the present closed-loop System is

$$\frac{C(z)}{R(z)} = \frac{G(z)}{1+GH(z)}$$

⇒ Some discrete-time closed-loop control Systems cannot be represented by $C(z)/R(z)$ because the input signal $R(s)$ cannot be Separated from the System dynamics.

★ Pulse transfer function of a Digital Controller

⇒ The pulse transfer function of a digital Controller may be obtained from the required input-output characteristics of the digital Controller.

⇒ Suppose the input to the digital Controller is $e(K)$ and the output $m(K)$.

⇒ In general, the output is given by the following type of difference equation:

$$m(K) + a_1 m(K-1) + a_2 m(K-2) + \cdots + a_n m(K-n)$$

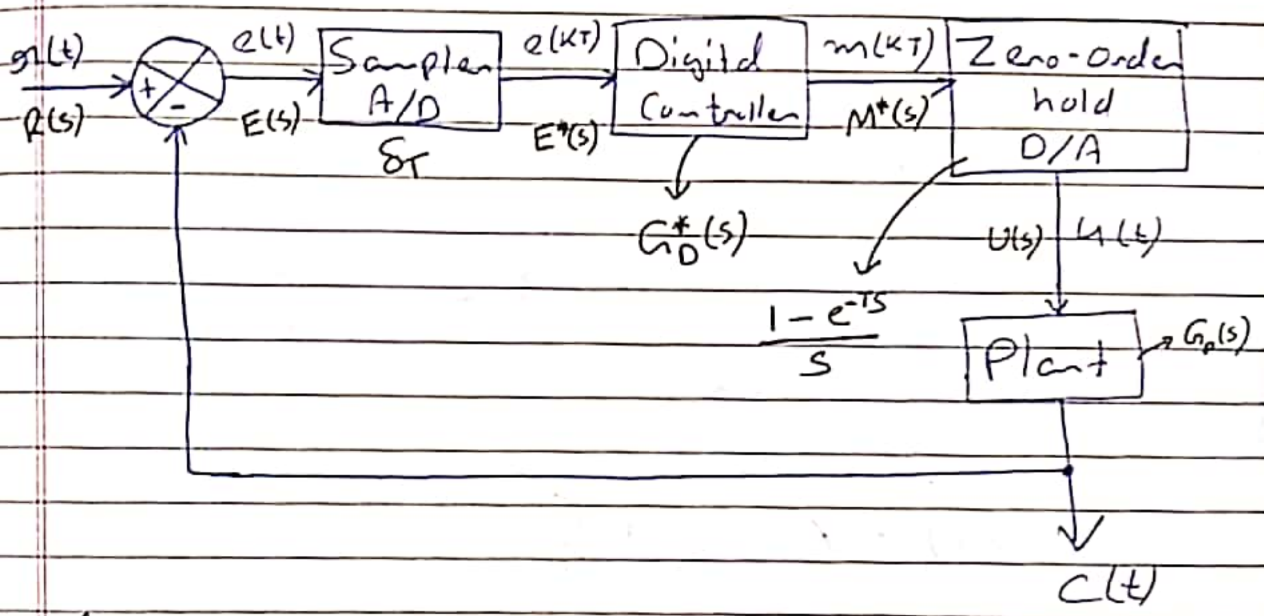
$$= b_0 e(K) + b_1 e(K-1) + \cdots + b_n e(K-n)$$

⇒ The Z transform of above equation is:

Pulse transfer function of digital controller

$$G_D(z) = \frac{M(z)}{E(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_n z^{-n}}{1 + a_1 z^{-1} + \cdots + a_n z^{-n}}$$

★ Closed-loop pulse transfer function of a Digital Control System



⇒ Let us define $\frac{1-e^{-TS}}{s} G_p(s) = G(s)$

$$\Rightarrow \quad C(s) = G(s)\, G_D^*(s)\, E^*(s)$$

$$\Rightarrow \quad C^*(s) = G^*(s)\, G_D^*(s)\, E^*(s)$$

$$\Rightarrow \quad C(z) = G(z)\, G_D(z)\, E(z)$$

$$\Rightarrow \quad E(z) = R(z) - C(z)$$

$$\Rightarrow \quad \boxed{\frac{C(z)}{R(z)} = \frac{G_D(z)\, G(z)}{1 + G_D(z)\, G(z)}}$$

Closed loop Pulse TF of digital control system

★ Pulse Transfer function of a Digital PID Controller

⇒ The PID Control action in analog Controller is given by:-

$$m(t) = K\left[e(t) + \frac{1}{T_i}\int_0^t e(t)\,dt + T_d\frac{de(t)}{dt}\right]$$

⇒ To obtain the pulse transfer function for the digital PID Controller, we may discretize above equation by approximating the integral term by the trapezoidal summation and the derivative term by a two point difference form. we obtain.

$$m(KT) = K\left[e(KT) + \frac{T}{T_i}\left\{\frac{e(0)+e(T)}{2} + \frac{e(T)+e(2T)}{2} + \cdots + \frac{e((K-1)T)+e(KT)}{2}\right\} + T_d\frac{e(KT)-e((K-1)T)}{T}\right]$$

$$\Rightarrow m(KT) = K\left[e(KT) + \frac{T}{T_i}\sum_{h=1}^{K}\frac{e((h-1)T)+e(hT)}{2} + \frac{T_d}{T}\left[e(KT)-e((K-1)T)\right]\right]$$

Let us define $\frac{e((h-1)T)+e(hT)}{2} = f(hT)$, $f(0)=0$

So, $\sum_{h=1}^{K} \frac{e((h-1)T)+e(hT)}{2} = \sum_{h=1}^{K} f(hT)$

Take the Z transfm of the equation:-

$$Z\left[\sum_{h=1}^{K} \frac{e((h-1)T)+e(hT)}{2}\right] = Z\left[\sum_{h=1}^{K} f(hT)\right]$$

$$= \frac{F(z)-f(0)}{1-z^{-1}}$$

$$\Rightarrow \frac{1}{1-z^{-1}} F(z)$$

$$F(z) = Z[f(hT)] = \frac{1+z^{-1}}{2} E(z)$$

$$\text{So } Z\left[\sum_{h=1}^{K}\left\{\frac{e((h-1)T)+e(hT)}{2}\right\}\right] = \frac{1+z^{-1}}{2(1-z^{-1})} E(z)$$

$$\Rightarrow M(z) = K\left[1 + \frac{T}{2T_i}\frac{1+z^{-1}}{1-z^{-1}} + \frac{T_d}{T}(1-z^{-1})\right]E(z)$$

$$\boxed{M(z) = \left[K_p + \frac{K_i}{1-z^{-1}} + K_D(1-z^{-1})\right]E(z)} \quad (1)$$

Pulse TF of digital PID Controller

$$\left\{ K_p = K - \frac{K_I}{2}, \quad K_I = \frac{KT}{T_i}, \quad K_D = \frac{KT_d}{T} \right\}$$

⇒ The pulse transfer function of the digital PID Controller given by Eq ① is commonly referred to as the positional form of the PID Control Scheme.

⇒ The other form commonly used in the digital PID Control scheme is referred to as the velocity form.

⇒ An advantage of the velocity-form PID Control scheme is to initialization is not necessary when the operation is switched from manual to automatic.

Note that, in digital Controllers, control laws can be implemented by software, and therefore the hardware restrictions of analog PID Controllers can be completely ignored.

3.6) Realization of Digital Controllers and Digital filters

⇒ In software realization we obtain computer programs for the digital computer involved.

⇒ In a hardware realization we build a special-purpose processor using such circuitry as digital adders, multipliers, and delay elements.

⇒ In the field of digital signal processing, a digital filter is a computational algorithm that converts an input sequence of numbers into an output sequence in such a way that the characteristics of the signal are changed in some prescribed fashion.

↳ In general term, a digital controller is a form of digital filter.

⇒ This section deals with the block diagram realization of digital filters using delay elements, adders, and multipliers.

↳ Such block diagram realizations can be used as a basis for a software or hardware design.

↳ In fact, once the block diagram realization is completed, the physical realization in hardware or software is straight forward.

⇒ The general form of the pulse transfer function between the output $Y(z)$ and input $X(z)$ is given by:-

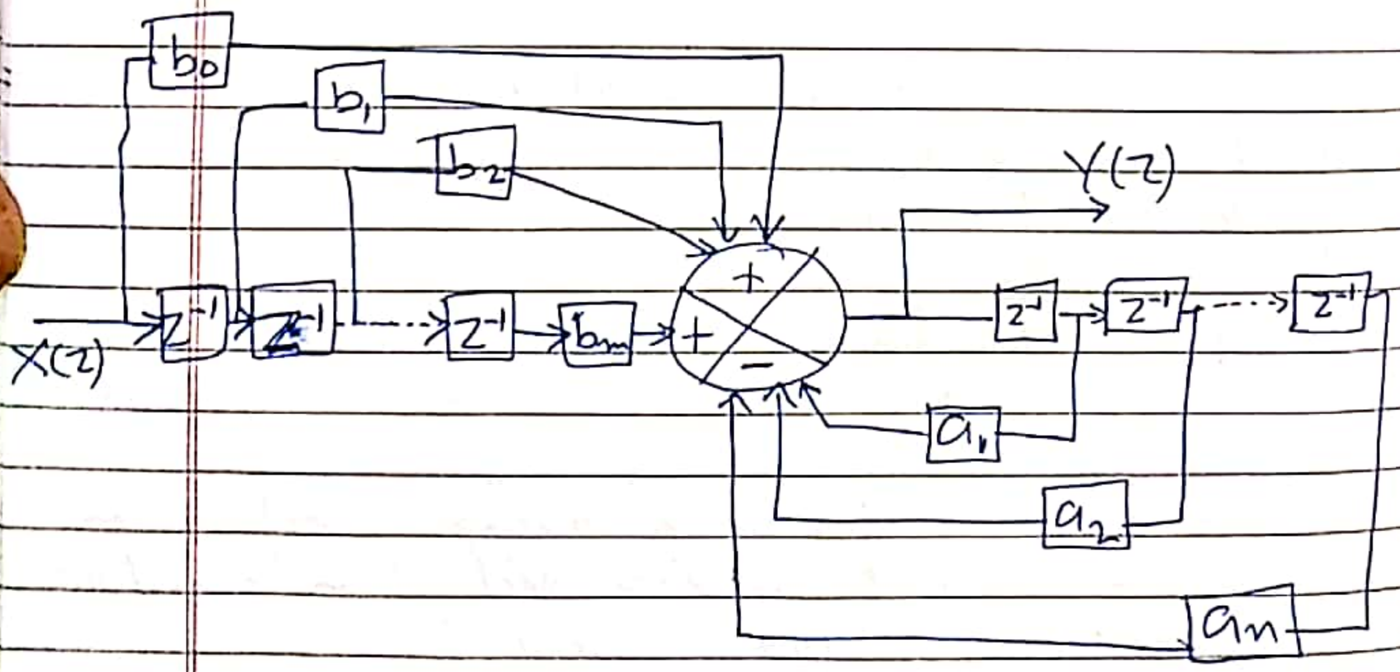
$$\boxed{G(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots b_m z^{-m}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots a_n z^{-n}}} \quad (1)$$

$\{n \geq m\}$

★ Direct Programming

⇒ Digital filter given by Eq (1) has n poles and m zeros.

$$Y(z) = -a_1 z^{-1} Y(z) - a_2 z^{-2} Y(z) - \cdots - a_n z^{-n} Y(z)$$
$$+ b_0 X(z) + b_1 z^{-1} X(z) + \cdots + b_m z^{-m} X(z)$$



⇒ This type of realization here is called direct programming.

⇒ The total number of delay element used in direct programming is m+n.

⇒ The number of delay elements used in direct programming can be reduced. In fact, the number of delay elements can be reduced from n+m to n.

★ Standard Programming

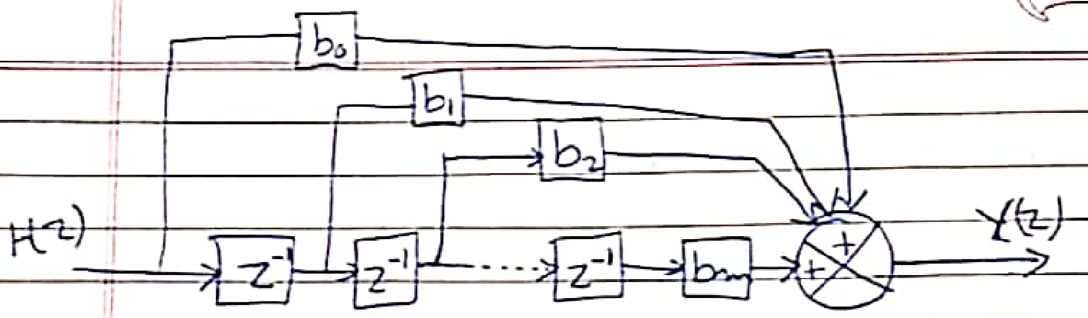
⇒ First, rewrite the pulse transfer function $Y(z)/X(z)$ as:-

$$\frac{Y(z)}{X(z)} = \frac{Y(z)}{H(z)} * \frac{H(z)}{X(z)}$$

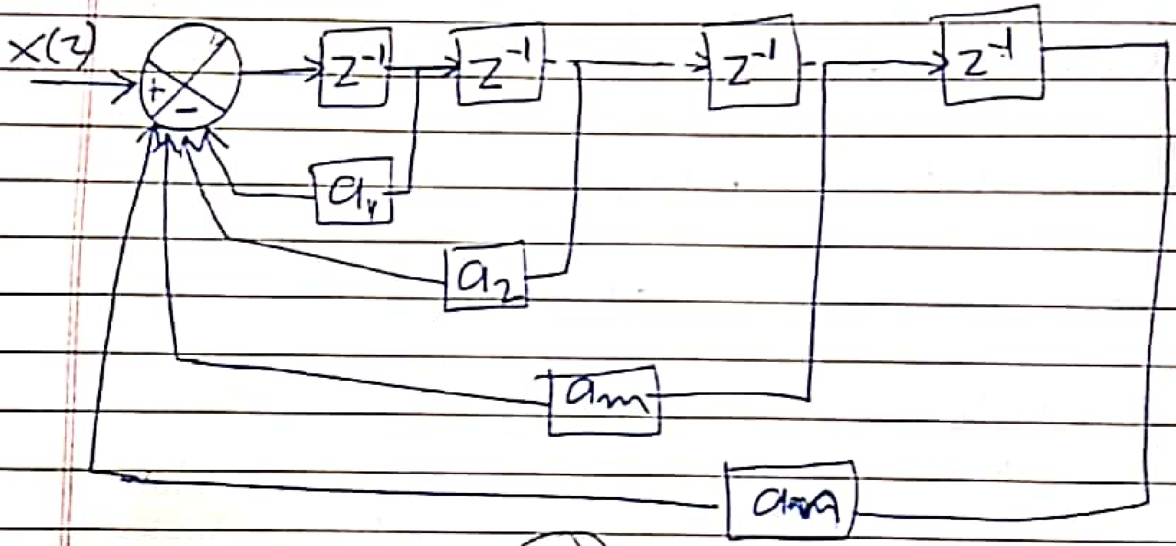
$$= \left(b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_m z^{-m}\right) * \left(\frac{1}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_n z^{-n}}\right)$$

$$\frac{Y(z)}{H(z)} = b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_m z^{-m}$$

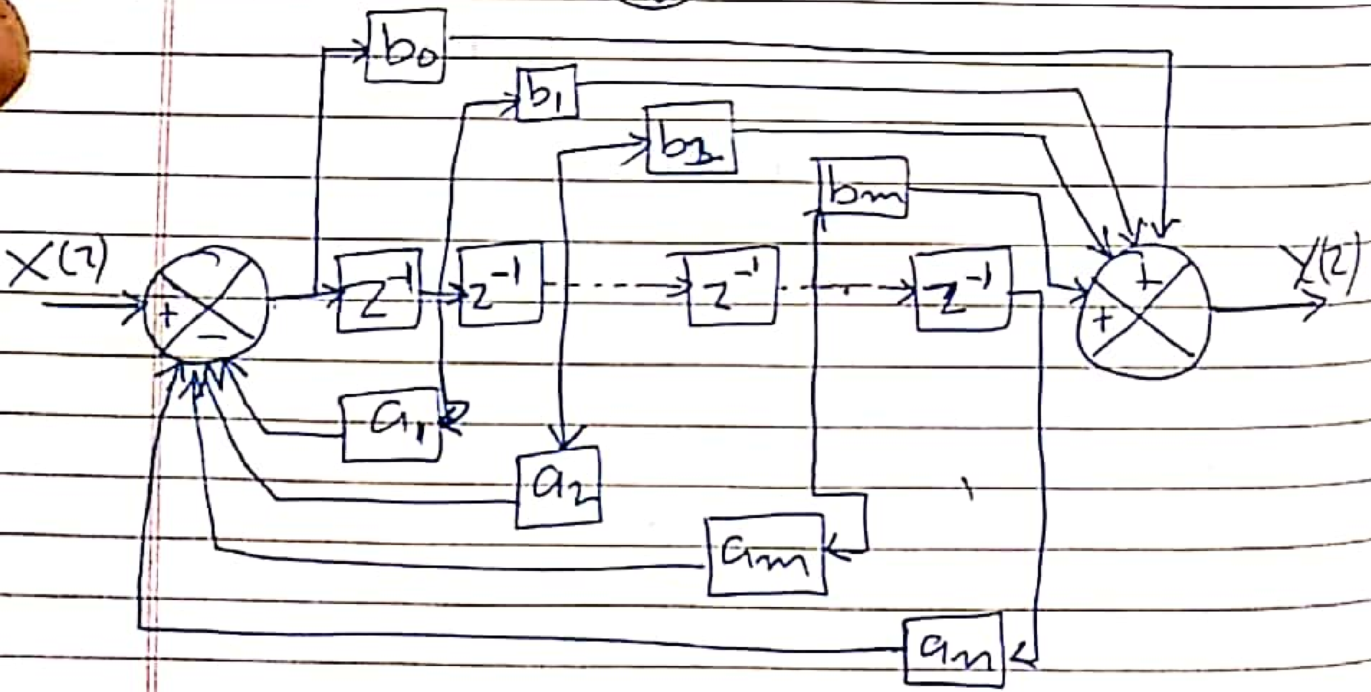
$$\frac{H(z)}{X(z)} = \frac{1}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_n z^{-n}}$$


b_0
b_1
b_2
$H(z)$
z^{-1}
z^{-1}
z^{-1}
b_m
$Y(z)$

$+$


$X(z)$
z^{-1}
z^{-1}
z^{-1}
z^{-1}
a_1
a_2
a_m
a_n

$\Downarrow$


b_0
b_1
b_2
b_m
$X(z)$
z^{-1}
z^{-1}
z^{-1}
z^{-1}
$Y(z)$
a_1
a_2
a_m
a_n

⇒ In realizing digital controllers or digital filters, it is important to have a good level of accuracy. Basically, three sources of errors effects the accuracy:-

1. The error due to the quantization of the input signal into a finite number of discrete level.

2. The error due to the accumulation of round off errors in the arithmetic operations in the digital ~~Signal~~ System.

3. The error due to quantization of the coefficients a_i and b_i of the pulse transfer function.

⇒ For decompositing higher-order pulse transfer functions in order to avoid the coefficient sensitivity problem, the following three approaches are commonly used.

1. Series Programming

2. Parallel programming

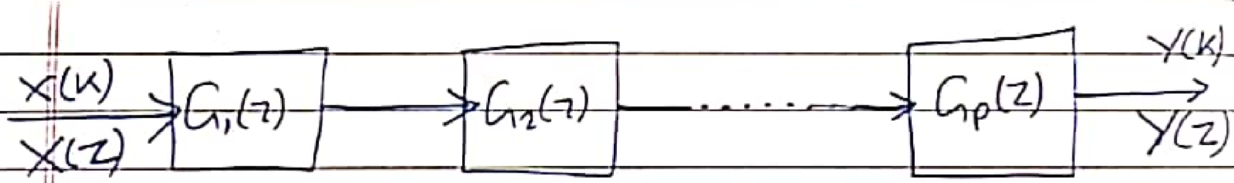
3. Ladder programming

1. Series Programming

⇒ The first approach used to avoid the Sensitivity problem is to implement the pulse transfer function $G(z)$ as a Series Connection of first-order and/or Second-order pulse transfer functions.

⇒ If $G(z)$ can be written as a product of pulse transfer function $G_1(z), G_2(z) \cdots G_p(z)$

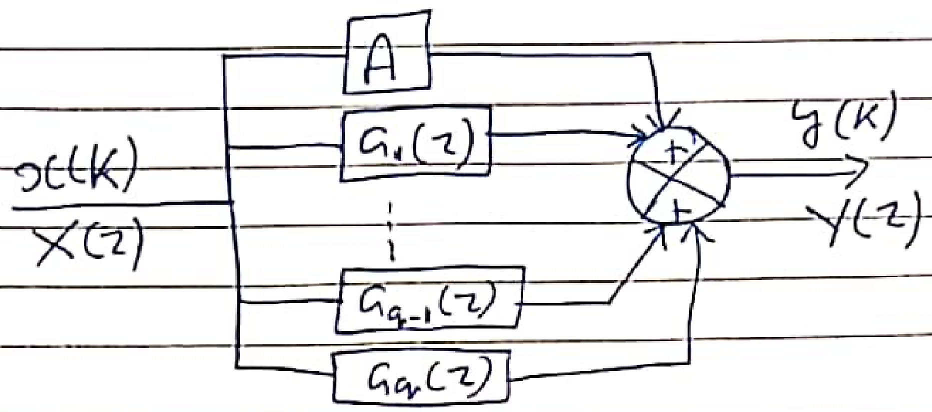
$$G_o(z) = G_1(z)\, G_2(z) \cdots G_p(z)$$



2. Parallel Programming

⇒ The Second approach to avoid the coefficient Sensitivity problem is to expand the pulse transfer function $G(z)$ into partial fractions.

$$G(z) = A + G_1(z) + G_2(z) \cdots + G_q(z)$$



3. Ladder Programming

⇒ The third approach to avoide the coefficient sensitivity problem is to implement a ladder structure, that is to expand the pulse transfer function $G(z)$ into the following continuous-fraction form and to program according to this eandition.

$$G(z) = A_0 + \cfrac{1}{B_1 z + \cfrac{1}{A_1 + \cfrac{1}{B_2 z \cfrac{1}{\ddots \; A_{n-1} \cfrac{1}{B_n z + \cfrac{1}{A_n}}}}}}$$

Let we define:

$$G_i^{(B)}(z) = \frac{1}{B_i z + G_i^{A}(z)} \qquad i = 1, 2 \cdots n-1$$

$$G_i^{(A)}(z) = \frac{1}{A_i + G_{i+1}^{(B)}(z)} \qquad i = 1, 2 \cdots n-1$$

$$G_n^{(B)}(z) = \frac{1}{B_n z + \frac{1}{A_n}}$$

⇒ Then we may write

$$\boxed{G(z) = A_0 + G_1^{(B)}(z)}$$

★ Infinite - Impulse Response Filter and Finite Impulse Response Filter

⇒ Digital filter may be classified according to the duration of the impulse response.

⇒ If $y(k) = -\{a_1 y(k-1) + a_2 y(k-2) \cdots - a_n y(k-n)\}$

$$+ \{b_0 x(k) + b_1 x(k-1) + \cdots + b_m x(k-m)\}$$

→ The impulse response of the digital filter shown above, where we assume not all a_i's are zero, has an infinite number of nonzero samples.

→ This type of digital filter is called an infinite-impulse response filter. (also called recursive filter).

⇒ If $y(k) =$ ~~$b_0 x$~~ $b_0 x(k) + b_1 x(k-1) + \cdots + b_m x(k-m)$

→ The impulse response of the digital filter defined above is limited to a finite number of samples defined over a finite range of time intervals.

→ This type of digital filter is called a finite-impulse response filter. (or non-recursive filter)