Appendix: Controllability & Observability

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★ The Concept of controllability and Observability

Controllability

→ Input do not act directly on the state but via state dynamics.

→ Can the inputs drive the system to any value in the state space in a finit time?

$$\dot{x} = Ax + Bu$$

Observability

→ States are not measured directly but instead impact the output via the output eqution:

$$y = \cancel{A}Cx + Du$$

→ Can we infer fully the initial state from the output and the input?

★ Controllability defination in discrete time

"A discrete-time linear system $x(k+1) = A(k)\,x(k) + B(k)u(k)$ is called controllable at $k=0$ if there exists a finite time k_1 such that for any initial state $x(0)$ and target state x_1, there exists a control sequence $\{u(k), k=0, \dots k_1\}$ that will transfer the system from $x(0)$ at $k=0$ to x_1 at $k=k_1$"

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★ Controllability of LTI Systems

$$\boxed{x(k+1) = Ax(k) + Bu(k)}$$

↓

$$x(n) = A^n x(0) + \sum_{k=0}^{n-1} A^{n-1-k} B u(k)$$

↓

$$x(n) - A^n x(0) = \underbrace{[B, AB, A^2B, \cdots A^{n-1}B]}_{P_d} \begin{bmatrix} u(n-1) \\ u(n-2) \\ \vdots \\ u(0) \end{bmatrix} \quad -(1)$$

⇒ Given any $x(n)$ and $x(0)$ in R^n

$[u(n-1) \cdots u(0)]^T$ can be solved if the columns of P_d span R^n.

⇒ Equivalently, system is controllable if P_d has rank n.

⇒ Also, no need to go beyond n: adding A^nB, $A^{n+1}B$, -- does not increase the rank of P_d.

{Cayley Hamilton Theorem}

★ Theorem (Controllability Theorem)

⇒ The n-dimensional m-input LTI system with

$$X(K+1) = AX(K) + Bu(K) \quad , A \in R^{n\times n} , B \in R^{n\times m}$$

is controllable if & only if either one of the following is satisfied.

① The $n\times nm$ controllability matrix

$$P_d = [B, AB, A^2B \cdots A^{n-1}B]$$

has rank n.

② The controllability grammian

$$W_{cd} = \sum_{K=0}^{K_1} A^K B B^T (A^T)^K$$

is nonsingular for some finite K_1.

★ Proof: From controllability matrix to grammian

⇒ P_d is full rank ⇒ $P_d P_d^T = \sum A^K B B^T (A^T)^K$

is non-singular.

⇒ A Solution to eq ① is

$$\begin{bmatrix} u(n-1) \\ u(n-2) \\ \vdots \\ u(0) \end{bmatrix} = P_d^{-1} [X(n) - A^n X(0)] = P_d^T (P_d P_d^T)^{-1} [X(n) - A^n X(0)]$$

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★ **Analysis: Controllability and Controllable canonical form**

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

⇒ Controllability matrix:

$$P_d = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -a_2 \\ 1 & -a_2 & -a_1 + a_2^2 \end{bmatrix}$$ has full rank.

⇒ System is controllable, canonical form is controllable.

★ **Analysis: Controllability grammian & Lyapunov Equation.**

$$W_{cd} = \sum_{K=0}^{K_1} A^K B B^T (A^T)^K$$

⇒ If A is Schur, K_1 can be set to infinity.

$$W_{cd} = \sum_{K=0}^{\infty} A^T \underbrace{B B^T}_{Q} (A^T)^K$$

⇒ This can be solved via the Lyapunov Eq.

$$A W_{cd} A^T - W_{cd} = -B B^T$$

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★ Analysis: Controllability & Similarity transform

$$\boxed{\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) + Du(k) \end{aligned}}$$

$$\Big\downarrow \; x = Tx^*$$

$$\boxed{\begin{aligned} x^*(k+1) &= \overbrace{T^{-1}AT}^{\tilde{A}}\, x^*(k) + \overbrace{T^{-1}B}^{\tilde{B}}\, u(k) \\ y(k) &= CTx^*(k) + Du(k) \end{aligned}}$$

⇒ Controllability matrix

$$P_d^* = [\tilde{B}, \tilde{A}\tilde{B}, \cdots \tilde{A}^{n-1}\tilde{B}]$$

$$= [T^{-1}B, T^{-1}AB, \cdots T^{-1}A^{n-1}B]$$

$$= T^{-1}P_d$$

⇒ Hence (A,B) Controllable $\Leftrightarrow$ $(T^{-1}AT, T^{-1}B)$ Controllable.

⇒ The controllability property is invariant under any coordinate transformation.

★ Popov-Belevitch-Hautus (PBH) Controllability test.

⇒ (A,B) is Controllable iff $\text{rank}[(A-\lambda I)\ B] = n \ \forall \lambda \in \mathbb{C}$

→ Rank $(A-\lambda I) = n$ except for eigenvalue λ.

→ So for controllability B needs to have some component in each eigenvector direction.

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★ Observability of LTI System

A discrete-time linear system

$$X(K+1) = A(K)X(K) + B(K)u(K)$$
$$Y(K) = C(K)X(K) + D(K)u(K)$$

is called observable at $K=0$ if there exists a finite time K_1 such that for any initial state $x(0)$, the knowledge of the input $\{u(K); K=0, \ldots K_1\}$ and $\{y(K); K=0, 1, \ldots K_1\}$ suffice to determine the state $x(0)$. Otherwise the system is said to be unobservable

★ Observability of LTI System

⇒ Let us start with the unforced system

$$\boxed{\begin{aligned} X(K+1) &= Ax(K) \\ y(K) &= Cx(K) \end{aligned}} \quad \begin{aligned} X &\in R^n \\ y &\in R^m \end{aligned}$$

$$\downarrow$$

$$x(K) = A^K X(0) \qquad y(K) = CA^K X(0)$$

$$\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(n-1) \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} X(0)$$

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⇒ If the linear matrix equation has a nonzero solution $X(0)$, the System is observable.

⇒ Generalizing to,

$$\boxed{\begin{aligned} X(K+1) &= Ax(K) + Bu(K) \\ y(K) &= Cx(K) + Du(K) \end{aligned}}$$

$$\boxed{\begin{aligned} X(K) &= A^K x(0) + \sum_{j=0}^{K-1} A^{K-1-j} Bu(j) \\ y(K) &= CA^K x(0) + \underbrace{C\sum_{j=0}^{K-1} A^{K-1-j} Bu(j) + Du(K)}_{y_{forced}(K)} \end{aligned}}$$

$$\underbrace{\begin{bmatrix} y(0) - y_{forced}(0) \\ \vdots \\ y(n-1) - y_{forced}(n-1) \end{bmatrix}}_{Y} = \underbrace{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}}_{Q_d} X(0)$$

⇒ $X(0)$ can be Solved if Q_d has rank n.

⇒ One way to write the solution is

$$X(0) = (Q_d^T Q_d)^{-1} Q_d^T Y$$

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$\Rightarrow$ also, no need to go beyond n in Q_d ; adding CA^n, CA^{n+1} --- does not increase the column rank of Q_d.

{Cayley Hamilton theorem}

★ **Theorem (Observability Theorem)**

System $x(K+1) = Ax(K) + Bu(K)$ $A \in R^{n\times n}$, $C \in R^{m\times n}$
$Y(K) = Cx(K) + Du(K)$
is observable if and only if either one of the following is satisfied.

1. The observability matrix $Q_d = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$ has full rank.

2. The observability grammian

$$W_{od} = \sum_{K=0}^{K_1} (A^T)^K C^T C A^K$$ is nonsingular for some finite K_1.

$$\boxed{Q_d^T Q_d = \sum_{K=0}^{n} (A^T)^K C^T C A^K}$$

★ Observability check

⇒ Analogous to the case in controllability, the observability property is invariant under any coordinate transformation.

$$(A, C) \text{ is observable} \Longleftrightarrow (T^{-1}AT, CT) \text{ is observable}$$

⇒ If A is Schur, K_1 can be set to ∞ in the observability grammian.

$$W_{od} = \sum_{K=0}^{\infty} (A^T)^K C^T C A^K$$

and we can compute by solving the Lyapunov equation:

$$\boxed{A^T W_{od} A - W_{od} = -C^T C}$$

⇒ The solution is nonsingular if and only if the system is observable.

★ Observability and observable canonical form

$$A = \begin{bmatrix} -a_2 & 1 & 0 \\ -a_1 & 0 & 1 \\ -a_0 & 0 & 0 \end{bmatrix} \qquad C = [1, 0, 0]$$

⇒ Observability matrix

$$Q_d = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -a_2 & 1 & 0 \\ a_2^2 - a_1 & -a_2 & 1 \end{bmatrix}$$ has full rank.

⇒ System in observable Canonical form is observable.

★ PBH test for observability.

⇒ (A, C) is observable if rank $\begin{bmatrix} A-\lambda I \\ C \end{bmatrix}$ has rank n.

→ Rank of $(A-\lambda I) = n$ except for eigenvalue λ.

→ So $\begin{bmatrix} A-\lambda I \\ C \end{bmatrix}$ should have rank n for all the eigenvalues.

Continuous time Case

★ Theorem (Controllability of Continuous-time System)

The n-dimensional m-input LTI System with $\dot{x} = Ax + Bu$, $A \in R^{n\times n}$, $B \in R^{n\times m}$ is Controllable if and only if either one of the following is Satisfied.

1. The $n \times nm$ controllability matrix

$$P = [B, AB, A^2B \cdots A^{n-1}B]$$ has rank n.

2. The Controllability grammian.

$$W_{c} = \int_0^t e^{A\tau} BB^T e^{A^T\tau} d\tau$$

is nonsingular for any $t > 0$

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★ Theorem (Observability of Continuous-time System)

System $\dot{x} = Ax + Bu$, $A \in R^{n\times n}$, $C \in R^{m\times n}$ is
$y = Cx + Du$
observable if and only if either one of the following is satisfied:

1. The $(mn)\times n$ observability matrix

$$Q = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \text{ has rank } n.$$

2. The observability grammian

$$W_{oc} = \int_0^t e^{A^T\tau} C^T C e^{A\tau} d\tau$$

is nonsingular for any $t > 0$.