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Spatial Descriptions

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★ Manipulator

→ Set of links connected through joints.

→ First link is fixed and is called Base.

→ Last link is called End-Effector.

Joints

→ Revolute joint (1 DOF)

"Allows rotation about a fixed axis"

→ Prismatic Joint (1DOF)

"Allows translation about a fixed axis"

Links {Rigid bodies}

→ n moving links & 1 fixed links.

★ Configuration Parameters

⇒ A set of parameters that describes the full configuration of the system.

Generalized coordinates

→ A set of independent configuration parameters.

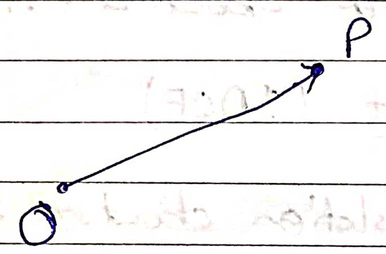
Degree of Freedom

→ Number of generalized coordinates.

Redundancy

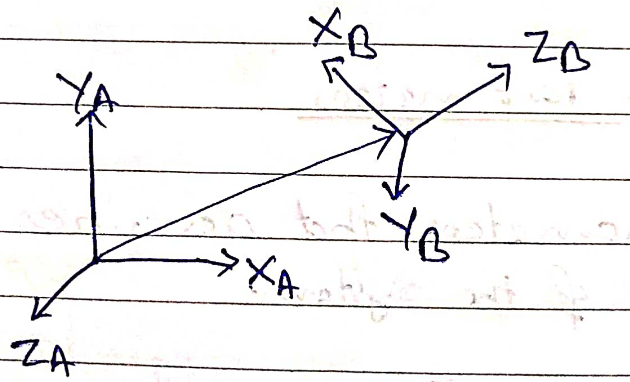
→ Number of degree of freedom of robot is greater than number of degree of freedom of end effector.

★ **Position of a point**



{Origin}

★ **Description of a Frame**



$$\boxed{{}^AP = {}^AR_B\,{}^BP + {}^AP_B}$$

$$\{B\} = \{{}^AR_B, {}^AP_B\}$$

⇒ Homogeneous Transform:

$$\begin{bmatrix} {}^AP \\ 1 \end{bmatrix} = \begin{bmatrix} {}^AR_B & {}^AP_B \\ 0 & 1 \end{bmatrix} \begin{bmatrix} {}^BP \\ 1 \end{bmatrix}$$

4x1 4x4 4x1

Rotation Matrix

Mapping — {Changing description from frame to frame}

Operators — {Moving points (within the same frame)}

$R_K(\theta): P_1 \rightarrow P_2$

$P_2 = R_K(\theta) P_1$

⇒ Inverse homogeneous transform

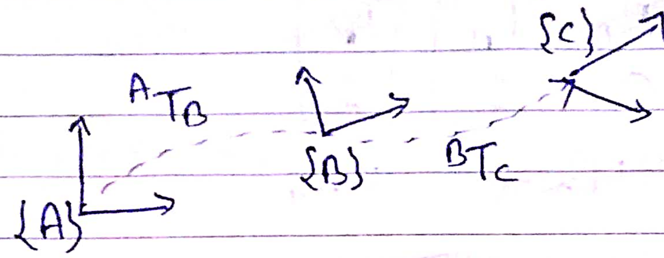
$${}^AT_B = \begin{bmatrix} {}^AR_B & {}^AP_B \\ 0 & 1 \end{bmatrix}$$

$${}^BT_A = \begin{bmatrix} {}^AR_B^T & -{}^AR_B^T\,{}^AP_B \\ 0 & 1 \end{bmatrix}$$ {Not transpose}

②

Spatial Description Cont.

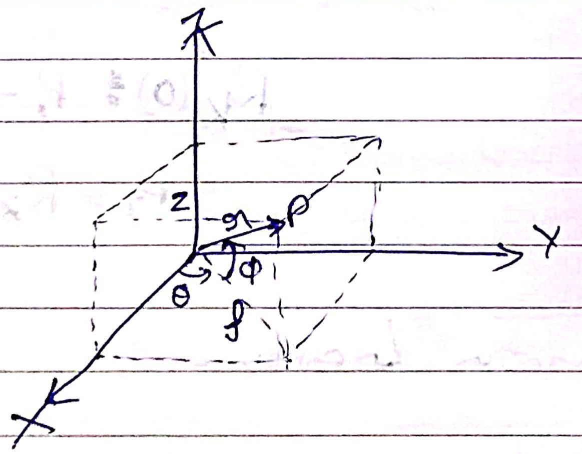
★ Compound Transformations



$$\boxed{^AT_C = {^AT_B}\,{^BT_C}}$$

$$^AT_C = \begin{bmatrix} {^AR_B}\,{^BR_C} & {^AR_B}\,{^BP_C} + {^AP_B} \\ 0 & 1 \end{bmatrix}$$

★ Position Representation



Cartesian: (x, y, z)

Cylindrical: (ρ, θ, z)

Spherical: (r, θ, ϕ)

★ Rotation Representation

① Rotation matrix { Complete representation of rotation }

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = [r_1 \; r_2 \; r_3]$$

$\|r_1\|_2 = \|r_2\|_2 = \|r_3\|_2 = 1$

$r_1 \cdot r_2 = r_1 \cdot r_3 = r_2 \cdot r_3 = 0$ $\{6 \text{ Constraints}\}$

~~② Direction Cosines~~

② Three angle representations

Euler Angles

- Fixed-axis rotation {12 possible representation}
- Relative Axis rotation {12 Possible representation}

★ XYZ Fixed axis rotation

Rotation about X roll (γ) → Rotation about Y Pitch (β) → Rotation about Z Yaw (α)

$$\boxed{R = R_z(\alpha) R_y(\beta) R_x(\gamma)}$$

$$\begin{bmatrix} C_\alpha & -S_\alpha & 0 \\ S_\alpha & C_\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_\beta & 0 & S_\beta \\ 0 & 1 & 0 \\ -S_\beta & 0 & C_\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_\gamma & -S_\gamma \\ 0 & S_\gamma & C_\gamma \end{bmatrix}$$

★ Inverse Problem (Given ${}^{A}R_{B}$ find (α, β, γ))

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} C_\alpha C_\beta & C_\alpha S_\beta S_\gamma - S_\alpha C_\gamma & C_\alpha S_\beta C_\gamma + S_\alpha S_\gamma \\ S_\alpha C_\beta & S_\alpha S_\beta S_\gamma + C_\alpha C_\gamma & S_\alpha S_\beta C_\gamma - C_\alpha C_\gamma \\ -S_\beta & C_\beta S_\gamma & C_\beta C_\gamma \end{bmatrix}$$

$\Rightarrow (C_\alpha C_\beta)^2 + (S_\alpha C_\beta)^2 = r_{11}^2 + r_{21}^2$

$\Rightarrow C_\beta^2 (C_\alpha^2 + S_\alpha^2) = r_{11}^2 + r_{21}^2$

$\Rightarrow C_\beta^2 = r_{11}^2 + r_{21}^2$

$\Rightarrow \boxed{C_\beta = \sqrt{r_{11}^2 + r_{21}^2}}$ if $C_\beta \geq 0 \Rightarrow \boxed{-\frac{\pi}{2} \leq \beta \leq \pi/2}$

$\Rightarrow -S_\beta = r_{31}$

$\Rightarrow \boxed{S_\beta = -r_{31}}$

$$\boxed{\beta = \text{atan2}\left(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2}\right)}$$

(using this we can find α & γ)

$\Rightarrow$ if $C_\beta = 0 \Rightarrow \beta = \pm \pi/2 \Rightarrow$ Singularity of representation

$\downarrow$

$S_\beta = \pm 1$

$$\cancel{R = \begin{pmatrix} 0 & -S(\alpha \mp \gamma) & C(\alpha \mp \gamma) \\ 0 & C(\alpha \mp \gamma) & S(\alpha \mp \gamma) \\ 1 & 0 & 0 \end{pmatrix}}$$

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⇒ ~~Only α-γ is defined.~~

⇒ If $C_\beta = 0$ & $S_\beta = +1$ $\{+\pi/2 = \beta\}$

$$^{A}R_B = \begin{pmatrix} 0 & -S_{(\alpha-\gamma)} & C_{(\alpha-\gamma)} \\ 0 & C_{(\alpha-\gamma)} & S_{(\alpha-\gamma)} \\ 1 & 0 & 0 \end{pmatrix}$$

⇒ Only $(\alpha-\gamma)$ is defined.

⇒ If $C_\beta = 0$, $S_\beta = -1$ $\{-\pi/2 = \beta\}$

$$^{A}R_B = \begin{pmatrix} 0 & -S_{(\alpha+\gamma)} & -C_{(\alpha+\gamma)} \\ 0 & C_{(\alpha+\gamma)} & -S_{(\alpha+\gamma)} \\ 1 & 0 & 0 \end{pmatrix}$$

⇒ Only $(\alpha+\gamma)$ is defined.

⇒ All three parameter representation has singularity somewhere.

$$\left.\begin{aligned} \alpha &= \operatorname{atan2}\left(\frac{r_{21}}{C_\beta}, \frac{r_{11}}{C_\beta}\right) \\ \gamma &= \operatorname{atan2}\left(\frac{r_{32}}{C_\beta}, \frac{r_{33}}{C_\gamma}\right) \end{aligned}\right\} \forall\, C_\beta \neq 0$$

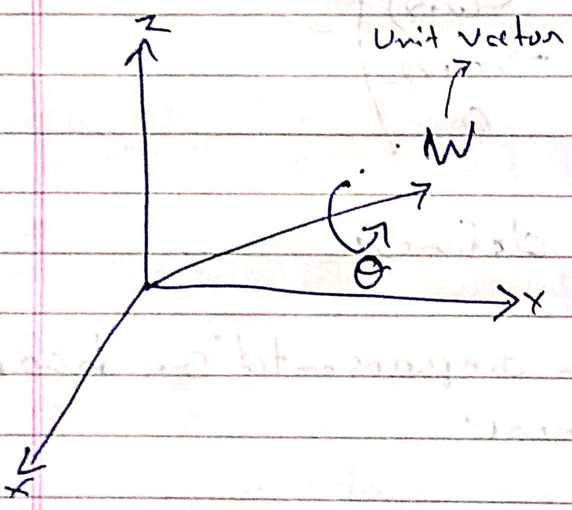
$\boxed{-\pi \leq \alpha \leq \pi}$ & $\boxed{-\pi \leq \gamma \leq \pi}$ $\boxed{-\frac{\pi}{2} \leq \beta \leq \frac{\pi}{2}}$

* ~~Equivalent angle-axis representation~~

* Euler Parameters

Euler's rotation theorem

↳ In three dimensional space, any displacement of a rigid body such that a point on the rigid body remains fixed is equivalent to a single rotation about some axis that runs through the fixed point.



$$\varepsilon_1 = W_x \sin \theta/2$$

$$\varepsilon_2 = W_y \sin \theta/2$$

$$\varepsilon_3 = W_z \sin \theta/2$$

$$\varepsilon_4 = \cos \theta/2$$

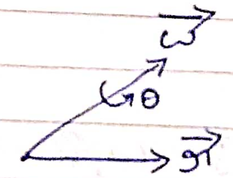
⇒ Normality Condition

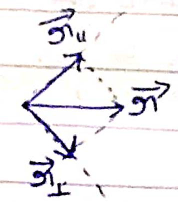
$|W| = 1$, $\boxed{\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2 = 1}$

ε : point on a unit hypersphere in 4-dimentional space.

~~Inverse Problem~~

$\Rightarrow$ Rotation of a vector about a unit vector





$$\vec{r} = \vec{r}_{\parallel} + \vec{r}_{\perp}$$

$$\boxed{\begin{aligned}\vec{r}_{\parallel} &= (\vec{r}\cdot\hat{\omega})\,\hat{\omega} \\ \vec{r}_{\perp} &= \vec{r} - \vec{r}_{\parallel}\end{aligned}}$$

$$\boxed{\vec{r}_{\theta} = \vec{r}_{\parallel} + \vec{r}_{\perp\theta}}$$

$$\vec{r}_{\perp\theta} = \|\vec{r}_{\perp}\|\cos\theta\,\hat{r}_{\perp} + \|\vec{r}_{\perp}\|\sin\theta\,(\hat{\omega}\times\hat{r}_{\perp})$$

$$\cos\theta\,\|\vec{r}_{\perp}\|\,\hat{r}_{\perp} \quad\quad \cancel{\|\vec{r}_{\perp}\|}\sin\theta\left(\hat{\omega}\times\frac{\vec{r}_{\perp}}{\cancel{\|\vec{r}_{\perp}\|}}\right)$$

$$\cos\theta\,\vec{r}_{\perp} \quad\quad \sin\theta\,(\hat{\omega}\times\vec{r}_{\perp})$$

$$\boxed{\vec{r}_{\perp\theta} = \cos\theta\,\vec{r}_{\perp} + \sin\theta\,(\hat{\omega}\times\vec{r}_{\perp})}$$

$\Rightarrow$ To find rotation matrix represented by rotation about $\vec{\omega}$ by amount θ:

- $\rightarrow$ We need to find X_θ Y_θ and Z_θ (i.e. Unit vector X, Y and Z rotated about $\vec{\omega}$ by amount θ)
- $\rightarrow$ So $R = [X_\theta \; Y_\theta \; Z_\theta]$

⇒ **finding X_θ**

⇒ Let $\vec{X} = \hat{i}$

⇒ & $\vec{\omega} = \omega_x\hat{i} + \omega_y\hat{j} + \omega_z\hat{k}$

$\vec{\omega}\cdot\vec{X} = \omega_x \qquad \vec{\omega}\times\vec{X} = +\omega_z\hat{j} - \omega_y\hat{k}$

$\vec{X}_{\parallel} = \omega_x\omega_x\hat{i} + \omega_x\omega_y\hat{j} + \omega_x\omega_z\hat{k}$

$\vec{X}_{\perp} = (1-\omega_x\omega_x)\hat{i} - \omega_x\omega_y\hat{j} - \omega_x\omega_z\hat{k}$

$$\vec{X}_{\perp\theta} = \cos\theta(1-\omega_x\omega_x)\hat{i}$$
$$-(\omega_x\omega_y\cos\theta + \omega_z\sin\theta)\hat{j}$$
$$+(-\omega_x\omega_z\cos\theta - \omega_y\sin\theta)\hat{k}$$

$\vec{X}_\theta = \vec{X}_{\parallel} + \vec{X}_{\perp\theta}$

$$\vec{X}_\theta = (\omega_x\omega_x(1-\cos\theta) + \cos\theta)\hat{i}$$
$$(\omega_x\omega_y(1-\cos\theta) + \omega_z\sin\theta)\hat{j}$$
$$(\omega_x\omega_z(1-\cos\theta) - \omega_y\sin\theta)\hat{k}$$

So $$X_\theta = \begin{bmatrix} \omega_x\omega_x(1-C_\theta) + C_\theta \\ \omega_x\omega_y(1-C_\theta) + \omega_z S_\theta \\ \omega_x\omega_z(1-C_\theta) - \omega_y S_\theta \end{bmatrix}$$

⇒ Similarly,

$$Y_\theta = \begin{bmatrix} \omega_x \omega_y (1-C_\theta) - \omega_z S_\theta \\ \omega_y \omega_y (1-C_\theta) + C_\theta \\ \omega_y \omega_z (1-C_\theta) + \omega_x S_\theta \end{bmatrix}$$

$$Z_\theta = \begin{bmatrix} \omega_x \omega_z (1-C_\theta) + \omega_y S_\theta \\ \omega_y \omega_z (1-C_\theta) - \omega_x S_\theta \\ \omega_z \omega_z (1-C_\theta) + C_\theta \end{bmatrix}$$

$$\Rightarrow \quad \varepsilon_1 = \omega_x \sin(\theta/2)$$

$$\boxed{\sin(\theta/2)^2 = \frac{1-\cos\theta}{2}}$$

$$\varepsilon_1^2 = \omega_x \omega_x \sin^2(\theta/2)$$

$$= \omega_x \omega_x \frac{(1-\cos\theta)}{2}$$

$$\Rightarrow \omega_x \omega_x (1-C_\theta) = 2\varepsilon_1^2 \quad \text{— (1)}$$

$$\varepsilon_4 = \cos \theta/2$$

$$\boxed{\frac{1+\cos\theta}{2} = \cos^2(\theta/2)}$$

$$\varepsilon_4^2 = \cos^2(\theta/2) = \frac{1+\cos\theta}{2}$$

$$\Rightarrow \cos(\theta) = 2\varepsilon_4^2 - 1 \quad \text{— (2)}$$

$$\Rightarrow \omega_x \omega_x (1-C_\theta) + C_\theta = 2\varepsilon_1^2 + 2\varepsilon_4^2 - 1$$

$\omega_x \omega_x (1-C_\theta) + C_\theta$

$$= 2\varepsilon_1^2 + 2\varepsilon_4^2 - 1$$

$$= 2\varepsilon_1^2 + 2\varepsilon_4^2 - 1 + (2\varepsilon_2^2 + 2\varepsilon_3^2 - 2\varepsilon_2^2 - 2\varepsilon_3^2)$$

$$= 2(\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2) - 1 - 2\varepsilon_2^2 - 2\varepsilon_3^2$$

$$= 1 - 2\varepsilon_2^2 - 2\varepsilon_3^2$$

$$\Rightarrow X_\theta(0) = 1 - 2\varepsilon_2^2 - 2\varepsilon_3^2$$

$\Rightarrow$ Similarly we can calculate all the terms of the Rotation matrix in terms of $\varepsilon_1\ \varepsilon_2\ \varepsilon_3 \ \& \ \varepsilon_4$.

$$R = \begin{bmatrix} 1-2\varepsilon_2^2-2\varepsilon_3^2 & 2(\varepsilon_1\varepsilon_2 - \varepsilon_3\varepsilon_4) & 2(\varepsilon_1\varepsilon_3 + \varepsilon_2\varepsilon_4) \\ 2(\varepsilon_1\varepsilon_2 + \varepsilon_3\varepsilon_4) & 1-2\varepsilon_1^2-2\varepsilon_3^2 & 2(\varepsilon_2\varepsilon_3 - \varepsilon_1\varepsilon_4) \\ 2(\varepsilon_1\varepsilon_3 - \varepsilon_2\varepsilon_4) & 2(\varepsilon_2\varepsilon_3 + \varepsilon_1\varepsilon_4) & 1-2\varepsilon_1^2-2\varepsilon_2^2 \end{bmatrix}$$

★ **Inverse Problem**

$$r_{11} + r_{22} + r_{33} = 3 - 4(\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2)$$
$$= 3 - 4(1-\varepsilon_4^2)$$
$$= 3 - 4 + 4\varepsilon_4^2$$
$$= 4\varepsilon_4^2 - 1$$

$$4\varepsilon_4^2 = 1 + r_{11} + r_{22} + r_{33}$$

$$\varepsilon_4 = \frac{1}{2}\sqrt{1 + r_{11} + r_{22} + r_{33}}$$

If $\varepsilon_4 \geq 0$
$\Rightarrow C_{\theta/2} \geq 0$
$\Rightarrow -\pi \leq \theta \leq \pi$

$$\varepsilon_1 = \frac{r_{32} - r_{23}}{4\varepsilon_4} \qquad \varepsilon_2 = \frac{r_{13} - r_{31}}{4\varepsilon_4} \qquad \varepsilon_3 = \frac{r_{21} - r_{12}}{4\varepsilon_4}$$

$\Rightarrow$ What if $\varepsilon_4 = 0$?

$\Rightarrow$ For all rotation one of the euler parameters is greater than or equal to $1/2$.
{Lemma}

$\Rightarrow$ Algorithm: Solve with respect to $\max_i \{\varepsilon_i\}$

- $\varepsilon_1 = \max_i \{\varepsilon_i\}$
  $$\varepsilon_1 = \frac{1}{2}\sqrt{r_{11} - r_{22} - r_{33} + 1}$$
- $\varepsilon_2 = \max_i \{\varepsilon_i\}$
  $$\varepsilon_2 = \frac{1}{2}\sqrt{-r_{11} + r_{22} - r_{33} + 1}$$
- $\varepsilon_3 = \max_i \{\varepsilon_i\}$
  $$\varepsilon_3 = \frac{1}{2}\sqrt{-r_{11} - r_{22} + r_{33} + 1}$$
- $\varepsilon_4 = \max_i \{\varepsilon_i\}$
  $$\varepsilon_4 = \frac{1}{2}\sqrt{1 + r_{11} + r_{22} + r_{33}}$$

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$\Rightarrow$ ε and $(-\varepsilon)$ represents same transformation

$$-\varepsilon_1 = -W_x \sin \theta/2 = W_x \sin(\pi + \theta/2) = W_x \sin\left(\frac{2\pi+\theta}{2}\right)$$

$$-\varepsilon_2 = -W_y \sin \theta/2 = W_y \sin\left(\frac{2\pi+\theta}{2}\right)$$

$$-\varepsilon_3 = -W_z \sin \theta/2 = W_z \sin\left(\frac{2\pi+\theta}{2}\right)$$

$$-\varepsilon_4 = -\cos \theta/2 = \cos(\pi + \theta/2) = \cos\left(\frac{2\pi+\theta}{2}\right)$$

$\Rightarrow$ $-\varepsilon$ represents rotation about same axis W as of ε by amount $2\pi + \theta$.

Quaternions

⇒ In mathematics, the quaternions are a number system that extends the complex number.

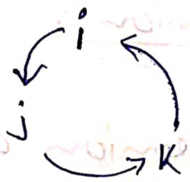
{ first described by William Rowan Hamilton }

$$q = a + bi + cj + dk$$

{ where,
$a, b, c, d \in \mathbb{R}$
i, j, k are the fundamental quaternion unit }

Quaternion Multiplication

$\times$	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	$-i$	-1



$$i^2 = j^2 = k^2 = ijk = -1$$

⇒ A quaternion is typically written as:

$$\boxed{q = S + V}$$

S → Scalar, V → Vector

⇒ Norm or length of Quaternion.

$$\|q\| = \sqrt{s^2 + v_1^2 + v_2^2 + v_3^2}$$

→ Product of two quaternion (Hamilton Product)

If we write q as $(s, \vec{v})$

$$q_1 \times q_2 = (s_1, \vec{v_1}) \times (s_2, \vec{v_2})$$

$$= \left(s_1 s_2 - \vec{v_1} \cdot \vec{v_2},\ s_1 \vec{v_2} + s_2 \vec{v_1} + \vec{v_1} \times \vec{v_2}\right)$$

⇒ Hamilton product is <u>not commutative</u> but <u>associative</u>.

$$q^{-1} = \frac{(s, -\vec{v})}{\|q\|^2}$$

★ **Unit Quaternion**

→ Quaternion with norm 1.

$$\|q\| = 1 \quad \text{—①}$$

$$q^{-1} = (s, -\vec{v}) \quad \text{—②}$$

$$q_1^{-1} q_2^{-1} = (q_2 q_1)^{-1} \quad \text{—③}$$

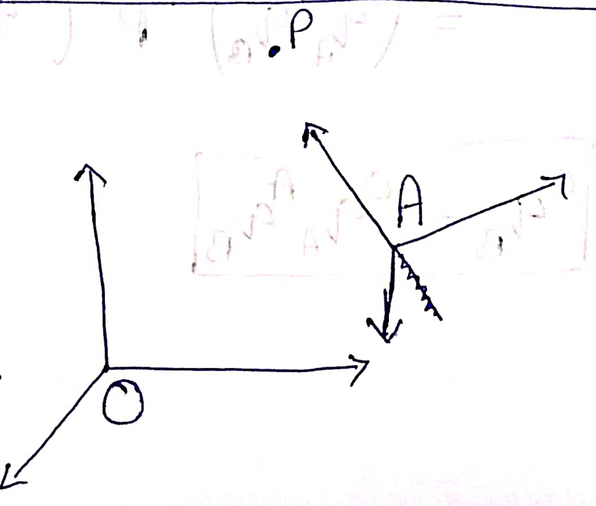
3D Rotation with Unit Quaternion

Let $q = s + \vec{v}$

where, $s = \cos\frac{\theta}{2}$ → angle of rotation

$\vec{v} = \hat{n}\sin\theta/2$ → rotation vector

$$\boxed{q = \cos\frac{\theta}{2} + \hat{n}\sin\theta/2}$$



⇒ Let A and O be two coordinate fra.

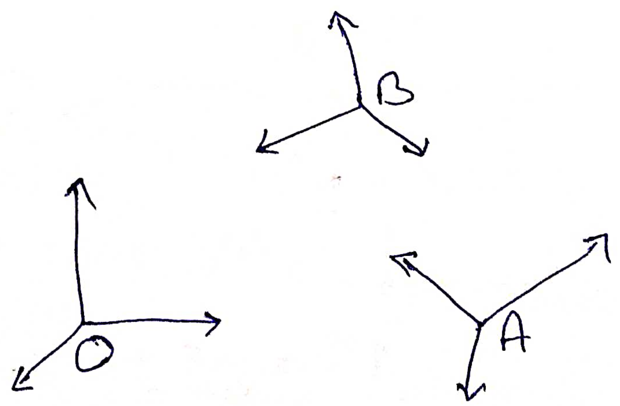
⇒ Let $^{O}q_A$ be transformation quaternion of A from O.

⇒ Let P be a general point

$$\boxed{{}^{O}P = {}^{O}q_A \, {}^{A}P \, {}^{O}q_A^{-1}}$$

{fact stated without proof}

★ Chaining



Given: $^{O}q_A$, $^{A}q_B$

To find: $^{O}q_B$

⇒ Let P be a general point :

$$^{O}p = {}^{O}q_A \, {}^{A}p \, {}^{O}q_A^{-1} \quad \text{—— (1)}$$

$$^{A}p = {}^{A}q_B \, {}^{B}p \, {}^{A}q_B^{-1} \quad \text{—— (2)}$$

⇒ using (1) and (2) :

$$^{O}p = {}^{O}q_A \, {}^{A}q_B \, {}^{B}p \, {}^{A}q_B^{-1} \, {}^{O}q_A^{-1}$$

$$= \left({}^{O}q_A \, {}^{A}q_B\right) {}^{B}p \left({}^{O}q_A \, {}^{A}q_B\right)^{-1}$$

$$\boxed{{}^{O}q_B = {}^{O}q_A \, {}^{A}q_B}$$