(A) & (B)

Appendix: Chapter-5

Jacobians: Velocities & Static force

Date

Page

★ Introduction

⇒ In this chapter:

→ We will examine the motions of linear & angular velocity of a rigid body and use these concept to analyze the motion of a manipulator.

→ We will also consider forces acting on a rigid body and then use it to study the application of static forces with manipulator.

★ Notation for time-varying position & Orientation

(#) Differentiation of position vectors

⇒ The velocity of a position vector can be thought of as the linear velocity of the point in space represented by the position vector.

$$ {}^{B}\left({}^{B}V_Q\right) = {}^{B}V_Q = \frac{d}{dt}\left({}^{B}Q\right) = \lim_{\Delta t \to 0} \frac{{}^{B}Q(t+\Delta t) - {}^{B}Q(t)}{\Delta t} $$

⇒ As with any vector, a velocity vector can be described in terms of any frame, and this

Date
Page

frame of reference is noted with a leading superscript.

$$^{A}({}^{B}V_Q) = \frac{{}^{A}d}{dt}\,{}^{B}Q$$

⇒ So we see that, in the general case, a velocity vector is associated with a point in space, but the numerical values describing the velocity of that point depend on two frames:

① → One with respect to which the differentiation was done

② → One in which the resulting velocity vector is expressed

$$^{A}({}^{B}V_Q) = {}^{A}R_B\,{}^{B}V_Q$$

⇒ Rather than considering a general point's velocity relative to an arbitrary frame, we will very often consider the velocity of the origin of a frame relative to some understood universal reference frame.

$$\boxed{V_C = {}^{U}V_{CORG}}$$

{frame C} {Universal reference frame U}

Date
Page

The angular velocity vector

⇒ Linear velocity describes an attribute of a point, angular velocity describes an attribute of a body.

→ We always attach a frame to the bodies we consider, so we can also think of angular velocity as describing rotational motion of a frame.

⇒ ${}^{A}\Omega_B$ describes the rotation of frame {B} relative to {A}.

→ Physically at any instant, the direction of ${}^{A}\Omega_B$ indicate the instantaneous axis of rotation of {B} relative to {A}.

→ And the magnitude of ${}^{A}\Omega_B$ indicate the speed of rotation.

⇒ Like any vector, an angular velocity vector may be expressed in any coordinate system, and so another leading superscript may be added.

→ Example ${}^{C}({}^{A}\Omega_B)$

$$\omega_C = {}^{U}\Omega_C$$

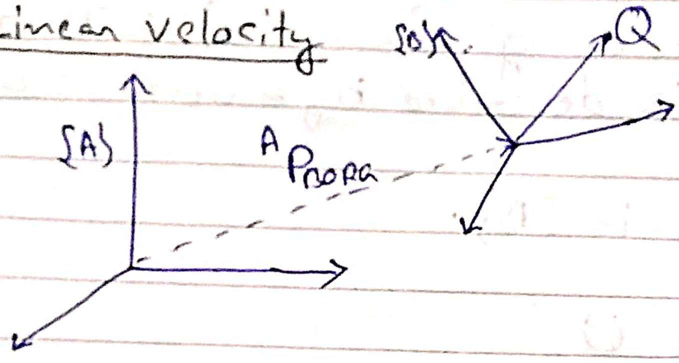
frame C

Universal frame

Date
Page

★ Linear and Rotational velocity of rigid body

(#) Linear velocity



→ {A} is a fixed frame.
→ {B} is attached to a rigid body.

$$^{A}Q = {}^{A}R_{B}\,{}^{B}Q + {}^{A}P_{BORG}$$

⇒ For the moment, we will assume that the orientation $^{A}R_{B}$ is not changing with time.

$$\boxed{^{A}V_{Q} = {}^{A}R_{B}\,{}^{B}V_{Q} + {}^{A}V_{BORG}}$$

(#) Rotational velocity

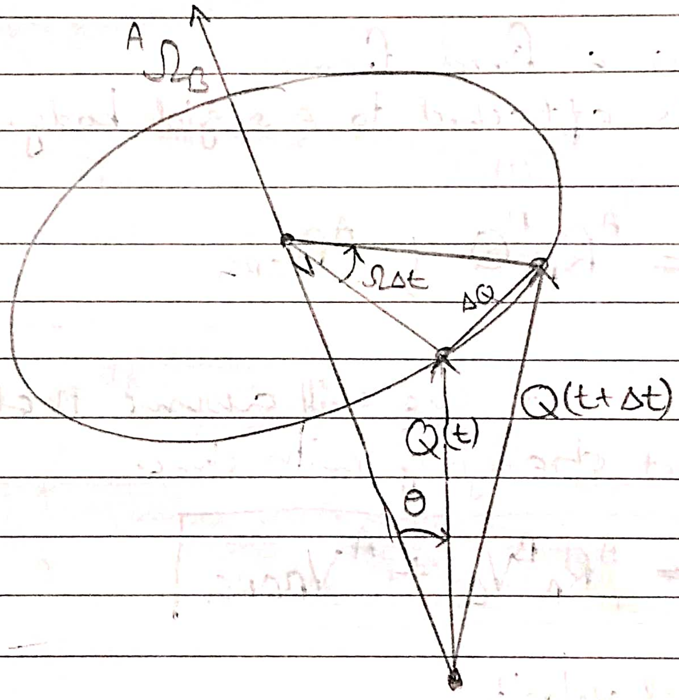
⇒ Now let us consider two frames with coincident origins and with zero linear relative velocity. (i.e. their origin will remain coincident for all time)

Date
Page

$$\boxed{{}^{A}Q = {}^{A}R_B\,{}^{B}Q}$$

⇒ The orientation of frame {B} with respect to frame {A} is described by a vector called ${}^{A}\Omega_B$.

⇒ Q is fixed in {B}.
↳ ${}^{B}V_Q = 0$



$$|\Delta Q| = (|{}^{A}Q| \sin\theta)(|{}^{A}\Omega_B| \Delta t)$$

⇒ ΔQ must be ⊥ to both ${}^{A}\Omega_B$ & ${}^{A}Q$.

⇒ These conditions on magnitude & direction immediately suggest vector cross product.

$$\boxed{{}^{A}V_Q = {}^{A}\Omega_B \times {}^{A}Q}$$

Date
Page

⇒ In the general case, the vector Q could also be changing with respect to frame {B}, so adding the component we have:

$$\boxed{{}^{A}V_{Q} = {}^{A}\left({}^{B}V_{Q}\right) + {}^{A}\Omega_{B} \times {}^{A}Q}$$

$$\Rightarrow \boxed{{}^{A}V_{Q} = {}^{A}R_{B}\,{}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}^{A}R_{B}\,{}^{B}Q}$$

Simultaneous linear and rotational velocity

$$\boxed{{}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}R_{B}\,{}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}^{A}R_{B}\,{}^{B}Q}$$

★ More on Angular Velocity

A property of the derivative of an orthogonal matrix

⇒ For any $n \times n$ orthonormal matrix R, we have,

$$R R^{T} = I_n$$

⇒ Differentiating yields:

$$\dot{R} R^{T} + R \dot{R}^{T} = O_n$$

⇒ Let $S = \dot{R} R^{T}$

$$S + S^{T} = O_n$$

So, S is skew-symmetric matrix.

Date
Page

⊕ Velocity of a point due to rotating reference frame

⇒ Consider a fixed vector ${}^{B}P$ unchanging with respect to frame {B}

$$\boxed{{}^{A}P = {}^{A}R_{B}\,{}^{B}P}$$

⇒ $${}^{A}\dot{P} = {}^{A}\dot{R}_{B}\,{}^{B}P$$

⇒ $${}^{A}V_{P} = {}^{A}\dot{R}_{B}\,{}^{B}P$$

⇒ $${}^{A}V_{P} = {}^{A}\dot{R}_{B}\,{}^{A}R_{B}^{-1}\,{}^{A}P$$

⇒ $$\boxed{{}^{A}V_{P} = {}^{A}S_{B}\,{}^{A}P} \Longleftrightarrow {}^{A}V_{P} = {}^{A}\Omega_{B} \times {}^{A}P$$

Angular Velocity matrix.

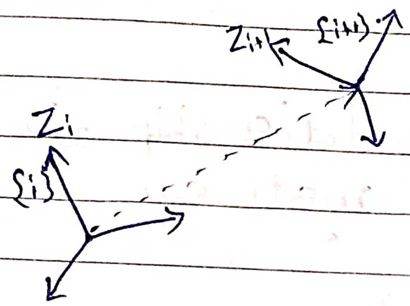
★ Motion of the links of a robot

→ We will always using link frame {0} as our reference frame.

→ V_i is the linear velocity of the origin of link frame {i}

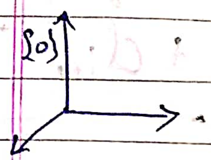
→ ω_i is the angular velocity of link frame {i}

★ Velocity propagation from link to link



⇒ Frame attached to link $\{i+1\}$

⇒ Frame attached to link $\{i\}$



⇒ Ground reference frame

⇒ Rotational velocities can be added when both ω vectors are written with respect to same frame.

$$\Rightarrow \quad {}^{i}\omega_{i+1} = {}^{i}\omega_i + {}^{i}R_{i+1}\underbrace{\dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}}_{\begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{i+1} \end{bmatrix}}$$

$$\Rightarrow \quad \boxed{{}^{i+1}\omega_{i+1} = {}^{i+1}R_i\,{}^{i}\omega_i + \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}}$$

⇒ The linear velocity of the origin of frame $\{i+1\}$ is same as that of the origin of frame $\{i\}$ plus new component caused by rotational velocity of link i.

$${}^{i}V_{i+1} = {}^{i}V_i + {}^{i}\omega_i \times {}^{i}P_{i+1}$$

Date
Page

$\Rightarrow$ $$\boxed{{}^{i+1}V_{i+1} = {}^{i+1}R_i \left({}^{i}V_i + {}^{i}\omega_i \times {}^{i}P_{i+1} \right)}$$

$\Rightarrow$ The corresponding relationship for the case that joint $i+1$ is prismatic are

$$\boxed{\begin{aligned} {}^{i+1}\omega_{i+1} &= {}^{i+1}R_i \, {}^{i}\omega_i \\ {}^{i+1}V_{i+1} &= {}^{i+1}R_i \left({}^{i}V_i + {}^{i}\omega_i \times {}^{i}P_{i+1} \right) + \dot{d}_{i+1} \, {}^{i+1}\hat{Z}_{i+1} \end{aligned}}$$

$\Rightarrow$ Applying these equations successively from link to link, we can compute ${}^{N}\omega_N$ and ${}^{N}V_N$.

★ Jacobians

$\Rightarrow$ The Jacobian is a multidimentional form of derivative

↳ We can think of the Jacobian as mapping velocities in X to those in Y

$$\boxed{\dot{Y} = J(X)\dot{X}}$$

$\Rightarrow$ In the field of robotics, we generally use Jacobians that relate joint velocities to Cartesian velocities of the tips of the arm.

$$\boxed{{}^{0}V = {}^{0}J(\Phi)\dot{\Phi}}$$ {where Φ is vector of joint angles}

Date
Page

⇒ **Note:** "For any given configuration of the manipulator, joint rates are related to velocity of the tip in a linear fashion"

⇒ In general case,

$$\Phi \in R^{6\times1}$$

$$^0V = \begin{bmatrix} ^0V \\ ^0\omega \end{bmatrix} \in R^{6\times1} \quad \text{where} \quad ^0V \in R^{3\times1} \;\&\; ^0\omega \in R^{3\times1}$$

⇒ Jacobian might also be found by directly differentiating the kinematic equations of the mechanism.

- → This is straight-forward for linear velocity
- → But there is no 3×1 orientation vector whose derivative is ω.

★ Changing a Jacobian's frame of reference

⇒ Given a Jacobian written in frame {B}, that is

$$\begin{bmatrix} ^BV \\ ^B\omega \end{bmatrix} = {}^BV = {}^BJ(\Theta)\dot{\Theta}$$

$$\Rightarrow \begin{aligned} {}^{A}V &= {}^{A}R_B\, {}^{B}V \\ {}^{A}\omega &= {}^{A}R_B\, {}^{A}\omega \end{aligned} \Rightarrow \begin{bmatrix} {}^{A}V \\ {}^{A}\omega \end{bmatrix} = \begin{bmatrix} {}^{A}R_B & 0 \\ 0 & {}^{A}R_B \end{bmatrix} \begin{bmatrix} {}^{B}V \\ {}^{B}\omega \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} {}^{A}V \\ {}^{A}\omega \end{bmatrix} = \begin{bmatrix} {}^{A}R_B & 0 \\ 0 & {}^{A}R_B \end{bmatrix} {}^{B}J(\Theta)\,\dot{\Theta}$$

So, $$\boxed{{}^{A}J(\Theta) = \begin{bmatrix} {}^{A}R_B & 0 \\ 0 & {}^{A}R_B \end{bmatrix} {}^{B}J(\Theta)}$$

★ **Singularities**

Is the Jacobian matrix invertible?

↳ If non-singular then,

$$\boxed{\dot{\Theta} = J^{-1}(\Theta)\, V}$$

⇒ Most manipulators have values of Θ where the Jacobian becomes singular.

↳ Such locations are called singularities of the mechanism

Singularity of Mechanism

- Workspace-boundary singularity
  - {Occurs when the manipulator is fully stretched out}
- Workspace-interior singularities
  - {Occurs by lining up of two or more joint axis}

Date
Page

★ Static forces in manipulator

⇒ Typically, the robot is pushing on something in the environment with the chain's force end (the end-effector) or is perhaps supporting a load at the hand.

↳ We wish to solve for the joint torques that must be acting to keep the system in static equilibrium

⇒ In considering static forces in a manipulator, we first lock all joints so that the manipulator becomes a structure.

↳ We then consider each link in this structure and write a force-moment balance relationship in terms of the link frames.

↳ Finally we compute what static torque must be acting about the joint axis in order for the manipulator to be in static equilibrium.

⇒ Here we will not be considering the forces on the links due to gravity.

Date
Page

⇒ Let,

f_i = force exerted on link i by link (i-1).

n_i = torque exerted on link i by link (i-1).

⇒ Summing the forces and setting them equal to zero, we have

$$ {}^{i}f_i - {}^{i}f_{i+1} = 0 \quad \text{——} \ (1) $$

⇒ Summing torques about the origin of frame {i} we have,

$$ {}^{i}n_i - {}^{i}n_{i+1} - {}^{i}P_{i+1} \times {}^{i}f_{i+1} = 0 \quad \text{——} \ (2) $$

⇒ The results can be written as :-

$$ {}^{i}f_i = {}^{i}f_{i+1} $$

$$ {}^{i}n_i = {}^{i}n_{i+1} + {}^{i}P_{i+1} \times {}^{i}f_{i+1} $$

$$ \boxed{{}^{i}f_i = {}^{i}R_{i+1}\,{}^{i+1}f_{i+1}} $$

$$ \boxed{{}^{i}n_i = {}^{i}R_{i+1}\,{}^{i+1}n_{i+1} + {}^{i}P_{i+1} \times {}^{i}f_{i}} $$

⇒ All components of the force & moment vectors are resisted by the structure of the mechanism itself, except for the torque about the joint axis.

Date
Page

⇒ To find the joint torque required to maintain the static equilibrium, the dot product of the joint-axis vector with the moment vector acting on the link is computed:

$$\boxed{\tau_i = {}^i n_i^T \, {}^i\hat{Z}_i}$$

⇒ In the case the joint i is prismatic, we compute the joint actuator force as

$$\boxed{\tau_i = {}^i f_i^T \, {}^i\hat{Z}_i}$$

★ Jacobians in the force domain

⇒ When forces act on a mechanism, work is done if the mechanism moves through a displacement.

⇒ The principle of virtual work allows us to make certain statements about the static case by allowing the amount of this displacement to go to an infinitesimal.

⇒ Work has the unit of energy, so it must be the same measured in any set of generalized coordinates.

⇒ We can equate the work done in Cartesian terms with the work done in joint-space terms.

Date
Page

$$F \cdot \delta X = \tau \cdot \delta\Theta$$

$$\Rightarrow F^T \delta X = \tau^T \delta\Theta$$

$$\Rightarrow F^T J \delta\Theta = \tau^T \delta\Theta \quad \{\text{as } \delta X = J\delta\Theta\}$$

$$\Rightarrow F^T J = \tau^T$$

$$\Rightarrow \boxed{\tau = J^T F}$$

⇒ When the Jacobian loses full rank, there are certain directions in which the end-effector cannot exert static force even if desired.

★ Cartesian Transformation of velocities and Static forces

$$\begin{bmatrix} {}^{B}V_B \\ {}^{B}\omega_B \end{bmatrix} = \underbrace{\begin{bmatrix} {}^{B}R_A & -{}^{B}R_A\,{}^{A}P_{BORG}\times \\ 0 & {}^{B}R_A \end{bmatrix}}_{{}^{B}T_{vA}} \begin{bmatrix} {}^{A}V_A \\ {}^{A}\omega_A \end{bmatrix}$$

$$\begin{bmatrix} {}^{A}F_A \\ {}^{A}N_A \end{bmatrix} = \underbrace{\begin{bmatrix} {}^{A}R_B & 0 \\ {}^{A}P_{BORG}\times {}^{A}R_B & {}^{A}R_B \end{bmatrix}}_{{}^{A}T_{fB}} \begin{bmatrix} {}^{B}F_B \\ {}^{B}N_B \end{bmatrix}$$

$$\boxed{{}^{A}T_{fB} = {}^{A}T_{vB}^{\,T}}$$