4B

Control System Analysis and Design by the Root Locus method

* Introduction

⇒ The basic characteristics of the transient response of a closed-loop System is closely related to the location of the closed loop poles.

→ If the System has a variable loop gain, then the location of the Closed loop poles depends on the value of the loop gain chosen.

→ So the designer must Know how the closed loop poles move in the s plane as the loop gain is varied.

If the gain adjustment alone does not yield a desired result, addition of a Compensator to the system will become necessary.

Root - Locus method

→ by W.R Evans

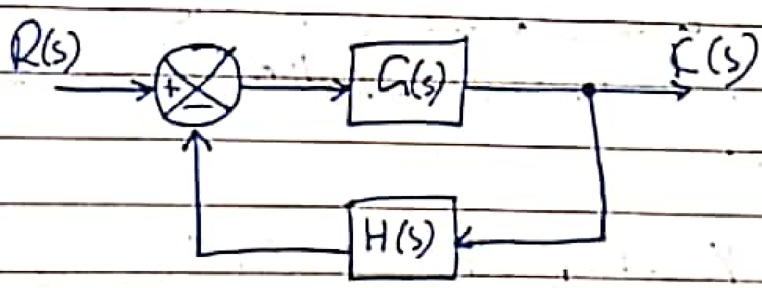
→ Roots of the characteristic equation are plotted for all values of a System Parameter.

⇒ Note: The parameter that varies is usually the gain, but any other variable of the open-loop transfer function may be used.

⇒ Root loci by use of MATLAB is very Simple, one may think sketching the root loci by hand is a waste of time and effort. However experience in Sketching the root loci by hand is invaluable for interpreting Computer generated root loci, as well as for getting a rough idea of the root loci very quickly.

★ Root Locus plot

Angle and Magnitude Condition



$$\frac{R(s)}{C(s)} = \frac{G(s)}{1+G(s)H(s)}$$

⇒ $1+G(s)H(s)$ is the Characteristic equation.

⇒ $G(s)H(s) = -1$

⇒ We assume that $G(s)H(s)$ is ratio of polynomial fns.

⇒ Since $G(s)H(s)$ is a Complex quantity, the above equation can be split into two equations by equating the angles and magnitudes on both side.

Angle Condition

$$\angle G(s)H(s) = \angle -1 = \pm(2K+1)180 \quad \{K = 0, 1, 2 - -\}$$

Magnitude Condition

$$|G(s)H(s)| = 1$$

⇒ The values of s that fulfill both the angle & magnitude conditions are the roots of the characteristic equation, or the closed loop poles.

⇒ In many cases, $G(s)H(s)$ involves a gain K, and the characteristic equation may be written as,

$$1 + \frac{K(s+Z_1)(s+Z_2)--(s+Z_m)}{(s+P_1)(s+P_2)---(s+P_n)} = 0$$

⇒ Then the root loci for the system are the loci of the closed loop poles as the gain K is varied from zero to infinity.

⇒ Note: To begin sketching the root loci of a system by the root locus method we must know the location of the poles and zeros of $G(s)H(s)$

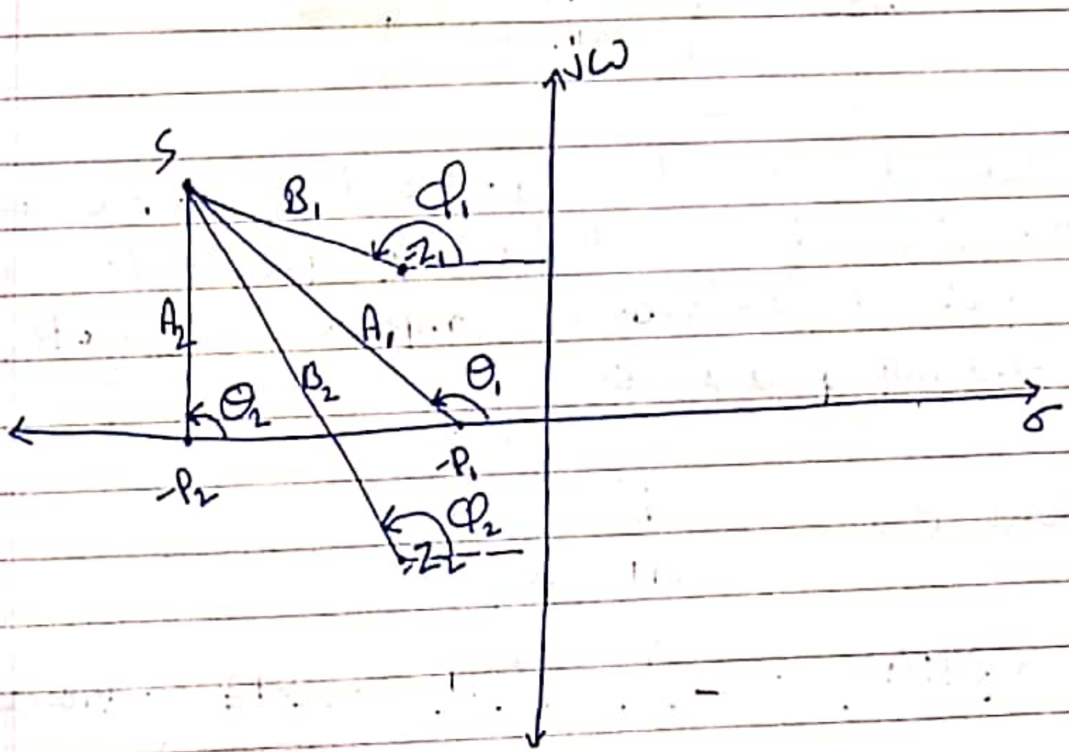
⇒ Angle of the complex quantity originating from the open loop poles and open loop zeros to the test point s are measured in the counterclockwise direction.

Let $G(s)H(s) = \dfrac{K(s+Z_1)(s+Z_2)\cdots(s+Z_m)}{(s+P_1)(s+P_2)\cdots(s+P_n)}$

$$\angle G(s)H(s) = \angle K(s+Z_1)(s+Z_2)\cdots(s+Z_m) - \angle (s+P_1)(s+P_2)\cdots(s+P_n)$$

$$= \{\angle s+Z_1 + \angle s+Z_2 + \cdots + \angle s+Z_m\} - \{\angle s+P_1 + \angle s+P_2 + \cdots \angle s+P_n\}$$

$$\Rightarrow \boxed{\angle G(s)H(s) = (\phi_1 + \phi_2 + \cdots \phi_m) - (\theta_1 + \theta_2 + \cdots \theta_n)}$$



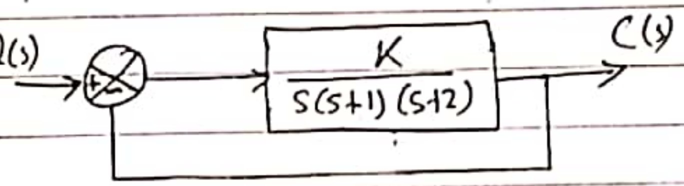
$$|G(s)H(s)| = \left| K(s+z_1)(s+z_2)\cdots(s+z_m) \right| \Big/ \left| (s+p_1)(s+p_2)\cdots(s+p_n) \right|$$

$$= \frac{K\,|s+z_1|\,|s+z_2|\cdots|s+z_m|}{|s+p_1|\,|s+p_2|\cdots|s+p_n|}$$

$$\Rightarrow \boxed{|G(s)H(s)| = \frac{K B_1 B_2 \cdots B_m}{A_1 A_2 \cdots A_n}}$$

$\Rightarrow$ Open-loop complex conjugate poles and complex conjugate zeros if any are always located symmetrically about the real axis, the root locus are always symmetrical with respect to this axis.

Example 6.1:



Sketch the root-locus plot and then determine the value of K such that the damping ratio ξ of a pair of dominant complex conjugate closed loop-poles is 0.5.

$$\Rightarrow G(s)H(s) = \frac{K}{s(s+1)(s+2)}$$

$$\angle G(s)H(s) = -\angle s - \angle s+1 - \angle s+2 = \pm 180(2k+1)$$

$$\{k = 0, 1, 2 \cdots\}$$

$$\left|\frac{K}{S(s+1)(s+2)}\right| = 1$$

// A typical procedure for sketching the root-locus plot is as follows.

1. <u>Determine the root locus on the real axis.</u>

<u>Note</u>

→ The location of open loop poles are indicated by crosses.

→ The location of open loop zeros are indicated by small circle.

⇒ Starting point of the root loci are open loop poles. (the point corresponding to ~~open loop poles~~ $K = 0$)

⇒ The number of individual root loci for this system is three, which is the same as the number of open-loop poles.

⇒ To determine the root loci on the real axis, we select a test points, s.

If the test point is on the positive real axis then,

$$\angle s = \angle s+1 = \angle s+2 = 0$$

→ Angle condition cannot be satisfied.

If test point is between $\boxed{0 \; \& \; -1}$:

$\angle s = 180 \quad \angle(s+1) = \angle s+2 = 0$

→ The angle Condition is <u>satisfied</u>.

If test point is between -1 & -2

$\angle s = \angle s+1 = 180 \; ; \; \angle s+2 = 0$

→ The angle Condition is <u>not satisfied</u>

If test point is between $\boxed{-2 \text{ and } -\infty}$

$\angle s = \angle s+1 = \angle s+2 = 180$

→ The angle Condition is <u>satisfied</u>.

2. <u>Determine the asymptotes of the root loci.</u>

The asymptotes of the root loci as s approaches infinity can be determined as follow

$$\lim_{s\to\infty} G(s)H(s) = \lim_{s\to\infty} \frac{K}{s(s+1)(s+2)} = \lim_{s\to\infty} \frac{K}{s^3}$$

<u>Angle Condition</u>: $-3\angle s = \pm 180^\circ (2K+1)$.

$$\Rightarrow \boxed{\angle s = \pm 60^\circ (2K+1) \quad \{K = 0, 1, 2 \cdots\}}$$

⇒ Since the angles repeats itself as K is varied, the distinct angles for the asymptotes are determined as 60°, −60° and 180°.

↳ Thus there are three asymptotes.

⇒ Before we can draw these asymptotes in the Complex plane, we must find the point where they intersect the real axis.

⇒ As $s \to \infty$ ⇒ $\angle s \cong \angle s+1 \cong \angle s+2$

So lets assume the asymptote to be at the avg of these three points.

$$\frac{(-2)+(-1)+0}{3} = -1$$

→ Thus the abscissa of the intersection of the asymptote and the real axis is -1.

3. <u>Determining the breakaway points</u>

⇒ Breakaway points are points where the root-locus branches originating from the poles at 0 and -1 break away from the real axis and move into the complex plane.

??

Let us write the characteristic equation as:-

$$f(s) = B(s) + KA(s) = 0$$

{A(s) & B(s) does not contain K}

$f(s)=0$ has multiple roots at point where,

$$\frac{df(s)}{ds}=0$$

"The breakaway point corresponds to a point in splane where multiple roots of the characteristic equation occur"

Suppose that $f(s)$ has multiple roots of order r, where $r \geq 2$.

Then, $f(s) = (s-s_1)^r (s-s_2) \cdots (s-s_n)$

$$\left.\frac{df(s)}{ds}\right|_{s=s_1} = 0$$

$$\frac{df(s)}{ds} = B'(s) + KA'(s) = 0$$

$$\Rightarrow K = -\frac{B'(s)}{A'(s)}$$

So, $f(s) = B(s) - \frac{B'(s)}{A'(s)} A(s) = 0$

$$\boxed{B(s)A'(s) - B'(s)A(s) = 0}$$

If equation above is solved for s, the points where multiple roots occur can be obtained.

$$K = -\frac{B(s)}{A(s)}$$

$$\frac{dK}{ds} = \frac{B'(s)A(s) - B(s)A'(s)}{A^2(s)}$$

If dK/ds is set equal to zero, we get the same equation.

⇒ Therefore, the break-away points can be simply determined from the roots of

$$\boxed{\frac{dK}{ds} = 0}$$

If at a point at which $dK/ds = 0$ the value of K takes a real positive value, then the point is an actual breakaway point.

$$1 + \frac{K}{s(s+1)(s+2)} = 0$$

$$K = -(s^3 + 3s^2 + 2s)$$

$$\frac{dK}{ds} = -(3s^2 + 6s + 2) = 0$$

$$\Rightarrow \quad s = -0.4226, \quad s = -1.5774$$

⇒ Since the breakaway point must lie on a root locus between 0 and −1, it is clear the $s = -0.4226$ corresponds to the actual breakaway point.

$$s = -0.4226 \Rightarrow K = 0.3849$$

4. <u>Determine the points where the root loci cross the imaginary axis.</u>

⇒ These point can be found out by using Routh's Stability Criterion.

$$s^3 + 3s^2 + 2s + K = 0$$

s^3	1	2
s^2	3	K
s^1	$(6-K)/3$	0
s^0	K	

⇒ The value of K that makes s^1 term in the first column zero is 6

⇒ The crossing points on the imaginary axis can then be found by solving the auxilary equation obtained from the s^2 row;

$$3s^2 + K = 3s^2 + 6 = 0$$

$$s = \pm\sqrt{2}\,j$$

So $\omega = \pm\sqrt{2}$ for $K=6$.

⇒ An alternate approach is to let $s = j\omega$ in the characteristic equation, equate both real and imaginary part to zero, and then solve for ω and K.

$$(j\omega)^3 + 3(j\omega)^2 + 2(j\omega) + K = 0$$

$$\Rightarrow \quad (K - 3\omega^2) + j(2\omega - \omega^3) = 0$$

$$K - 3\omega^2 = 0 \qquad 2\omega - \omega^3 = 0$$

So $\quad \omega = \pm\sqrt{2} \quad \forall\ K = 6$

$\omega = 0 \quad \forall\ K_1 = 0$.

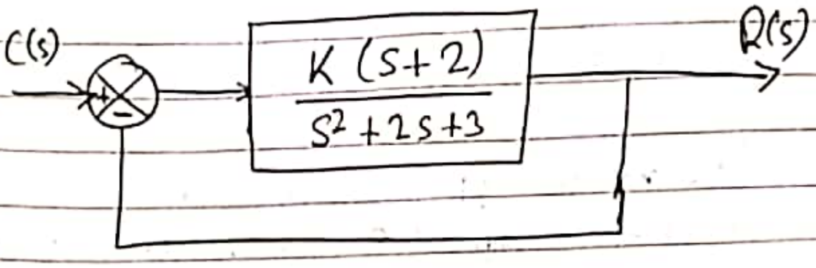
5. Choose a test point in the broad neighborhood of the $j\omega$ axis and the origin

6. Draw the root loci based on the information obtained in the foregoing steps.

7. Determine a pair of dominant complex-conjugate closed-loop poles such that the damping ratio ξ is 0.5.

⇒ Closed-loop poles with $\xi = 0.5$ lie on the lines passing through the origin and making the angle $\pm\cos^{-1}\xi = \pm\cos^{-1}0.5 = \pm 60°$ with the negative real axis.

Example 6.2:



$$G(s) = \frac{K(s+2)}{s^2+2s+3} \qquad H(s) = 1$$

$$G(s)H(s) = K\frac{s+2}{s^2+2s+3}$$

$$\boxed{1 + K\frac{s+2}{s^2+2s+3} = 0}$$ {Characteristic equation}

Poles : $s = -1 \pm j\sqrt{2}$

Zeros : $s = -2$

1. Determine the root loci on the real axis.

$\forall s \in (-1\ \infty)$

$$\phi_1 + \phi_2 = 0 \quad \& \quad \theta = 0$$

$$-(\phi_1 + \phi_2) + \theta = 0$$

hence not possible.

$\forall s \in (-2\ -1)$

$$\phi_1 + \phi_2 = 0 \quad \& \quad \phi = 0$$

$$\phi - (\theta_1 + \theta_2) = 0$$

hence not possible.

$\forall\ S \in (-\infty\ -2)$

$\theta_1 + \theta_2 = 0 \quad \phi = 180$

$\phi - (\theta_1 + \theta_2) = 180$

Hence possible.

$\lim_{s \to \infty} G(s)H(s) = \frac{K}{s}$

Angle Condition.

$\angle s - 2\angle s = \pm 180\,(2K+1)$

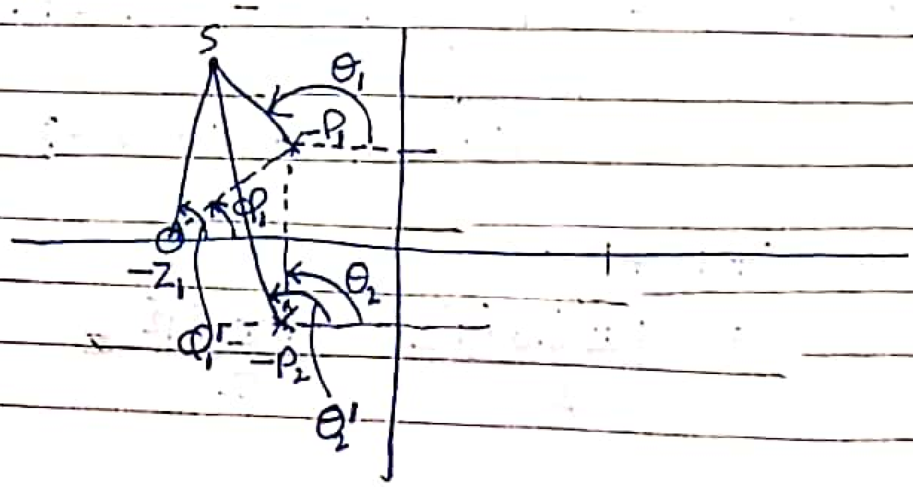
$\{K = 0, 1, 2 \cdots\}$

$\Rightarrow \angle s = \pm 180\,(2K+1)$

Asymptote at $\theta = -180°$

2. Determine the angle of departure from the Complex- conjugate open-loop poles.

$\Rightarrow$ The presence of pair of complex conjugate open loop poles requires the determination of the angle of departure from these poles.



⇒ If we choose a test point and move it in the very vicinity of the Complex open-loop pole at $s = -P_1$.

↳ We find that the sum of the angular Contributions from the pole at $S = P_2$ and zero at $S = -Z_1$ to the test point Can be Considered remaining the same

$$\phi_1' - (\theta_1 + \theta_2') = \pm 180\,(2K+1)$$

$$\theta_1 = 180 - \theta_2' + \phi_1' = 180 - \theta_2 + \phi_1$$

$$\theta_1 = 180 - \underbrace{90 + 55}_{-35} = 145^\circ.$$

⇒ Since the root locus is symmetric about the real axis, the angle of departure from the pole at $S = -P_2$ is -145°.

3. <u>Determine the break in point</u>

$$K = -\frac{s^2+2s+3}{s+2}$$

$$\frac{dK}{ds} = 0 = (s+2)(2s+2) - (s^2+2s+3)$$

$$\downarrow$$

$$2s^2+6s+4 - s^2 - 2s - 3$$

$$\Rightarrow s^2+4s+1 = 0$$

$$s = \frac{-4 \pm \sqrt{16-4}}{2} = \frac{-4 \pm \sqrt{12}}{2}$$

Scanned by CamScanner

$S = -3.732 \quad \forall \quad K = 5.4641$

★ General Rules for Constructing Root Loci

First obtain the characteristic equation

$$1 + G(s)H(s) = 0$$

Then rearrange this equation so that the parameter of interest appears as the multiplying factor in the form:

$$1 + K\frac{(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)} = 0$$

{In present discussion, we assume that the parameter of interest is the gain K, where $K>0$. (If $K<0$, which corresponds to the positive-feedback case).}

1. Locate the poles and zeros of $G(s)H(s)$ on the s plane. The root-locus branches start from open-loop poles and terminate at zeros (finite zeros or zeros at infinity).

Note that the root loci are symmetrical about the real axis of the S plane, because the complex poles and complex zeros occur only in conjugate pairs.

2. Determine the root loci on the real axis.

Root loci on the real axis are determined by open-loop poles & zeros lying on it.

The Complex-conjugate poles & zeros of the open loop transfer function have no effect on the location of the root loci on the real axis because the angle contribution of a pair of complex-conjugate poles or complex-conjugate zeros is 360° on the real axis.

3. Determine the asymptotes of root loci

If the test point s is located far from the origin, then the angle of each complex quantity may be considered same.

→ Open loop poles & zeros cancel each other effect.

$$\text{Angle of asymptotes} = \frac{\pm 180(2K+1)}{n-m} \quad (K=0,1,2\ldots)$$

n = Number of finite poles of $G(s)H(s)$

m = number of finite zeros of $G(s)H(s)$

Scanned by CamScanner

⇒ All the asymptotes intersect at a point on the real axis.

$$G(s)H(s) = \frac{K[s^m + (z_1+z_2+z_3+\cdots+z_m)s^{m-1} + \cdots + z_1z_2\cdots z_m]}{s^n + (P_1+P_2+\cdots P_n)s^{n-1} + \cdots + P_1P_2\cdots P_n}$$

⇒ If a test point is located very far from the origin, then by dividing the denominator by the numerator, it is possible to write $G(s)H(s)$ as

$$G(s)H(s) = \frac{K}{s^{n-m} + [(P_1+P_2+\cdots P_n) - (z_1+z_2+\cdots z_m)]s^{n-m-1} + \cdots}$$

$$= \frac{K}{\left[s + \frac{(P_1+P_2+\cdots P_n) - (z_1+z_2+\cdots+z_m)}{n-m}\right]^{n-m}}$$

⇒ The abscissa of the intersection of the asymptotes and the real axis is then obtained by setting the denominator of the right-hand side of above equation equal to zero and solving for s.

$$s = -\frac{(P_1+P_2+\cdots+P_n) - (z_1+z_2+\cdots+z_m)}{n-m}$$

4. Find the breakaway and break in point

⇒ Breakaway and break in points can be determined from the roots of

$$\frac{dK}{ds} = 0$$

⇒ The breakaway points and break in points must be the roots of above equation, but not all roots are breakaway or breakin point.

⇒ If the two roots $s = s_1$ & $s = -s_1$ of above equation are a complex-conjugate pair and if it is not certain whether they are on root loci, then it is necessary to check the corresponding K value.

5. Determine the angle of departure (angle of arrival) of the root locus from a complex pole (at a complex zero)

6. Find the points where the root loci may cross the imaginary axis.

7. Taking a series of test points in the broad neighborhood of the origin of the s plane sketch the root loci.

8. Determine closed loop-poles.

★ Cancellation of poles of G(s) with Zeros of H(s)

⇒ If the denominator of G(s) and the numerator of H(s) involve common factors, then the corresponding open-loop poles & zeros will cancel each other, reducing the degree of the characteristic equation by one or more.

↳ The root-locus plot of G(s)H(s) does not show all the roots of the characteristic equation, only the roots of the reduced equation.

★ Constant ξ Loci and Constant ω_n Loci

$$\xi = \cos\phi$$

lines of constant damping ratio ξ are radial lines passing through the origin.

Distance of the pole from the origin is determined by undamped natural frequency ω_n. The constant ω_n loci are circles.