# Projective Geometry

{Homogeneous Coordinates}

## Pinhole Camera Properties

- **Line-preserving:** straight lines are mapped to straight lines
- **Not length-preserving:** size of objects is inverse proportional to the distance
- **Not angle-preserving:** Angles between lines change

- Parallel lines may intersect.

## Central Questions in Photogrammetry

- Relationship between the object in the scene and the object in the image
- Relationship between points in the image and the rays to the object
- Orientation of the camera in the scene
- Inferring the geometry of an object or a scene given an image

## Information Loss Caused by the Projection of the Camera

- A camera **projects from** the **3D** world **to** a **2D** image
- This causes a **loss of information**
- 3D information can only be recovered if additional information is available
  - Multiple images
  - Details about the camera
  - Known size of objects

## Vanishing Points

- Parallel lines are not parallel anymore
- All mapped parallel lines intersect in a vanishing point
- The vanishing point is the "point at infinity" for the parallel lines
- Every direction has exactly one vanishing point

## Projective Geometry Motivation

- Cameras generate a projected image of the world
- **Euclidian geometry is suboptimal to describe the central projection**
- In Euclidian geometry, the math can get difficult
- Projective geometry is an alternative algebraic representation of geometric objects and transformations

## Homogeneous Coordinates

- H.C. are a system of coordinates used in projective geometry
- Formulas involving H.C. are often simpler than in the Cartesian world
- **Points at infinity can be represented using finite coordinates**
- **A single matrix can represent affine and projective transformations**

## Notation

**Point** $\mathcal{X}$ (or $\mathcal{Y}$ or $\mathcal{P}$)
- in homogeneous coordinates $\mathbf{x}$
- in Euclidian coordinates $\boldsymbol{x}$

**Line** $\mathcal{l}$ (or $\mathcal{m}$)
- in homogeneous coordinates $\mathbf{l}$

**Plane** $\mathcal{A}$
- in homogeneous coordinates $\mathbf{A}$

**2D** vs. **3D** space
- lowercase = 2D; capitalized = 3D

## Homogeneous Coordinates

**Definition**

The representation $\mathbf{x}$ of a geometric object is **homogeneous** if $\mathbf{x}$ and $\lambda\mathbf{x}$ represent the same object for $\lambda \neq 0$

**Example**

$$\mathbf{x} = \lambda \mathbf{x} \qquad\qquad x \neq \lambda x$$

homogeneous — Euclidian

## Homogeneous Coordinates

- H.C. use a n+1 dimensional vector to represent the same (n-dim.) point
- Example for $\mathbb{R}^2/\mathbb{P}^2$

$$x = \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \mathbf{x} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} = w \begin{bmatrix} u/w \\ v/w \\ 1 \end{bmatrix} = \begin{bmatrix} u/w \\ v/w \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

## Definition

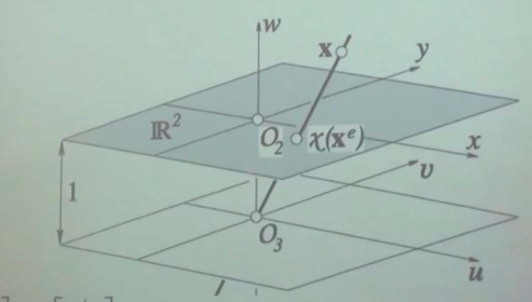
- Homogeneous Coordinates of a point $\chi$ in the plane $\mathbb{R}^2$ is a 3-dim. vector

$$\chi: \quad \mathbf{x} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} \quad \text{with } |\mathbf{x}|^2 = u^2 + v^2 + w^2 \neq 0$$

- it corresponds to Euclidian coordinates

$$\chi: \quad x = \begin{bmatrix} u/w \\ v/w \end{bmatrix} \quad \text{with } w \neq 0$$

## From Homogeneous to Euclidian Coordinates




$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} u/w \\ v/w \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} u/w \\ v/w \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

## 3D Points

Analogous for points in 3D Euclidian space $\mathbb{R}^3$

homogeneous — Euclidian

$$\mathbf{X} = \begin{bmatrix} U \\ V \\ W \\ T \end{bmatrix} = \begin{bmatrix} U/T \\ V/T \\ W/T \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} U/T \\ V/T \\ W/T \end{bmatrix}$$

## Representations of Lines

- Hesse normal form (DE: Hesse form) (angle $\phi$, distance $d$)

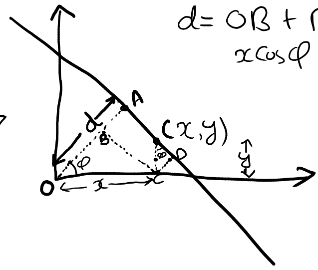
$$x\cos\phi + y\sin\phi - d = 0$$

- Intercept form (DE: Achsenabschnittsform)

$$\frac{x}{x_0} + \frac{y}{y_0} = 1 \quad \text{or} \quad \frac{x}{x_0} + \frac{y}{y_0} - 1 = 0$$

- Standard form (DE: implizite Form)

$$ax + by + c = 0$$




$d = OB + BA$

$x\cos\varphi + CD$

$\hookrightarrow y\sin\varphi$

$\boxed{x\cos\varphi + y\sin\varphi - d = 0}$

## Representations of Lines

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \text{ (point)} \qquad \mathbf{l} = \begin{bmatrix} \cos\phi \\ \sin\phi \\ -d \end{bmatrix} \text{ (Hesse)}$$

$$\mathbf{l} = \begin{bmatrix} \frac{1}{x_0} \\ \frac{1}{y_0} \\ -1 \end{bmatrix} \text{ (intercept)} \qquad \mathbf{l} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \text{ (standard)}$$

$$\boxed{\mathbf{x} \cdot \mathbf{l} = \mathbf{x}^\mathsf{T}\mathbf{l} = \mathbf{l}^\mathsf{T}\mathbf{x} = 0}$$

## Definition

- Homogeneous Coordinates of a line $\ell$ in the plane is a 3-dim. vector

$$\ell: \quad \mathbf{l} = \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix} \quad \text{with } |\mathbf{l}|^2 = l_1^2 + l_2^2 + l_3^2 \neq 0$$

- it corresponds to Euclidian representation

$$l_1 x + l_2 y + l_3 = 0$$

## Intersecting Lines

- Given two lines $\ell, m$ expressed in H.C., we look for the intersection $x = \ell \cap m$
- Find the point $x = [x, y]^T$ through the following system linear equations

$$\begin{bmatrix} \mathbf{l} \cdot \mathbf{x} \\ \mathbf{m} \cdot \mathbf{x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \boxed{x = l \times m}$$

$$\begin{bmatrix} l_1 & l_2 \\ m_1 & m_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -l_3 \\ -m_3 \end{bmatrix}.$$

## Intersecting Lines

- The intersection of two lines in H.C. is

$$\boxed{x = \ell \cap m: \quad \mathbf{x} = \mathbf{l} \times \mathbf{m}}$$

- **Simple way for computing the intersection of two lines using H.C.**

## Points at Infinity

- It is possible to **explicitly** model infinitively distant points **with finite coordinates**

$$x_\infty: \quad \mathbf{x}_\infty = \begin{bmatrix} u \\ v \\ 0 \end{bmatrix}$$

- We can **maintain the direction** to that infinitively distant point
- Great tool when working with cameras as they are bearing-only sensors

## Intersection at Infinity

- All lines $\ell$ with $\ell \cdot x_\infty = 0$ pass through $x_\infty$
- This means $[u, v] \cdot [\cos\phi, \sin\phi] = 0$
- This hold for any line $\mathbf{m} = [\cos\phi, \sin\phi, *]^T$ i.e. for any line that is parallel to $\ell$
- This can also be seen by

$$\mathbf{l} \times \mathbf{m} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \times \begin{bmatrix} a \\ b \\ d \end{bmatrix} = \begin{bmatrix} bd - bc \\ ac - ad \\ ab - ab \end{bmatrix} = \begin{bmatrix} bd - bc \\ ac - ad \\ 0 \end{bmatrix}$$

**All parallel lines meet at one point at infinity!**

## Infinitively Distant Objects

- Infinitively distant point

$$x_\infty: \quad \mathbf{x}_\infty = \begin{bmatrix} u \\ v \\ 0 \end{bmatrix}$$

- The infinitively distant line is the **ideal line**

$$\ell_\infty: \quad \mathbf{l}_\infty = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

- $\ell_\infty$ can be interpreted as the horizon

## Infinitively Distant Objects

- All points at infinity lie on the line at infinity called the **ideal line** given by

$$\mathbf{x}_\infty \cdot \mathbf{l}_\infty = \begin{bmatrix} u \\ v \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0$$

- The ideal line can be seen as the horizon

## Analogous for 3D Objects

- 3D point

$$\mathbf{X} = \begin{bmatrix} U \\ V \\ W \\ T \end{bmatrix} = \begin{bmatrix} U/T \\ V/T \\ W/T \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} U/T \\ V/T \\ W/T \end{bmatrix}$$

- Plane

$$\mathbf{A} = \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix}$$

## Point on a Plane

- Via the scalar product, we can again test if a point lies on a plane

$$\mathbf{A} \cdot \mathbf{X} = \mathbf{A}^{\mathsf{T}}\mathbf{X} = \mathbf{X}^{\mathsf{T}}\mathbf{A} = 0$$

- which is based on

$$AX + BY + CZ + D = 0 \quad \text{or} \quad N \cdot X - S = 0$$

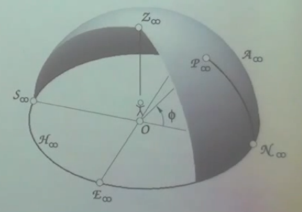
## 3D Objects at Infinity

- 3D point

$$\mathbf{P}_\infty = \begin{bmatrix} U \\ V \\ W \\ 0 \end{bmatrix}$$

- Plane

$$\mathbf{A}_\infty = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$



## Transformations

- A projective transformation is an invertible linear mapping

$$\mathbf{X}' = \mathrm{H}\mathbf{X}$$

## Important 3D Transformations

- General projective mapping

$$\mathbf{X}' = \mathrm{H}\mathbf{X}$$

- Translation: 3 parameters
(3 translations)

$$\mathrm{H} = \lambda \begin{bmatrix} I & t \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix} \qquad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad t = \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix} \quad \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

homogeneous property



## Important 3D Transformations

- Rotation: 3 parameters
(3 rotation)

$$\mathrm{H} = \lambda \begin{bmatrix} R & \mathbf{0} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix}$$

- Rigid body transformation: 6 params
(3 translation + 3 rotation)

$$\mathrm{H} = \lambda \begin{bmatrix} R & t \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix}$$

## Important 3D Transformations

- Similarity transformation: 7 params
(3 trans + 3 rot + 1 scale)

$$\mathrm{H} = \lambda \begin{bmatrix} mR & t \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix}$$

(angle-preserving)

- Affine transformation: 12 parameters
(3 trans + 3 rot + 3 scale + 3 sheer)

$$\mathrm{H} = \lambda \begin{bmatrix} A & t \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix}$$

(not angle-preserving but parallel lines remain parallel)

## Important 3D Transformations

- Projective transformation: 15 params.

$$\mathrm{H} = \lambda \begin{bmatrix} A & t \\ a^{\mathsf{T}} & 1 \end{bmatrix}$$

affine transformation + 3 parameters

- These 3 parameters are the projective part and they are the reason that **parallel lines may not stay parallel**

## Transformations Hierarchy

projective transformation (8/15 parameters)
↓
affine transformation (6/12 parameters)
↓
similarity transformation (4/7 parameters)
↓
rigid body/motion transformation (3/6 parameters)
↓
translation (2/3 parameters) | rotation (1/3 parameters)

## Inverting and Chaining

- Inverting a transformation

$$\mathbf{X}' = \mathrm{H}\mathbf{X}$$
$$\mathbf{X} = \mathrm{H}^{-1}\mathbf{X}'$$

- Chaining transformations via matrix products (not commutative)

$$\mathbf{X}' = \mathrm{H}_1\mathrm{H}_2\mathbf{X} \neq \mathrm{H}_2\mathrm{H}_1\mathbf{X}$$