2

Control System Analysis in State Space

★ State-Space Representations of Transfer-Function System

(1) State-Space Representations in Canonical Forms

⇒ Consider a system defined by:

$$\overset{n}{y} + a_1 \overset{n-1}{y} + \cdots a_{n-1}\dot{y} + a_n y = b_0 \overset{n}{u} + b_1 \overset{n-1}{u} + \cdots + b_{n-1}\dot{u} + b_n u$$

where u is the input and y is the output.

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}$$

Controllable Canonical Form

(a)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 \cdots & 1 \\ -a_n & -a_{n-1} & \cdots & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \left[b_n - a_n b_0 \,\vdots\, b_{n-1} - a_{n-1} b_0 \,\vdots\, \cdots \,\vdots\, b_1 - a_1 b_0 \right] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0 u$$

⇒ The Controllable Canonical form is important in discussing the pole-placement approach to control system design.

Observable Canonical form

b

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_n - a_n b_0 \\ b_{n-1} - a_{n-1} b_0 \\ \vdots \\ b_1 - a_1 b_0 \end{bmatrix} u$$

$$y = [0\ 0 \cdots 0\ 1] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0 u$$

Diagonal Canonical Form

c

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1} s + b_n}{(s+P_1)(s+P_2)\cdots(s+P_n)}$$

All distinct roots

$$= b_0 + \frac{C_1}{s+P_1} + \frac{C_2}{s+P_2} + \cdots \frac{C_n}{s+P_n}$$

$\Rightarrow$ The diagonal canonical form of the state space representation of this system is given by :-

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -P_1 & & & 0 \\ & -P_2 & & \\ & & \ddots & \\ 0 & & & -P_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0 u$$

Jordan Canonical Form

u

⇒ Case where denominator polynomial contains multiple roots.

Suppose,

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1} s + b_n}{(s+p_1)^3 + (s+p_4) + (s+p_5) \cdots (s+p_n)}$$

$$= b_0 + \frac{c_1}{(s+p_1)^3} + \frac{c_2}{(s+p_1)^2} + \frac{c_3}{s+p_1} + \frac{c_4}{s+p_4} + \cdots \frac{c_n}{s+p_n}$$

⇒ State-space representation in the Jordan Canonical form is given by:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \left[\begin{array}{ccc|ccc} -p_1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & -p_1 & 1 & & & \\ 0 & 0 & -p_1 & 0 & \cdots & 0 \\ \hline 0 & \cdots & 0 & -p_4 & & 0 \\ \vdots & & \vdots & & \ddots & \\ 0 & \cdots & 0 & 0 & & -p_n \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + b_0 u$$

★ Eigenvalues of an n×n Matrix A

⇒ The eigenvalues of an n×n matrix A are the roots of the characteristic equation

$$|\lambda I - A| = 0$$

★ Diagonalization of n×n Matrix

⇒ If n×n matrix A with distinct eigenvalues is given by

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & \cdots & \cdots & -a_1 \end{bmatrix}$$

the transformation $X = PZ$, where

$$P = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \cdots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \cdots & \lambda_n^2 \\ \vdots & \vdots & \vdots & & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \cdots & \lambda_n^{n-1} \end{bmatrix}$$

{where $\lambda_1, \lambda_2 \cdots \lambda_n$ = n distinct Eigenvalues of A}

will transform $P^{-1}AP$ into the diagonal matrix.

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

⇒ If the matrix A involves multiple eigenvalues, then diagonalization is impossible.

⇒ Eigenvalues of A and those of $P^{-1}AP$ are identical. We shall prove this for a general case in what follows. (1)

★ Invariance of Eigen Values

To prove the invariance of the eigenvalues under a linear transformation, we must show that the Characteristic polynomials $|\lambda I - A|$ and $|\lambda I - P^{-1}AP|$ are identical.

$$\begin{aligned}
|\lambda I - P^{-1}AP| &= |\lambda P^{-1}P - P^{-1}AP| \\
&= |P^{-1}(\lambda I - A)P| \\
&= |P^{-1}||\lambda I - A||P| \\
&= |P^{-1}||P||\lambda I - A| \\
&= |P^{-1}P||\lambda I - A| \\
&= |\lambda I - A|
\end{aligned}$$

★ Nonuniqueness of Set of State Variables

Let $x_1, x_2 \cdots x_n$ are a set of State Variables.

⇒ Then we may take as another set of State Variables any set of functions.

$$\begin{aligned}
\hat{x}_1 &= X_1(x_1, x_2 \cdots x_n) \\
\hat{x}_2 &= X_2(x_1, x_2 \cdots x_n) \\
&\vdots \\
\hat{x}_n &= X_n(x_1, x_2 \cdots x_n)
\end{aligned}$$

Provided that, for every set of values $\hat{x}_1, \hat{x}_2 \cdots \hat{x}_n$, there corresponds to a unique set of values $x_1, x_2 \cdots x_n$ and Vice-Versa

$\Rightarrow$ Thus, if x is a state vector, then $\hat{x}$, where

$$\boxed{\hat{x} = Px}$$

is also a state vector, provided the matrix P is nonsingular.

$\Rightarrow$ Different state vectors convey the same information about the system behavior.

★ Solving the time-invariant State Equation

Solving Homogeneous State Equations

$\Rightarrow$ Before we solve vector-matrix differential equations, let us review the solution of the scalar differential equation.

$$\dot{x} = ax$$

$\Rightarrow$ In solving this equation, we may assume a solution $x(t)$ of the form.

$$x(t) = b_0 + b_1 t + b_2 t^2 + \cdots b_K t^K + \cdots$$

$\Rightarrow$ By substituting this assumed solution we obtain :-

$$b_1 + 2b_2 t + 3b_3 t^2 + \cdots + K b_K t^{K-1} + \cdots$$
$$= a(b_0 + b_1 t + b_2 t^2 + \cdots b_K t^K + \cdots)$$

$\Rightarrow$ If the assumed solution is to be true solution, the above equation must hold for any t.

$$b_1 = a b_0$$
$$b_2 = \frac{1}{2} a b_1 = \frac{1}{2} a^2 b_0$$

$$b_3 = \frac{1}{3} a b_2 = \frac{1}{6} a^3 b_0$$

$$\vdots \quad \vdots \quad \vdots$$

$$b_k = \frac{1}{k!} a^k b_0$$

$\Rightarrow$ The value of b_0 is determined by substituting $t=0$ in above equation.

$$b_0 = x(0)$$

$$\Rightarrow x(t) = \left(1 + at + \frac{1}{2!} a^2 t^2 + \cdots + \frac{1}{k!} a^k t^k + \cdots\right) x(0)$$

$$\Rightarrow \boxed{x(t) = e^{at} x(0)}$$

※ We shall now solve the vector-matrix differential equation.

$$\boxed{\dot{X} = AX}$$

X = n vector
A = n×n vector

$\Rightarrow$ By analogy with the scalar case, we assume that the solution is in the form of a vector power series in t.

(Vectors)

$$x(t) = b_0 + b_1 t + b_2 t^2 + \cdots + b_k t^k + \cdots$$

$\Rightarrow$ By substituting this assumed solution into the vector differential equation we get:-

$$b_1 + 2b_2 t + 3b_3 t^2 + \cdots + k b_k t^{k-1} + \cdots$$
$$= A(b_0 + b_1 t + b_2 t^2 + \cdots + b_k t^k + \cdots)$$

$\Rightarrow$ By equating the like powers of t on both side, we obtain,

$$b_1 = A b_0$$
$$b_2 = \frac{1}{2} A b_1 = \frac{1}{2} A^2 b_0$$
$$\vdots \quad \vdots \quad \vdots$$

$$b_K = \frac{1}{K!} A^K b_0$$

So $X(t) = \underbrace{\left(I + At + \frac{1}{2!}A^2t^2 + \cdots + \frac{1}{K!}A^Kt^K + \cdots\right)}_{e^{At}\ \{\text{Matrix Exponential}\}} X(0)$

$$\Rightarrow \boxed{X(t) = e^{At} X(0)}$$

Matrix Exponential

2

$$\boxed{e^{At} = \sum_{K=0}^{\infty} \frac{A^K t^K}{K!}}$$

{At Converges for all finite t.}

$$\frac{de^{At}}{dt} = A + A^2t + \frac{A^3t^2}{2!} + \cdots \frac{A^K t^{K-1}}{(K-1)!} + \cdots$$

$$= A\left[I + At + \frac{A^2t^2}{2!} + \cdots \frac{A^{K-1}t^{K-1}}{(K-1)!} + \cdots\right]$$

$$\boxed{\frac{de^{At}}{dt} = Ae^{At} = e^{At}A}$$

$$\boxed{e^{A(t+s)} = e^{At}e^{As}}$$

Proof

$$e^{At}e^{As} = \left(\sum_{K=0}^{\infty} \frac{A^K t^K}{K!}\right)\left(\sum_{K=0}^{\infty} \frac{A^K s^K}{K!}\right)$$

$$= \sum_{K=0}^{\infty} A^K \left(\sum_{i=0}^{\infty} \frac{t^i s^{K-i}}{i!\,(K-i)!}\right) = \sum_{K=0}^{\infty} A^K \frac{(t+s)^K}{K!}$$

$$= e^{A(t+s)}$$

$\Rightarrow$ In particular if $s=-t$ then

$$e^{At}e^{-At} = e^{-At}e^{At} = e^{A(t-t)} = I$$

$\Rightarrow$ Thus the inverse of e^{At} is e^{-At}. $\Rightarrow \boxed{(e^{At})^{-1} = e^{-At}}$

$\Rightarrow$ Since the inverse of e^{At} always exists, e^{At} is non singular.

$$\Rightarrow \boxed{e^{(A+B)t} = e^{At}e^{Bt}} \quad \{\text{if } AB = BA\}$$

$$\Rightarrow \boxed{e^{(A+B)t} \neq e^{At}e^{Bt}} \quad \{\text{if } AB \neq BA\}$$

Laplace Transform approach to the Solution of Homogeneous State Equation

$$\dot{x}(t) = Ax(t)$$

$$sX(s) - x(0) = AX(s) \quad \{X(s) = \mathcal{L}[x(t)]\}$$

$$X(s) = (sI-A)^{-1}x(0)$$

$$x(t) = \mathcal{L}^{-1}\{(sI-A)^{-1}x(0)\}$$

$$\Rightarrow \boxed{(sI-A)^{-1} = \frac{I}{s} + \frac{A}{s^2} + \frac{A^2}{s^3} + \cdots}$$

$$\Rightarrow \mathcal{L}^{-1}\{(sI-A)^{-1}\} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots$$

$$= e^{At} \Rightarrow \boxed{x(t) = e^{At}x(0)}$$

★ State-Transition matrix

We can write solution of the homogeneous state equation

$$\dot{x} = Ax$$

as $\boxed{x(t) = \phi(t)x(0)}$ {State transition matrix}

Where $\phi(t)$ is an $n\times n$ matrix and is the unique solution of

$$\boxed{\dot{\phi}(t) = A\,\phi(t)} \qquad \phi(0) = I$$

$$x(0) = \phi(0)\,x(0) = x(0)$$

$$\phi(t) = e^{At} = \mathcal{L}^{-1}\{(sI - A)^{-1}\}$$

$$\boxed{\phi^{-1}(t) = e^{-At} = \phi(-t)}$$

⇒ $\phi(t)$ is called the state

⇒ If the eigenvalues $\lambda_1, \lambda_2 \ldots \lambda_n$ of the matrix A are distinct, then $\phi(t)$ will contain the n exponentials.

$$e^{\lambda_1 t}, e^{\lambda_2 t} \ldots e^{\lambda_n t}$$

⇒ In particular, if the matrix A is diagonal then,

$$\phi(t) = e^{At} = \begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix}$$

⇒ If there is a multiplicity in the eigenvalues then $\phi(t)$ will contain, in addition to the exponentials $e^{\lambda_1 t}, e^{\lambda_2 t} \ldots e^{\lambda_n t}$, terms like $te^{\lambda_1 t}$ and $t^2 e^{\lambda_1 t}$.

★ Properties of State-Transition Matrices

1. $\phi(0) = e^{A0} = I$
2. $\phi(t) = e^{At} = (e^{-At})^{-1} = [\phi(-t)]^{-1}$ or $\boxed{\phi^{-1}(t) = \phi(-t)}$
3. $\phi(t_1 + t_2) = \phi(t_1)\phi(t_2) = \phi(t_2)\phi(t_1)$
4. $[\phi(t)]^n = \phi(nt)$
5. $\phi(t_2 - t_1)\phi(t_1 - t_0) = \phi(t_2 - t_0) = \phi(t_1 - t_0)\phi(t_2 - t_1)$

Example 9-5: Obtain the state-transition matrix $\phi(t)$ of the following system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\Rightarrow$ $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

$$\boxed{\phi(t) = e^{At} = \mathcal{L}^{-1}[(SI-A)^{-1}]}$$

$$SI - A = \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} S & -1 \\ 2 & S+3 \end{bmatrix}$$

$$(SI-A)^{-1} = \cancel{\frac{A}{(S+2)(S+1)}} \frac{1}{|SI-A|} \text{Adj}(SI-A)$$

$$= \frac{1}{S(S+3)+2} \begin{bmatrix} S+3 & -2 \\ +1 & S \end{bmatrix}^T$$

$$= \frac{1}{(S+1)(S+2)} \begin{bmatrix} S+3 & 1 \\ -2 & S \end{bmatrix}$$

$$= \begin{bmatrix} \frac{(S+3)}{(S+1)(S+2)} & \frac{1}{(S+1)(S+2)} \\ \frac{-2}{(S+1)(S+2)} & \frac{S}{(S+1)(S+2)} \end{bmatrix}$$

$$\varphi(t) = \mathcal{L}^{-1}\left[(sI-A)^{-1}\right]$$

$$= \begin{bmatrix} 2e^{-t}-e^{-2t} & e^{-t}-e^{-2t} \\ -2e^{-t}+2e^{-2t} & -e^{-t}+2e^{-2t} \end{bmatrix}$$

★ **Solving Nonhomogeneous State Equation**

We shall begin by Considering the Scalar Case:-

$$\dot{x} = ax + bu$$

$$\dot{x} - ax = bu$$

⇒ Multiplying both side by e^{-at}, we obtain

$$e^{-at}\left[\dot{x}(t) - ax(t)\right] = \frac{d}{dt}\left[e^{-at}x(t)\right] = e^{-at}bu(t)$$

⇒ Integrating this equation between 0 and t gives

$$e^{-at}x(t) - x(0) = \int_0^t e^{-at}bu(t)\,dt$$

$$\boxed{x(t) = e^{at}x(0) + e^{at}\int_0^t e^{-a\tau}bu(\tau)\,d\tau}$$

Response to the Initial Condition

Response to the input $u(t)$

⇒ Let us Consider the nonhomogeneous state equation described by

$$\boxed{\dot{X} = AX + Bu}$$

Where, X = n-Vector
u = r-Vector
A = n×n Constant matrix
B = n×r Constant matrix

$$\dot{x}(t) - Ax(t) = Bu(t)$$

$$\Rightarrow \quad e^{-At}\left[\dot{x}(t) - Ax(t)\right] = \frac{d}{dt}\left[e^{-At}x(t)\right] = e^{-At}Bu(t)$$

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)} Bu(\tau)\, d\tau$$

$$\Rightarrow \boxed{x(t) = \phi(t)x(0) + \int_0^t \phi(t-\tau)\, Bu(\tau)\, d\tau}$$

★ Laplace Transform Approach to the Solution of Nonhomogeneous State Equation

$$\dot{x} = Ax + BU$$

$$\Rightarrow sX(s) - x(0) = AX(s) + BU(s)$$

$$\Rightarrow (sI - A)X(s) = x(0) + B(U)(s)$$

$$X(s) = (sI-A)^{-1}x(0) + (sI-A)^{-1}BU(s)$$

$$X(s) = \mathcal{L}[e^{At}]x(0) + \mathcal{L}[e^{At}]\, BU(s)$$

$$x(t) = e^{At}x(0) + \underbrace{\int_0^t e^{A(t-\tau)} Bu(\tau)\, d\tau}_{\text{Convolution Integral}}$$

Solution in Terms of $x(t_0)$

⇒ Thus far we have assumed that initial time to be zero. If however, the initial time is given by t_0 instead of 0 then :-

$$\boxed{x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)} Bu(\tau)\, d\tau}$$

★ Some useful Results in Vector matrix Analysis

Cayley - Hamilton Theorem

Consider an $n \times n$ matrix A and its characteristic equation:

$$|\lambda I - A| = \lambda^n + a_1 \lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0$$

The Cayley-Hamilton theorem states that the matrix A satisfies its own characteristic equation

$$A^n + a_1 A^{n-1} + \cdots + a_{n-1} A + a_n I = 0$$

⇒ To prove this theorem note that $\mathrm{adj}(\lambda I - A)$ is a Polynomial in λ of degree $n-1$

$$\mathrm{adj}(\lambda I - A) = B_1 \lambda^{n-1} + B_2 \lambda^{n-2} + \cdots + B_{n-1}\lambda + B_n$$

where $B_1 = I$ since,

$$(\lambda I - A)\,\mathrm{adj}(\lambda I - A) = [\mathrm{adj}(\lambda I - A)](\lambda I - A) = |\lambda I - A|\, I$$

we obtain

$$|\lambda I - A| I = I\lambda^n + a_1 I \lambda^{n-1} + \cdots + a_n I \lambda + a_n I$$

$$= (\lambda I - A)(B_1 \lambda^{n-1} + B_2 \lambda^{n-2} + \cdots B_{n-1}\lambda + B_n)$$

$$= (B_1 \lambda^{n-1} + B_2 \lambda^{n-2} + \cdots + B_{n-1}\lambda + B_n)(\lambda I - A)$$

⇒ If A is substituted for λ in this last equation, then clearly $\lambda I - A$ become zero.

we obtain

$$\boxed{A^n + a_1 A^{n-1} + \cdots + a_{n-1} A + a_n I = 0}$$

Minimal polynomial

3

"The least degree polynomial having A as a root is called the minimal polynomial"

⇒ Minimal polynomial of an $n \times n$ matrix A is defined as the polynomial $\phi(\lambda)$ of least degree

$$\phi(\lambda) = \lambda^m + a_1 \lambda^{m-1} \cdots + a_{m-1}\lambda + a_m \quad \{m \leq n\}$$

Such that $\phi(A) = 0$

⇒ The minimal polynomial plays an important role in the computation of polynomials in an $n \times n$ matrix.

⇒ Let us suppose that $d(\lambda)$, a polynomial in λ, is the greatest common divisor of all the elements of $adj(\lambda I - A)$.

↳ We can show that if the coefficient of the highest-degree term in λ of $d(\lambda)$ is chosen as 1, then the minimal polynomial $\phi(\lambda)$ is given by

$$\phi(\lambda) = \frac{|\lambda I - A|}{d(\lambda)}$$

Minimal polynomial $\phi(\lambda)$ of an $n \times n$ matrix A can be determined by the following procedure:-

1. Form $adj(\lambda I - A)$ and write the elements of $adj(\lambda I - A)$ as factored polynomial in λ.

2. Determine $d(\lambda)$ as the greatest common divisor of all the elements of $adj(\lambda I - A)$. Choose the coefficient of the highest degree term in λ of $d(\lambda)$ to be 1. If there is no common divisor, $d(\lambda) = 1$.

3. The minimal polynomial:-

$$\boxed{\phi(\lambda) = \frac{|\lambda I - A|}{d(\lambda)}}$$

★ Matrix Exponential e^{At}

Matlab provides a simple way to compute e^{AT}, where T is a constant.

Computation of e^{At}: Method 1

If matrix A can be transformed into a diagonal form, then e^{At} can be given by

$$e^{At} = Pe^{Dt}P^{-1} = P\begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix} P^{-1}$$

where P is a diagonalizing matrix for A.

⇒ If matrix A can be transformed into a Jordan canonical form, then e^{At} can be given by

$$e^{At} = Se^{Jt}S^{-1}$$

{S = transformation matrix that transforms matrix A into a Jordan Canonical form J}

Computation of e^{At}: Method 2

⇒ The second method of computing e^{At} uses the Laplace transform approach.

$$e^{At} = \mathcal{L}^{-1}\left[(sI - A)^{-1}\right]$$

Computation of e^{At}: Method 3

⇒ This method is based on Sylvester's interpolation method.

Case 1: Minimal polynomial of A Involves Only Distinct Roots

(4)

⇒ We shall assume that the degree of the minimal Polynomial of A is m.

⇒ By using Sylvester's interpolation formula, it can be shown that e^{At} can be obtained by solving the following determinant equation.

$$\begin{vmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{m-1} & e^{\lambda_1 t} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{m-1} & e^{\lambda_2 t} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & \lambda_m & \lambda_m^2 & \cdots & \lambda_m^{m-1} & e^{\lambda_m t} \\ I & A & A^2 & \cdots & A^{m-1} & e^{At} \end{vmatrix} = 0$$

$\left\{ \lambda_1, \lambda_2 \cdots \lambda_m \right.$ are roots of minimal Polynomial $\left. \right\}$

⇒ Solving above equation for e^{At} is the same as:

$$\boxed{e^{At} = \alpha_0(t) I + \alpha_1(t) A + \alpha_2(t) A^2 + \cdots + \alpha_{m-1} A^{m-1}}$$

where $\alpha_K(t)$ can be determined by solving the following set of m equations.

$$\alpha_0(t) + \alpha_1(t)\lambda_1 + \alpha_2(t)\lambda_1^2 + \cdots + \alpha_{m-1}(t)\lambda_1^{m-1} = e^{\lambda_1 t}$$

$$\alpha_0(t) + \alpha_1(t)\lambda_2 + \alpha_2(t)\lambda_2^2 + \cdots + \alpha_{m-1}(t)\lambda_2^{m-1} = e^{\lambda_2 t}$$

$$\vdots$$

$$\alpha_0(t) + \alpha_1(t)\lambda_m + \alpha_2(t)\lambda_m^2 + \cdots + \alpha_{m-1}(t)\lambda_m^{m-1} = e^{\lambda_m t}$$

Case 2: Minimal Polynomial of A Involves Multiple Roots

⇒ As an example, consider the case where the minimal polynomial of A involves three equal roots ($\lambda_1 = \lambda_2 = \lambda_3$) and has other roots ($\lambda_4, \lambda_5 \cdots \lambda_m$) that are all distinct.

⇒ By applying Sylvester's interpolation formula it can be shown that e^{At} can be obtained from the following determinant equation.

$$\begin{vmatrix} 0 & 0 & 1 & 3\lambda_1 & \cdots & \frac{(m-1)(m-2)}{2}\lambda_1^{m-3} & \frac{t^2}{2}e^{\lambda_1 t} \\ 0 & 1 & 2\lambda_1 & 3\lambda_1^2 & \cdots & (m-1)\lambda_1^{m-2} & te^{\lambda_1 t} \\ 1 & \lambda_1 & \lambda_1^2 & \lambda_1^3 & \cdots & \lambda_1^{m-1} & e^{\lambda_1 t} \\ 1 & \lambda_4 & \lambda_4^2 & \lambda_4^3 & \cdots & \lambda_4^{m-1} & e^{\lambda_4 t} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & \lambda_m & \lambda_m^2 & \lambda_m^3 & \cdots & \lambda_m^{m-1} & e^{\lambda_m t} \\ I & A & A^2 & A^3 & \cdots & A^{m-1} & e^{At} \end{vmatrix} = 0$$

⇒ Note that if the minimal polynomial of A is not found, it is possible to substitute the characteristic polynomial for the minimal polynomial.

★ Linear Independence of Vector

⇒ The vectors $X_1, X_2 \cdots X_m$ are said to be linearly independent if

$$C_1X_1 + C_2X_2 + C_3X_3 + \cdots C_mX_m = 0$$

where $C_1, C_2 \cdots C_m$ are constant, implise that

$$C_1 = C_2 = C_3 = \cdots = C_m = 0$$

⇒ The Vectors $x_1, x_2 \cdots x_n$ are said to be linearly dependent if and only if x_i can be expressed as a linear combination of x_j $(j = 1, 2 \cdots n,\ j \neq i)$

$$x_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} c_j x_j \quad \left\{ \text{for some set of Constants } c_j \right\}$$

★ Controllability

Controllable ⇒ A System is said to be Controllable at time t_0 if it is possible by means of an Unconstrained Control Vector to transfer the System from any initial State $x(t_0)$ to any other state in a finite interval of time.

Observable ⇒ A System is said to be observable at time t_0 if, with the System in state $x(t_0)$, it is Possible to determine this State from the observation of the output over a finite time interval.

⇒ The Concept of Controllability and Observability were introduced by Kalman.

Complete State Controllability of Continuous-Time System

⇒ Consider the Continuous-time System

$$\boxed{\dot{x} = Ax + Bu}$$

where, x = State Vector (n-Vector)
u = Control Signal (scalar)
A = $n \times n$ matrix
B = $n \times 1$ matrix

⇒ The system described above is said to be state controllable at $t = t_0$ if it is possible to construct an unconstrained control signal that will transfer an initial state to any final state in a finite time interval $t_0 \leq t \leq t_1$.

↳ If every state is controllable, then the system is said to be completely state controllable.

⇒ Without any loss of generality, we can assume that the final state is the origin of the state space and that the initial time is zero.

⇒ The solution of above equation is :-

$$X(t) = e^{At} x(0) + \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau$$

$$X(0) = -\int_0^{t_1} e^{-A\tau} Bu(\tau) d\tau$$

$$e^{-A\tau} = \sum_{K=0}^{n-1} \alpha_K(\tau) A^K$$

So $$X(0) = -\sum_{K=0}^{n-1} A^K B \int_0^{t_1} \alpha_K(\tau) u(\tau) d\tau$$

Let us put $$\beta_K = \int_0^{t_1} \alpha_K(\tau) u(\tau) d\tau$$

So $$X(0) = -\sum_{K=0}^{n-1} A^K B \beta_K$$

$$= -\left[B \,\vdots\, AB \,\vdots\, \cdots \,\vdots\, A^{n-1}B\right] \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \vdots \\ \beta_{n-1} \end{bmatrix}$$

$\Rightarrow$ If the System is Completely State Controllable, then given any initial State $x(0)$ above equation must be satisfied.

$\hookrightarrow$ This requires that the rank of the $n \times nr$ matrix

$$[B : AB : \cdots : A^{n-1}B] \text{ be } n.$$

// Condition for State Controllability :-

The System given by

$$\dot{x} = Ax + Bu$$

is Completely State Controllable if and only if the vectors $B, AB \cdots A^{n-1}B$ are linearly independent, or the $n \times n$ matrix

$$[B : AB : \cdots : A^{n-1}B] \text{ is of rank } n.$$

// The result just obtained can be extended to the case where the Control Vector u is r-dimensional.

If the System is described by

$$\dot{x} = Ax + BU$$

then it can be proved that the Condition for Complete State Controllability is that the $n \times nr$ matrix

$$[B : AB : \cdots : A^{n-1}B]$$

be of rank n, or Contain n linear independent Column Vectors.

// The matrix

$$[B : AB : \cdots : A^{n-1}B]$$

is commonly called the Controllability matrix.

Alternative Form of the Condition for Complete State Controllability

Consider a System defined by

$$\dot{X} = AX + BU$$

Where, X = State Vector (n-Vector)
U = Control Vector (r-Vector)
A = $n \times n$ matrix
B = $n \times r$ matrix

$\Rightarrow$ If the ~~eigenvalue~~ eigenvector of A are distinct, then it is Possible to find a transformed matrix P Such that

$$P^{-1}AP = D = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

Note: If the eigenvalues of A are distinct, then the eigenvectors of A are distinct; however Converse is not true.

Let us define

$$X = PZ$$

$$\Rightarrow (\dot{PZ}) = A(PZ) + BU$$

$$\Rightarrow \dot{Z} = P^{-1}APZ + P^{-1}BU \quad \text{—} (1)$$

Let $P^{-1}B = F = (f_{ij})$

We can rewrite Eq ① as :-

$$\dot{z}_1 = \lambda_1 z_1 + f_{11}u_1 + f_{12}u_2 + \cdots + f_{1r}u_r$$
$$\dot{z}_2 = \lambda_2 z_2 + f_{21}u_1 + f_{22}u_2 + \cdots + f_{2r}u_r$$
$$\vdots$$
$$\dot{z}_n = \lambda_n z_n + f_{n1}u_1 + f_{n2}u_2 + \cdots f_{nr}u_r$$

⇒ If the elements of any one row of the $n\times r$ matrix F are all zero, then the corresponding state variable cannot be controlled by any of the u_i.

"The condition of complete state controllability is that if the eigenvectors of A are distinct, then the system is completely state controllable if and only if no row of $P^{-1}B$ has all zero element"

⇒ If the A matrix does not possess distinct eigenvectors, then diagonalization is impossible.
↳ In such case we may transform A into a Jordan canonical form.

⇒ Suppose that we can find a transformation matrix S such that

$$S^{-1}AS = J$$

If we define new state vector z by

$$X = SZ$$

So, $\dot{Z} = S^{-1}ASZ + S^{-1}BU$

$\Rightarrow \dot{Z} = JZ + S^{-1}BU$

"The System is Completely State Controllable if and only if :-

1) No two Jordan blocks in J are associated with the same eigen values.

2) The elements of any row of $S^{-1}B$ that Correspond to the last row of each Jordan block are not all zero.

3) The elements of row of $S^{-1}B$ that Corresponds to distinct eigenvalues are not all zero.

Condition for Complete State Controllability in the S plane

"Necessary and Sufficient Condition for Complete state Controllability is that no Cancellation occur in the transfer function or transfer matrix. If Cancellation occure, the System Cannot be Controlled in the direction of the Canceled mode"

Output Controllability

5

$\Rightarrow$ In the practical design of a Control System, we may want to Control the output rather than the State of the System.

⇒ For this reason, it is desirable to define separately complete output controllability.

⇒ Consider the system defined by

$$\dot{X} = AX + BU$$
$$Y = CX + DU$$

Where, X = State Vector (n-Vector)
U = Control Vector (r-Vector)
Y = Output Vector (m-Vector)
A = $n \times n$ matrix
B = $n \times r$ matrix
C = $m \times n$ matrix
D = $m \times r$ matrix

"The System described above is said to be completely Output Controllable if it is possible to construct an unconstrained control vector $u(t)$ that will transfer any given initial output $Y(t_0)$ to any final output $y(t_1)$ in a finite time interval $t_0 \leq t \leq t_1$."

" The System described above is completely Output Controllable if and only if the $m \times (n+1)r$ matrix.

$$\left[CB \,\vdots\, CAB \,\vdots\, CA^2B \,\vdots\, \cdots \,\vdots\, CA^{n-1}B \,\vdots\, D\right]$$

is of rank m.

<u>Un Controllable System</u> ⇒ System which has a Subsystem that is physically disconnected from the input

Stabilizability ⇒ For a partially Controllable System, if the UnControllable modes are stable and the ~~stable~~ unstable modes are Controllable, the System is said to be Stabilizable.

★ Observability

⇒ Consider an unforced System described by the following equations:-

$$\dot{x} = Ax \quad -①$$
$$y = Cx \quad -②$$

$x \Rightarrow$ State Vector (n-Vector)
$y \Rightarrow$ Output Vector (m-Vector)
$A \Rightarrow n \times n$ matrix
$C \Rightarrow m \times n$ matrix

⇒ System is <u>Completely observable</u> if :-

"Every State $x(t_0)$ Can be determined from the observation of $y(t)$ over a finite time interval, $t_0 \leq t \leq t_1$"

⇒ In this Section we treat only linear, time-invariant System. Therefore, without loss of generality, we can assume that $t_0 = 0$.

Importance of Concept of Observability

↳ In practice, the difficulty encountered with State feedback Control is that some of the State Variables are not accessible for direct measurement, with the result it become necessary to estimate the unmeasurable State Variables in order to Construct the Control signals.

If we describe System by :-

$$\boxed{\dot{x} = Ax + Bu} \Rightarrow x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)} Bu(\tau)d\tau$$

$$\boxed{Y = Cx + Du} \Rightarrow y(t) = Ce^{At}x(0) + C\int_0^t e^{A(t-\tau)} Bu(\tau)d\tau + Du$$

Since the matrix A, B, C, & D are Known and $u(t)$ is also Known, the last two terms on the right hand side of this last equation are known quantity.

⇒ Hence, for investigating a necessary & sufficient condition for Complete observability, it suffices to consider the System described by Equation ① & ②

Complete Observability of Continuous-time System

⇒ The output Vector $y(t)$ is :-

$$y(t) = Ce^{At}x(0)$$

(degree of characteristic polynomial)

We Know, $$e^{At} = \sum_{K=0}^{n-1} \alpha_K(t) A^K$$

So, $$y(t) = \sum_{K=0}^{n-1} \alpha_K(t)\, CA^K x(0)$$

$$y(t) = \alpha_0(t) Cx(0) + \alpha_1(t) CAx(0) + \dots + \alpha_{n-1}(t)\, CA^{n-1}x(0)$$

⇒ It Can be Shown that for Complete observability requires the rank of this matrix of $mn \times n$ to be n.

$$\begin{bmatrix} C \\ \hline CA \\ \hline \vdots \\ \hline CA^{n-1} \end{bmatrix}$$

Condition for Complete Observability

The System described by eq ① & ② is Completely Observable if & only if the $n \times nm$ matrix

$$\left[C^T \,\vdots\, A^T C^T \,\vdots\, \cdots \,\vdots\, (A^T)^{n-1} C^T \right]$$

is of rank n or has n linearly independent Column vectors. This matrix is called the <u>observability matrix</u>.

Conditions for Complete Observability in the S plane

"The necessary & sufficient Conditions for Complete observability is that no Cancellation occur in the transfer function or transfer matrix"

Alternate Form of Condition for Complete Observability

⇒ Consider the System described by equation ① & ②.

Let $P^{-1}AP = D$ $\left\{ \begin{array}{l} D \Rightarrow \text{Diagonal matrix} \\ P \Rightarrow \text{Transformation matrix} \end{array} \right\}$

Let $x = Pz$

$$\Rightarrow \quad \dot{z} = P^{-1}APz = Dz$$

$$y = CPz$$

Hence, $y(t) = CPe^{Dt}z(0)$

$$y(t) = CP \begin{bmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{bmatrix} z(0) = CP \begin{bmatrix} e^{\lambda_1 t} z_1(0) \\ e^{\lambda_2 t} z_2(0) \\ \vdots \\ e^{\lambda_n t} z_n(0) \end{bmatrix}$$

⇒ The System is Completely observable if none of the Columns of the $m \times n$ matrix CP Consists of all zero element.

⇒ If the matrix A cannot be transformed into a diagonal matrix, then by use of a Suitable transformation matrix S, we can transform A into a Jordan Canonical form.

★ **Principle of Duality**

⇒ Consider the System S_1 described by

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

⇒ And the dual System S_2 defined by

$$\dot{z} = A^T z + C^T v$$
$$n = B^T z$$

⇒ The principle of duality states that the system S_1 is Completely state Controllable (observable) if and only if system S_2 is Completely observable (state Controllable).

For System S1

State Controllability

$[B \vdots AB \vdots \cdots \vdots A^{n-1}B]$ → rank n

State observability

$[C^T \vdots A^T C^T \vdots \cdots \vdots (A^T)^{n-1} C^T]$ → Rank n

For Dual System S2

State Controllability

$[C^T \vdots A^T C^T | \cdots | (A^T)^{n-1} C^T]$ → Rank n

State observability

$[B \vdots AB | \cdots | A^n B]$ → rank n

Scanned by CamScanner