Lecture 5
Transfer Function

★ Exponential Input

⇒ Let us choose as elementary input $u(t) = e^{st}$, where $s \in \mathbb{C}$ is a Complex number.

⇒ If s is real, then u is a simple exponential.

⇒ If $s = j\omega$ is imaginary, then the elementary input must be accompanied by the "conjugate"

$$u^*(t) + u(t) = e^{j\omega t} + e^{-j\omega t} = 2\cos(\omega t)$$

↳ If s is imaginary, then $u(t) = e^{st}$ must be understood as a "half" of a sinusoidal signal.

⇒ If $s = \sigma + j\omega$ then

$$u(t) + u^*(t) = 2e^{\sigma t}\cos(\omega t)$$

↳ Input u is a half of a sinusoid with exponentially-changing amplitude.

⇒ For unit step input $u(t) = e^{st}$, set $s = 0$.

So $u(t) = 1 \quad \forall t \geq 0$

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★ Output response to elementary inputs

$\Rightarrow$ $y(t) = Ce^{At}x(0) + C\int_0^t e^{A(t-\tau)} Bu(\tau)\,d\tau + Du(t)$

$\Rightarrow$ Plug in $u(t) = e^{st}$

$\Rightarrow$ $y(t) = \quad '' \quad + C\int_0^t e^{A(t-\tau)} B e^{s\tau}\,d\tau + De^{st}$

$$C\int_0^t e^{At} e^{-A\tau} B e^{s\tau}\,d\tau$$

$$Ce^{At}\left[\int_0^t e^{(sI-A)\tau}\,d\tau\right]B$$

$\Rightarrow$ If $(sI-A)$ is invertible (i.e. s is not an eigenvalue of A), then

$$\boxed{y(t) = Ce^{At}x(0) + Ce^{At}(sI-A)^{-1}\left[e^{(sI-A)t} - I\right]B + De^{st}}$$

$$\boxed{y(t) = \underbrace{Ce^{At}\left[x(0) - (sI-A)^{-1}B\right]}_{\text{Transient response}} + \underbrace{\left[C(sI-A)^{-1}B + D\right]e^{st}}_{\text{Steady-state response}}}$$

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⇒ If the system is asymptotically stable, the transient response will converge to zero.

⇒ The steady state response to an input $u(t) = e^{st}$ can be written as:

$$y_{ss} = G(s)\, e^{st} \quad , \quad G(s) = C(sI - A)^{-1}B + D$$

$$, \quad G(s) \in \mathbb{C}$$

⇒ The function $G: s \to G(s)$ is known as the transfer function.

★ <u>Frequency response</u>

⇒ Consider the case in which the input is a sinusoidal signal $(s = \pm j\omega)$

$$u(t) = e^{j\omega t} + e^{-j\omega t} = 2\cos(\omega t)$$

⇒ The output is

$$y(t) = G(j\omega)\, e^{j\omega t} + G(-j\omega)\, e^{-j\omega t}$$

$$= M e^{j\phi} e^{j\omega t} + M e^{-j\phi} e^{-j\omega t}$$

$$= 2M \cos(\omega t + \phi)$$

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⇒ The output is another sinusoid of frequency ω, the magnitude of which is $M = |G(j\omega)|$ times the magnitude of the input, and with phase leading the input by $\angle G(j\omega)$.

⇒ Using linearity and the solution of $u(t) = e^{st}$ we can find the solution to more complex input.

★ From State-space to Transfer Function (SISO)

$$\boxed{g(s) = C(sI - A)^{-1}B + d}$$

⇒ If A is diagonal, with eigenvalues $\lambda_1, \lambda_2 \dots \lambda_n$ this simply becomes:

$$\boxed{g(s) = \frac{c_1 b_1}{s - \lambda_1} + \dots + \frac{c_n b_n}{s - \lambda_n} + d}$$

⇒ In general, the transfer function is a rational function of the form

$$g(s) = \frac{b_{n-1} s^{n-1} + b_{n-2} s^{n-2} + \dots + b_0}{s^n + a_{n-1} s^{n-1} + \dots + a_0} + d$$

★ From Transfer function to State Space (SISO)

⇒ Given a transfer function $g(s)$, there are many state-space models (A, B, C, D) such that $g(s) = C(sI - A)^{-1}B + D$.

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⇒ If the transfer function is written as a partial fraction expansion of the form

$$g(s) = \frac{P_1}{s-\lambda_1} + \frac{P_2}{s-\lambda_2} + \cdots + \frac{P_n}{s-\lambda_n} + d$$

then a realization is,

$$A = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \quad B = \begin{bmatrix} \sqrt{P_1} \\ \sqrt{P_2} \\ \vdots \\ \sqrt{P_n} \end{bmatrix}$$

$$C = [\sqrt{P_1} \; \sqrt{P_2} \; \cdots \; \sqrt{P_n}] \quad D = d$$

⇒ In the general case,

$$g(s) = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0} + d$$

You can verify that the following is a minimal realization of $g(s)$

$$A \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 1 \\ -a_0 & -a_1 & \cdots & \cdots & -a_{n-1} \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$C = [b_0 \; b_1 \; \cdots \; b_{n-1}] \quad D = [d]$$

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★ The Laplace Transform

⇒ Can all input $u(t)$ be expressed as a (infinite) sum of complex exponentials?

↳ Yes! The tool for this is the Laplace transform and the inverse Laplace transform.

$$\boxed{\mathcal{L}[u] = U(s) = \int_0^{\infty} u(t)\, e^{-st}\, dt}$$

⇒ The inverse laplace transform is denoted

$$\mathcal{L}^{-1}[U] = u$$

⇒ So,

$$\boxed{u(t) = \frac{1}{2\pi j} \lim_{\omega \to \infty} \int_{\sigma - j\omega}^{\sigma + j\omega} U(s)\, e^{st}\, ds}$$

⇒ So using the above expression we can convert any arbitrary input $u(t)$ as an (infinite) sum of complex exponentials.

⇒ So we can write output as:

$$y(t) = \frac{1}{2\pi j} \lim_{\omega \to \infty} \int_{\sigma - j\omega}^{\sigma + j\omega} g(s)\, U(s)\, e^{st}\, ds$$

$$\Rightarrow \quad y(t) = \mathcal{L}^{-1}[g(s)U(s)](t)$$

$$\Rightarrow \quad \mathcal{L}[y(t)] = g(s)U(s)$$

$$\Rightarrow \quad \boxed{Y(s) = g(s)U(s)}$$