Discrete-time Linear Quadratic Optimal Control

$$\boxed{\begin{aligned} X(K+1) &= AX(K) + BU(K) \\ Y(K) &= CX(K) + DU(K) \end{aligned}}$$

Performance Index

$$J = \frac{1}{2} X^T(N) S X(N) + \frac{1}{2} \sum_{K=K_0}^{N-1} X^T(K) \overset{C^TC}{Q} X(K) + U^T(K) R U(K)$$

Optimal Control Law

$$U(K) = -\left[R + B^T P(K+1) B\right]^{-1} B^T P(K+1) A X(K)$$

$$J_0^o = \frac{1}{2} X^T(0) P(0) X(0)$$

$$\boxed{P(K) = A^T P(K+1) A + Q - A^T P(K+1) B\left[R + B^T P(K+1) B\right]^{-1} B^T P(K+1) A}$$

$$\boxed{P(N) = S}$$

{Riccati Equation}

⇒ Steady state LQ assumptions:

- → (A, B) Controllable
- → (A, C) Observable

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★ Form finite-horizon to infinite-horizon LQ

⇒ In the case of $N \to \infty$ it turns out that

→ (A,B) is controllable ⇒ guaranteed convergence of $P(K)$ to a bounded P_s.

★ Infinite-horizon discrete-time LQ for LTI System

⇒ System dynamics:

$$X(K+1) = AX(K) + Bu(K) \quad , X(K_0) = X_0$$

⇒ Performance index:

$$J = \frac{1}{2} \sum_{K=K_0}^{\infty} \left\{ X^T(K) Q X(K) + u^T(K) R u(K) \right\} \quad Q \geq 0 \; R > 0$$

⇒ Optimal state-feedback control law:

$$u^o(K) = -\underbrace{\left[R + B^T P_s B\right]^{-1} B^T P_s A}_{K_s} X(K)$$

⇒ Algebraic Riccati equation:

$$P_s = A^T P_s A + Q - A^T P_s B \left[R + B^T P_s B\right]^{-1} B^T P_s A$$

- (A,B) is controllable ⇒ P_s & K_s bounded
- (A,C) is observable ⇒ $P_s > 0$ with $u^o(K) = -K_s X(s)$ & $Q = C^T C$

(A, B) Controllable & (A, C) observable $(Q = C^TC)$

$$\Downarrow$$

Guaranteed closed-loop asymptotic stability for

$$x(k+1) = (A - BK_s)x(k) = A_{cl}x(k)$$

$$\Rightarrow x_0^T P_s x_0 = \sum_{k=0}^{\infty} \{ x^T(k) Q x(k) + u^T(k) R u(k) \}$$

$u(k) = -K_s x(k)$

$Q = C^TC$

$R = R^{1/2^T} R^{1/2}$

$$x^T(k) C^T C x(k) + x^T(k) K_s^T R K_s x(k)$$

$$x^T(k) \left\{ C^TC + K_s^T (R^{1/2})^T R^{1/2} K_s \right\} x(k)$$

$$x^T(k) \begin{bmatrix} C \\ R^{1/2}K_s \end{bmatrix}^T \begin{bmatrix} C \\ R^{1/2}K_s \end{bmatrix} x(k)$$

$A_{cl}^k x_0$

$$x_0^T (A_{cl}^k)^T \begin{bmatrix} C \\ R^{1/2}K_s \end{bmatrix}^T \begin{bmatrix} C \\ R^{1/2}K_s \end{bmatrix} A_{cl}^k x_0$$

$$x_0^T P_s x_0 = x_0^T \underbrace{\left(\sum_{K=0}^{\infty} (A_{cl}^K)^T \begin{bmatrix} C \\ R^{1/2} K_s \end{bmatrix}^T \begin{bmatrix} C \\ R^{1/2} K_s \end{bmatrix} A_{cl}^K \right)}_{W_{cl}} x_0$$

$$\Rightarrow P_s = W_c$$

$\Rightarrow$ W_{cl} is the observability gramian for

$$x(K+1) = A_{cL}\, x(K)$$
$$\tilde{y}(K) = \begin{bmatrix} C \\ R^{1/2} K_s \end{bmatrix} x(K)$$

$\Rightarrow$ If the original system is observable, the new system is also observable.

$$\Rightarrow P_s = W_c > 0$$

$\Rightarrow$ Closed-loop stability of

$$x(K+1) = A_{cl}\, x(K)$$
$$\tilde{y}(K) = \begin{bmatrix} C \\ R^{1/2} K_s \end{bmatrix} x(K)$$

$$P_s = A^T P_s A + Q - \underline{A^T P_s B [R + B^T P_s B]^{-1} B^T P_s A}$$

$$\downarrow$$

$$\underbrace{A^T P_s B [R + B^T P_s B]^{-1}}_{K_s^T} [R + B^T P_s B] \underbrace{[R + B^T P_s B]^{-1} B^T P_s A}_{K_s}$$

$$\downarrow$$

$$K_s^T [R + B^T P_s B] K_s$$

$$P_s = (A-BK_s)^T P_s (A-BK_s) + 2A^T P_s BK_s - K_s^T B^T P_s BK_s + C^T C$$
$$- K_s^T (R + B^T P_s B) K_s$$

$$P_s = (A-BK_s)^T P_s (A-BK_s) + \underbrace{C^T C + K_s^T R K_s}_{\geq 0}$$

$$C^T C + K_s^T R K_s = \begin{bmatrix} C \\ R^{1/2} K_s \end{bmatrix}^T \begin{bmatrix} C \\ R^{1/2} K_s \end{bmatrix} \geq 0$$

⇒ So the System is stable in the Sense of Lyapunov using Lyapunov equation.

★ <u>Theorem</u> (An Extension of Lyapunov theory based on observability)

"If we find from a Lyapunov equation $A^T P A - P = -Q$ where $P > 0$, $Q = C^T C \geq 0$ and (A, C) is observable, then the System $x(k+1) = Ax(k)$ is asymptotically stable."
