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CHAPTER 1

Introduction to Discrete-Time Control System

Introduction to Discrete-Time Control System

1.1) Introduction

"The Current trend toward digital rather than analog control of dynamic system is mainly due to the availability of low-cost digital Computers and the advantage found in working with digital Signals rather than Continuous-time signal"

⇒ The process of representing a variable by a set of distinct values is called quantization, and the resulting values are called quantized value.

Analog Signal ⇒ An analog Signal is a Signal defined over a continuous range of time whose amplitude can assume a Continuous range of values.

Discrete time Signal ⇒ A discrete time signal is a signal defined only at discrete instants of time.

Continuous time Signal ⇒ A Signal defined over a Continuous range of time. ; Analog Signal

Note: The amplitude may assume a continuous range of values or may assume a finite number of distinct value.

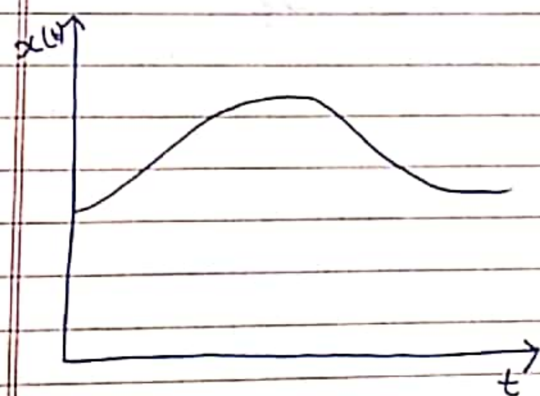
Continuous time quantized Signal

<u>Sampled data Signal</u> ⇒ In discrete time Signal, if the amplitude can assume a continuous range of values, then the signal is called a Sampled-data signal.

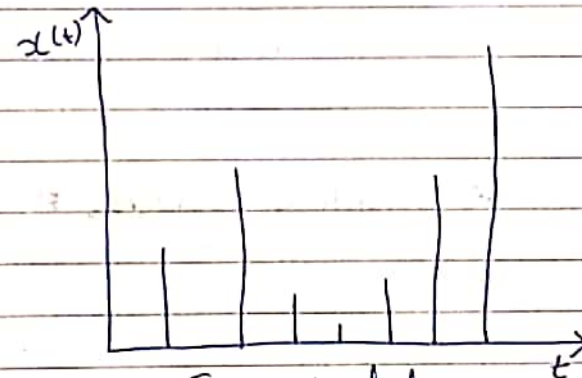
{Signals in numerically Coded form}

<u>digital signal</u> ⇒ discrete time signal with quantized amplitude.

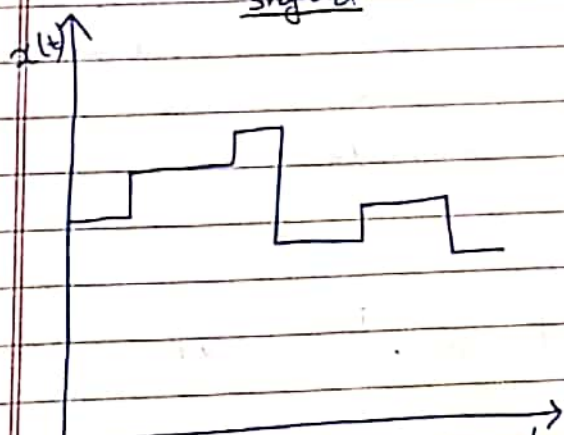
<u>Note</u>: The use of the digital Controller requires quantization of signals both in amplitude and in time.



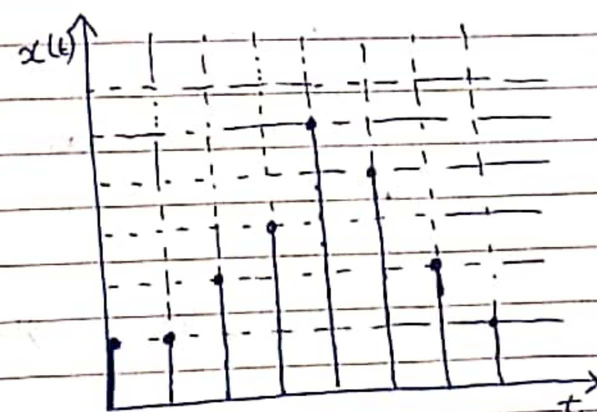
Continuous time Analog Signal



Sampled-data Signal



Continuous time quantized Signal



digital Signal

⇒ A time-invariant *System* is one in which the coefficients in the differential equation or difference equation do not vary with time.

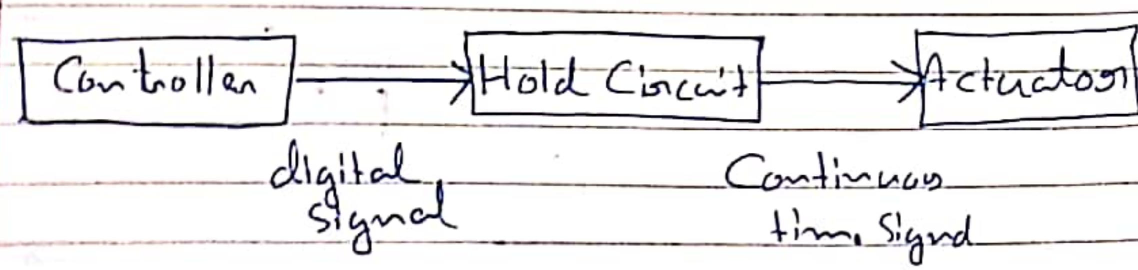
Discrete Time Control System ⇒ Control system in which one or more variables can change at discrete instants of time.

↳ These instants, which we shall denote by KT or t_K $(K=0,1,2,\cdots)$

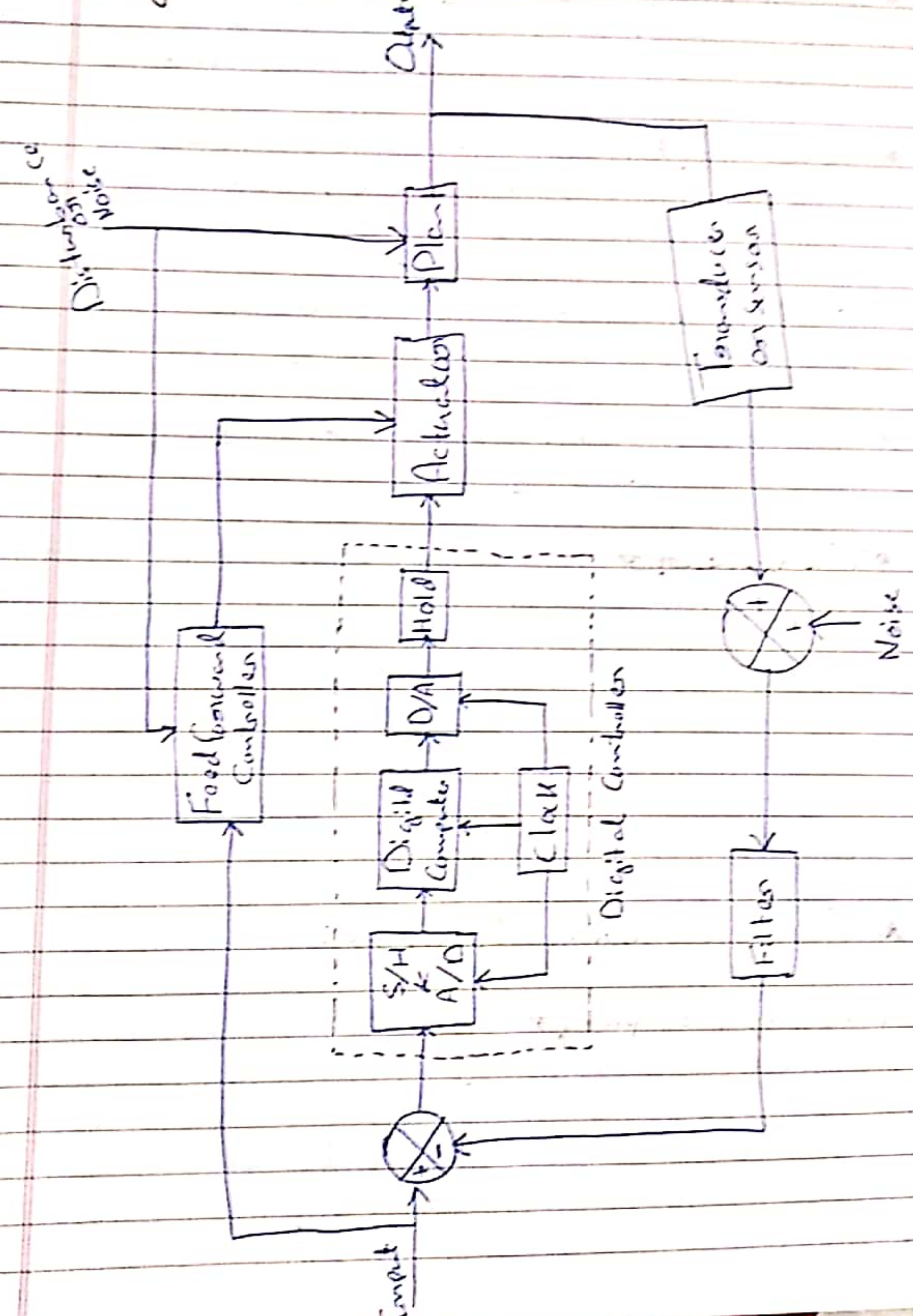
↳ The time interval between two discrete instants is taken to be sufficiently short that the data for the time between them can be approximated by simple interpolation.

Sampling process ⇒ The Sampling of a Continuous-time signal replaces the original continuous-time signal by a sequence of values at discrete time points.

⇒ The Sampling process is usually followed by a quantization process. In the quantization process the sampled analog amplitude is replaced by a digital amplitude.



1.2) Digital Control System



Sampling or discretization ⇒ Operation that transforms continuous-time signal to discrete time data is called discretization or sampling.

data hold ⇒ Operation that transforms discrete-time data in a continuous time signal is called data hold.

⇒ Analog to digital (A/D) conversion process is called Coding or encoding.

⇒ Digital to analog (D/A) conversion process is called decoding.

Transducer ⇒ It is a device that converts an input signal into an output signal of another form.

eg ⇒ Pressure signal into voltage output

- Analog Transducer
- Sampled-data transducer
- digital transducer

★ Type of Sampling Operation :-

1) Periodic Sampling ⇒ The sampling instants are equally spaced or $t_k = KT$ $(K = 0, 1, 2 \ldots)$

→ It is most conventional type.

2) Multiple-order Sampling ⇒ The pattern of the t_k's is repeated periodically; that is $t_{k+r} - t_k$ is Constant $\forall k$.

3) Multiple-rate Sampling ⇒ In a control system having multiple loops, the largest time constant involved in one loop may be quite different from that in other loops.

⇒ It may be advisable to sample slowly in a loop involving a large time constant, while in a loop involving only small time constants, the sampling rate must be fast.

⇒ Thus, a digital control system may have different sampling periods in different feedback paths or may have multiple sampling rate.

4) Random Sampling ⇒ The sampling instants are random or t_k is a random variable.

1.3) Quantizing & Quantization error

⇒ The main functions involved in analog to digital conversion are sampling, amplitude quantizing and coding.

⇒ The sampling period and quantizing levels affect the performance of digital control system. So they must be determined carefully.

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Quantization level (Q) ⇒ It is defined as the range between two adjacent decision points and is given by:-

$$Q = \frac{FSD}{2^n}$$

$\left\{ \begin{array}{l} FSD = \text{Full scale deflection} \\ n = \text{number of bits} \end{array} \right.$

Quantization error ⇒ Digital output can assume only a finite number of levels, so any A/D Conversion involve Quantization error.

- Quantization error varies between 0 & $\pm \frac{1}{2}Q$.
- The uncertainity present in the quantization process is called quantization noise.
- The quantization error e(t) is the difference between the input signal and the quantized output:-

$$e(t) = x(t) - y(t)$$

⇒ For a small quantization level Q, the nature of the quantization is similar to that of random noise. An, in effect, the quantization process acts as a source of random noise.

⇒ Suppose that the quantization level Q is small and we assume that the quantization error e(t) is distributed uniformly between $-\frac{1}{2}Q$ and $\frac{1}{2}Q$ and that this error acts as a white noise.

⇒ Then the variance σ^2 of the quantization noise is

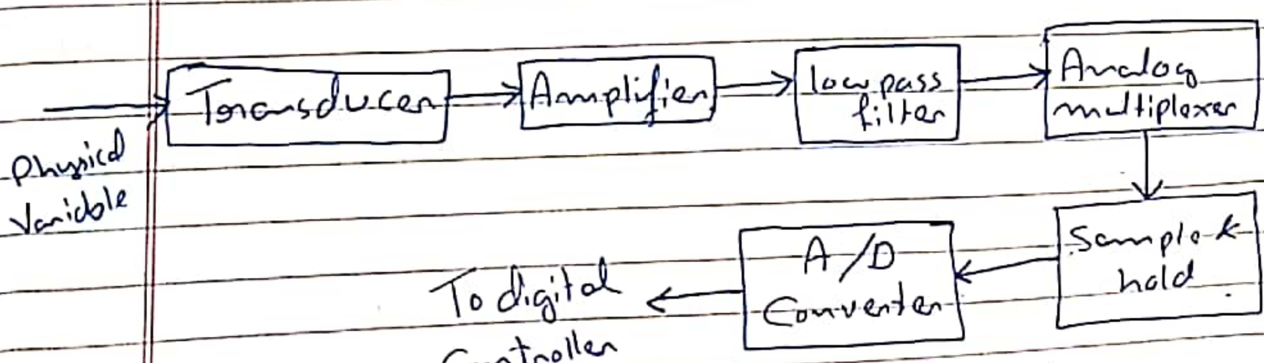
$$\sigma^2 = E\left[e(t) - \overline{e(t)}\right]^2 = \frac{1}{Q}\int_{-Q/2}^{Q/2} \varepsilon^2 \, d\varepsilon = \frac{Q^2}{12}$$

{Assuming Uniform Probability distribution}

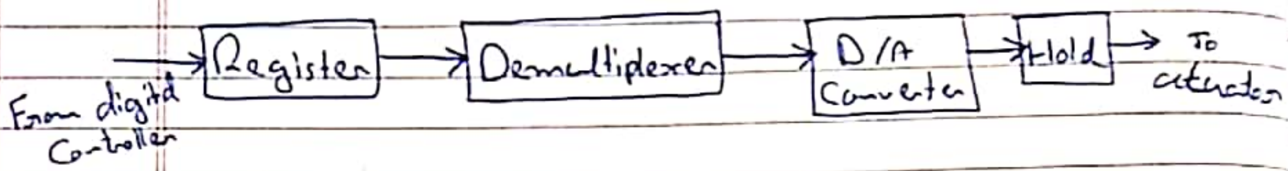
1.4) Data Acquisition, Conversion, and Distribution System

⇒ The signal conversion that takes place in the digital control system involves the following operations:-

1. Multiplexing and demultiplexing
2. Sample and hold
3. Analog to digital conversion (quantizing & encoding)
4. Digital to analog conversion (decoding)



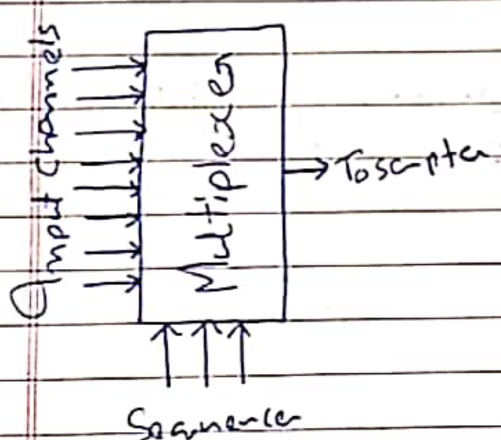
Block diagram of a data-acquisition System

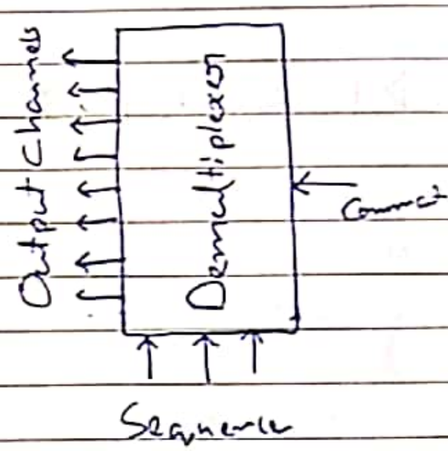


Block diagram of data distribution

The low pass filter that follows the amplifier attenuates the high-frequency signal components such as noise signals.

★ Analog Multiplexer ★ Demultiplexer



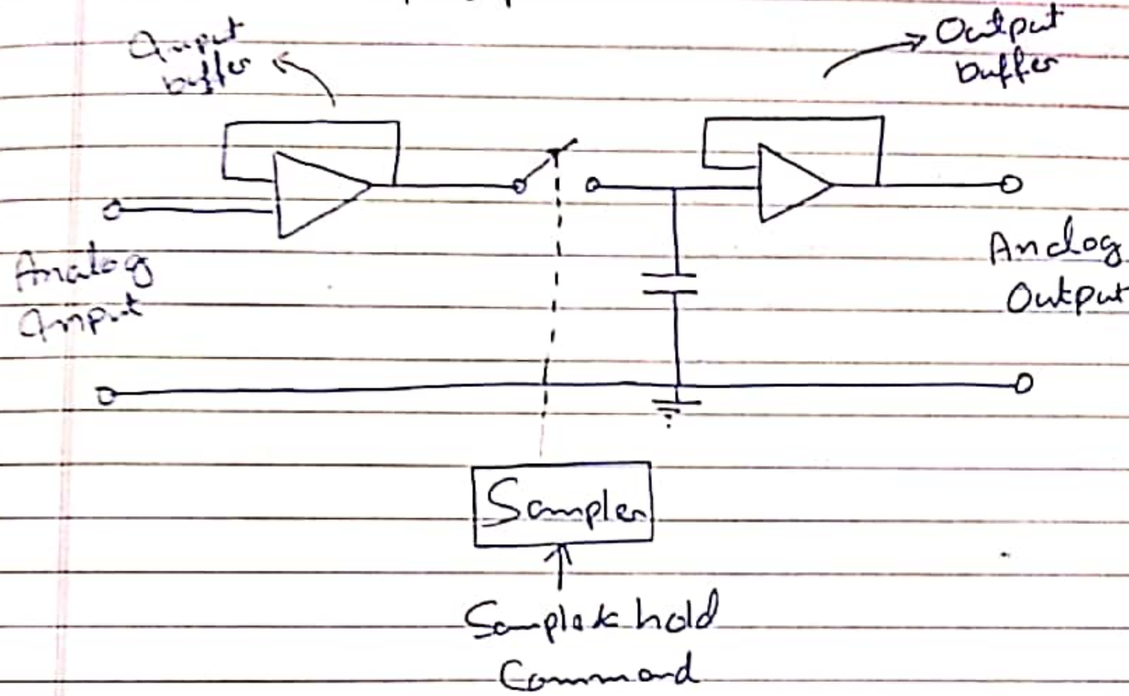


★ Sample and Hold Circuit

A Sampler in a digital System converts an analog Signal into a train of amplitude-modulated pulses. The hold circuit holds the value of the Sampled pulse signal over a Specified period of time.

⇒ Commercially, sample & hold circuits are available in a single unit, Known as a Sample and hold (S/H).

⇒ In practice, sampling duration is very short compared with the sampling period T.



→ Input voltage is acquired and then stored on a high quality capacitor with low leakage and low dielectric absorption characteristics.

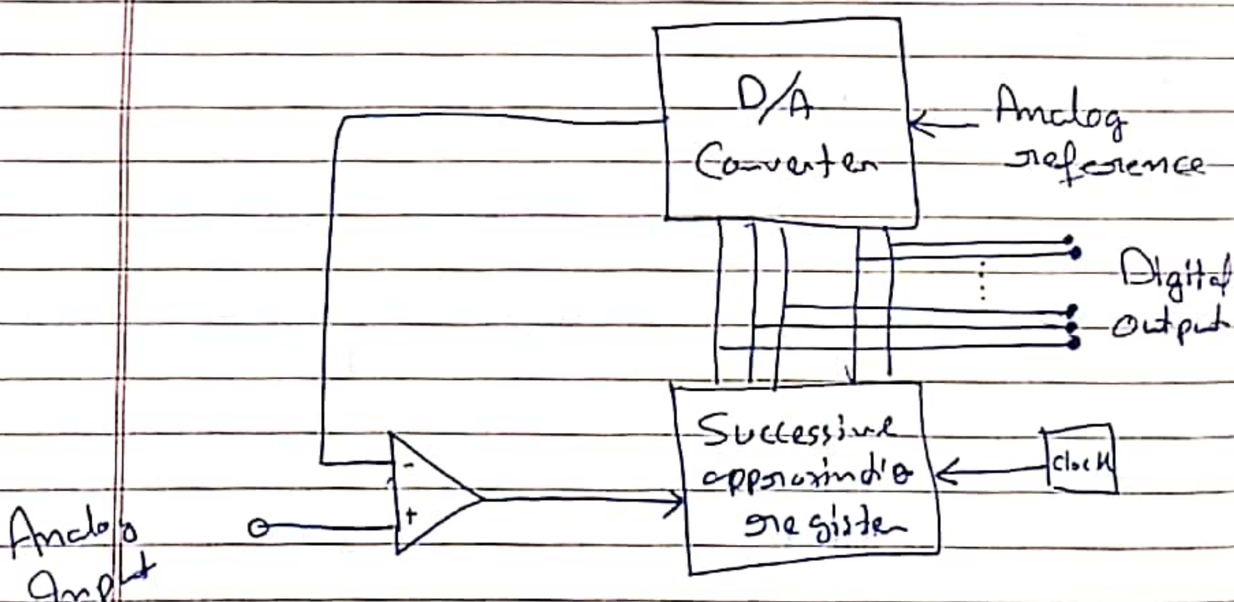
→ The time interval during which the switching takes place is called aperture time.

→ Input & Output buffer must have high impedance.

★ Types of Analog to Digital (A/D) Converters

- → Successive-approximation type ← Fast & most frequently used
- → Integrating type
- → Counter type ← Simplest
- → Parallel type

⇒ In any particular application, the conversion speed, accuracy, size and cost are the main factor to be considered in choosing the type of A/D converter.

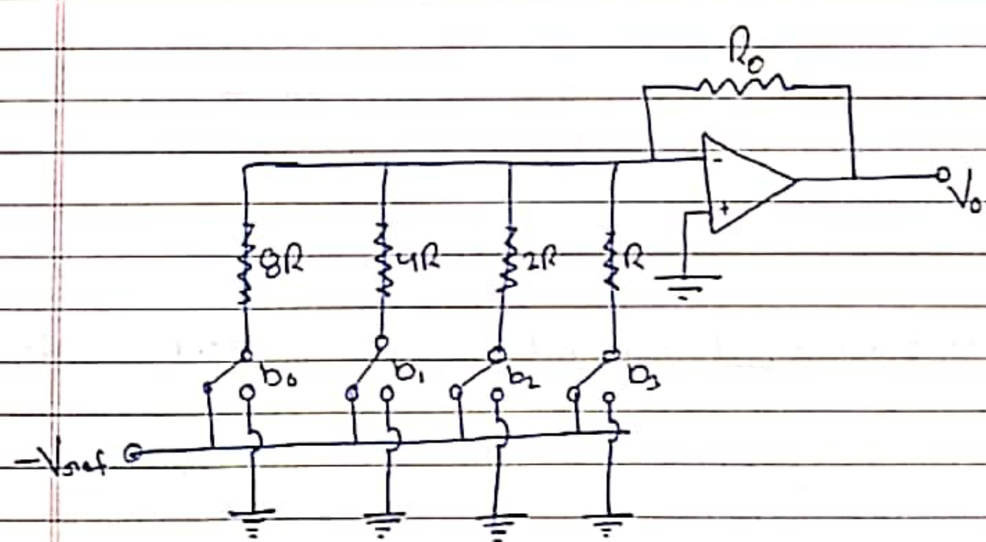


Successive-approximation type A/D Converter

* Digital to analog Converter

⇒ Two method are commonly used for digital to analog Conversion :-

- → Weighted resistor. → [Simple but accuracy is not good]
- → R-2R Ladder network. → [Little more Complicated but is more accurate]



D/A Converter using Weighted resistors

⇒ b_0, b_1, b_2 & b_3 Can either be zero or one.

$$\boxed{V_0 = \frac{R_0}{R}\left(b_3 + \frac{b_2}{2} + \frac{b_1}{4} + \frac{b_0}{8}\right)V_{ref}}$$

⇒ as the number of bits is increased the range of resistance values becomes large and Consequently the accuracy becomes poor.

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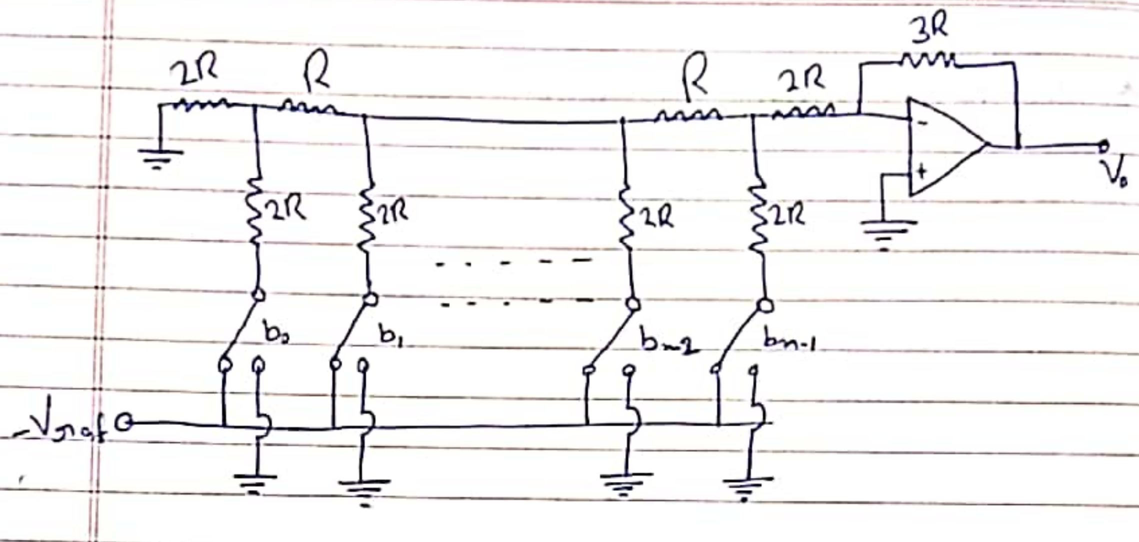


n-Bit D/A Converter using an R-2R ladder

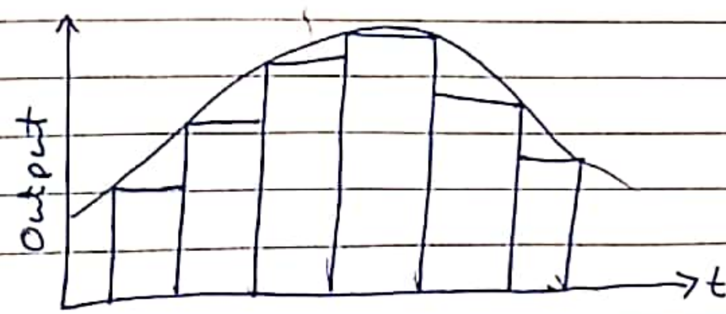
⇒ All resistance involved are either R or 2R. This means that a high level of accuracy can be achieved.

$$V_o = \frac{1}{2}\left(b_{n-1} + \frac{1}{2}b_{n-2} + \text{—} \frac{1}{2^{n-1}}b_0\right)V_{ref}$$

★ **Reconstructing the input Signal by Hold Circuits**

⇒ The Sampling operation produces an amplitude -modulated Pulse signal. The function of hold ~~signal~~ Circuit is to reconstruct the analog signal by filling the Space between.

Zero-order hold ⇒ The hold circuit that produces such a staircase waveform is called a zero order hold.

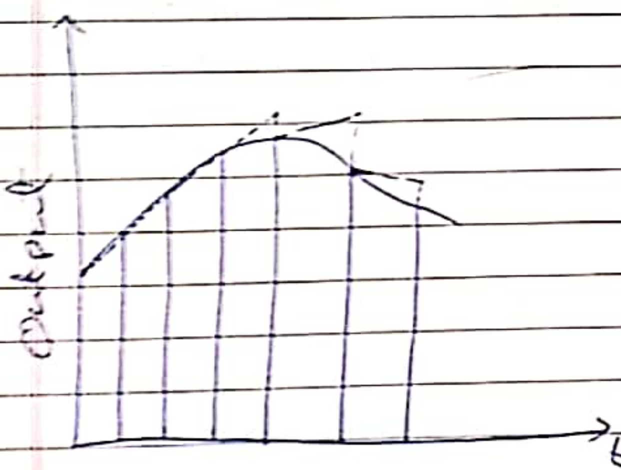


Higher Order hold Circuit

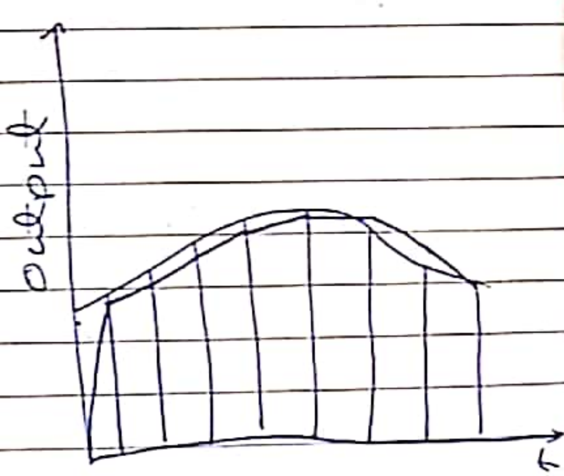
- → First order hold Circuit
- → Second order hold Circuit

Interpolative first order hold

→ It reconstructs the original signal much more accurately but there is delay of 1 Sampling Period.



First Order hold



Interpolative First order hold

CHAPTER 2

The Z Transform

2
The Z Transform

2.1) Introduction

⇒ The role of z transform in discrete-time systems is similar to that of the Laplace transform in continuous-time system.

⇒ In a linear discrete-time control system, a linear difference equation characterizes the dynamics of the system.

★ **Discrete-Time Signals**

→ The z transform applies to the continuous time signal $x(t)$, sampled signal $x(KT)$ and the number sequence $x(K)$.

2.2) The z transform

⇒ In considering the z transform of a time function $x(t)$, we consider only the sampled values of $x(t)$, that is $x(0), x(T), x(2T)$ --- where T is the sampling period.

$$X(z) = Z[x(t)] = Z[x(KT)] = \sum_{K=0}^{\infty} x(KT) z^{-K}$$

⇒ For a sequence of number $x(K)$, the z transform is defined by ÷

$$X(z) = Z[x(K)] = \sum_{K=0}^{\infty} x(K) z^{-K}$$

⇒ The z transform defined above is referred to as the one-sided z-transform.

→ For the one-sided z transform, we assume $x(t) = 0 \ \forall \ t < 0$ or $x(k) = 0 \ \forall \ k < 0$.

→ Z is a Complex Variable.

⇒ The z transform of $x(t)$, where $-\infty < t < \infty$ or of $x(k)$, where k takes integer values ($k = 0, \pm 1, \pm 2 \dots$) is defined by

$$X(z) = Z[x(t)] = Z[x(kT)] = \sum_{k=-\infty}^{\infty} x(kT) z^{-k}$$

$$X(z) = Z[x(k)] = \sum_{k=-\infty}^{\infty} x(k) z^{-k}$$

⇒ The z transform defined above is referred to as the two-sided z transform.

(Note: In this book, only the one-sided z transform is considered in detail.)

$$X(z) = x(0) + x(T) z^{-1} + x(2T) z^{-2} + \dots + x(kT) z^{-k} + \dots$$

2.3) Z - Transforms of Elementary Functions

⇒ In sampling a discontinuous function $x(t)$, we assume that the function is continuous from the right; that is if discontinuity occurs at $t=0$, then we assume that $x(0)$ is equal to $x(0+)$ rather than to the average at the discontinuity, $[x(0-)+x(0+)]/2$.

Unit-Step function

$$x(t) = \begin{cases} 1 & 0 \leq t \\ 0 & t < 0 \end{cases}$$

$$X(z) = \sum_{k=0}^{\infty} 1 \times z^{-k} = 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots$$

$$\Rightarrow \frac{X(z) - 1}{1/z} = 1 + \frac{1}{z} + \frac{1}{z^2} + \cdots = X(z)$$

$$\Rightarrow X(z) - 1 = \frac{x(z)}{z} \Rightarrow \boxed{x(z) = \frac{z}{z-1}}$$

$\{\forall\ |z| > 1\}$

→ It is not necessary to specify the region of z over which $X(z)$ is convergent. It suffices to know that such a region exists.

→ The z transform $X(z)$ of a time function $x(t)$ obtained in this way is valid throughout the z plane except at poles of $X(z)$

Unit-Ramp Function

$$x(t) = \begin{cases} t & 0 \le t \\ 0 & t < 0 \end{cases}$$

$$x(KT) = KT \quad \forall\, t = 0, 1, 2 \cdots$$

$$X(z) = Z[t] = \sum_{K=0}^{\infty} x(KT) z^{-K} = T \sum_{K=0}^{\infty} K z^{-K}$$

$$\Rightarrow T\left(\frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \cdots\right)$$

$$X(z) = \frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \cdots$$

$$\Rightarrow \frac{X(z)}{z} = \frac{1}{z^2} + \frac{2}{z^3} + \frac{3}{z^4} + \cdots$$

$$\Rightarrow X(z) - z^{-1}X(z) = z^{-1} + z^{-2} + z^{-3} + \cdots$$

$$\Rightarrow X(z)\left(1 - z^{-1}\right) = \frac{1}{1 - z^{-1}}$$

$$\Rightarrow \boxed{X(z) = \frac{1}{(1 - z^{-1})^2}}$$

$$\boxed{X(z) = \frac{Tz}{(z-1)^2}}$$

Polynomial Function a^k

$$x(k) = \begin{cases} a^k & k = 0, 1, 2 \cdots \\ 0 & k < 0 \end{cases}$$

$$X(z) = \sum_{K=0}^{\infty} x(k) z^{-K} = 1 + \frac{a}{z} + \frac{a^2}{z^2} + \frac{a^3}{z^3} + \cdots$$

$$\Rightarrow \boxed{X(z) = \frac{z}{z-a}}$$

Exponential Function

$$x(t) = \begin{cases} e^{-at} & 0 \leq t \\ 0 & t < 0 \end{cases}$$

$$x(KT) = e^{-aKT} \quad \forall K = 0, 1, 2 \cdots$$

$$X(z) = \sum_{K=0}^{\infty} \frac{e^{-aKT}}{z^K}$$

$$= 1 + \frac{e^{-aT}}{z} + \frac{e^{-2aT}}{z^2} + \frac{e^{-3aT}}{z^3} + \cdots$$

$$\boxed{X(z) = \frac{z}{z - e^{-aT}}}$$

Sinusoidal Function

$$x(t) = \begin{cases} \sin\omega t & 0 \leq t \\ 0 & t < 0 \end{cases}$$

$$e^{j\omega t} = \cos\omega t + j\sin\omega t$$

$$e^{-j\omega t} = \cos\omega t - j\sin\omega t$$

$$\Rightarrow \sin\omega t = \frac{1}{2j}\left(e^{j\omega t} - e^{-j\omega t}\right)$$

$$X(z) = Z[\sin\omega t] = Z\left[\frac{1}{2j}\left(e^{j\omega t} + e^{-j\omega t}\right)\right]$$

$$\Rightarrow \frac{1}{2j}\left(\frac{1}{1-e^{j\omega T}z^{-1}} - \frac{1}{1-e^{-j\omega T}z^{-1}}\right)$$

$$\Rightarrow \frac{1}{2j} \times \frac{1-e^{-j\omega T}z^{-1} - 1 + e^{j\omega T}z^{-1}}{z^{-2}+1-(e^{j\omega T}+e^{-j\omega T})z^{-1}}$$

$$\Rightarrow \frac{z^{-1}\sin\omega T}{1-2z^{-1}\cos\omega T + z^{-2}}$$

$$\Rightarrow \boxed{Z[\sin\omega t] = \frac{z\sin\omega T}{z^2-2z\cos\omega T+1}}$$

Similarly $\boxed{Z[\cos\omega t] = \frac{z^2 - z\cos\omega T}{z^2-2z\cos\omega T+1}}$

Example 2.2: $X(s) = \frac{1}{s(s+1)}$

$$\Rightarrow x(t) = 1-e^{-t} \qquad 0 \leq t$$

$$X(z) = Z[1-e^{-t}] = \frac{1}{1-z^{-1}} - \frac{1}{1-e^{-T}z^{-1}}$$

$$\Rightarrow X(z) = \frac{(1-e^{-T})z}{(z-1)(z-e^{-T})}$$

2.4) Important Properties and Theorems of the Z transform

Multiplication by a Constant

$$Z[a x(t)] = a Z[x(t)] = a X(z)$$

Linearity of the Z Transform

If $x(k) = \alpha f(k) + \beta g(k)$

then, $X(z) = \alpha F(z) + \beta G(z)$

Multiplication by a^k

$$Z[a^k x(k)] = X(a^{-1}z)$$

Proof

$$Z[a^k x(k)] = \sum_{k=0}^{\infty} a^k x(k) z^{-k}$$

$$= \sum_{k=0}^{\infty} x(k) (a^{-1}z)^{-k}$$

$$= X(a^{-1}z)$$

Shifting Theorem (also reffered to as Real translation theorem)

$$\Rightarrow \boxed{Z[x(t-nT)] = z^{-n}X(z)}$$

$$\left\{ \text{where } X(z) = Z[x(t)] \right\}$$

&

$$\Rightarrow \boxed{Z[x(t+nT)] = z^{n}\left[X(z) - \sum_{k=0}^{n-1} x(kT)z^{-k}\right]}$$

{where n is zero or a positive integer}

Proof

$$\star \; Z[x(t-nT)] = \sum_{k=0}^{\infty} x(kT-nT)\, z^{-k}$$

$$= z^{-n} \sum_{k=0}^{\infty} x(kT-nT)\, z^{-(k-n)}$$

Let $m = k-n$

$$\Rightarrow Z[x(t-nT)] = z^{-n}\left[\sum_{m=-n}^{\infty} x(mT)\, z^{-m}\right]$$

Since $x(mT) = 0 \;\forall\; m < 0$ so we may change lower limit from $m=-n$ to $m=0$.

$$Z[x(t-nT)] = z^{-n} \sum_{m=0}^{\infty} x(mT) z^{-m} = z^{-n}X(z)$$

$$\star \; Z[x(t+nT)] = \sum_{k=0}^{\infty} x(kT+nT)\, z^{-k}$$

$$= z^{n} \sum_{k=0}^{\infty} x(kT+nT)\, z^{-(k+n)}$$

$$\Rightarrow z^{n}\left[\sum_{K=0}^{\infty} x(KT+mT)\, z^{-(K+m)} + \sum_{K=0}^{m-1} x(KT)\, z^{-K} - \sum_{K=0}^{m-1} x(KT)\, z^{-K}\right]$$

$$\Rightarrow z^{n}\left[\sum_{K=0}^{\infty} x(KT)\, z^{-K} - \sum_{K=0}^{m-1} x(KT)\, z^{-K}\right]$$

$$= z^{m}\left[X(z) - \sum_{K=0}^{m-1} x(KT)\, z^{-K}\right]$$

$$Z[x(K+1)] = z\left[X(z) - \sum_{K=0}^{0} x(K)\, z^{-K}\right]$$

$$\Rightarrow Z[x(K+1)] = zX'(z) + z\,x(0)$$

$$Z[x(K+2)] = z^{2}X(z) - z^{2}x(0) - z\,x(1)$$

Similarly,

$$\boxed{Z[x(K+m)] = z^{m}X(z) - z^{m}x(0) - z^{m-1}x(1) \ldots - z\,x(m-1)}$$

{where m is a positive number}

<u>Note</u>: Multiplication of the z transform $X(z)$ by z has the effect of advancing the signal $x(KT)$ by one step and that multiplication of the z transform $X(z)$ by z^{-1} has the effect

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of delaying the signal $x(kT)$ by one step.

Example 2.3:

$$Z[1(t-T)] = z^{-1} Z[1(t)] = z^{-1} \times \frac{1}{1-z^{-1}} = \frac{1}{z-1}$$

$$Z[1(t-2T)] = z^{-2} Z[1(t)] = \frac{z^{-2}}{1-z^{-1}}$$

Example 2.4:

$$f(k) = \begin{cases} a^{k-1} & k = 1, 2, 3 \ldots \\ 0 & k \leq 0 \end{cases}$$

let $g(k) = \begin{cases} a^k & k = 0, 1, 2 \ldots \\ 0 & k < 0 \end{cases}$

$$f(k) = g(k-1)$$

$$Z[f(k)] = Z[g(k-1)] = z^{-1} Z[g(k)]$$

$$= z^{-1} \times \frac{z}{z-a}$$

$$\Rightarrow Z[f(k)] = \frac{1}{z-a}$$

Example 2.5:

$$y(k) = \sum_{h=0}^{k} x(h) \qquad k = 0, 1, 2 \ldots$$

{$x(h)$ is a function}

$$y(k) = 0 \quad \forall\ k < 0$$

$$\Rightarrow \quad y(k) = x(0) + x(1) + x(2) + \cdots x(k)$$

$$y(k-1) = x(0) + x(1) + x(2) + \cdots x(k-1)$$

$$y(k) - y(k-1) = x(k)$$

$$Z[y(k) - y(k-1)] = Z[x(k)]$$

$$\cancel{Y(z) - Y(k-1) = X(k)}$$

$$\cancel{\Rightarrow Y(z) - z^{-1}Y(z)}$$

$$\Rightarrow Y(z) - z^{-1}Y(z) = X(z)$$

$$\Rightarrow Y(z)\{1 - z^{-1}\} = X(z)$$

$$\Rightarrow Y(z) = \frac{X(z)}{1 - z^{-1}} \quad \{\text{where } X(z) = Z[x(k)]\}$$

★ Complex Translation Theorem

If $x(t)$ has the z transform $X(z)$, then the z transform of $e^{-at}x(t)$ can be given by $X(ze^{aT})$. This is known as the complex translation theorem.

Proof

$$Z\{e^{-at}x(t)\} = \sum_{k=0}^{\infty} x(kT)e^{-akT}z^{-k}$$

$$= \sum_{k=0}^{\infty} x(kT)(ze^{aT})^{-k}$$

$$= X(ze^{aT})$$

Example 2-6: Given the z transforms of $\sin\omega t$ and $\cos\omega t$, obtain the z transform of $e^{-at}\sin\omega t$ and $e^{-at}\cos\omega t$, respectively by using the Complex translation theorem.

$\Rightarrow$

$$Z[\sin\omega t] = \frac{z^{-1}\sin\omega T}{1 - 2z^{-1}\cos\omega T + z^{-2}}$$

So

$$Z[e^{-at}\sin\omega t] = \frac{e^{-aT}z^{-1}\sin\omega T}{1 - 2e^{-aT}z^{-1}\cos\omega T + e^{-2aT}z^{-2}}$$

Similarly

$$Z[e^{-at}\cos\omega t] = \frac{1 - e^{-aT}z^{-1}\cos\omega T}{1 - 2e^{-aT}z^{-1}\cos\omega T + e^{-2aT}z^{-2}}$$

Example 2-7: Z transform of te^{-at}

$$Z[te^{-at}] = \frac{Te^{-aT}z^{-1}}{(1 - e^{-aT}z^{-1})^2}$$

★ Initial Value theorem: If $x(t)$ has the z transform $X(z)$ and if $\lim_{z\to\infty} X(z)$ exists, then the initial value $x(0)$ of $x(t)$ or $x(k)$ is given by:

$$\boxed{x(0) = \lim_{z\to\infty} X(z)}$$

Proof:

$$X(z) = \sum_{k=0}^{\infty} x(k)z^{-k} = x(0) + x(1)z^{-1} + x(2)z^{-2} + \cdots$$

So $\lim_{z\to\infty} X(z) = x(0)$

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★ Final Value Theorem: Suppose that $x(k)$, where $x(k)=0$ for $k<0$, has the z transform $X(z)$ and that all the poles of $X(z)$ lie inside the unit circle, with the possible exception of a simple pole at $z=1$.

⇒ Then the final value of $x(k)$, that is, the value of $x(k)$ as k approaches infinity can be given by:

$$\boxed{\lim_{k\to\infty} x(k) = \lim_{z\to 1}\left[(1-z^{-1})X(z)\right]}$$

Proof

$$Z[x(k)] = X(z) = \sum_{k=0}^{\infty} x(k)z^{-k}$$

$$Z[x(k-1)] = z^{-1}X(z) = \sum_{k=0}^{\infty} x(k-1)z^{-k}$$

$$\sum_{k=0}^{\infty} x(k)z^{-k} - \sum_{k=0}^{\infty} x(k-1)z^{-k} = (1-z^{-1})X(z)$$

→ Taking limit as $z \to 1$.

$$\lim_{z\to 1}(1-z^{-1})X(z) = [\cancel{x(0)} - \overset{0}{x(-1)}] + [\cancel{x(1)} - \cancel{x(0)}]$$
$$[\cancel{x(2)} - \cancel{x(1)}] + \dots [x(\infty) - \cancel{x(\infty-1)}]$$
$$= x(\infty)$$

So $\lim_{k \to \infty} x(k) = \lim_{z \to 1} \left[(1-z^{-1}) X(z)\right]$

* Example 2.9: $X(z) = \frac{1}{1-z^{-1}} - \frac{1}{1-e^{-aT}z^{-1}}$ $a>0$

$$x(\infty) = \lim_{z \to 1} \left(\frac{1-z^{-1}}{1-z^{-1}} - \frac{1-z^{-1}}{1-e^{-aT}z^{-1}} \right) = 1-0 = 1$$

2.5) The Inverse Z transform

⇒ The notation for the inverse z transform is Z^{-1}.

⇒ The inverse z transform of $X(z)$ yields the corresponding time sequence $x(k)$

↳ Inverse z transform of $X(z)$ yields a unique $x(k)$, but does not yield a unique $x(t)$.

⇒ There are four methods for obtaining the inverse z-transform are commonly available :-

1. Direct division method
2. Computational method
3. Partial-fraction method
4. Inversion integral method

+ (Referring to z-transform table)

⇒ In obtaining the inverse z transform, we assume, as usual, that the time sequence $x(kT)$ or $x(k)$ is zero for $k < 0$.

★ **Poles and Zeros in the z plane**

In engineering applications of the z-transform method, $X(z)$ may have the form:-

$$X(z) = \frac{b_0 z^m + b_1 z^{m-1} + \cdots + b_m}{z^n + a_1 z^{n-1} + \cdots a_n} \quad (m \leq n)$$

or

$$X(z) = \frac{b_0 (z-z_1)(z-z_2)\cdots(z-z_m)}{(z-p_1)(z-p_2)\cdots(z-p_n)}$$

⇒ The location of the poles and zeros of $X(z)$ determine the characteristics of $x(k)$, the sequence of values or numbers.

⇒ In control engineering and signal processing $X(z)$ is frequently expressed as a ratio of polynomials in z^{-1} as follows:-

$$X(z) = \frac{b_0 z^{-(n-m)} + b_1 z^{-(n-m+1)} + \cdots + b_m z^{-n}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_n z^{-n}}$$

Where z^{-1} is interpreted as the unit delay operator.

① Direct Division Method

⇒ In the direct division method we obtain the inverse z-transform by expanding $X(z)$ into an infinite power series in z^{-1}.

→ This method is useful when it is difficult to obtain the closed-form expression for the inverse z transform.

or

→ It is ~~diff~~ desired to find only the first several terms of $x(k)$.

$$X(z) = \sum_{k=0}^{\infty} x(kT) z^{-k}$$

$$= x(0) + x(T) z^{-1} + x(2T) z^{-2} + \cdots + x(kT) z^{-k} + \cdots$$

or

$$X(z) = \sum_{k=0}^{\infty} x(k) z^{-k}$$

$$= x(0) + x(1) z^{-1} + x(2) z^{-2} + \cdots + x(k) z^{-k} + \cdots$$

⇒ If $X(z)$ is given in the form of a rational function, the expansion into an infinite power series in increasing powers of z^{-1} can be accomplished by simply dividing the numerator by the denominator, where both the numerator & denominator of $X(z)$ are written in increasing power of z^{-1}.

Example 2.10: $X(z) = \dfrac{10z + 5}{(z-1)(z-0.2)}$

$$\Rightarrow X(z) = \frac{10z^{-1} + 5z^{-2}}{1 - 1.2z^{-1} + 0.2z^{-2}}$$

$$
\begin{array}{r l}
 & 10z^{-1} + 17z^{-2} + 18.4z^{-3} + 18.68z^{-4} + \cdots \\
\hline
1 - 1.2z^{-1} + 0.2z^{-2} \;\big) & 10z^{-1} + 5z^{-2} \\
 & 10z^{-1} - 12z^{-2} + 2z^{-3} \\
\hline
 & 17z^{-2} - 2z^{-3} \\
 & 17z^{-2} - 20.4z^{-3} + 3.4z^{-4} \\
\hline
 & 18.4z^{-3} - 3.4z^{-4} \\
 & 18.4z^{-3} - 22.08z^{-4} + 3.68z^{-5} \\
\hline
 & 18.68z^{-4} - 3.68z^{-5} \\
 & 18.68z^{-4} - 22.416z^{-5} + 3.7z^{-6}
\end{array}
$$

Thus, $X(z) = 10z^{-1} + 17z^{-2} + 18.4z^{-3} + 18.68z^{-4} + \cdots$

$$
\left.
\begin{aligned}
x(0) &= 0 \\
x(1) &= 10 \\
x(2) &= 17 \\
x(3) &= 18.4 \\
x(4) &= 18.68 \\
&\vdots
\end{aligned}
\right\}
$$

(B) Computational Method

Two Computational approaches to obtain inverse Z-transform:

1. Matlab approach
2. Difference equation approach.

⇒ Consider a System $G(z)$ defined by

$$G(z) = \frac{0.4673z^{-1} - 0.3393z^{-2}}{1 - 1.5327z^{-1} + 0.6607z^{-2}} \quad \text{——} (1)$$

⇒ For finding the inverse z transform, we utilize the Kronecker delta function $\delta_0(KT)$

$$\delta_0(KT) = 1, \ \forall K=0$$
$$= 0 \ \forall K \neq 0$$

⇒ Assume that $x(k)$ the input to the system $G(z)$, is the Kronecker delta input.

$$x(K) = \begin{cases} 1 & \forall K=0 \\ 0 & \forall K \neq 0 \end{cases}$$

The z transform of the Kronecker delta input is

$$X(z) = 1.$$

⇒ Using the Kronecker delta input Eq (1) can be written as:

$$G(z) = \frac{Y(z)}{X(z)} = \frac{0.4673z - 0.3393}{z^2 - 1.5327z + 0.6607}$$

Matlab Approach

⇒ To obtain the inverse z transform of $G(z)$ with matlab, we proceed as follows:

→ Enter the numerator & denominator as follows

num = [0 0.4673 -0.3393]
den = [1 . -1.5327 0.6607]

→ Enter the Kronecker delta input

x = [1 zeros(1, 40)]

→ Then enter the Command

Y = filter(num, den, x)

→ to obtain the response Y(K) from K=0 to K=40

Difference Equation Approach

$$(z^2 - 1.5327z + 0.6607)Y(z) = (0.4673z - 0.3393)X(z)$$

⇒ We convert this equation into the difference equation as follows:

$$y(k+2) - 1.5327y(k+1) + 0.6607y(k) = 0.4673x(k+1) - 0.3393x(k)$$

$$\left\{ \begin{array}{l} \text{where } x(0)=1 \text{ \& } x(k)=0 \ \forall\, k \neq 0 \\ \text{\& } y(k)=0 \ \forall\, k<0 \end{array} \right\}$$

⇒ Putting $K = -2$ we get :-

$$y(0) = 0$$

⇒ Putting $K = -1$ we get :-

$$y(1) = 0.4673$$

Similarly all values can be found!

(C) Partial-Fraction-Expansion Method

⇒ The partial-Fraction expansion method presented here, which is parallel to the partial-fraction expansion method used in Laplace transformation is widely used in routine problem involving Z transforms.

(D) Inversion Integral Method

$$Z^{-1}[X(z)] = x(KT) = x(K) = \frac{1}{2\pi j}\oint_C X(z)\, z^{K-1}\, dz$$

where C is a circle with the center at the origin of the z plane such that all poles of $X(z)z^{K-1}$ are inside it.

⇒ The equation for giving the inverse z transform in terms of residues can be derived by using theory of complex variables. It can be obtained as follows:-

$$x(KT) = x(K) = K_1 + K_2 + K_3 + \dots + K_m$$

$$= \sum_{i=1}^{m} \left[\underset{\text{Pole } z}{\text{residue of } X(z)z^{K-1} \text{ at}} = z_i \text{ of } X(z)z^{K-1} \right]$$

Where $K_1, K_2 \cdots K_m$ denote the residues of $X(z)z^{k-1}$ at poles $z_1, z_2 \cdots z_m$ respectively.

→ If the denominator of $X(z)z^{k-1}$ contains a simple pole $z = z_i$ then the corresponding residue K is given by

$$K = \lim_{z \to z_i} \left[(z - z_i) X(z) z^{k-1} \right]$$

→ If $X(z)z^{k-1}$ contains a multiple pole z_j of order q, then the residue K is given by

$$K = \frac{1}{(q-1)!} \lim_{z \to z_j} \frac{d^{q-1}}{dz^{q-1}} \left[(z - z_j)^q X(z) z^{k-1} \right]$$

⇒ It should be noted that the inversion integral method, when evaluated by residues, is a very simple technique for obtaining the inverse z transform, provided that $X(z)z^{k-1}$ has no poles at the origin $z = 0$.

⇒ If however $X(z)z^{k-1}$ has a simple pole or a multiple pole at $z = 0$, then calculations may become cumbersome and the partial-fraction-expansion method may prove to be simple to ~~be simpler~~ apply.

Example 2-16: $X(z) = \dfrac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})}$

$$X(z)\, e^{k-1} = \frac{z^k(1-e^{-aT})}{(z-1)(z-e^{-aT})}$$

for $k = 0, 1, 2 \ldots$ $X(z) z^{k-1}$ has two simple poles $z = z_1 = 1$ & $z = z_2 = e^{-aT}$.

$$K_1 = \lim_{z \to 1} \left[(z-1) \frac{z^k(1-e^{-aT})}{(z-1)(z-e^{-aT})} \right] = 1$$

$$K_2 = \lim_{z \to e^{-aT}} \left[(z-e^{-aT}) \times \frac{z^k(1-e^{aT})}{(z-1)(z-e^{-aT})} \right] = -e^{-akT}$$

Hence, $\boxed{x(kT) = K_1 + K_2 = 1 - e^{-akT}}$

2.6) Z Transform method for Solving difference Equation

⇒ Difference equations can be solved easily by use of a digital computer, provided the numerical values of all coefficients and parameters are given.

⇒ However, closed-form expressions of $x(k)$ cannot be obtained from the computer solution, except for very special case.

⇒ The usefulness of the z-transform method is that it enables us to obtain the closed-form expression of $x(k)$.

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Consider the linear time-invariant discrete time System characterized by the following linear difference equation:

$$x(K) + a_1 x(K-1) + \cdots + a_n x(K-n)$$
$$= b_0 u(K) + b_1 u(K-1) + \cdots + b_n u(K-n)$$

Where $u(K)$ & $x(K)$ are the System's input & output respectively, at the K^{th} iteration.

Let $Z[x(K)] = X(z)$

Example 2-18

$$x(K+2) + 3x(K+1) + 2x(K) = 0$$

$$\{x(0) = 0, x(1) = 1\}$$

$$\Rightarrow Z[x(K+2)] + 3Z[x(K+1)] + 2Z[x(K)] = 0$$

$$\Rightarrow \left(z^2 X(z) - z^2 x(0) - z x(1)\right) + 3\left(zX(z) - z x(0)\right)$$

$$+ 2X(z) = 0$$

$$X(z) = \frac{z}{z+1} - \frac{z}{z+2} = \frac{1}{1+z^{-1}} - \frac{1}{1+2z^{-1}}$$

$$x(K) = (-1)^K - (-2)^K \quad \forall K = 0, 1, 2 \ldots$$

2.7) Concluding Comments

"With the z transform method, linear time-invariant difference equations can be transformed into algebraic equations. This facilitates the transient response analysis of the digital control system"

CHAPTER 3

Z Plane Analysis of Discrete Time Control System

3

Z-plane Analysis of Discrete time Control Systems

⇒ The z-transform method is particularly useful for analyzing and designing Single input Single output linear time Invariant discrete time Control Systems.

⇒ The main advantage of the z transform method is that it enables the engineer to apply conventional Continuous-time design methods to discrete-time Systems that may be partly discrete time and partly continuous time

3.2) Impulse Sampling and Data Hold

★ Impulse Sampling

⇒ We shall consider a fictious Sampler Commonly Called an impulse sampler.

⇒ The output of this Sampler is considered to be a train of impulse that begin with t=0, with the Sampling period equal to T and the strength of each impulse equal to the Sampled value of the Continuous-time Signal at the Corresponding Sampling instant.

⇒ So Impulse-Sampled output is a sequence of impulses, with the strength of each impulse equal to the magnitude of $x(t)$ at the Corresponding instant of time.

Sampled Signal

$$\delta(t) = \begin{cases} 1 & t=0 \\ 0 & t \neq 0 \end{cases}$$

$$x^*(t) = \sum_{K=0}^{\infty} x(KT)\,\delta(t-KT) \quad \text{——} (1)$$

⇒ We shall define a train of unit impulse as :-

$$\delta_T(t) = \sum_{K=0}^{\infty} \delta(t-KT)$$

⇒ The sampler output is equal to the product of the continuous-time input $x(t)$ and the train of unit impulses $\delta_T(t)$.

⇒ Next consider the Laplace transform of Eq (1).

$$X^*(s) = \mathcal{L}[x^*(t)] = x(0)\,\mathcal{L}[\delta(t)] + x(T)\,\mathcal{L}[\delta(t-T)] + x(2T)\,\mathcal{L}[\delta(t-2T)] + \cdots$$

$$= x(0) + x(T)e^{-TS} + x(2T)e^{-2TS} + \cdots$$

$$X^*(s) = \sum_{K=0}^{\infty} x(KT)\,e^{-KTS}$$

If we define e^{TS} as Z

then above equation become

$$X^*(s)\Big|_{s=\frac{\ln Z}{T}} = \sum_{K=0}^{\infty} x(KT)\,Z^{-K}$$

⇒ It is the Z transform of the sequence $x(0), x(T), x(2T)\cdots$ generated from $x(t)$ at $t=KT$ where $K=0,1,2\cdots$

⇒ Hence we may write

$$X^*(s)\Big|_{s=\frac{\ln z}{T}} = X(z)$$

Summary: If the Continuous-time Signal $x(t)$ is impulse sampled in a periodic manner, mathematically the Sampled Signal may be represented by :-

$$x^*(t) = \sum_{K=0}^{\infty} x(t)\,\delta(t-KT)$$

⇒ The Laplace transform of the impulse-sampled Signal $x^*(t)$ has been shown to be the Same as the z transform of signal $x(t)$ if e^{Ts} is defined as z.

★ Data-Hold Circuits

⇒ Data-hold is a process of generating a Continuous-time Signal $h(t)$ from a discrete-time Sequence $x(KT)$.

~~⇒ A hold Circuit Converts the Sampled Signal into a Continuous-time Signal, which approximately~~

⇒ The Signal $h(t)$ during the time interval $KT \leq t \leq (K+1)T$ may be approximated by a polynomial in τ as follows :-

$$h(KT+\tau) = a_n\tau^n + a_{n-1}\tau^{n-1} + \cdots + a_1\tau + a_0$$

{where $0 \leq \tau \leq T$

$$\Rightarrow h(KT+\tau) = a_n\tau^n + a_{n-1}\tau^{n-1} + \cdots + a_1\tau + x(KT)$$

{as $h(KT) = x(KT)$}

⇒ If the data hold circuit is an n^{th} order polynomial extrapolator, it is called an n^{th} order hold.

{The n^{th} order hold uses the past $n+1$ discrete data $x((K-n)T)$, $x((K-n+1)T)$ - - - $x(KT)$ the generate a signal $h(KT+\tau)$.}

⇒ It will be seen later that the TF G_h of the Zero-order hold may be given by

$$G_h = \frac{1-e^{-TS}}{S}$$

Zero Order Hold

$$\boxed{h(KT+\tau) = x(KT) \quad \forall\ 0 \leq \tau < T}$$

Let $H'(t)$ be Heaviside Unit step function

$$H'(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

$$h(t) = x(0)\left[H'(t) - H'(t-T)\right] + x(T)\left[H'(t-T) - H'(t-2T)\right] + x(2T)\left[H'(t-2T) - H'(t-3T)\right] + \cdots$$

$$h(t) = \sum_{K=0}^{\infty} x(KT)\left[H'(t-KT) - H'(t-(K+1)T)\right]$$

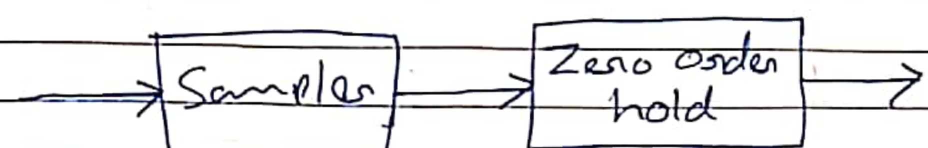
$$\mathcal{L}\left[h(t)\right] = H(s) = \sum_{K=0}^{\infty} x(KT)\left(\frac{e^{-KTs} - e^{-(K+1)Ts}}{s}\right)$$

$$\Rightarrow \frac{1-e^{-Ts}}{s} \sum_{K=0}^{\infty} x(KT)e^{-KTs}$$

$$X^*(s) = \sum_{K=0}^{\infty} x(KT)e^{-KTs}$$

So $\boxed{G_{ho}(s) = \frac{1-e^{-Ts}}{s}}$

{Transfer function of zero order hold}



First Order Hold

Although we do not use first order holds in control systems, it is worthwhile to see what the TF of first order holds may look like.

$$h(kT+\tau) = a_1\tau + x(kT)$$

$$\{ 0 \le \tau \le T \;\; \& \;\; k = 0, 1, 2 \dots \}$$

$$h((k-1)T) = x((k-1)T)$$

$$\Rightarrow h((k-1)T) = -a_1T + x(kT) = x((k-1)T)$$

$$a_1 = \frac{1}{T}\left(x(kT) - x((k-1)T)\right)$$

So
$$\boxed{h(kT+\tau) = x(kT) + \frac{x(kT) - x((k-1)T)}{T}\tau}$$

$$h(t) = \left(x(0) + \frac{x(0)-x(-T)}{T}t\right)\left(H'(t) - H'(t-T)\right)$$

$$+ \left(x(1) + \frac{\{x(1)-x(0)\}}{T}t\right)\left(H'(t-T) - H'(t-2T)\right)$$

$$+ \dots$$

$$\cancel{\mathcal{L}[h(t)] = H(s) = \left(\frac{x(0)}{s} + \frac{x(0)-x(-T)}{Ts^2}\right)e^{-0Ts} - \left(\frac{x(0)e^{-Ts}}{s} + \frac{(x(0)-x(-T))e^{-Ts}}{Ts^2}\right)}$$

$$\cancel{- + \dots}$$

$$H(s) = \frac{1-e^{-TS}}{TS^2}\left(x(0)-x(-T)\right) + \frac{1-e^{-TS}}{S}x(0)$$

$$+ \text{———}$$

$$H(s) = \sum_{K=0}^{\infty} \frac{1-e^{-TS}}{TS^2}\left(x(K)-x(K-1)\right) + \frac{1-e^{-TS}}{S}x(K)$$

$$H(s) = \sum_{K=0}^{\infty}\left[\frac{x(KT)}{S} + \frac{x(KT)-x((K-1)T)}{TS^2}\right]e^{-KTS}$$

$$-\left[\frac{x(KT)}{S} + \frac{x(KT)-x((K-1)T)}{TS^2}\right]e^{-(K-1)TS}$$

$$\Rightarrow \sum_{K=0}^{\infty}\left[\frac{e^{-KTS}-e^{-(K+1)TS}}{S}x(KT)\right.$$

$$\left. + \frac{e^{-KTS}-e^{-(K+1)TS}}{TS^2}\left(x(KT)-x((K-1)T)\right)\right]$$

$$\Rightarrow \cancel{\sum_{K=0}^{\infty}} \frac{1-e^{-TS}}{S}\sum_{K=0}^{\infty} x(KT)e^{-KTS}$$

$$+ \frac{1-e^{-TS}}{TS^2}\left(\sum_{K=0}^{\infty} x(KT)e^{-KTS} - \sum_{K=0}^{\infty} x((K-1)T)e^{-KTS}\right)$$

$$\Rightarrow \frac{1-e^{-TS}}{S}X(z) + \frac{1-e^{-TS}}{TS^2}\left(X(z) - \frac{1}{z}X(z)\right)$$

$$\boxed{z = e^{TS}}$$

$$H(s) = X(z) * \left[\frac{1-e^{-TS}}{s} + \frac{1-e^{-TS}}{Ts^2}\left(1-e^{-TS}\right) \right]$$

$$\left(\frac{1-e^{-Ts}}{s}\right)^2 \left[\frac{s}{1-e^{-Ts}} + \frac{1}{T} \right]$$

$$\Rightarrow h(t) = \sum_{K=0}^{\infty} \left\{ \left(\overbrace{x(KT)}^{A} + \overbrace{\left(\frac{x(KT) - x((K-1)T)}{T} \right)}^{B} \tau \right) \left(H(t-KT) - H(t-(K+1)T) \right) \right\}$$

$$\Rightarrow \mathcal{L}[h(t)] = \sum_{K=0}^{\infty} \left\{ x(KT)\, \mathcal{L}[H(t-KT)] + \frac{x(KT) - x((K-1)T)}{T} \mathcal{L}[t H(t-KT)] \right.$$

$$\left. - \left(x(KT)\, \mathcal{L}[H(t-(K+1)T)] + \frac{x(KT) - x((K-1)T)}{T} \mathcal{L}[t H(t-(K+1)T)] \right) \right\}$$

$$\Rightarrow H(s) = \sum_{K=0}^{\infty} \left[\left(\frac{A e^{-KTS}}{s} + \frac{B}{s^2} e^{-KTS} \right) - \left(\frac{A e^{-(K+1)TS}}{s} + \frac{B}{s^2} e^{-(K+1)TS} \right) \right]$$

$$= \frac{1-e^{-TS}}{s} \sum A e^{-KTS} + \frac{1-e^{-TS}}{s^2} \sum B e^{-KTS}$$

$$\Rightarrow \frac{1-e^{-TS}}{s} \sum_{K=0}^{\infty} x(KT) e^{-KTS} + \frac{1-e^{-TS}}{Ts^2} \left(\sum x(KT) e^{-KTS} - \sum x((K-1)T) e^{-KTS} \right)$$

$$X^*(s) \left\{ \frac{1-e^{-TS}}{s} + \frac{1-e^{-TS}}{Ts^2} \left(1 - e^{-TS}\right) \right\}$$

$$G_{H1}(s) = \left(\frac{1-e^{-TS}}{S}\right)^2 \left(\frac{TS+1}{T}\right)$$

$$\boxed{G_{h1}(s) = \frac{1-e^{-TS}}{TS^2}\left(\frac{TS+1-e^{-TS}}{T}\right)}$$??

3.3) Obtaining the z transform by the Convolution Integral method

$$x(t) \xrightarrow{\delta_T} x^*(t)$$ Impulse Sampler

$$x^*(t) = \sum_{K=0}^{\infty} x(t)\,\delta(t-KT) = x(t)\sum_{K=0}^{\infty}\delta(t-KT) \quad \text{—①}$$

$$\mathcal{L}\left\{\sum_{K=0}^{\infty}\delta(t-KT)\right\} = \sum_{K=0}^{\infty} e^{-KTS} = \frac{1}{1-e^{-TS}} \quad \text{—②}$$

$$X^*(s) = \mathcal{L}[x^*(t)] = \mathcal{L}\left[x(t)\sum_{K=0}^{\infty}\delta(t-KT)\right] \quad \text{—③}$$

$$\mathcal{L}[f(t)g(t)] = \int_0^{\infty} f(t)g(t)e^{-st}\,dt$$

$$= \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(p)\,G(s-p)\,dp$$

Where $F(s)$ & $G(s)$ are Laplace transform of $f(t)$ & $g(t)$ respectively

Let $f(t) = x(t)$

& $g(t) = \sum_{K=0}^{\infty} \delta(t-KT)$

$$\boxed{X^*(s) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} X(P)\,\frac{1}{1-e^{-T(s-P)}}\,dP}$$

where the integration is along the line $c-j\infty$ to $c+j\infty$ and this line is parallel to the imaginary axis in the P plane and separates the poles of $X(P)$ from those of $\frac{1}{1-e^{-T(s-P)}}$.

⇒ Such an integral can be evaluated in terms of Residues by forming a closed contour consisting of a line from $c-j\infty$ to $c+j\infty$ and a semicircle of infinite radius in the left or right half plane, provided that the integral along the added semicircle is a constant.

$$X^*(s) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} X(P)\frac{1}{1-e^{-T(s-P)}}\,dP$$

$$\Rightarrow \frac{1}{2\pi j}\oint \frac{X(P)}{1-e^{-T(s-P)}}\,dP - \frac{1}{2\pi j}\int_{\Gamma} \frac{X(P)}{1-e^{-T(s-P)}}\,dP$$

{where Γ is a semicircle of infinite radius in the left or right half of P plane}

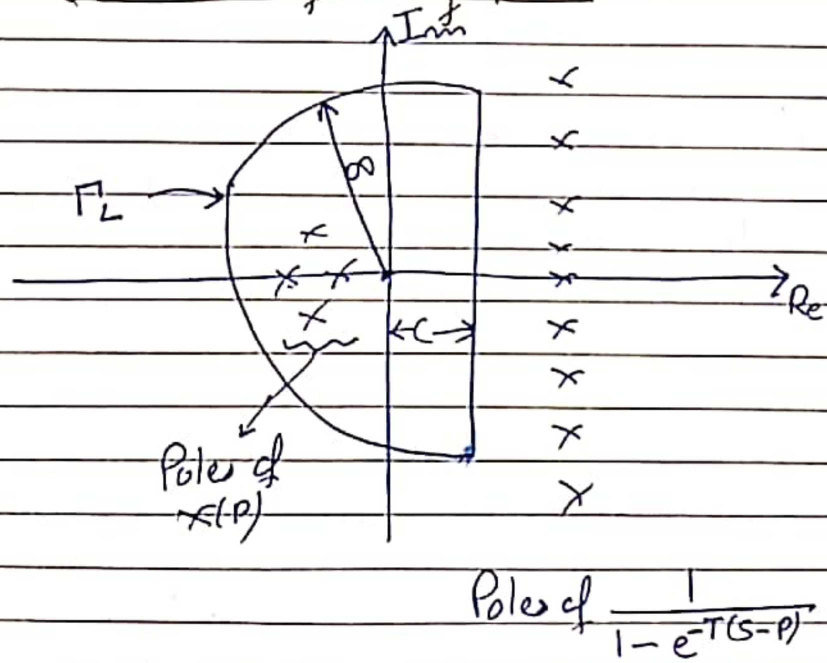
⇒ There are two ways to evaluate this integral: One using an infinite semicircle in the left-half plane and the other an infinite semicircle in the right half plane.

⇒ In our analysis here, we assume that the poles of $X(s)$ lie in the left half-plane and $X(s)$ can be expressed as a ratio of polynomials in s

$$X(s) = \frac{Q(s)}{P(s)} \quad \left\{ P(s) \ \& \ Q(s) \text{ are polynomial in } S \right\}$$

⇒ We assume $P(s)$ is of higher degree in s than $Q(s)$. $\boxed{\lim_{s\to\infty} X(s) = 0}$

Evaluation of Convolution Integral in the Left half plane



⇒ If the denominator of $X(s)$ is of higher degree in s than the numerator, the integral along Γ_L vanishes.

$$X^*(s) = \frac{1}{2\pi j} \oint \frac{X(P)}{1 - e^{-T(s-P)}} dP$$

⇒ This integral is equal to the sum of the residues of $X(P)$ in the closed contour.

$$X^*(s) = \sum \left[\text{residue of } \frac{X(P)}{1-e^{-T(s-P)}} \text{ at poles of } X(P) \right]$$

⇒ By substituting z for e^{Ts} and changing complex variable notation from P to S, we obtain.

$$X^*(z) = \sum \left[\text{residue of } \frac{X(s)z}{z-e^{Ts}} \text{ at poles of } X(s) \right]$$

⇒ Assuming that $X(s)$ has poles $S_1, S_2 \text{---} S_m$.

→ If a pole at $s=s_j$ is a simple pole, then the corresponding residue K_j is

$$K_j = \lim_{s \to s_j} \left[(s-s_j) \frac{X(s)z}{z-e^{Ts}} \right]$$

→ If a pole at $s=s_i$ is a multiple pole of order n_i, then the residue K_i is

$$K_i = \frac{1}{(n_i-1)!} \lim_{s \to s_i} \frac{d^{n_i-1}}{ds^{n_i-1}} \left[(s-s_i)^{n_i} \frac{X(s)z}{z-e^{Ts}} \right]$$

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Example 3-1: $X(s) = \frac{1}{s^2(s+1)}$

Obtain $X(z)$ by use of the Convolution integral in the left half plane.

$\Rightarrow$ $X(s)$ has double pole at $s=0$
& simple pole at $s=-1$

$$X(z) = \sum \left[\text{residue of } \frac{X(s)}{z-e^{Ts}} \text{ at pole of } X(s) \right]$$

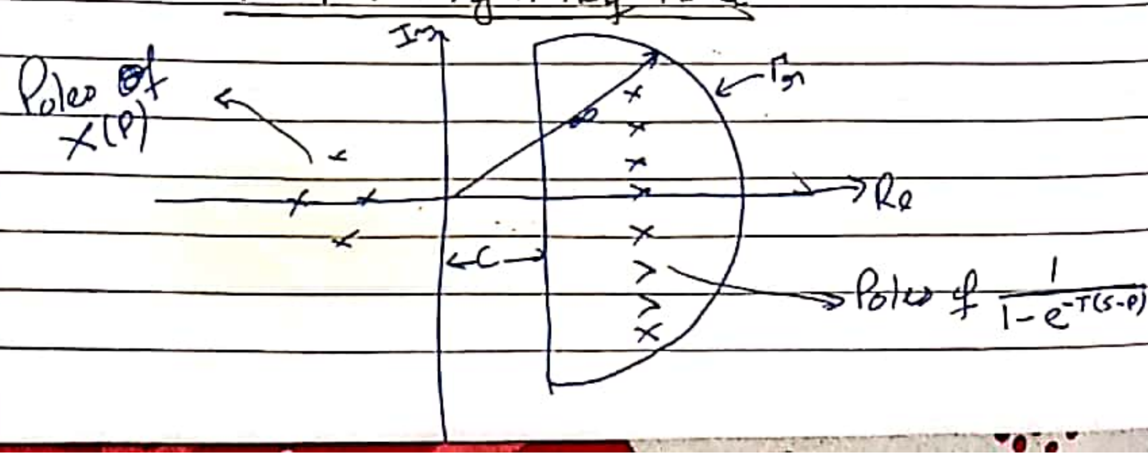
$$K_0 = \frac{1}{(2-1)!} \lim_{s\to 0} \frac{d}{ds} \left[\frac{s^2}{s^2(s+1)} \frac{z}{z-e^{Ts}} \right]$$

$$= \lim_{s\to 0} \frac{-z[z-e^{Ts} + (s+1)(-T)e^{Ts}]}{(s+1)^2 (z-e^{Ts})^2} = \frac{-z(z-1-T)}{(z-1)^2}$$

$$K_{-1} = \lim_{s\to -1} \left[\cancel{(s+1)} \frac{1}{s^2\cancel{(s+1)}} \frac{z}{z-e^{Ts}} \right] = \frac{z}{z-e^{-T}}$$

$$X(z) = \frac{z}{z-e^{-T}} - \frac{z(z-1-T)}{(z-1)^2}$$

Evaluation of Convolution Integral in the Right Half-Plane



⇒ The closed contour encloses all poles of $\frac{1}{1-e^{-T(s-p)}}$, but it does not enclose any poles of $X(p)$.

$$X^*(s) = \frac{1}{2\pi j}\oint \frac{X(p)}{1-e^{-T(s-p)}}dp - \frac{1}{2\pi j}\int_{\Gamma_1} \frac{X(p)}{1-e^{-T(s-p)}}dp$$

⇒ In evaluating the integrals on the right-half side of above equation we need to consider two cases separately:-

→ One case where denominator of $X(s)$ is two or more degree higher in s than the numerator; ~~and~~

→ And another case where the denominator of $X(s)$ is only one degree higher in s than the numerator.

Case 1: $X(s)$ has a denominator two or more degree higher in s than the numerator.

$$\lim_{s\to\infty} sX(s) = x(0+) = 0$$

⇒ Then the integral along Γ_1 is zero. Thus in present case

$$\frac{1}{2\pi j}\int_{\Gamma_1} \frac{X(p)}{1-e^{-T(s-p)}}dp = 0$$

Thus, ~~X(s)~~

$$X^*(s) = \frac{1}{T}\sum_{K=-\infty}^{\infty} X(s+j\omega_s K)$$

$$\Rightarrow X(z) = \frac{1}{T}\sum_{K=-\infty}^{\infty} X(s+j\omega_s K)\Big|_{s=\frac{\ln z}{T}}$$

⇒ It is very tedious to obtain z transform expressions of commonly encountered functions by this method.

<u>Case 2:</u> X(S) has a denominator one degree higher in s than the numerator.

⇒ For this case $\lim_{s\to\infty} sX(s) = x(0+) \neq 0 < \infty$ and the integral along Γ_R is not zero.

⇒ It can be shown that the contribution of the integral along Γ_R is $-\frac{1}{2}x(0+)$.

$$\frac{1}{2\pi j}\int_{\Gamma_R} \frac{X(p)}{1-e^{-T(s-p)}}\,dp = -\frac{1}{2}x(0+)$$

So $$X^*(s) = \frac{1}{T}\sum_{K=-\infty}^{\infty} X(s+j\omega_s K) + \frac{1}{2}x(0+)$$

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Example 3.2: So that $x^*(s)$ is periodic with period $2\pi/\omega_s$:

$$X^*(s) = \frac{1}{T}\sum_{h=-\infty}^{\infty} X(s+j\omega_s h) + \frac{1}{2}x(0+)$$

$$\Rightarrow X^*(s+j\omega_s k) = \frac{1}{T}\sum_{h=-\infty}^{\infty} X(s+j\omega_s k+j\omega_s h) + \frac{1}{2}x(0+)$$

Let $k+h=m$

$$\Rightarrow X^*(s+j\omega_s k) = \frac{1}{T}\sum_{m=-\infty}^{\infty} X(s+j\omega_s m) + \frac{1}{2}x(0+)$$

$$= X^*(s)$$

Therefore,

$$X^*(s) = X^*(s \pm j\omega k) \quad k=0,1,2\ldots$$

⇒ Thus $X^*(s)$ is periodic, with period $2\pi/\omega_s$. This means that if a function $X(s)$ has a pole at $s=s_1$ in the s plane, then $X^*(s)$ has poles at $s=s_1 \pm j\omega_s k$ $(k=0,1,2\ldots)$

★ Obtaining Z Transforms of Function Involving the term $(1-e^{-Ts})/s$.

⇒ Suppose the transfer function $G(s)$ follows the zero-order hold.

$$X(s) = \frac{1-e^{-Ts}}{s}G(s)$$

$$\Rightarrow X(s) = (1-e^{-Ts})\frac{G(s)}{s} = (1-e^{-Ts})\,G_1(s)$$

$$\left\{\text{where } G_1(s) = \frac{G(s)}{s}\right\}$$

Consider the function $X_1(s) = e^{-Ts}G_1(s)$

$$\Rightarrow x_1(t) = \int_0^t g_0(t-\tau)\,g_1(\tau)\,d\tau$$

$$\begin{cases} g_0(t) = \mathcal{L}^{-1}[e^{-Ts}] = \delta(t-T) \\ g_1(t) = \mathcal{L}^{-1}[G_1(s)] \end{cases}$$

$$x_1(t) = \int_0^t \delta(t-T-\tau)\,g_1(\tau)\,d\tau$$

$$= g_1(t-T)$$

$$\Rightarrow Z[g_1(t)] = G_1(z)$$

$$Z[x_1(t)] = Z[g_1(t-T)] = z^{-1}G_1(z)$$

$$\Rightarrow X(z) = Z[G_1(s) - e^{-Ts}G_1(s)]$$

$$= Z[g_1(t)] - Z[x_1(t)]$$

$$= G_1(z) - z^{-1}G_1(z)$$

$$= (1-z^{-1})\,G_1(z)$$

$$\boxed{X(z) = Z[X(s)] = (1 - z^{-1})\, Z\left[\frac{G(s)}{s}\right]}$$

Example 3.3 $X(s) = \frac{1 - e^{-Ts}}{s} \frac{1}{s+1}$

$$X(z) = Z\left[\frac{1 - e^{-Ts}}{s} \times \frac{1}{s+1}\right]$$

$$= (1 - z^{-1})\, Z\left[\frac{1}{s(s+1)}\right]$$

$$= (1 - z^{-1})\, Z\left[\frac{1}{s} - \frac{1}{s+1}\right]$$

$$\frac{1}{1 - z^{-1}} \qquad \frac{1}{1 - e^{-T} z^{-1}}$$

$$\Rightarrow \boxed{X(z) = \frac{(1 - e^{-T}) z^{-1}}{1 - e^{-T} z^{-1}}}$$

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3.4> Reconstructing Original Signal From Sampled Signals

Sampling Theorem: If ω_s, defined as $2\pi/T$, where T is the sampling period, is greater than $2\omega_1$

$$\omega_s > 2\omega_1$$

where ω_1 is the highest-frequency component present in the continuous-time signal $x(t)$ then the signal $x(t)$ can be reconstructed completly from the sampled signal $x^*(t)$.

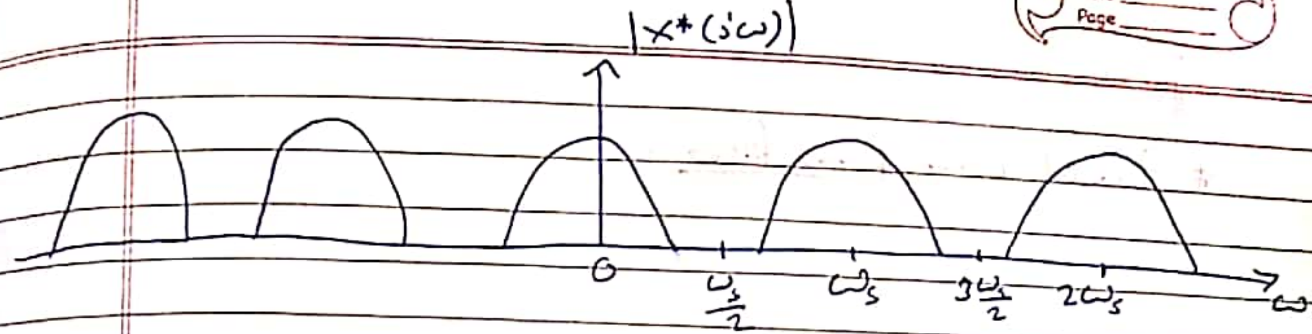
⇒ To show the validity of sampling theorem, we need to find the frequency spectrum of the sampled signal $x^*(t)$

⇒ To obtain the frequency spectrum, we substitute $j\omega$ for s in $X^*(s)$.

$$X^*(j\omega) = \frac{1}{T}\sum_{K=-\infty}^{\infty} X(j\omega + j\omega_s k)$$

⇒ Thus, the process of impulse modulation of the continuous-time signal produces a series of side bands since $X^*(s)$ is periodic with period $2\pi/\omega_s$.

→ If a function $X(s)$ has a pole at $s = s_1$ then $X^*(s)$ has poles at $s = s_1 \pm j\omega_s k$.



⇒ Each plot of $|X^*(j\omega)|$ versus ω consists of $|X(j\omega)|/T$ repeated every $\omega_s = 2\pi/T$ rad/sec.

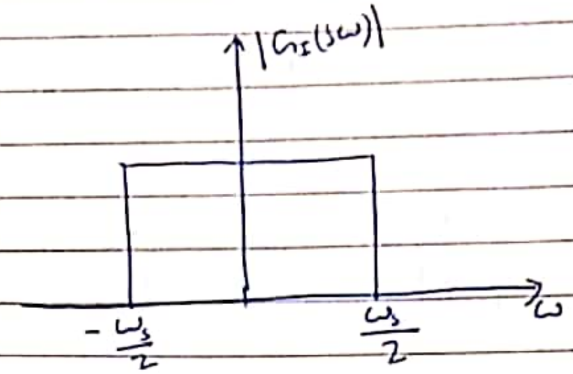
⇒ In the frequency spectrum of $|X^*(j\omega)|$ the component $|X(j\omega)|/T$ is called the primary component, and the other components, $|X(j\omega \pm \omega_s k)|/T$ are called complementary component.

→ If $\omega_s > 2\omega_1$ no two components of $X^*(j\omega)$ will overlap, and the sampled frequency spectrum will be repeated every ω_s rad/sec.

→ If $\omega_s < 2\omega_1$, the original signal of $|X(j\omega)|$ no longer appears in the plot of $|X^*(j\omega)|$ versus ω because of the superposition of the spectra.

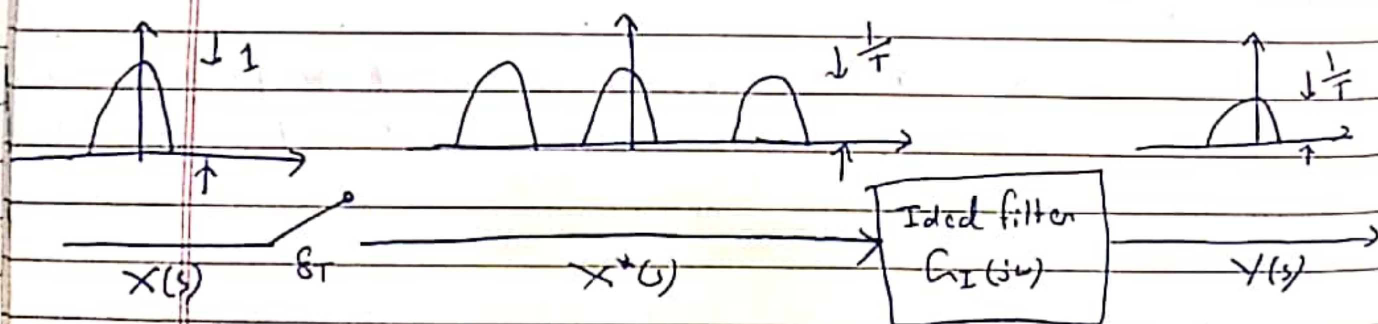
→ Therefore, we see that the continuous-time signal $x(t)$ can be reconstructed from the impulse-sampled signal $x^*(t)$ by filtering if and only if $\omega_s > 2\omega_1$.

Ideal Low-pass filter



⇒ The magnitude of the ideal filter is unity over the frequency range $-\frac{1}{2}\omega_s \leq \omega \leq \frac{1}{2}\omega_s$ and is zero outside this frequency range.

⇒ Such an ideal filter reconstructs the continuous-time signals before and after ideal filtering.



Ideal Low pass filter is not physically Realizable

$$G_I(j\omega) = \begin{cases} 1 & -\frac{1}{2}\omega_s \leq \omega \leq \frac{1}{2}\omega_s \\ 0 & \text{elsewhere} \end{cases}$$

$$g_I(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G_I(j\omega)\, e^{j\omega t}\, d\omega$$

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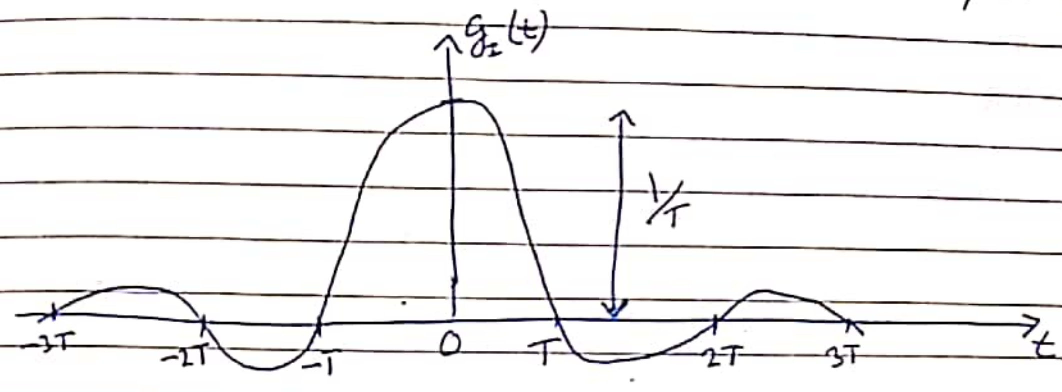
$$g_I(t) = \frac{1}{2\pi}\int_{-\omega_s/2}^{\omega_s/2} e^{j\omega t}\, d\omega = \frac{1}{T}\frac{\sin(\omega_s t/2)}{(\omega_s t/2)}$$



⇒ Notice that the response extends from $t = -\infty$ to $t = \infty$. This implies that there is a response $\forall\, t < 0$ to a unit impulse at $t = 0$. This can not be true in real physical world. Hence, such an ideal filter is physically unrealizable.

⇒ So it is no possible, in practice, to exactly reconstruct a continuous time signal from the sampled signal, no matter what sampling frequency is chosen.

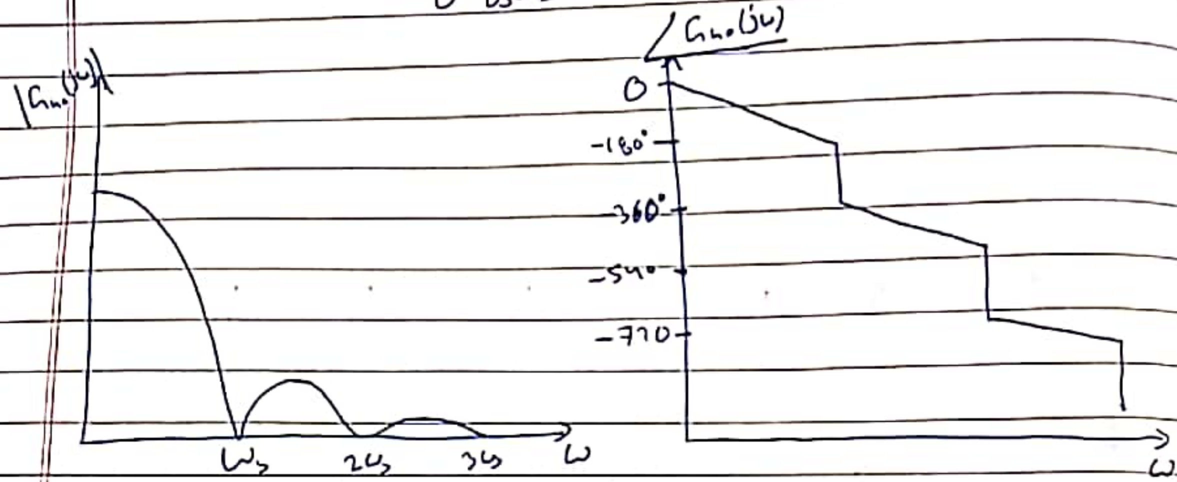
Frequency-Response Characteristics of the Zero-Order Hold

$$G_{ho}(s) = \frac{1 - e^{-Ts}}{s}$$

$$G_{ho}(j\omega) = \frac{1 - e^{-Tj\omega}}{j\omega} = T\frac{\sin(\omega T/2)}{\omega T/2} e^{-\frac{Tj\omega}{2}}$$

$$\left|G_{ho}(j\omega)\right| = T\left|\frac{\sin(\omega T/2)}{\omega T/2}\right|$$

$$\angle G_{ho}(j\omega) = \underbrace{\angle \sin\frac{\omega T}{2}}_{0^\circ \text{ or } \pm 180^\circ} - \frac{\omega T}{2}$$



⇒ The frequency spectrum of the output of the zero order hold includes complementary components, since the magnitude characteristics shows that the magnitude of $G_{ho}(j\omega)$ is not zero for $|\omega| > \frac{1}{2}\omega_s$, except at points where $\omega = \pm\omega_s$, $\omega = \pm 2\omega_s$ ---.

⇒ In the phase curve there are phase discontinuities of $\pm 180^\circ$ at frequency points that are multiples of ω_s.

Folding: The phenomenon of the overlap in the frequency spectrum is known as folding.

→ The frequency $\frac{1}{2}\omega_s$ is called folding frequency or Nyquist frequency ω_N.

Aliasing: The phenomenon that the frequency component $n\omega_s - \omega_2$, where n is an integer shows up at frequency ω_2 when the signal $x(t)$ is sampled is called ~~al~~ aliasing.

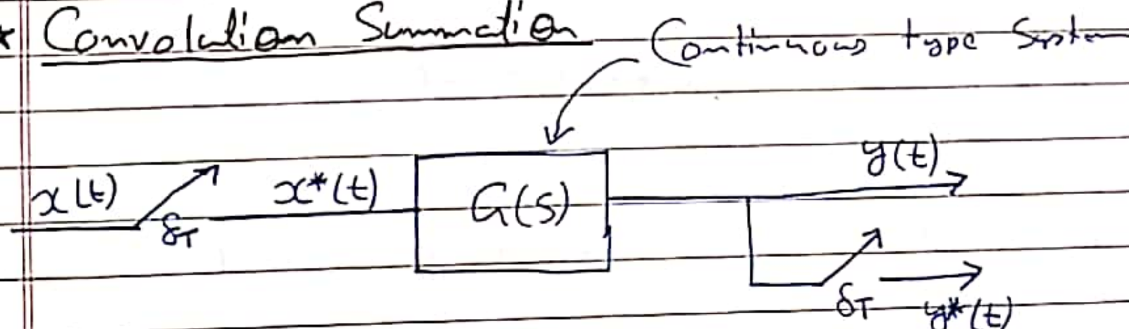
↳ The frequency $n\omega_s \pm \omega_2$ is called an alias of ω_2

3.5 The Pulse Transfer function

⇒ The transfer function for the continuous-time system relates the Laplace transform of the continuous-time output to that of the continuous time input.

⇒ Pulse transfer function relates the Z transform of the output at the sampling instants to that of the sampled ~~output~~ input.

★ **Convolution Summation**



Let $x(t) = 0 \;\; \forall \;\; t < 0$

$$Z[y(t)] = Y(z) = \sum_{k=0}^{\infty} y(kT) z^{-k}$$

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⇒ For a continuous-time system, it is a well-known fact that the output $y(t)$ of the system is related to the input $x(t)$ by the convolution integral

$$y(t) = \int_0^t g(t-\tau)x(\tau)d\tau = \int_0^t x(t-\tau)g(\tau)d\tau$$

$$\left\{ g(t) = \text{Impulse response function of the System or weighting function of the System.} \right\}$$

⇒ For discrete-time system we have a Convolution summation, which is similar to the convolution integral.

$$x^*(t) = \sum_{K=0}^{\infty} x(KT)\delta(t-KT)$$

$$y(t) = \sum_{h=0}^{K} g(t-hT)x(hT) \quad \{0 \leq t \leq KT\}$$

⇒ The value of the output $y(t)$ at the s

$$\boxed{y(KT) = \sum_{h=0}^{K} x(KT-hT)\,g(hT)}$$

⇒ This Summation is called Convolution Summation.

⇒ Simplified notation of Convolution Summation.

$$\boxed{y(KT) = x(KT) * g(KT)}$$

⇒ Above equation can be taken from 0 to K without changing the value of the Summation.

$$y(KT) = \sum_{h=0}^{\infty} g(KT-hT)\,x(hT) = \sum_{h=0}^{\infty} x(KT-hT)\,g(hT)$$

→ If $G(s)$ is a ratio of polynomials in s and if the degree of the denominator polynomial exceeds the degree of the numerator polynomial only by 1 the output $y(t)$ is discontinuous.

→ If the degree of the denominator polynomial exceeds that of the numerator polynomial by 2 or more, however output is Continuous.

★ Pulse transfer function

$$y(KT) = \sum_{h=0}^{\infty} g(KT-hT)\,x(hT) \quad \forall\ K = 0,1,2\ldots$$

$$\left\{ \cancel{g(KT-hT) = 0 \ \forall\ h > K} \right\}$$

⇒ The Z transform of $y(KT)$ become

$$y(kT) = \sum_{h=0}^{\infty} g(kT - hT)\,x(hT)$$

$$Y(z) = \sum_{k=0}^{\infty} y(kT)\, z^{-k}$$

$$\Rightarrow \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} g(kT-hT)\,x(hT)\,z^{-k}$$

$\{m = k-h\}$

$$\Rightarrow \sum_{m=0}^{\infty} \sum_{h=0}^{\infty} g(mT)\,x(hT)\,z^{-(m+h)}$$

$$\Rightarrow \sum_{m=0}^{\infty} g(mT)\,z^{-m} \sum_{h=0}^{\infty} x(hT)\,z^{-h}$$

$$\Rightarrow G(z)\,X(z)$$

Z transform of g(t)   Z transform of x(t)

$$\Rightarrow \boxed{G(z) = \frac{Y(z)}{X(z)}}$$

Pulse transfer function

$$X(z) \longrightarrow \boxed{G(z)} \longrightarrow Y(z)$$

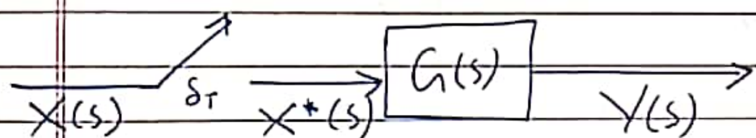
$\Rightarrow$ $G(z)$ is also the z transform of the System's response to the Kronecker delta input:

$$x(KT) = \delta_0(KT) \begin{cases} 1 & \forall K = 0 \\ 0 & \forall K \neq 0 \end{cases}$$

Z transform of the Kronecker delta input is

$$X(z) = \sum_{K=0}^{\infty} x(KT) z^{-K} = 1$$

★ **Starred Laplace transform of the Signal Involving Both ordinary and Starred Laplace Transform**



$$Y(s) = G(s) X^*(s)$$

$$Y^*(s) = [G(s) X^*(s)]^* = G^*(s) X^*(s)$$

$$y(t) = \mathcal{L}^{-1}[G(s) X^*(s)]$$

$$= \int_0^t g(t-\tau) x^*(\tau) d\tau$$

$$= \int_0^t g(t-\tau) \sum_{K=0}^{\infty} x(\tau) \delta(\tau - KT) d\tau$$

$$\Rightarrow \sum_{K=0}^{\infty} \int_0^t g(t-\tau) x(\tau) \delta(\tau - KT)$$

$$\Rightarrow \sum_{K=0}^{\infty} g(t-KT)x(KT)$$

$$Y(z) = Z[y(t)]$$

$$= \sum_{n=0}^{\infty}\left[\sum_{K=0}^{\infty} g(nT-KT)x(KT)\right]z^{-n}$$

$$\{m=n-K\} \qquad = \sum_{m=0}^{\infty} g(mT)z^{-m} \sum_{K=0}^{\infty} x(KT)z^{-K}$$

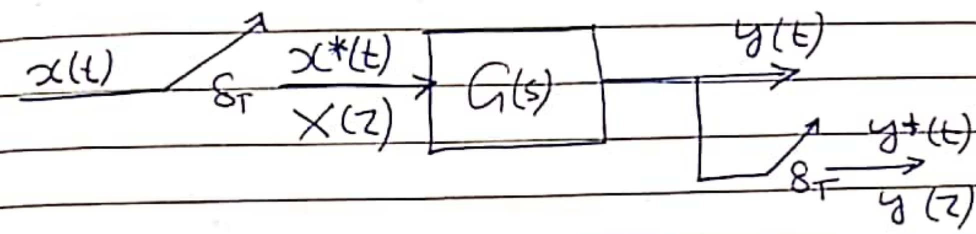
$$\Rightarrow G(z)X(z)$$

⇒ Since the Z transform can be understood as the Starred Laplace transform with e^{TS} replaced by Z, the Z transform may be considered as to be a Shorthand notation for the Starred Laplace transform

⇒ It is noted that Systems become periodic under Starred Laplace transform operations. Such periodic System are generally more Complicated to analyze than the original non periodic ones but the former may be analyzed without difficulty if carried out in the Z plane.

★ General procedure for obtaining pulse transfer function

Example 3-4



$$G(s) = \frac{1}{s+a}$$

$$Z\left[\frac{1}{s+a}\right] = \frac{1}{1-e^{-aT}z^{-1}} = G(z)$$

Example 3-5

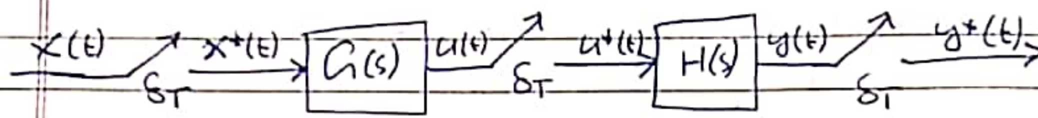
$$G(s) = \frac{1-e^{-Ts}}{s}\,\frac{1}{s(s+1)}$$

$$G(z) = Z\left[\frac{1-e^{-Ts}}{s}\,\frac{1}{s(s+1)}\right]$$

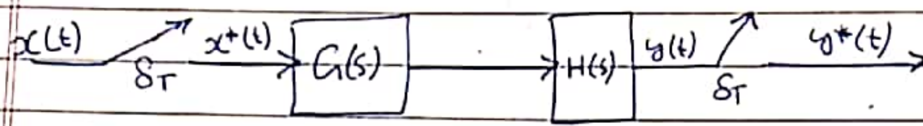
$$= (1-z^{-1})\,Z\left[\frac{1}{s^2(s+1)}\right]$$

$$G(z) = (1-z^{-1})\left[\frac{Tz^{-1}}{(1-z^{-1})^2} - \frac{1}{1-z^{-1}} + \frac{1}{1-e^{-T}z^{-1}}\right]$$

* Pulse transfer function of Cascaded Element



(a)



(b)

(a) $U(s) = G(s)\,X^*(s)$

$Y(s) = H(s)\,U^*(s)$

$\Rightarrow U^*(s) = G^*(s)\,X^*(s)$

$Y^*(s) = H^*(s)\,U^*(s)$

$\Rightarrow Y^*(s) = G^*(s)\,H^*(s)\,X^*(s)$

$\Rightarrow Y(z) = G(z)\,H(z)\,X(z)$

$$\frac{Y(z)}{X(z)} = G(z)\,H(z)$$

(b) $Y(s) = G(s)\,H(s)\,X^*(s)$

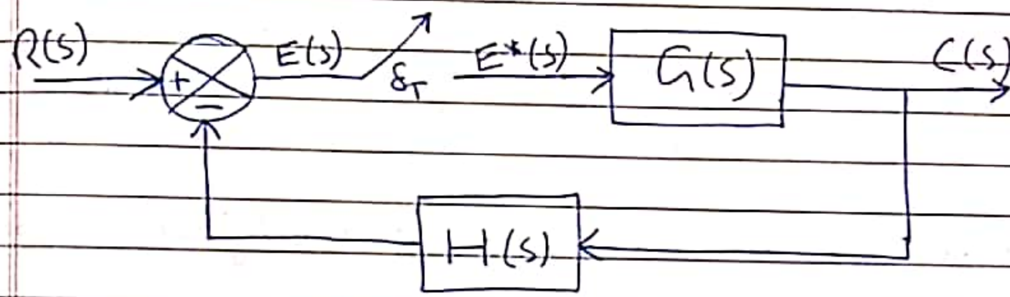
$Y^*(s) = \boxed{GH(s)}^*\,X^*(s)$

$\Rightarrow Y(z) = GH(z)\,X(z)$

$$GH(z) = Z[G(s)H(s)] = \frac{Y(z)}{X(z)}$$

$$Z[G(s)H(s)] \neq G(z)H(z)$$

★ Pulse transfer function of Closed-loop System



$$E(s) = R(s) - H(s)C(s)$$
$$C(s) = G(s)E^*(s)$$

$$\Rightarrow E(s) = R(s) - H(s)G(s)E^*(s)$$

$$\Rightarrow E^*(s) = R^*(s) - GH^*(s)E^*(s)$$

$$\Rightarrow E^*(s) = \frac{R^*(s)}{1 + GH^*(s)}$$

$$C^*(s) = G^*(s)E^*(s)$$

$$\Rightarrow C^*(s) = \frac{G^*(s)R^*(s)}{1 + GH^*(s)}$$

$$\Rightarrow C(z) = \frac{G(z)R(z)}{1 + GH(z)}$$

⇒ The pulse transfer function for the present closed-loop System is

$$\frac{C(z)}{R(z)} = \frac{G(z)}{1+GH(z)}$$

⇒ Some discrete-time closed-loop control Systems cannot be represented by $C(z)/R(z)$ because the input signal $R(s)$ cannot be Separated from the System dynamics.

★ Pulse transfer function of a Digital Controller

⇒ The pulse transfer function of a digital Controller may be obtained from the required input-output characteristics of the digital controller.

⇒ Suppose the input to the digital Controller is $e(K)$ and the output $m(K)$.

⇒ In general, the output is given by the following type of difference equation:

$$m(K) + a_1 m(K-1) + a_2 m(K-2) + \cdots + a_n m(K-n)$$

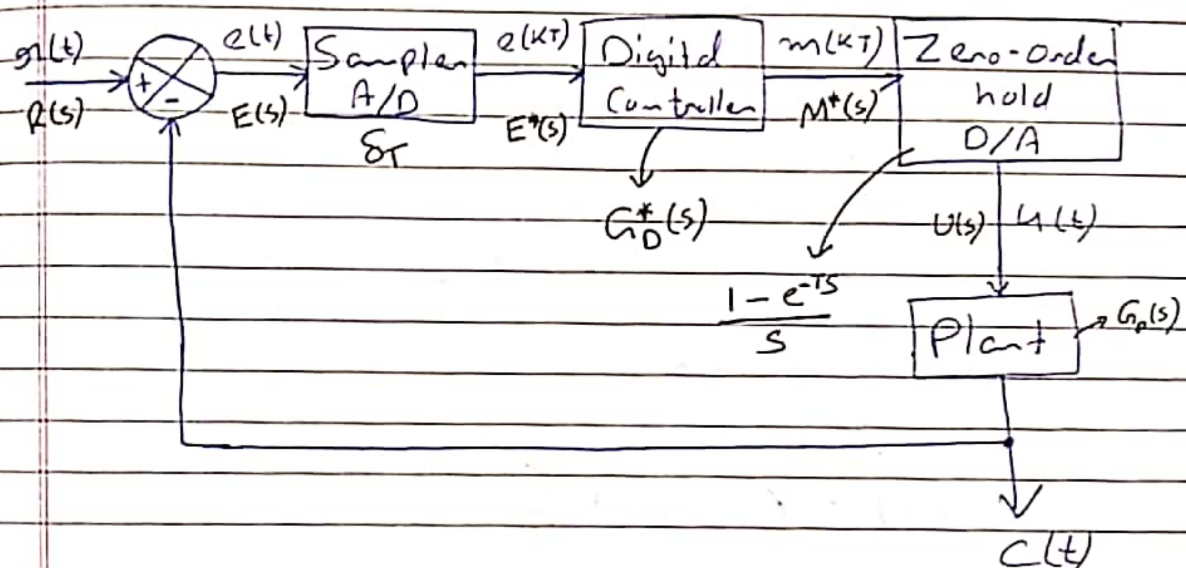
$$= b_0 e(K) + b_1 e(K-1) + \cdots + b_n e(K-n)$$

⇒ The Z transform of above equation is :-

Pulse transfer function of digital controller

$$G_D(z) = \frac{M(z)}{E(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_n z^{-n}}{1 + a_1 z^{-1} + \cdots + a_n z^{-n}}$$

★ Closed-loop pulse transfer function of a Digital Control System



⇒ Let us define $\frac{1-e^{-Ts}}{s} G_p(s) = G(s)$

$\Rightarrow \quad C(s) = G(s)\, G_D^*(s)\, E^*(s)$

$\Rightarrow \quad C^*(s) = G^*(s)\, G_D^*(s)\, E^*(s)$

$\Rightarrow \quad C(z) = G(z)\, G_D(z)\, E(z)$

$\Rightarrow \quad E(z) = R(z) - C(z)$

$$\Rightarrow \quad \boxed{\frac{C(z)}{R(z)} = \frac{G_D(z)\, G(z)}{1 + G_D(z)\, G(z)}}$$

Closed loop Pulse TF of digital control system

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★ Pulse Transfer function of a Digital PID Controller

⇒ The PID Control action in analog Controllers is given by :-

$$m(t) = K\left[e(t) + \frac{1}{T_i}\int_0^t e(t)\,dt + T_d \frac{de(t)}{dt}\right]$$

⇒ To obtain the pulse transfer function for the digital PID Controller, we may discretize above equation by approximating the integral term by the trapezoidal summation and the derivative term by a two point difference form. we obtain.

$$m(KT) = K\left[e(KT) + \frac{T}{T_I}\left\{\frac{e(0)+e(T)}{2} + \frac{e(T)+e(2T)}{2} + \cdots + \frac{e((K-1)T)+e(KT)}{2}\right\} + T_d\,\frac{e(KT)-e((K-1)T)}{T}\right]$$

$$\Rightarrow m(KT) = K\left[e(KT) + \frac{T}{T_I}\sum_{h=1}^{K}\frac{e((h-1)T)+e(hT)}{2} + \frac{T_d}{T}\left[e(KT) - e((K-1)T)\right]\right]$$

Let us define $\dfrac{e((h-1)T)+e(hT)}{2} = f(hT)$, $f(0)=0$

So, $$\sum_{h=1}^{K} \frac{e((h-1)T)+e(hT)}{2} = \sum_{h=1}^{K} f(hT)$$

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$$Z\left[\sum_{h=1}^{K} \frac{e((h-1)T)+e(hT)}{2}\right] = Z\left[\sum_{h=1}^{K} f(hT)\right]$$

$$= \frac{F(z)-f(0)}{1-z^{-1}}$$

$$\Rightarrow \frac{1}{1-z^{-1}} F(z)$$

$$F(z) = Z\left[f(hT)\right] = \frac{1+z^{-1}}{2} E(z)$$

$$\text{So } Z\left[\sum_{h=1}^{K} \left\{\frac{e((h-1)T)+e(hT)}{2}\right\}\right] = \frac{(1+z^{-1})}{2(1-z^{-1})} E(z)$$

$$\Rightarrow M(z) = K\left[1 + \frac{T}{2T_i}\frac{1+z^{-1}}{1-z^{-1}} + \frac{T_d}{T}(1-z^{-1})\right] E(z)$$

Pulse TF of digital PID Controller

$$\boxed{M(z) = \left[K_P + \frac{K_I}{1-z^{-1}} + K_D(1-z^{-1})\right] E(z)} \quad (1)$$

$$\left\{ K_P = K - \frac{K_I}{2} \qquad K_D = \frac{K T_d}{T} \qquad K_I = \frac{KT}{T_i} \right\}$$

⇒ The pulse transfer function of the digital PID Controller given by Eq ① is commonly referred to as the positional form of the PID Control Scheme.

⇒ The other form commonly used in the digital PID Control scheme is referred to as the velocity form.

⇒ An advantage of the velocity-form PID Control scheme is to initialization is not necessary when the operation is switched from manual to automatic.

Note that in digital Controllers, control laws can be implemented by software, and therefore the hardware restrictions of analog PID Controllers can be completely ignored.

3.6) Realization of Digital Controllers and Digital filters

⇒ In software realization we obtain computer programs for the digital computer involved.

⇒ In a hardware realization we build a special-purpose processor using such circuitry as digital adders, multipliers, and delay elements.

⇒ In the field of digital signal processing, a digital filter is a computational algorithm that converts an input sequence of numbers into an output sequence in such a way that the characteristics of the signal are changed in some prescribed fashion.

↳ In general term, a digital controller is a form of digital filter.

⇒ This section deals with the block diagram realization of digital filters using delay elements, adders, and multipliers.

↳ Such block diagram realizations can be used as a basis for a software or hardware design.

↳ In fact, once the block diagram realization is completed, the physical realization in hardware or software is straight forward.

⇒ The general form of the pulse transfer function between the output $Y(z)$ and input $X(z)$ is given by:-

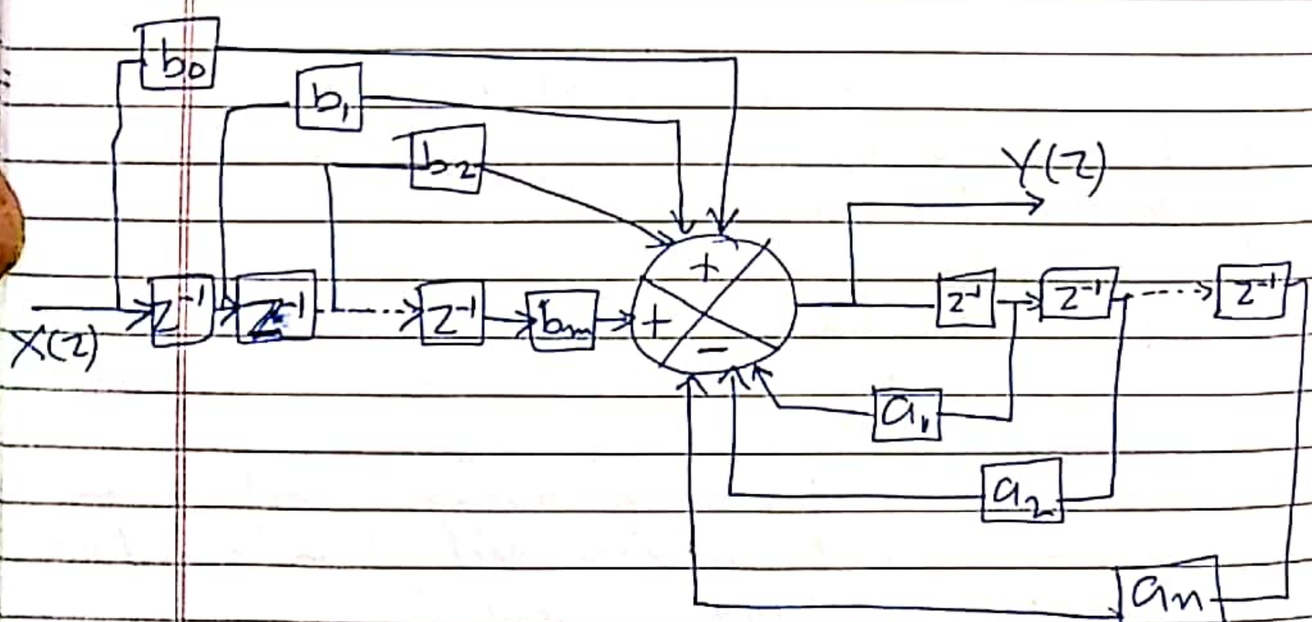
$$\boxed{G(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots b_m z^{-m}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots a_n z^{-n}}} \quad \text{---} (1)$$

$\{n \geq m\}$

★ Direct Programming

⇒ Digital filter given by Eq (1) has n poles and m zeros.

$$Y(z) = -a_1 z^{-1} Y(z) - a_2 z^{-2} Y(z) - \cdots - a_n z^{-n} Y(z)$$
$$+ b_0 X(z) + b_1 z^{-1} X(z) + \cdots + b_m z^{-m} X(z)$$



⇒ This type of realization here is called direct programming.

⇒ The total number of delay element used in direct programming is m+n.

⇒ The number of delay elements used in direct programming can be reduced. In fact, the number of delay elements can be reduced from n+m to n.

★ Standard Programming

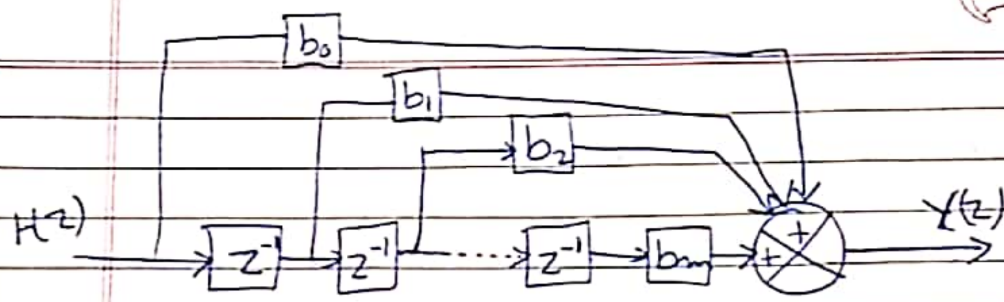
⇒ First, rewrite the pulse transfer function $Y(z)/X(z)$ as:-

$$\frac{Y(z)}{X(z)} = \frac{Y(z)}{H(z)} * \frac{H(z)}{X(z)}$$

$$= \left(b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_m z^{-m}\right) * \left(\frac{1}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_n z^{-n}}\right)$$

$$\frac{Y(z)}{H(z)} = b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_m z^{-m}$$

$$\frac{H(z)}{X(z)} = \frac{1}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_n z^{-n}}$$



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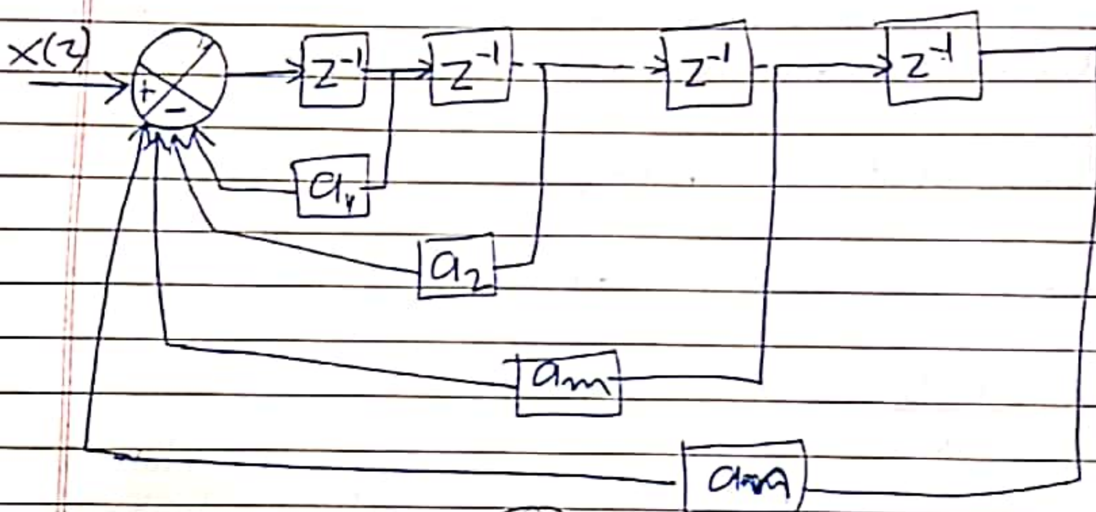

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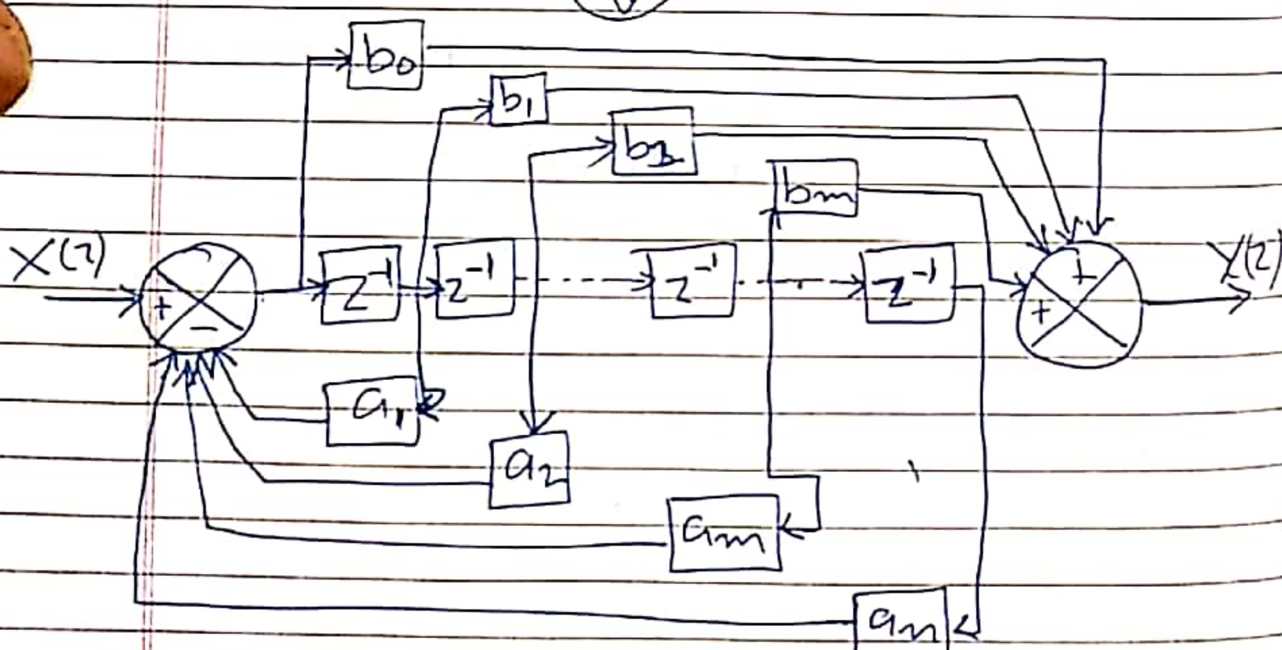


⇒ In realizing digital controllers or digital filters, it is important to have a good level of accuracy. Basically, three sources of errors effects the accuracy:-

1. The error due to the quantization of the input Signal into a finite number of discrete level.

2. The error due to the accumulation of round off errors in the arithmetic operations in the digital ~~Signal~~ System.

3. The error due to quantization of the coefficients a_i and b_i of the pulse transfer function.

⇒ For decompositing higher-order pulse transfer functions in order to avoid the coefficient sensitivity problem, the following three approaches are commonly used.

1. Series Programming

2. Parallel programming

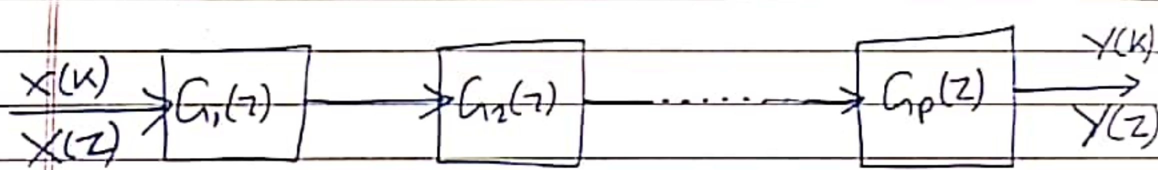
3. Ladder programming

1. Series Programming

⇒ The first approach used to avoid the Sensitivity problem is to implement the pulse transfer function $G(z)$ as a Series Connection of first-order and/or Second-order pulse transfer functions.

⇒ If $G(z)$ can be written as a product of pulse transfer function $G_1(z), G_2(z) \cdots G_p(z)$

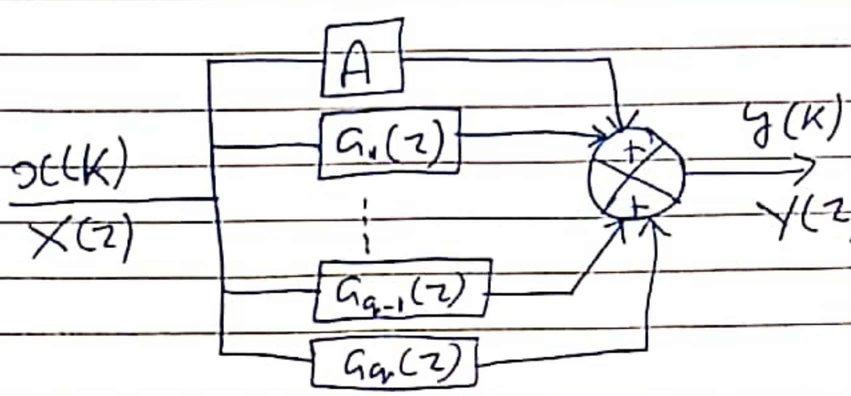
$$G_{\bullet}(z) = G_1(z)\, G_2(z) \cdots G_p(z)$$



2. Parallel Programming

⇒ The Second approach to avoid the Coefficient Sensitivity problem is to expand the pulse transfer function $G(z)$ into partial fractions.

$$G(z) = A + G_1(z) + G_2(z) \cdots + G_q(z)$$



3. Ladder Programming

⇒ The third approach to avoide the Coefficient Sensitivity problem is to implement a ladder Structure, that is to expand the pulse transfer function $G(z)$ into the following continuous-fraction form and to program according to this eandition.

$$G(z) = A_0 + \cfrac{1}{B_1 z + \cfrac{1}{A_1 + \cfrac{1}{B_2 z \cfrac{1}{\vdots \; A_{n-1} \cfrac{1}{B_n z + \cfrac{1}{A_n}}}}}}$$

Let we define:-

$$G_i^{(B)}(z) = \frac{1}{B_i z + G_i^{A}(z)} \qquad i = 1, 2 \cdots n-1$$

$$G_i^{(A)}(z) = \frac{1}{A_i + G_{i+1}^{(B)}(z)} \qquad i = 1, 2 \cdots n-1$$

$$G_n^{(B)}(z) = \frac{1}{B_n z + \frac{1}{A_n}}$$

⇒ Then we may write

$$\boxed{G(z) = A_0 + G_1^{(B)}(z)}$$

★ Infinite - Impulse Response Filter and Finite Impulse Response Filter

⇒ Digital filter may be classified according to the duration of the impulse response.

⇒ If $y(k) = -\{a_1 y(k-1) + a_2 y(k-2) \cdots - a_n y(k-n)\} + \{b_0 x(k) + b_1 x(k-1) + \cdots + b_m x(k-m)\}$

→ The impulse response of the digital filter shown above, where we assume not all a_i's are zero, has an infinite number of nonzero samples.

→ This type of digital filter is called an infinite-impulse response filter. (also called recursive filter).

⇒ If $y(k) = b_0 x(k) + b_1 x(k-1) + \cdots + b_m x(k-m)$

→ The impulse response of the digital filter defined above is limited to a finite number of samples defined over a finite range of time intervals.

→ This type of digital filter is called a finite-impulse response filter. (or non-recursive filter)

CHAPTER 4

Design of Discrete Time Control System By Conventional Method

4

Design of discrete-time Control System by Conventional method

4.1) Introduction

⇒ Conventionally three different design methods for SISO discrete-time control system:

i) Root Locus technique using pole-zero configurations in the z plane

ii) Frequency response method in the ω plane.

iii) Analytical method in which we attempt to obtain a desired behavior of the closed-loop system by manipulating the pulse transfer function of digital controller.

4.2) Mapping between the s plane and the z plane

⇒ The absolute stability and relative stability of a linear time-invariant continuous time closed-loop control system are determined by the locations of the closed-loop poles in the s plane.

⇒ Since the complex variables z and s are related by $z = e^{TS}$, the pole and zero locations in the z plane are related to the pole & zero locations in the s plane.

⇒ Therefore, the stability of the linear time-invariant discrete-time closed loop system can be determined in terms of the locations of the poles of the closed loop pulse transfer function.

⇒ Dynamic behavior of the discrete-time control system depends on the sampling period T. Therefore change in the sampling period T modifies the pole and zero locations in the z plane and causes the response behavior to change.

★ Mapping of the Left half of S plane into the Z plane

⇒ Pole in S plane can be located in the Z plane through the transformation $Z = e^{TS}$.

$$S = \sigma + j\omega$$

$$\Rightarrow Z = e^{T(\sigma + j\omega)} = e^{T\sigma} e^{jT\omega} = e^{T\sigma} * e^{j(T\omega + 2\pi K)}$$

⇒ From above we see that poles and zeros in the s plane, where frequencies differ in integral multiple of the sampling frequency $2\pi/T$, are mapped into the same locations in the z plane.

↳ This means that there are infinite many values of s for each value of z.

⇒ Since σ is negative in the left half of the s plane, the left half of the s plane corresponds to

$$|Z| = e^{T\sigma} < 1$$

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⇒ The $j\omega$ axis in the S plane Corresponds to $|z|=1$. That is, the imaginary axis in the S plane (the line $\sigma=0$) Corresponds to the unit circle in the Z plane, and the interior of the unit circle Corresponds to the left half of the S plane.

★ Primary Strip and Complementary strips

$$\boxed{\angle z = T\omega}$$

⇒ Consider a representative point on the $j\omega$ axis in the S plane. As this point moves from $-\frac{1}{2}\omega_s j$ to $j\frac{1}{2}\omega_s$ on the $j\omega$ axis where ω_s is the sampling frequency, we have $|z|=1$, and $\angle z$ Varies from $-\pi$ to π in the Counterclockwise direction in the Z plane.

⇒ As the representative point moves from $j\frac{1}{2}\omega_s$ to $j\frac{3}{2}\omega_s$ on the $j\omega$ axis, the Corresponding point in the Z plane traces out the unit circle once in the Counterclockwise direction.

⇒ From this analysis, it is clear that each strip of width ω_s in the left half of the S plane maps into the inside of the unit circle in the Z plane.

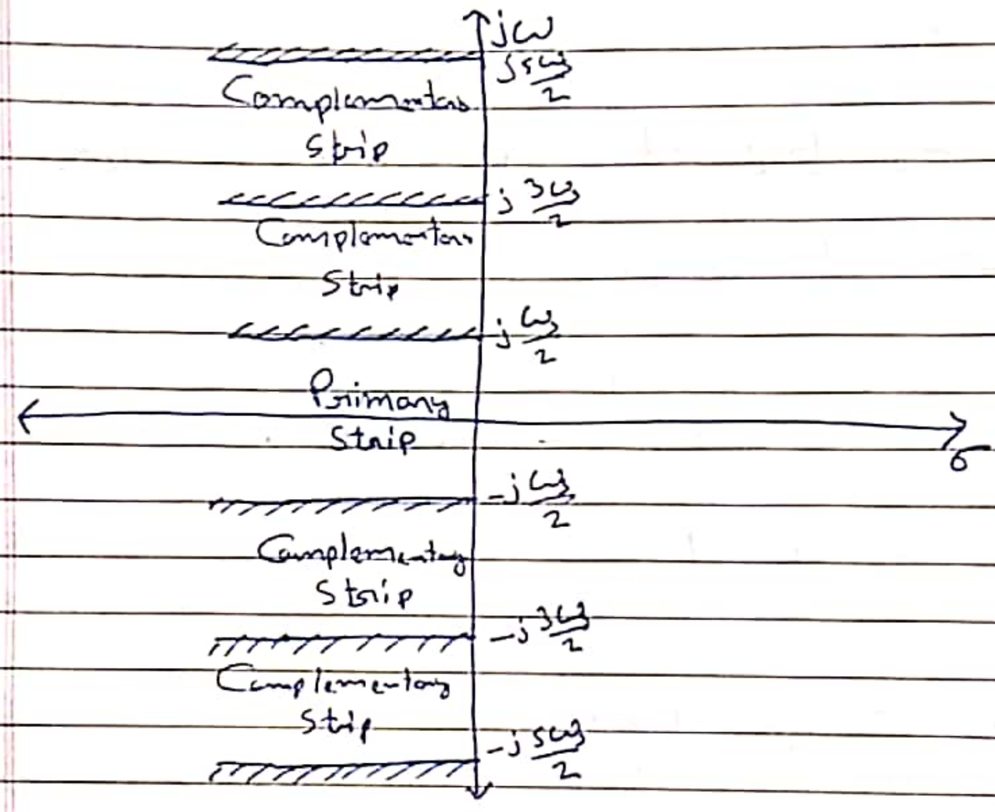
⇒ This implies that the left half of the S plane may be divided into an infinite number of periodic Strips.

* Primary strip ⇒ $j\omega = \left(-j\frac{1}{2}\omega_s \text{ to } j\frac{1}{2}\omega_s\right)$

$\ldots \left(j\frac{1}{2}\omega_s \text{ to } -j\frac{3}{2}\omega_s\right),$

* Complementary strips $\Rightarrow \left(j\frac{1}{2}\omega_s \text{ to } j\frac{3}{2}\omega_s\right)$

$, \left(j\frac{3}{2}\omega_s \text{ to } j\frac{5}{2}\omega_s\right) \ldots$



$\Rightarrow$ A point in the Z plane corresponds to an infinite number of points in the S plane, although a point in the S plane corresponds to a single point in the Z plane.

$\Rightarrow$ If the sampling frequency is at least twice as fast as the highest-frequency component involved in the system, then every point in the unit circle in the z plane represents frequencies between $-\frac{1}{2}\omega_s$ and $\frac{1}{2}\omega_s$.

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★ Constant-Attenuation Loci

A Constant-attenuation line (σ = Constant) in the S plane maps into a Circle of radius $Z = e^{T\sigma}$ Centered at the origin in the Z plane.

★ Settling time t_s

⇒ The setting time is determined by the value of attenuation σ of the dominant Closed loop poles.

⇒ If the settling time is specified, it is possible to draw a line $\sigma = \sigma_1$ in the S plane Corresponds to a given Settling time.

⇒ The region to the left of the line $\sigma = \sigma_1$ in the S plane Corresponds to the inside of a Circle with radius $e^{-\sigma_1 T}$ in the Z plane.

★ Constant-Frequency Loci

⇒ A Constant-frequency locus $\omega = \omega_1$ in the S plane is mapped into a radial line of Constant angle $T\omega_1$ (in radians) in the Z plane.

★ Constant-Damping Ratio Loci

⇒ A Constant damping ratio line (a radial line) in the S plane is mapped into a Spiral in the Z plane.

⇒ This can be seen as follows :-

$$S = -\xi\omega_n + j\omega_n\sqrt{1-\xi^2} = -\xi\omega_n + j\omega_d$$

$$\{\omega_d = \omega_n\sqrt{1-\xi^2}\}$$

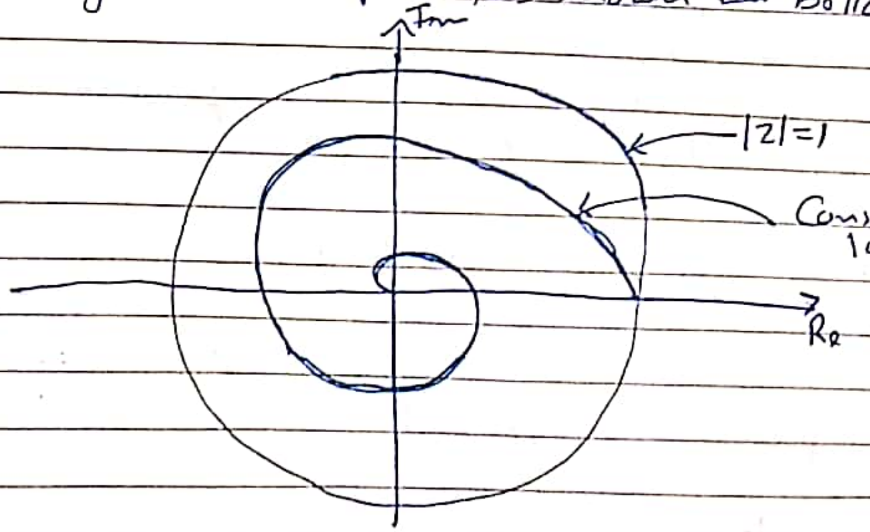
$$Z = e^{TS} = \exp(-\xi\omega_n T + j\omega_d T)$$

$$= \exp\left(\frac{-2\pi\xi\omega_d}{\omega_s\sqrt{1-\xi^2}} + j2\pi\frac{\omega_d}{\omega_s}\right)$$

$$|Z| = \exp\left(\frac{-2\pi\xi}{\omega_s\sqrt{1-\xi^2}}\omega_d\right)$$

$$\angle Z = 2\pi\frac{\omega_d}{\omega_s}$$

$\Rightarrow$ Thus, the magnitude of z decreases and the angle of z increases linearly as ω_d increases, and the locus in the Z-plane becomes a logarithmic spiral, as shown ~~ba~~ bellow.



⇒ Thus the Spiral can be graduated in terms of a normalized frequency ω_d/ω_s.

⇒ Constant ξ loci are normal to the Constant ω_n loci in the s plane.

↳ In the z plane mapping, Constant ω_n loci intersect Constant ξ spirals at right angles.

⇒ A mapping such as this, which preserves both the size and the sense of angles, is called a Conformal mapping.

4.3> Stability Analysis of Closed-loop System in the Z-plane

⇒ Consider the following closed-loop pulse-transfer function system:

$$\frac{C(z)}{R(z)} = \frac{G(z)}{1+GH(z)}$$

⇒ The Stability of the System defined above as well as of other types of discrete-time control systems, may be determined from the location of the closed loop poles in the z plane, or the roots of the characteristic equation

$$\boxed{P(z) = 1 + GH(z) = 0}$$

as follows :-

1. For the System to be Stable, the Closed-loop Poles or the roots of the characteristic equation must lie within the unit circle in the Z plane.
   → Any closed-loop pole outside the unit circle makes the System unstable.

2. If a simple pole lie at $z=1$, then the System becomes critically stable. Also the System becoms critically stable if a single pair of Conjugate complex poles lies on the unit circle in the Z plane.
   → Any multiple closed-loop pole on the unit circle makes the System unstable.

3. Closed loop zeros donot affect the absolute stability and therefor may be located anywhere in the Z plane.

★ Methods for Testing Absolute Stability

⇒ Three Stability test can be applied directly to the characteristic equation $P(z)=0$ without solving for the roots.

1. Schur-Cohn Stability test

2. Jury stability test.

3. Bilinear transformation coupled with the Routh stability criterion.

1) The Jury Stability test

Let the characteristic equation $P(z)$ is a polynomial in z as follows:

$$P(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$$

$$\{a_0 > 0\}$$

⇒ Elements in the first row consist of the coefficients in $P(z)$ arranged in the ascending order of powers of z.

⇒ The elements in the second row consist of the coefficients of $P(z)$ arranged in the descending order of power of z.

⇒ The elements for rows 3 through $2n-3$ are given by the following determinants.

$$b_k = \begin{vmatrix} a_n & a_{n-1-k} \\ a_0 & a_{k+1} \end{vmatrix} \qquad k = 0, 1, 2 \ldots n-1$$

$$c_k = \begin{vmatrix} b_{n-1} & b_{n-2-k} \\ b_0 & b_{k+1} \end{vmatrix} \qquad k = 0, 1, 2 \ldots n-2$$

$$\vdots$$

$$q_k = \begin{vmatrix} p_3 & p_{2-k} \\ p_0 & p_{k+1} \end{vmatrix} \qquad k = 0, 1, 2$$

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⇒ Element of any even-numbered row are simply the reverse of the immediately preceding odd-numbered row.

Stability criterion by the Jury test

1. $|a_n| < a_0$

2. $P(z)\big|_{z=1} > 0$

3. $P(z)\big|_{z=-1} \begin{cases} > 0 & \forall n \text{ even} \\ < 0 & \forall n \text{ odd} \end{cases}$

4. $|b_{n-1}| > |b_0|$

   $|c_{n-2}| > |c_0|$

   $\vdots \qquad \vdots$

   $|q_2| > |q_0|$

Example 4.3: $P(z) = a_0 z^4 + a_1 z^3 + a_2 z^2 + a_3 z + a_4$

Row	z^0	z^1	z^2	z^3	z^4
1	a_4	a_3	a_2	a_1	a_0
2	a_0	a_1	a_2	a_3	a_4
3	b_3	b_2	b_1	b_0	
4	b_0	b_1	b_2	b_3	
5	c_2	c_1	c_0		

$P(1) = a_0 + a_1 + a_2 + a_3 + a_4 > 0$ —— (a)

$P(-1) = a_0 - a_1 + a_2 - a_3 + a_4 > 0$ —— (b)

Example 4.4: $P(z) = z^4 - 1.2z^3 + 0.07z^2 + 0.3z - 0.08 = 0$

	z^0	z^1	z^2	z^3	z^4
1	1	-1.2	0.07	0.3	-0.08
2	-0.08	0.3	0.07	-1.2	1
3	~~0.204~~ -0.99	~~0.0756~~ -1.176	~~-1.176~~ 0.076	~~-0.99~~ 0.204	
4	~~-0.99~~ 0.204	~~-1.176~~ 0.0756	~~0.0756~~ -1.176	~~0.204~~ -0.99	
5	0.938	1.148	0.16506		
6	0.16506	1.148	0.938		

$P(1) = 0.09$ ✓

$P(-1) = 1.85$ ✓

		z^0	z^{-1}	z^2	z^3	z^4
a	1	-0.08	0.3	0.07	-1.2	1
	2	1	-1.2	0.07	0.3	-0.08
b	3	-0.99	1.176	-0.0756	-0.204	
	4	-0.204	-0.0756	1.176	-0.99	
c	5	0.94	-1.18	0.3147		
	6	0.3147	-1.18	0.94		

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2) Stability Analysis by Use of the Bilinear Transformation and Routh Stability

⇒ The method requires transformation from the Z plane to another complex plane, the ω plane.

→ The amount of Computation required is much more than that required in the Jury stability Criterion.

⇒ The bilinear transformation defined by

$$Z = \frac{\omega + 1}{\omega - 1} \quad \text{or} \quad \omega = \frac{Z+1}{Z-1}$$

⇒ This transformation maps the unit circle in the Z plane into the left half of the ω plane.

$$P(z) = a_0 Z^n + a_1 Z^{n-1} + \cdots a_{n-1} Z + a_n = 0$$

$$\Rightarrow a_0 \left(\frac{\omega+1}{\omega-1}\right)^n + a_1 \left(\frac{\omega+1}{\omega-1}\right)^{n-1} + \cdots + a_{n-1}\left(\frac{\omega+1}{\omega-1}\right) + a_n = 0$$

⇒ Multiplying both side by $(\omega-1)^n$ we obtain.

$$Q(\omega) = b_0 \omega^n + b_1 \omega^{n-1} + \cdots + b_{n-1}\omega + b_n = 0$$

⇒ Once we transform $P(z)=0$ into $Q(\omega)=0$, it is possible to apply the Routh stability Criterion in the Same manner as in Continuous-time system.

4.4> Transient and Steady-State Response Analysis

"Absolute stability is a basic requirement of all Control system. In addition, good relative stability and steady-state accuracy are also required of any Control System, whether Continuous time or discrete time"

★ Transient Response Specifications

⇒ The transient response of a System to a unit-step input depends on the initial Conditions. For Convenience in Comparing transient responses of various system, it is a common practice to use the standard initial condition; the System is at rest initially and the output and all its time-derivatives are zero.

1. Delay time t_d
2. Rise time t_r
3. Peak time t_p
4. Maximum overshoot M_p
5. Settling time t_s

⇒ Let us assume that the Sampling theorem is satisfied and no frequency folding occurs.

⇒ Consider the discrete-time Control System defined by :-

$$\frac{C(z)}{R(z)} = \frac{b_0 z^n + b_1 z^{n-1} + \cdots + b_n}{z^n + a_1 z^{n-1} + \cdots + a_n}$$

Where $R(z)$ is the z transform of the input and $C(z)$ is the z transform of the output.

⇒ The transient response of such a system to to the Kronecker delta input, step input, ramp input, and so on, can be obtained easily by use of Matlab.

★ Steady-State Error Analysis

⇒ The steady-state performance of a stable control system is generally judged by the steady-state error due to step, ramp, and acceleration input.

⇒ Consider the continuous time control system whose open-loop transfer function $G(s)H(s)$ is given by

$$G(s)\,H(s) = \frac{K(T_a s+1)(T_b s+1)\cdots(T_m s+1)}{s^N (T_1 s+1)(T_2 s+1)\cdots(T_p s+1)}$$

⇒ It is customary to classify the system according to the number of integrators in the open-loop transfer function.

↳ A system is said to be of type 0, type 1, type 2 ··· if $N=0$, $N=1$, $N=2 \cdots$ respectively.

System	Steady state Error to		
	Step Input	Ramp input	acceleration input
Type 0	finite	∞	∞
Type 1	0	finite	∞
Type 2	0	0	finite

⇒ However, increasing the type number aggravates the stability problem.

↳ A compromise between steady-state accuracy and relative stability is always necessary.

⇒ The concepts of static error constants can be extended to the discrete-time control system, as discussed in what follows.

⇒ Discrete-time control system can be classified according to the number of open-loop poles at $z=1$.

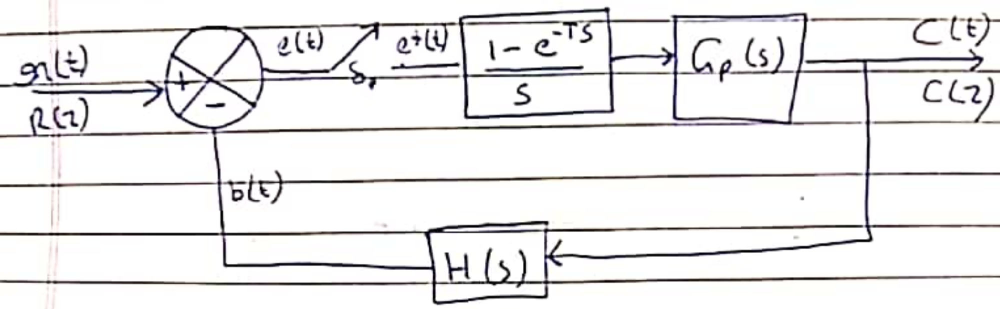
⇒ Suppose open-loop transfer function is given by

$$\text{Open-loop pulse transfer function} = \frac{1}{(z-1)^N}\frac{B(z)}{A(z)}$$

Where $B(z)/A(z)$ contains neither a pole nor a zero at $z=1$.

⇒ The N is the type of the system.

⇒ Consider the discrete-time control system shown below:



$$e(t) = r(t) - b(t)$$

$$\lim_{K\to\infty} e(KT) = \lim_{z\to 1}\left[(1-z^{-1})E(z)\right]$$

$$G(z) = (1-z^{-1})\, Z\left[\frac{G_p(s)}{s}\right]$$

$$GH(z) = (1-z^{-1})\, Z\left[\frac{G_p(s)\,H(s)}{s}\right]$$

$$\frac{C(z)}{R(z)} = \frac{G(z)}{1+GH(z)}$$

$$\Rightarrow E(z) = R(z) - B(z) = R(z) - GH(z)E(z)$$

$$\Rightarrow E(z) = \frac{R(z)}{1+GH(z)}$$

$$e_{ss} = \lim_{z\to 1}\left[(1-z^{-1})\frac{R(z)}{1+GH(z)}\right]$$

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★ Static Position Error Constant

For unit-step input $r(t) = H(t)$

$$R(z) = \frac{1}{1-z^{-1}}$$

$$e_{ss} = \lim_{z \to 1} \left[(1-z^{-1}) \frac{1}{1+GH(z)} \frac{1}{1-z^{-1}} \right] = \lim_{z\to 1} \frac{1}{1+GH(z)}$$

⇒ We define the static position error constant K_p as

$$\boxed{K_p = \lim_{z \to 1} GH(z)}$$

$$\boxed{e_{ss} = \frac{1}{1+K_p}}$$

★ Static velocity Error constant

$r(t) = t\,H(t)$

$$\Rightarrow R(z) = \frac{T z^{-1}}{(1-z^{-1})^2}$$

$$e_{ss} = \lim_{z \to 1} \left[(1-z^{-1}) \frac{1}{1+GH(z)} \frac{T z^{-1}}{(1-z^{-1})^2} \right] = \lim_{z\to 1} \frac{T}{(1-z^{-1})GH(z)}$$

Let $\boxed{K_v = \lim_{z \to 1} \frac{(1-z^{-1})GH(z)}{T}}$

$$\boxed{e_{ss} = \frac{1}{K_v}}$$

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★ Static Acceleration Error Constant

$$r(t) = \frac{1}{2} t^2 H(t)$$

$$R(z) = \frac{T^2 (1 + z^{-1}) z^{-1}}{2(1 - z^{-1})^3}$$

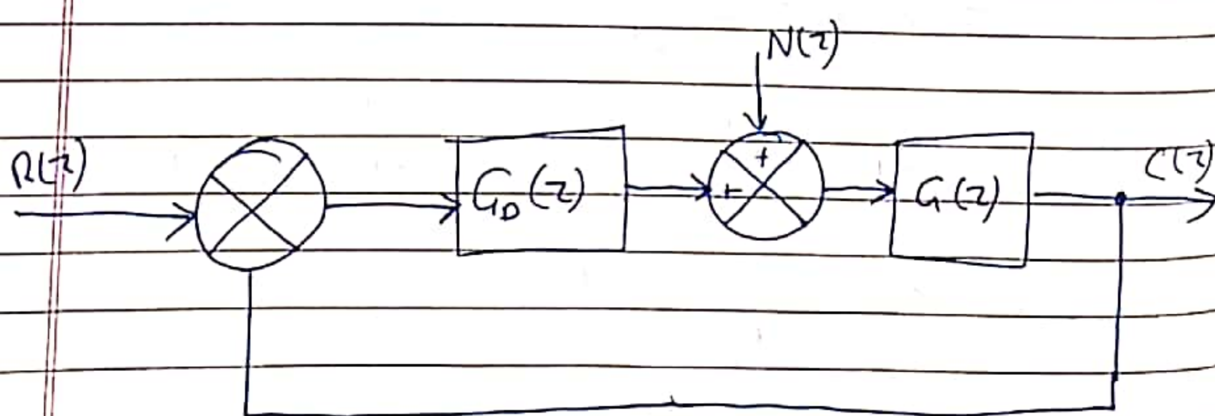
$$e_{ss} = \lim_{z \to 1} \left[(1 - z^{-1}) \frac{1}{1 + GH(z)} \frac{T^2 (1 + z^{-1}) z^{-1}}{2(1 - z^{-1})^3} \right]$$

$$= \lim_{z \to 1} \frac{T^2}{(1 - z^{-1})^2 GH(z)}$$

Let $\boxed{K_a = \frac{(1 - z^{-1})^2 GH(z)}{T^2}}$

$$\boxed{e_{ss} = \frac{1}{K_a}}$$

★ Response to Disturbances



$$\frac{C(z)}{N(z)} = \frac{G(z)}{1+G_D(z)G(z)} \quad \{\text{assuming } R(z)=0\}$$

If $|G_D(z)G(z)| \gg 1$

$$\Rightarrow \frac{C(z)}{N(z)} \simeq \frac{1}{G_D(z)}$$

$$E(z) = R(z) - C(z) = -C(z) = -\frac{N(z)}{G_D(z)}$$

$\Rightarrow$ Thus the larger the gain $G_D(z)$ is, the smaller the error $E(z)$.

$\Rightarrow$ Note that the point where the disturbance enters the system is very important in adjusting the gain of $G_D(z)G(z)$.

4.5) Design based on the Root locus Method

⇒ In addition to the transient response characteristic of a given System, it is often necessary to investigate the effects of the System gain and/or Sampling period on the absolute and relative stability of the Closed-loop System.

↳ For such purpose the root-locus method proves to be very useful.

⇒ The root-locus method developed for Continuous-time systems Can be extended to discrete-time System without modification, except that the stability boundary is changed from the $j\omega$ axis in the S plane to the unit circle in the Z plane.

* Angle and Magnitude Condition

$$1 + G(z)H(z) = 0 \quad \text{or} \quad 1 + GH(z) = 0$$

$$\Rightarrow \quad 1 + F(z) = 0$$

$F(z)$ → $G(z)H(z)$ or $GH(z)$

→ It is open loop pulse transfer function.

$$\Rightarrow \quad \boxed{F(z) = -1} \rightarrow |F(z)| = 1$$

$$\rightarrow \angle F(z) = \pm 180^\circ (2K+1) \quad \forall K = 0, 1, 2 \cdots$$

⇒ The value of z that fulfill both the angle and magnitude conditions are the roots of the characteristic equation, or the closed-loop poles.

General procedure for Constructing Root Loci

1. Obtain the characteristic equation

$$1 + F(z) = 0$$

and then rearrange this equation as given bellow.

$$1 + K\frac{(z+z_1)(z+z_2)\cdots(z+z_m)}{(z+P_1)(z+P_2)\cdots(z+P_n)} = 0$$

↳ Locate open loop poles and zeros in z plane.

2. Find the starting points and terminating points of the root loci.

↳ Starting point ⇒ Open loop poles $(K=0)$

↳ Terminating point $(K=\infty)$ ⇒ (Open loop zero) or (open loop zero at ∞)

3. Determine the root loci on the real axis.

↳ In constructing the root loci on the real axis, choose a test point on it and try to satisfy the angle condition.

$$\angle F(z) = \pm 180(2K+1) \quad \forall K = 0, 1, 2 \ldots$$

4. Determine the asymptotes of the root loci.

→ If the test point z is located far from the origin, then the angles of all the complex quantities may be considered the same.

→ One open-loop zero and one open-loop pole then each cancel the effects of the other.

$$\text{Angle of Asymptote} = \frac{\pm 180^\circ (2N+1)}{n-m} \quad N = 0, 1, 2 \cdots$$

n = {Number of finite Poles}, m = {number of finite zeros}

⇒ All the asymptotes intersect on the real axis. The point at which they do so is obtained as follows.

$$F(z) = \frac{K\left[z^m + (z_1+z_2+\cdots z_m)z^{m-1} + \cdots + z_1 z_2 z_3 \cdots z_m\right]}{z^n + (P_1+P_2+\cdots P_m)z^{n-1} + \cdots + P_1 P_2 \cdots P_n}$$

$$= K \times \frac{1}{z^{n-m} + \left[(P_1+P_2\cdots+P_n) - (z_1+z_2\cdots z_n)\right] z^{n-m-1} + \cdots}$$

for a large value of z above equation can be approximated as :-

$$F(z) \simeq \frac{K}{\left[z + \frac{(P_1+P_2+\cdots P_n)-(z_1+z_2+\cdots z_m)}{n-m}\right]^{n-m}}$$

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⇒ If the abscissa of the intersection of the asymptotes and the real axis is denoted by $-\sigma_a$ then

$$-\sigma_a = -\frac{(P_1+P_2+\cdots P_n)-(Z_1+Z_2+\cdots+Z_m)}{n-m}$$

5. Find the breakaway points and break in points.

→ If a root locus lies between two adjacent open-loop poles on the real axis, then there exists at least one breakaway point between the two poles.

→ If the root locus lies between two adjacent zeros, on the real axis (one zero may be at ∞), then there always exists at least one break in point between the two zeros.

If the characteristic equation is

$$1 + K\frac{B(z)}{A(z)} = 0$$

$$\Rightarrow K = -\frac{A(z)}{B(z)}$$

⇒ The breakaway and break in point can be determined from the roots of

$$\boxed{\frac{dK}{dz} = 0}$$

6. Determine the angle of departure (or angle of arrival) of the root loci from the complex poles (or at the complex zeros).

↳ Angle can be found out by applying angle condition at that particular pole or zero.

7. Find the point where the root loci crosses the imaginary axis.

8. Any point on the root loci is a possible closed-loop pole.

↳ Value of K can be found by satisfying magnitude condition.

★ **Cancellation of Poles of $G(z)$ With Zeros of $H(z)$**

⇒ If $F(z) = G(z)H(z)$ and the denominator of $G(z)$ and the numerator of $H(z)$ involve common factor then the corresponding open loop poles and zeros will cancel each other, reducing the degree of the characteristic equation by one or more.

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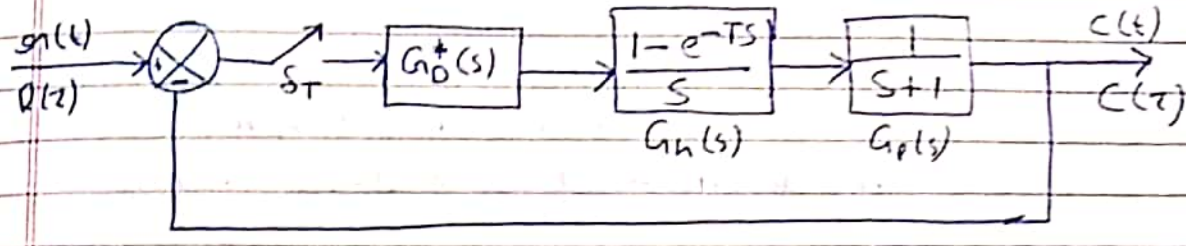
* Root locus Diagrams of Digital Control System



Let $G_D(z)$ be one integral type Controller.

$$G_D(z) = \frac{K}{1-z^{-1}} = K\frac{z}{z-1}$$

$$Z\left[G_h(s)\,G_p(s)\right] = Z\left[\frac{1-e^{-TS}}{S}\,\frac{1}{S+1}\right]$$

$$\boxed{Z\left[G_h(s)\,G_p(s)\right] = \frac{1-e^{-T}}{z-e^{-T}}}$$

$$G(z) = G_D(z)\,Z\left[G_h(s)\,G_p(s)\right] = \frac{Kz}{z-1}\,\frac{1-e^{-T}}{z-e^{-T}}$$

$\Rightarrow$ $1 + G(z) = 0$ {Characteristic equation}

$$\Rightarrow \boxed{1 + \frac{Kz\,(1-e^{-T})}{(z-1)(z-e^{-T})} = 0}$$

* Effect of Sampling period T on transient Response Characteristics

"For a given value of gain K, increasing the sampling period T will make the discrete-time control system less stable and eventually make it ~~stable~~ unstable.

⇒ A Rule of thumb :-

→ Sample eight to ten times during a Cycle of the damped Sinusoidal oscillations of the output of the closed-loop System, if it is underdamped.

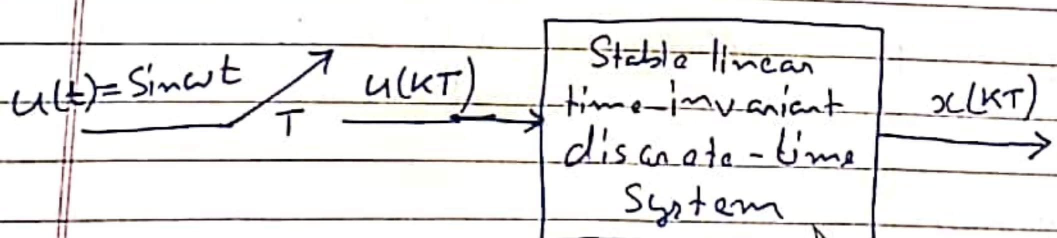
→ For overdamped system, Sample eight to ten times during the rise time in the Step response.

4.6) Design based on the Frequency-Response Method

⇒ In performing frequency-response tests on a discrete-time System, it is important that the System have a low-pass filter before the Sampler So that sidebands are filtered out.

★ Response of a Linear Time-Invariant Discrete-Time System to a Sinusoidal Input

⇒ Consider the Stable linear time-invariant discrete-time control system as shown.



$u(t) = \sin \omega t$

$u(kT) = \sin k\omega T$

$U(z) = Z[\sin k\omega T]$

$$U(z) = \frac{z \sin \omega T}{(z-e^{j\omega T})(z-e^{-j\omega T})}$$

$$X(z) = G(z)\,U(z) = G(z)\frac{z \sin \omega T}{(z-e^{j\omega T})(z-e^{-j\omega T})}$$

$$= \frac{az}{z-e^{j\omega T}} + \frac{\bar{a}z}{z-e^{-j\omega T}} + \left[\text{terms due to poles of } G(z)\right]$$

$$a = G(z)\frac{\sin \omega T}{z-e^{-j\omega T}}\Bigg|_{z=e^{j\omega T}} = \frac{G(e^{j\omega T})}{2j}$$

$$\bar{a} = -\frac{G(e^{-j\omega T})}{2j}$$

Let us define, $G(e^{j\omega T}) = Me^{j\theta}$

& $G(e^{-j\omega T}) = Me^{-j\theta}$

So $$X(z) = \frac{Me^{j\theta}}{2j}\left(\frac{z}{z-e^{j\omega T}}\right) - \frac{Me^{-j\theta}}{2j}\left(\frac{z}{z-e^{-j\omega T}}\right)$$

$$+ \left[\text{terms due to poles of } G(z)\right]$$

$$x(kT) = \frac{M}{2j}\left[e^{jk\omega T}e^{j\theta} - e^{-jk\omega T}e^{-j\theta}\right]$$

$$+ Z^{-1}\left[\text{terms due to poles of } G(z)\right]$$

⇒ Since the system $G(s)$ has been assumed to be stable, all transient response terms will disappear at steady state and will get the following steady-state response $x_{ss}(kT)$

$$x_{ss}(kT) = \frac{M}{2j}\left[e^{j(k\omega T+\theta)} - e^{-j(k\omega T+\theta)}\right]$$

$$\boxed{\begin{aligned} x_{ss}(kT) &= M \sin(k\omega T+\theta) \\ M &= |G(e^{j\omega T})| \\ \theta &= \angle G(e^{j\omega T}) \end{aligned}}$$

⇒ The function $G(e^{j\omega T})$ is commonly called the <u>Sinusoidal pulse transfer function</u>.

$$e^{j(\omega + \frac{2\pi}{T})T} = e^{j\omega T}$$

↳ Sinusoidal pulse transfer function $G(e^{j\omega T})$ is periodic, with the period equal to T.

★ Bilinear Transformation and the w plane

⇒ Before we can advantageously apply over-well developed frequency response methods to the analysis and design of discrete-time Control Systems, Certain modifications in the z plane approach are necessary.

→ We transform the pulse transfer function in the z plane into that in the w plane.

→ The transformation, commonly called the w transformation, a bilinear transformation, is defined by

$$\left\{ w = \frac{2}{T}\frac{z-1}{z+1} \right\} \quad z = \frac{1 + (T/2)w}{1 - (T/2)w}$$

{Sampling period}

⇒ Through the z transform and the w transformation, the primary strip of the left half of the s plane is first mapped into the inside of the unit Circle in the z plane and then mapped into the entire left half of the w plane.

⇒ Once the pulse transfer function G(z) is transformed into G(w) by means of w transformation.

→ It may be treated as a Conventional TF in w.

→ Conventional frequency response techniques Can then be used in w plane.

⇒ As noted earlier, ν represents the fictious frequency. By replacing ω by $j\nu$ conventional frequency-response techniques may be used to draw the Bode diagram for the transfer function in ω.

⇒ Although the ω plane resembles the S plane geometrically, the frequency axis is in the ω plane is distorted.

$$\omega\Big|_{\omega = j\nu} = j\nu = \frac{2}{T}\frac{z-1}{z+1}\Big|_{z=e^{j\omega T}} = \frac{2}{T} j \tan\frac{\omega T}{2}$$

$$\nu = \frac{2}{T}\tan\frac{\omega T}{2}$$

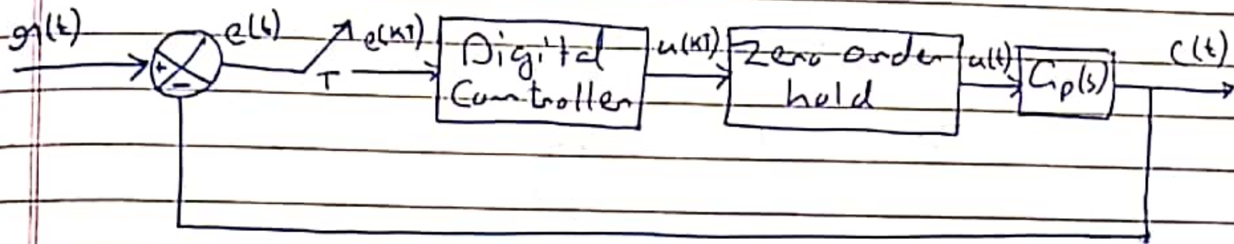
4.7 ⇒ Analytical Design Method

⇒ The main reason why the control actions of analog controllers are limited is that there are physical limitations in pneumatic, hydraulic, and electronic components.

↳ Such limitations may be completely ignored in designing digital controllers.

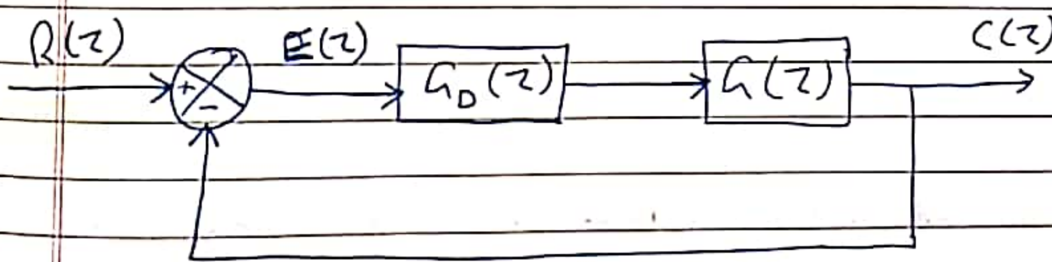
⇒ If the response of a closed-loop control system to a step input exhibits the minimum possible settling time, no steady-state error, and no ripples between the sampling instants, then this type of response is commonly called a deadbeat response.

* Design of Digital Controllers for Minimum Setting time with Zero Steady State error



Let us define the Z transform of the plant that is preceded by the zero-order hold as G(z)

$$G(z) = Z\left[\frac{1-e^{-Ts}}{s} G_P(s)\right]$$



Let the desired closed-loop pulse transfer function as F(z).

$$F(z) = \frac{C(z)}{R(z)} = \frac{G_D(z)G(z)}{1 + G_D(z)G(z)}$$

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⇒ Since it is required that the System exhibit a finite Settling time with zero Steady-State error, the System must exhibit a finite impulse response.

↳ Hence, the desired closed loop pulse transfer function must be of the following form:-

$$F(z) = \frac{a_0 z^N + a_1 z^{N-1} + \dots + a_N}{z^N}$$

$$\Rightarrow a_0 + a_1 z^{-1} + a_2 z^{-2} + \dots a_N z^{-N}$$

$$\{N \geq n\}$$

Order of the System

$$\Rightarrow G_D(z) = \frac{F(z)}{G(z)[1 - F(z)]}$$

⇒ The designed System must be physically realizable.

↳ The Conditions for physical realizability may be stated as follows:-

1. The order of the numerator of $G_D(z)$ must be equal to or lower than the order of the denominator.

2. If the plant $G_p(s)$ involves a transportation lag e^{-Ls}, then the designed closed loop System

must involve at least the same magnitude of the transportation Lag.

3. If $G(z)$ is expanded into a series in z^{-1}, the lowest-power term of the series expansion of $F(z)$ in z^{-1} must be at least as larger as that of $G(z)$.

⇒ In addition to the physical realizability conditions we must pay attention to the stability aspect of the system.

↳ Specifically, we must avoid canceling an unstable pole of the plant by a zero of the digital controller.

↳ If such a cancellation is attempted any error in the pole-zero cancellation will diverge as time elapses and the system will become unstable.

⇒ Let us investigate what will happen to the closed-loop pulse transfer function $F(z)$ if $G(z)$ involves an unstable pole:

$$G(z) = \frac{G_1(z)}{z-\alpha} \qquad \{|\alpha| \geq 1\}$$

→ It does not involves terms that cancel with $(z-\alpha)$

So $\dfrac{C(z)}{R(z)} = \dfrac{G_D(z)G(z)}{1 + G_D(z)G(z)} = \dfrac{G_D(z)\dfrac{G_1(z)}{z-\alpha}}{1 + G_D(z)\dfrac{G_1(z)}{z-\alpha}} = F(z)$

$$1 - F(z) = \frac{1}{1 + G_D(z)\dfrac{G_1(z)}{z-\alpha}} = \frac{(z-\alpha)}{(z-\alpha) + G_D(z)G_1(z)}$$

$\Rightarrow$ Since we require that no zero of $G_D(z)$ cancel the unstable pole of $G(z)$ at $z=\alpha$.

↳ We must have that $1-F(z)$ must have $z=\alpha$ as a zero.

Summarize

1. Since the digital Controller $G_D(z)$ should not Cancel unstable poles of $G(z)$, all unstable poles of $G(z)$ must be included in $1-F(z)$ as Zeros.

2. Zeros of $G(z)$ that lie inside the unit circle may be Canceled with poles of $G_D(z)$. However, Zeros of $G(z)$ that lie on or outside the unit circle must not be Canceled with poles of $G_D(z)$. Hence, all zeros of $G(z)$ that lies on or outside the unit circle must be included in $F(z)$ as zero.

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$\Rightarrow$ $E(z) = R(z) - C(z)$

$= R(z)\{1 - F(z)\}$

$$R(z) = \frac{1}{1-z^{-1}} \quad \{\text{if } r(t) = H(t)\}$$

$$R(z) = \frac{T z^{-1}}{(1-z^{-1})^2} \quad \{\text{if } r(t) = tH(t)\}$$

$$R(z) = \frac{T^2 z^{-1}(1+z^{-1})}{2(1-z^{-1})^3} \quad \{\text{if } r(t) = \frac{1}{2}t^2 H(t)\}$$

$\Rightarrow$ Thus in general Z transform of such time domain polynomial inputs may be written as

$$R(z) = \frac{P(z)}{(1-z^{-1})^{q+1}}$$

$P(z) \rightarrow$ Polynomial in z^{-1}

$$\Rightarrow E(z) = \frac{P(z)[1-F(z)]}{(1-z^{-1})^{q+1}}$$

$\Rightarrow$ To ensure that the System reaches steady state in finite number of sampling periods and maintain zero steady-state error, $E(z)$ must be a polynomial in z^{-1} with finite number of term.

$$1 - F(z) = (1-z^{-1})^{q+1} N(z)$$

$N(z) \rightarrow$ Polynomial in z^{-1}

$$E(z) = P(z)\, N(z)$$

⇒ From the preceding analysis, the pulse transfer function of the digital controller can be determined as follows:-

↳ By letting $F(z)$ satisfy the physical realizability and stability conditions.

$$G_D(z) = \frac{F(z)}{G(z)\,(1-z^{-1})^{a+1}\,N(z)}$$

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