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Analysis of MIMO System

(1) ★ Norms

⇒ A norm is a mathematical tool to measure "lengths".

⇒ Let X be a vector space. $\|\cdot\|: X \rightarrow R$ is a norm if it satisfies the following properties:

① Non-negativity: $\|x\| \geq 0$

② Positive definiteness: $\|x\| = 0 \Leftrightarrow x = 0$

③ Homogeneity: $\|\alpha x\| = |\alpha| \|x\| \quad \forall \alpha \in \mathbb{C}$

④ Triangular inequality: $\|x + y\| \leq \|x\| + \|y\| \quad \forall x, y \in X$

★ Vectors norm

Let $x = [x_1 \cdots x_n]^T \in \mathbb{C}^n$

① P-th norm

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \quad \forall p \in N^+ = \{1, 2, \cdots\}$$

② Two norm (Euclidian norm)

$$\|x\|_2 = \sqrt{\sum_{i=1}^{n} |x_i|^2}$$

② Infinity norm

$$\|x\|_\infty = \max_i |x_i|$$

★ Matrix norms

⇒ Let $G \in \mathbb{C}^{l \times m}$, with elements $g_{i,j}$

① Forobenius matrix norm

$$\|G\|_F = \sqrt{\sum_{i,j} |g_{i,j}|^2}$$

② Max element norm

$$\|G\|_{max} = \max_{i,j} |g_{i,j}|$$

★ Induced matrix norms

↳ Quantify the maximum gain (amplification) of output vector for any possible input direction.

① Induced p-norm

$$\|G\|_{i,p} = \max_{\omega \neq 0} \frac{\|G\omega\|_p}{\|\omega\|_p}$$

② Induced 2-norm

$$\|G\|_{i,2} = \max_{\omega \neq 0} \frac{\|G\omega\|_2}{\|\omega\|_2} = \sqrt{\max_j |\lambda_j(G^*G)|}$$

③ Induced infinity norm

$$\|G\|_{i,\infty} = \max_i \left(\sum_j |g_{i,j}| \right)$$

⇒ All matrix norm satisfy:

$$\boxed{\|AB\| \leq \|A\|\|B\|}$$

★ Signal norms

Let $e(t) = [e_1(t) \cdots e_n(t)]^T$, $e_i(t) \in C, \forall i = 1, \dots n$

① P-th norm

$$\|e(t)\|_p = \left(\int_{-\infty}^{\infty} \sum_{i=1}^{n} |e_i(\tau)|^p \, d\tau\right)^{\frac{1}{p}}$$

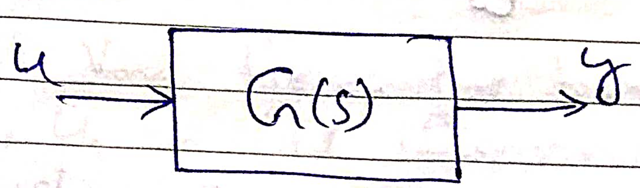
② Two norm ("Energy of the Signal")

$$\|e(t)\|_2 = \sqrt{\int_{-\infty}^{\infty} \sum_{i=1}^{n} |e_i(\tau)|^2 \, d\tau}$$

③ Infinity norm

$$\|e(t)\|_\infty = \sup_{\tau}\left(\max_{i} |e_i(\tau)|\right)$$

★ System norms



① 2-norm

→ A measure of the combination of system gains in all direction over all frequencies.

→ Does not respect the multiplicative property.

$$\|G(s)\|_2 = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} \|G(j\omega)\|_F^2 \, d\omega}$$

② Infinity norm

→ A measure of the peak of the maximum singular value (i.e. the biggest amplification the system may bring at any frequency for any input direction.

→ Respects the multiplicative property

$$\|G(s)\|_\infty = \sup_{\omega} \|G(j\omega)\|_{i,2}$$

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② MIMO Analysis (directionality)

★ **Singular value Decomposition (SVD)**

⇒ Let $G \in C^{l \times m}$

⇒ It can always be written as

$$G = U \Sigma V^*$$

Where:

$$U = [u_1 \cdots u_l] \in C^{l \times l}$$

$$V = [v_1 \cdots v_m] \in C^{m \times m}$$

{~~Orthonormal~~} Unitary

$$\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{with} \quad \Sigma_1 = \begin{bmatrix} \sigma_1 & & 0 \\ & \sigma_2 & \\ 0 & & \ddots \; \sigma_p \end{bmatrix}$$

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_p \quad \{P \text{ is rank of } G\}$$

$$\sigma_i = \sqrt{\lambda_i (G^* G)}$$

⇒ $v_1, v_2 \cdots v_m$ forms the basis of input direction and $u_1, u_2 \cdots u_l$ forms the basis of output direction.

⇒ Singular values

↳ Are directional gains along each input/output direction.

$$G = U \Sigma V^*$$

$$\Rightarrow GV = U\Sigma$$

$$\Rightarrow G v_i = \sigma_i u_i$$

$$\sigma_i = \frac{\|G v_i\|_2}{\|u_i\|_2}$$

★ SVD observations

⇒ For any input, the input output's two-norm ratio is bounded by the singular values.

$$\underline{\sigma}(G) \leq \frac{\|Gu\|_2}{\|u\|_2} \leq \bar{\sigma}(G)$$

Condition number

↳ Coarse measure of how "directional" a MIMO plant is

$$K(G) = \frac{\bar{\sigma}(G)}{\underline{\sigma}(G)} \geq 1$$

⇒ The higher the condition number is, the more difficult it is (typically) to control a MIMO plant.

★ SVD Computation recipe

1. Find eigenvalues of G^*G

2. Calculate singular values as the roots of the eigenvalues found in ①.
   ↳ Name the singular values in decreasing order such that $\sigma_1 \geq \sigma_2 \geq \cdots$

3. Build the singular value matrix, recalling it has same dimension of G.
   ↳ Place the singular values on the main diagonal in decreasing order.

4. Compute the right singular vectors V_i as eigen vectors of G^*G
   ↳ Make matrix unitary (vector orthogonal) by using Gram-Schmid if necessary.

5. Compute the left singular vectors as

$$\boxed{u_i = \frac{1}{\sigma_i} G V_i}$$

* Poles and zeros direction

ⓐ Zero

$$0 = G(z) = \begin{bmatrix} \underset{\sim}{u_1} & \underset{\sim}{u_2} \end{bmatrix} \begin{bmatrix} \sigma_1 > 0 & 0 \\ 0 & \sigma_2 = 0 \end{bmatrix} \begin{bmatrix} v_1 & \underset{\sim}{v_2} \end{bmatrix}^*$$

Output direction   Input direction

ⓑ Poles

$$G(p+\varepsilon) \begin{bmatrix} \underset{\sim}{u_1} & u_2 \end{bmatrix} \begin{bmatrix} \sigma_1 \gg 0 & 0 \\ 0 & \sigma_2 \ll \sigma_1 \end{bmatrix} \begin{bmatrix} \underset{\sim}{v_1} & v_2 \end{bmatrix}^*$$

$\varepsilon \to 0$

Output direction.   Input direction

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