Appendix: Stability Part 1

★ **Equilibrium state**

⇒ For nth order unforced system

$$\dot{x} = f(x,t) \quad , x(t_0) = x_0$$

an equilibrium state/point x_e is one such that

$$\boxed{f(x_e, t) = 0 \quad \forall t}$$

★ **Equilibrium state of a linear System**

⇒ For a linear System:

$$\dot{x}(t) = A(t)x(t) \quad , x(t_0) = x_0$$

- Origin $x_e = 0$ is always an equilibrium state
- When $A(t)$ is singular, multiple equilibrium states exist.

★ **Continuous function**

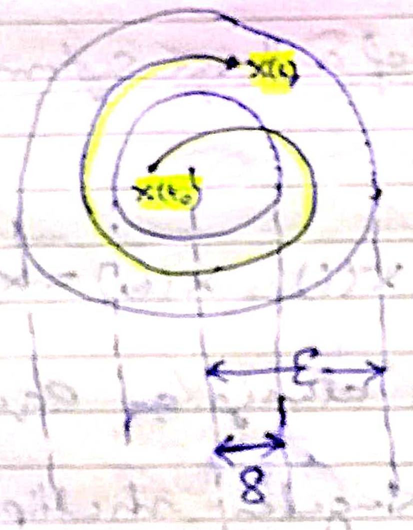
The function $f: R \to R$ is continuous at x_0 if $\forall \epsilon > 0$, there exists a $\delta(x_0, \epsilon) > 0$ such that

$$|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon$$

* Lyapunov's defination of Stability

→ The equilibrium state of $\dot{x} = f(x,t)$ is stable in the sense of Lyapunov if for all $\varepsilon > 0$, and t_0, there exist $\delta(\varepsilon, t_0) > 0$ Such that,

$$\boxed{\|x(t_0)\|_2 < \delta \text{ gives } \|x(t)\|_2 < \varepsilon \quad \forall\, t \geq t_0}$$



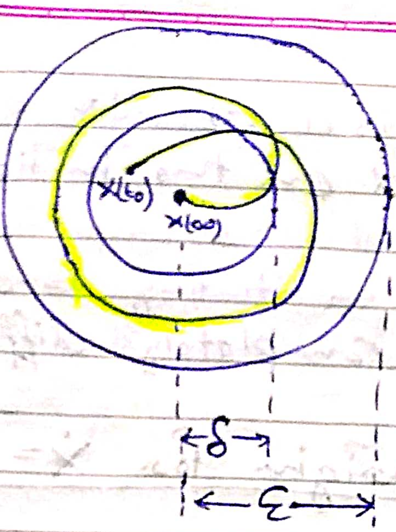
* Asymptotic stability

→ The equilibrium state of $\dot{x} = f(x,t)$ is asymptotically stable if:

- → It is stable in the sense of Lyapunov
- → For all $\varepsilon > 0$ and t_0 there exist $\delta(\varepsilon, t_0) > 0$ Such that

$$\boxed{\|x(t_0)\|_2 < \delta \text{ gives } \lim_{t \to \infty} x(t) = 0}$$

47:00



★ Stability of LTI Systems: method of eigenvalue / pole locations

⇒ The stability of the equilibrium point for $\dot{x} = Ax$ or $x(k+1) = Ax(k)$ can be concluded immediately based on the eigen value λ's of A.

→ The response $e^{At}x(0)$ involves modes such as $e^{\lambda t}$, $te^{\lambda t}$, $e^{\sigma t}\cos(\omega t)$, $e^{\sigma t}\sin(\omega t)$

↳ $e^{\sigma t} \to 0$ if $\sigma < 0$ $\quad Re(\lambda) < 0$

$e^{\lambda t} \to 0$ if $\lambda < 0$

→ The response $A^k x(k_0)$ involves modes such as λ^k, $k\lambda^{k-1}$, $r^k \cos(k\theta)$, $r^k \sin(k\theta)$

↳ $\lambda^k \to 0$ if $|\lambda| < 1$

$r^k \to 0$ if $|r| = |\sqrt{\sigma^2+\omega^2}| = |\lambda| < 1$

⇒ Double pendulum is a chaotic system because it is highly sensitive on the initial conditions.

↳ Tiny change in starting conditions will result in a completely different motion.

★ **Stability of the origin for $\dot{x} = Ax$**

① Unstable

↳ $Re(\lambda_i) > 0$ for some λ_i

↳ $Re(\lambda_i) \leq 0$ $\forall \lambda_i$ but for repeated λ_m on the imaginary axis with multiplicity m, Nullity $(A - \lambda_m I) < m$ (Jordan form)

② Lyapunov stable

↳ $Re(\lambda_i) \leq 0$ $\forall$ all λ_is and $\forall$ repeated λ_m on the imaginary axis with multiplicity m Nullity$(A - \lambda_m I) = m$ (Diagonal form)

③ Asymptotically stable

↳ $Re\{\lambda_i\} < 0$ $\forall$ λ_i (A is then called Hurwitz)

{Such matrix are called Hurwitz matrix}

Page No.
Date: / /

★ Stability of the Origin for $x(k+1) = f(x(k), k)$

⇒ For stability of equilibrium point x_e:

$$\boxed{f(x_e, K) = x_e \;\; \forall K}$$

★ Stability of the Origin for $X(K+1) = Ax(K)$

① Unstable

→ $|\lambda_i| > 1$ for sum λ_i

→ $|\lambda_i| \leq 1 \;\forall \lambda_i$ but for a repeated λ_m on the unit circle with multiplicity m, $\text{nullity}(A - \lambda_m I) < m$

(Jordan form)

② Lyapunov stable

→ $|\lambda_i| \leq 1 \;\forall \lambda_i$ but for any repeated λ_m on the unit circle with multiplicity. $\text{nullity}(A - \lambda_m I) = m$

(diagonal form)

③ Asymptotically stable

→ $|\lambda_i| < 1 \;\forall \lambda_i$ {Such matrix are called Schur matrix}

Appendix: Stability Part 2

★ Lyapunov's approach to stability

⇒ The direct method of Lyapunov to stability problem:

→ No need for explicit solutions to system responses

→ An "energy" perspective

→ Fit for general dynamic system (linear/non-linear time-invariant/time-varying)

★ Stability from an energy view point

⇒ Consider unforced, time-varying, nonlinear systems:

$$\dot{x}(t) = f(x(t), t) \;,\; x(t_0) = x_0$$

$$x(k+1) = f(x(k), k) \;,\; x(k_0) = x_0$$

⇒ Assume the origin is an equilibrium state

⇒ Energy function ⇒ Lyapunov function

A scalar function of x and t (or x and k in discrete-time case)

⇒ Goal is to relate properties of the state through the Lyapunov function.

★ Important properties of a symmetric matrix

① Eigenvalues of symmetric matrices are all real.

↳ Eigenvalues of skew-symmetrics are all imaginary or zero.

② All eigenvalues of an orthogonal matrix has a magnitude of 1.

Matrix Structure	analogy in Complex plane
Symmetric	real line
Skew-Symmetric	Imaginary line
Orthogonal	Unit Circle

★ The spectral theorem for symmetric matrices

→ When $A \in R^{n \times n}$ has n distinct eigenvalues, we can do diagonalization $A = U \Lambda U^{-1}$.

↳ The Spectral theorem significantly simplifies the result when A is symmetric.

Theorem: Symmetric eigenvalue decomposition (SED)

⇒ $\forall : A \in \mathbb{R}^{n\times n}$, $A^T = A$, there always exist $\lambda_i \in \mathbb{R}$ and $u_i \in \mathbb{R}^n$ s.t.

$$A = \sum_{i=1}^{n} \lambda_i u_i u_i^T = U \Lambda U^T$$

→ λ_i's ⇒ Eigenvalues of A
→ $U = [u_1, u_2 \cdots u_n]$ is orthogonal : $U^T U = U U^T = I$
→ $\Lambda = \text{diagonal}(\lambda_1, \lambda_2 \cdots \lambda_n)$

★ Theorem: Eigenvalues of symmetric matrices

⇒ If $A = A^T \in \mathbb{R}^{n\times n}$, then the eigenvalues of A satisfy

$$\lambda_{max} = \max_{x\in\mathbb{R}^n, x\neq 0} \frac{x^T A x}{\|x\|_2^2}$$

$$\lambda_{min} = \min_{x\in\mathbb{R}^n, x\neq 0} \frac{x^T A x}{\|x\|_2^2}$$

Proof

$$A = \sum_{i=1}^{n} \lambda_i u_i u_i^T \qquad \{u_i\}_{i=1}^{n} \text{ forms orthonormal a basis}$$

$$x = \sum_{i=1}^{n} \alpha_i u_i$$

$$\{\underset{\lambda_{max}}{\lambda_1} \geq \lambda_2 \cdots \geq \underset{\lambda_{min}}{\lambda_n}\}$$

Page No.:
Date: / /

$$x^T A x = (\alpha_1 u_1^T + \alpha_2 u_2^T + \cdots)(\lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T + \cdots)(\alpha_1 u_1 + \alpha_2 u_2 + \cdots)$$

$$\Rightarrow (\alpha_1 u_1^T + \alpha_2 u_2^T + \cdots)(\alpha_1 \lambda_1 u_1 + \alpha_2 \lambda_2 u_2 + \cdots)$$

$$\Rightarrow \lambda_1 \alpha_1^2 + \lambda_2 \alpha_2^2 + \cdots$$

$$\frac{x^T A x}{\|x\|_2^2} = \frac{\lambda_1 \alpha_1^2 + \lambda_2 \alpha_2^2 + \cdots + \lambda_n \alpha_n^2}{\alpha_1^2 + \alpha_2^2 + \cdots + \alpha_n^2}$$

$$\boxed{\lambda_n \leq \frac{x^T A x}{\|x\|_2^2} \leq \lambda_1}$$

$\alpha_2 = \alpha_3 = \cdots = \alpha_n = 0$ & $\alpha_1 \neq 0$

$\alpha_n \neq 0$ & $\alpha_1 = \alpha_2 = \cdots = \alpha_{n-1} = 0$

★ **Positive definite matrices**

A Symmetric matrix $P \in R^{n \times n}$ is called positive-definite, written $P > 0$, if $x^T P x > 0 \ \forall x \neq 0 \in R^n$.

- P is called positive-semidefinite written $P \geq 0$ if $x^T P x \geq 0 \ \forall x \neq 0 \in R^n$.
- Similarly negative definite & negative semidefinite

⇒ We often use the following necessary and Sufficient Conditions to check if a Symmetric matrix P is positive (semi) definit or not:

① $P > 0$ $(P \geq 0)$ ⇔ The leading ~~positive~~ principle ~~(semi) de~~ minors defined below are positive (non-negative).

⇒ The leading principle minors of $P = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix}$

are defined as:

→ P_{11}

→ $\begin{vmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{vmatrix}$

→ $\begin{vmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{vmatrix}$

② $P > 0$ $(P \geq 0)$ ⇔ P can be decomposed as $P = N^T N$ where N is non singular (Singular)

$$P = U \Lambda U^T = \underbrace{U \begin{bmatrix} \sqrt{\lambda_1} & & \\ & \sqrt{\lambda_2} & \\ & & \sqrt{\lambda_n} \end{bmatrix}}_{N^T} \underbrace{\begin{bmatrix} \sqrt{\lambda_1} & & \\ & \sqrt{\lambda_2} & \\ & & \sqrt{\lambda_n} \end{bmatrix} U^T}_{N}$$

Appendix: Stability Part 3

⇒ The equilibrium point 0 of $\dot{x}(t) = f(x(t), t)$, $x(t_0) = x_0$ is:

- → Stable in the sense of Lyapunov if
  - → $\exists$ locally PD function $V(x,t)$ (called Lyapunov function)
  - → Such that $\dot{V}(x,t) \leq 0 \ \forall \ t > t_0$ & $\|x\| < r$ for some $r > 0$.
- → Asymptotically stable if $\underbrace{\dot{V}(x,t) < 0}_{\text{negative definite (ND)}} \ \forall \ x \neq 0$

★ Linear System ($\dot{x} = Ax$)

⇒ A good candidate for Lyapunov function is quadratic:

$$V(x) = x^T P x, \quad P = P^T$$

→ Where P is positive definite

$$\dot{V}(x) = \dot{x}^T P x + x^T P \dot{x}$$

$$= (Ax)^T P x + x^T P (Ax)$$

$$= x^T (A^T P + PA) x$$

$$\underbrace{A^T P + PA \leq 0}_{\text{Stable in the sense of Lyapunov}} \quad \text{or} \quad \underbrace{< 0}_{\text{Asymptotic Stable}}$$

{Lyapunov Equation}

$\Rightarrow$ $\dot{x} = Ax$, $A \in \mathbb{R}^{n\times n}$ Origin is asymptotically stable iff $\exists\, Q > 0$ $A^T P + PA = -Q$ has a positive definit (PD) solution $P > 0$, $P^T = P$.

Proof

$$\dot{x} = Ax \Rightarrow x(t) = e^{At} x(0) \to 0$$

$$\dot{V}(x) = x^T (A^T P + PA) x = -x^T Q x$$

$$V(x,t)\Big|_0^\infty = \int_0^\infty \dot{V}(x)\, dt$$

$$x^T(\infty) P x(\infty) - x^T(0) P x(0) = \int_0^\infty -x^T Q x\, dt$$

$$x^T(0) P x(0) = \int_0^\infty x^T(0) e^{A^T t} Q e^{At} x(0)\, dt$$

$$\Rightarrow \boxed{P = \int_0^\infty e^{A^T t} Q e^{At}\, dt}$$

$\Rightarrow$ $Q > 0 \Rightarrow Q = N^T N$, where N is nonsingular

$$\int_0^\infty e^{A^T t} Q e^{At} dt = \int e^{A^T t} N^T N e^{At} dt$$

$x^T(0)$ $x(0)$ $x^T(0)$ $x(0)$

$$\Rightarrow \int_0^\infty (Ne^{At}x(0))^T \underbrace{(Ne^{At}x(0))}_{n(t)} dt$$

$$\Rightarrow \int_0^\infty \|n(t)\|^2 dt \geq 0$$

$x^T(0) P x(0) \geq 0$

$x^T(0) P x(0) = 0$ only if $x(0) = 0$

★ Solving $A^T P + PA = -Q$ (2×2 case):

Let $Q = [q_1 \; q_2]$

$P = [P_1 \; P_2]$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$A^T [P_1 \; P_2] + [P_1 \; P_2] A = [-q_1 \; -q_2]$$

① First Column:

$$A^T P_1 + [P_1 \; P_2] \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} = -q_1$$

② Second Column:

$$A^T P_2 + [P_1 \; P_2] \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} = -q_2$$

Page No.
Date: / /

⇒ Stacking 1^{st} & 2^{nd} Columns:

$$\begin{cases} A^T P_1 + P_1 a_{11} + P_2 a_{21} = -q_1 \\ A^T P_2 + P_1 a_{12} + P_2 a_{22} = -q_2 \end{cases}$$

$$\begin{bmatrix} A^T & 0 \\ 0 & A^T \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} + \begin{bmatrix} a_{11}I & a_{21}I \\ a_{21}I & a_{22}I \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} -q_1 \\ -q_2 \end{bmatrix}$$

$$\begin{bmatrix} A^T + a_{11}I & a_{21}I \\ a_{12}I & A^T + a_{22}I \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} -q_1 \\ q_2 \end{bmatrix}$$

Fact → Is invertable iff $\lambda_i(A) + \lambda_j(A) \neq 0$

★ Procedures of Lyapunov's direct method

⇒ Given a matrix A, select an arbitrary positive definite symmetric matrix Q (eg I)

⇒ Find the Solution matrix P to the Lyapunov equation:

$$A^T P + PA = -Q$$

⇒ If a Solution P cannot be found, A is not Hurwitz

⇒ If a solution is found

→ If $P > 0$ then A is Hurwitz.

→ If P is not positive definite, then A has at least one eigenvalue with a positive real part.

★ <u>Instability theorem</u>

⇒ The previous theorems only provide sufficient but not necessary conditions.

→ failure to find a Lyapunov function does not imply instability.

⇒ <u>Theorem:</u> The equilibrium state 0 of $\dot{x} = f(x)$ is unstable if there exists a function $W(x)$ such that

→ $\dot{W}(x)$ is PD locally.

$$\{\dot{W}(x) > 0 \ \forall \ |x| < r \text{ for some } r\}$$

→ $W(0) = 0$

→ There exist states x arbitrarily close to the origin such that $W(x) > 0$

Page No.
Date: / /

* Discrete-time Case

⇒ For the discrete-time ~~case~~ System:

$$\boxed{x(k+1) = Ax(k)}$$

⇒ We consider a quadratic Lyapunov function candidate:

$$V(x) = x^T P x \quad , \quad P = P^T > 0$$

⇒ Compute $\Delta V(x)$ along the trajectory state

$$V(x(k+1)) - V(x(k)) = \Delta V(x(k))$$

$$\Rightarrow x^T(k+1) P x(k+1) - x^T(k) P x(k)$$

$$\downarrow$$

$$x^T(k) A^T P A x(k)$$

$$\Rightarrow x^T(k) [A^T P A - P] x(k)$$

$$\downarrow$$

{Gives DT Lyapunov stability theorem for LTI System}

Page No.
Date: / /

★ DT Lyapunov stability theorem

For system $x(k+1) = Ax(k)$ with $A \in R^{n\times n}$, the origin is asymptotically stable if and only if $\exists\, Q > 0$, such that the discrete-time Lyapunov equation

$$\boxed{A^T P A - P = -Q}$$

has a unique positive definite solution $P > 0$ $P^T = P$.

⇒ Solution to the DT Lyapunov equation, when asymptotic stability holds (A is Schur) comes from the following:

$$X(k+1) = AX(k) \qquad V(x(k)) = x^T(k) P x(k)$$

$$\begin{pmatrix} V(x(k+1)) - \cancel{V(x(k))} \\ + \\ \cancel{V(x(k))} - \cancel{V(x(k-1))} \\ \vdots \\ \cancel{V(x(1))} - V(x(0)) \end{pmatrix} = \sum_{k=0}^{K} x^T(k)\,[A^T P A - P]\, x(k)$$

$$\cancel{V(x(\infty))}^{\,0} - V(x(0)) = \sum_{k=0}^{\infty} x^T(k) \underbrace{[A^T P A - P]}_{-Q} x(k)$$

$$= -\sum_{k=0}^{\infty} x^T(0)\,(A^T)^k Q A^k x(0)$$

$$x^T(0) P\, x(0) = \sum_{k=0}^{\infty} x^T(0)\,(A^T)^k Q A^k x(0)$$

$$P = \sum_{K=0}^{\infty} (A^T)^K Q A^K$$

$\Rightarrow$ Discrete time Lyapunov operator $L_A = A^T P A - P$ is invertible if and only if for all i, j $(\lambda_A)_i (\lambda_A)_j \neq 1$.