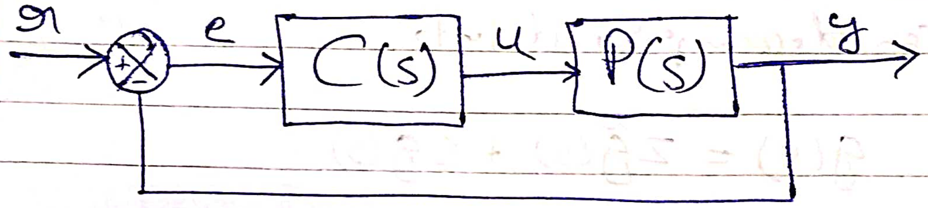
Lecture 7

Analysis of feedback systems I: internal stability (root Locus)

★ **Standard feedback configuration**



⇒ Open Loop gain: $L(s) = P(s)C(s)$

⇒ Complementary sensitivity
↳ (Closed-loop) transfer function from r to y.

$$T(s) = \frac{L(s)}{1+L(s)}$$

⇒ Sensitivity:
↳ (Closed-loop) transfer function from r to e.

$$S(s) = \frac{1}{1+L(s)}$$

★ **How to determine closed-loop stability?**

⇒ Design $C(s)$ in such a way that all the poles of $T(s)$ have negative real part

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⇒ All of classical control can be summarized in "exploit the knowledge of the loop gain L(s) to figure out the properties of the Closed-loop transfer function T(s) and S(s) with the least effort possible"

★ Classical methods for feedback control

① Root Locus

-> Quick assessment of control design feasibility. The insights are Correct and clean.

-> Can only be used for finite-dimentional Systems.

-> Difficult to do sophisicated design.

-> Hard to represent uncertainity.

② Nyquist plot

-> The most authoritative closed-loop stability test.

-> It can always be used (finite or infinite dimensional system)

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→ Easy to represent uncertainity.

→ Difficult to draw and to use for Sophisticated design.

③ Bode plots

→ Potentially misleading results unless the System is open-loop stable and minimum-phase.

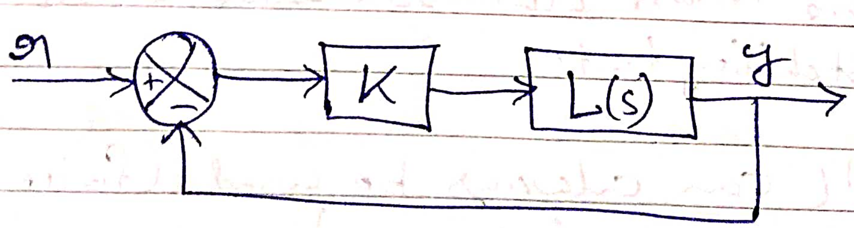
→ Easy to represent uncertainity.

→ Easy to draw, this is the tool of choice for sophisticated design.

★ Evan's Root Locus method

⇒ Invented in the late 40's by Walter R. Evans

⇒ Useful to study how the roots of a polynomial change as a function of a scalar parameter e.g the gain.



$$\text{Loop gain} = K\,L(s) = K\frac{N(s)}{D(s)} = K\frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$

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⇒ The sensitivity function is

$$S(s) = \frac{1}{1+KL(s)} = \frac{D(s)}{D(s)+KN(s)}$$

⇒ The closed-loop poles are the solutions of the characteristic equation

$$\boxed{D(s) + KN(s) = 0}$$

★ <u>The root locus rules</u>

⇒ Since the degree of $D(s) + KN(s)$ is the same as the degree of $D(s)$, the number of closed-loop poles is the same as the number of open-loop poles.

⇒ For $K \to 0$ $\quad D(s) + KN(s) \approx D(s)$

↳ The closed-loop poles approaches the open loop poles.

⇒ For $K \to \infty$ $\quad D(s) + KN(s) \approx KN(s)$

↳ The closed-loop poles approaches the open-loop zeros.

↳ If the degree of $N(s)$ is smaller, then the "excess" closed-loop poles goes to infinity.

★ The angle and magnitude rule

⇒ Let us rewrite the closed-loop characteristic equation as:

$$\frac{N(s)}{D(s)} = \frac{-1}{K}$$

- **The angle rule**

⇒ Take the argument on both sides:

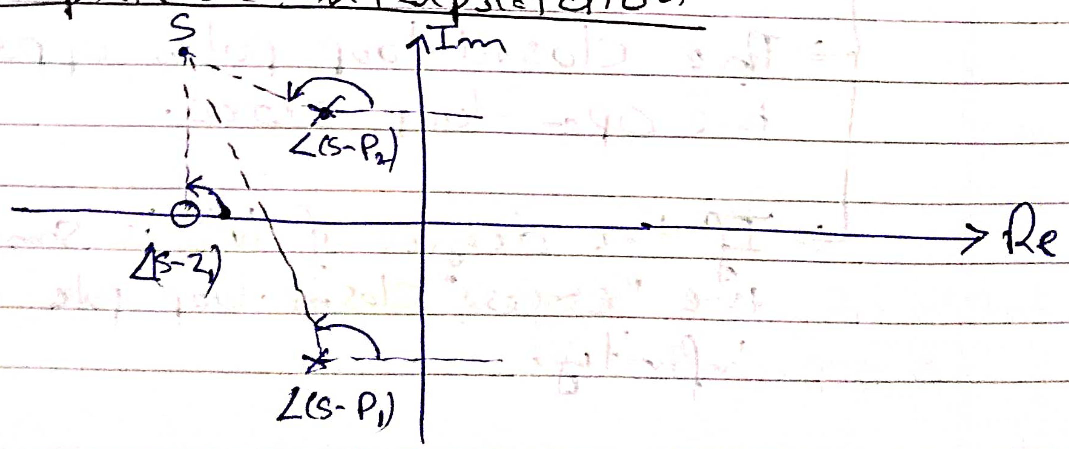
$$\angle(s-z_1) + \angle(s-z_2) + \cdots + \angle(s-z_m) - \angle(s-p_1) - \angle(s-p_2) - \cdots - \angle(s-p_n) = \begin{cases} 180^\circ & \text{if } K>0 \\ 0 & \text{if } K<0 \end{cases}$$

- **The magnitude rule**

⇒ Take the argument on both sides:

$$\frac{|s-z_1|\cdot|s-z_2|\cdots|s-z_m|}{|s-p_1|\cdot|s-p_2|\cdots|s-p_n|} = \frac{1}{|K|}$$

★ Graphical Interpretation



⇒ All points on the complex plane that could potentially be a closed-loop pole have to satisfy the angle condition.

⇒ All points on the real axis are on the root locus.

↳ Some corresponds to negative K rootlocus and some corresponds to positive K rootlocus.

★ <u>Asymptotes</u>

⇒ When $K \to \infty$ and there are more open-loop poles than zeros, from the magnitude condition, the excess closed-loop poles will have to go to infinity.

⇒ Since this is the complex plane we need to identify "in which direction" they go toward infinity".

⇒ If we "zoom out" sufficiently far, the contributions from all the finite open loop poles and zeros will all be approximately equal to $\angle s$.

↳ So angle rule can be approximated as:

$$(m-n)\angle s = -\angle K \pm q360^\circ$$
$$= \angle -K \pm q360^\circ$$

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$$\boxed{\angle s = \frac{\angle -K \pm q\,360}{n-m}}$$

$$\left\{ \angle s = \frac{180 \pm q\,360}{(n-m)} \right\}$$

$\Rightarrow$ These asymptotes meet in a "center of mass" laying on the real axis at

$$\boxed{S_{com} = \frac{\sum_{i=1}^{n} P_i - \sum_{j=1}^{m} Z_j}{n-m}}$$