Contents

1 Chapter 1

Trying to do a brief overview of some of the terms and important concepts in the *"Invitation to Metatheory"*

A reminder: Doing metatheory is about *relations*. We are not studying Metatheory but using it to talk about theories.

Vocabulary	Symbol	Definition
Propositional Signature	Σ	Collection of Eleme
Context	$\phi_1 \ldots \phi_n$ OR Δ, Γ	Finite collection of
Derivability	$\vdash$	Smallest relation b
		Syntatically entails
Proof	$\Delta \vdash \phi$	
Provable	$\vdash \phi$ OR $\top$	
Not Provable	$\bot OR \not\top$	
Interpretation/Valuation	f: $\Sigma \rightarrow$ {true, false}	
Propositional Theory	T ; $\Sigma \wedge \Delta \in \Sigma$	
Tarski Truth	v(ϕ) = 1 $\rightarrow$ ϕ is true in v	v = interp. of Σ an
Model	v(ϕ) = 1 $\forall$ $\phi \in \Delta$	v is a model of Δ v
Consistent	Δ has at least one model v	
Inconsistent	Δ has no models v	
Proof of Relative Cons.		X theory $\rightarrow$ model
		Y = Boolean Algeb
Semantically entails	v(ϕ) = 1 $\forall$ $\phi \in \Delta \rightarrow \Delta \vDash \phi$	Δ = set of Σ sent.;
	Δ semantically entails ϕ	
Soundness	$\Delta \vdash \phi \rightarrow \Delta \vDash \phi$	If proof then truth
Completeness	$\Delta \vDash \phi \rightarrow \Delta \vdash \phi$	If truth table then
Compactness	$\forall$ $\Delta_F \subseteq \Delta$ cons.$\rightarrow$ Δ cons.	Δ set of sentences.
Complete Theory	$\Delta cons. \wedge \forall \phi \in \Sigma, \Delta \vDash \phi \vee \Delta \vDash \not\phi$	T consists of axiom
Deductive Closure	Cn(T)=T $\implies$ $\{Sent(\Sigma)\}$	T theory in Σ
Reconstrual	f: $\Sigma \rightarrow$ Sent($\Sigma\prime$)	
Substitution	$\forall f : \Sigma \rightarrow \Sigma\prime, \phi \vdash \psi \rightarrow f(\phi) \vdash f(\psi)$	Where f is a recons
Translation	T $\vdash \phi \rightarrow$ T$\prime \vdash f(\phi)$	f is a special recons
Equality of Trans.	$T\prime \vdash f(p) \iff g(p) \forall p \in \Sigma$	f $\simeq$ g; f, g trans. fr
Homotopy Equiv.	$\exists$ f: T$\rightarrow$ T$\prime$ $\wedge$ g: T$\prime \rightarrow$ T s.t. gf $\simeq 1_{\mathrm{T}}$ $\wedge$ fg $\simeq 1_{\mathrm{T}\prime}$	

Two Propositional Constants are defined by identity and are dependent on the signature they are a part of.

Bare Set is related to the Leibniz's identity of indiscernibles which states that there cannot be separate objects or enitites that have all their properites in common. What this means, is essentially that a Bare Set for a Propositional Signature just means that there can be separate objects with all the same common properties. Nothing stands closer together in relation

The Possible set of Sentences that can be formed from Σ is called the set Sent(Σ). These rules are the following:

1. $\phi \in \Sigma \rightarrow \phi \in$ Sent(Σ)

2. $\phi \in \text{Sent}(\Sigma) \rightarrow (\neg\ \phi) \in \text{Sent}(\Sigma)$

3. $\phi \in \text{Sent}(\Sigma) \wedge \psi \in \text{Sent}(\Sigma) \rightarrow (\phi \wedge \psi) \in \text{Sent}(\Sigma)$, $(\phi \vee \psi) \in \text{Sent}(\Sigma)$, $\wedge\ (\phi \rightarrow \psi) \in \text{Sent}(\Sigma)$

4. Nothing is in Sent(Σ) unless it enters one of the previous clauses

Sentence: Finite string of symbols and contains finitely many propositional constants.

Unions are commutative in Contexts.

1.1 Proof by Induction

1. Show that some property of interest P, holds of the elements of Σ

2. Show that [P holds of $\phi \rightarrow$ P holds of ψ]

3. Show if P holds of ϕ and $\psi \rightarrow$ P holds of $\phi \wedge \psi$, $\phi \vee \psi$, and $\phi \rightarrow \psi$

We allow for empty contexts as well

1.2 Rules for Turnstile

1. $\vdash$ is closed under the previously given logical clauses

2. $\Delta \vdash \phi$ and $\Delta \subseteq \Delta_0$, then $\Delta_0 \vdash \phi$. (**Monotonicity**)

$\Delta\ ,\ \phi \vdash \bot \rightarrow \Delta \vdash \not\phi$

Semantics fundamentally differs from Syntax by introducing the concept of Truth and false; Essentially, introducing a world

Interpretation v of Σ extends to a function v: Sent(Σ) $\rightarrow$ {0,1} by the following clauses:

1. $\text{v}(\neg\ \phi) = \iff v(\phi) = 0$

2. $\text{v}(\phi \wedge \psi) = 1 \iff v(\phi) = 1 \wedge \text{v}(\psi) = 1.$

3. $\text{v}(\phi \vee \psi) = 1 \iff v(\phi) = 1 \vee \text{v}(\psi) = 1$

4. $\text{v}(\phi \rightarrow \psi) = \text{v}(\neg\ \phi \vee \psi).$

Sent(Σ) could be thought of as simply sentences using the vocabulary of the Σ to create sentences without being interpreted. When we make it into a function/interpretation v where the set of sentences is mapped to a set {1,0}, we could say that it then endows the senteces/symbols with meaning due to our judgement on these sentences. However, it is important to note that the domain of predicate logic interpretation must be a **set**. End stop. This is something that can be demonstrated using set theory. The World is not a consequence of set theory ∴ the world is not a set. They are of two different qualities.

Abstract	Concrete
Set	World

Can write T instead of Δ, but the formulation of Σ affects T.
e.g. p would be different in Σ = {p} than in Σ = {p,q}

A Concept defined for sets of sentences such as consistency can also apply to Theories but, *again*, it must be emphasized that a theory is always the same:

$$T = \Sigma, \Delta \in \Sigma$$

Where Σ is a Propositional Sig. and Δ are a set of sentences in that signature.

1.3 Exercise 1.3.8

Question For Δ a set of Σ sentences and for Σ sentences ϕ and ψ, if Δ, ϕ $\vDash \psi$ then $\Delta \vDash \phi \to \psi$

Assume Δ , $\phi \vDash \psi$ and that ∃ v ∈ Σ s.t. ∀ χ ∈ Δ : v(χ) = 1 and v($\phi \to \psi$) = 0

By the material conditional we can do the following move

This thereby leaves us with the following result:

v(ϕ) = 1 and v(ψ) = 0

However, we assumed Δ, $\phi \vDash \psi$ which is impossible if $\psi = 0$. Therefore, there is a contradiction. Q.E.D □

Another Solution: Suppose that Δ, $\phi \vDash \psi$. We want to show that any model v of Δ is such that v($\phi \to \psi$) = 1, i.e.,

$$v(\not\phi) = 1 or v(\psi) = 1$$

So, given any such interpretation v. Either (i) v(ϕ) = 1 or (ii) v(¬ ϕ) = 1. If (i), then v is a model of Δ, ϕ, since it is a model of Δ. Hence by our original assumption and the definition of semantic entailment v(ψ) = 1 and we are done, since () **follows. If (ii), then we are also done, since** () follows. □

1.4 Exercise 1.3.9 **

Show that $\Delta \vDash \phi \iff \Delta \cup \{\neg\ \phi\ \}$ is inconsistent. Here $\Delta \cup \{\neg\ \phi\}$ is the theory consisting of $\neg\ \phi$ and all sentences in Δ.

Suppose that $\Delta \vDash \phi$ and $\Delta \cup \{\neg\ \phi\}$ is consistent. This means that we can find a model v for Δ s.t. v(χ)=1 $\forall\ \chi \in \Delta$. This would mean that both v(ϕ) and v($\neg\ \phi$) = 1. However, this isn't possible given the fact that one of the clauses of an interpretation is that v($\neg\ \phi$)=1 $\iff v(\phi) = 1$ contradicting our initial assumption.□

1.5 Parameters for Reconstruals

1. for p in Σ, $\bar{f}(p) = f(p)$
2. $\forall\ \phi$, $\bar{f}(\not\phi) = \not\bar{f}(\phi)$
3. binary circles can be taken out to function f

1.6 Exercise 1.3.16 **

Let T$\prime$ = {p} $\in \Sigma\prime$ and $\Sigma\prime$ = {p,q}. Show that T$\prime$ is not complete.

Proof by Contradiction: Assume that T$\prime$ is complete, this means that T$\prime$ is consistent(Δ has at least one model) and that $\forall\phi \in \Sigma\prime; \Delta \vDash \phi \vee \Delta \vDash \not\phi$. However, T$\prime$ only describes p, so although we can say $\Delta \vDash p$, we cannot say anything about q in relation to Δ which means we can't say that $\Delta \vDash q$ or $\Delta \vDash q$

1.7 Exercise 1.3.17

Show that Cn(Cn(T)) = Cn(T).

Cn(T) is a deductive closure which is the set of Σ sentences that is implied by T.

1.8 Exercise 1.4.7

Prove that if v is a model of T$\prime$, and f:T→ T$\prime$ is a translation, then $v \circ f$ is a model of T. Here $v \circ f$ is the interpretation of Σ optained by applying f first, and then applying v.

1.9 Lingering Questions:

What does it mean for Δ to have more than one model?

2 Week 2

2.1 Sets is a Category (29)

Terms	Symbology	Definition
Category		Objects and arrows
Monomorphism	g: Z$\rightarrow$ X	fg=fh$\rightarrow$ g=h
	h: Z$\rightarrow$ X	where f: X$\rightarrow$ Y
Epimorphism	g: Y$\rightarrow$ Z	gf=hf, then g=h
	h: Y$\rightarrow$ Z	
Isomorphism	$\exists$ g:Y$\rightarrow$ X	gf=1_X and fg=1_Y
		X$\simeq$ Y
THEORIES		
Conservative	T'$\vdash$ $f(\phi)\rightarrow$ T$\vdash$ ϕ	$\forall$ $\phi \in$ Sent(Σ)
Ess. surj.	T'$\vdash$ $\phi \iff f(\psi)$	$\forall\phi \in \Sigma'; \exists\psi \in \Sigma$

We are going to be looking at the general theory of categories. To do this we are going to look at Set Theory as an example. In this case, the theory of sets has objects called sets and relations called functions.

What does it mean to say a set is category? It means the following

1. Every function f has a domain set $d_0 f$ and a codomain set $d_1 f$.
   (a) $f : X \rightarrow Y$indicates that $d_0 f = X$ $d_1 f = Y$
2. Compatible functions can be composed.
   (a) e.g. can have transitivity f: x$\rightarrow$ y g: y$\rightarrow$ z $f \circ g : x \rightarrow z$
   (b) These compositions can also be associative
   (c) $h \circ (g \circ f) = (h \circ g) \circ f)$
3. $\forall$ X | X is a set, there is a function $1_X : X \rightarrow X$ that acts as a left and right identity relative to composition.

In short, there are objects, arrows, composition with transitivity and associativity, and identity which should conform to equation.

Containment or $\in$ is *not* a primitive notion of the Elemental Theory of the Category of Sets

2.2 Chapter 3.1 On the Category of Propositional Theories

Th is the category of propositional Theories
f:T→ T'
g:T→ T'
f and g are equal, f$\simeq$ g, just in case T' $\vdash f(\phi) \iff g(\phi)$ $\forall$ $\phi \in$ Sent(Σ)

2.3 Proving that conservative translations are monomorphic

Suppose first that f is conservative with two granslations g, h from T" T s.t. $f \circ g = f \circ h$.