Intermediate Logic Notes

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Contents

1 Introduction to Course 09/12/19

1.1 Administrative stuff

Sign up for precept; Thursday times

1. Contemporary symbolic logic; Can start proving things; can write explicitly steps in math.

2. Ask questions about Logic can do; Metalogic/Metamathematics
    (a) Logic $\sim$ Phi 201
    (b) Metalogic $\sim$ Phi 312
        i. Completeness and Soundness Proofs

Emphasis is on *perseverance* Strongly encourage collaboration; Straight on definitions. Struggle Math safe and not complete(Hilbert) or open and complete(Godel).

First couple of weeks is about propositional logic. theories in propositional logic $\sim$ boolean algebra $\sim$ Stone topological space

Predicate logic; Completeness, soundness

1.2 Theories in the wild

- Evolutionary Biology; Theories can be facts; Theory=System of beliefs to understand the world.
- Relativity
- Neuroscience
- Folk Psycholgy; (think have understanding of thoughts and feelings)

1.3 Regimented Theories (math)

- Peano arithmetic
- Group theory
- Field Theory
- Theory of Topological Spaces
- Set Theory (Most general)
- Elemental Thoery of the Category Sets

1.4 How to decide which theory to believe

Not a class on epistemology. Question before knowledge. Suppose need two people, German and French. Both say Einstein's theories of relativity in their own languages. This is a demonstration of equivalence.

1.4.1 Metaconcept on how Theories Relate

Idea that some theories can be equivalent. e.g. Schrodinger(Wave mechanics) and Heisenberg(matrix mechanics). Bohr realizes that they are saying the same thing. Neumann proves that both theories are equivalent.

e.g. Folk psychology is *reducible* to neuroscience.

Theory of heat; Thermodynamics is reducible to statistical mechanics.

1.4.2 Explicating Terms

What we would like is precision in terms. Math likes to swallow things wwe already know and be rigorous in definition.

- **Continuous:**
- **Equivilant**
- **Reducible**
- **Function**
- **infinite**

Regimented Theory are theories in First Order Logic. Propositional Theories is subcollection. Structure of Bolean Algebras $\sim$ Stone Topological Space

2 Rigorous idea of the Theory 09/17/19

2.1 Pset 1 DUE DATE DEADLINE: *<2019-09-23 Mon>*

2.2 Notion of the Theory; How to make it rigorous

Define Theory in first-order logic. Subclass of Propositional Theories (Think of this as a network). Need to understand the form, *relations between theories.* Can study closeness of theories?

Primary Focus: Relation of theories, that we can schematize by $\rightarrow$ with $\rightarrow$ meaning *translation*

Equivalent: Translation between theories

Reduction: Some kind of translation.

1. What is Propositional Logic
2. Define notion of translation between theories

2.3 Propositional Logic

1. Grammar
2. Proof Theory
3. Semantics (Representation Theory)

Hans is already using a Metatheory that is similar to Set Theory. Logic has a grammar; A set of Symbols.

$\Sigma = \{\text{p}\}$ where Σ is a non logical symbol

Logical Symbols don't have meaning, just help to make sentences;

1. $\wedge$
2. $\vee$
3. $\rightarrow$
4. etc.

Definition: Let Str(Σ) be the set of finite strings of Symbols: Logical + Σ.

Literal Identity of Symbols. Sameness means same pattern. $\sim$ Platonic

ϕ is being used a metatheoretical tool.

All the stuff below is Grammar

Definition: The set Sent(Σ) is defined inductively by the following

1. $\forall\ \phi \in \Sigma$, we have $\phi \in \text{Sent}(\Sigma)$
2. $\phi \in \text{Sent}(\Sigma) \rightarrow \neg\ \phi \in \text{Sent}(\Sigma)$
3. $[\phi \in \text{Sent}(\Sigma) \wedge \psi \in \text{Sent}(\Sigma)] \implies \phi \rightarrow \psi \in \text{Sent}(\Sigma)$.
4. $\wedge$
5. $\vee$
6. No infinitely long sentences.

There is at least one inductive set in the theory that we are looking at.

2.4 Proof Theory

$\Delta \vdash \phi$

Δ, Γ = Γ, Δ where the comma acts as union

1. $\phi \vdash \phi$ Rule of Assumptions.
2. $\Delta \vdash \phi \wedge \Delta_0 \subseteq \Delta \rightarrow \Delta_0 \vdash \phi$

$$\frac{\Gamma \vdash \phi \rightarrow \psi \quad \Delta \vdash \phi}{\Gamma\ ,\ \Delta \vdash \psi}$$

$$\frac{\Gamma\ ,\ \phi \vdash \psi}{\Gamma \vdash \phi \rightarrow \psi}$$

$$\frac{\Gamma \vdash \phi \vee \psi \quad \Delta\ ,\ \phi \vdash \chi \qquad \Theta\ ,\ \psi \vdash \chi}{\Gamma\ ,\ \Delta\ ,\ \Theta \vdash \chi}$$

$\perp = \mathit{contradiction}$

$$\frac{\Gamma\ ,\ \phi \vdash \perp}{\Gamma \vdash \phi}$$

2.5 Semantics

Definition: A Σ -valuation v is a function from Σ to {0,1}

Fact Each valuation v extends uniquely to a function $\bar{v}$: Sent(Σ) $\rightarrow$ {0,1} s.t. $\bar{v}$ (ϕ) = v(ϕ), all $\phi \in \Sigma$
$\bar{v}$ ($\phi \wedge \psi$) = min { $\bar{v}(\phi)$, $\bar{v}(\psi)$ }
$\bar{v}$ ($\phi \vee \psi$) = max{ " "}
$\bar{v}$ ($\neg$ ϕ) = 1 - $\bar{v}(\phi)$

Definition: ϕ

- contingent: One valuation = 0 and one valuation = 1; At least
- Tautologies: v(ϕ) = 1
- Inconsistent: v(ϕ) = 0

Definition: $\Delta \models \phi \iff \forall$ valuations v , v(ψ) = 1 , all $\psi \in \Delta$, $\rightarrow$ v(ϕ) = 1.

$\models$ = Semantically implies

Fact: $\phi \models \psi \wedge \psi \models \chi \rightarrow \psi \models \chi$.

3 Theorems and Theory 09/19/19

Sent(Σ) has a p or a q.
Sentences finitely long Context: set of sentences
, = union
ϕ .. ϕ sentences individ.

$\Delta \vdash \phi$; Proven inductively

$\Delta \vDash \phi$; defined by valuation using 1,0 over universal; Not effectively decidable; n elementary sentences (p, q , . . .) 2^{n} evaluations.

Soundness: If $\Delta \vdash \phi$ then $\Delta \vDash \phi$

predicate built up inductively.

Completeness: If $\Delta \vDash \phi$ the $\Delta \vdash \phi$

Second much more difficult than the first; Says that there is a proof.

Consistency: $\exists$ v; v(ϕ)=1, $\forall$ $\phi \in \Delta$

Compactness:

1. If every finite subset Δ_0 of Δ is consistent(There is a valuation that is true), Then Δ is consistent.
2. *Corollary*: If $\Delta \vDash \phi$ then there is a finite subset $\Delta_0 \subseteq \Delta s.t. \Delta_0 \vDash \phi$.
3. Why? If $\Delta \vDash \phi$ then Δ , $\neg$ ϕ is inconsistent.
4. Hence there is a finite subset $\Delta_0 \subseteq \Delta s.t. \Delta_0$, $\not\phi$ is inconsistent $\Delta_0 \vDash \phi$
5. $P1$There is more that 1 number.
6. $P2$There is more than 2 numbers.

$\vdots$

C There are infinitely many numbers

Set of axioms form a theory.

Propositional Theory requires a language(choose how many symbols/propositional constants; Need to choose a signature)

Empty Theory:

$\Sigma = \{$p$\}$	T = 0 with no axioms	Will only get tautologies
$\Sigma = \{$p, q$\}$	T = 0 with no axioms	
$\Sigma = \{$q$\}$	T= 0 with no axioms	

Theory is not a set; contains a signature and a set of axioms. Choice of language and some axioms.

$T \vdash \phi$ If you add the axioms of T then you can derive ϕ.

Definition v is a *model* of T. if v(ϕ) = 1, $\forall$ $\phi \in$ T. A valuation that makes all the sentences in T true.

Definition T is consistent if T has a model

$\Sigma = p_0, p_1, ...$

T $= p_0 \rightarrow p_1, p_0 \rightarrow p_2, ...$

Is there a model for T? Any valuation where p_0 is false there is infinite possibilities for a model. Any valuation where p_0 is true only has one possibility to make a model because everything must be true.

Definition A theory T is **complete** if for any $\phi \in$ Sent(Σ) either T $\vdash \phi$ or $T \vdash \not\phi$

Empty theories are incomplete as it doesn't tell you anything about the signature;

E.g. $\Sigma = \{$p$\}$; T = 0

To make it complete make T = $\{$p$\}$; There is just one valuation.

This new T has exactly one model. Show that T $\vdash \phi$ and $T \vdash \not\phi$

new T $\tilde{}$ $T_v = \phi \in Sent(\Sigma)|v(\phi) = 1$

$$T_v \vdash \phi$$

or

$$T_v \vdash \not\phi$$

complete and consistent has only one theory.

Properties of Theories:

- Complete
- having finitely many axioms
  - Isn't a big deal, infinitely increasing
- Σ is finite
- How do we decide what is interesting? To decide must be in relation to relations between theories.

Relations between theories

Definition A reconstrual of Σ in Σ_0 is a function $f : \Sigma \rightarrow Sent(\Sigma_0)$

Fact: A reconstrual f extends uniquely to a function $f : Sent(\Sigma) \rightarrow Sent(\Sigma_0) s.t. f(\not\phi) = \not f(\phi)$ and so on for other logical connectives

Definition f is a /translation from T to T′. f: T→ T′ iff for all $\phi \in$ Sent(Σ) if T $\vdash \phi$ then T′ $\vdash f(\phi)$

$\Sigma = p_0, p_1, ...$ T $= 0$

T $\vdash \phi$

$\vDash \phi$

T$\vDash f(\phi)$; Reconstruals always take

Definition a T-atom; Atom relative to a Theory.

$$\begin{array}{ll} \top & \\ & \psi \\ \phi & \text{Nothing lower than it} \\ \bot & \end{array}$$

Atom for a theory is a sentence that is consistnent for a theory,

4 Structure of Propositional Theories 09/24/19

How to organize propositional Theories. Within Proposition Theories we can have T $\rightarrow$ T'. (translation) T".

Another example is to replace propositional theories with sets. Applies to anything in pure mathematics such as groups, ... ,.etc . Let's continue with Sets though:

A$\rightarrow$ B. (Function from A to B, is A x B) C

An assignment of elements in B to elements of A.

1. A $\rightarrow$ (f) B$\rightarrow$ (g)C$\rightarrow$ (h) D
   (a) $(g \circ f)(a) = g(f(a))$ where $\circ$ is composition
   (b) $h \circ (g \circ f) = (h \circ g) \circ f$
   (c) $\circ$ is associative

$1_A(a) = a_1 \forall a \in A$ (identity)

C$\rightarrow$ (g)A$\rightarrow$ (f) B

$\uparrow$

1_A

$f \circ 1_A = f$

$1_A \circ g = g$

Everything you say is a category; Objects/things and arrows, composition, associative, and identity

Examples of categories

- Groups

- Vectors
- Topological spaces
- Boolean Algebra
- Rings
- Sheathes on X
- Categories; Must start on Categories of small categories;

Equivilance properties

- Symmetry
- Reflexivity
- Transitivity

4.1 Category of theories

- Objects: Proof Theory; T | (Σ, T) $\implies T \vdash \phi$ and T'$\vdash f(\phi)$
- Arrows:
  - reconstrual f; T → T'
  - reconstrual g; T' → T
  - T ∈ Σ ; T' ∈ Σ′
  - f ≃ g iff T' ⊢ $f(\phi) \iff g(\phi)$
    - Reflexivity is obvious
    - Symmetry is true by nature of $\iff$
- Define Composition in this category
- $T_0 \to (f)(f')T \to (g)(g')T'$
  - $[g] \circ [f] = [g \circ f]$
  - Need to show that If f ≃ f' and g≃ g' then $g \circ f \simeq g' \circ f'$
  - Let $\phi \in Sent(\Sigma_0)$. Show T ⊢ $g(f(\phi)) \iff g'(f'(\phi))$
  - T ⊢ $f(\phi) \iff f'(\phi)$
  - Since g≃ g', T' ⊢ $g(f'(\phi)) \iff g'(f'(\phi))$ (1)

- g(f(ϕ) $\iff$ $f'(\phi)$) = g(f(ϕ)) $\iff$ $g(f'(\phi))$
    - Literal equality of sentences; Definition of reconstrual
  - Since g is a translation, T' $\vdash g(f(\phi)) \iff g(f'(\phi))$ (2)
  - T' $\vdash g(f(\phi)) \iff g'(f'(\phi))$

- What is the Identity 1_T?
  - Identity is strict, only takes you to the same thing

How to decide definition of a category thoeries is c

4.2 Special kinds of Arrows

- Monomorphism
  - $f : A \to B$ if for any two arrows $g : X \to A$ and $h : X \to A$ if $f \circ g = f \circ h$ then $g = h$
  - monic
- Epimorphism
  - f:A$\to$ B is epi iff $\forall$ g: B$\to$ C and h: B$\to$ C. If $g \circ f = h \circ f$ then g= h
  - If f is epi then f is onto. F is injective i.e. doesnt cover whole range
    - function $g(y_0) = 0$ and $h(y_0) = 1$
    - $g \circ f = h \circ f$
- Isomorphism
  - Suppose we have two objects A and B; with f A$\to$ B and g B$\to$ A
  - $gf = 1_A$
  - $fg = 1_B$

These don't always correspond set theoretic notions. How do these relate?

Term	Definition
monomorphism/monic	f:A$\to$ B if for any two arrows g: X $\to$ A and h: X$\to$ A if f^ g= f ^ h then g

- Every isomorphism is a monomorphism and every iso is an epi
- Try to do inverse; should fail.