Fourier Series & Unit-2

Any arbitrary real or complex $x(t)$ which is periodic with fundamental time period T_0 can be expressed as a sum of sinusoids of period T_0 and its harmonic components

Fourier series can be expressed in two form.

1) Trigonomentric fourier series
2) Exponential fourier "

Trigonometric fourier Series:

A periodic signal $x(t)$ with fundamental perid T_0 (fundamental frequency $\omega_0 = 2\pi/T_0$) can be expressed as following trigonometric series

$$x(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos n\omega_0 t + b_n \sin n\omega_0 t]$$

a_0 = DC component or average value of $x(t)$
a_n = Coeff of nth harmonic component of $x(t)$
b_n = " " " " of $x(t)$

$$a_0 = \frac{1}{T_0}\int_{T_0} x(t)\, dt$$

$$a_n = \frac{2}{T_0}\int_{T_0} x(t) \cdot \cos n\omega_0 t \cdot dt$$

$$b_n = \frac{2}{T_0}\int_{T_0} x(t) \cdot \sin n\omega_0 t \, dt$$

$$\sin\omega_0 t = \frac{1}{2j}\left[e^{j\omega_0 t} - e^{-j\omega_0 t}\right]$$

$$\cos\omega_0 t = \frac{1}{2}\left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right]$$

Complex expontial signal: $x(t) = e^{-j\omega_0 t}$

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

C_n: Exp. fours coeff $= \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jn\omega_0 t}\, dt$

Relationship b/w trigonometric and complex exponential fourier series coefficients.

$$C_0 = a_0 \qquad a_n = C_n + C_n^*$$

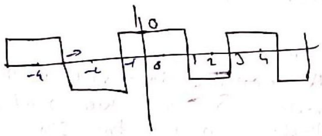
$$C_n = \frac{1}{2}[a_n - jb_n]$$

$$C_{-n} = \frac{1}{2}[a_n + jb_n]$$

Dirichlets condition for the existence of fourier Series

1. The function $x(t)$ must be absolutely integrable over on period
2. The function $x(t)$ must have finite number of maxima and minima
3. The function $x(t)$ must have finite number of discontinuities and each of these discontinuities must be finite

Ex



$T_0 = 4, \quad \omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{4} = \pi/2$

$x(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos n\omega_0 t + b_n \sin n\omega_0 t]$

$a_0 = \frac{1}{T_0}\int_{T_0} x(t)\,dt = \frac{1}{4}\int_{-1}^{3} x(t)\,dt$

$= \frac{1}{4}\left[\int_{-1}^{1} 1\,dt + \int_{1}^{3} (-1)\,dt\right]$

$= \frac{1}{4}\left[(1-(-1)) - (3-1)\right] = 1/4[2-2]$

$= 0$

$a_n = \frac{2}{T_0}\int_{T_0} x(t)\cos n\omega_0 t\,dt = \frac{2}{4}\int^{3} x(t)\cos n(\frac{\pi}{2})t\,dt$

$= \frac{1}{2}\left[\int_{-1}^{1} 1\cdot\cos(\frac{n\pi}{2}t)\,dt + \int_{1}^{3}(-1)\cos(\frac{n\pi}{2})t\,dt\right]$

$= \frac{1}{2}\left[\sin(\frac{n\pi}{2}t)\Big]_{-1}^{+1} - \frac{2}{n\pi}\sin(\frac{n\pi}{2}t)\Big]_{1}^{3}\right]$

$= \frac{1}{n\pi}\left[\sin\frac{n\pi}{2} - \sin(\frac{-n\pi}{2}) - \sin\frac{3n\pi}{2} + \sin\frac{n\pi}{2}\right]$

$a_n = \frac{1}{n\pi}\left[\sin\frac{n\pi}{2} + \sin\frac{n\pi}{2}\right] = \frac{4}{n\pi}\sin(\frac{n\pi}{2})$

$n = \text{even} \Rightarrow a_n = 0$

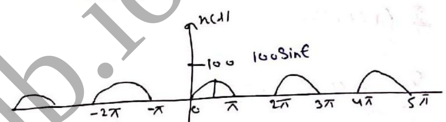
$n = \text{odd} \Rightarrow a_n = \frac{4}{n\pi}\sin(\frac{n\pi}{2})$

$b_n = \frac{2}{4}\int_{-1}^{3} x(t)\sin\frac{n\pi}{2}t\,dt = \frac{1}{2}\left[\int_{-1}^{1}\sin(\frac{n\pi}{2})t\,dt - \int_{1}^{3}\sin(\frac{n\pi}{2})t\,dt\right]$

$b_n = 0$

$x(t) = \sum_{n=1}^{\infty}\frac{4}{n\pi}\sin(\frac{n\pi}{2})\cdot\cos\frac{n\pi}{2}t \quad ; n = \text{odd}$

Ex :-



Find the trigonometric fourier series for given function $x(t)$

Sol^n $T_0 = 2\pi$

$\omega_0 = \frac{2\pi}{T_0} = 1$ { Period for integration

$a_0 = \frac{1}{T_0}\int_{T_0} x(t)\,dt = \frac{1}{2\pi}\int_0^{2\pi} x(t)\,dt = \frac{1}{2\pi}\left[\int_0^{\pi} 100\sin t\,dt + \int_{\pi}^{2\pi} 0\,dt\right]$

$= \frac{100}{\pi}$

$a_n = \frac{2}{T_0}\int_{T_0} x(t)\cos n\omega_0 t\,dt = \frac{2}{2\pi}\int_0^{\pi} 100\sin t\cdot\cos nt\,dt$

$= \frac{100}{\pi}\int_0^{\pi}\sin t\cos nt\,dt$

$= \frac{100}{\pi}\int_0^{\pi}\frac{1}{2}[\sin(t+nt) + \sin(t-nt)]\,dt$

$= \frac{50}{\pi}\left[\int_0^{\pi}\sin(n+1)t\,dt + \int_0^{\pi}\sin(1-n)t\,dt\right]$

$= \frac{50}{\pi}\left[-\frac{\cos(n+1)t}{(n+1)}\Big]_0^{\pi} - \frac{\cos(1-n)t}{1-n}\Big]_0^{\pi}\right]$

$= \frac{50}{\pi}\left[-\left\{\frac{\cos(n+1)\pi - 1}{n+1}\right\} - \left\{\frac{\cos(1-n)\pi - 1}{1-n}\right\}\right]$

$= \frac{50}{\pi}\left[-\frac{(-1)^{n+1}}{n+1} - \frac{(-1)^{n-1} - 1}{1-n}\right] \quad \begin{cases}(-1)^{n+1} \\ n = 0 = -1 \\ n = 1 = 1 \\ n = 2 = -1\end{cases}$

for n; even

$$a_n = \frac{50}{\pi}\left[\frac{-2}{n+1} - \frac{-2}{1-n}\right]$$

$$= \frac{-200}{\pi(n^2-1)}$$

for n = 0

$b_n = 0$

$$b_n = \frac{2}{T_0}\int x(t)\, \sin \cdot$$

$$b_n = \frac{2}{T_0}\int_{T_0} x(t) \sin n\omega_0 t\, dt$$

$$= \frac{2}{2\pi}\int_0^{\pi} 100 \sin t \sin nt\, dt$$

$$= \frac{100}{\pi}\int_0^{\pi} \sin t \sin nt\, dt$$

$$= \frac{50}{\pi}\left[\frac{\sin(n-1)\pi - \sin 0}{n-1} - \frac{\sin(n+1)\pi - \sin 0}{n+1}\right]$$

for n = 1

$$= \frac{50}{\pi}\left[\frac{0}{0}\right] \qquad \left\{ \sin(n-1)\pi = 0,\ n = 1 \right.$$

$b_n = 0;\ n \neq 1$

for n = 1

$$b_n = \frac{100}{\pi}\int_0^{\pi} \sin^2 t\, dt = \frac{50}{\pi}$$

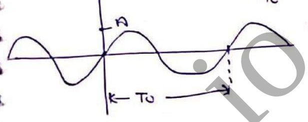
fourier series ...

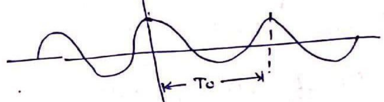
$$x(t) = \frac{100}{\pi} + \sum_{n=1}^{\infty} \frac{200}{\pi(n^2-1)} \cos nt + 50 \sin t$$

Wave form Symmetry:-

A function $x(t)$ can be an even or an odd function

$x(t) = A\sin(\omega_0 t + \theta)$ $\qquad \frac{2\pi}{T_0} = \omega$ $\qquad\qquad x(t) = A\cos(\omega_0 t + \theta)$





$\Downarrow$ Mirror image about Y-axis

$\Downarrow$ Different

$\boxed{x(-t) = -x(t)}$

odd function/signal

$\Downarrow$ Mirror image

$\Downarrow$ Same as $x(t)$

$\boxed{x(-t) = x(t)}$

even function/signal

Any arbitrary signal $x(t)$ can be broken into its even and odd components i.e. $x(t) = x_e(t) + x_o(t)$

$$x_e(t) = \frac{1}{2}[x(t) + x(-t)], \qquad x_o(t) = \frac{1}{2}[x(t) - x(-t)]$$

i) Even or Mirror Symmetry:-

If a function is even, $x(t) = \frac{1}{2}[x(t) + x(-t)]$

$$a_0 = \frac{1}{T_0}\int_{T_0} x(t)\, dt$$

$$= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\, dt \Rightarrow \frac{1}{T_0}\left[\int_{-T_0/2}^{0} x(t)\, dt + \int_0^{T/2} x(t)\, dt\right]$$

Replacing $t = -t$ $\quad t_0 = -T_0/2$ $\qquad$ original limit $\quad t \to -t \quad LL = -t = T_0/2$, $\ t = T_0/2$

$$= \frac{1}{T_0}\left[\int_{T_0/2}^{0} x(-t)(-dt) + \int_0^{T_0/2} x(t)\, dt\right]$$

$\Downarrow$ exchange the limits

$$= \frac{1}{T_0}\left[\int_0^{T_0/2} x(-t)\cdot(-dt) + \int_0^{T_0/2} x(t)\, dt\right] \qquad \left\{ \int_a^b x(t)\, dt = -\int_b^a x(t)\, dt \right.$$

$$= \frac{1}{T_0}\left[\int_0^{T_0/2} x(-t)\, dt + \int_0^{T_0/2} x(t)\, dt\right]$$

$$= \frac{1}{T_0}\left[\int_0^{T_0/2} x(-t) + x(t)\right]dt = \frac{1}{T_0}\int_0^{T_0/2} 2x(t)\,dt$$

$$\boxed{a_0 = \frac{2}{T_0}\int_0^{T_0/2} x(t)\,dt}$$ For even function.

→ Original Limit have reduced to $T_0/2$ from T_0

$$a_n = \frac{2}{T_0}\int_{T_0} x(t)\cos n\omega_0 t\,dt$$

$$= \frac{2}{T_0}\left[\int_{-T_0/2}^{0} x(t)\cdot\cos n\omega_0 t\,dt + \int_0^{T_0/2} x(t)\cdot\cos n\omega_0 t\,dt\right]$$

$$= \frac{2}{T_0}\left[-\int_0^{T_0/2} x(-t)\cos n\omega_0 t\,(-dt) + \int_0^{T_0/2} x(t)\cdot\cos n\omega_0 t\,dt\right] \quad \{\cos(-\theta) = \cos\theta\}$$

$$= \frac{2}{T_0}\left[\int_0^{T_0/2}[x(-t)+x(t)]\cos n\omega_0 t\,dt\right]$$

$$= \frac{2}{T_0}\int_0^{T_0/2} 2x(t)\cos n\omega_0 t\,dt$$

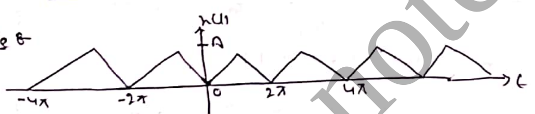
$$\boxed{a_n = \frac{4}{T_0}\int_0^{T_0/2} x(t)\cos n\omega_0 t\,dt}$$ For even function

Similarly for b_n..

$$b_n = \frac{2}{T_0}\int x(t)\sin n\omega_0 t\,dt$$

$$\boxed{b_n = 0}$$

<u>Example</u>:



Determine the trigonometric fourier series for above.

<u>Solⁿ</u>: $T_0 = 2\pi$

$\omega_0 = 1$

$$x(t) = \begin{cases} \frac{A}{\pi}t & 0 \le t \le \pi \\ -\frac{A}{\pi}t & \pi \le t \le 2\pi \end{cases}$$

<u>ODD Symmetry</u>:

$$x(t) = \frac{1}{2}[x(t) - x(-t)]$$

$a_0 = 0$

$a_n = 0$

$$b_n = \frac{4}{T_0}\int_0^{T_0/2} x(t)\sin n\omega_0 t\,dt$$

$$\boxed{x(t) = \sum_{n=1}^{\infty} b_n \sin n\omega_0 t}$$

→ $x(t)$ is an even function, so using even symmetry property:-

$$a_0 = \frac{2}{T_0}\int_0^{T_0/2} x(t)\,dt$$

$$a_n = \frac{4}{T_0}\int_0^{T_0/2} x(t)\cos n\omega_0 t\,dt$$

$$b_n = 0$$

$$a_0 = \frac{2}{2\pi}\int_0^{\pi}\frac{A}{\pi}t\,dt = \frac{2A}{\pi^2}[\pi^2 - 0]$$

$$= \frac{A}{2}$$

$$a_n = \frac{4}{2\pi}\int_0^{\pi}\frac{A}{\pi}t\cos nt\,dt = \frac{2A}{\pi}\int_0^{\pi} t\cdot\cos nt\,dt$$

$$= \frac{2A}{\pi^2}\left[t\cdot\frac{\sin nt}{n}\Big|_0^{\pi} - \int_0^{\pi} 1\cdot\frac{\sin nt}{n}\,dt\right]$$

$$= \frac{2A}{\pi^2}\left[\left\{\frac{\pi\sin n\pi - 0}{n}\right\} + \left\{\frac{\cos nt}{n^2}\right\}_0^{\pi}\right]$$

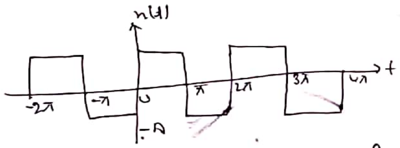
$$= \frac{2A}{\pi^2}\left[0 + \frac{1}{n^2}(\cos n\pi - \cos 0)\right]$$

$$= \frac{2A}{n^2\pi^2}\left[(-1)^n - 1\right] = \begin{cases} \frac{-4A}{\pi^2 n^2} & n = \text{odd} \\ 0 & n = \text{even} \end{cases}$$

$$x(t) = \frac{A}{2} + \sum_{n=1}^{\infty}\left(\frac{-4A}{\pi^2 n^2}\right)\cos nt \quad (n = odd)$$

$$\boxed{x(t) = \frac{A}{2} - \frac{4A}{\pi^2}\sum_{n=odd}^{\infty}\frac{\cos nt}{n^2}}$$

Ex-2 :-



Find the trigonometric fourier series for given x(t)

Sol^n: x(t) is an odd function, so using odd symmetry property —

$a_0 = 0$

$a_n = 0$

$b_n = \frac{4}{T_0}\int_0^{T_0/2} x(t)\sin n\omega_0 t\,dt$

$T_0 = 2\pi$, $\omega_0 = 1$

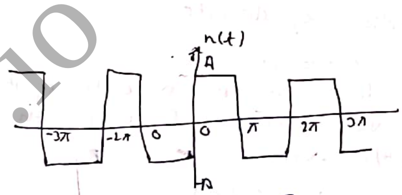
$$x(t) = \begin{cases} A, & 0 \le t \le \pi \\ -A, & \pi \le t \le 2\pi \end{cases}$$

$$b_n = \frac{2A}{n\pi}\left[1-(-1)^n\right] = \begin{cases} 4A/n\pi, & n = odd \\ 0, & n = even \end{cases}$$

Exponential fourier Series :-

$x(t) = \sum_{n=\infty} C_n$

Ex:-



Sol: $T_0 = 2\pi$ $\omega_0 = 1$ $x(t) = \begin{cases} A, & 0 \le t \le \pi \\ -A, & \pi \le t \le 2\pi \end{cases}$

$$C_n = \frac{1}{2\pi}\int_0^{2\pi} x(t)e^{-jnt}dt$$

$$= \frac{1}{2\pi}\left[A\int_0^{\pi} e^{-jnt}dt - A\int_{\pi}^{2\pi} e^{-jnt}dt\right]$$

$C_n = -j\frac{2A}{n\pi}$; $n = odd$

$= 0$, $n = even$

Avg or DC value $C_0\,(n=0) = \frac{1}{T_0}\int_{T_0} x(t)dt = \frac{1}{2\pi}\left[\int_0^{\pi} A\,dt + \int_{\pi}^{2\pi} -A\,dt\right]$

$$= \frac{A}{2\pi}\left[(\pi-0)-(2\pi-\pi)\right] \Rightarrow 0$$

fourier Series $x(t) = \sum_{n=odd}^{\infty} -j\frac{2A}{n\pi}\cdot e^{-jnt}$

$$\boxed{x(t) = -j\frac{2A}{\pi}\sum_{n=odd}^{\infty}\frac{e^{jnt}}{n}}$$

Discrete fourier Spectrum :-

The exponential fourier series of a period function is given as:

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} \quad \text{, where } C_n = \frac{1}{T_0}\int_{T_0} x(t)\cdot e^{-jn\omega_0 t}\,dt$$

So for a given function $x(t)$, its fourier coefficients C_n are function of discrete frequency $n\omega_0$ (i.e. $\omega_0, 2\omega_0, 3\omega_0$)

The discrete fourier spectrum of a plot of its Fourier Coefficients C_n versus discrete frequency (Harmonics) $n\omega$ where $n = 0, \pm1, \pm2 \dots$

Fourier Spectrum has 2 parts.

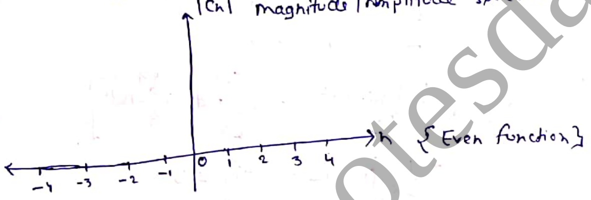
1. Magnitude or Amplitude Spectrum Mag(C_n) vs n
2. Phase Spectrum ; Ang(C_n) vs n

C_n = Complex form $\rightarrow \underbrace{a+jb}$

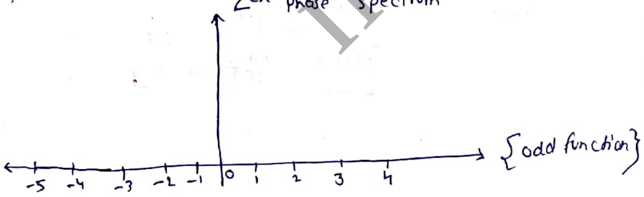
$C_n = a + jb = re^{\theta}$

$|C_n| = r = \sqrt{a^2+b^2}$

$\angle C_n = \theta = \tan^{-1}(b/a)$



{Even function}



{odd function}

Ex:- Plot the mag & phase spectrum for the previous example.

from previous calculation

$C_n = -j\frac{2A}{n\pi}$; n = odd

Complex function $= 0$; n = even

at $n = 0$

$C_0 = 0$

$|C_n| = \sqrt{0 + \left(\frac{-2A}{n\pi}\right)^2}$

$= \frac{2A}{n\pi}$

$\angle C_n = \tan^{-1}\left(\frac{-2\pi/n\pi}{0}\right)$

$= \tan^{-1}(-\infty)$

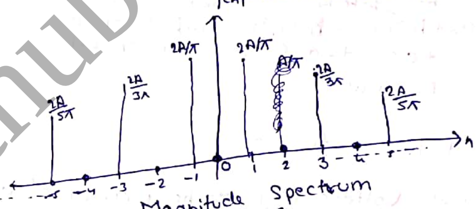
$= -\tan^{-1}(\infty)$

$|C_n| \rightarrow$ absolute $\angle C_n = -\pi/2$

$\angle C_n = -\pi/2$; $n \rightarrow +ve$

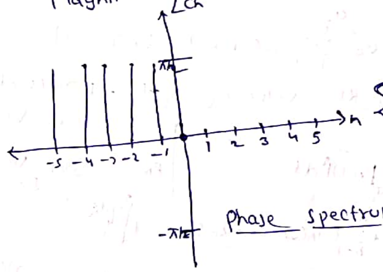
$\angle C_n = \pi/2$; $n \rightarrow -ve$

$\angle C_n = 0$; $n = 0$



} even function

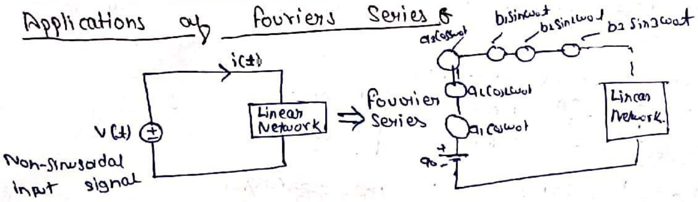
Magnitude Spectrum



{ odd function always origin

Phase spectrum

Applications of fouriers Series :



Any orbitrary signal v(t) can be represented as a linear combination of its harmonic components.

$$V(t) = a_0 + \sum_{n=1}^{\infty} (a_n\cos n\omega_0 t + b_n \sin n\omega_0 t)$$

1st harmonic, 2nd harmonic, 3rd harmonic.

$$= a_0 + b_1\sin\omega_0 t + b_2\sin(2\omega_0 t) + b_3\sin(3\omega_0 t) + \cdots$$
$$+ a_1\cos\omega_0 t + a_2\cos(2\omega_0 t) + a_3\sin(3\omega_0 t) + \cdots$$

$a_0 \rightarrow$ Avg or DC value

$i(t) = \frac{V(t)}{Z} \rightarrow$ Depend on freq. ω

Any Non-sinusoidal Voltage V(t) can be represented using fourier Series as --

$$V(t) = a_0 + \sum_{n=1}^{\infty} (a_n\cos n\omega_0 t + b_n \sin n\omega_0 t)$$

Effective value or RMS value of voltage.

$x(t) = A\sin(\omega_0 t + \theta)$ — A: Peak Value

$\text{RMS} = \frac{\text{Peak Value}}{\sqrt{2}}$

$$V_{eff} = V_{rms} = \sqrt{\frac{1}{T}\int_T |v(t)|^2 dt}$$

$$= \left\{\frac{1}{T}\int_0^T \left|a_0 + \sum_{n=1}^{\infty}(a_n\cos n\omega_0 t + b_n \sin n\omega_0 t)\right|^2\right\}^{1/2}$$

$$= \left\{a_0^2 + \frac{a_1^2}{2} + \frac{a_2^2}{2} + \cdots + \frac{b_1^2}{2} + \frac{b_2^2}{2} + \cdots\right\}^{1/2}$$

$$V_{rms} = \left\{A_0^2 + \frac{1}{2}\left\{A_1^2 + A_2^2 + A_3^2 + \cdots\right\}\right\}^{1/2}$$

$$A_n = a_n^2 + b_n^2$$

$$V_{rms} = \left\{V_0 + \frac{1}{2}(V_1^2 + V_2^2 + V_3^2 + \cdots)\right\}^{1/2}$$

$$= \left\{V_0 + \left(\frac{V_1}{\sqrt{2}}\right)^2 + \left(\frac{V_2}{\sqrt{2}}\right)^2 + \cdots\right\}^{1/2}$$

$$\boxed{V_{rms} = \left\{V_0^2 + V_{rms1}^2 + V_{rms2}^2 + V_{rms3}^2 + \cdots\right\}^{1/2}}$$

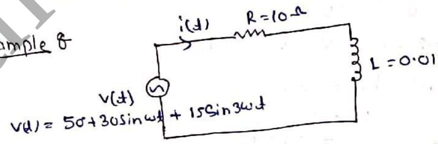
V_0 = Average or DC value of V(t)

V_{rms1} = RMS value of V(t) for 1st harmonic

V_{rms2} = " " " " 2nd "

$$\boxed{I_{rms} = \left\{I_0^2 + I_{rms1}^2 + I_{rms2}^2 + I_{rms3}^2 + \cdots\right\}^{1/2}}$$

Example :



Determine the current i(t) and Average Power..

Solⁿ :- $V(t) = 10\sin 2t$

$V_{rms} = \frac{10}{\sqrt{2}}$ $\quad i(t) = \frac{V_{rms}}{Z}$

$V(t) = 50 + 30\sin\omega t + 15\sin(3\omega)t$

Compare the above signal with trigonometric series.

Step-I : $a_0 = 50$, $V_1 = 30$, $V_2 = 0$, $V_3 = 15$, $V_4 = V_5 = V_6 = \cdots = 0$

$$V_{rms} = \left\{V_0^2 + \frac{V_1^2}{2} + \frac{V_3^2}{2}\right\}^{1/2} = \left\{50 + \frac{30^2}{2} + \frac{15^2}{2}\right\}$$
$$= 55.34 \text{ volts}$$

Step-II : Calculate impedance for all harmonic Components separately :—

Imp $\quad Z = R + j(X_L - X_C)$ $\quad X_L = \omega L$, $X_C = \frac{1}{\omega C}$

For DC or Avg. Component $(\omega = 0)$

$Z_0 = 10 + j0$

$= 10\ \Omega$

$I_0 = \frac{V_0}{Z_0} = \frac{50}{10} = 5A$

for first Harmonic, $\omega = 250$ rad/s

$Z_1 = R + j\omega L$

$= 10 + j250 \times 0.01$

$= 10 + j2.5$

$= 10.308 \angle 14.03^\circ \, \Omega$

$i_1(t) = \frac{V_1}{|Z_1|} \sin(\omega t - \theta_1)$

$= \frac{30}{10.308} \sin(\overset{\omega t}{\cancel{250}} - 14.03^\circ)$

$= 2.91 \sin(\omega t - 14.03^\circ)$ Amp

Current at Second Harmonic –

$X_L = 2\omega L$ $\quad V_2 = \infty$ $\quad i_2(t) = 0$

Current for third harmonic

$V_3 = 15$ $\quad \omega = 3 \times \omega_0$

$= 3 \times 250$

$= 750$ rad/sec

$Z_3 = 10 + j \times 750 \times 0.01$

$= 10 + j7.5$ $\quad |Z_3| = 12.5$

$= 12.5 \angle 36.86^\circ$ $\quad \theta_3 = 36.86$

$i_3(t) = \frac{V_3}{|Z_3|} \sin(3\omega t - \theta_3) = \frac{15}{12.5} \sin(3\omega t - 36.36^\circ)$

$= 1.2 \sin(3\omega t - 36.86) \, A$

Since given network is a linear Network,

Total current $i(t) = i_0 + i_1(t) + i_3(t)$

$= 5 + 2.91 \sin(\omega t - 14.03) + 1.2 \sin(3\omega t - 36.86)$

$I_{rms} = I_{eff} = \sqrt{I_0^2 + \frac{I_1^2}{2} + \frac{I_3^2}{2}}$

$= \sqrt{25 + \frac{2.91^2}{2} + \frac{1.2^2}{2}}$

$I_{rms} = 5.473 \, A$

Average Power. $P_{av} = I_{rms}^2 \cdot R$

$= 5.473^2 \times 10$

$= 299.54$ Volts

Fourier transform

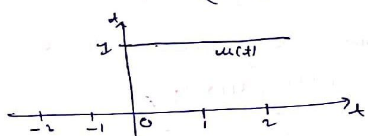
Basic Signals:– A signal is a function of one or more independent variable ~~vertical~~

$x = f(t)$

Unit Step Functions :- $u(t)$

The unit step function $u(t)$ is defined as

$$u(t) = \begin{cases} 1 & t \ge 0 \\ 0 & t < 0 \end{cases}$$

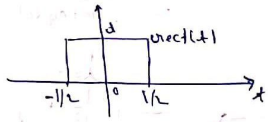


also known as the Heaviside step function. Alternate definitions of value exactly at zero, such as 1/2.

Unit Rectangle :- (Gate function)

Unit rectangle signal

$$rect(t) = \begin{cases} 1 & \text{if } |t| \le 1/2 \\ 0 & \text{otherwise} \end{cases}$$



Combinations of unit steps to create other signals
The offset rectangular signal.

$$x(t) \begin{cases} 0 & t \geq 1 \\ 1 & 0 \leq t < 1 \\ 0 & t < 0 \end{cases}$$

Can be written as

$$x(t) = u(t) - u(t-1)$$

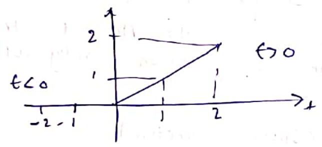
Unit Ramp :-

The unit ramp is defined as

$$r(t) = \begin{cases} t & t \geq 0 \\ 0 & t < 0 \end{cases}$$

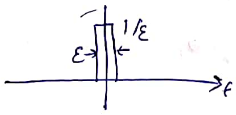
The unit ramp is the integral of the unit step

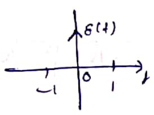
$$r(t) = \int_{-\infty}^{t} u(\tau)\,d\tau$$



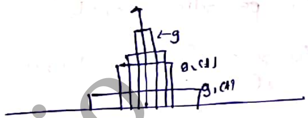
Unit Impulse function :

(Dirac's) delta function or impulse' δ is an idealization of a signal that
· is very large near $t=0$
· is very small away from $t=0$
· has integral 1





the exact shape of the function doesn't matter
ϵ is small (which depends on context)

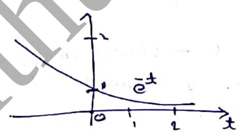


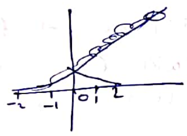
Exponential Signals

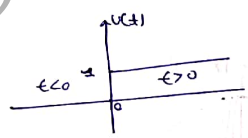
An exponential signal is given by

$$x(t) = e^{\alpha t}$$

α → +ve → growing function
→ -ve → decaying func.



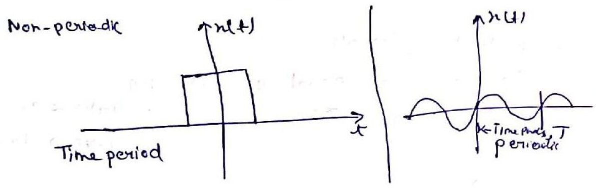




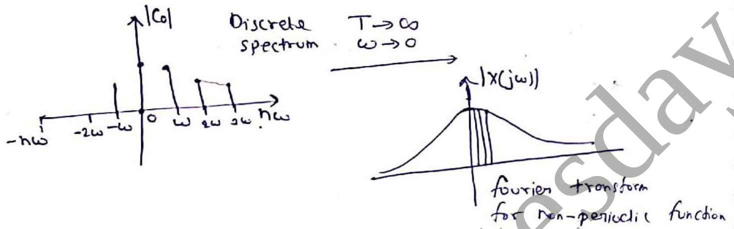
$\{ -\infty \leq t \leq 0 \rightarrow 0$

Fourier - transform: Fourier transform is an extension or generalization of fourier series for the class of Non-periodic signals.

→ Example of Non-Periodic function - step, ramp, rectangular, Exp.



→ When time period $T \to \infty$, the freq $\omega \to 0$ (very small), then Discrete spectrum (with freq. variable) is converted to a Continuous spectrum (with freq. variable ω)



The F.T transform a time-domain function $x(t)$ (time) into corresponding frequency domain $X(\omega)$ $\{X(j\omega)\}$ to study a signal/function in frequency domain.

Definition: The fourier transform of a time-domain function

fourier transform $X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$

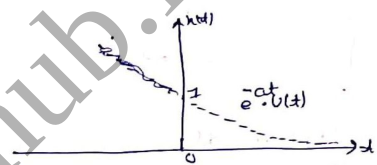
$\left[\begin{array}{l} \text{Exp F.S. Coeff} \\ C_n = \frac{1}{T}\int_{T} x(t)\, e^{-jn\omega t}\, dt \end{array} \right.$

$T \to \infty$

Fourier Transform Pair:

Time domain function	$x(t) \xrightarrow{F} X(j\omega)$	Frequency Domain
Freq Domain function	$X(j\omega) \xrightarrow[F^{-1}]{\text{inverse fourier function}} x(t)$	Time Domain function

Eg:- Find fourier transform of given function
$x(t) = e^{-at}\, u(t)$ → Time Domain



Fourier transform $X(j\omega) = \int_{-\infty}^{\infty} x(t)\cdot e^{-j\omega t}\, dt$

$= \int_{-\infty}^{\infty} e^{-at}\, u(t)\cdot e^{-j\omega t}\, dt$

$= \int_{-\infty}^{0} e^{-at}\, u(t)\, e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-at}\, u(t)\, e^{-j\omega t}\, dt$

$= \int_{0}^{\infty} e^{-(a+j\omega)t}\, dt = \frac{-1}{a+j\omega}\left[e^{-(a+j\omega)t}\right]_0^{\infty}$

$= \frac{-1}{a+j\omega}\left[e^{-\infty} - e^{0}\right]$

$= \frac{-1}{a+j\omega}\left[0 - 1\right]$

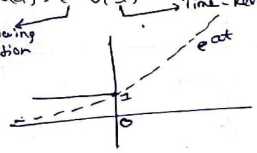
$X(j\omega) = \frac{1}{a+j\omega}$

$x(t) = e^{-at}u(t) \xleftrightarrow{F} X(j\omega) = \frac{1}{a+j\omega}$

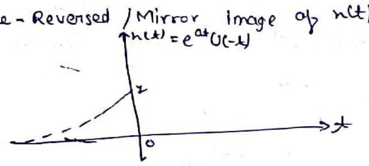
Example-2: $x(t) = e^{at}u(-t)$ → Time-Reverse unit step function

Growing function

Sol^n



$x(-t)$ = Time-Reversed / Mirror image of $x(t)$



$$F\{x(t)\} = F\{e^{at}u(-t)\} = \int_{-\infty}^{\infty} x(t)\cdot e^{-j\omega t}\,dt.$$

$$= \int_{-\infty}^{\infty} e^{at}u(-t)\cdot e^{-j\omega t}\,dt$$

$$= \int_{-\infty}^{0} e^{at}u(-t)\cdot e^{-j\omega t}\,dt + \int_{0}^{\infty} e^{at}\underbrace{u(-t)}_{0}\cdot e^{-j\omega t}\,dt$$

$$= \int_{-\infty}^{0} e^{(a-j\omega)t}\,dt.$$

$$= \frac{1}{a-j\omega}\left[e^{0} - e^{-\infty}\right]$$

$$= \frac{1}{a-j\omega}(1-0) \Rightarrow \frac{1}{a-j\omega}$$

$$F\{e^{at}u(-t)\} = \frac{1}{a-j\omega}$$

$$e^{at}u(-t) \xleftrightarrow{F} \frac{1}{a-j\omega}$$

Fourier Transform Pairs

	$x(t)$	$X(j\omega)$
1.	$e^{-at}u(t)$	$\frac{1}{a+j\omega}$
2.	$e^{at}u(-t)$	$\frac{1}{a-j\omega}$
3.	$e^{-a\lvert t\rvert}$	$\frac{2a}{a^2+\omega^2}$
4.	$A\,\mathrm{rect}\left(\frac{t}{T}\right)$	$\frac{2A}{\omega}\sin(\omega T/2) = A\cdot T\,\mathrm{Sinc}(\omega T/2)$
5.	$\delta(t)$	1
6.	1	$2\pi\,\delta(\omega)$
7.	$\mathrm{Sgn}(t)$	$\frac{2}{j\omega}$
8.	$u(t)$	$\pi\,\delta(\omega) + \frac{1}{j\omega}$

Magnitude and Phase Spectrum of Fourier Transform

$X(j\omega) = a + jb$
$\quad = |X(j\omega)| e^{j\angle X(j\omega)}$

Plot of $|X(j\omega)|$ w.r.t freq $\omega \rightarrow$ Mag. Spectrum
Plot of $\angle X(j\omega)$ w.r.t freq $\omega \rightarrow$ Phase Spectrum

Ex: Plot the spectrum of function in example-I.

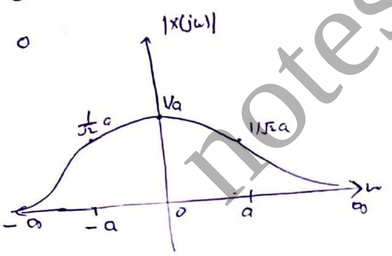
Sol: $e^{-at}u(t) \xleftrightarrow{F} \frac{1}{a+j\omega} = X(j\omega)$ → Complex function

$$\boxed{|X(j\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}}$$

$\angle X(j\omega) = \angle Num - \angle Denom$
$= \tan^{-1}(0/1) - \tan^{-1}\left(\frac{\omega}{a}\right)$
$= 0 - \tan^{-1}(\omega/a)$

$$\boxed{\angle X(j\omega) = 0 - \tan^{-1}(\omega/a)}$$

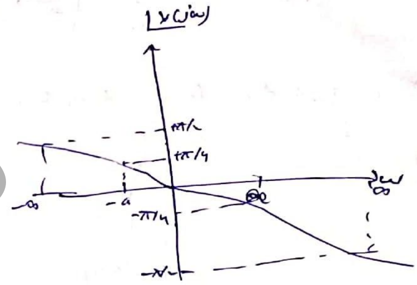
ω	$X(j\omega)$
0	$1/a$
a	$1/\sqrt{2}a$
$-a$	$1/\sqrt{2}a$
$+\infty$	0
$-\infty$	0



Mag Spectrum

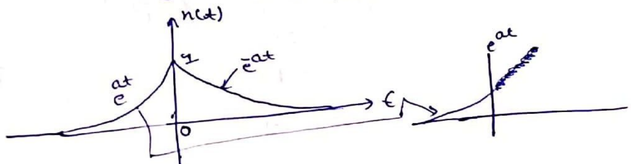
Phase Spectrum

ω	$\angle X(j\omega)$
0	0
$+a$	$-\pi/4$
$-a$	$\pi/4$
$+\infty$	$-\pi/2$
$-\infty$	$+\pi/2$



Ex-2: $x(t) = e^{-a|t|}$

Soln. $= \begin{cases} e^{-at} & t<0\,;\ -ve \\ e^{-at} & t>0\,;\ +ve \end{cases}$



$e^{-a|t|} = x(t) = e^{at}u(-t) + e^{-at}\cdot u(t)$

$F\{e^{-a|t|}\} = F\{e^{at}u(-t) + e^{-at}u(t)\}$

$= F\{e^{at}u(-t)\} + F\{e^{-at}u(t)\}$

$= \int_{-\infty}^{\infty} e^{at}u(-t)\cdot e^{-j\omega t}\,dt + \int_{-\infty}^{\infty} e^{-at}u(t)\cdot e^{-j\omega t}\,dt$

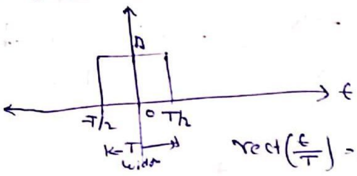
$= \int_{-\infty}^{0} e^{(a-j\omega)t}\,dt + \int_{0}^{\infty} e^{-(a+j\omega)t}\,dt$

$= \frac{1}{a-j\omega} + \frac{1}{a+j\omega} \Rightarrow \frac{2a}{a^2+\omega^2}$

$$F\{e^{-a|t|}\} = \frac{2a}{a^2+\omega^2} \quad e^{-a|t|} \xleftrightarrow{F} \frac{2a}{a^2+\omega^2}$$

Ex:4 Find the fourier transform of a rectangular puls function given as:-

Soln:-



$$rect\left(\frac{t}{T}\right) = \begin{cases} A & |t| < T/2 \\ 0 & |t| > T/2 \end{cases}$$

$$F\{rect(t/T)\} = \int_{-\infty}^{\infty} A\,rect\left(\frac{t}{T}\right) \cdot e^{-j\omega t}\,dt$$

$$= A\int_{-T/2}^{T/2} e^{-j\omega t}\,dt = A\left\{\frac{e^{-j\omega t}}{-j\omega}\right\}_{-T/2}^{T/2}$$

$$= \frac{A}{\omega}\left[\frac{e^{-j\omega T/2} - e^{j\omega T/2}}{-j}\right]$$

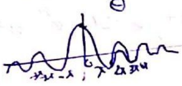
$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$= \frac{2A}{\omega}\sin(\omega T/2) \quad \text{Ans}$$

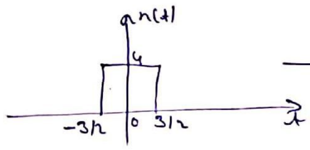
$$\text{Sinc} = \frac{\sin\theta}{\theta}$$

$$= \frac{A \cdot T}{\omega/2}\,\frac{\sin(\omega T/2)}{T}$$

$$= A \cdot T\left\{\frac{\sin(\omega T/2)}{\omega T/2}\right\}$$



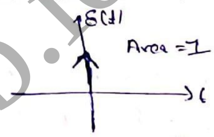
$$\boxed{F\,rect\left(\frac{t}{T}\right) = AT\,\text{Sinc}(\omega T/2)}$$



$$\longrightarrow X(j\omega) = 4 \cdot 3\sin(\omega \cdot 3/2)$$
$$= 12\,\text{Sinc}(3\omega/2)$$

Ex5: Find the fourier transform of a Impulse function (Dirac-Delta function)

Sol:



Area = 1 $\Rightarrow \int_{-\infty}^{\infty} \delta(t)\,dt = 1$ & $\delta(t) = 0$, $t \neq 0$

$$F\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t) \cdot e^{-j\omega t}\,dt$$

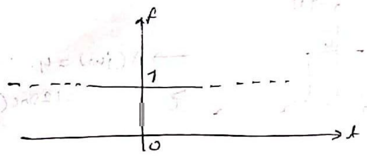
Using Property of impulse function:-

$$\int_{-\infty}^{\infty} x(t) \cdot \delta(t) \cdot dt = x(t)\Big|_{t=0} = x(0)$$

$$= e^{-j\omega t}\Big|_{t=0} = e^0 = 1$$

$$\boxed{F\{\delta(t)\} = 1 \quad \text{or} \quad \delta t \xleftrightarrow{F} 1}$$

Ex :



Dirichlet's conditions for the existence of fourier transform :

- The function $x(t)$ must be absolutely integrable -

$$\int_{-\infty}^{\infty} |x(t)|\, dt \le \infty$$

- The function $x(t)$ must have finite no of maxima & minima points in finite interval of time
- The function $x(t)$ must have finite no. of discontinuities and each of these discontinuous

$$x(t) = 1$$

$$\int_{-\infty}^{\infty} |x(t)|\, dt = \int_{-\infty}^{\infty} 1 . dt = (t)_{-\infty}^{\infty} = \infty$$

→ The given function $x(t) = 1$ is not absolutely integrable.
→ So, its fourier transform can be calculated directly from eq^n of FT

→ Instead of finding the FT, of $x(t) = 1$, we will calculate the inverse FT of impulse function $\delta(\omega)$ in freq domain

→ What is Inverse fourier Transform :

$$x(t) \xleftrightarrow{F} X(j\omega)$$

$$X(j\omega) \xleftrightarrow{F^{-1}} x(t)$$

Inverse fourier transform of a freq domain function $X(j\omega)$ is defined as :

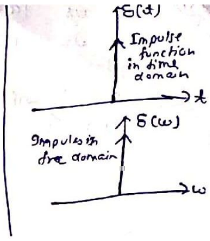
$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$$

Now, the fourier transform of $x(t) = 1$ is calculated as follows

$$F^{-1}\{\delta(\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega)\, e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi} \left[e^{j\omega t} \right]_{\omega = 0}$$

$$= \frac{1}{2\pi}$$



$$F^{-1}\{\delta(\omega)\} = \frac{1}{2\pi}$$

take the F.T of above equation -

$$F\{F^{-1}(\delta(\omega))\} = F\left\{\frac{1}{2\pi}\right\}$$

$$= \frac{1}{2\pi} F\{1\}$$

$$\delta(\omega) = \frac{1}{2\pi} F\{1\}$$

$$F\{1\} = 2\pi \cdot \delta(\omega)$$

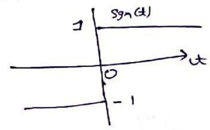
$$\boxed{1 \xleftrightarrow{F} 2\pi\, \delta(\omega)}$$

similarly

$$5 \xleftrightarrow{F} 5 \cdot 2\pi\, \delta(\omega)$$

Ex. 7 :- Find fourier transform of Signum function -

Sol^n :

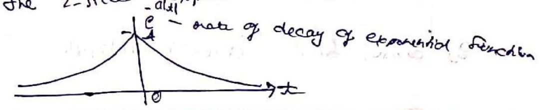


$$\Rightarrow sgn(t) = \begin{cases} +1 & t \ge 0 \\ -1 & t \le 0 \end{cases}$$

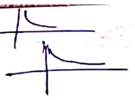
$$\int_{-\infty}^{\infty} |sgn(t)|\, dt = \int_{-\infty}^{\infty} 1\, dt = (t)_{-\infty}^{\infty} = \infty$$

The given function $sgn(t)$ is not absolutely integrable So its FT can't be calculated directly.

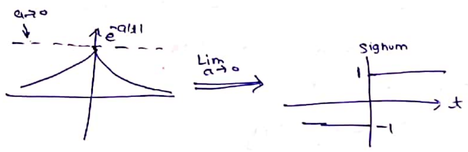
→ Consider the 2-sided exponential Signal $e^{-a|t|}$



If a is large $\rightarrow$ decay is fast
If a is small $\rightarrow$ decay slowly



$$\lim_{a\to 0} \left\{ e^{-a|t|} \right\}$$



$$\text{sgn}(t) = \lim_{a\to 0} \left\{ \underbrace{e^{-at}u(t)}_{\text{Right Side}} - \underbrace{e^{at}u(-t)}_{\text{Left Side}} \right\}$$

$$F\{\text{sgn}(t)\} = F\left\{ \lim_{a\to 0} \left(e^{-at}u(t) - e^{at}u(-t) \right) \right\}$$

$$= \lim_{a\to 0} \left[F\left(e^{-at}u(t)\right) - F\left(e^{at}u(-t)\right) \right]$$

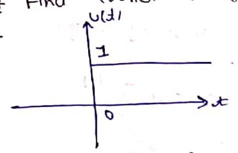
$$= \lim_{a\to 0} \left[\frac{1}{a+j\omega} - \frac{1}{a-j\omega} \right]$$

$$= \frac{1}{j\omega} - \frac{1}{-j\omega} = \frac{2}{j\omega}$$

$$\boxed{F\{\text{sgn}(t)\} = \frac{2}{j\omega} \; ; \; \text{sgn}(t) \xleftrightarrow{F} 2/j\omega}$$

Ex: Find fourier transform of unit step function $u(t)$.

Sol:

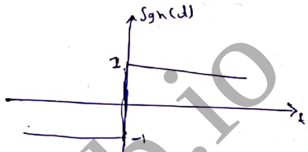


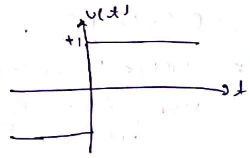
$$u(t) = \begin{cases} 1 ; & t \geq 0 \\ 0 ; & t \leq 0 \end{cases}$$

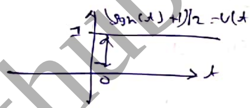
Is above function absolutely integrable

$$\int_{-\infty}^{\infty} |x(t)| dt = \int_0^{\infty} 1 \, dt = (t)_0^{\infty} = \infty$$

Since the given function is not absolutely integrable, the F.T can't be calculated directly.







$$\boxed{u(t) = \frac{1}{2}\left[1 + \text{sgn}(t)\right]}$$

$$F\{u(t)\} = F\left\{\frac{1}{2}\left(1 + \text{sgn}(t)\right)\right\} = \frac{1}{2} F\{1 + \text{sgn}(t)\}$$

$$= \frac{1}{2}\left\{ F\{1\} + F\{\text{sgn}(t)\} \right\}$$

$$= \frac{1}{2\pi}\left\{ 2\pi\delta(\omega) + \frac{2}{j\omega} \right\} = \pi\delta(\omega) + \frac{1}{j\omega}$$

$$\boxed{F\{u(t)\} = \pi\delta(\omega) + \frac{1}{j\omega} \; ; \; u(t) \xleftrightarrow{F} \pi\delta(\omega) + \frac{1}{j\omega}}$$

~~Ex:~~

Properties of fourier Transform:

1. Linearity:-
$$x_1(t) \xleftrightarrow{F} X_1(j\omega)$$
$$x_2(t) \xleftrightarrow{F} X_2(j\omega)$$
then, $a x_1(t) + b x_2(t) \xleftrightarrow{F} a X_1(j\omega) + b X_2(j\omega)$

2. Time Shifting :- If $x(t) \xleftrightarrow{F} X(j\omega)$
then $x(t - t_0) \xleftrightarrow{F} e^{-j\omega t_0} X(j\omega)$

Ex: $x(t) = e^{-a(t-2)} u(t-2)$

Sol: we know ---

$$e^{-at}u(t) \xleftrightarrow{F} \frac{1}{a+j\omega}$$

Using time shift property ---

Using time-shift property.

$$e^{-a(t-2)}u(t-2) \xleftrightarrow{F} \frac{1}{a+j\omega} \cdot e^{-j\omega(2)}$$

$$= \frac{e^{-2j\omega}}{a+j\omega}$$

Ex:- $F\{u(t-1)\} = \cancel{\pi\delta(\omega)+\frac{1}{j\omega}}$

Sol^n: $u(t) \xleftrightarrow{F} \pi\delta(\omega) + \frac{1}{j\omega}$

using time shift property

$$u(t-1) \xleftrightarrow{F} \left(\pi\delta(\omega) + \frac{1}{j\omega}\right) \cdot e^{-j\omega}$$

$$= e^{-j\omega}\left(\pi\delta(\omega) + \frac{1}{j\omega}\right)$$

Similarly:

$$u(t+1) \xleftrightarrow{F} e^{j\omega}\left(\pi\delta(\omega) + \frac{1}{j\omega}\right)$$

3:- **Differentiation in time:**

$$\frac{d}{dt}x(t) \xleftrightarrow{F} j\omega \cdot X(j\omega)$$

4- Differentiation in frequency

$$t \cdot x(t) \xleftrightarrow{F} j \cdot \frac{d}{d\omega} X(j\omega)$$

5- Frequency shifting $e^{\pm j\omega_0 t} x(t) \xleftrightarrow{F} X[j(\omega \mp \omega_0)]$ freq

6- Time-scaling

$$x(t) \xleftrightarrow{F} X(j\omega)$$

$$x(at) \xleftrightarrow{F} \frac{1}{|a|} \cdot X(j(\omega/a))$$

Constant

7. Time-Reversal:-

$$x(t) \xleftrightarrow{F} X(j\omega)$$

$$x(-t) \xleftrightarrow{F} X(-j\omega)$$

Properties of

Examples on Properties of F.T:

① $x(t) = t \cdot e^{-at}u(t)$

$$e^{-at}u(t) \xleftrightarrow{F} \frac{1}{a+j\omega}$$

Now using Diff in freq domain property

$$t \cdot e^{-at}u(t) \xleftrightarrow{F} j\frac{d}{d\omega}\left[\frac{1}{a+j\omega}\right]$$

$$F\{t \cdot e^{-at}u(t)\} = j\left[\frac{(a+j\omega)\cdot 0 - 1\cdot j}{(a+j\omega)^2}\right]$$

$$= \frac{-j^2}{(a+j\omega)^2}$$

$$= \frac{1}{(a+j\omega)^2}$$

$$F\{t \cdot e^{-at}u(t)\} = \frac{1}{(a+j\omega)^2}$$

Ex.2:- Find the F.T of $\frac{d}{dt}u(t)$

Sol^n: Using diff in time domain property-

$$F\left\{\frac{d}{dt}u(t)\right\} = j\omega \cdot F\{u(t)\} = j\omega\left\{\frac{1}{j\omega} + \pi\delta(\omega)\right\}$$

$$= 1 + j\omega\,\pi\delta(\omega)$$

Ex.3: $x(t) = e^{-j\omega_0 t}$ (Complex function)

Sol^n: $x(t) = 1 \cdot e^{-j\omega_0 t}$

we know, $F\{1\} = 2\pi\,\delta(\omega)$

Using frequency shifting property—

$$F\{1 \cdot e^{-j\omega_0 t}\} = 2\pi\,\delta(\omega+\omega_0)$$

$$e^{-j\omega_0 t} \xleftrightarrow{F} 2\pi\,\delta(\omega+\omega_0)$$

Ex.4: $x(t) = e^{j\omega_0 t}$

$$e^{j\omega_0 t} \xleftrightarrow{F} 2\pi\,\delta(\omega-\omega_0)$$

Ex 5: $x(t) = \cos\omega_0 t$ ..

Sol^n Using Euler's identity $\cos\omega_0 t = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$

Imp

$$F\{\cos\omega_0 t\} = \frac{1}{2} F\{e^{j\omega_0 t} + e^{-j\omega_0 t}\}$$

Using Linearity property ---

$$= \frac{1}{2}\{F(e^{j\omega_0 t}) + F(e^{-j\omega_0 t})\}$$

using freq shifting property - $\begin{cases} e^{j\omega_0 t} \longleftrightarrow 2\pi\delta(\omega-\omega_0) \\ e^{-j\omega_0 t} \xleftrightarrow{F} 2\pi\delta(\omega+\omega_0) \end{cases}$

$$F\{\cos\omega_0 t\} = \frac{1}{2}\{2\pi\delta(\omega-\omega_0) + 2\pi\delta(\omega+\omega_0)\}$$

$$\boxed{\cos\omega_0 t \xleftrightarrow{F} \pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)}$$

Ex 6: $x(t) = \sin\omega_0 t$

Ex 6: $x(t) = c$

Properties of FT:

8. Convolution Property:

Convolution: $x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) \cdot h(t-\tau) \cdot d\tau$

$x(t) = e^{-at} u(t)$

$x(t) = e^{-bt} u(t)$

$$x(t) * h(t) \xleftrightarrow{F} X(j\omega) \cdot H(j\omega)$$

Proof: L.H.S $F\{x(t) * h(t)\} = \int_{-\infty}^{\infty} x(t) * h(t)\, e^{-j\omega_0 t} dt$

Now using formula for convolution

$$F\{x(t) * h(t)\} = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\tau) \cdot h(t-\tau) \cdot d\tau\right] \cdot e^{-j\omega_0 t} dt$$

Changing the order of integration ---

$$\int_{-\infty}^{\infty} x(\tau) \left[\int_{-\infty}^{\infty} h(t-\tau) \cdot e^{-j\omega_0 t} dt\right] d\tau$$

Now $\int_{-\infty}^{\infty} h(t) \cdot e^{-j\omega_0 t} dt = H(j\omega) \rightarrow$ Fourier transform of $h(t)$

$$h(t) \xleftrightarrow{F} H(j\omega)$$

$$h(t-\tau) \xleftrightarrow{F} e^{-j\omega\tau} H(j\omega)$$

$$F\{x(t) * h(t)\} = \int_{-\infty}^{\infty} x(\tau) \cdot H(j\omega) \cdot e^{-j\omega\tau} \cdot d\tau$$

$$= H(j\omega) \int_{-\infty}^{\infty} x(\tau) \cdot e^{-j\omega\tau} d\tau$$

$$= H(j\omega) \cdot X(j\omega)$$

$F\{x(\tau)\} = X(j\omega)$.

Ex: Find the F.T of $x(t) = x_1(t) * x_2(t)$, where $x_1(t) = e^{-3t}u(t)$ & $x_2(t) = e^{2t}u(t)$

$$F\{x(t)\} = F\{x_1(t) * x_2(t)\}$$

Using convolution theorem --

$$= F\{x_1(t)\} \cdot F\{x_2(t)\}$$

$$= F\{e^{-3t}u(t)\} \cdot F\{e^{2t}u(t)\}$$

$$F\{e^{-3t}u(t)\} = \frac{1}{3+j\omega}, \quad F\{e^{2t}u(t)\} = \frac{1}{-2+j\omega}$$

$$F\{x_1(t) * x_2(t)\} = \frac{1}{3+j\omega} \cdot \frac{1}{j\omega-2}$$

$$= \frac{1}{(3+j\omega)(j\omega-2)}$$

9 ⇒ Parseval's Theorem :- Total energy of a function $x(t)$:-

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$

The total energy of a function can be calculated by integrating the energy per unit frequency i.e. $\frac{|X(j\omega)|^2}{2\pi}$ for, over all frequency range.

Proof :-

$$L.H.S = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

In general, if $x(t)$ is a complex function $|x(t)|^2 = x(t) \cdot x(t)^*$

$$= \int_{-\infty}^{\infty} x(t) \cdot x(t)^* dt$$

using the eq^n of inverse FT $\left\{ x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) \cdot e^{j\omega t} d\omega \right\}$

$$x^*(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\omega) \cdot e^{-j\omega t} d\omega$$

$$= \int_{-\infty}^{\infty} x(t) \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\omega) \cdot e^{j\omega t} d\omega\right] dt$$

changing the order of integration

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} dt\right] d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\omega)\, X(j\omega) \cdot d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega = R.H.S. \quad \text{Proved}$$

10 Duality Property :- (Interchanging of independent variable of time & frequency domain function)

time-domain $\qquad$ Freq domain

$$x(t) \xleftrightarrow{F} X(j\omega)$$

$$X(t) \xleftrightarrow{F} x(-\omega)2\pi$$

Ex: Find the FT of function $x(t) = \frac{1}{a^2+t^2}$ using duality property

Sol^n :- I method

$$F\{x(t)\} = \int_{-\infty}^{\infty} \frac{1}{(a^2+t^2)} \cdot e^{-j\omega t} dt$$

II method :- Duality Property

$$x(t) \xleftrightarrow{F} X(j\omega)$$

$$X(t) \longleftrightarrow x(-\omega)2\pi$$

$$x(t) = \frac{1}{a^2+t^2} \quad \Big\{ \quad X(\omega) = \frac{1}{a^2+\omega^2}$$

we know that the FT pair $\quad e^{-a|t|} \xleftrightarrow{F} \frac{2a}{a^2+\omega^2}$ → const

$$e^{-a|t|} \longleftrightarrow \frac{2a}{a^2+\omega^2}$$

$t \to -\omega$

$$\frac{2a}{a^2+t^2} \longleftrightarrow e^{-a|-\omega|} \cdot 2\pi = e^{-a\omega} \cdot 2\pi$$

$$\frac{1}{a^2+t^2} \xleftrightarrow{F} \frac{2\pi}{2a} \cdot e^{-a\omega} = \frac{\pi}{a} \cdot e^{-a\omega}$$

Inverse fourier transform :

$$x(t) \xleftrightarrow{F} X(j\omega)$$
$$X(j\omega) \xleftrightarrow{F^{-1}} x(t)$$

$$F\{x(t)\} = X(j\omega)$$
$$F^{-1}\{X(j\omega)\} = x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) \cdot e^{j\omega t} d\omega$$

Ex : Find inverse FT of $\frac{1}{5+j\omega}$

Soln :- we know,

$$e^{-at}u(t) \xleftrightarrow{F} \frac{1}{a+j\omega}$$
$$\frac{1}{a+j\omega} \xleftrightarrow{F^{-1}} e^{-at}u(t)$$
$$\frac{1}{5+j\omega} \xleftrightarrow{F^{-1}} e^{-5t}u(t)$$
$$F^{-1}\left\{\frac{1}{5+j\omega}\right\} = e^{-5t}u(t)$$