Change of Number Bases

What are Number Bases?

When converting to different number bases, one is expressing numbers in different numeration systems. These numeration systems, or bases, use different digits and have different place values depending on the base being used. Place values are powers of a given base. The ten-based decimal system, which seems most natural to us, is constructed in powers of ten.

Place value	1000_{ten}	100_{ten}	10_{ten}	1_{ten}
Exponent	10^3	10^2	10^1	10^0
Decimal value	1000	100	10	1

Base nine is constructed in powers of nine.

Place value	1000_{nine}	100_{nine}	10_{nine}	1_{nine}
Exponent	9^3	9^2	9^1	9^0
Decimal value	729	81	9	1

In the ten-based decimal system, we have the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. There is no single digit for *ten*. The quantity *ten* is represented by writing the numeral 10, which means, "1 ten and 0 ones." These two individual digits represent one number. Base nine only has the digits 0, 1, 2, 3, 4, 5, 6, 7, and 8. The single digit 9 does not exist in base nine. *Nine* in base nine would be, "1 nine and 0 ones," or 10_{nine}, just as *ten* is, "1 ten and 0 ones," or 10, in base ten. Letters are used to serve as extra digits in bases greater than ten.

Reading Numerals

It is important to be able to read and verbalize numbers in other numeration systems. Any number without a subscript is assumed to be in base ten. For example, a number in base ten can be properly written as 56_{ten}, but 56 is assumed to be in base ten. Numbers in other bases are not verbalized the same way as base ten. For example: 56_{seven} should be vocalized as, "five six base seven." It should not be read as, "fifty-six," because this implies base ten.

The Difference Between Bases

Looking at the place values of the different systems side by side can help visualize the difference between the different number bases.

Example:

	Fourth Power X^4	**Third Power** X^3	**Second Power** X^2	**First Power** X^1	**Zero Power** X^0
Base Ten	10,000	1,000	100	10	1
Base Nine	6,561	729	81	9	1
Base Eight	4,096	512	64	8	1
Base Seven	2,401	343	49	7	1
Base Six	1,296	216	36	6	1
Base Five	625	125	25	5	1

Notice that the traditional *ten's* place in the base ten would be the *five's* place in base five or the *seven's* place in base seven. Similarly, the *hundred's* place in base ten would be the *twenty-five's* place in base five or the *thirty-six's* place in base six, etc.

Converting from Other Bases to Base Ten

By using multiplication and the place values of the original base, one can convert from any number base to base ten.

Example: Convert 52031_{six} to base ten.

The first step is to find out what quantity each place value represents. Since the number is in base six, it can be represented this way

5	**2**	**0**	**3**	**1**
6^4	6^3	6^2	6^1	6^0

Then, multiply each number by its place value using expanded notation.

$$(5\times6^4)+(2\times6^3)+(0\times6^2)+(3\times6^1)+(1\times6^0)\rightarrow$$

$$(5\times1{,}296)+(2\times216)+(0\times36)+(3\times6)+(1\times1)\rightarrow$$

$$(6480)+(432)+(0)+(18)+(1)=6931$$

Therefore, 52031_{six} is the same as 6,931 in base ten.

Converting from Base Ten to any Other Base.

When changing from base ten to another number base, there are two different methods. The first method involves dividing by the place values of the desired base.

Example: Convert 407 to base four.

Step 1: Set up a basic template to visually convert from base ten to any other base:

X^5	X^4	X^3	X^2	X^1	X^0

Step 2: Replace each X with the desired base using as many number places as necessary.

4^5	4^4	4^3	4^2	4^1	4^0

Step 3: Solve for what each place represents. Reminder: any number to the zero power is 1.

4096	**256**	**64**	**16**	**4**	**1**
4^5	4^4	4^3	4^2	4^1	4^0

Step 4: To convert, divide the original number by the largest place value that is still smaller than the original number, which in this case is 256.
$407 \div 256 = 1$ with a remainder of 151

0					
4096	256	64	16	4	1
4^5	4^4	4^3	4^2	4^1	4^0

Step 5: Record the non-remainder portion of the answer above the place value divided by, and use the remainder in the next step.

0	**1**				
4096	256	64	16	4	1
4^5	4^4	4^3	4^2	4^1	4^0

Step 6: Divide the remainder from the previous step by the next smaller place value. Repeat this step for each of the remaining place values.
$151 \div 64 = 2$ with a remainder of 23

0	**1**	**2**			
4096	256	64	16	4	1
4^5	4^4	4^3	4^2	4^1	4^0

Step 7: $23 \div 16 = 1$ with a remainder of 7

0	**1**	**2**	**1**		
4096	256	64	16	4	1
4^5	4^4	4^3	4^2	4^1	4^0

Step 8: $7 \div 4 = 1$ with a remainder of 3

Step 9: $3 \div 1 = 3$ with no remainders

0	**1**	**2**	**1**	**1**	**3**
4096	256	64	16	4	1
4^5	4^4	4^3	4^2	4^1	4^0

Finally, $407 = 12113_{four}$

The second method involves using a division method which keeps track of remainders and progressively divides by the desired base.

Example: Convert 407 to base four.

Divide the number you want to convert by the desired base, and keep track of remainders in a separate column. Proceed by dividing the non-remainder part of the dividend by the desired base until the result is less than one.

$$407 \div 4 = 101 \longrightarrow \text{Remainder } 3$$
$$101 \div 4 = 25 \longrightarrow \text{Remainder } 1$$
$$25 \div 4 = 6 \longrightarrow \text{Remainder } 1$$
$$6 \div 4 = 1 \longrightarrow \text{Remainder } 2$$
$$1 \div 4 = 0 \quad \text{Remainder } 1$$

To get the answer, write the numerals in the remainder column from bottom to top.

Therefore, $407 = 12113_{\text{four}}$. Notice that while converting, <u>only the non-remainder part of the dividend is used in the next step</u>; remainders are recorded in a separate column as whole numbers and not as decimals. As a result, the final equation equals 0 with a remainder of 1 and not $\frac{1}{4}$. Since the final dividend is 0 with a remainder, division is no longer necessary.

Converting from One Non-Decimal Number to another Non-decimal Number

To convert from one non-decimal base to another, convert first to base ten, then to the desired base. For example, to convert 13_{four} to base six, convert from base four to base ten first, and then convert the base ten number to base six. $13_{\text{four}} = 7_{\text{ten}} = 11_{\text{six}}$

Bases Greater Than Base Ten

Working with bases greater than base ten represents a special problem. In bases over ten, there are not enough Arabic numerals to represent all the digits needed. Therefore, letters are used to represent numbers above the digit 9. For example: in hexadecimal, or base sixteen, the letters A, B, C, D, E, and F are used to represent the numbers 10 – 15 as shown in the table below.

Base Ten	Base 16 (Hexadecimal)
1	**1**
2	**2**
3	**3**
4	**4**
5	**5**
6	**6**
7	**7**
8	**8**
9	**9**
10	**A**
11	**B**
12	**C**
13	**D**
14	**E**
15	**F**
16	**10**

Example: Convert 6903 to Base 16

Step 1: Set up a template to convert

4096	256	16	1
16^3	16^2	16^1	16^0

Step 2: $6903 \div 4096 = 1$ with a remainder of 2807

1			
4096	256	16	1
16^3	16^2	16^1	16^0

Step 3: $2807 \div 256 = 10$ with a remainder of 247.
Using the chart on the left you can see that the letter A represents the number 10 in hexadecimal notation.

1	**A**		
4096	256	16	1
16^3	16^2	16^1	16^0

Step 4: $247 \div 16 = 15$ with a remainder of 7
15 in hexadecimal notation is represented with the letter F

1	**A**	**F**	
4096	256	16	1
16^3	16^2	16^1	16^0

Step 5: $7 \div 1 = 7$ with no remainder

1	**A**	**F**	**7**
4096	256	16	1
16^3	16^2	16^1	16^0

Therefore, $6903 = 1AF7_{\text{sixteen}}$

Practice Problems

1. Convert 17 to base five
2. Convert 890 to base three
3. Convert 642_{seven} to base ten
4. Convert 10011_{two} to base ten
5. Convert 587_{nine} to base six
6. Convert 210_{three} to base eight
7. Convert 45 to base sixteen
8. Convert 97,093 to base sixteen

Solutions

1. 32_{five}
2. 1012222_{three}
3. 324
4. 19
5. 2124_{six} $(484_{ten} \longrightarrow 2124_{six})$
6. 25_{eight} $(21_{ten} \longrightarrow 25_{eight})$
7. $2D_{sixteen}$
8. $17B45_{sixteen}$