DTS104TC
NUMERICAL METHODS

LECTURE 5

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CONTENTS

- Numerical Differentiation and Integration
- The Trapezoidal Rule
- Simpson’s Rules

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NUMERICAL DIFFERENTIATION AND INTEGRATION

- Calculus is the mathematics of change. Especially most engineering problems continuously deal with systems and processes that change, calculus is an essential tool.
- Standing in the heart of calculus are the mathematical concepts of *differentiation* and *integration*:

$$\frac{\Delta y}{\Delta x} = \frac{f(x_i + \Delta x) - f(x_i)}{\Delta x}$$

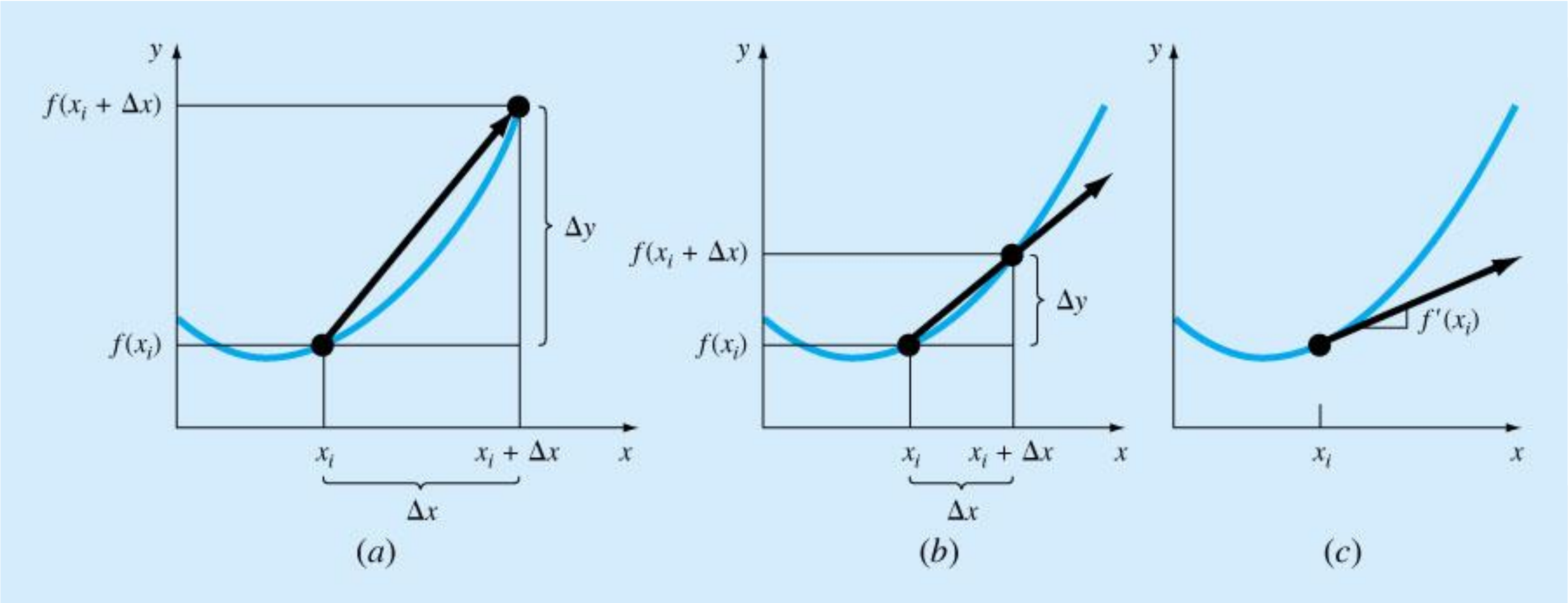
$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x_i + \Delta x) - f(x_i)}{\Delta x}$$

Differentiation

$$I = \int_a^b f(x)dx$$

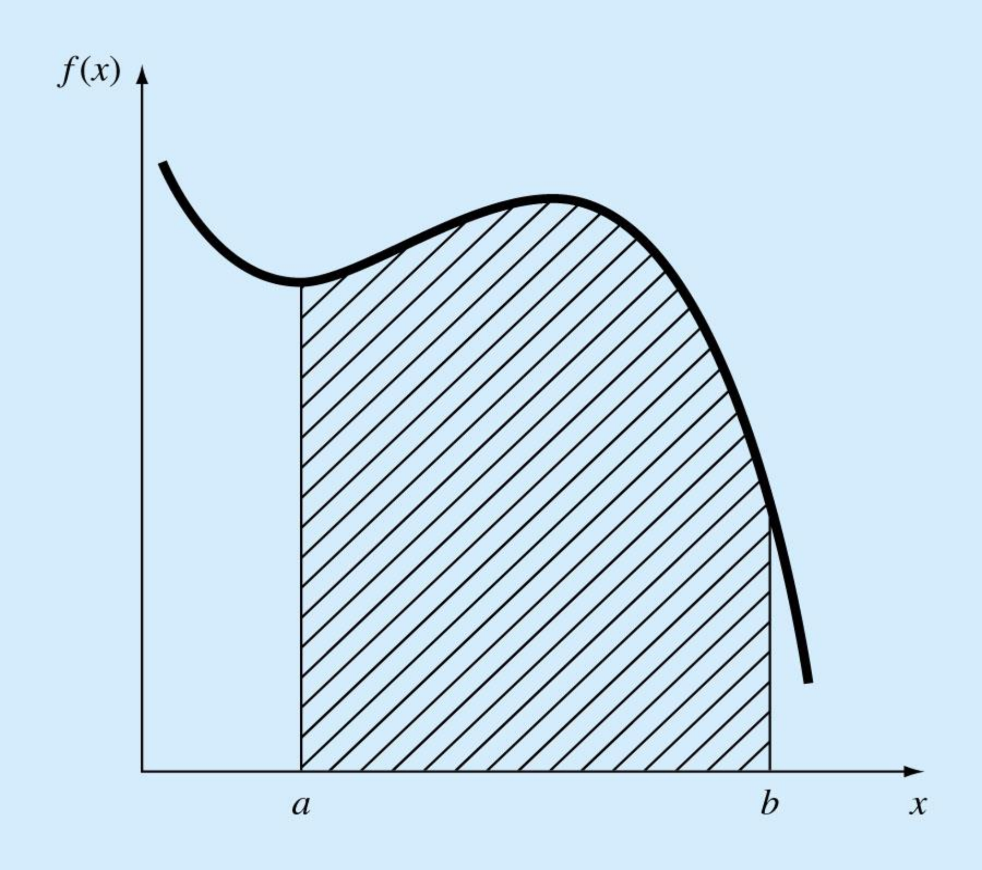
Integration

THE GRAPHICAL DEFINITION OF A DERIVATIVE



As Δx approaches zero in going from (a) to (c), the difference approximation becomes a derivative

THE INTEGRAL OF F(X) BETWEEN THE LIMITS X = A TO B.



The integral is equivalent to the area under the curve

TYPES OF FUNCTIONS TO BE INTEGRATED

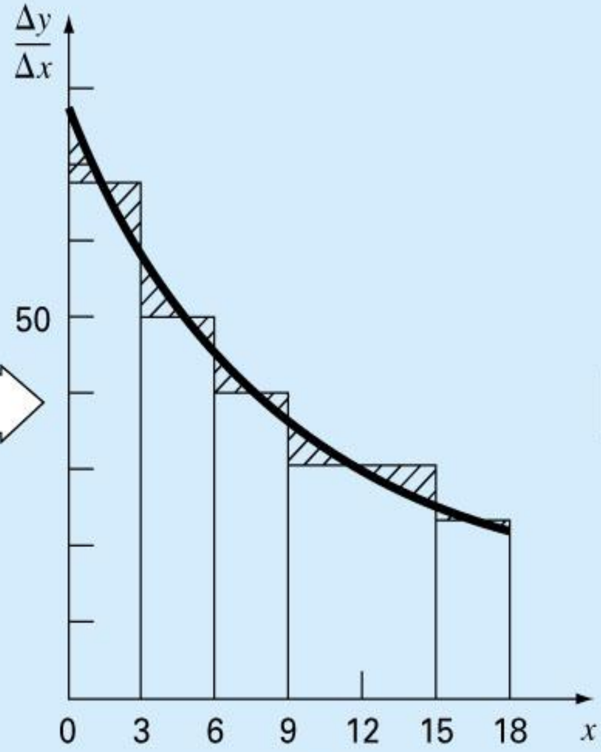
The function to be differentiated or integrated will typically be in one of the following three forms:

- A simple continuous function such as polynomial, an exponential, or a trigonometric function.
- A complicated continuous function that is difficult or impossible to differentiate or integrate directly.
- A tabulated function where values of x and $f(x)$ are given at a number of discrete points, as is often the case with experimental or field data.

EQUAL AREA DIFFERENTIATION

x	y	$\Delta y/\Delta x$
0	0	
		66.7
3	200	
		50
6	350	
		40
9	470	
		30
15	650	
		23.3
18	720	

(*a*)

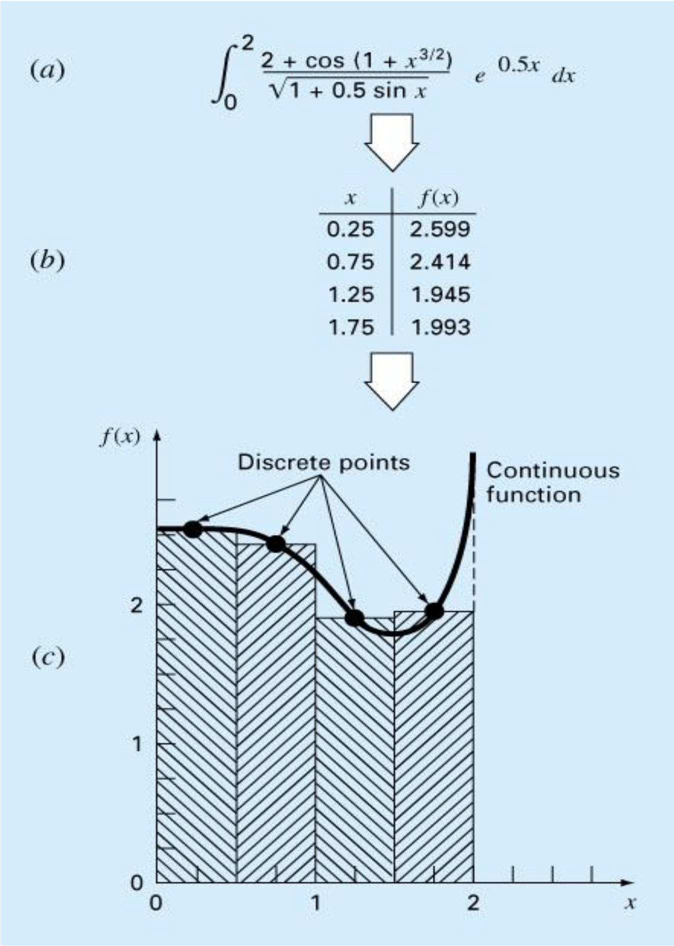


(*b*)

x	dy/dx
0	76.50
3	57.50
6	45.00
9	36.25
15	25.00
18	21.50

(*c*)

INTEGRATING A FUNCTION



← A complicated, continuous function.

← Table of discrete values of *f*(*x*) generated from the function.

← Use of a numerical method (the strip method here) to estimate the integral on the basis of the discrete points.

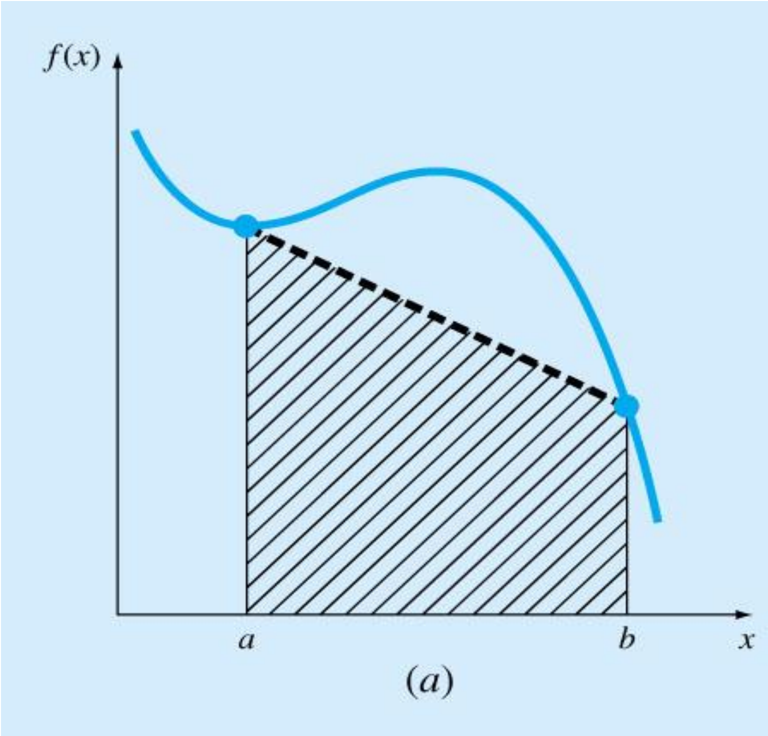
NEWTON-COTES INTEGRATION FORMULAS

- The *Newton-Cotes formulas* are the most common numerical integration schemes.
- They are based on the strategy of replacing a complicated function or tabulated data with an approximating function that is easy to integrate: a polynomial:

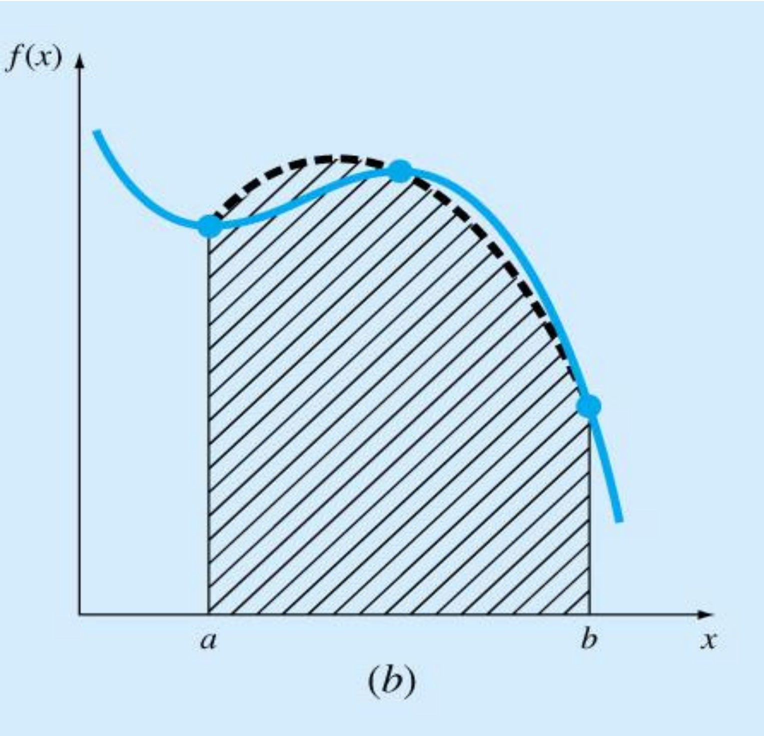
$$I = \int_a^b f(x)dx \cong \int_a^b f_n(x)dx$$

$$f_n(x) = a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} + a_n x^n$$

THE APPROXIMATION OF AN INTEGRAL BY THE AREA UNDER SIMPLE POLYNOMIALS

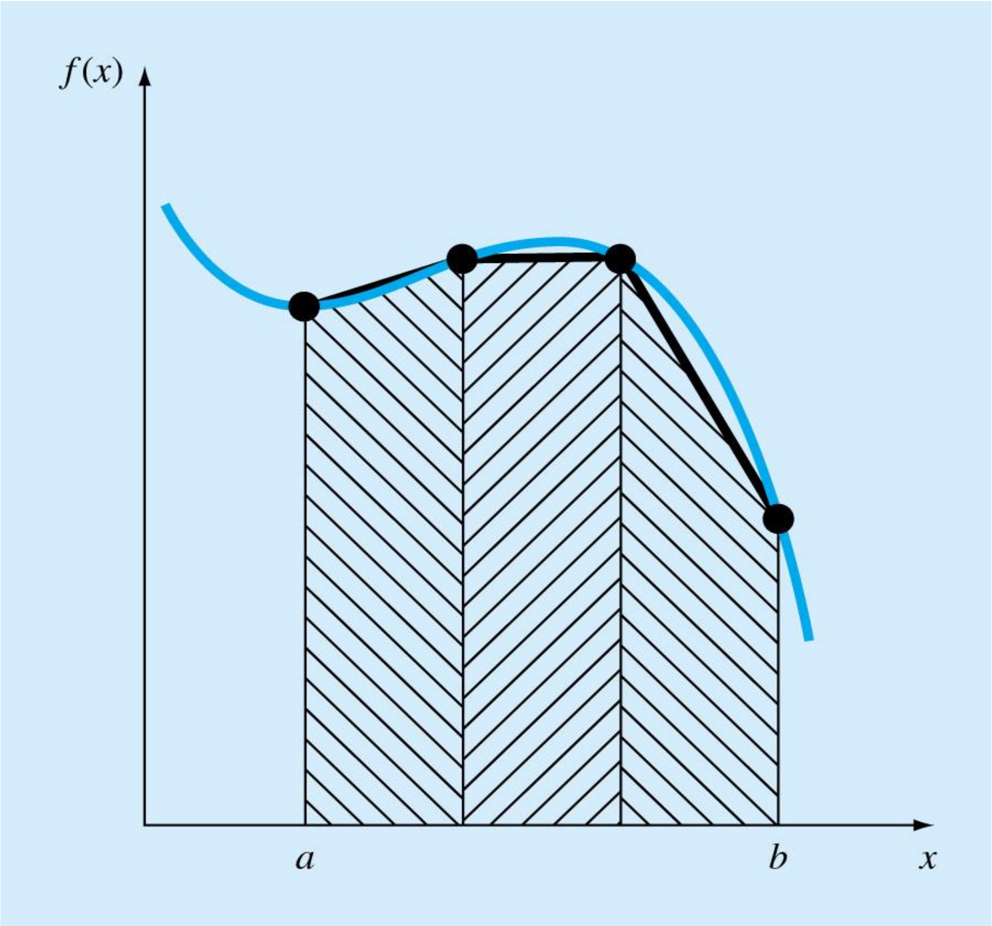


straight line (1st order)



parabola (2nd order)

THE APPROXIMATION OF AN INTEGRAL BY THE AREA UNDER THREE STRAIGHT-LINE SEGMENTS.



THE TRAPEZOIDAL RULE

- The *Trapezoidal rule* is the first of the Newton-Cotes closed integration formulas, corresponding to the case where the polynomial is first order:

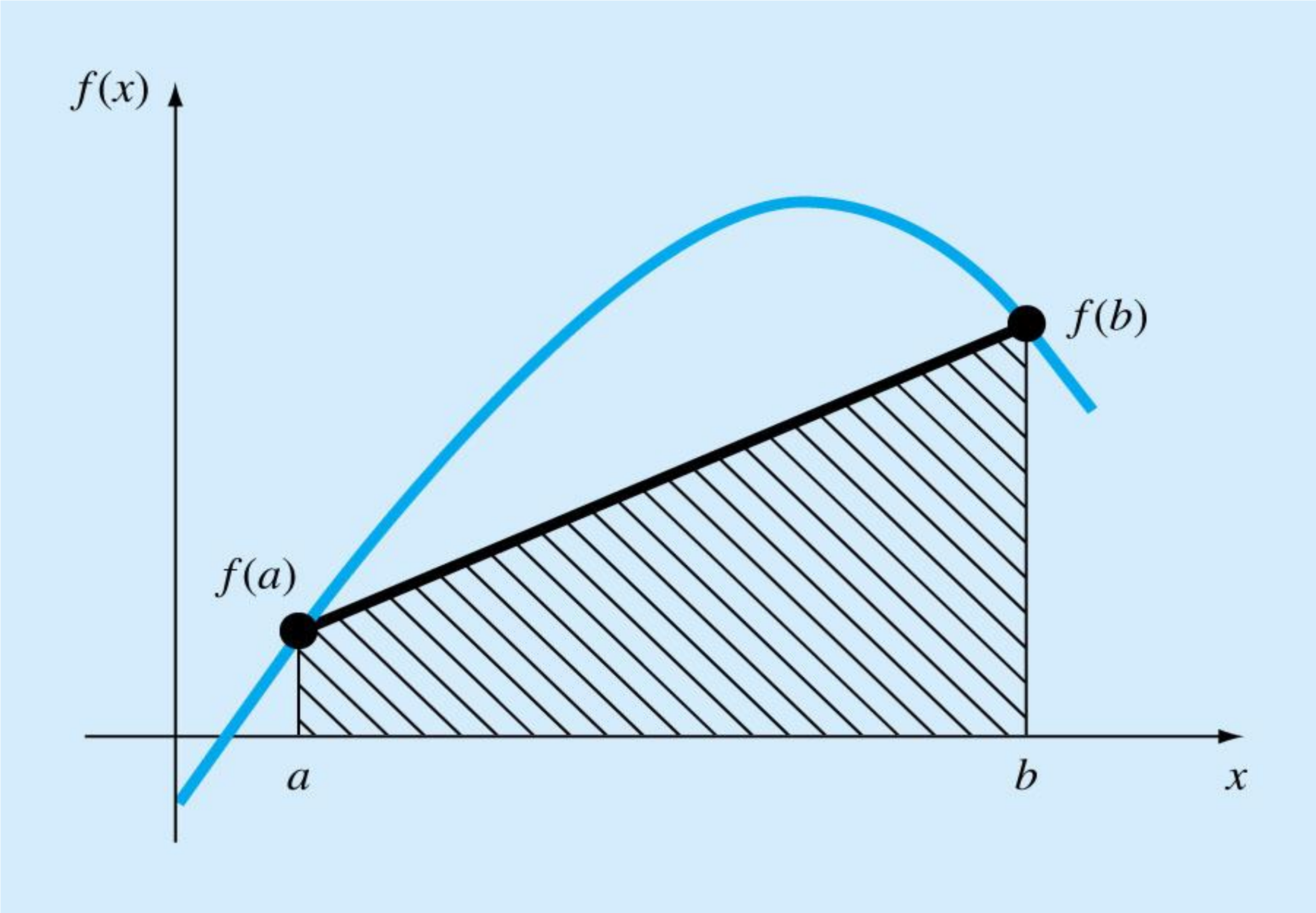
$$I = \int_a^b f(x)dx \cong \int_a^b f_1(x)dx$$

- The area under this first order polynomial is an estimate of the integral of $f(x)$ between the limits of a and b:

$$I = (b-a)\frac{f(a)+f(b)}{2}$$

The trapezoidal rule

GRAPHICAL DEPICTION OF THE TRAPEZOIDAL RULE.



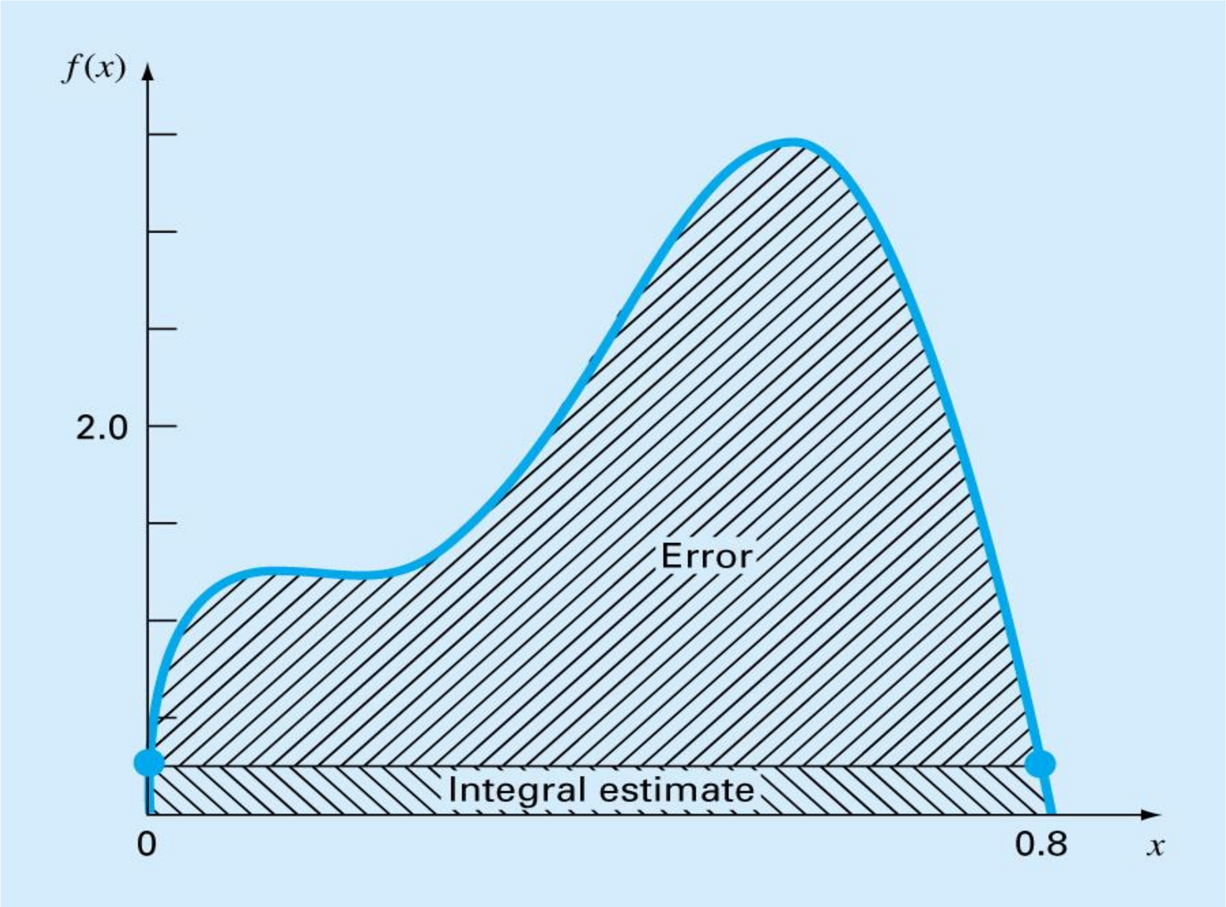
ERROR OF A SINGLE APPLICATION OF THE TRAPEZOIDAL RULE

- When we employ the integral under a straight line segment to approximate the integral under a curve, error may be substantial:

$$E_t = -\frac{1}{12} f''(\xi)(b-a)^3$$

- where ξ lies somewhere in the interval from *a* to *b*.

SOMETIMES THE TRAPEZOIDAL RULE CAN YIELD A LARGE ERROR



Use of a single application of the trapezoidal rule to approximate the integral of $f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$ from $x = 0$ to 0.8.

MULTIPLE APPLICATION TRAPEZOIDAL RULE

- One way to improve the accuracy of the trapezoidal rule is to divide the integration interval from a to b into a number of segments and apply the method to each segment.
- The areas of individual segments can then be added to yield the integral for the entire interval.

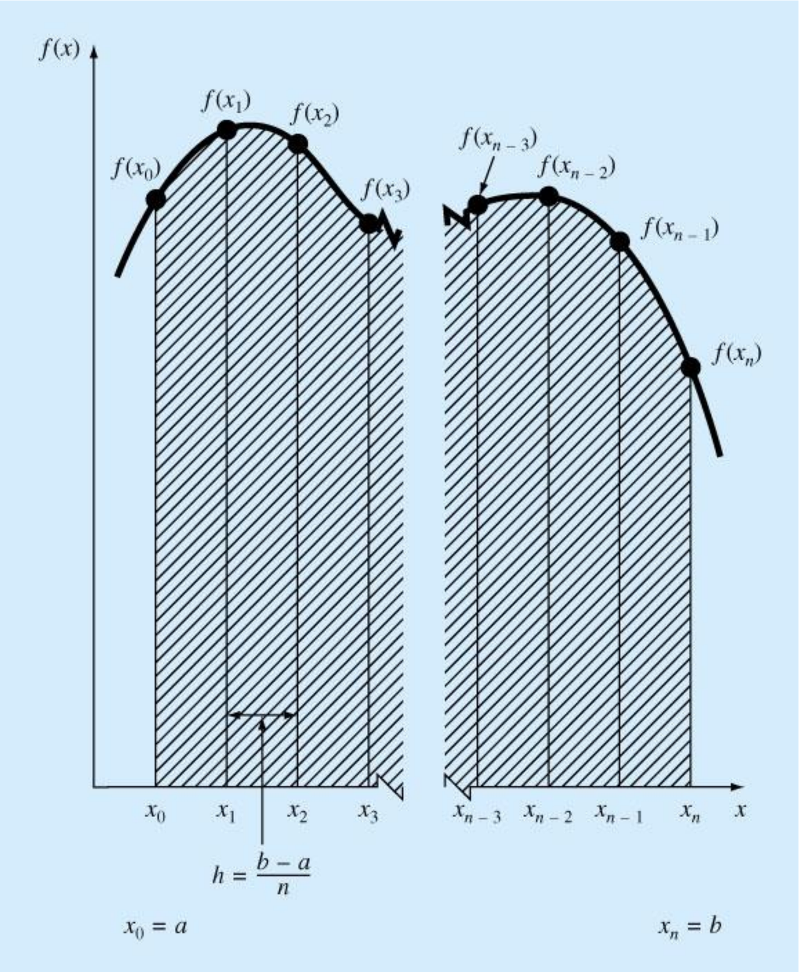
$$h = \frac{b-a}{n} \qquad a = x_0 \quad b = x_n$$

$$I = \int_{x_0}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \cdots + \int_{x_{n-1}}^{x_n} f(x)dx$$

- Substituting the trapezoidal rule for each integral yields:

$$I = h\frac{f(x_0) + f(x_1)}{2} + h\frac{f(x_1) + f(x_2)}{2} + \cdots + h\frac{f(x_{n-1}) + f(x_n)}{2}$$

MULTIPLE APPLICATION TRAPEZOIDAL RULE



ERROR OF MULTIPLE APPLICATION TRAPEZOIDAL RULE

- An error for multiple-application trapezoidal rule can be obtained by summing the individual errors for each segment:

$$\sum f''(\xi_i) \cong n\bar{f}''$$

$$E_a = -\frac{(b-a)^3}{12n^2}\bar{f}''$$

- Thus, if the number of segments is doubled, the truncation error will be quartered.

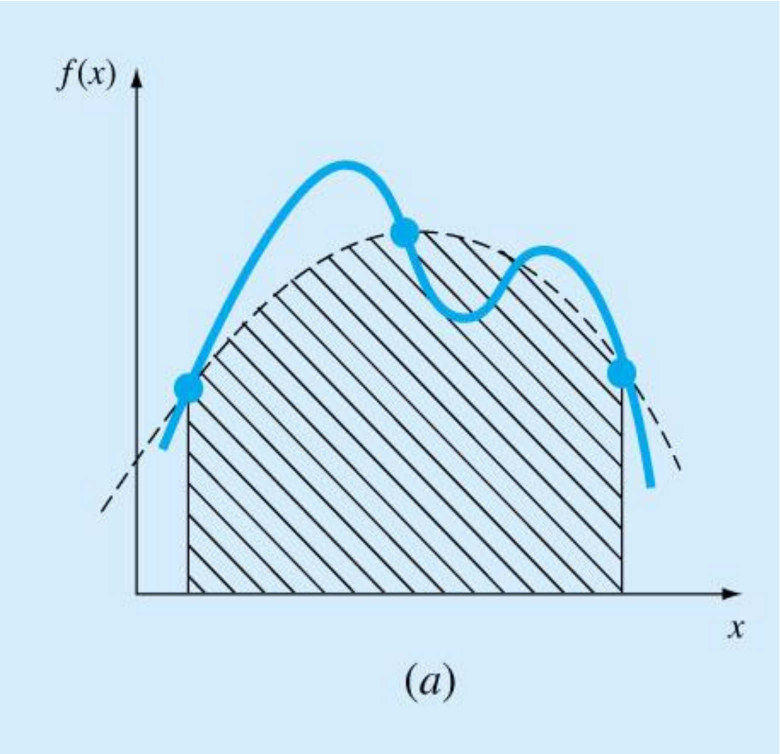
SIMPSON'S RULES

- More accurate estimate of an integral is obtained if a high-order polynomial is used to connect the points. The formulas that result from taking the integrals under 2^{nd} & 3^{rd}-order polynomials are called *Simpson's rules*.

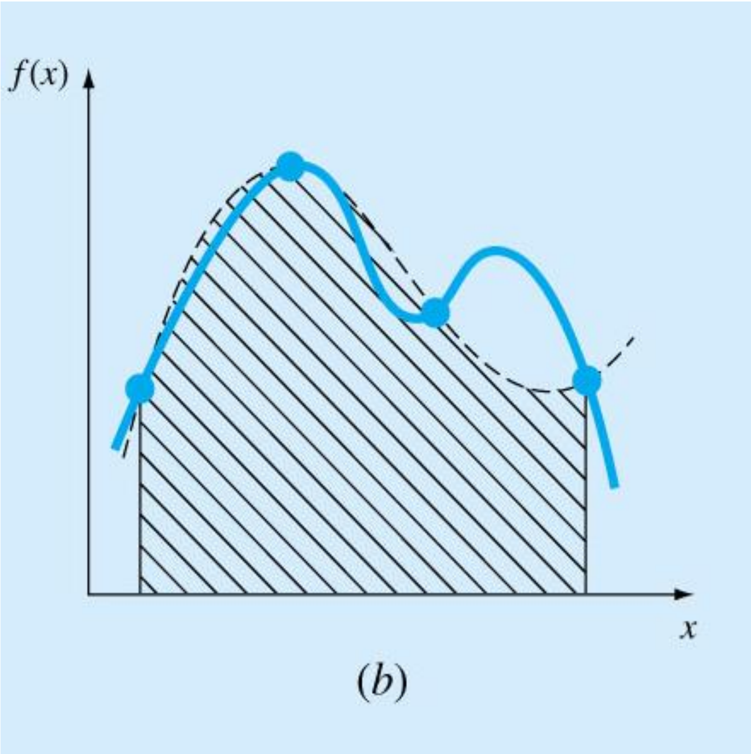
Simpson's 1/3 Rule:

- Results when a second-order interpolating polynomial is used.

SIMPSON'S RULES



Simpson's 1/3 Rule



Simpson's 3/8 Rule

SIMPSON'S 1/3 RULE

$$I = \int_a^b f(x)dx \cong \int_a^b f_2(x)dx$$

$$\text{where } a = x_0 \;\; \& \;\; b = x_2$$

$$I = \int_{x_0}^{x_2} \left[\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2) \right] dx$$

$$\boxed{I \cong \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2) \right]} \qquad h = \frac{b-a}{2}$$

Simpson's 1/3 Rule

- Single segment application of Simpson's 1/3 rule has a truncation error of:

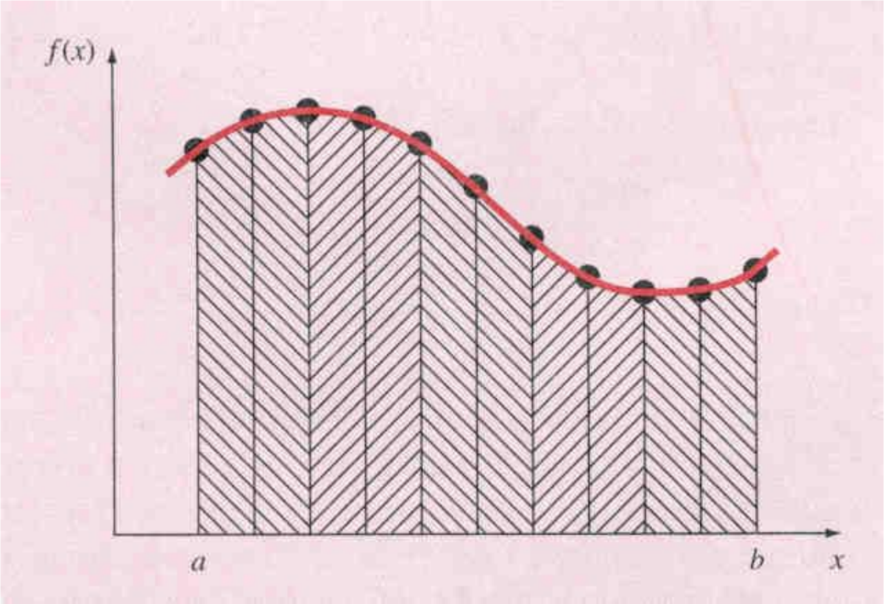
$$E_t = -\frac{(b-a)^5}{2880} f^{(4)}(\xi) \qquad a \prec \xi \prec b$$

- **Simpson's 1/3 rule is more accurate than trapezoidal rule.**

THE MULTIPLE-APPLICATION SIMPSON’S 1/3 RULE

- Just as the trapezoidal rule, Simpson’s rules can be improved by dividing the integration interval into a number of segments of equal width.
- Yields accurate results and considered superior to trapezoidal rule for most applications.
- *However*, it is limited to cases where values are equispaced.
- *Further*, it is limited to situations where there are an even number of segments and odd number of points.

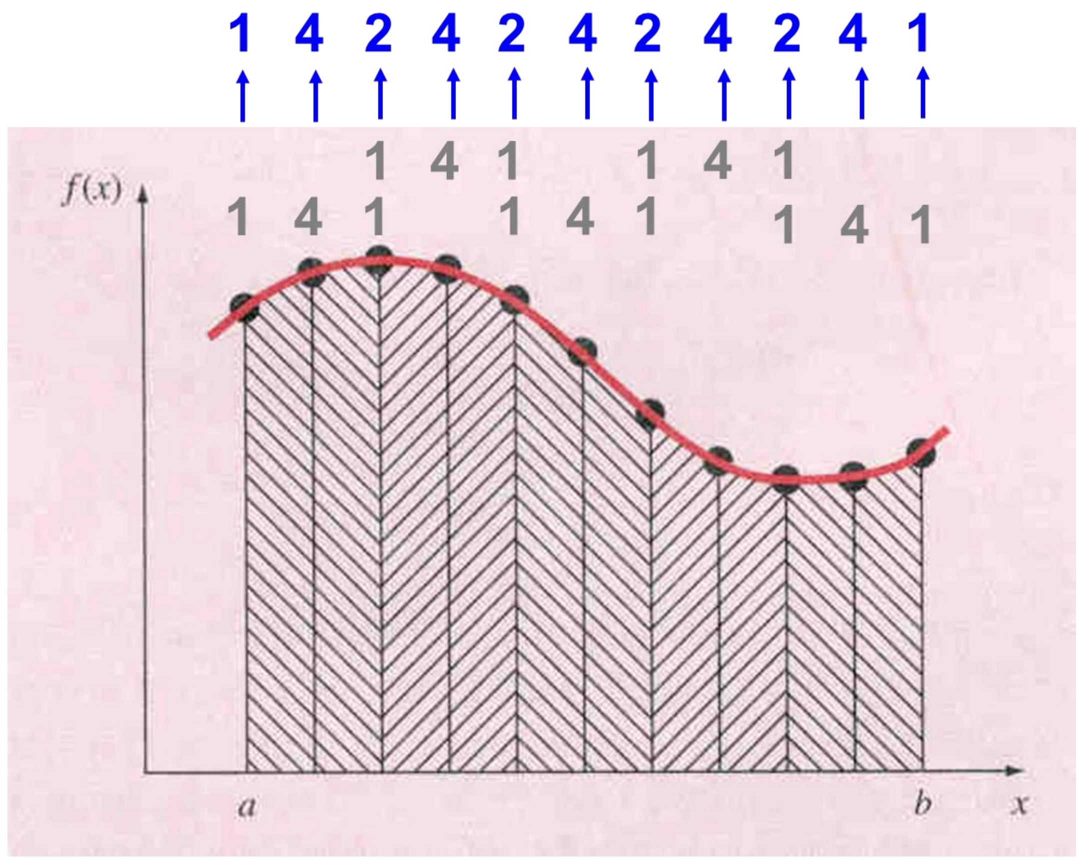
THE MULTIPLE-APPLICATION SIMPSON'S 1/3 RULE



Can only be used for even number of segments

$$I \cong 2h\frac{f(x_0)+4f(x_1)+f(x_2)}{6}+2h\frac{f(x_2)+4f(x_3)+f(x_4)}{6}$$
$$+\cdots+2h\frac{f(x_{n-2})+4f(x_{n-1})+f(x_n)}{6}$$

WEIGHTS



THE MULTIPLE-APPLICATION SIMPSON'S 1/3 RULE

$$I \cong \underbrace{(b-a)}_{\text{Width}} \underbrace{\frac{f(x_0) + 4\sum_{i=1,3,5}^{n-1} f(x_i) + 2\sum_{j=2,4,6}^{n-2} f(x_j) + f(x_n)}{3n}}_{\text{Average height}}$$

$$E_a = -\frac{(b-a)^5}{180n^4}\bar{f}^{(4)}$$

SIMPSON'S 3/8 RULE

- An odd-segment-even-point formula used in conjunction with the 1/3 rule to permit evaluation of both even and odd numbers of segments.

$$I = \int_a^b f(x)dx \cong \int_a^b f_3(x)dx$$

$$I \cong \frac{3h}{8}\left[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3) \right]$$

$$h = \frac{(b-a)}{3}$$

$$E_t = -\frac{(b-a)^5}{6480} f^{(4)}(\xi)$$

More accurate

SIMPSON'S 1⁄3 AND 3⁄8 RULES

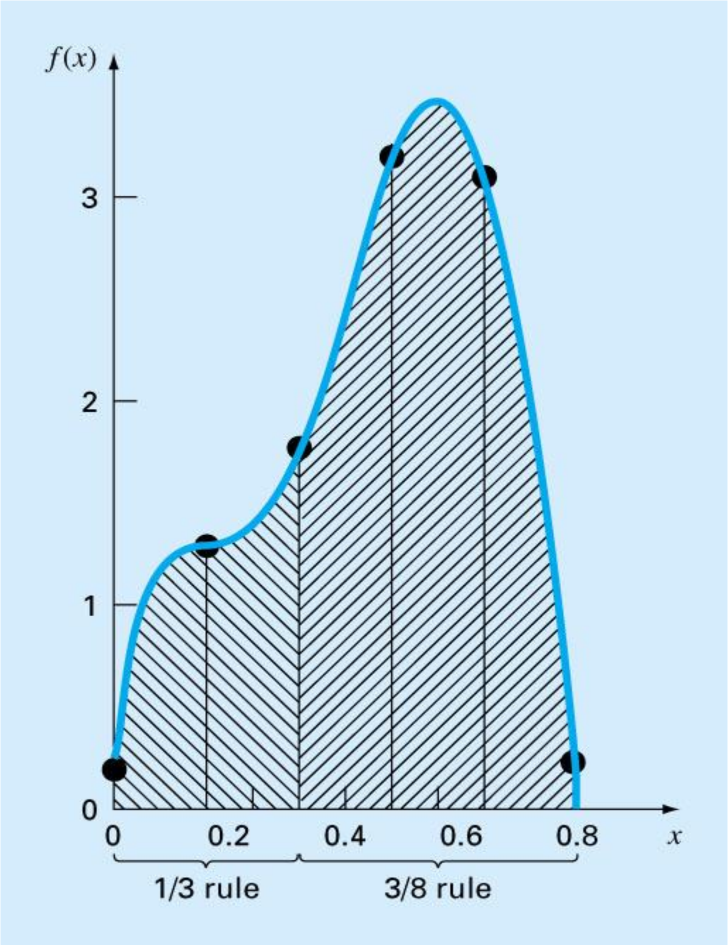


Illustration of how Simpson's 1⁄3 and 3⁄8 rules can be applied in tandem to handle multiple applications with odd numbers of intervals.