DTS104TC
NUMERICAL METHODS

LECTURE 6

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CONTENTS

- Integration of Equations
- Romberg integration
- Gauss quadrature

INTEGRATION OF EQUATIONS

Functions to be integrated numerically are in two forms:

- *A table of values*. We are limited by the number of points that are given.
- *A function*. We can generate as many values of $f(x)$ as needed to attain acceptable accuracy.

- Will focus on two techniques that are designed to analyze functions:
  - *Romberg integration.*
  - *Gauss quadrature.*

ROMBERG INTEGRATION

- Is based on successive application of the trapezoidal rule to attain efficient numerical integrals of functions.

Richardson’s Extrapolation

- Uses two estimates of an integral to compute a third and more accurate approximation.

ROMBERG INTEGRATION

- The estimate and error associated with a multiple-application trapezoidal rule can be represented generally as

$$I = I(h) + E(h)$$

$$h = (b-a)/n$$

$$I(h_1) + E(h_1) = I(h_2) + E(h_2)$$

$$n = (b-a)/h$$

$$E \cong \frac{b-a}{12} h^2 \bar{f}''$$

Assumed constant regardless of step size

$$\frac{E(h_1)}{E(h_2)} \cong \frac{h_1^2}{h_2^2}$$

$$E(h_1) \cong E(h_2)\left(\frac{h_1}{h_2}\right)^2$$

I =exact value of integral

I(h) =the approximation from an *n* segment application of trapezoidal rule with step size *h*

E(*h*) =the truncation error

ROMBERG INTEGRATION

$$I(h_1)+E(h_2)\left(\frac{h_1}{h_2}\right)^2 \cong I(h_2)+E(h_2)$$

$$E(h_2)\cong \frac{I(h_1)-I(h_2)}{1-\left(h_1/h_2\right)^2}$$

$$I = I(h_2)+E(h_2)$$

$$\left.I \cong I(h_2)+\frac{1}{\left(h_1/h_2\right)^2-1}\left[I(h_2)-I(h_1)\right]\right\}\text{ Improved estimate of the integral}$$

ROMBERG INTEGRATION

	$\frac{4}{3}I_2 - \frac{1}{3}I_1$	$\frac{16}{15}I_2 - \frac{1}{15}I_1$	$\frac{64}{63}I_2 - \frac{1}{63}I_1$
0.172800	1.367467	1.640533	1.640533
1.068800	1.623467	1.640533	
1.484800	1.639467		
1.600800			

$$I_{j,K} \cong \frac{4^{k-1} I_{j+1,k-1} - I_{j,k-1}}{4^{k-1} - 1}$$

ROMBERG ERROR ESTIMATE

$$\frac{4}{3}I_2 - \frac{1}{3}I_1$$

$$\begin{array}{ccc} 0.172800 & \longrightarrow & 1.367467 \\ & \nearrow & \\ 1.068800 & & \end{array}$$

$$\left|\varepsilon_a\right| = \left|\frac{1.367467 - 1.068800}{1.367467}\right| \times 100\% = 21.8\%$$

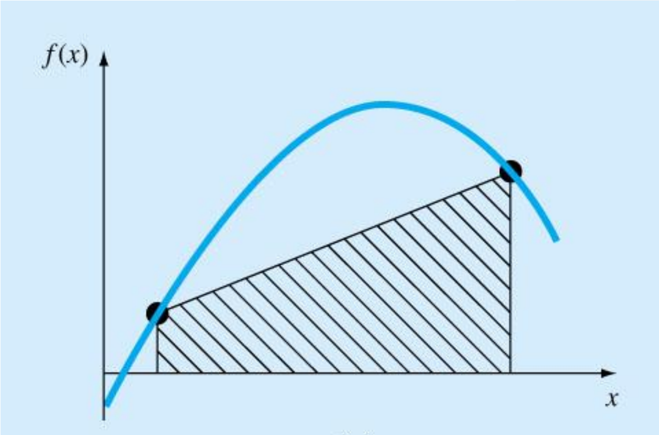
$$\left|\varepsilon_a\right| = \left|\frac{I_{1,k} - I_{2,k-1}}{I_{1,k}}\right| \times 100\%$$

GAUSS QUADRATURE

- *Gauss quadrature* implements a strategy of positioning any two points on a curve to define a straight line that would balance the positive and negative errors.
- Hence the area evaluated under this straight line provides an improved estimate of the integral.

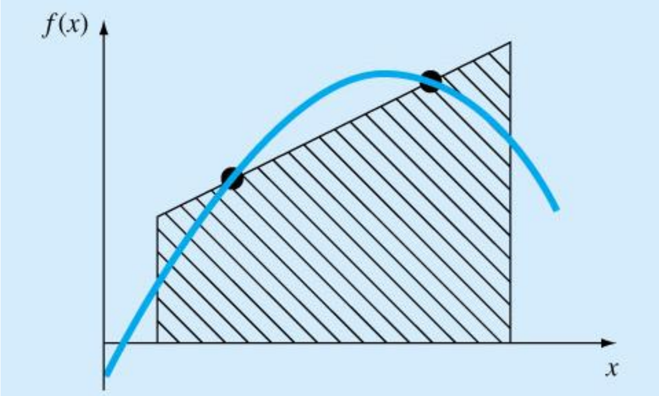
GAUSS QUADRATURE

(a) Graphical depiction of the trapezoidal rule as the area under the straight line joining fixed end points.



(a)

(b) Improved integral estimate by taking the area under the straight line passing through two intermediate points. By positioning these points wisely, the positive and negative errors are balanced, and an improved integral estimate results.



(b)

METHOD OF UNDETERMINED COEFFICIENTS

$$I \cong c_0 f(a) + c_1 f(b)$$

- The trapezoidal rule yields exact results when the function being integrated is a constant or a straight line, such as *y*=1 and *y*=*x*:

$$c_0 + c_1 = \int_{-(b-a)/2}^{(b-a)/2} 1\, dx$$

$$-c_0 \frac{b-a}{2} + c_1 \frac{b-a}{2} = \int_{-(b-a)/2}^{(b-a)/2} x\, dx$$

$$\left.\begin{aligned} c_0 + c_1 &= b-a \\ -c_0 \frac{b-a}{2} + c_1 \frac{b-a}{2} &= 0 \end{aligned}\right\} \text{Solve simultaneously}$$

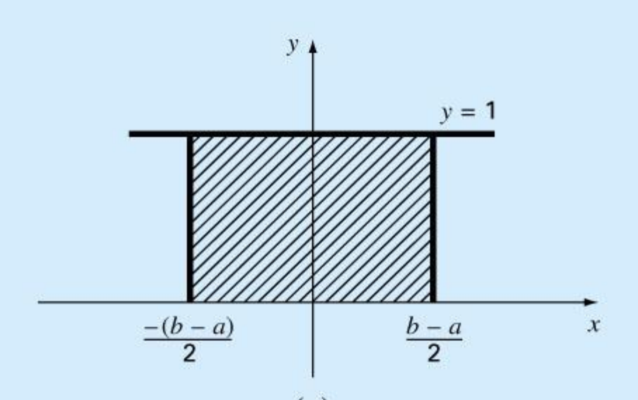
$$c_0 = c_1 = \frac{b-a}{2}$$

$$\boxed{I = \frac{b-a}{2} f(a) + \frac{b-a}{2} f(b)}$$

Trapezoidal rule

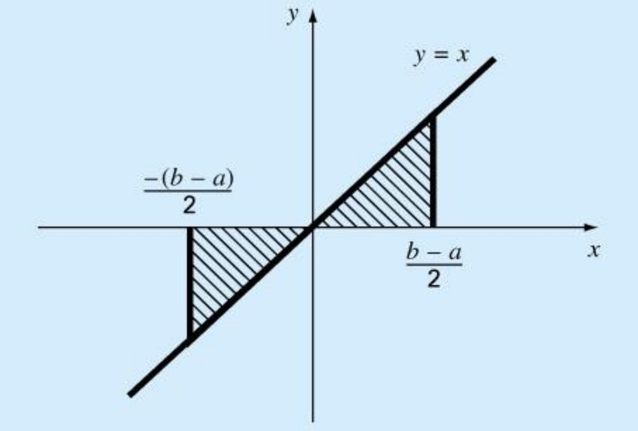
INTEGRALS THAT SHOULD BE EVALUATED EXACTLY BY THE TRAPEZOIDAL RULE

(a) a constant



(a)

(b) a straight line



(b)

DERIVATION OF THE TWO-POINT GAUSS-LEGENDRE FORMULA

- The object of Gauss quadrature is to determine the equations of the form

$$I \cong c_0 f(x_0) + c_1 f(x_1)$$

- However, in contrast to trapezoidal rule that uses fixed end points *a* and *b*, the function arguments x_0 and x_1 are not fixed end points but unknowns.
- Thus, *four unknowns* to be evaluated require *four conditions*.
- First two conditions are obtained by assuming that the above eqn. for *I* fits the integral of a *constant* and a *linear* function exactly.
- The other two conditions are obtained by extending this reasoning to a *parabolic* and a *cubic* functions.

DERIVATION OF THE TWO-POINT GAUSS-LEGENDRE FORMULA

$$\left.\begin{aligned}
c_0 f(x_0) + c_1 f(x_1) &= \int_{-1}^{1} 1 \, dx = 2 \\
c_0 f(x_0) + c_1 f(x_1) &= \int_{-1}^{1} x \, dx = 0 \\
c_0 f(x_0) + c_1 f(x_1) &= \int_{-1}^{1} x^2 \, dx = \frac{2}{3} \\
c_0 f(x_0) + c_1 f(x_1) &= \int_{-1}^{1} x^3 \, dx = 0
\end{aligned}\right\} \text{Solved simultaneously}$$

$$c_0 = c_1 = 1$$

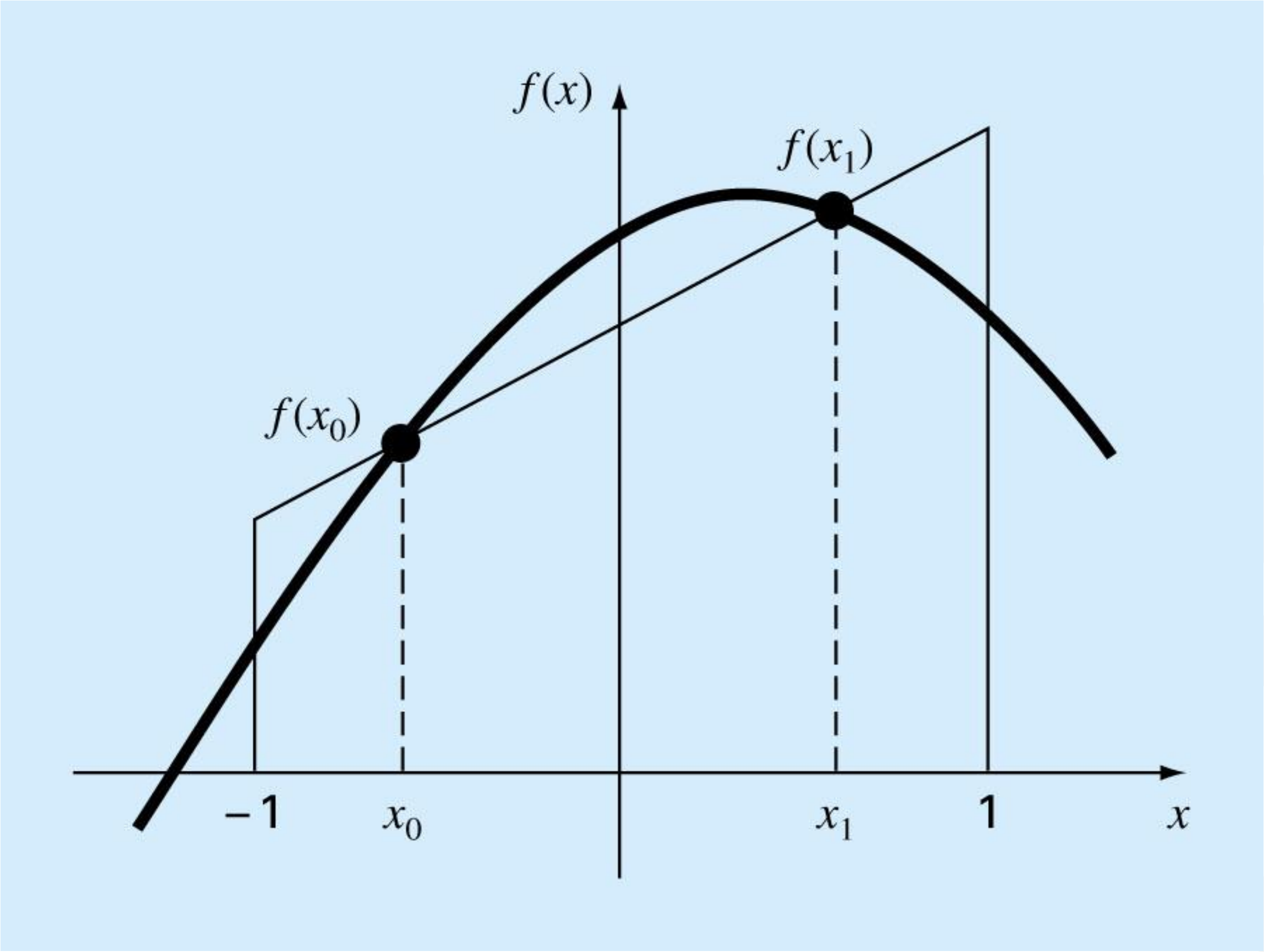
$$x_0 = -\frac{1}{\sqrt{3}} = -0.5773503\ldots$$

$$x_1 = \frac{1}{\sqrt{3}} = -0.5773503\ldots$$

$$\boxed{I \cong f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)}$$

Yields an integral estimate that is third order accurate

GRAPHICAL DEPICTION OF THE TWO-POINT GAUSS-LEGENDRE FORMULA



TWO-POINT GAUSS-LEGENDRE FORMULA

- Notice that the integration limits are from −1 to 1. This was done for simplicity and make the formulation as general as possible.
- A simple change of variable is used to translate other limits of integration into this form.
- Provided that the higher order derivatives do not increase substantially with increasing number of points (n), Gauss quadrature is superior to Newton-Cotes formulas.

$$E_t = \frac{2^{2n+3}\left[(n=1)!\right]^4}{(2n+3)\left[(2n+2)!\right]^3} f^{(2n+2)}(\xi)$$

Error for the Gauss-Legendre formulas

IMPROPER INTEGRALS

- Improper integrals can be evaluated by making a change of variable that transforms the infinite range to one that is finite,

$$\int_a^b f(x)\,dx = \int_{1/b}^{1/a} \frac{1}{t^2} f\left(\frac{1}{t}\right) dt \qquad ab > 0$$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{-A} f(x)\,dx + \int_{-A}^{b} f(x)\,dx$$

- where –*A* is chosen as a sufficiently large negative value so that the function has begun to approach zero asymptotically at least as fast as $1/x^2$.