Solution to Exercises
Lesson 3 - Random variables and their distributions

Exercise 1. Consider a bag containing five balls, with the balls numbered 1 through to 5. Three balls are drawn from the bag without replacement.

(a) What is the probability the balls numbered 1, 2 and 3 are drawn from the bag, in the order 1, 2, 3?

(b) What is the probability the balls numbered 1, 2 and 3 are drawn in any order?

(c) Complete the table below from the probabilities of the sum of the 3 balls drawn from the bag.

sum of balls	6	7	8	9	10	11	12
probability							$\frac{1}{10}$

Answer:

(a) We pick three balls from five, and the order matters. This procedure amounts to variations. The total number of possible ways should be

$$P_5^3 = \frac{5!}{(5-3)!} = 60,$$

i.e., $|\Omega| = 60$, where Ω is the sample space.

The required order only gives us a single outcome in Ω, thus by using the Fundamental formula the probability is $1/60$.

(b) This time the three balls are 1, 2 and 3 but can be picked in any order. All together we have 3! ways to pick those three particular balls. By using the Fundamental formula again we have the required probability to be

$$\frac{3!}{P_5^3} = \frac{3!}{60} = \frac{1}{10}.$$

(c) Using the arguments in part (b), it is straightforward to show that the probability of drawing (x, y, z) with $(1 \leq x < y < z \leq 5)$ in any order is $1/10$.
Therefore we enumerate all the possible outcomes in the sample space

$$\Omega_1 = \begin{Bmatrix} (1,2,3), & (1,2,4), & (1,2,5), & (1,3,4), & (1,3,5), \\ (1,4,5), & (2,3,4), & (2,3,5), & (2,4,5), & (3,4,5) \end{Bmatrix}.$$

each of which has the chance $1/10$ to occur. Now denote the $(\omega_1, \omega_2, \omega_3)$ as the possible outcomes, we are interested in the collection

$$\{(\omega_1, \omega_2, \omega_3) : \ \omega_1 + \omega_2 + \omega_3 = x\},$$

where $x \in M := \{6, 7, 8, 9, 10, 11, 12\}$. Define a random variable $X : \Omega_1 \to M$ such that

$$X(\omega_1, \omega_2, \omega_3) := \omega_1 + \omega_2 + \omega_3.$$

Then by Fundamental formula we can compute the probabilities desired:

$$\mathbb{P}(X = 6) = \mathbb{P}(\{(1,2,3)\}) = \frac{|\{(1,2,3)\}|}{|\Omega_1|} = \frac{1}{10},$$

$$\mathbb{P}(X = 7) = \mathbb{P}(\{(1,2,4)\}) = \frac{|\{(1,2,4)\}|}{|\Omega_1|} = \frac{1}{10},$$

$$\mathbb{P}(X = 8) = \mathbb{P}(\{(1,2,5),(1,3,4)\}) = \frac{|\{(1,2,5),(1,3,4)\}|}{|\Omega_1|} = \frac{2}{10} = \frac{1}{5},$$

$$\mathbb{P}(X = 9) = \mathbb{P}(\{(1,3,5),(2,3,4)\}) = \frac{|\{(1,3,5),(2,3,4)\}|}{|\Omega_1|} = \frac{2}{10} = \frac{1}{5},$$

$$\mathbb{P}(X = 10) = \mathbb{P}(\{(1,4,5),(2,3,5)\}) = \frac{|\{(1,4,5),(2,3,5)\}|}{|\Omega_1|} = \frac{2}{10} = \frac{1}{5},$$

$$\mathbb{P}(X = 11) = \mathbb{P}(\{(2,4,5)\}) = \frac{|\{(2,4,5)\}|}{|\Omega_1|} = \frac{1}{10},$$

$$\mathbb{P}(X = 12) = \mathbb{P}(\{(3,4,5)\}) = \frac{|\{(3,4,5)\}|}{|\Omega_1|} = \frac{1}{10}.$$

sum of balls	6	7	8	9	10	11	12
probability	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{10}$

Exercise 2. Consider two fair dice A and B. Die A is 6-sided and is numbered 1 through to 6 whilst die B is 4-sided and is numbered 1 though to 4. Both dice are rolled. Let $X = A + B$. Find the probability mass function and probability distribution function of X.

Answer: First let us figure out the sample space of the face numbers of the two dice as

$$\begin{aligned}\Omega =& \{(i,j) : i \in \{1,\dots,6\}, j \in \{1,2,3,4\}\}\\ =& \left\{\begin{matrix} (1,1), & (1,2), & (1,3), & (1,4), \\ (2,1), & (2,2), & (2,3), & (2,4), \\ (3,1), & (3,2), & (3,3), & (3,4), \\ (4,1), & (4,2), & (4,3), & (4,4), \\ (5,1), & (5,2), & (5,3), & (5,4), \\ (6,1), & (6,2), & (6,3), & (6,4), \end{matrix}\right\},\end{aligned}$$

each of which has the same probability to occur, i.e., with probability $\dfrac{1}{|\Omega|} = \dfrac{1}{24}$. Now denote the (ω_1, ω_2) as the possible outcomes, we are interested in the collection

$$\{(\omega_1, \omega_2) : \omega_1 + \omega_2 = x\},$$

where $x \in M := \{2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Define a random variable $X : \Omega_1 \to M$ such that

$$X(\omega_1, \omega_2) := \omega_1 + \omega_2.$$

Then by Fundamental formula we can compute the probabilities desired:

$$\mathbb{P}(X = 2) = \mathbb{P}(\{(1,1)\}) = \frac{|\{(1,1)\}|}{|\Omega|} = \frac{1}{24},$$

$$\mathbb{P}(X = 3) = \mathbb{P}(\{(1,2),(2,1)\}) = \frac{|\{(1,2),(2,1)\}|}{|\Omega|} = \frac{2}{24} = \frac{1}{12},$$

$$\mathbb{P}(X = 4) = \mathbb{P}(\{(1,3),(2,2),(3,1)\}) = \frac{|\{(1,3),(2,2),(3,1)\}|}{|\Omega|} = \frac{3}{24} = \frac{1}{8},$$

$$\mathbb{P}(X = 5) = \mathbb{P}(\{(1,4),(2,3),(3,2),(4,1)\}) = \frac{|\{(1,4),(2,3),(3,2),(4,1)\}|}{|\Omega|} = \frac{4}{24} = \frac{1}{6},$$

$$\mathbb{P}(X = 6) = \mathbb{P}(\{(2,4),(3,3),(4,2),(5,1)\}) = \frac{|\{(2,4),(3,3),(4,2),(5,1)\}|}{|\Omega|} = \frac{4}{24} = \frac{1}{6},$$

$$\mathbb{P}(X = 7) = \mathbb{P}(\{(3,4),(4,3),(5,2),(6,1)\}) = \frac{|\{(3,4),(4,3),(5,2),(6,1)\}|}{|\Omega|} = \frac{4}{24} = \frac{1}{6},$$

$$\mathbb{P}(X = 8) = \mathbb{P}(\{(4,4),(5,3),(6,2)\}) = \frac{|\{(4,4),(5,3),(6,2)\}|}{|\Omega|} = \frac{3}{24} = \frac{1}{8},$$

$$\mathbb{P}(X = 9) = \mathbb{P}(\{(5,4),(6,3)\}) = \frac{|\{(5,4),(6,3)\}|}{|\Omega|} = \frac{2}{24} = \frac{1}{12},$$

$$\mathbb{P}(X = 10) = \mathbb{P}(\{(6,4)\}) = \frac{|\{(6,4)\}|}{|\Omega|} = \frac{1}{24}.$$

Thus we can construct the probability mass function and distribution function table as follows:

x	2	3	4	5	6	7	8	9	10
$\mathbb{P}(X = x)$	$\frac{1}{24}$	$\frac{1}{12}$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{8}$	$\frac{1}{12}$	$\frac{1}{24}$

and

x	2	3	4	5	6	7	8	9	10
$F_X(x) = \mathbb{P}(X \le x)$	$\frac{1}{24}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{5}{12}$	$\frac{7}{12}$	$\frac{3}{4}$	$\frac{7}{8}$	$\frac{23}{24}$	1

Exercise 3. The random variable Z has the probability mass function below.

Z	0	1	2	3
$f_Z(z)$	0.2	0.16	0.41	a

(a) What is the value of a?

(b) What is $\mathbb{P}(1 \le Z < 3)$?

(c) What is $F_Z(1.7)$?

Answer:

(a) By the normalization property of probability mass function, we shall have
$$\sum_{z\in\{0,1,2,3\}} f_Z(z) = 0.2 + 0.16 + 0.41 + a = 1.$$
Thus $a = 0.23$.

(b)
$$\mathbb{P}(1 \le Z < 3) = f_Z(1) + f_Z(2) = 0.16 + 0.41 = 0.57.$$

(c)
$$F_Z(1.7) = \mathbb{P}(z \le 1.7) = \mathbb{P}(Z=0)+\mathbb{P}(Z=1) = f_Z(0)+f_Z(1) = 0.2+0.16 = 0.36.$$

Exercise 4. The probability mass function of random variable Y is
$$f_Y(y) := \begin{cases} \frac{ky}{y^2+1}, & \text{if } y \in \{2,3\}; \\ \frac{ky}{2(y^2-1)}, & \text{if } y \in \{4,5\}; \\ 0, & \text{otherwise.} \end{cases}$$
Find the value of the constant k.

Answer: Note that $\sum_y f_Y(y) = 1$. Hence,
$$k\left(\frac{2}{5} + \frac{3}{10} + \frac{4}{2\times 15} + \frac{5}{2\times 24}\right) = 1.$$
Then $k = \dfrac{16}{15}$.

Exercise 5. Let X be a discrete random variable and
$$f(x) := \begin{cases} C(4-|x-4|), & \text{if } x \in \{1,2,3,4,5,6,7\}, \\ 0, & \text{otherwise,} \end{cases}$$
where C is a real-valued constant.

(a) Find C such that f is a probability mass function;

(b) Assuming that f is the probability mass function of X, calculate the mean, variance, and standard deviation of X, rounding your answers to 3rd decimal places if necessary.

(c) Determine the probability $P(1 < X \le 5)$.

Answer:

(a) In order f to be a probability mass function, it must be non-negative and normalized (i.e., its values need to add to 1). We have $f(1) = C$, $f(2) = 2C$, $f(3) = 3C$, $f(4) = 4C$, $f(5) = 3C$, $f(6) = 2C$, $f(7) = C$. They all must be non-negative numbers, therefore the first requirement is satisfied if $C \geq 0$. Furthermore,

$$f(1) + \cdots + f(7) = C(1 + 2 + 3 + 4 + 3 + 2 + 1) = 1$$

gives $C = \dfrac{1}{16}$. This ensures that f is a probability mass function (p.m.f).

(b) The mean of the random variable with this p.m.f is

$$\mathbb{E}[X] = 1 \cdot \frac{1}{16} + 2 \cdot \frac{2}{16} + 3 \cdot \frac{3}{16} + 4 \cdot \frac{4}{16} + 5 \cdot \frac{3}{16} + 6 \cdot \frac{2}{16} + 7 \cdot \frac{7}{16} = 4.$$

Furthermore,

$$\mathbb{E}[X^2] = 1^2 \cdot \frac{1}{16} + 2^2 \cdot \frac{2}{16} + 3^2 \cdot \frac{3}{16} + 4^2 \cdot \frac{4}{16} + 5^2 \cdot \frac{3}{16} + 6^2 \cdot \frac{2}{16} + 7^2 \cdot \frac{1}{16} = 18.5,$$

hence,

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = 2.5$$

and the standard deviation is

$$\sqrt{\mathrm{Var}(X)} \approx 1.581.$$

(c) To calculate the required probability we write

$$\begin{aligned}
\mathbb{P}(1 < X \leq 5) &= \mathbb{P}(X = 2) + \mathbb{P}(X = 3) + \mathbb{P}(X = 4) + \mathbb{P}(X = 5) \\
&= f(2) + f(3) + f(4) + f(5) \\
&= \frac{2}{16} + \frac{3}{16} + \frac{4}{16} + \frac{3}{16} = 0.75.
\end{aligned}$$

Exercise 6. A component of a particular assessment is a multiple choice test consisting of two problems. The first has three possible answers, the second has five. The student did not revise much before the test, and chooses at random one answer from each of the problems independently. What is the expected number of correct answers? Is random answering a good strategy?

Answer: Let X be the random variable counting the total number of correct answers given in the two questions. Its possible values are $M = \{0, 1, 2\}$. In the first question there are 3 choices of which 1 is right and 2 are wrong; in the second question there are 5 choices of which 1 is right and 4 are wrong. Denote by X_1 the number of correct answers for question 1, and by X_2 the number of correct answers for question 2; both X_1 and X_2 take the possible values 0 or 1. The probability of picking the correct answer in question 1 is therefore $\mathbb{P}(X_1 = 1) = 1/3$, and it is $\mathbb{P}(X_2 = 1) = 1/5$ in question 2. Hence, by the independence of X_1 and X_2,

$$\begin{aligned}\mathbb{P}(X = 0) =&\mathbb{P}(X_1 + X_2 = 0) = \mathbb{P}(X_1 = 0, X_2 = 0) = \mathbb{P}(X_1 = 0)\,\mathbb{P}(X_2 = 0)\\ =&\frac{2}{3} \cdot \frac{4}{5} = \frac{8}{15},\end{aligned}$$

$$\begin{aligned}\mathbb{P}(X = 1) =&\mathbb{P}(X_1 + X_2 = 1) = \mathbb{P}(\{X_1 = 1, X_2 = 0\} \cup \{X_1 = 0, X_2 = 1\})\\ =&\mathbb{P}(X_1 = 1, X_2 = 0) + \mathbb{P}(X_1 = 0, X_2 = 1)\\ =&\mathbb{P}(X_1 = 1)\,\mathbb{P}(X_2 = 0) + \mathbb{P}(X_1 = 0)\,\mathbb{P}(X_2 = 1)\\ =&\frac{2}{3} \cdot \frac{1}{5} + \frac{1}{3} \cdot \frac{4}{5} = \frac{6}{15},\end{aligned}$$

$$\mathbb{P}(X = 2) = \mathbb{P}(X_1+X_2 = 2) = \mathbb{P}(X_1 = 1, X_2 = 1) = \mathbb{P}(X_1 = 1)\,\mathbb{P}(X_2 = 1) = \frac{1}{3}\cdot\frac{1}{5} = \frac{1}{15}.$$

This gives for the expected number of correct answers

$$\mathbb{E}[X] = 0 \cdot \mathbb{P}(X = 0) + 1 \cdot \mathbb{P}(X = 1) + 2 \cdot \mathbb{P}(X = 2) = \frac{8}{15} \simeq 53\%.$$

This number is less than 1 coming to a mark just under 27% for each problem, so even with 40% passing threshold this is a poor strategy.

Exercise 7. The probability that Ms. Brown will sell a piece of property at a profit of \$3,000 is 3/20, the probability that she will sell it at a profit of \$1,500 is 7/20, the probability that she will break even is 7/20, and the probability that she will lose \$1,500 is 3/20. What is her expected profit?

Answer: Let X denote the profit Ms. Brown can get from selling a piece of property. From the question we know that

$$\mathbb{P}(X = 3000) = \frac{3}{20},\ \mathbb{P}(X = 1500) = \frac{7}{20},\ \mathbb{P}(X = 0) = \frac{7}{20},\ \mathbb{P}(X = -1500) = \frac{3}{20}.$$

By the definition of expectation we have

$$\begin{aligned}\mathbb{E}[X] =&3000 \cdot \mathbb{P}(X = 3000) + 1500 \cdot \mathbb{P}(X = 1500)\\ =&\frac{7}{20} + 0 \cdot \mathbb{P}(X = 0) - 1500 \cdot \mathbb{P}(X = -1500)\\ =&3000 \cdot \frac{3}{20} + 1500 \cdot \frac{7}{20} + 0 \cdot \frac{7}{20} - 1500 \cdot \frac{3}{20}\\ =&\frac{1}{20}(9000 + 10500 - 4500)\\ =&750.\end{aligned}$$

Exercise 8. Assume that a game of chance is considered fair, or equitable, if each player's expectation is equal to zero. If someone pays us \$10 each time that we roll a 3 or a 4 with a balanced die, how much should we pay that person when we roll a 1, 2, 5 or 6 to make the game equitable?

Answer: Let X denote how much a person can generate from a fair game. From the question we know

$$\mathbb{P}(X = 10) = \frac{|\{3, 4\}|}{|\{1, 2, 3, 4, 5, 6\}|} = \frac{1}{3},$$

and suppose this person will lose a if we roll 1, 2, 5, 6, then

$$\mathbb{P}(X = -a) = \frac{|\{1, 2, 5, 6\}|}{|\{1, 2, 3, 4, 5, 6\}|} = \frac{2}{3}.$$

Since this is a fair game, which implies a zero expectation, i.e.,

$$0 = \mathbb{E}[X] = 10 \cdot \mathbb{P}(X = 10) + (-a) \cdot \mathbb{P}(X = -a) = 10 \cdot \frac{1}{3} + (-a) \cdot \frac{2}{3}.$$

Thus we can see to make this game fair we need $a = 5$.

Exercise 9. The probability that a fluorescent bulb burns for at least 500 hours is 0.90. Of 8 such bulbs, find the probability that

(a) all 8 burn for at least 500 hours,

(b) 7 burn for at least 500 hours.

Answer: We can think of 'success' when the light bulb burns, and 'failure' when it burns out. Each bulb can therefore be described as a Bernoulli variable with $p = 0.9$. Since in the set of 8 the light bulbs work or fail independently, we can think of the set as an 8-fold succession of independent Bernoulli trials, i.e., of a $B(8, 0.9)$ variable. Then by probability mass function of Binomial variable we have

$$\mathbb{P}(\text{all 8 burn}) = \binom{8}{8} \cdot 0.9^8 \cdot 0.1^0 = 0.43046;$$

$$\mathbb{P}(\text{only 7 burn}) = \binom{8}{7} \cdot 0.9^7 \cdot 0.1^1 = 0.3826.$$

Exercise 10. Samuel Pepys being about to place a bet asked Newton which of the following events is more likely

A. at least one six when 6 dice are thrown

B. at least a double six when 12 dice are thrown

C. at least a triple six when 18 dice are thrown.

Newton gave the correct answer. What is that? Round your answers to 3 decimal places.

Answer: Using the independence of rolls, we have

$$\mathbb{P}(A) = 1 - \mathbb{P}(A^c) = 1 - \left(\frac{5}{6}\right)^6 \approx 0.665.$$

Alternatively we can view this as a success/failure game in 6 independent trials, that is, a binomial variable $B(6, \frac{1}{6})$, i.e., $n = 6$, $p = 6$, $1 - p = \frac{5}{6}$. Then A^c means no success and 6 failures, so

$$\mathbb{P}(A) = 1 - \binom{6}{0}\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^6,$$

i.e., the same answer as above. We continue to view the experiment in this way since it is easy to calculate the probabilities of the remaining two events. For B we have a $B(12, \frac{1}{6})$ experiment:

$$\begin{aligned}\mathbb{P}(B) =&1 - \mathbb{P}(\{\text{no } 6\}) - \mathbb{P}(\{\text{one } 6\})\\ =&1 - \binom{12}{0}\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^{12} - \binom{12}{1}\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^{11}\\ \approx&0.619.\end{aligned}$$

For C we have a $B(18, \frac{1}{6})$ experiment:

$$\begin{aligned}\mathbb{P}(C) =&1 - \mathbb{P}(\{\text{no } 6\}) - \mathbb{P}(\{\text{one } 6\}) - \mathbb{P}(\{\text{two 6s}\})\\ =&1 - \binom{18}{0}\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^{18} - \binom{18}{1}\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^{17} - \binom{18}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^{16}\\ \approx&0.959.\end{aligned}$$

So Newton's answer should be the item C.

Exercise 11. The following gambling game, known as the wheel of fortune (or chuk-a-luck), is quite popular at many carnivals and gambling casinos: A player bets on one of the numbers 1 through 6. Three dice are then rolled, and if the number bet by the player appears i times, $i = 1, 2, 3$, then the player wins i unites; on the other hand, if the number bet by the player does not appear on any of the dice, then the player loses 1 unit. Is this game fair to the player? (Actually the game is played by spinning a wheel that comes to rest on a slot labeled by three of the numbers 1 through 6, but it is mathematically equivalent to the dice version.)

Answer: If we assume that the dice are fair and act independently of each other, then the number of times that the number bet appears is a binomial random variable with parameter $n = 3$ and $p = 1/6$, i.e., $B(3, 1/6)$. Hence denote the player's winnings in the game by X, we have

$$\mathbb{P}(X = -1) = \binom{3}{0}\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^3 = \frac{125}{216},$$

$$\mathbb{P}(X = 1) = \binom{3}{1}\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^2 = \frac{75}{216},$$

$$\mathbb{P}(X=2) = \binom{3}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^1 = \frac{15}{216},$$

$$\mathbb{P}(X=3) = \binom{3}{3}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^0 = \frac{1}{216}.$$

In order to determine whether or not this is a fair game for the player, let us determine $\mathbb{E}[X]$. From the preceding probabilities we obtain

$$\mathbb{E}[X] = (-1)\cdot\frac{125}{216} + 1\cdot\frac{75}{216} + 2\cdot\frac{15}{216} + 3\cdot\frac{1}{216} = -\frac{17}{216} \neq 0.$$

So the game is not fair to the player.

Exercise 12. Suppose that a particular trait (such as eye color or left handedness) of a person is classified on the basis of one pair of genes and suppose that d represents a dominant gene and r a recessive gene. Thus a person with dd genes is pure dominance, one with rr is pure recessive, and one with rd is hybrid. The pure dominance and the hybrid are alike in appearance. Children receive 1 gene from each parent. If, with respect to a particular trait, 2 hybrid parents have a total 4 children, what is the probability that 3 out of 4 children have the outward appearance of the dominant gene?

Answer: If we assume that each child is equally likely to inherit either of 2 genes from each parent, the probabilities that the child of 2 hybrid parents will have dd, rr, or rd pairs of genes are, respectively, 1/4, 1/4 and 1/2. Hence as an offspring will have the outward appearance of the dominant gene if its gene pair is either dd or rd, it follows that the number of such children is binomially distributed with parameters $n = 4$ and $p = 1/4 + 1/2 = 3/4$, i.e., $B(4, 3/4)$. Thus the desired probability is

$$\binom{4}{3}\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^1 = \frac{27}{64}.$$

Exercise 13. (See also Theorem 3.2.3, (2) as a particular case when $Y \equiv 1$.) Let X and Y be two discrete random variables, prove that if X and Y are independent, then for any constants a, $b \in \mathbb{R}$,

$$\mathrm{Var}(aX + bY) = a^2\mathrm{Var}(X) + b^2\mathrm{Var}(Y).$$

Hint: Two random variables X and Y are said to be independent if

$$\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y).$$

Answer: Let $\mathbb{E}(X) = m$ and $\mathbb{E}(Y) = n$. Since

$$\mathrm{Var}(aX + bY) = \mathbb{E}[(aX + bY)^2] - [\mathbb{E}(aX + bY)]^2,$$

we compute the two terms separately in the right hand side of the quality above.

$$
\begin{aligned}
\mathbb{E}[(aX+bY)^2] &= \mathbb{E}(a^2X^2 + 2abXY + b^2Y^2) \\
&\overset{\text{Linearity}}{=} a^2\mathbb{E}(X^2) + 2ab\mathbb{E}(XY) + b^2\mathbb{E}(Y) \\
&\overset{\text{Indep.}}{=} a^2\mathbb{E}(X^2) + 2ab\mathbb{E}(X)\mathbb{E}(Y) + b^2\mathbb{E}(Y)
\end{aligned}
$$

$$
\mathbb{E}(aX+bY) \overset{\text{Linearity}}{=} a\mathbb{E}(X) + b\mathbb{E}(Y) = am + bn.
$$

Thus,

$$
\begin{aligned}
LHS &= a^2\mathbb{E}(X^2) + 2abmn + b^2\mathbb{E}(Y^2) - (am+bn)^2 \\
&= a^2[\mathbb{E}(X^2) - m^2] + b^2[\mathbb{E}(Y^2) - n^2] \\
&= a^2\mathrm{Var}(X) + b^2\mathrm{Var}(Y) = RHS.
\end{aligned}
$$

Exercise 14. (See also Exercise 3.3.1) Prove that if X follows Geometric distribution with parameter p, then

$$
\mathbb{E}[X] = \frac{1}{p}, \quad \mathrm{Var}(X) = \frac{1-p}{p^2}.
$$

Answer: To determine $\mathbb{E}[X]$, let $q = 1-p$, we have

$$
\begin{aligned}
\mathbb{E}[X] &\overset{\text{def.}}{=} \sum_{i=1}^{\infty} iq^{i-1}p = \sum_{i=1}^{\infty}(i-1+1)q^{i-1}p \\
&= \sum_{i=1}^{\infty}(i-1)q^{i-1}p + \sum_{i=1}^{\infty} q^{i-1}p \\
&= \sum_{j=0}^{\infty} jq^jp + 1, \quad \text{letting } j = i-1 \\
&= q\sum_{i=1}^{\infty} jq^{j-1}p + 1 \\
&= q\mathbb{E}[X] + 1.
\end{aligned}
$$

Hence, $p\mathbb{E}[X] = 1$ yielding the result

$$
\mathbb{E}[X] = 1/p.
$$

To determine $\mathrm{Var}(X)$, let us first compute $\mathbb{E}[X^2]$. We have

$$
\begin{aligned}
\mathbb{E}[X^2] &\overset{\text{def.}}{=} \sum_{i=1}^{\infty} i^2 q^{i-1}p = \sum_{i=1}^{\infty}(i-1+1)^2 q^{i-1}p \\
&= \sum_{i=1}^{\infty}(i-1)^2 q^{i-1}p + \sum_{i=1}^{\infty} 2(i-1)q^{i-1}p + \sum_{i=1}^{\infty} q^{i-1}p \\
&= \sum_{j=0}^{\infty} j^2 q^j p + 2\sum_{j=0}^{\infty} j q^j p + 1, \quad \text{letting } j = i-1 \\
&= \sum_{j=1}^{\infty} j^2 q^j p + 2\sum_{j=1}^{\infty} j q^j p + 1 \\
&= q\sum_{j=1}^{\infty} j^2 q^{j-1} p + 2q\sum_{j=1}^{\infty} j q^{j-1} p + 1 \\
&= q\mathbb{E}[X^2] + 2q\mathbb{E}[X] + 1.
\end{aligned}
$$

Using $\mathbb{E}[X] = 1/p$, the equation for $\mathbb{E}[X^2]$ yields

$$p\mathbb{E}[X^2] = \frac{2q}{p} + 1.$$

Hence,

$$\mathbb{E}[X^2] = \frac{2q+p}{p^2} = \frac{q+1}{p^2}$$

giving the result

$$\mathrm{Var}(X) = \frac{q+1}{p^2} - \frac{1}{p^2} = \frac{q}{p^2} = \frac{1-p}{p^2}.$$

Exercise 15. The number of trucks X arriving on any one day at a truck depot in a certain city is a Poisson distributed r.v. Suppose that the average number of X is 12. What is the probability that on a given day fewer than nine trucks will arrive at this depot?

Answer: It is given that $X \sim Poi(\lambda)$, with $\lambda = 12$. The required probability thus is

$$\mathbb{P}(X < 9) = \sum_{i=0}^{8} \mathbb{P}(X = i) = \sum_{i=0}^{8} \frac{12^i}{i!} e^{12} \approx 0.1550.$$

Exercise 16. Let $N(t)$ denote the number of earthquakes during the time interval $(0, t]$, for $t > 0$, in the western portion of the United States. It is known that the $N(t)$ is a Poisson r.v. with parameter λt, for any $t > 0$. Let t be of 1 week as the unit of time and $\lambda = 2$.

(a) Find the probability that at least 3 earthquakes occur during the next 2 weeks.

(b) Find the probability distribution of the time, starting from now, until the next earthquake.

Answer:

(a) The average number of occurrences during 2 weeks is $2\lambda = 2 \cdot 2 = 4$. Thus $N(2) \sim Poi(4)$. Then we have

$$\begin{aligned}\mathbb{P}(N(2) \geq 3) &= 1 - \mathbb{P}(N(2) = 0) - \mathbb{P}(N(2) = 1) - \mathbb{P}(N(2) = 2)\\ &= 1 - e^{-4} - 4 \cdot e^{-4} - \frac{4^2}{2} e^{-4} \approx 0.76.\end{aligned}$$

(b) Let X denote the amount of time (in weeks) until the next earthquake. Because X will be greater than t if and only if no events occur within the next t unite of time, we have that

$$\mathbb{P}(X > t) = \mathbb{P}(N(t) = 0) = e^{-2t}.$$

Then we have

$$F(t) = \mathbb{P}(X \leq t) = 1 - \mathbb{P}(X > t) = 1 - e^{-2t}.$$

Exercise 17. The lifetime of a component $\mathcal{C}$ is an random variable X with the probability density function (p.d.f) given by

$$f(t) = \begin{cases} k(2t - t^2) & \text{if } 0 \leq t < 2 \\ 0 & \text{otherwise} \end{cases}$$

where k is a real valued constant.

(a) Determine the value k.

(b) Determine the cumulative distribution function (c.d.f) $F(x)$ of X.

(c) Determine the probability $\mathbb{P}(X \leq 1)$.

(d) Determine the probability $\mathbb{P}(X \leq \frac{1}{2})$.

(e) Determine the probability $\mathbb{P}(X \geq \frac{3}{2})$.

(f) Determine the conditional probability $\mathbb{P}(X \geq \dfrac{3}{2} | X \geq 1)$.

Answer:

(a) Since f is a p.d.f, $\displaystyle\int_{\mathbb{R}} f(t)\mathrm{d}t = 1$. That is

$$\begin{aligned}&\int_0^2 k(2t - t^2)\,\mathrm{d}t = 1\\ \Longleftrightarrow\ & k\left[t^2 - \frac{1}{3}t^3\right]_0^2 = 1\\ \Longleftrightarrow\ & k\left(4 - \frac{1}{3} \times 8 - 0\right) = 1\\ \Longleftrightarrow\ & \frac{4}{3}k = 1\\ \Longleftrightarrow\ & k = \frac{3}{4}.\end{aligned}$$

So we obtain that $k = 3/4$.

(b) From Part (a), the p.d.f f can be rewritten as follows

$$f(t) = \begin{cases} \frac{3}{4}(2t - t^2) & \text{if } 0 \le t < 2 \\ 0 & \text{otherwise} \end{cases}$$

By the definition of a c.d.f for a continuous random variable,

$$F(x) = \int_{-\infty}^{x} f(t)\mathrm{d}t.$$

The value of $F(x)$ will be different if x is located in different positions. There are 3 cases.

- If $x \le 0$, then $F(x) = 0$.
- If $0 < x \le 2$, then

$$F(x) = \int_{-\infty}^{x} f(t)\mathrm{d}t = \int_{0}^{x} \frac{3}{4}(2t - t^2)\mathrm{d}t = \frac{3}{4}\left[t^2 - \frac{1}{3}t^3\right]_0^x = \frac{3}{4}\left(x^2 - \frac{1}{3}x^3\right).$$

- If $x > 2$, then

$$F(x) = \int_{-\infty}^{x} f(t)\mathrm{d}t = \int_{0}^{2} \frac{3}{4}(2t - t^2)\mathrm{d}t = 1.$$

In conclusion, the c.d.f $F(x)$ is

$$F(x) = \begin{cases} 0 & \text{if } x \le 0 \\ \frac{3}{4}\left(x^2 - \frac{1}{3}x^3\right) & \text{if } 0 < x \le 2 \\ 1 & \text{if } x > 2. \end{cases}$$

(c)

$$\mathbb{P}(X \le 1) = F(1) = \frac{3}{4}\left(1^2 - \frac{1}{3}\cdot 1^3\right) = \frac{1}{2}.$$

(d)

$$\mathbb{P}(X \le \frac{1}{2}) = F\left(\frac{1}{2}\right) = \frac{3}{4}\left[\left(\frac{1}{2}\right)^2 - \frac{1}{3}\cdot\left(\frac{1}{2}\right)^3\right] = \frac{15}{96}.$$

(e)

$$\mathbb{P}(X \ge \frac{3}{2}) = 1 - \mathbb{P}(X < \frac{3}{2}) = 1 - F\left(\frac{3}{2}\right) = 1 - \frac{3}{4}\left[\left(\frac{3}{2}\right)^2 - \frac{1}{3}\left(\frac{3}{2}\right)^3\right] = 1 - \frac{27}{32} = \frac{5}{32}.$$

(f)

$$\mathbb{P}(X \ge \frac{3}{2} | X \ge 1) = \frac{\mathbb{P}(X \ge 1 \text{ and } X \ge \frac{3}{2})}{\mathbb{P}(X \ge 1)} = \frac{\mathbb{P}(X \ge \frac{3}{2})}{\mathbb{P}(X \ge 1)}.$$

By using the results obtained in Part (c) and (e), we have

$$\mathbb{P}(X \ge \frac{3}{2} | X \ge 1) = \frac{\frac{5}{32}}{1 - \frac{1}{2}} = \frac{5}{16}.$$

Exercise 18. The lifetime of an insect is an random variable X with c.d.f $F(x)$ given by

$$F(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1-(x+1)e^{-x} & \text{if } x > 0. \end{cases}$$

(a) Determine, rounding to the second decimal place, the probability $\mathbb{P}(X \leq 1)$.

(b) Prove that the random variable X admits an p.d.f $f(t)$ and determine it.

Answer:

(a)

$$\mathbb{P}(X \leq 1) = F(1) = 1-(1+1)e^{-1} = 1-\frac{2}{e} \approx 0.26.$$

(b) We have $\lim_{x\to\infty} F(x) = 1$. The function F is continuous on $(0, +\infty)$, the derivative $F'(x)$ exists and is continuous. The p.d.f f is thus $f(t) = F'(t)$ for any $x \in (0, +\infty)$. So the p.d.f of the random variable X is

$$f(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ te^{-t} & \text{if } t > 0. \end{cases}$$

Exercise 19. The lifetime of an organism is an random variable X with p.d.f

$$f(t) = \begin{cases} k(10t^2 - t^3) & \text{if } 0 \leq t \leq 10 \\ 0 & \text{otherwise} \end{cases}$$

where k is a real valued constant.

(a) Determine k.

(b) Determine $F(x)$, the c.d.f of X.

(c) Calculate the probability $\mathbb{P}(X \geq 1)$.

(d) Calculate the probability $\mathbb{P}(1 \leq X \leq 2)$.

(e) Calculate the conditional probability $\mathbb{P}(1 \leq X \leq 2 | X \geq 1)$, rounding your answer to the fourth decimal place.

(f) Determine $\mathbb{E}(X)$.

(g) Determine $\mathbb{E}\left(X^2\right)$.

(h) Determine the variance $\mathrm{Var}(X)$ of X.

Answer:

(a) We should have

$$1 = \int_{\mathbb{R}} f(t)\mathrm{d}t = k\int_0^{10}(10t^2 - t^3)\mathrm{d}t = k\left[\frac{10}{3}t^3 - \frac{1}{4}t^4\right]_0^{10} = \frac{2500}{3}k,$$

which gives $k = \dfrac{3}{2500}$.

(b) From Part (a), the p.d.f f can be rewritten as follows

$$f(t) = \begin{cases} \frac{3}{2500}(10t^2 - t^3) & \text{if } 0 \le t \le 10 \\ 0 & \text{otherwise} \end{cases}$$

By the definition of a c.d.f for a continuous random variable,

$$F(x) = \int_{-\infty}^{x} f(t)\mathrm{d}t.$$

The value of $F(x)$ will be different if x is located in different positions. There are 3 cases.

- If $x \le 0$, then $F(x) = 0$.
- If $0 < x \le 10$, then

$$\begin{aligned} F(x) &= \int_{-\infty}^{x} f(t)\mathrm{d}t = \int_0^x \frac{3}{2500}(10t^2 - t^3)\mathrm{d}t \\ &= \frac{3}{2500}\left[\frac{10}{3}t^3 - \frac{1}{4}t^4\right]_0^x = \frac{3}{2500}\left(\frac{10}{3}x^3 - \frac{1}{4}x^4\right) \\ &= \frac{1}{250}x^3 - \frac{3}{10000}x^4. \end{aligned}$$

- If $x > 2$, then

$$F(x) = \int_{-\infty}^{x} f(t)\mathrm{d}t = \int_0^{10} \frac{3}{2500}(10t^2 - t^3)\mathrm{d}t = 1.$$

In conclusion, the c.d.f $F(x)$ is

$$F(x) = \begin{cases} 0 & \text{if } x \le 0 \\ \frac{1}{250}x^3 - \frac{3}{10000}x^4 & \text{if } 0 < x \le 10 \\ 1 & \text{if } x > 10. \end{cases}$$

(c)

$$\begin{aligned} \mathbb{P}(X \ge 1) &= 1 - \mathbb{P}(X < 1) \\ &= 1 - F(1) \\ &= 1 - \left(\frac{1}{250} - \frac{3}{10000}\right) \\ &= 1 - \frac{37}{10000} = \frac{9963}{10000} = 0.9963. \end{aligned}$$

(d)

$$\mathbb{P}(1 \leq X \leq 2) = F(2) - F(1) = \frac{8}{250} - \frac{3 \times 16}{10000} - \frac{37}{10000} = \frac{235}{10000} = 0.0235.$$

(e) By using the results that we obtained in Parts (c) and (d),

$$\mathbb{P}(1 \leq X \leq 2 | X \geq 1) = \frac{\mathbb{P}(1 \leq X \leq 2)}{\mathbb{P}(X \geq 1)} = \frac{0.0235}{0.9963} \approx 0.0236.$$

(f)

$$\begin{aligned}
\mathbb{E}(X) &= \int_{-\infty}^{+\infty} t f(t) \mathrm{d}t \\
&= \frac{3}{2500} \int_0^{10} t(10t^2 - t^3) \mathrm{d}t \\
&= \frac{3}{2500} \int_0^{10} (10t^3 - t^4) \mathrm{d}t \\
&= \frac{3}{2500} \left[\frac{10}{4} t^4 - \frac{1}{5} t^5 \right]_0^{10} = \frac{3 \times 5000}{2500} = 6.
\end{aligned}$$

(g)

$$\begin{aligned}
\mathbb{E}(X^2) &= \int_{-\infty}^{+\infty} t^2 f(t) \mathrm{d}t \\
&= \frac{3}{2500} \int_0^{10} t^2(10t^2 - t^3) \mathrm{d}t \\
&= \frac{3}{2500} \int_0^{10} (10t^4 - t^5) \mathrm{d}t \\
&= \frac{3}{2500} \left[\frac{10}{5} t^5 - \frac{1}{6} t^6 \right]_0^{10} = \frac{3}{2500} \times 10^5 \left(2 - \frac{10}{6} \right) = 40.
\end{aligned}$$

(h) By using the results in Part (f) and (g), we have

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2 = 40 - 6^2 = 4.$$

Exercise 20. The concentration of an urinary metabolite is an random variable X with p.d.f defined by

$$f(t) = \begin{cases} k(1-t) & \text{if } 0 \leq t \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Draw the curve of the function f and calculate k.

(b) Determine the expression of the c.d.f $F(x)$ of the random variable X by the position of x in comparison to 0 and 1. Draw the curve of the function F.

(c) Evaluate $\mathbb{P}(0 < X < 1)$.

(d) Calculate $\mathbb{E}(X)$.

(e) Calculate $\mathrm{Var}(X)$.

Let Y be another random variable given by

$$Y = 1 - X.$$

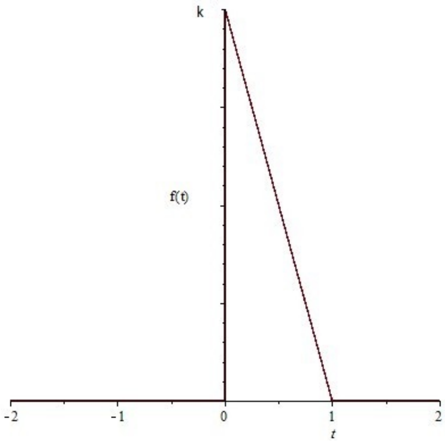
(f) Calculate $\mathbb{E}(Y)$.

(g) Calculate $\mathrm{Var}(Y)$.

(h) Calculate $\mathbb{P}\left(\frac{1}{3} \leq Y \leq \frac{2}{3}\right)$.

Answer:

(a) The curve of the p.d.f f is shown below:



Since $\displaystyle\int_{-\infty}^{+\infty} f(t)\mathrm{d}t = \frac{k}{2}$, (the area of the triangle), we obtain $k = 2$.

(b) $F(x) = \displaystyle\int_{-\infty}^{x} f(t)\mathrm{d}t.$

Since $\displaystyle\int_{0}^{x} 2(1-t)\mathrm{d}t = \left[2t - t^2\right]_0^2 = 2x - x^2$, we have

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 2x - x^2 & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } x > 1. \end{cases}$$

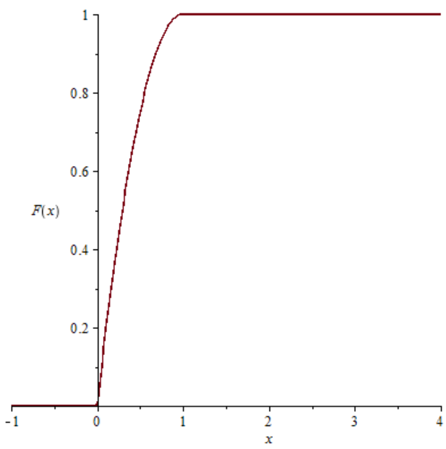
The curve of F is shown in the figure below:

(c) There are two methods to evaluate $\mathbb{P}(0 < X < 1)$.
Using c.d.f,

$$\mathbb{P}(0 < X < 1) = F(1) - F(0) = 2 - 1^2 - 0 = 1.$$

Or using the curve of the p.d.f f drawn in Part (a),

$$\mathbb{P}(0 < X < 1) = \text{ area of the triangle } = \frac{k}{2} = 1.$$



(d)

$$\mathbb{E}(X) = \int_{-\infty}^{+\infty} x f(x)\mathrm{d}x = \int_0^1 2x(1-x)\mathrm{d}x = \left[x^2 - \frac{2x^3}{3}\right]_0^1 = 1-\frac{2}{3} = \frac{1}{3}.$$

(e)

$$\mathbb{E}(X^2) = \int_{-\infty}^{+\infty} x^2 f(x)\mathrm{d}x = \int_0^1 2x^2(1-x)\mathrm{d}x = \left[\frac{2x^3}{3} - \frac{2x^4}{4}\right]_0^1 = \frac{1}{6}.$$

The variance of X can be obtained by

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2 = \frac{1}{6} - \frac{1}{9} = \frac{1}{18}.$$

We can also use the definition:

$$\mathrm{Var}(X) = \mathbb{E}\left[\left(X - \frac{1}{3}\right)^2\right] = \int_0^1 \left(x - \frac{1}{3}\right)^2 (2-2x)\mathrm{d}x,$$

but the calculation is more complicated:

$$\int_0^1 \left(-2x^3 + \frac{10}{3}x^2 - \frac{14}{9}x + \frac{2}{9}\right)\mathrm{d}x = -\frac{2}{4} + \frac{10}{9} - \frac{14}{18} + \frac{2}{9} = \frac{1}{18}.$$

(f) We use $\mathbb{E}(aX + bY) = a\mathbb{E}(X) + b\mathbb{E}(Y)$ to obtain

$$\mathbb{E}(Y) = \mathbb{E}(1) - \mathbb{E}(X) = 1 - \mathbb{E}(X) = \frac{2}{3}.$$

(g) We use $\mathrm{Var}(aX + b) = a^2\mathrm{Var}(X)$ to obtain

$$\mathrm{Var}(Y) = (-1)^2\mathrm{Var}(X) = \mathrm{Var}(X) = \frac{1}{18}.$$

(h) The event

$$\left\{\frac{1}{3} \le Y \le \frac{2}{3}\right\} = \left\{\frac{1}{3} \le 1 - X \le \frac{2}{3}\right\} = \left\{\frac{1}{3} \le X \le \frac{2}{3}\right\},$$

so we obtain

$$\mathbb{P}\left(\frac{1}{3} \le Y \le \frac{2}{3}\right) = \mathbb{P}\left(\frac{1}{3} \le X \le \frac{2}{3}\right) = F\left(\frac{2}{3}\right) - F\left(\frac{1}{3}\right) = \frac{1}{3}.$$

Exercise 21. Suppose that X is a uniformly distributed random variable over the interval $(-10, 10)$. The functions $f(t)$ and $F(x)$ are respectively the p.d.f and c.d.f of X. Determine $f(t)$, $F(x)$, $\mathbb{E}(X)$ and $\mathrm{Var}(X)$.

Answer: By the definition of a uniform distribution, the p.d.f $f(t)$ of the random variable X is given by

$$f(t) = \begin{cases} \frac{1}{20} & \text{if } t \in (-10, 10) \\ 0 & \text{otherwise.} \end{cases}$$

The c.d.f $F(x)$ of the random variable X is given by

$$F(x) = \begin{cases} 0 & \text{if } x \leq -10 \\ \frac{x+10}{20} & \text{if } x \in (-10, 10) \\ 1 & \text{otherwise.} \end{cases}$$

$\mathbb{E}(X) = \dfrac{-10+10}{2} = 0$ and

$\mathrm{Var}(X) = \mathbb{E}[(X - \mathbb{E}(X))^2] = \dfrac{(-10-10)^2}{12} = \dfrac{100}{3}.$

Exercise 22. The lifetime of a radioactive atom is an exponential random variable with parameter $\theta = 1/2$.

(a) Determine the expectation $\mathbb{E}(X)$ of X.

(b) Determine the variance $\mathrm{Var}(X)$ of X.

(c) For $x \geq 0$, what is the expression of the c.d.f, $F(x)$, of X?

(d) Determine the probability $\mathbb{P}(X \geq 1)$, rounding your answer to the fourth decimal place.

(e) Determine the probability $\mathbb{P}(1 \leq X \leq 2)$, rounding your answer to the fourth decimal place.

(f) Determine the conditional probability $\mathbb{P}(X \leq 2 | X \geq 1)$, rounding your answer to the fourth decimal place.

Answer:

(a) $\mathbb{E}(X) = \theta = \dfrac{1}{2} = 0.5.$

(b) $\mathrm{Var}(X) = \theta^2 = \dfrac{1}{4} = 0.25.$

(c) If $x \geq 0$, $F(x) = 1 - e^{-2x}$.

(d) $\mathbb{P}(X \geq 1) = 1 - \mathbb{P}(X < 1) = 1 - F(1) = 1 - (1 - e^{-2}) = e^{-2} \approx 0.1353.$

(e)

$$\begin{aligned} \mathbb{P}(1 \leq X \leq 2) = F(2) - F(1) = 1 - e^{-4} - (1 - e^{-2}) &= e^{-2} - e^{-4} \\ &\approx 0.1170. \end{aligned}$$

(f) $\mathbb{P}(X \leq 2 | X \geq 1) = \dfrac{\mathbb{P}(1 \leq X \leq 2)}{\mathbb{P}(X \geq 1)} = \dfrac{e^{-2} - e^{-4}}{e^{-2}} = 1 - e^{-2} \approx 0.8647.$

Exercise 23. At a particular point of the electric grid the time of a voltage drop is an exponential random variable. The mean time elapsed between consecutive voltage drops is 20 days.

(a) What is the probability that there is at least a 30 days gap between consecutive voltage drops?

(b) What is the probability that within 30 days at least two voltage drops occur?

Answer:

(a) Since the time gaps (in units of days) are exponential random variables we have for its p.d.f

$$f_\theta(x) := \begin{cases} \frac{1}{\theta} e^{-x/\theta}, & \text{if } x \in [0, \infty) \\ 0, & \text{otherwise.} \end{cases}$$

The parameter θ coincides with the mean, therefore $\theta = 20$ so the non-vanishing part of the p.d.f is

$$f_\theta(x) = \frac{1}{20} e^{-\frac{x}{20}}.$$

We are interested in the probability $\mathbb{P}(X \geq 30)$. We obtain

$$\mathbb{P}(X \geq 30) = \int_{30}^{\infty} \frac{1}{20} e^{-\frac{x}{20}} \, \mathrm{d}x = e^{-1.5} \approx 0.2231.$$

(b) The number of events within given time gaps are Poisson random variables. They have probability mass function

$$f_\lambda(k) := \begin{cases} \frac{\lambda^k}{k!} e^{-\lambda}, & \text{if } k \in \mathbb{N} \cup \{0\}, \\ 0, & \text{otherwise.} \end{cases}$$

Here we have

$$\lambda = \frac{30}{20} = 1.5,$$

Thus the non-vanishing part of the p.m.f is

$$f_\lambda(k) = \frac{1.5^k}{k!} e^{-1.5}.$$

We are interested in the probability $\mathbb{P}_\lambda(X \geq 2)$. We have

$$\begin{aligned} \mathbb{P}(X \geq 2) &= 1 - \mathbb{P}(X < 2) \\ &= 1 - \mathbb{P}(X = 0) - \mathbb{P}(X = 1) \\ &= 1 - \frac{1.5^0}{0!} e^{-1.5} - \frac{1.5^1}{1!} e^{-1.5} \\ &\approx 0.4422. \end{aligned}$$

Exercise 24. A loom stops automatically as soon as a thread breaks. Let X be the time elapsed from the start of the weaving process to the first stop caused by this failure. Assuming that X is of exponential distribution with a mean of 2.5 hours, what is the probability that on an average 8 hours workday the machine does not fail?

Answer: Since the mean of the exponential distribution equals θ, we have $\theta = 2.5$. Thus the probability distribution function is

$$F(x) := \begin{cases} 1 - e^{-0.4x}, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0. \end{cases}$$

Therefore,

$$\mathbb{P}(X > 8) = 1 - \mathbb{P}(X \leq 8) = 1 - F(8) = 1 - (1 - e^{-3.2}) = e^{-3.2} \approx 0.0408.$$

Exercise 25. An component has lifetime X, which is an exponential random variable with parameter $\theta = 250$.

(a) Determine $\mathbb{E}(X)$.

(b) Determine $\mathrm{Var}(X)$.

(c) Evaluate the probability that the lifetime of the component is > 100, rounding your answer to the second decimal place.

(d) Determine the conditional probability $\mathbb{P}(X > 200|X > 100)$, rounding your answer to the second decimal place.

(e) Determine the smallest integer n such that $\mathbb{P}(X > n) \leq 0.05$.

Answer:

(a) $\mathbb{E}(X) = \theta = 250$.

(b) $\mathrm{Var}(X) = \theta^2 = 250^2 = 62500$.

$$F(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ 1 - e^{-0.004x}, & \text{if } x > 0. \end{cases}$$

(c)

$$\mathbb{P}(X > 100) = 1 - \mathbb{P}(X \leq 100) = 1 - F(100) = e^{-0.4} \approx 0.67.$$

(d) $\mathbb{P}(X > 200|X > 100) = \dfrac{\mathbb{P}(X > 200)}{\mathbb{P}(X > 100)} = \dfrac{e^{-0.8}}{e^{-0.4}} \approx 0.67.$

(e)

$$
\begin{aligned}
\mathbb{P}(X > n) \leq 0.05 &\Longleftrightarrow 1 - F(n) \leq 0.05 \\
&\Longleftrightarrow e^{-0.004n} \leq 0.05 \\
&\Longleftrightarrow -0.004n \leq \ln(0.05) \\
&\Longleftrightarrow n \geq \frac{-\ln(0.05)}{0.004} \approx 748.933
\end{aligned}
$$

Therefore, the smallest integer n such that $\mathbb{P}(X > n) \leq 0.05$ is 749.

Exercise 26. The weight of an egg in a farm follows normal distribution with mean 65 and standard deviation 10, denoted by $\mathrm{Normal}(65, 100)$.

(a) Determine the probability $\mathbb{P}(X < 50)$.

(b) Determine the probability $\mathbb{P}(X > 80)$.

(c) Determine the probability $\mathbb{P}(X > 80 | X \geq 65)$.

Answer: Let $Z = \dfrac{X - 65}{10}$, then Z is a standard normal distributed random variable, i.e., $Z \sim \mathrm{Normal}(0, 1)$.

Let Φ be the c.d.f of the standard normal distribution. In order to determine the following probabilities, we need to use the table of values of the function Φ in the Appendix.

(a)

$$
\begin{aligned}
\mathbb{P}(X < 50) =&\mathbb{P}\left(Z < \frac{50 - 65}{10}\right) \\
=&\mathbb{P}(Z < -1.5) \\
=&\Phi(-1.5) \\
=&1 - \Phi(1.5) \approx 1 - 0.9332 \approx 0.0668,
\end{aligned}
$$

where the value $\Phi(1.5)$ is found in the table of Appendix.

(b) We have

$$
\begin{aligned}
\mathbb{P}(X > 80) =&\mathbb{P}\left(Z > \frac{80 - 65}{10}\right) \\
=&\mathbb{P}(Z > 1.5) \\
=&1 - \mathbb{P}(Z \leq 1.5) \\
=&1 - \Phi(1.5) \\
\approx&1 - 0.9332 \approx 0.0668,
\end{aligned}
$$

where the value $\Phi(1.5)$ is found in the table of Appendix.

(c) We have

$$\begin{aligned}\mathbb{P}(X \geq 65) =& \mathbb{P}\left(Z \geq \frac{65-65}{10}\right)\\ =&\mathbb{P}(Z \geq 0)\\ =&1-\mathbb{P}(Z<0)\\ =&1-\Phi(0)=1-0.5=0.5.\end{aligned}$$

Using the result in Part (b), we obtain

$$\mathbb{P}(X>80|X \geq 65)=\frac{\mathbb{P}(X>80)}{\mathbb{P}(X \geq 65)} \approx \frac{0.0668}{0.5} \approx 0.1336.$$

Exercise 27. In certain population, the rate of glucose in the body of an individual is a random variable X, which follows a normal distribution with mean m (g/l) and standard deviation σ (g/l), denoted by Normal(m, σ^2).

It follows from a study performed on a large number of independent people that

$$\mathbb{P}(X \leq 1)=\mathbb{P}(X \geq 0.8)=0.84.$$

(a) Determine m and σ, rounding your answers to the second decimal place.

In the following question, use the values m and σ that you obtained in Part (a).

(b) Determine the probability $\mathbb{P}(X \geq 1.1)$.

Answer: Let $Z=\dfrac{X-m}{\sigma}$, then $Z \sim$ Normal$(0,1)$.

(a) From the given equality

$$\mathbb{P}(X \leq 1)=\mathbb{P}(X \geq 0.8)=0.84,$$

we have

$$1-\mathbb{P}(X \leq 1)=1-\mathbb{P}(X \geq 0.8)=1-0.84=0.16,$$

or

$$\mathbb{P}(X>1)=\mathbb{P}(X<0.8)=0.16,$$

which follows the p.d.f $f(x)$ of the random variable X is symmetric to the vertical line $x=\dfrac{1+0.8}{2}=0.9$ in the xy-plane. Moreover, it is known the function f is symmetric to $x=m$. So we have $m=0.9$.

To find σ, we use for instance $\mathbb{P}(X \leq 1)=0.84$. We have

$$\mathbb{P}(X \leq 1)=\mathbb{P}(Z \leq \frac{1-m}{\sigma})=\Phi\left(\frac{1-m}{\sigma}\right)=0.84.$$

So

$$\frac{1-m}{\sigma}=\Phi^{-1}(0.84),$$

which implies $\frac{1-m}{\sigma} \approx 1$ from the table in Appendix. And $m=0.9$, so we have $\sigma \approx 1-0.9=0.1$.

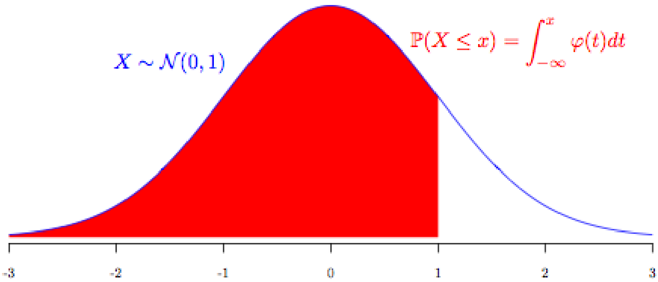
(b)

$$
\begin{aligned}
\mathbb{P}(X \geq 1.1) =& \mathbb{P}\left(Z \geq \frac{1.1 - m}{\sigma}\right)\\
=& \mathbb{P}\left(Z \geq \frac{1.1 - 0.9}{0.1}\right)\\
=& \mathbb{P}(Z \geq 2) = 1 - \mathbb{P}(Z < 2)\\
=& 1 - \Phi(2) \approx 1 - 0.9772 = 0.0228,
\end{aligned}
$$

where the value $\Phi(2)$ is found in the table of Appendix.

Appendix

The values of c.d.f of the standard normal distribution



	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

Figure 1: Area $\Phi(x)$ under the standard normal curve to the left of x.