Xi'an Jiaotong-Liverpool University
西交利物浦大学

MTH113TC Introduction to Probability and Statistics — Tutorial 4

Year: 2020/21 Week: 3 **Based on Chapter 4**

1. If S^2 is the variance of a random sample of size n, show that it can be written as

$$S^2 = \frac{1}{n\,(n-1)}\left[n\sum_{i=1}^{n} X_i^2 - \left(\sum_{i=1}^{n} X_i\right)^2\right].$$

Let x_i be the number of fish caught by the i^{th} fisherman in a random sample of six fishermen. Using the fact that $\sum x_i^2 = 171$ and $\sum x_i = 31$, find the variance of the data.

Answer: $\frac{13}{6}$

2. Suppose $X_1, X_2, \cdots, X_n$ are independent and identically distributed random variables from a normal population with mean μ and variance σ^2. The mean and variance of the random sample are respectively $\bar{X}$ and S^2. Show that

(i) $\mathrm{E}\left(\bar{X}\right) = \mu$,

(ii) $\mathrm{Var}\left(\bar{X}\right) = \frac{\sigma^2}{n}$, and

(iii) $\mathrm{E}\left(S^2\right) = \sigma^2$.

Given the random variables $Z \sim N\left(0, 1^2\right)$, $V \sim \chi^2_n$ are such that Z and V are independent, then the random variable T defined as

$$T = \frac{Z}{\sqrt{V/n}}$$

is t-distributed with n degrees of freedom, i.e. $T \sim t_n$. Use this fact to prove that

(iv) $\frac{\bar{X}-\mu}{S/\sqrt{n}} \sim t_{n-1}$.

3. With the use of appropriate definitions, show that

(i) If $X \sim F_{n,m}$, then $X^{-1} \sim F_{m,n}$.

(ii) If $T \sim t_n$, then $T^2 \sim F_{1,n}$.

4. Traveling between two campuses of a university via shuttle bus takes, on average, 28 minutes with a standard deviation of 5 minutes. In a given week, a bus transported passengers 40 times. What is the probability that the average transport time was more than 30 minutes?

Answer: 0.0057

5. Suppose X is a random variable with mean μ and variance $\sigma^2 = 1.0$. Suppose also that a random sample of size n is to be taken and $\bar{x}$ is to be used as an *estimate* of μ. When the data are taken and the sample mean is measured, we wish it to be within 0.05 unit of the true mean with probability of at least 0.99, i.e.

$$P\left(\left|\bar{X} - \mu\right| < 0.05\right) \geq 0.99.$$

What is the minimum sample size required?

6. The computer of manufacturer A have a mean lifetime of 6.5 years and a standard deviation of 0.9 year, while those of manufacturer B have a mean lifetime of 6.0 years and a standard deviation of 0.8 year. What is the probability that a random sample of 36 tubes from manufacturer A will have a mean lifetime that is at least 1 year more than the mean lifetime of a sample of 49 tubes from manufacturer B?

   Answer: 0.0040

7. In a factory, a filling machine is used to fill cartons with a liquid product. The filling machine is required to operate under the specification of 9 ± 1.5 oz. If any carton is produced with weight outside these bounds, it is considered to be defective. It is hoped that at least 99% of cartons will meet these specifications.

   (i) With the conditions $\mu = 9$ and $\sigma = 1$, what proportion of cartons from the process is defective?

   (ii) If changes are made to reduce variability, what must σ be reduced to in order to meet the specifications with probability 0.99?

   Assume a normal distribution for the weight.

   Answer: (i) 0.1336 (ii) 0.5825

8. Find k such that $P\left(k < T < -1.761\right) = 0.045$ for a random sample of size 15 and random variable $T = \frac{\bar{X}-\mu}{s/\sqrt{n}}$.

   Answer: -2.977

9. If S_1^2 and S_2^2 represent the variances of independent random samples of size $n_1 = 25$ and $n_2 = 31$, taken from normal populations with variances $\sigma_1^2 = 10$ and $\sigma_2^2 = 15$ respectively, find

   $$P\left(S_1^2/S_2^2 > 1.26\right).$$

   Answer: 0.05

10. Consider the following data:

17	62	15	65
28	51	24	65
39	41	35	15
39	32	36	37
40	21	44	37
59	13	44	56
12	54	64	59

   (a) Construct a frequency distribution of 6 classes with class width 10.

   (b) Construct a histogram.

   (c) Construct an ogive.

   (d) Construct a stem-and-leaf plot.

11. Consider the following frequency distribution:

Class	Frequency
$0 < 10$	8
$10 < 20$	10
$20 < 30$	13
$30 < 40$	12
$40 < 50$	6

   (a) Construct a relative frequency distribution.

   (b) Construct a cumulative frequency distribution.

   (c) Construct a cumulative relative frequency distribution.

12. A sample of 20 financial analysts was asked to provide forecasts of earnings per share of a corporation for next year. The results are summarized in the following table:

Forecast ($ per share)	Number of Analysts
$9.95 < 10.45$	2
$10.45 < 10.95$	8
$10.95 < 11.45$	6
$11.45 < 11.95$	3
$11.95 < 12.45$	1

   (a) Construct the histogram.

   (b) Determine the relative frequencies.

   (c) Determine the cumulative frequencies.

   (d) Determine and interpret the relative cumulative frequencies.

13. Regulatory agencies and the U.S. Congress are recognizing both the values and emerging issues for small firms as the Sarbanes-Oxley Act of 2002 (SOX) has been implemented. On April 23, 2006, the Advisory Committee on Smaller Public Companies issued a final report to the Security and Exchange Commission assessing the impact of SOX on smaller public companies (Final Report 2006). A random sample of CEOs, CFOs, and board members of small, medium, and large firms were surveyed and their opinions of the overall impact of SOX on their firm were:

Impact of Sox	Small Firms	Medium Firms	Large Firms
Little or no impact	17	13	6
Moderate to very major impact	13	41	22

   Construct a cluster bar chart of these findings.

14. Four types of checking accounts are offered by one bank. Suppose that recently a random sample of 300 bank customers was surveyed and asked several questions. It was found that 60% of the respondents preferred Easy Checking; 12%, Intelligent Checking; 18%, Super Checking; and the remainder, Ultimate Checking. Of those who selected Easy Checking, 100 were females; one-third of the respondents who selected Intelligent Checking were males; half of the respondents who selected Super Checking were males; and 80% of respondents who selected Ultimate Checking were males.

    (a) Describe the data with a cross table.

    (b) Describe the data graphically with a stacked bar chart.