DTS104TC
NUMERICAL METHODS

LECTURE 9

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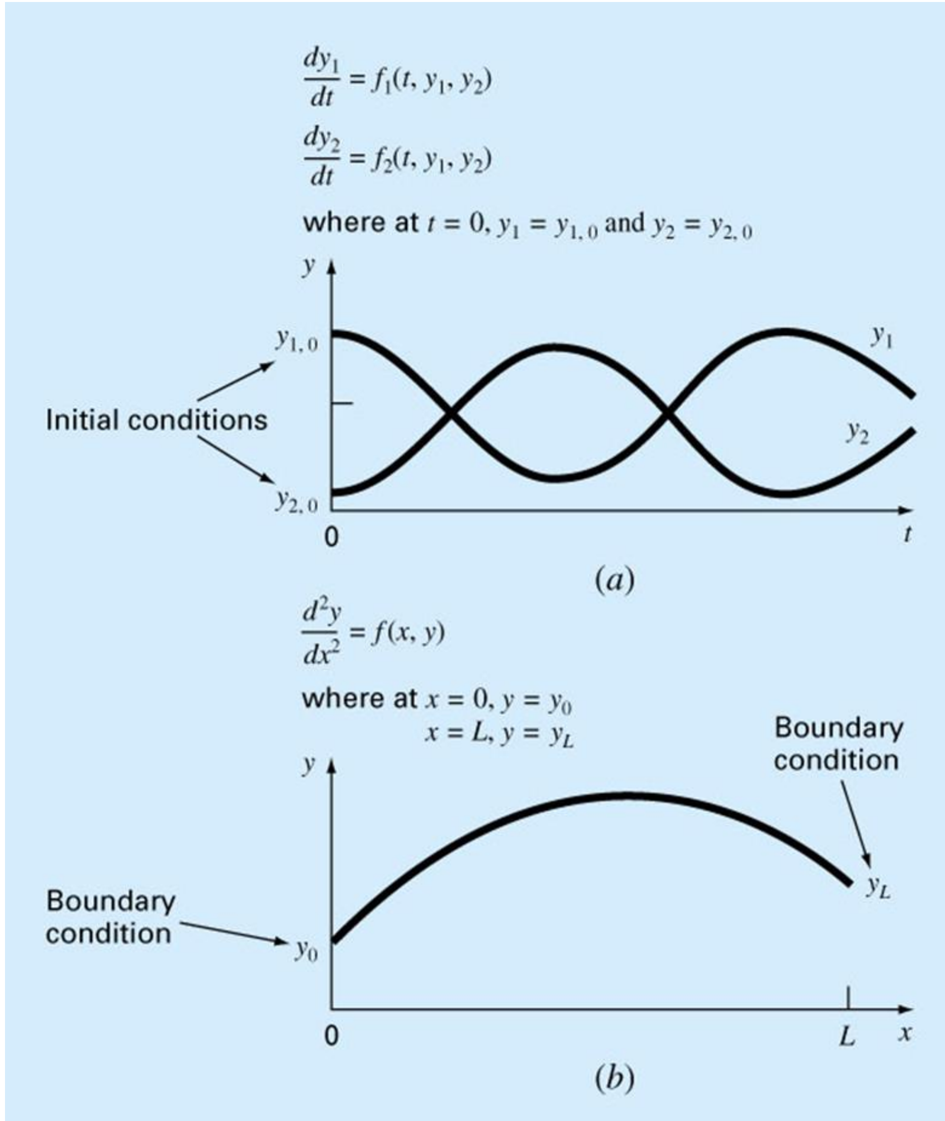
CONTENTS

- Boundary-Value
- Eigenvalue Problems

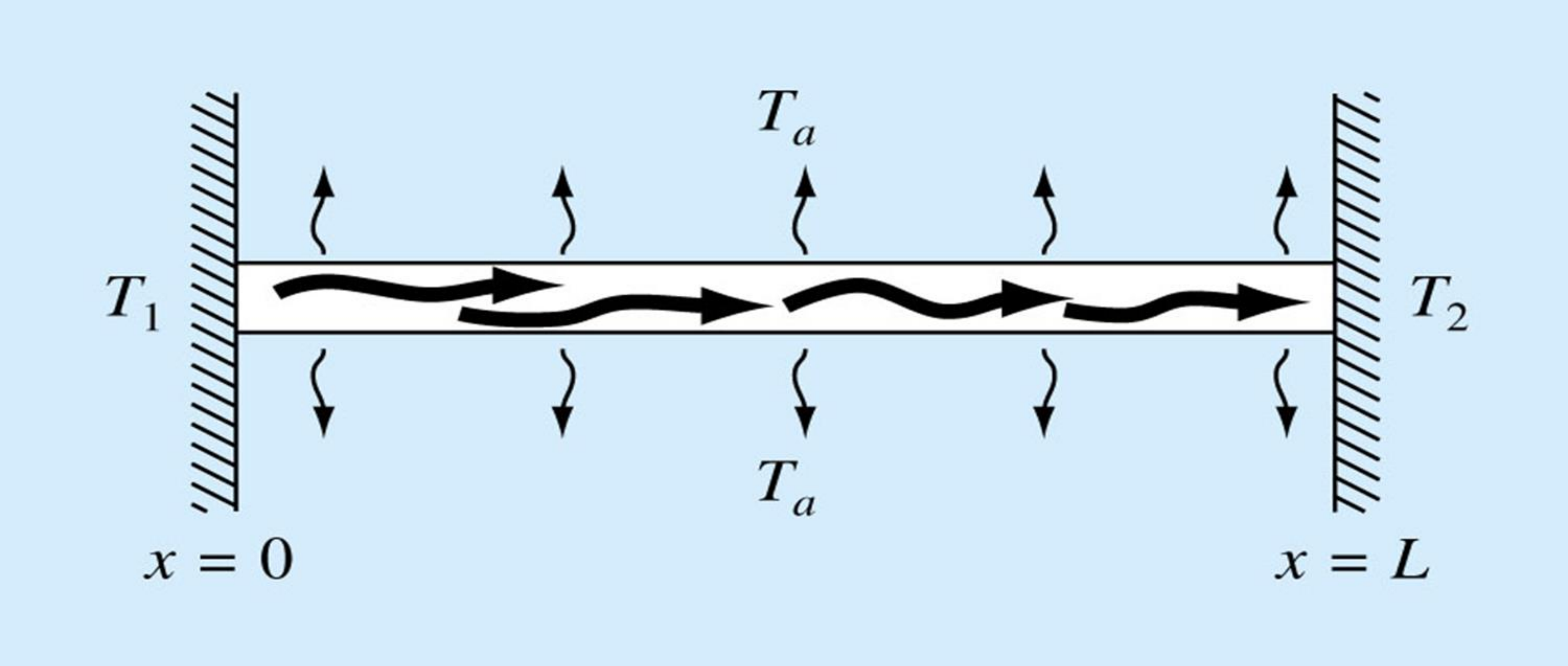
INITIAL-VALUE AND BOUNDARY-VALUE PROBLEMS

- An ODE is accompanied by auxiliary conditions. These conditions are used to evaluate the integral that result during the solution of the equation. An n^{th} order equation requires n conditions.
- If all conditions are specified at the same value of the independent variable, then we ***have an initial-value problem.***
- If the conditions are specified at different values of the independent variable, usually at extreme points or boundaries of a system, then we have a ***boundary-value problem.***

(A) INITIAL-VALUE VERSUS (B) BOUNDARY-VALUE PROBLEMS



EXAMPLE: HEAT TRANSFER PROBLEM



A noninsulated uniform rod positioned between two bodies of constant but different temperature.
For this case $T_1 > T_2$ and $T_2 > T_a$.

EXAMPLE: HEAT TRANSFER PROBLEM

$$\frac{d^2T}{dx^2} + h'(T_a - T) = 0$$

(Heat transfer coefficient)

Parameter values:

$$T_a = 20, \; L = 10\,m, \; h' = 0.01\,m^{-2}$$

$$\left.\begin{array}{l} T(0) = T_1 = 40 \\ T(L) = T_2 = 200 \end{array}\right\} \textbf{Boundary Conditions}$$

Analytical Solution:

$$T = 73.4523e^{0.1x} - 53.4523e^{-0.1x} + 20$$

THE SHOOTING METHOD

- Converts the boundary value problem to initial-value problem. A trial-and-error approach is then implemented to solve the initial value approach.
- For example, the 2nd order equation can be expressed as two first order ODEs:

$$\frac{dT}{dx} = z$$

$$\frac{dz}{dx} = h'(T - T_a)$$

- An initial value is guessed, say $z(0)$=10.
- The solution is then obtained by integrating the two 1st order ODEs simultaneously.

THE SHOOTING METHOD

- Using a 4th order RK method with a step size of 2:

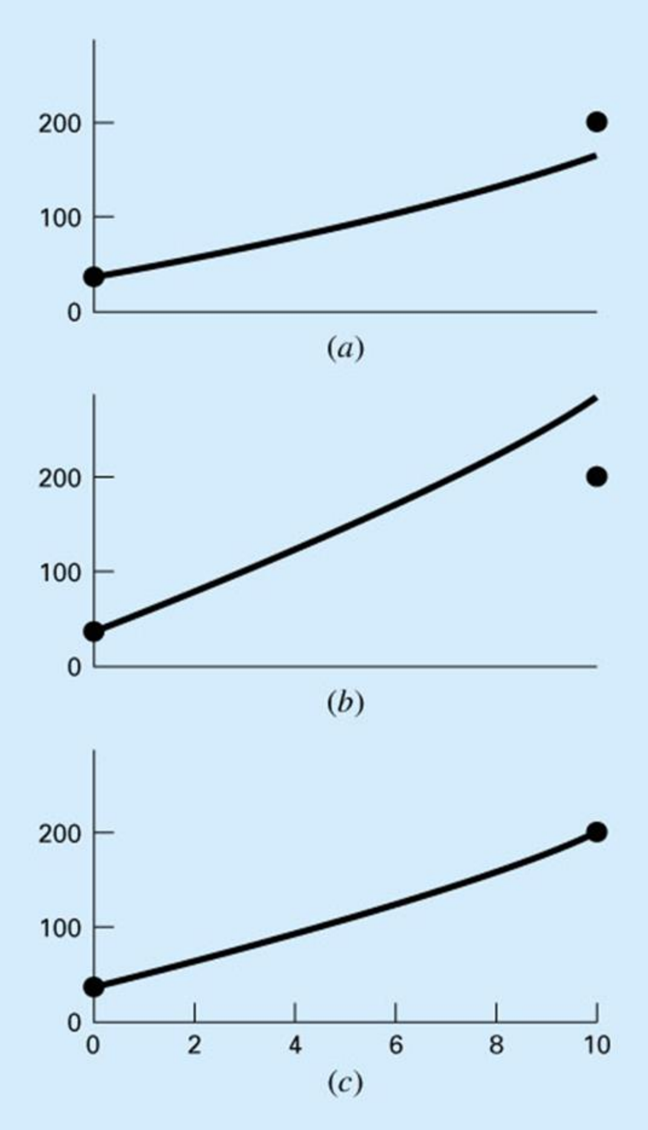
$$T\left(10\right)=168.3797.$$

- This differs from $T(10)=200$. Therefore a new guess is made, $z(0)=20$ and the computation is performed again.

$$z\left(0\right)=20 \qquad T\left(10\right)=285.8980$$

- Since the two sets of points, $(z, T)_1$ and $(z, T)_2$, are linearly related, a linear interpolation formula is used to compute the value of $z(0)$ as 12.6907 to determine the correct solution.

THE SHOOTING METHOD



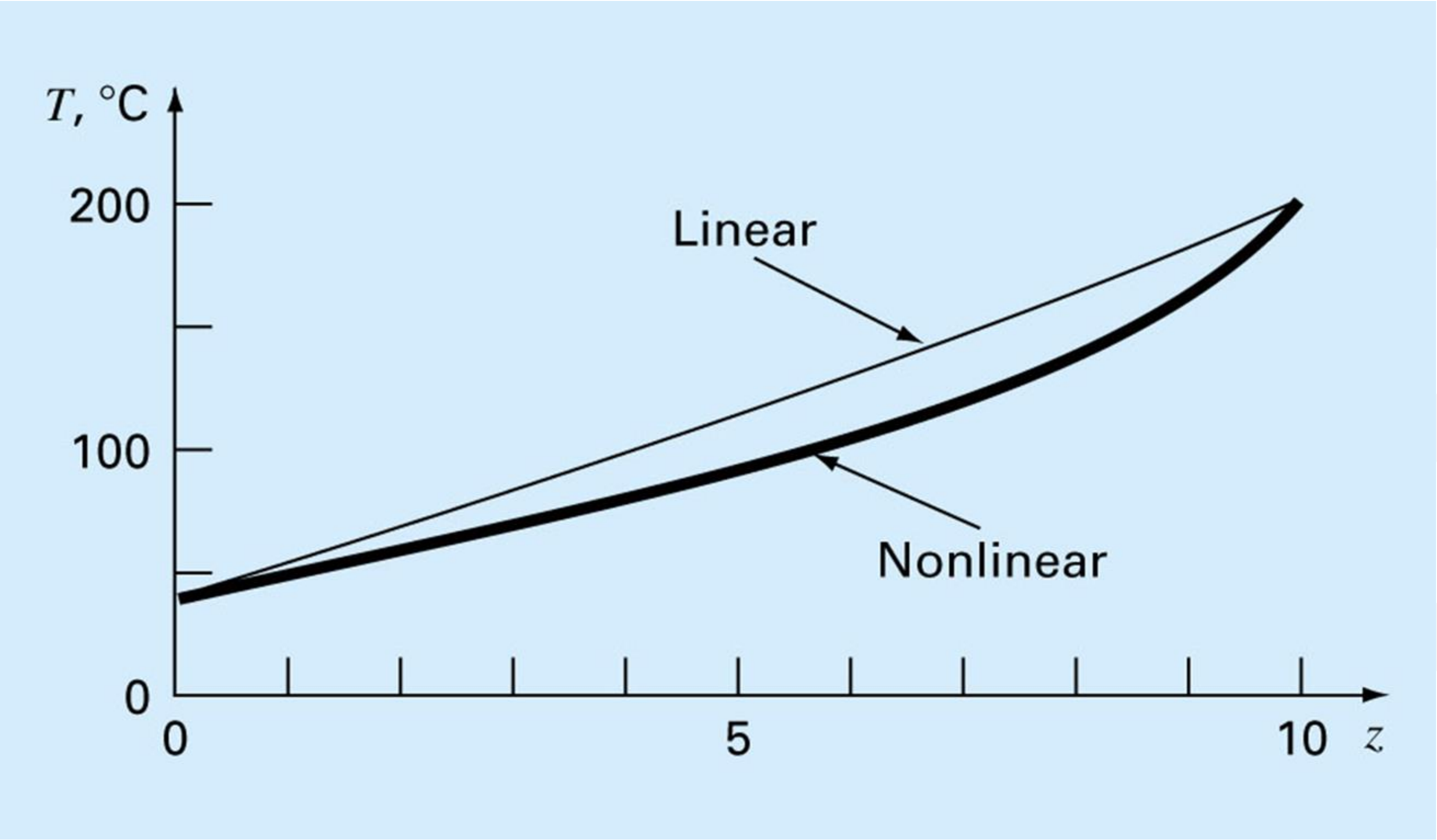
THE SHOOTING METHOD FOR NONLINEAR ODEs

- For a nonlinear problem an approach involves recasting it as a roots problem,

$$T_{10} = f(z_0)$$
$$200 = f(z_0)$$
$$g(z_0) = f(z_0) - 200$$

- Driving this new function, $g(z_0)$, to zero provides the solution.

THE SHOOTING METHOD FOR NONLINEAR ODEs



FINITE DIFFERENCE METHODS

- The most common alternatives to the shooting method.
- Finite differences are substituted for the derivatives in the original equation.

$$\frac{d^2T}{dx^2} = \frac{T_{i+1} - 2T_i + T_{i-1}}{\Delta x^2}$$

$$\frac{T_{i+1} - 2T_i + T_{i-1}}{\Delta x^2} - h'(T_i - T_a) = 0$$

$$-T_{i-1} + (2 + h'\Delta x^2)T_i - T_{i+1} = h'\Delta x^2 T_a$$

- Finite differences equation applies for each of the interior nodes. The first and last interior nodes, T_{i-1} and T_{i+1}, respectively, are specified by the boundary conditions.
- Thus, a linear equation transformed into a set of simultaneous algebraic equations can be solved efficiently.

EIGENVALUE PROBLEMS

- Special class of boundary-value problems that are common in engineering involving vibrations, elasticity, and other oscillating systems.
- Eigenvalue problems are of the general form:

$$
\begin{array}{cccc}
(a_{11}-\lambda)x_1 + & a_{12}x_2 + & \cdots + & a_{1n}x_n = 0 \\
a_{21}x_1 + & (a_{22}-\lambda)x_2 + & \cdots + & a_{2n}x_n = 0 \\
\vdots & \vdots & \vdots & \vdots \\
a_{n1}x_1 + & a_{n2}x_2 + & \cdots + & (a_{nn}-\lambda)x_n = 0
\end{array}
$$

- or, expressed in matrix form,

$$\big[[A]-\lambda[I]\big]\{X\}=0$$

EIGENVALUE PROBLEMS

- λ is the unknown parameter called the *eigenvalue* or

characteristic value.

- A solution $\{X\}$ for such a system is referred to as an *eigenvector*.

- The determinant of the matrix $\left[[A]-\lambda[I]\right]$ must equal zero for

for nontrivial solutions to be possible.

- Expanding the determinant yields a polynomial in λ.

- The roots of this polynomial are the solutions to the eigenvalues.

POLYNOMIAL METHOD

- When dealing with complicated systems or systems with heterogeneous properties, analytical solutions are often difficult or impossible to obtain.
- Numerical solutions to such equations may be the only practical alternatives.
- These equations can be solved by substituting a central finite-divided difference approximation for the derivatives.
- Writing this equation for a series of nodes yields a homogeneous system of equations.
- Expansion of the determinant of the system yields a polynomial, the roots of which are the eigenvalues.

POWER METHOD

- An iterative approach that can be employed to determine the largest eigenvalue.
- To determine the largest eigenvalue the system must be expressed in the form:

$$[A]\{X\} = \lambda\{X\}$$

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