DTS104TC
NUMERICAL METHODS

LECTURE 10

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CONTENTS

PARTIAL DIFFERENTIAL EQUATIONS

Linear, second-order partial differential equations in two variables can be classified.

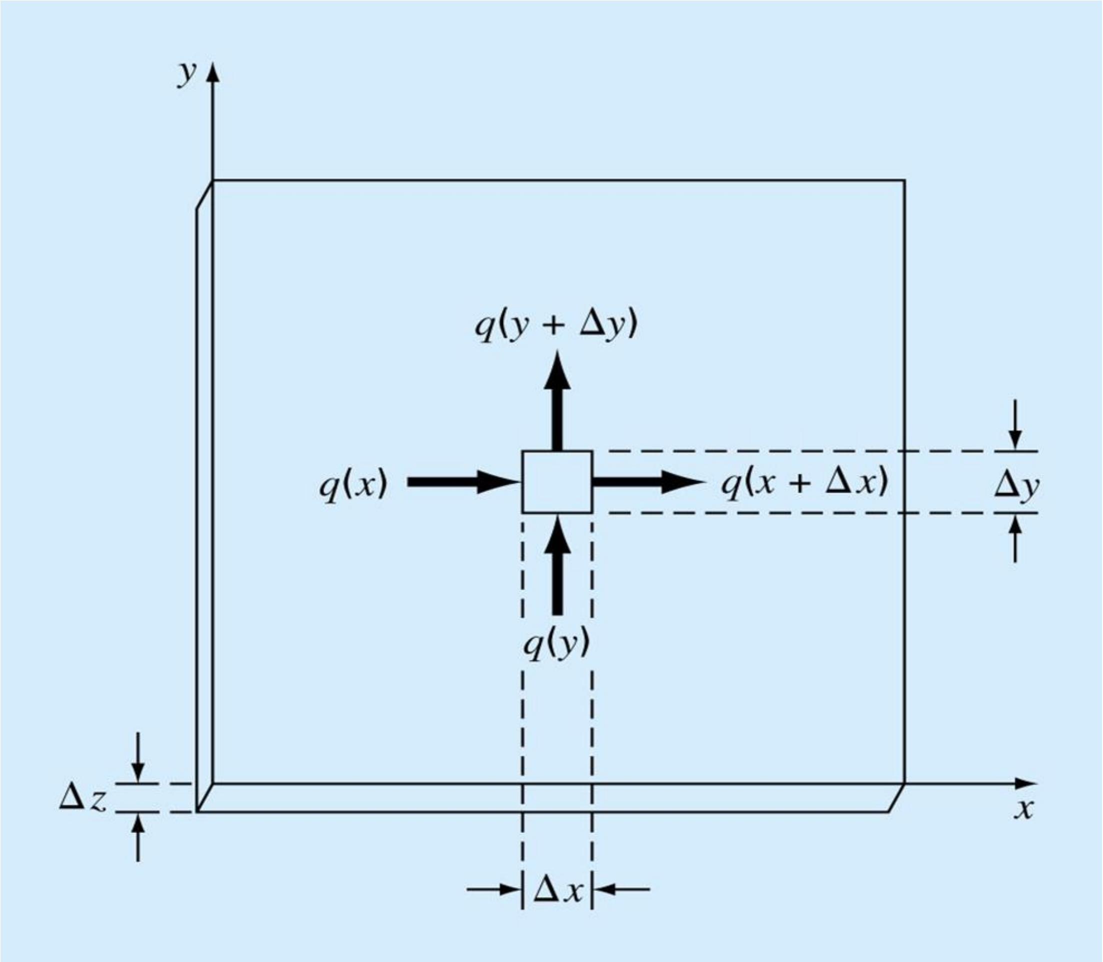
$B^2 - 4AC$	Category	Example
< 0	Elliptic	Laplace equation (steady state with two spatial dimensions), $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$
$= 0$	Parabolic	Heat-conduction equation (time variable with one spatial dimension), $\frac{\partial T}{\partial t} = k' \frac{\partial^2 T}{\partial x^2}$
> 0	Hyperbolic	Wave equation (time variable with one spatial dimension), $\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 y}{\partial t^2}$

FINITE DIFFERENCE: ELLIPTIC EQUATIONS

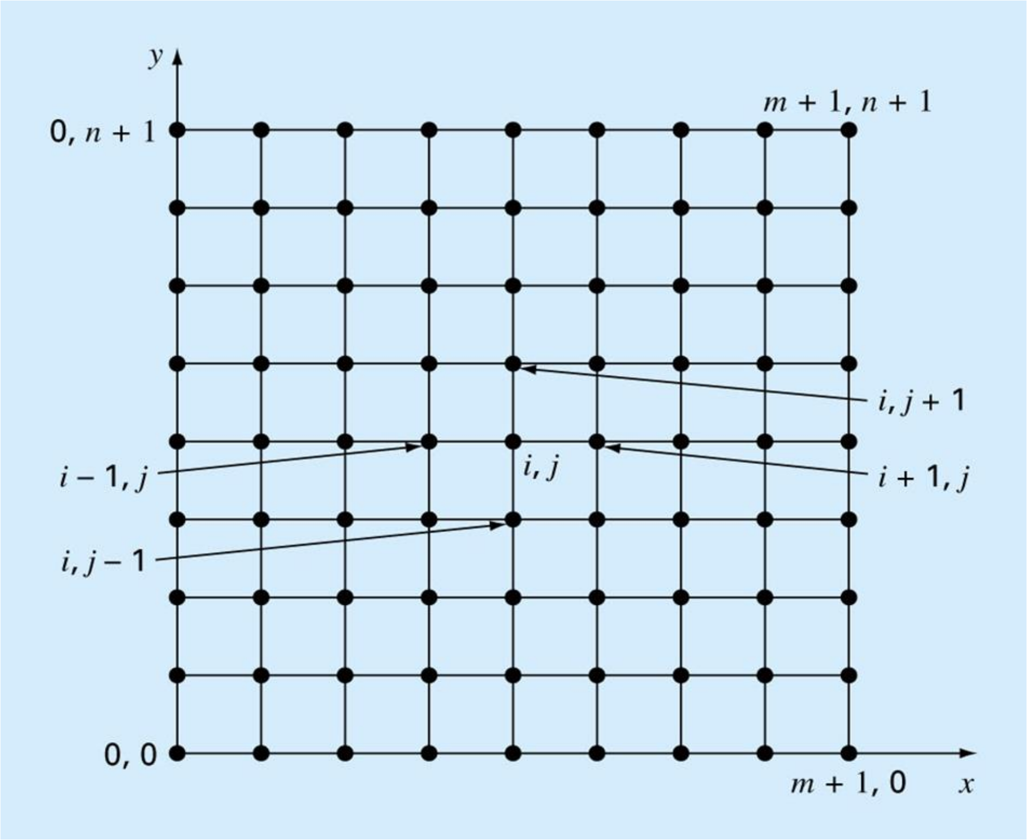
Solution Technique

- Elliptic equations are typically used to characterize steady-state, boundary value problems.
- For numerical solution of elliptic PDEs, the PDE is transformed into an algebraic difference equation.
- Because of its simplicity and general relevance to most areas of engineering, we will use a heated plate as an example for solving elliptic PDEs.

A HEAT BALANCE FOR AN ELEMENT IN THIN PLATE



A FINITE DIFFERENCE GRID



A finite-difference grid for the solution of elliptic PDEs in two independent variables, such as the Laplace equation.

THE LAPLACIAN DIFFERENCE EQUATION

The *Laplace equation*:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

- Finite−difference approximations for 2nd derivatives

$$\frac{\partial^2 T}{\partial x^2} = \frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{\Delta x^2} \qquad O(\Delta x^2)$$

$$\frac{\partial^2 T}{\partial y^2} = \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{\Delta y^2} \qquad O(\Delta y^2)$$

- Substitute into Laplace equation to give

$$\frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{\Delta x^2} + \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{\Delta y^2} = 0$$

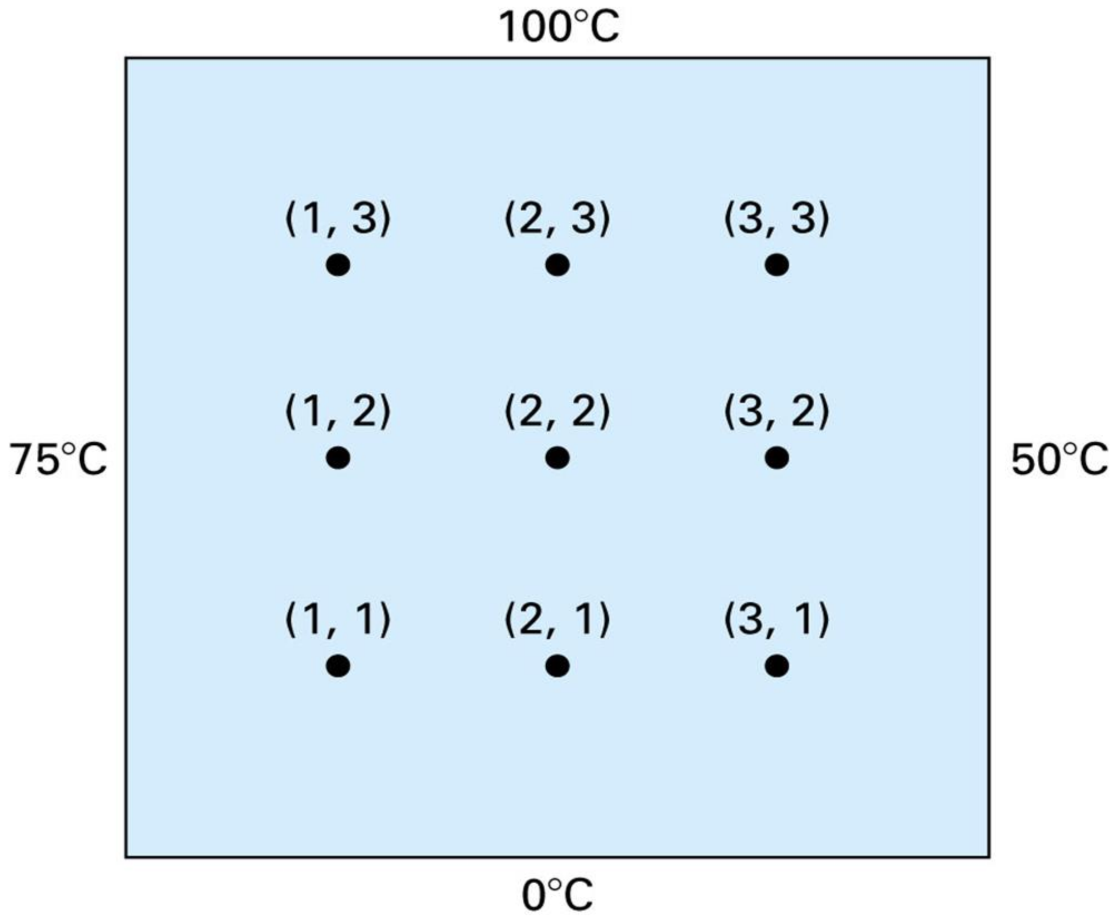
THE LAPLACIAN DIFFERENCE EQUATION

- If $\Delta x = \Delta y$, the result is the *Laplacian difference equation*

$$T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1} - 4T_{i,j} = 0$$

- This applies to all interior nodes of the grid.

GRID NUMBERING SCHEME



BOUNDARY CONDITIONS

- In addition, boundary conditions along the edges must be specified to obtain a unique solution.
- The simplest case is where the temperature at the boundary is set at a fixed value, *Dirichlet boundary condition*.
- A balance for node (1,1) is:

$$T_{21} + T_{01} + T_{12} + T_{10} - 4T_{11} = 0$$

$$T_{01} = 75$$

$$T_{10} = 0$$

$$-4T_{11} + T_{12} + T_{21} = -75$$

- Similar equations can be developed for other interior points to result a set of simultaneous equations.

SIMULTANEOUS EQUATIONS

- The result is a set of nine simultaneous equations with nine unknowns:

$$
\begin{array}{cccccccccl}
4T_{11} & -T_{21} & & -T_{12} & & & & & & =75 \\
-T_{11} & +4T_{21} & -T_{13} & & -T_{22} & & & & & =0 \\
 & -T_{21} & +4T_{31} & & & -T_{32} & & & & =50 \\
-T_{11} & & & +4T_{12} & -T_{22} & & -T_{13} & & & =75 \\
 & -T_{21} & & -T_{12} & +4T_{22} & -T_{32} & & -T_{23} & & =0 \\
 & & -T_{31} & & -T_{22} & +4T_{32} & & & -T_{33} & =50 \\
 & & & -T_{12} & & & +4T_{13} & -T_{23} & & =175 \\
 & & & & -T_{22} & & -T_{13} & +4T_{23} & -T_{33} & =100 \\
 & & & & & -T_{32} & & -T_{23} & +4T_{33} & =150
\end{array}
$$

THE LIEBMANN METHOD

- Most numerical solutions of Laplace equation involve systems that are very large.
- For larger size grids, a significant number of terms will be zero.
- For such sparse systems, most commonly employed approach is *Gauss-Seidel*, which when applied to PDEs is also referred as *Liebmann's method*.

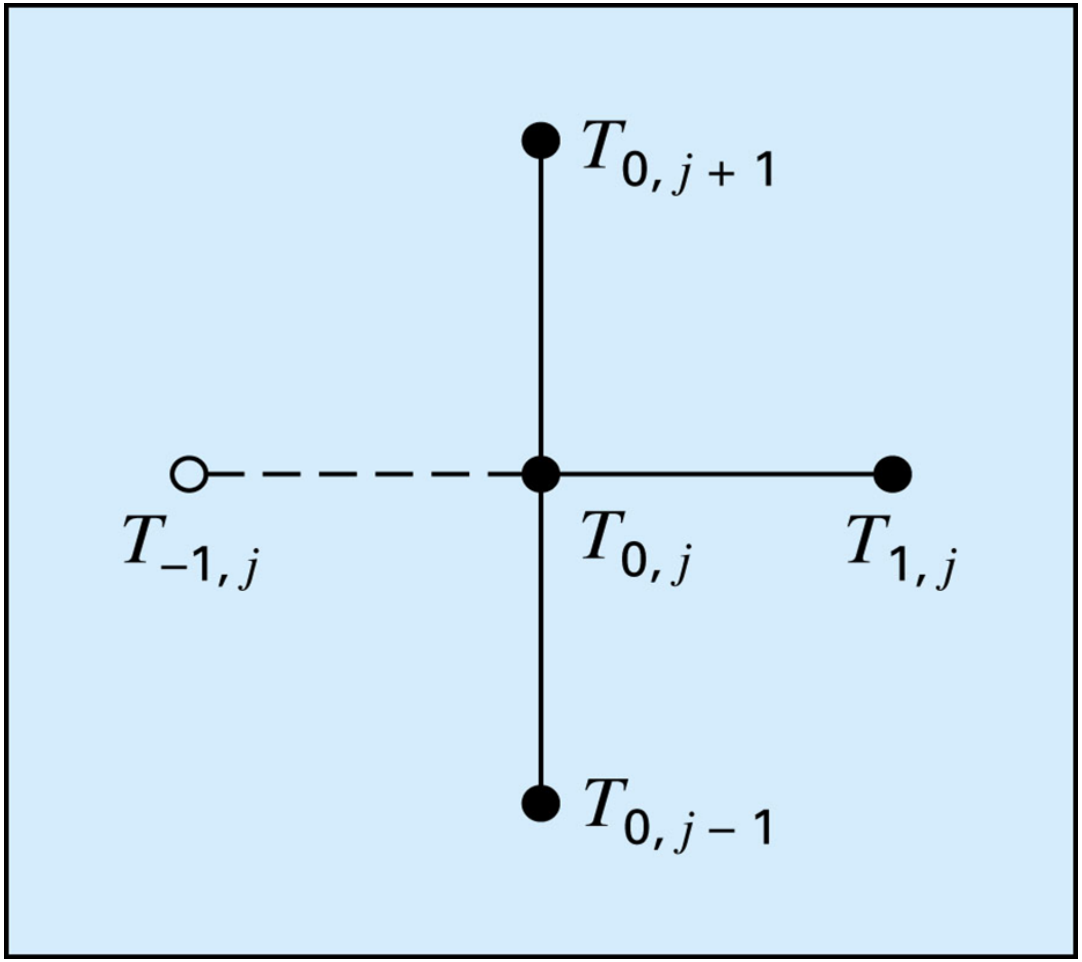
BOUNDARY CONDITIONS

- We will address problems that involve boundaries at which the derivative is specified and boundaries that are irregularly shaped.

Derivative Boundary Conditions

- Known as a *Neumann boundary condition*.
- For the heated plate problem, heat flux is specified at the boundary, rather than the temperature.
- If the edge is insulated, this derivative becomes zero.

BOUNDARY CONDITION AT LOWER LEFT CORNER



DERIVATIVE BOUNDARY CONDITION

$$T_{1,j} + T_{-1,j} + T_{0,j+1} + T_{0,j-1} - 4T_{0,j} = 0$$

$$\frac{\partial T}{\partial x} \cong \frac{T_{1,j} - T_{-1,j}}{2\Delta x}$$

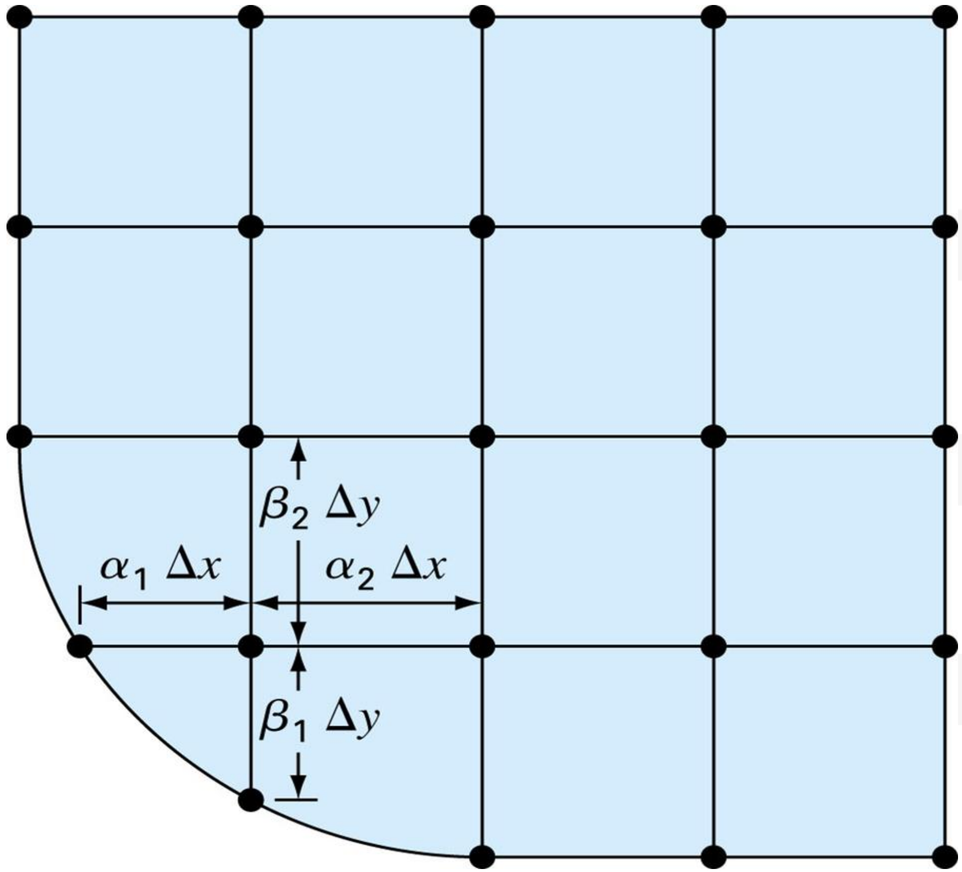
$$T_{-1,j} = T_{1,j} - 2\Delta x \frac{\partial T}{\partial x}$$

$$2T_{1,j} - 2\Delta x \frac{\partial T}{\partial x} + T_{0,j+1} + T_{0,j-1} - 4T_{0,j} = 0$$

- Thus, the derivative has been incorporated into the balance.
- Similar relationships can be developed for derivative boundary conditions at the other edges.

IRREGULAR BOUNDARIES

- Many problems exhibit irregular boundaries.



IRREGULAR BOUNDARIES

- First derivatives in the *x* direction can be approximated as:

$$\left(\frac{\partial T}{\partial x}\right)_{i-1,i} \cong \frac{T_{i,j}-T_{i-1,j}}{\alpha_1 \Delta x}$$

$$\left(\frac{\partial T}{\partial x}\right)_{i,i+1} \cong \frac{T_{i+1,j}-T_{i,j}}{\alpha_2 \Delta x}$$

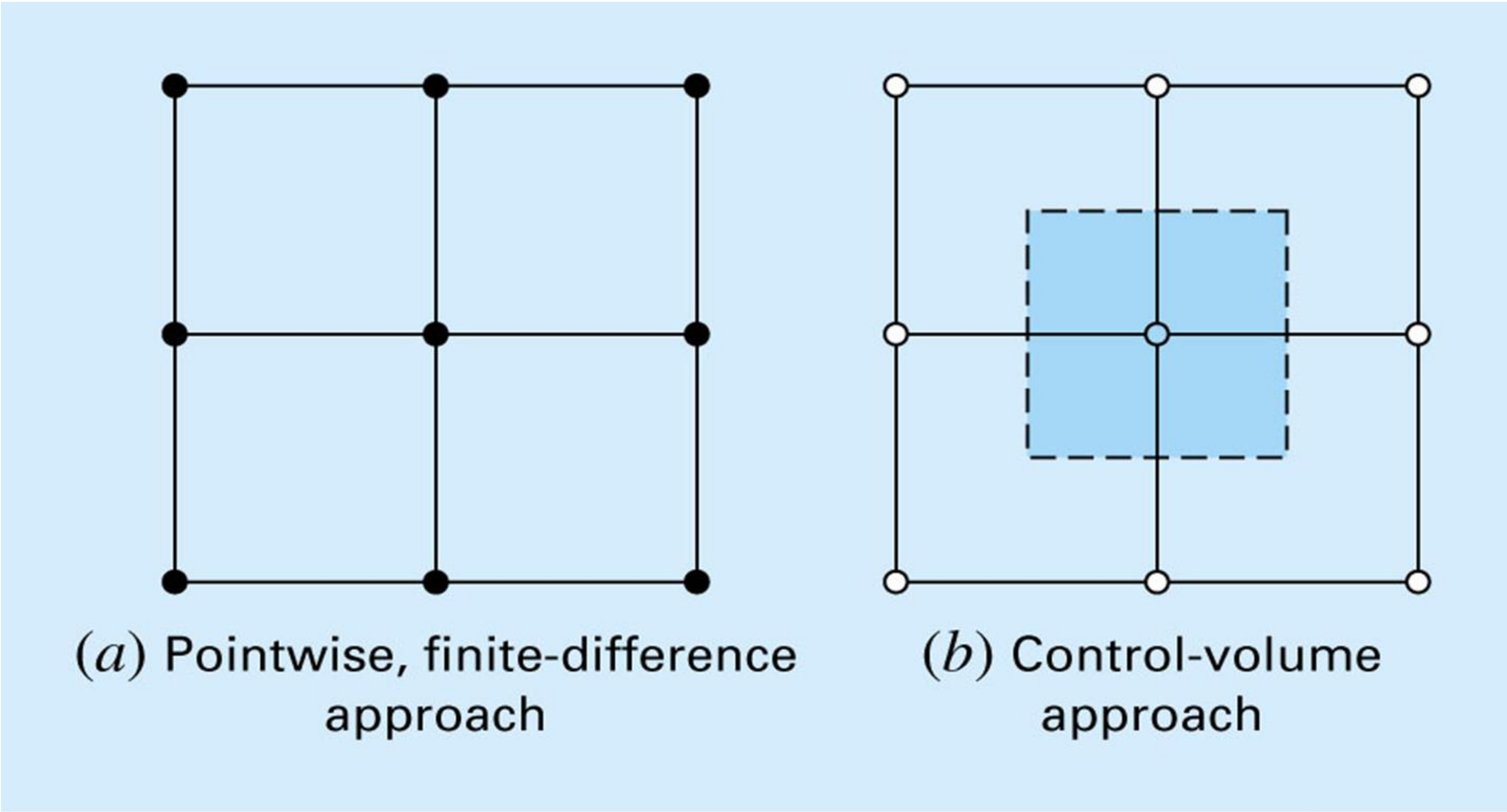
$$\frac{\partial^2 T}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right) = \frac{\left(\frac{\partial T}{\partial x}\right)_{i,i+1} - \left(\frac{\partial T}{\partial x}\right)_{i-1,i}}{\frac{\alpha_1 \Delta x + \alpha_2 \Delta x}{2}}$$

$$\frac{\partial^2 T}{\partial x^2} = 2\frac{\frac{T_{i,j}-T_{i-1,j}}{\alpha_1 \Delta x} - \frac{T_{i+1,j}-T_{i,j}}{\alpha_2 \Delta x}}{\frac{\alpha_1 \Delta x + \alpha_2 \Delta x}{2}}$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{2}{\Delta x^2}\left[\frac{T_{i-1,j}-T_{i,j}}{\alpha_1(\alpha_1+\alpha_2)} + \frac{T_{i+1,j}-T_{i,j}}{\alpha_2(\alpha_1+\alpha_2)}\right]$$

- A similar equation can be developed in the *y* direction.

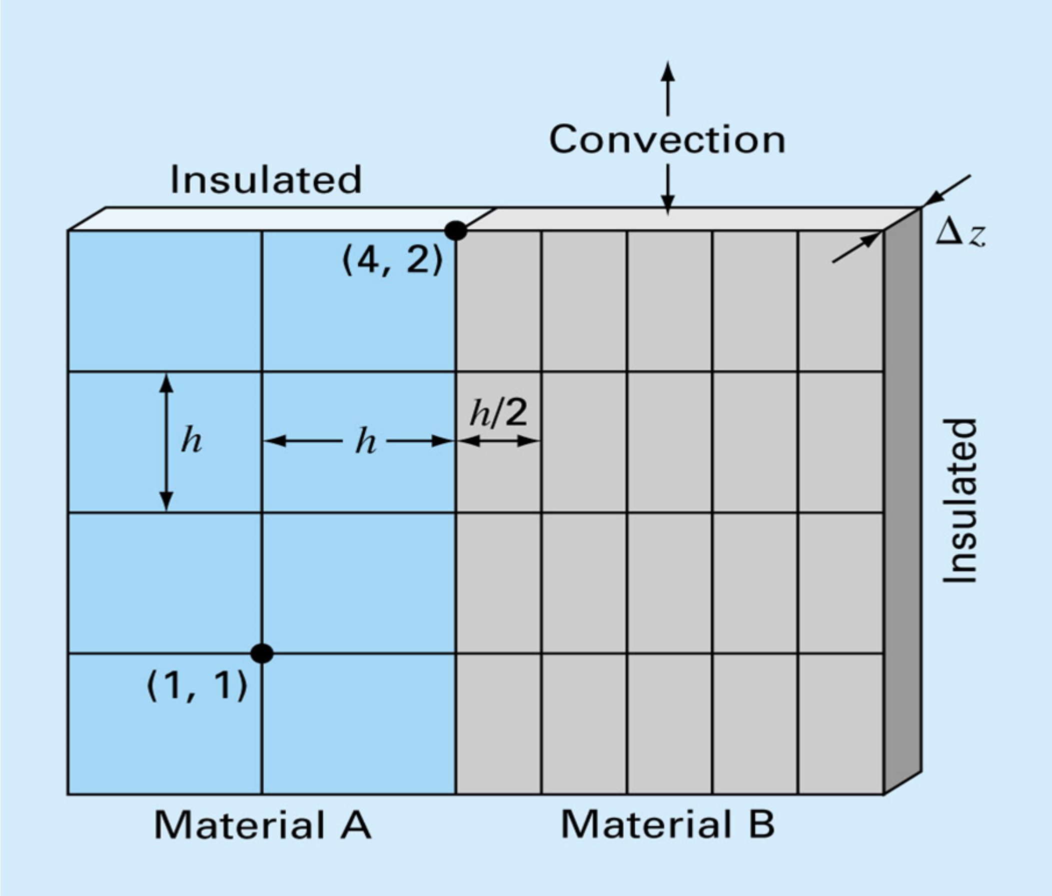
CONTROL-VOLUME APPROACH



CONTROL-VOLUME APPROACH

- The *control-volume approach* resembles the point-wise approach in that points are determined across the domain.
- In this case, rather than approximating the PDE at a point, the approximation is applied to a volume surrounding the point.

EXAMPLE: A HEATED PLATE WITH UNEQUAL GRID SPACING, TWO MATERIALS, AND MIXED BOUNDARY CONDITIONS.

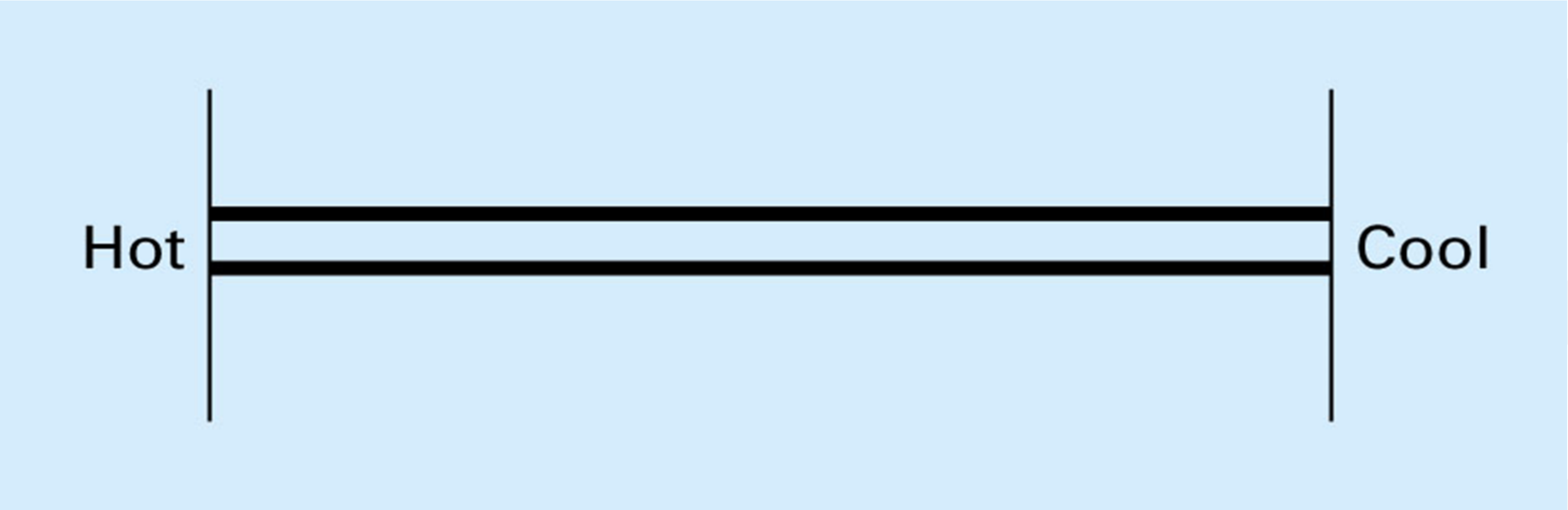


FINITE DIFFERENCE: PARABOLIC EQUATIONS

- Parabolic equations are employed to characterize time-variable (*unsteady-state*) problems.

- Conservation of energy can be used to develop an *unsteady-state* energy balance for the differential element in a long, thin insulated rod.

AN INSULATED THIN ROD



A thin rod, insulated at all points except at its ends.

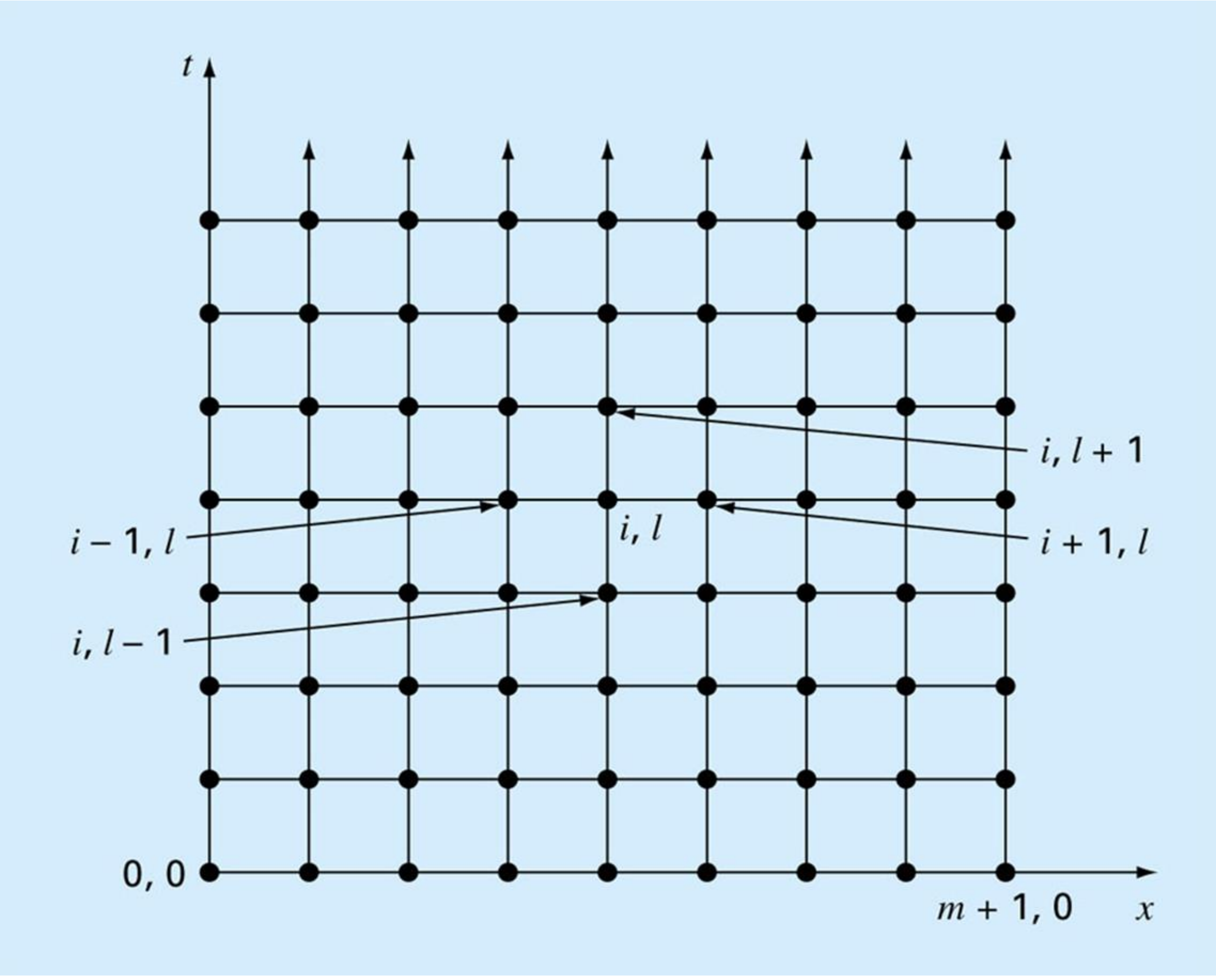
FINITE DIFFERENCE APPROACHES FOR PARABOLIC PDES

- Energy balance together with Fourier's law of heat conduction yields the *heat-conduction equation*:

$$k \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}$$

- Just as elliptic PDEs, parabolic equations can be solved by substituting finite divided differences for the partial derivatives, we can use a similar approach for parabolic equations.
- But, in contrast to elliptic PDEs, we must now consider changes in time as well as in space.
- Parabolic PDEs are temporally open-ended and involve new issues such as stability.

FINITE DIFFERENCE GRID FOR PARABOLIC PDES



EXPLICIT METHODS

- The heat conduction equation requires approximations for the second derivative in space and the first derivative in time:

$$\frac{\partial^2 T}{\partial x^2} = \frac{T_{i+1}^{l} - 2T_i^{l} + T_{i-1}^{l}}{\Delta x^2}$$

$$\frac{\partial T}{\partial t} = \frac{T_i^{l+1} - T_i^{l}}{\Delta t}$$

$$k\frac{T_{i+1}^{l} - 2T_i^{l} + T_{i-1}^{l}}{(\Delta x)^2} = \frac{T_i^{l+1} - T_i^{l}}{\Delta t}$$

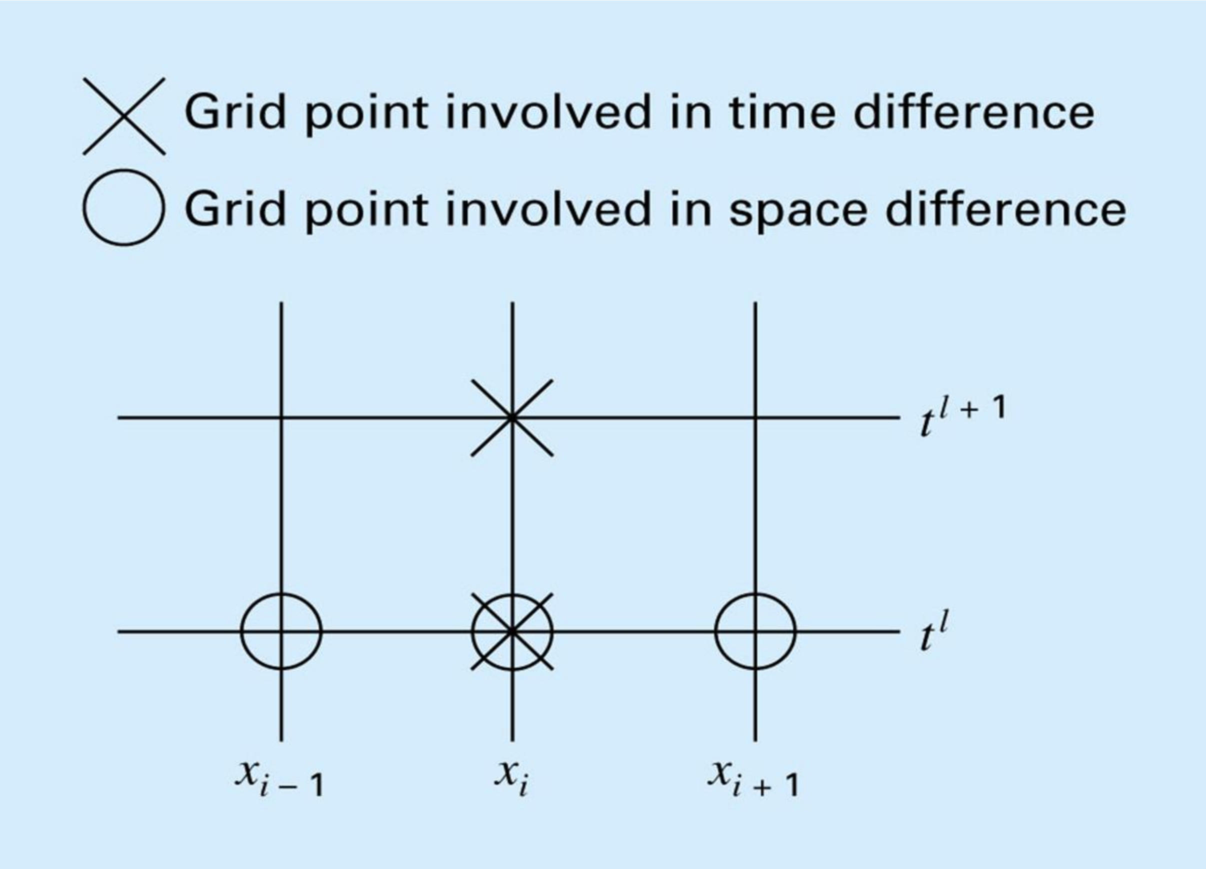
$$\boxed{T_i^{l+1} = T_i^{l} + \lambda\left(T_{i+1}^{l} - 2T_i^{l} + T_{i-1}^{l}\right)}$$

$$\lambda = \frac{k\Delta t}{(\Delta x)^2}$$

EXPLICIT METHODS

- This equation can be written for all interior nodes on the rod.
- It provides an explicit means to compute values at each node for a future time based on the present values at the node and its neighbors.

EXPLICIT METHODS



CONVERGENCE AND STABILITY

- Convergence means that as Dx and Dt approach zero, the results of the finite difference method approach the true solution.
- Stability means that errors at any stage of the computation are not amplified but are attenuated as the computation progresses.
- The explicit method is both convergent and stable if

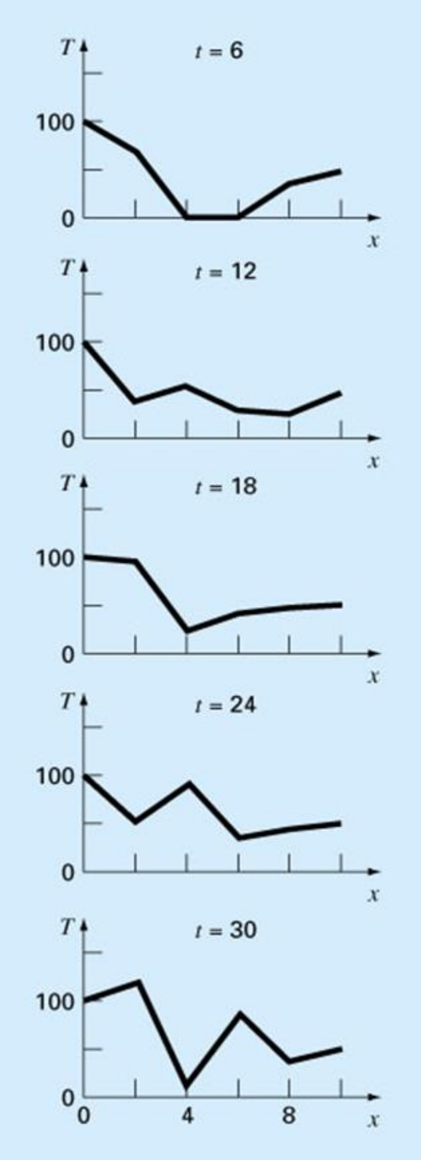
$$\lambda \leq 1/2$$

or

$$\Delta t \leq \frac{1}{2}\frac{\Delta x^2}{k}$$

INSTABILITY OF THE EXPLICIT METHOD

An example of instability caused by violating the stability criterion for the explicit scheme. The solution undergoes progressively increasing oscillations which will continue to deteriorate as the computation proceeds.



DERIVATIVE BOUNDARY CONDITIONS

- As was the case for elliptic PDEs, derivative boundary conditions can be readily incorporated into parabolic equations.

$$T_0^{l+1} = T_0^l + \lambda\left(T_1^l - 2T_0^l + T_{-1}^l\right)$$

- Thus an imaginary point is introduced at *i*= −1, providing a vehicle for incorporating the derivative boundary condition into the analysis.

A SIMPLE IMPLICIT METHOD

- Implicit methods overcome difficulties associated with explicit methods at the expense of somewhat more complicated algorithms.
- In implicit methods, the spatial derivative is approximated at an advanced time, $l + 1$:

$$\frac{\partial^2 T}{\partial x^2} \cong \frac{T_{i+1}^{l+1} - 2T_i^{l+1} + T_{i-1}^{l+1}}{\left(\Delta x\right)^2}$$

- which is second-order accurate.

A SIMPLE IMPLICIT METHOD

Substitution into the heat conduction equation gives

$$k\frac{T_{i+1}^{l+1}-2T_i^{l+1}+T_{i-1}^{l+1}}{\left(\Delta x\right)^2}=\frac{T_i^{l+1}-T_i^l}{\Delta t}$$

which can be expressed as

$$-\lambda T_{i-1}^{l+1}+\left(1+2\lambda\right)T_i^{l+1}-\lambda T_{i+1}^{l+1}=T_i^l \qquad (1)$$

This eqn. applies to all but the first and the last interior nodes, which must be modified to reflect the boundary conditions. For case where temperatures at the ends of bar are given, the boundary condition at the left end of bar (i = 0) is

$$T_0^{l+1}=f_0\left(t^{l+1}\right)$$

A SIMPLE IMPLICIT METHOD

- which can be substituted into Eq. (1) with i = 1 to give

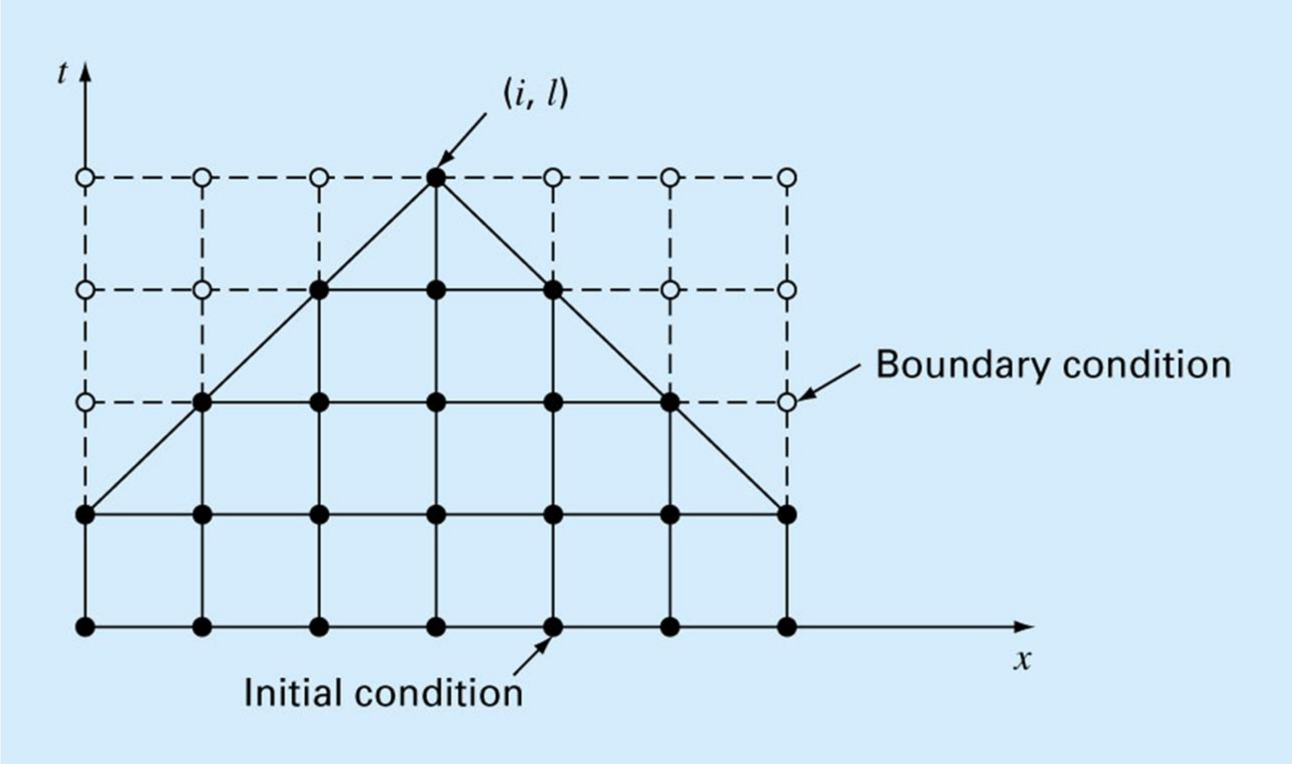
$$(1+2\lambda)T_1^{l+1} - \lambda T_2^{l+1} = T_1^l + f_0\left(t^{l+1}\right)$$

- Similarly, for the last interior node ($i = m$)

$$(1+2\lambda)T_m^{l+1} - \lambda T_{m-1}^{l+1} = T_m^l + f_{m+1}\left(t^{l+1}\right)$$

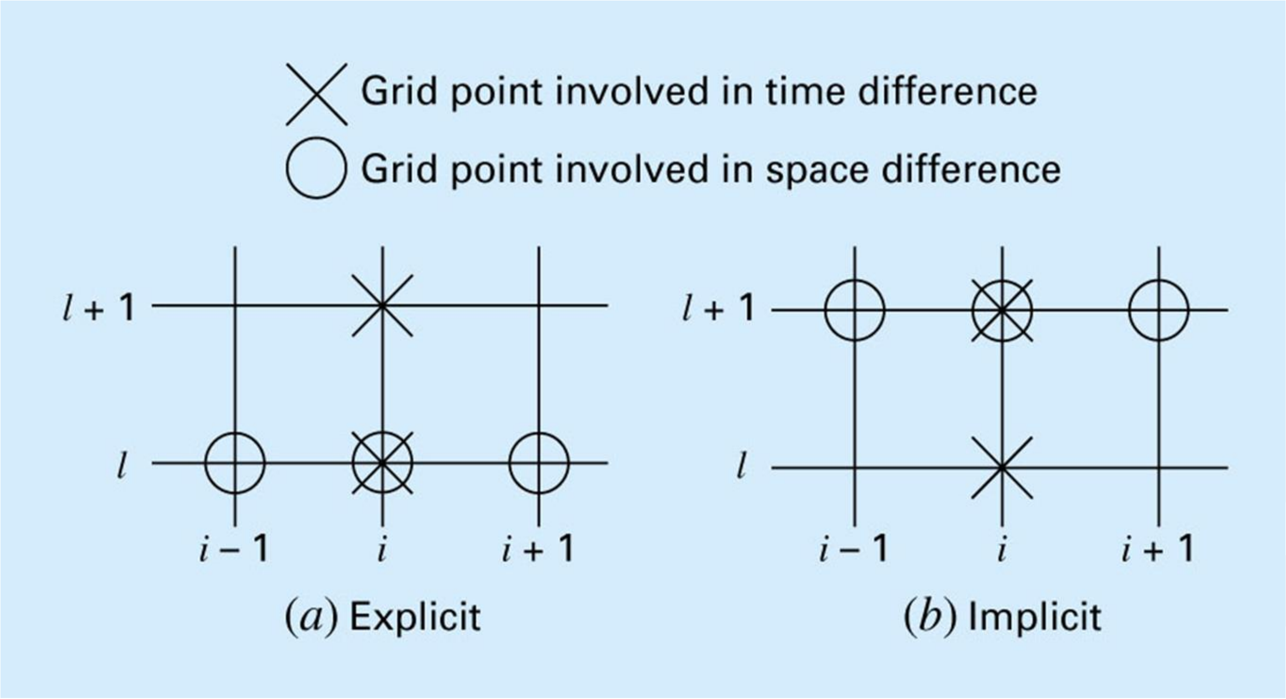
- The result is m linear algebraic equations with m unknowns

THE ROOT CAUSE OF INSTABILITY OF EXPLICIT FINITE DIFFERENCE SCHEMES



The effect of other nodes on the finite difference approximation at node (*i*, *l*) using an explicit finite-difference scheme.

COMPUTATIONAL MOLECULES FOR (A) EXPLICIT AND (B) IMPLICIT METHODS.



THE CRANK-NICOLSON METHOD

- Provides an alternative implicit scheme that is second order accurate both in space and time.
- To provide this accuracy, difference approximations are developed at the midpoint of the time increment:

$$\frac{\partial T}{\partial t} \cong \frac{T_i^{l+1} - T_i^l}{\Delta t}$$

$$\frac{\partial^2 T}{\partial x^2} \cong \frac{1}{2}\left[\frac{T_{i+1}^l - 2T_i^l + T_{i-1}^l}{(\Delta x)^2} + \frac{T_{i+1}^{l+1} - 2T_i^{l+1} + T_{i-1}^{l+1}}{(\Delta x)^2}\right]$$

PARABOLIC EQUATIONS IN TWO SPATIAL DIMENSIONS

- For two dimensions the heat-conduction equation can be applied more than one spatial dimension:

$$\frac{\partial T}{\partial t} = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

- As with the one-dimensional case, both explicit and implicit schemes have been developed
- An *explicit* solution can be obtained by substituting finite-differences approximations for the partial derivatives. However, this approach is limited by a stringent stability criterion, thus increases the required computational effort.

PROBLEM WITH IMPLICIT SCHEMES IN TWO DIMENSIONS

- The direct application of *implicit* methods leads to solution of *m x n* simultaneous equations.
- When written for two or three dimensions, these equations lose the valuable property of being tridiagonal and require very large matrix storage and computation time.

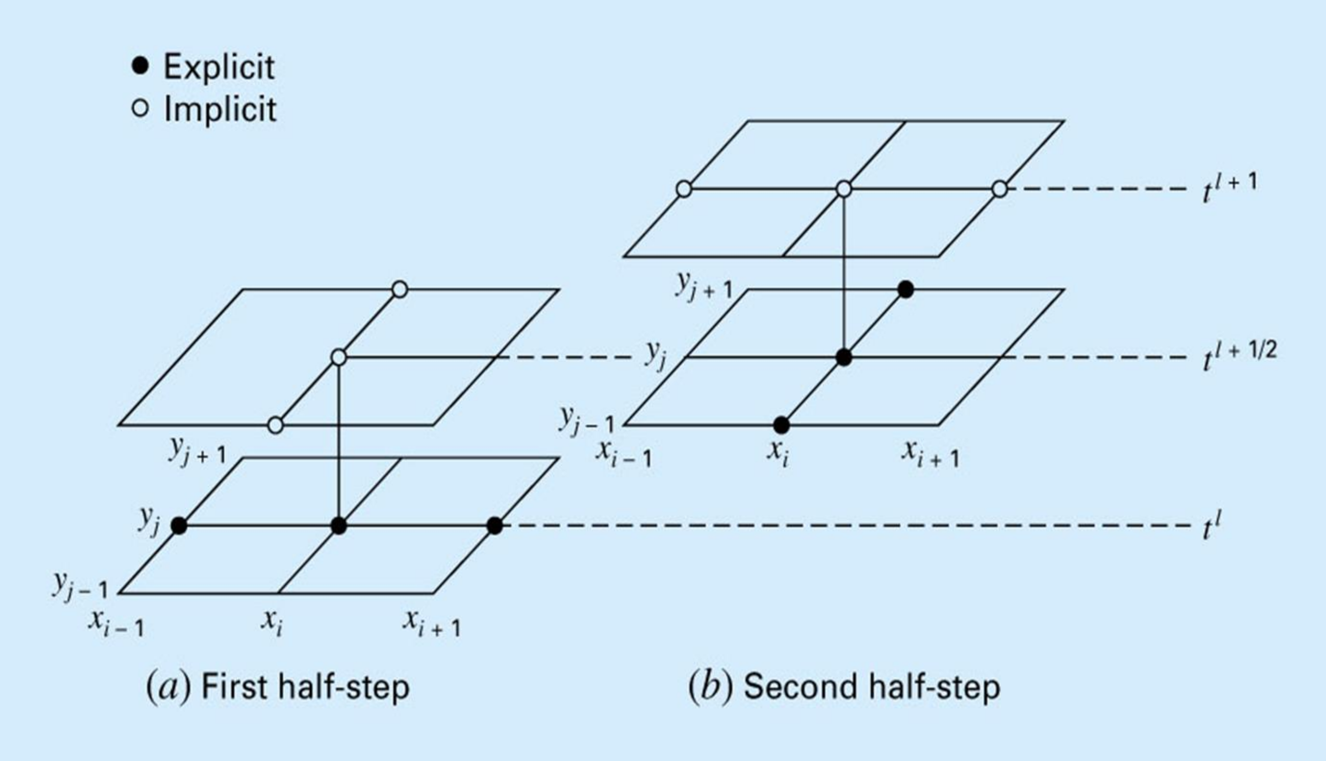
THE ADI SCHEME

- The alternating-direction implicit, or ADI, scheme provides a means for solving parabolic equations in two spatial dimensions using tridiagonal matrices.
- Each time increment is executed in two steps.
- For the first step, heat conduction equation is approximated by:

$$\frac{T_{i,j}^{l+1/2} - T_{i,j}^{l}}{\Delta t / 2} = k\left[\frac{T_{i+1,j}^{l} - 2T_{i,j}^{l} + T_{i-1,j}^{l}}{\left(\Delta x\right)^2} + \frac{T_{i,j+1}^{l+1/2} - 2T_{i,j}^{l+1/2} + T_{i,j-1}^{l+1/2}}{\left(\Delta y\right)^2}\right]$$

- Thus the, approximation of partial derivatives are written explicitly, that is at the base point t^l where temperatures are known. Consequently only the three temperature terms in each approximation are unknown.

THE ADI SCHEME



The two half-steps used in implementing the ADI scheme for solving parabolic equations in two spatial dimensions.

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