MTH113TC: Intro. to Probability and Statistics

Lesson 3 - Random variables and their distributions - Part II

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3. Continuous random variables

Continuous random variables (1)

In this lesson we will learn some random variables whose set of possible values is uncountable.

Definition

Let $M \subseteq \mathbb{R}$ be a interval or a union of intervals and Ω be a sample space. A function $X : \Omega \to M$ is called a **continuous random variable** if there exists an integrable function $f_X : \mathbb{R} \to \mathbb{R}$ such that

$$f_X(x) \geq 0, \forall x \in M,$$

$$\int_{-\infty}^{+\infty} f_X(x)\mathrm{dx} = 1; \tag{1}$$

for which we have

$$F_X(x) := \mathbb{P}(X \leq x) = \int_{-\infty}^{x} f_X(y)\mathrm{dy}.$$

Continuous random variables (2)

The function f_X is called the **probability density function (p.d.f) of** X and F_X is called the **cumulative distribution function (c.d.f) of** X.

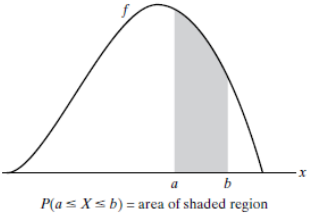


Figure: Probability density function f.

Continuous random variables (3)

Proposition

Let f_X be a p.d.f of a continuous random variable X. Then $\forall a < b \in M$.

$$\mathbb{P}(a \leq X \leq b) = F_X(b) - F_X(a) = \int_a^b f_X(x)\mathrm{dx}.$$

In particular, $\mathbb{P}(X = a) = 0$ for all $a \in M$, and

$$\mathbb{P}(a \leq X \leq b) = \mathbb{P}(a \leq X < b) = \mathbb{P}(a < X \leq b) = \mathbb{P}(a < X < b).$$

Continuous random variables (4)

Proof.

$$\begin{aligned}\mathbb{P}(a \le X \le b) &= \mathbb{P}(X \le b) - \mathbb{P}(X \le a)\\ &= F_X(b) - F_X(a)\\ &= \int_{-\infty}^{b} f_X(x)\mathrm{dx} - \int_{-\infty}^{\mathrm{a}} \mathrm{f_X(x)dx}\\ &= \int_{a}^{b} f_X(x)\mathrm{dx}.\end{aligned}$$

Clearly, the event $\{X = a\}$ is the same as $a \le X \le a$.
Choosing $a = b$ above, we obtain
$\mathbb{P}(X = a) = \int_a^a f_X(x)\mathrm{dx} = 0$. So
$\mathbb{P}(a \le X < b) = \mathbb{P}(a \le X \le b) - \mathbb{P}(X = b) = \mathbb{P}(a \le X \le b)$
etc. □

Continuous random variables (5)

Remark

From the proposition above, whenever the c.d.f F_X is differentiable, we have the following relation

$$f_X(x) = \frac{\mathrm{d}\, F_X(x)}{\mathrm{d}\, x} = F_X'(x).$$

3.1. Statistic characteristic of continuous random variables

Definition (Expectation/Mean)

Let X be a continuous random variable with p.d.f f_X . The real number

$$\mathbb{E}[X] := \int_{-\infty}^{+\infty} x f_X(x)\mathrm{dx}$$

is called the **expectation (or expected value, mean)** of X whenever the integral exists.

Remark:

(1) More generally, given a continuous random variable X and a function $h : M \to \mathbb{R}$. The expectation of $h(X)$ is defined as

$$\mathbb{E}[h(X)] := \int_{-\infty}^{+\infty} h(x) f_X(x)\mathrm{dx}.$$

(2) For instance, if we choose $h(X) = aX + b$, where $a, b \in \mathbb{R}$ are constants. Then we have

$$\begin{aligned}
\mathbb{E}(aX+b) &\overset{\text{def.}}{=} \int_{-\infty}^{+\infty} (ax+b) f_X(x)\mathrm{dx} \\
&= a\int_{-\infty}^{+\infty} x f_X(x)\mathrm{dx} + \mathrm{b} \underbrace{\int_{-\infty}^{+\infty} f_X(x)\mathrm{dx}}_{=1 \text{ by } (1)} \\
&= a\mathbb{E}[X] + b\cdot 1 = a\mathbb{E}(X) + b.
\end{aligned}$$

Or more generally, we have the **linear property** of taking expectations, i.e.,

$$\mathbb{E}[ah(X) + bg(Y)] = a\mathbb{E}[h(X)] + b\mathbb{E}[g(Y)],$$

where h, g are two given functions and X, Y are two continuous random variables.

Definition (Variance and Standard deviation)

Let X be a continuous random variable with p.d.f f_X. The positive real number

$$\mathsf{Var}(X) := \mathbb{E}[(X - \mathbb{E}[X])^2] = \int_{-\infty}^{+\infty} (x - \mathbb{E}[X])^2 f_X(x) \mathrm{dx}$$

is called the **variance** of X whenever the integral exists. The square root of the variance $\sqrt{\mathsf{Var}(X)}$ is called the **standard deviation** of X.

Remark

Similarly as for a discrete random variable, an alternative formula for Var(X) in continuous case is as follows:

$$\text{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \int_{-\infty}^{+\infty} x^2 f_X(x)\mathrm{dx} - \left(\int_{-\infty}^{+\infty} x f_X(x)\mathrm{dx}\right)^2$$

As for a discrete random variable, the variance of a continuous random variable also has the following property (prove it, as exercise):

Theorem

Let X be a continuous random variable. For any $a, b \in \mathbb{R}$,

$$\mathit{Var}(aX + b) = a^2 \mathit{Var}(X).$$

Example

Find the constant C for the following p.d.f with $k > 0$,

$$f(x) := \begin{cases} C\mathrm{e}^{-kx}, & \text{if } x \geq 0, \\ 0, & \text{otherwise,} \end{cases}$$

and the expectation and variance.

Solution: We should have

$$1 = \int_{-\infty}^{+\infty} f(x)\mathrm{dx} = C\int_0^{+\infty} \mathsf{e}^{-kx}\mathrm{dx} = C\left[\frac{-\mathsf{e}^{-kx}}{k}\right]_0^{+\infty} = \frac{C}{k},$$

which gives $C = k$. For expectation we have by integration by parts

$$\begin{aligned}\mathbb{E}[X] &= \int_{-\infty}^{+\infty} xf(x)\mathrm{dx} = \int_0^{+\infty} kx\mathsf{e}^{-kx}\mathrm{dx} = -\int_0^{+\infty} x\mathrm{d}(\mathsf{e}^{-\mathrm{kx}}) \\ &= \Big[(-x)\mathsf{e}^{-kx}\Big]_0^{+\infty} - \frac{1}{k}\int_0^{+\infty}\mathsf{e}^{-kx}\mathrm{d}(-\mathrm{kx}) \\ &= \Big[(-x)\mathsf{e}^{-kx}\Big]_0^{+\infty} - \frac{1}{k}\Big[\mathsf{e}^{-kx}\Big]_0^{+\infty} \\ &= 0 - \left(-\frac{1}{k}\right) = \frac{1}{k}.\end{aligned}$$

Finally, we have

$$\begin{aligned}
\mathbb{E}[X^2] &= \int_{-\infty}^{+\infty} x^2 f(x)\mathrm{dx} = \int_0^{+\infty} kx^2 e^{-kx}\mathrm{dx} \\
&= -\int_0^{+\infty} x^2 \mathrm{d}(e^{-\mathrm{kx}}) \\
&= \Big[(-x^2)e^{-kx}\Big]_0^{+\infty} - \int_0^{+\infty} e^{-kx}\mathrm{d}(-\mathrm{x}^2), \text{ using IBP} \\
&= \Big[(-x^2)e^{-kx}\Big]_0^{+\infty} + 2\int_0^{+\infty} x e^{-kx}\mathrm{dx} \\
&= \Big[(-x^2)e^{-kx}\Big]_0^{+\infty} + \frac{2}{k}\int_0^{+\infty} kx e^{-kx}\mathrm{dx} \\
&= \Big[(-x^2)e^{-kx}\Big]_0^{+\infty} + \frac{2}{k}\mathbb{E}[X] = 0 + \frac{2}{k}\cdot\frac{1}{k} = \frac{2}{k^2}.
\end{aligned}$$

Thus, $\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \dfrac{2}{k^2} - \dfrac{1}{k^2} = \dfrac{1}{k^2}.$

3.2. Specific types of continuous random variables

3.2.1. Uniform distribution

Uniform distribution (1)

Definition (Uniform distribution)

Let $M = (a, b) \subseteq \mathbb{R}$. The random variable $X : \Omega \to (a, b)$ is said to be **uniformly** distributed over the interval (a, b) if its p.d.f is given by

$$f_X(x) := \begin{cases} \dfrac{1}{b-a}, & \text{if } x \in (a, b) \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

Note that Equation (2) is a density function, since $f_X(x) \geq 0$ and

$$\int_{-\infty}^{+\infty} f_X(x)\mathrm{dx} = \int_a^b \frac{1}{b-a}\mathrm{dx} = \frac{1}{b-a}(b-a) = 1.$$

Uniform distribution (2)

Theorem

Let X be a uniformly distributed random variable over the interval (a, b). Then

(1) *Its c.d.f is given by*

$$F_X(x) := \begin{cases} 0, & \text{if } x \leq a, \\ \dfrac{x-a}{b-a}, & \text{if } x \in (a,b), \\ 1, & \text{otherwise.} \end{cases}$$

(2) *For expectation and variance we have*

$$\mathbb{E}[X] = \frac{a+b}{2}, \quad Var(X) = \frac{(a-b)^2}{12}.$$

Uniform distribution (3)

Proof:

(1) If $x \leq a$, then $F_X(x) = 0$.
If $x \in (a, b)$, we have

$$F_X(x) = \int_{-\infty}^{x} f_X(y)\mathrm{dy} = \underbrace{\int_{-\infty}^{a} f_X(y)\mathrm{dy}}_{=0} + \int_{a}^{x} f_X(y)\mathrm{dy}$$

$$= \int_{a}^{x} \frac{1}{b-a}\mathrm{dy} = \frac{x-a}{b-a}.$$

Uniform distribution (4)

If $x \geq b$,

$$F_X(x) = \int_{-\infty}^{x} f_X(y)\mathrm{dy}$$
$$= \underbrace{\int_{-\infty}^{a} f_X(y)\mathrm{dy}}_{=0} + \int_{a}^{b} f_X(y)\mathrm{dy} + \underbrace{\int_{b}^{x} f_X(y)\mathrm{dy}}_{=0}$$
$$= \int_{a}^{b} \frac{1}{b-a}\mathrm{dy} = 1.$$

(2) By the definition of expectation we have

$$\mathbb{E}[X] = \int_{-\infty}^{+\infty} x f_X(x)\mathrm{dx} = \frac{1}{b-a}\int_{a}^{b} x\mathrm{dx}$$
$$= \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b = \frac{1}{b-a}\frac{b^2-a^2}{2} = \frac{a+b}{2}.$$

Uniform distribution (5)

Moreover,

$$\begin{aligned}\mathbb{E}[X^2] &= \int_{-\infty}^{+\infty} x^2 f_X(x)\mathrm{dx} = \frac{1}{b-a}\int_a^b x^2\mathrm{dx} \\ &= \frac{1}{b-a}\left[\frac{x^3}{3}\right]_a^b = \frac{1}{b-a}\frac{b^3-a^3}{3} = \frac{a^2+ab+b^2}{3}.\end{aligned}$$

Therefore,

$$\mathsf{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \frac{(a-b)^2}{12}.$$

Uniform distribution (6)

Example

If X is uniformly distributed over $(0, 10)$, calculate the probability that (a) $X < 3$, (b) $X > 6$, and (c) $3 < X < 8$.

Solution:

(a) $\mathbb{P}(X < 3) = \int_0^3 \frac{1}{10}\mathrm{dx} = \frac{3}{10}$.

(b) $\mathbb{P}(X > 6) = \int_6^{10} \frac{1}{10}\mathrm{dx} = \frac{4}{10}$.

(c) $\mathbb{P}(3 < X < 8) = \int_3^8 \frac{1}{10}\mathrm{dx} = \frac{1}{2}$.

3.2.2. Exponential random variables

Exponential random variables (1)

Definition (Exponential random variable)

Let $M = (0, +\infty)$ and $\theta > 0$. The random variable $X : \Omega \to M$ is said to be **exponential** distributed with parameter θ if its p.d.f is given by

$$f_X(x) := \begin{cases} \frac{1}{\theta}\mathrm{e}^{-x/\theta}, & \text{if } x \in [0, +\infty), \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

Note that Equation (3) is a density function, since $f_X(x) \geq 0$ and

$$\int_{-\infty}^{+\infty} f_X(x)\mathrm{dx} = \frac{1}{\theta}\int_0^{+\infty} \mathrm{e}^{-x/\theta}\mathrm{dx} \overset{y=x/\theta}{=} \int_0^{+\infty} \mathrm{e}^{-y}\mathrm{dy} = 1.$$

Exponential random variables (2)

This distribution very often occurs in practice as description of the time elapsing between unpredictable events (such as telephone calls, earthquakes, emissions of radioactive particles, and arrival of buses, and so on).

Theorem

Let X be an exponential distributed random variable with parameter θ. We have

(1) *Its c.d.f is given by* $F_X(x) = \begin{cases} 1 - e^{-x/\theta}, & \textit{if } x > 0, \\ 0, & \textit{otherwise.} \end{cases}$

(2) *For expectation and variance, we have*

$$\mathbb{E}[X] = \theta, \quad \mathit{Var}(X) = \theta^2.$$

Exponential random variables (3)

Proof:

(1) For the c.d.f, we have if $x \leq 0$, then $F_X(x) = 0$. If $x > 0$,

$$F_X(x) := \int_{-\infty}^{x} f_X(y)\mathrm{dy} = \int_0^x e^{-y/\theta}\mathrm{d(y/\theta)}$$

$$\overset{u=y/\theta}{=} \int_0^{x/\theta} e^{-u}\mathrm{du} = \left[-e^{-u}\right]_0^{x/\theta} = 1 - e^{-x/\theta}.$$

(2) For the expectation, using integration by parts, we have

$$\mathbb{E}[X] = \int_{-\infty}^{+\infty} x f_X(x)\mathrm{dx} = \theta \int_0^{+\infty} \frac{x}{\theta} e^{-x/\theta}\mathrm{d(x/\theta)}$$

$$\overset{y=x/\theta}{=} \theta \int_0^{+\infty} y e^{-y}\mathrm{dy}$$

$$= \theta \left(\underbrace{\left[-ye^{-y}\right]_0^{+\infty}}_{=0} + \underbrace{\int_0^{+\infty} e^{-y}\mathrm{dy}}_{=1} \right) = \theta.$$

Exponential random variables (4)

By a similar argument,

$$\begin{aligned}\mathbb{E}[X^2] = & \int_{-\infty}^{+\infty} x^2 f_X(x)\mathrm{dx} = \theta^2 \int_0^{+\infty} \frac{x^2}{\theta^2} e^{-x/\theta} \mathrm{d(x/\theta)} \\ \overset{y=x/\theta}{=} & \theta^2 \int_0^{+\infty} y^2 e^{-y} \mathrm{dy} \\ = & \theta^2 \left(\underbrace{\left[-y^2 e^{-y}\right]_0^{+\infty}}_{=0} + 2 \underbrace{\int_0^{+\infty} y e^{-y} \mathrm{dy}}_{=1} \right) = 2\theta^2.\end{aligned}$$

Thus,

$$\mathrm{Var}(X) = 2\theta^2 - \theta^2 = \theta^2.$$

And so the standard deviation is θ.

Exponential random variables (5)

Example

In a storm the time elapsed between two consecutive thunderbolts is an exponential random variable. Its standard deviation is 1 minute.

(1) What is the probability that the time gap between two thunderbolts is at most 2 mins?

(2) What is the probability that the time gap between two thunderbolts is at least 1 min?

Exponential random variables (6)

Solution: Denote by X the time gap in minutes between two consecutive thunderbolts. Using Theorem in Page 27, which says that both the mean and the standard deviation of an exponential random variable is θ, we conclude that its value is 1.

Hence, the c.d.f is

$$F_X(x) = 1 - e^{-x}, \quad x > 0.$$

Thus,

(1) $\mathbb{P}(X \leq 2) = F_X(2) = 1 - e^{-2} \approx 0.8647.$

(2)

$$\begin{aligned}\mathbb{P}(X \geq 1) =& 1 - \mathbb{P}(X < 1) = 1 - F_X(1) = 1 - (1 - e^{-1}) \\ =& e^{-1} \approx 0.3679.\end{aligned}$$

3.2.3. Normal random variables

Normal random variables (1)

Definition (Normal random variables)

We say that X is a **normal random variable or X is normally distributed**, with parameters μ and σ^2 if the p.d.f of X is given by

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}, \quad \text{for } -\infty < x < \infty.$$

This p.d.f is a bell-shaped curve that is symmetric about $x = \mu$ (See the figure below).

Normal random variables (2)

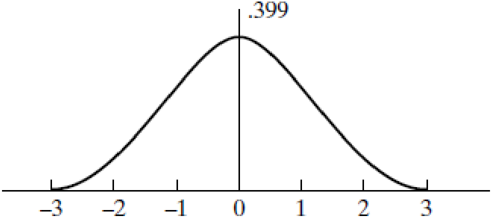


Figure: Normal density function: $\mu = 0$, $\sigma = 1$.

Normal random variables (3)

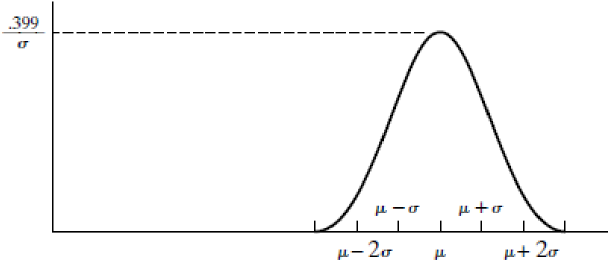


Figure: Normal density function: arbitrary μ, σ.

Normal random variables (4)

Remark:

(1) To prove that $f_X(x)$ is indeed a p.d.f, we need to show that

$$\frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{+\infty} e^{-(x-\mu)^2/2\sigma^2}\mathrm{dx} = 1.$$

Making the substitution $y = (x-\mu)/\sigma$,

$$LHS = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-y^2/2}\mathrm{dy}.$$

Hence, we must show that

$$\int_{-\infty}^{+\infty} e^{-y^2/2}\mathrm{dy} = \sqrt{2\pi}. \tag{4}$$

Toward this end, let $I = \int_{-\infty}^{+\infty} e^{-y^2/2}\mathrm{dy}$. Then

Normal random variables (5)

$$I^2 = \int_{-\infty}^{+\infty} e^{-y^2/2}\mathrm{dy} \int_{-\infty}^{+\infty} e^{-x^2/2}\mathrm{dx}$$
$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-(y^2+x^2)/2}\mathrm{dy}\,\mathrm{dx}.$$

We now evaluate the double integral by means of a change of variables to polar coordinates. (That is, let $x = r\cos\theta$, $y = r\sin\theta$, and $\mathrm{dydx} = r\,\mathrm{d}\theta\,\mathrm{dr}$.) Thus,

$$I^2 = \int_0^{+\infty} \int_0^{2\pi} e^{-r^2/2} r\,\mathrm{d}\theta\,\mathrm{dr}$$
$$= 2\pi \int_0^{+\infty} r e^{-r^2/2}\,\mathrm{dr}$$
$$= -2\pi \left[e^{-r^2/2}\right]_0^{+\infty} = 2\pi.$$

Hence, $I = \sqrt{2\pi}$, and the result is proved.

Normal random variables (6)

(2) If X is normally distributed with parameters μ and σ^2, then we use the shorthand notation:

$$X \sim \text{Normal}(\mu, \sigma^2).$$

In the particular case, when $X \sim \text{Normal}(0, 1)$, X is called a **standard normal** r.v..

Normal random variables (7)

Theorem

If $X \sim$ *Normal*(μ, σ^2)*, then*

$$\mathbb{E}[X] = \mu,$$

$$Var(X) = \sigma^2.$$

Proof: By the definition, the p.d.f. of X is given by

$$F_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}.$$

Let $y := \dfrac{x-\mu}{\sigma}$, then

Normal random variables (8)

$$\begin{aligned}\mathbb{E}[X] &= \int_{-\infty}^{+\infty} x f_X(x)\mathrm{dx} \\ &= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{+\infty} x e^{-\frac{(x-\mu)^2}{2\sigma^2}} \mathrm{dx} \\ &= \frac{\sigma}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{+\infty} (\sigma y + \mu) e^{-\frac{y^2}{2}} \mathrm{dy} \\ &= \frac{1}{\sqrt{2\pi}} \left[\sigma \int_{-\infty}^{+\infty} y e^{-\frac{y^2}{2}} \mathrm{dy} + \mu \int_{-\infty}^{+\infty} e^{-\frac{y^2}{2}} \mathrm{dy}\right] = \mu.\end{aligned}$$

Since $y \mapsto y e^{-\frac{y^2}{2}}$ is an odd function over $\mathbb{R}$, so the first integration in the brackets above equals 0, the second integration equals $\sqrt{2\pi}$, due to (4).
Thus,

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mu^2.$$

Normal random variables (9)

And using IBP, we have

$$
\begin{aligned}
\mathbb{E}(X^2) &= \int_{-\infty}^{+\infty} x^2 f_X(x)\mathrm{dx} = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{+\infty} x^2 \mathsf{e}^{-\frac{(x-\mu)^2}{2\sigma^2}} \mathrm{dx} \\
&= \frac{\sigma}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{+\infty} (\sigma y + \mu)^2 \mathsf{e}^{-\frac{y^2}{2}} \mathrm{dy} \\
&= \left(\frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} y^2 \mathsf{e}^{-\frac{y^2}{2}} \mathrm{dy} \right) + \mu^2 \\
&= \frac{\sigma^2}{\sqrt{2\pi}} \left(-\underbrace{\left[y\mathsf{e}^{-\frac{y^2}{2}} \right]_{-\infty}^{+\infty}}_{=0} + \underbrace{\int_{-\infty}^{+\infty} \mathsf{e}^{-\frac{y^2}{2}} \mathrm{dy}}_{=\sqrt{2\pi}} \right) + \mu^2 \\
&= \sigma^2 + \mu^2.
\end{aligned}
$$

Hence, we have $\mathsf{Var}(X) = \sigma^2$.

Normal random variables (10)

Remark:

(1) Denote the c.d.f of a standard normal random variable by Φ. That is, $\forall x \in \mathbb{R}$,

$$\Phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} \mathrm{dy}.$$

The values of $\Phi(x)$ with $x \geq 0$ are given in the table of Page 48.
The values of $\Phi(x)$ can be obtained from the relationship

$$\Phi(-x) = 1 - \Phi(x), \quad \forall x \in \mathbb{R}. \tag{5}$$

Equation (5) follows from the symmetry of the standard normal density (the proof is left as an exercise). This equation states that if $Z \sim$ Normal$(0, 1)$, then

$$\mathbb{P}(Z \leq -x) = \mathbb{P}(Z > x).$$

Normal random variables (11)

(2) Let $Y = aX + b$, with two non-null constant $a, b \in \mathbb{R}$. If $X \sim \text{Normal}(\mu, \sigma^2)$, then

$$Y \sim \text{Normal}(a\mu + b, a^2\sigma^2).$$

To prove this statement, suppose that $a > 0$. (The proof when $a < 0$ is similar.) Let F_Y denote the c.d.f of Y.

$$\begin{aligned} F_Y(x) &= \mathbb{P}(Y \leq x) \\ &= \mathbb{P}(aX + b \leq x) \\ &= \mathbb{P}(X \leq \frac{x-b}{a}) \\ &= F_X(\frac{x-b}{a}), \end{aligned}$$

where F_X is the c.d.f. of X.

Normal random variables (12)

By differentiation, the p.d.f of Y is then

$$f_Y(x) = \frac{1}{a} f_X(\frac{x-b}{a}) = \frac{1}{\sqrt{2\pi} a\sigma} \exp\left\{-\frac{(\frac{x-b}{a}-\mu)^2}{2\sigma^2}\right\}$$
$$= \frac{1}{\sqrt{2\pi} a\sigma} \exp\left\{-\frac{[x-(a\mu+b)]^2}{2(a\sigma)^2}\right\},$$

which shows that $Y \sim \text{Normal}(a\mu + b, a^2\sigma^2)$ from definition.

Normal random variables (13)

(3) An important implication of (2) is that if $X \sim \text{Normal}(\mu, \sigma^2)$, then $Z := (X - \mu)/\sigma \sim \text{Normal}(0, 1)$.
And the c.d.f of X at any point a can be expressed as

$$
\begin{aligned}
F_X(a) := \mathbb{P}(X \leq a) =& \mathbb{P}(\frac{X - \mu}{\sigma} \leq \frac{a - \mu}{\sigma}) \\
=& \mathbb{P}(Z \leq \frac{a - \mu}{\sigma}) = \Phi\left(\frac{a - \mu}{\sigma}\right).
\end{aligned}
$$

Example

If X is a normal random variable with parameters $\mu = 3$ and $\sigma^2 = 9$, find
(a) $\mathbb{P}(2 < X < 5)$; (b) $\mathbb{P}(X > 0)$; (c) $\mathbb{P}(|X - 3| > 6)$.

Normal random variables (14)

Solution: Let $Z := \frac{X-3}{3}$.

(a)

$$
\begin{aligned}
&\mathbb{P}(2 < X < 5) \\
=&\mathbb{P}\left(\frac{2-3}{3} < \frac{X-3}{3} < \frac{5-3}{3}\right) \\
=&\mathbb{P}\left(-\frac{1}{3} < Z < \frac{2}{3}\right) \\
=&\Phi\left(\frac{2}{3}\right) - \Phi\left(-\frac{1}{3}\right) \qquad \text{since } Z \sim \text{Normal}(0,1) \\
=&\Phi\left(\frac{2}{3}\right) - \left[1 - \Phi\left(\frac{1}{3}\right)\right] \\
\approx&0.3779.
\end{aligned}
$$

Normal random variables (15)

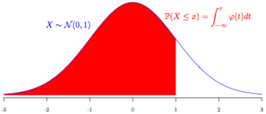
(b)

$$\mathbb{P}(X > 0) = \mathbb{P}\left(\frac{X-3}{3} > \frac{0-3}{3}\right) = \mathbb{P}(Z > -1)$$
$$= 1 - \mathbb{P}(Z \leq -1), \quad \text{since } Z \sim \text{Normal}(0,1)$$
$$= 1 - \Phi(-1) = 1 - [1 - \Phi(1)] = \Phi(1) \approx 0.8413.$$

(c)

$$\mathbb{P}(|X-3| > 6) = \mathbb{P}(X - 3 < -6 \text{ or } X - 3 > 6)$$
$$= \mathbb{P}(X > 9) + \mathbb{P}(X < -3)$$
$$= \mathbb{P}\left(\frac{X-3}{3} > \frac{9-3}{3}\right) + \mathbb{P}\left(\frac{X-3}{3} < \frac{-3-3}{3}\right)$$
$$= \mathbb{P}(Z > 2) + \mathbb{P}(Z < -2)$$
$$= 1 - \Phi(2) + \Phi(-2), \quad \text{since } Z \sim \text{Normal}(0,1)$$
$$= 2[1 - \Phi(2)] \approx 0.0456.$$

Normal random variables (16)



	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

The end of Lesson 3 - Part II