MTH113TC: Intro. to Probability & Statistics

Lesson 2 - Conditional probability & independence

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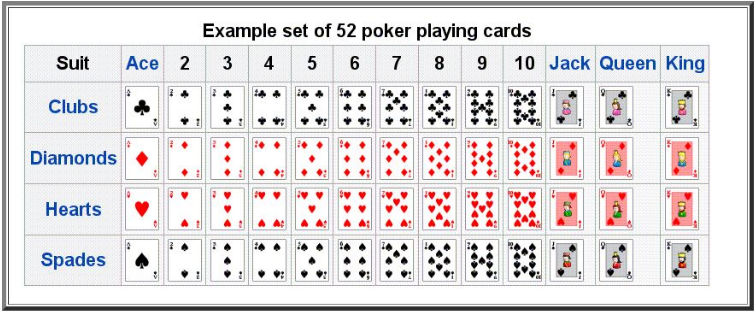
Lesson 2: Conditional probability & independence I

1. Conditional probability
2. The multiplication principle
3. The law of total probability
4. Bayes' formula
5. Independent events

1. Conditional probability



Conditional probability (1)

Example

Suppose you play bridge with three of your friends. After the cards are shuffled, the deck of 52 cards is dealt so that each player receives a hand of 13 cards.

- Before you look at your deal, your judgement about the probability that your partner has the ace of hearts is $1/4$.
- Suppose you look at your deal and discover the ace of hearts among your cards. The probability that your partner has it is now 0. If you do not have it in your deal, then the probability that your partner has it goes up to $1/3$.
- Your judgement of the probabilities differs because it is conditioned by the extra information whether you do or do not have the ace of hearts.

Conditional probability (2)

Example

Consider a degree programme in a mathematical department located in China with the following student numbers:

	1st year	2nd year	total
Overseas	10	18	28
China	15	20	35
total	25	38	63

By using the Fundamental formula of probability

$$\mathbb{P}(\text{overseas student}) = \frac{|\{\text{overseas students}\}|}{|\{\text{students}\}|} = \frac{28}{63} \approx 0.44.$$

$$\mathbb{P}(\text{Chinese student}) = \frac{|\{\text{Chinese students}\}|}{|\{\text{students}\}|} = \frac{35}{63} \approx 0.56.$$

Conditional probability (3)

Definition (Conditional probability)

Suppose A and B are two events in the sample space Ω and $\mathbb{P}(B) \neq 0$. The probability given by

$$\mathbb{P}(A|B) := \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} \tag{1}$$

is called the **conditional probability of event A given (or condition on) event B**.

Example (Revisited the previous example)

If we select a second year student at random, what is the chance that the student is from China?

Conditional probability (4)

Solution:

The required probability is given by:

$$\mathbb{P}(\text{China}|\text{ 2nd year student})$$

$$:=\frac{|\{\text{2nd year Chinese student}\}|}{|\{\text{2nd year students}\}|}=\frac{20}{38}\approx 0.53.$$

Example (Reinterpretation of Example in Page 4)

$$\mathbb{P}(\text{your partner has ace of hearts}|\text{ before you look at your cards})=\frac{1}{4}$$

$$\mathbb{P}(\text{your partner has ace of hearts}|\text{ after you notice you have it})=0;$$

$$\mathbb{P}(\text{your partner has ace of hearts}|\text{after you know you haven't})$$

$$=\frac{1}{3}.$$

Conditional probability (5)

Theorem

For a fixed event $B \subseteq \Omega$, *such that* $\mathbb{P}(B) \neq 0$, *we have* $\widetilde{\mathbb{P}}(A) := \mathbb{P}(A|B)$ *is a probability defined on the sample space* $\widetilde{\Omega} = B$, *i.e.,*

(1) *For any event* E *(*$\subseteq \widetilde{\Omega}$*)*, $0 \leq \widetilde{\mathbb{P}}(A) \leq 1$;

(2) $\widetilde{\mathbb{P}}(\widetilde{\Omega}) = 1$ *and* $\widetilde{\Omega}$ *is the unique event which satisfies this identity.*

(3) *For any sequence of pairwise mutually exclusive events* $E_1, E_2, \ldots$ *(that is, events for which* $E_i \cap E_j = \emptyset$ *when* $i \neq j$*) in* $\widetilde{\Omega}$,

$$\widetilde{\mathbb{P}}\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \widetilde{\mathbb{P}}(E_i).$$

Conditional probability (6)

Proof:

(1) Non-negativity of $\widetilde{\mathbb{P}}(E)$ follows both of the numerator and the denominator are non-negative. Since $E \cap B \subseteq B$, the monotonic property, implies $\mathbb{P}(E \cap B) \leq \mathbb{P}(B)$. Thus,

$$0 \leq \widetilde{\mathbb{P}}(E) := \mathbb{P}(E|B) = \frac{\mathbb{P}(E \cap B)}{\mathbb{P}(B)} \leq \frac{\mathbb{P}(B)}{\mathbb{P}(B)} = 1.$$

(2) By the definition, we have

$$\widetilde{\mathbb{P}}(\widetilde{\Omega}) = \frac{\mathbb{P}(B \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(B)}{\mathbb{P}(B)} = 1,$$

where $\widetilde{\Omega} = B$ is the only event which satisfies the identity above.

Conditional probability (7)

(3) Let E_1, E_2, $\ldots$ be a sequence of pairwise mutually exclusive events, i.e., $E_i \cap E_j = \emptyset$ if $i \neq j$. Then the sequence of events $E_1 \cap B$, $E_2 \cap B$, $\ldots$ is also pairwise mutually exclusive, i.e., $(E_i \cap B) \cap (E_j \cap B) = \emptyset$, if $i \neq j$. Then

$$\begin{aligned}
\widetilde{\mathbb{P}}\left(\bigcup_{i=1}^{\infty} E_i\right) &\overset{\text{def}}{=} \frac{\mathbb{P}\left[\left(\bigcup_{i=1}^{\infty} E_i\right) \cap B\right]}{\mathbb{P}(B)} \overset{\text{distr.}}{=} \frac{\mathbb{P}\left[\bigcup_{i=1}^{\infty}(E_i \cap B)\right]}{\mathbb{P}(B)} \\
&\overset{\text{Ax.3}}{=} \frac{\mathbb{P}(E_1 \cap B)}{\mathbb{P}(B)} + \frac{\mathbb{P}(E_2 \cap B)}{\mathbb{P}(B)} + \cdots \\
&= \sum_{i=1}^{\infty} \mathbb{P}(E_i | B) \\
&= \sum_{i=1}^{\infty} \widetilde{\mathbb{P}}(E_i)
\end{aligned}$$

2. The multiplication principle

The multiplication principle (1)

Corollary (Multiplication principle)

The defining formula (1) for conditional probability can be rewritten to obtain the **the multiplication principle**

$$\mathbb{P}(A \cap B) = \mathbb{P}(A|B)\,\mathbb{P}(B).$$

Remark

We can continue this process as a **chain rule**:

$$\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A|B \cap C)\,\mathbb{P}(B \cap C) = \mathbb{P}(A|B \cap C)\,\mathbb{P}(B|C)\,\mathbb{P}(C).$$

The multiplication principle (2)

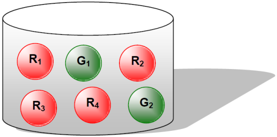
Example

Suppose that two balls are to be randomly drawn, one after another, from a container holding 4 **red** balls and 2 **green** balls (See the figure). Under the scenario of sampling without replacement, calculate the probability of the events

(1) $A = \{$first ball is **red**$\}$,

(2) $B = \{$second ball is **red**$\}$,

(3) $A \cap B = \{$first ball is **red** AND second ball is **red**$\}$.



The multiplication principle (3)

Solution:

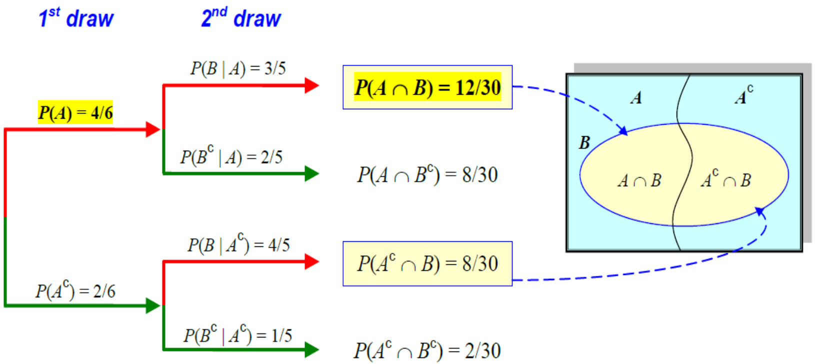


Figure: illustration of the solution

The multiplication principle (4)

We can calculate the probability $\mathbb{P}(B)$ by applying the rule of total probability (Lesson 1) and adding the two "boxed" values above, i.e.

$$\begin{aligned}\mathbb{P}(B) =& \mathbb{P}(A \cap B) + \mathbb{P}(A^c \cap B)\\ =& \mathbb{P}(B|A)\mathbb{P}(A) + \mathbb{P}(B|A^c)\mathbb{P}(A^c)\\ =& \frac{3}{5} \times \frac{4}{6} + \frac{4}{5} \times \frac{2}{6} = \frac{2}{3}.\end{aligned}$$

Note: This last formula, which can be written as

$$\mathbb{P}(B) = \mathbb{P}(B|A)\mathbb{P}(A) + \mathbb{P}(B|A^c)\mathbb{P}(A^c),$$

is known as the **law of total probability**, and is a useful tool in **Bayes' formula** (see next sections).

(As an exercise, list the $6 \times 5 = 30$ outcomes in the sample space of this experiment, and use "brute force" to solve it.)

3. The law of total probability

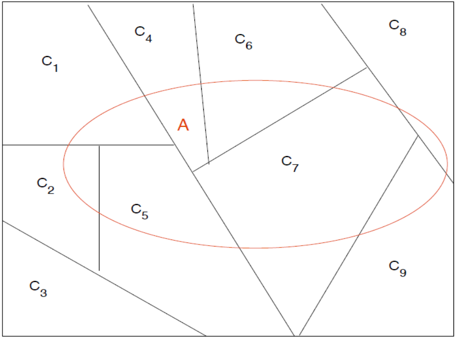
The law of total probability (1)

Definition

A **partition** of the sample space Ω is a finite collection of events C_1, C_2, $\cdots$, C_n in Ω such that

(1) $C_i \cap C_j = \emptyset$, for any $i \neq j$ and $i, j = 1, 2, \cdots n$.

(2) $C_1 \cup C_2 \cup \cdots \cup C_n = \Omega$.



The law of total probability (2)

Remark

(1) We can refer to the figure above for illustration of the partition. The events $\{C_1, \cdots, C_9\}$ is a partition of the sample space and every outcome $\omega\,(\in \Omega)$ belongs to *exactly* one of the C_i.

(2) In the figure above, the event A can be written as the union $(A \cap C_1) \cup \cdots \cup (A \cap C_9)$ of mutually exclusive events.

The law of total probability (3)

Theorem (Law of total probability)

Let $\mathbb{P}$ be a probability on Ω and let $\{C_1, C_2, \cdots, C_n\}$ be a partition of Ω chose so that $\mathbb{P}(C_i) \neq 0$ for all $i = 1, 2, \ldots, n$. Then for any event $A \subseteq \Omega$,

$$\begin{aligned}\mathbb{P}(A) =& \mathbb{P}(A|C_1)\,\mathbb{P}(C_1) + \cdots + \mathbb{P}(A|C_n)\,\mathbb{P}(C_n) \\ =& \sum_{i=1}^{n} \mathbb{P}(A|C_i)\,\mathbb{P}(C_i).\end{aligned} \tag{2}$$

The law of total probability (4)

Sketch of proof: Because $\{C_1, C_2, \cdots, C_n\}$ is a partition,

$$\{(A \cap C_1), (A \cap C_2), \ldots, (A \cap C_n)\}$$

is a group of pairwise mutually exclusive events. By the distributive laws of events, their union is the event A. Using the generalization of the rule of total probability, we can write

$$\mathbb{P}(A) = \mathbb{P}(A \cap C_1) + \mathbb{P}(A \cap C_2) + \cdots + \mathbb{P}(A \cap C_n). \quad (3)$$

Finish by using the multiplication principle,

$$\mathbb{P}(A \cap C_i) = \mathbb{P}(A|C_i)\,\mathbb{P}(C_i), \quad i = 1, 2 \ldots, n$$

and substituting into (3) to obtain the identity (2).

The law of total probability (5)

Remark

The most frequent use of the law of total probability comes in the case of a partition of the sample space into 2 events, say $\{B, B^c\}$. In this case, the law of total probability becomes the identity

$$\mathbb{P}(A) = \mathbb{P}(A|B)\,\mathbb{P}(B) + \mathbb{P}(A|B^c)\,\mathbb{P}(B^c).$$

See the figure below for illustration.

The law of total probability (6)

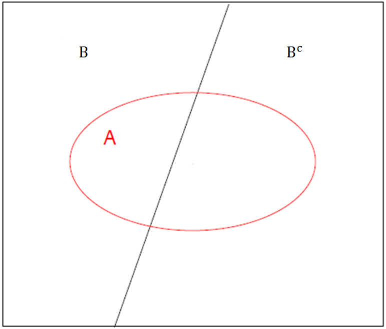


Figure: A partition into 2 events B and B^c.

The law of total probability (7)

Example

There are two urns, each containing coloured balls. In Urn I there are 2 white and 3 blue balls, in Urn II there are 3 white and 4 blue ones. A ball is drawn at random from Urn I and put into Urn II, and then a ball is drawn at random from Urn II and examined. What is the probability this ball is blue? We may suppose that picking out a ball of any colour from any cup is equally likely.

<u>Solution</u>:

Let $E := \{\text{2nd ball is blue}\}$, $F := \{\text{1st ball is blue}\}$. Then

$$\begin{aligned} F &= \{\text{the No. of balls in Urn II is increased by a blue ball}\} \\ &= \{\text{the Urn II contains 3 white and 5 blue balls}\} \end{aligned}$$

and

The law of total probability (8)

$$\begin{aligned} F^c &= \{\text{the No. of balls in Urn II is increased by a white ball}\} \\ &= \{\text{the Urn II contains 4 white and 4 blue balls}\}. \end{aligned}$$

Hence we have

$$\begin{aligned} \mathbb{P}(E|F) =& \mathbb{P}(E|\{\text{the Urn II contains 3 white and 5 blue balls}\}) \\ =& \frac{5}{8}, \end{aligned}$$

$$\begin{aligned} \mathbb{P}(E|F^c) =& \mathbb{P}(E|\{\text{the Urn II contains 4 white and 4 blue balls}\}) \\ =& \frac{1}{2}. \end{aligned}$$

Moreover, $\mathbb{P}(F) = 3/5$ and $\mathbb{P}(F^c) = 1 - 3/5 = 2/5$. By the law of total probability,

$$\mathbb{P}(E) = \mathbb{P}(E|F)\,\mathbb{P}(F) + \mathbb{P}(E|F^c)\,\mathbb{P}(F^c) = \frac{5}{8} \times \frac{3}{5} + \frac{1}{2} \times \frac{2}{5} = \frac{23}{40}.$$

4. Bayes' formula

Bayes' formula (1)

Theorem (Bayes' formula)

Let $\{C_1, C_2, \ldots, C_n\}$ be a partition of Ω and $A \subseteq \Omega$ is an event such that $\mathbb{P}(A|C_i) \neq 0$ for $i = 1, 2, \ldots, n$. Then

$$\mathbb{P}(C_i|\, A) = \frac{\mathbb{P}(A|\, C_i)\, \mathbb{P}(C_i)}{\sum_{j=1}^{n} \mathbb{P}(A|\, C_j)\mathbb{P}(C_j)}.$$

Proof.

By using the multiplication principle and the law of total probability (2), we have

$$\mathbb{P}(C_i|\, A) = \frac{\mathbb{P}(C_i \cap A)}{\mathbb{P}(A)} = \frac{\mathbb{P}(A|\, C_i)\, \mathbb{P}(C_i)}{\mathbb{P}(A)} = \frac{\mathbb{P}(A|\, C_i)\, \mathbb{P}(C_i)}{\sum_{j=1}^{n} \mathbb{P}(A|\, C_j)\, \mathbb{P}(C_j)}.$$

$\square$

Bayes' formula (2)

Example (CFA, Reading #8, 2015)

You are forecasting the sales of a building materials supplier by assessing the expansion plans of its largest customer, a homebuilder. You estimate the probability that the customer will increase its orders for building materials to 25%. If the customer does increase its orders from this supplier, you estimate the probability that the homebuilder will start a new development at 70%. If the customer does not increase its orders from the supplier, you estimate only a 20% chance that it will start the new development. Later, you find out that the homebuilder will start the new development. In light of this new information, what is your new (updated) probability that the builder will increase its orders from this supplier?

Bayes' formula (3)

Solution: Set the following events:

$$I = \{\text{homebuilder increases order from the supplier}\},$$

$$N = \{\text{homebuilder starts new development}\}.$$

Then

$$I^c = \{\text{homebuilder does not increase order from the supplier}\}$$

and $N^c = \{\text{homebuilder does not start new development}\}$.
Moreover,
$\mathbb{P}(I) = 0.25 \Longrightarrow \mathbb{P}(I^c) = 1 - \mathbb{P}(I) = 1 - 0.25 = 0.75.$
It is given that $\mathbb{P}(N|\, I) = 0.7$ and $\mathbb{P}(N|\, I^c) = 0.2$.

Bayes' formula (4)

We need to find the value of $\mathbb{P}(I|N)$.
Using Bayes' formula,

$$
\begin{aligned}
\mathbb{P}(I|\,N) &= \frac{\mathbb{P}(N|\,I)\,\mathbb{P}(I)}{\mathbb{P}(N|\,I)\,\mathbb{P}(I)+\mathbb{P}(N|\,I^c)\,\mathbb{P}(I^c)} \\
&= \frac{0.7\times 0.25}{0.7\times 0.25+0.2\times 0.75} \\
&= \frac{0.175}{0.175+0.15} \\
&= \frac{0.175}{0.325}\approx 0.5385.
\end{aligned}
$$

Bayes' formula (5)

Example Let A be the event that an individual tests positive for some disease and C be the event that the person actually has the disease. We can perform clinical trials to estimate the probability that a randomly chosen individual tests positive given that they have the disease,

$$\mathbb{P}(\{\text{tests positive} |\, \text{has the disease}\}) = \mathbb{P}(A|\, C),$$

by taking the individuals with the disease and playing the test. The Public Health Department gives us the following information:

- A test for the disease yields a positive result 90% of the time when the disease is present.
- A test for the disease yields a positive result 1% of the time when the disease is not present.
- One person in 1,000 has the disease.

Bayes' formula (6)

What is the probability one person has the disease given that the test yields a positive result, that is

$$\mathbb{P}(\{\text{has the disease} | \text{ tests positive}\}) = \mathbb{P}(C|\,A)?$$

<u>Solution</u>: From the information given by the Public Health Department, we have $\mathbb{P}(A|\,C) = 90\%$, $\mathbb{P}(A|\,C^c) = 1\%$ and $\mathbb{P}(C) = 1/1000$. We want to evaluate the probability $\mathbb{P}(C|\,A)$. For this, we can apply Bayes' formula

$$\begin{aligned}\mathbb{P}(C|\,A) &= \frac{\mathbb{P}(A|\,C)\,\mathbb{P}(C)}{\mathbb{P}(A|\,C)\,\mathbb{P}(C) + \mathbb{P}(A|\,C^c)\,\mathbb{P}(C^c)} \\ &= \frac{0.9 \times 0.001}{0.9 \times 0.001 + 0.01 \times (1 - 0.001)} \\ &= \frac{0.0009}{0.01089} \approx 0.0826.\end{aligned}$$

5. Independent events

Independent events (1)

Definition

Two events A and B are said to be **independent** if either:

$$\mathbb{P}(A|\, B) = \mathbb{P}(A), \quad \text{i.e. } \mathbb{P}(B|A) = \mathbb{P}(B),$$

or equivalently,

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B) \tag{4}$$

More generally, a finite collection of events $\{A_1, A_2, \ldots, A_n\}$ are called **independent**, if

$$\mathbb{P}(\bigcap_{k\in I} A_k) = \prod_{k\in I} \mathbb{P}(A_k),$$

for every subset $I \subseteq \{1, 2, \ldots, n\}$.

Independent events (2)

Remark

$A,\ B$ **disjoint** (i.e. $\mathbb{P}(A \cap B) = 0$) $\Longleftrightarrow$

If either event occurs, then the other <u>cannot</u> occur;

$A,\ B$ **independent** (i.e. $\mathbb{P}(A|\, B) = \mathbb{P}(A)$) $\Longleftrightarrow$

If either event occurs, this gives <u>no information</u> about the other.

Independent events (3)

Example

Let A and B be two independent events. Show that the following pairs of events are independent.

(1) A^c and B.

(2) A^c and B^c.

Solution:

(1) By the rule of total probability and definition of independent events, we have

$$\begin{aligned}\mathbb{P}(A^c \cap B) &= \mathbb{P}(B) - \mathbb{P}(A \cap B) \\ &\overset{\text{def.}}{=} \mathbb{P}(B) - \mathbb{P}(A)\,\mathbb{P}(B) \\ &= (1 - \mathbb{P}(A))\,\mathbb{P}(B) \\ &= \mathbb{P}(A^c)\,\mathbb{P}(B).\end{aligned}$$

Independent events (4)

(2) The second claim can be done in the same fashion.

$$\begin{aligned}
\mathbb{P}(A^c \cap B^c) &= \mathbb{P}(A^c) - \mathbb{P}(A^c \cap B) \\
&\overset{\text{def.}}{=} \mathbb{P}(A^c) - \mathbb{P}(A^c)\,\mathbb{P}(B) \\
&= \mathbb{P}(A^c)\,(1 - \mathbb{P}(B)) \\
&= \mathbb{P}(A^c)\,\mathbb{P}(B^c).
\end{aligned}$$

Example

On average, a faulty switch works on every 12 trials. Assume that trials are made independently. What is the probability that the switch works on the third trial (but not before)?

Independent events (5)

Solution: The probability that the switch works is $1/12$ and that it does not is $1 - 1/12 = 11/12$. Let
$A = \{\text{not work on the 1st}\}$,
$B = \{\text{not work on the 2nd}\}$,
$C = \{\text{works on the 3rd}\}$.
Since the successive attempts to use the switch are independent, then

$$
\begin{aligned}
&\mathbb{P}(\{\text{works on the 3rd trial for the 1st time}\}) \\
=&\mathbb{P}(A \cap B \cap C) \\
=&\mathbb{P}(A) \times \mathbb{P}(B) \times \mathbb{P}(C) \\
=&\frac{11}{12} \times \frac{11}{12} \times \frac{1}{12} \\
=&\frac{11^2}{12^3} \approx 0.07.
\end{aligned}
$$

The end of Lesson 2