Combinatorial Analysis

1 The basic principle of counting

Theorem 1.1 (The basic principle of counting)**.** *Suppose that 2 experiments are to be performed. Then if experiment 1 can result in any one of m possible outcomes and if, for each outcome of experiment 1, there are n possible outcomes of experiment 2, then together there are mn possible outcomes of the two experiments.*

Proof. The basic principle may be proven by enumerating all the possible outcomes of the two experiments; that is,

$$\begin{array}{cccc} (1,1), & (1,2), & \cdots, & (1,n) \\ (2,1), & (2,2), & \cdots, & (2,n) \\ \vdots & & & \\ (m,1), & (m,2), & \cdots, & (m,n) \end{array}$$

where we say that the outcome is (i,j) if experiment 1 results in its ith possible outcome and experiment 2 then results in its jth possible outcome. Hence, the set of possible outcomes consists of m rows, each containing n elements. There are in total mn possible outcomes. This proves the results. □

Example 1. A small community consists of 10 women, each of whom has 3 children. If one woman and one of her child are to be chosen as mother and child of the year, how many different choices are possible?

Solution: By regarding the choice of the woman as the outcome of the first experiment and the subsequent choice of one of her children as the outcome of the second experiment, we see from the basic counting principle there are $10 \times 3 = 30$ possible choices.

Theorem 1.2 (The generalized basic principle of counting)**.** *If r experiments that are to be performed are such that the 1st one may result in any of n_1 possible outcomes; and if, for each of these n_1 possible outcomes, there n_2 possible outcomes of the 2nd experiment; and if, for each of the possible outcomes of the first two experiments, there are n_3 possible outcomes of the 3rd experiment; and if $\ldots$, then there is a total of $n_1 n_2 \cdots n_r$ possible outcomes of the r experiments.*

Example 2. A college planning committee consists of 3 freshmen, 4 sophomore, 5 juniors, and 2 seniors. A subcommittee of 4, consisting of 1 person from each class, is to be chosen. How many different subcommittees are possible?

Solution: We may regard the choice of a subcommittee as the combined outcome of the 4 separate experiments of choosing a single representative from each of the classes. It then follows from the generalized basic principle of counting, there are $3 \times 4 \times 5 \times 2 = 120$ possible subcommittees.

Example 3. How many different 7-place license plates are possible if the first 3 places are to be occupied by letters and the final 4 by numbers?

Solution: By the generalized counting principle, the answer is $26 \times 26 \times 26 \times 10 \times 10 \times 10 \times 10 = 175,760,000$

Example 4. In the previous example, how many license plates would be possible if repetition among letters or numbers were prohibited?

Solution: In this case, there would be $26 \times 25 \times 24 \times 10 \times 9 \times 8 \times 7 = 78,624,000$ possible license plates.

2 Some basic concepts on sets

Definition 1 (finite set)**.** If a set S contains a finite number of elements, then the set S is called a **finite set**.

Eg. $S := \{2, 4, 6, 8, 10\}$ is a finite set.

Remark 2.1. We have the following convention to write a set:

(1) If a set contains several same elements, then this set is usually written as the set containing distinguish elements without repetition of the same element. For example,

$$\{1, 4, 4, 5, 6, 7, 7, 7\} = \{1, 4, 5, 6, 7\}.$$

(2) If one set contains numbers only, enumerate all the elements in **ascending** order. For example, we usually write the unordered set $\{2, 3, 1, 5, 4\}$ as $\{1, 2, 3, 4, 5\}$.

(3) If one set contains letters, enumerate all the elements in **alphabetic** order. For example, we usually write the unordered set $\{F, D, S, E, J\}$ as $\{D, E, F, J, S\}$.

Definition 2 (Countable set)**.** A set S is **countable** if there exists an injective function f from S to the natural numbers $\mathbb{N} = \{0, 1, 2, 3, \cdots\}$.

For example,
- The even number set $\{0, 2, 4, 6, 8, \cdots\}$ is a countable set.
- The real number set $\mathbb{R}$ is NOT a countable set.

Definition 3 (Cardinality)**.** The number of elements of a set S is called the **cardinality** of S, usually denoted by $|S|$.

Example 5. Find the cardinality of the following sets:

(1) $S_1 := \{2, 3, 5\}$;

(2) $S_2 := \{2, 2, 3, 5\}$;

(3) $S_3 := \{\{1\}, \{1, 2\}, \{1, 2, 3\}\}$;

(4) $S_4 := \{\{1\}, E, 6, \{1, \{2, 4\}\}\}$.

Answer:

(1) $|S_1| = 3$;

(2) $|S_2| = 3$;

(3) $|S_3| = 3$;

(4) $|S_4| = 4$.

Definition 4 (Empty set)**.** The **empty set** $\emptyset$ is a set containing no element.

Eg. $\{n \in \mathbb{N} : n \neq n\} = \emptyset$.

Definition 5 (Power set)**.** The set of all subsets of a given set S is called its **power-set**, denoted by 2^S, and defined as

$$2^S := \{A : A \subseteq S\}.$$

Remark 2.2. (1) The power set 2^S contains both $\emptyset$ and S.

(2) Eg. $2^{\{1,2\}} = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$, and $\left|2^{\{1,2\}}\right| = 4 = 2^{|S|}$.

(3) $\left|2^S\right| = 2^{|S|}$.

Proof of Remark 2.2, (3). Suppose we have $S := \{a_1, a_2, \cdots, a_r\}$, with $|S| = r$. To find the cardinality of power-set, it is equivalent to counting how many subsets of S are there. We could assign to any arbitrary subset, name it Q, a unique sequence of r elements $(x_1, x_2, \cdots, x_r)$ such that if $a_k \in Q$, for $1 \leq k \leq r$, we would have $x_k = 1$, otherwise, $x_k = 0$. For instance, $\{a_1, a_3\}$, one subset of S, will be assigned with $(1, 0, 1, 0, \cdots, 0)$. Another example is $(1, 1, 1, \cdots, 1)$ will represent S itself since it contains all the elements in S. Now we could define the direct product

$$P := \{(x_1, x_2, \cdots, x_r) : x_1 \in \{0,1\},\, x_2 \in \{0,1\}, \cdots,\, x_r \in \{0,1\}\}.$$

As each element of P has one-to-one correspondence with subsets of S, the cardinality of P represents how many subsets of S. Thus

$$|P| = |2^S| = |\{0,1\}|^r = 2^r = 2^{|S|}.$$

□

Definition 6. Let S_1 and S_2 be two finite sets. The set

$$S_1 \times S_2 := \{(x_1, x_2) : x_1 \in S_1, x_2 \in S_2\}$$

is called the **direct product** of S_1 and S_2.
More generally, for the given n sets S_1, S_2, $\cdots$, S_n the direct product is defined as

$$S_1 \times S_2 \times \cdots \times S_n = \{(x_1, x_2, \cdots, x_n) : x_1 \in S_1, x_2 \in S_2, \cdots, x_n \in S_n\}.$$

Example 6. (1) Suppose that we toss a coin twice with two faces called respectively 'Head' and 'Tail'. The possible outcomes that we could obtain by the end of the toss can be represented by any element in the set $\{H,T\}\times\{H,T\} = \{(H,H), (H,T), (T,H), (T,T)\}$, where H stands for 'Head' and T stands for 'Tail'.

(2) Given $S_1 := \{1, 3, 5\}$ and $S_2 := \{2, 4\}$, then

$$S_1 \times S_2 = \{(1,2), (1,4), (3,2), (3,4), (5,2), (5,4)\},$$
$$S_2 \times S_1 = \{(2,1), (2,3), (2,5), (4,1), (4,3), (4,5)\}.$$

According to the basic principle of counting Theorem 1.2,

$$|S_1 \times S_2| = |S_1| \times |S_2| = 6 = |S_2| \times |S_1| = |S_2 \times S_1|.$$

We can see, although $S_1 \times S_2$ and $S_2 \times S_1$ represent different products, their cardinalities are equal.

3 Permutations

Definition 7. Let S be a set and $|S| = n$, then an ordered arrangement of the n elements in S is called a **permutation of** S. The number of the possible permutations of S is denoted by P_n, and it is qual to

$$P_n = n! := n \cdot (n-1) \cdot \ldots \cdot 1.$$

The symbol '!' is called **factorial**.

Remark 3.1. (1) There are n choices to place the 1st element, $(n-1)$ choices to place the 2nd element, $\cdots$, and 1 choice to place the last element. So according to the basic principle of counting, the total number of all permutations of S is $n \cdot (n-1) \ldots \cdot 1 = n!$.

(2) Note that $0! = 1$.

Example 7. How many possible ways to arrange the 12 students below in a row?

Answer: There are in total $P_{12} = 12! = 12 \times 11 \times \cdots \times 2 \times 1 = 479,001,600$ possible ways to arrange the students in a row.

3.1 Permutation without repetition

Definition 8. In the most general terms, a **permutation without repetition** is an ordered list of elements selected from some set and the selected elements CAN NOT be repeated. **Order counts**.
The total number of permutations of r elements selected from a set of n elements ($r \leq n$) without repetition is denoted by P_n^r, which is given by the formula

$$P_n^r = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-r+1) = \frac{n!}{(n-r)!}.$$

Remark 3.2. (1) There are n choices to place the 1st element, $(n-1)$ choices to place the 2nd element, ..., and $(n-r+1)$ choices to place the rth element. So according to the basic principle of counting, the total number of permutations of r elements selected from a set of n elements without repetition is $n \cdot (n-1) \cdot \ldots \cdot (n-r+1)$.

(2) The permutation of a set S defined in Definition 7 is a special case of permutation without repetition where we select all the elements in S, i.e. $r = n$. In order to simplify the notation, we usually write P_n instead of P_n^n in this case.

Example 8. Here some example permutations without repetition:

(1) $(3, 1, 2)$ and $(2, 1, 3)$ are both distinct permutations of the set $\{1, 2, 3\}$, without repetition allowed.

(2) $(5, 6)$ is a 2-permutation of the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, without repetition allowed.

(3) (a, b, c), (b, c, a), (b, e, f) are distinct 3-permutation of the set $\{a, b, c, d, e, f, g\}$, without repetition allowed.

Example 9. Among the 9 cards from the Spade Ace to the Spade 9 showing below, how many ordered arrangement of 4 cards can be formed?

Answer: The arrangement is a permutation of 4 members chosen from a set of 9, without repetition. Thus, the total number of possible permutations is:

$$P_9^4 = 9 \times 8 \times 7 \times 6 = 3,024$$

Example 10. Consider a lottery in which 6 balls are consecutively drawn at random from an urn without replacement containing 99 balls, each printed with a unique number 1, ..., 99. What are the total number of possible outcomes of this draw?

Answer: Each draw is a permutation of 6 numbers chosen from a set of 99, without repetition. Thus, the total number of possible permutations is:

$$\begin{aligned} P_{99}^6 &= 99 \times 98 \times 97 \times 96 \times 95 \times 94 \\ &= 806,781,064,320. \end{aligned}$$

This is a very large number, over 800 billion possible permutations. (As a prelude to the next chapter, think of trying to guess the exact permutation that is chosen. What would be your probability of guessing correctly?)

3.2 Permutation with repetition

Definition 9. A permutation with repetition is an ordered list of elements selected from some set, in which some selected elements are repeated a specific of times. **Order counts**.

Let $S := \{a_1, a_2, \cdots, a_k\}$ be a given set which contains k elements. The total number of permutations of n elements with repetition in which the 1st element a_1 is repeated n_1 times, the 2nd element a_2 is repeated n_2 times, ..., and the kth element a_k is repeated n_k times, such that $n_1 + n_2 + \cdots + n_k = n$, is denoted by $P_n^{(n_1, n_2, \cdots, n_k)}$ and is given by the multinomial coefficient:

$$P_n^{(n_1, n_2, \cdots, n_k)} = \frac{n!}{n_1! \cdot n_2! \cdot \ldots \cdot n_k!}.$$

Remark 3.3. Imagine we need to select n objects with k categories ($k \le n$), and each category contains n_i number of the same objects, $i = 1, 2, \cdots, k$. So we have obviously $n = n_1 + n_2 + \cdots + n_k$. There are $n!$ possible permutations of n objects. Among all these permutations, there are $n_1!$ permutations for the 1st category that we don't distinguish, there are $n_2!$ permutations for the 2nd category that we don't distinguish, etc. In the $n!$ permutations, we count $n_1! \cdot n_2! \cdot \ldots \cdot n_k!$ times more.

Example 11 (Anagrams)**.** An *anagram* is direct word switch or word play, the result of rearranging the letters of a word or phrase to produce a new word or phrase, using all the original letters exactly once. For example,

$$\text{Tom Marvolo Riddle} \;\to\; \text{I am Lord Voldemort}$$

How many so called anagrams (or letter rearrangements) can be formed by using the letters of the word TALL?

Solution: We have $n = 4$ letters in total. Thus we could define a set S as $S := \{\text{T, A, L}\}$. In the original word, 'T' is featured once, 'A' is featured once while 'L' twice, i.e.,

$$\begin{aligned} n_1 &= 1 \ \text{T's}, \\ n_2 &= 1 \ \text{A's}, \\ n_3 &= 2 \ \text{L's}, \end{aligned}$$

By the definition of permutation with repetition, the total number of anagrams that can be formed is

$$P_4^{(1,1,2)} = \frac{4!}{1!\,1!\,2!} = 12.$$

Example 12. How many anagrams of the word MISSISSIPPI are possible?

Solution: We have $n = 11$ letters in total:

$$\begin{aligned} n_1 &= 1 \ \text{M's}, \\ n_2 &= 4 \ \text{I's}, \\ n_3 &= 4 \ \text{S's}, \\ n_4 &= 2 \ \text{P's}. \end{aligned}$$

Thus, the total number of permutations is

$$P_{11}^{(1,4,4,2)} = \frac{11!}{1!\,4!\,4!\,2!} = 34,650.$$

Example 13. How many different ways can we place 5 red balls, 3 green balls and 2 blue balls next to each other?

Solution: Here we need to arrange in total $n = 5 + 3 + 2 = 10$ balls with $n_1 = 5$ repetition of blues balls, $n_2 = 3$ repetition of green balls and $n_3 = 2$ repetition of blue balls. So the total number of ways to arrange the 10 balls (permutation of 10 objects with repetition) is

$$P_{10}^{(5,3,2)} = \frac{10!}{5! \cdot 3! \cdot 2!} = 2,520.$$

4 Combinations

Definition 10. Let S be a finite set with $|S| = n$. An unordered subset of S of k $(\leq n)$ elements is called a ***k*-combination** of the set S. **Order DOES NOT count.**

Example 14. The 3-combinations of the set $\{1, 2, 3, 4, 5\}$ are

$$\begin{array}{lllll} \{1,2,3\} & , & \{1,2,4\} & , & \{1,2,5\} \\ \{1,3,4\} & , & \{1,3,5\} & , & \\ \{1,4,5\} & , & & & \\ \{2,3,4\} & , & \{2,3,5\} & , & \\ \{2,4,5\} & , & & & \\ \{3,4,5\} & . & & & \end{array}$$

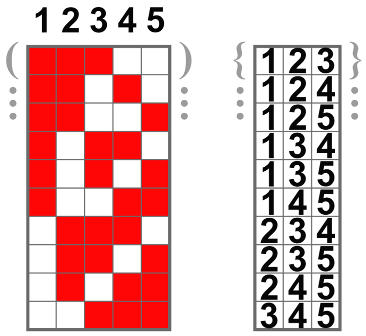

Figure 1: An other representation of 3-combination of the set $\{1, 2, 3, 4, 5\}$ from Wikipedia.

The figure below shows an illustration of the 3-combinations of the set $\{1, 2, 3, 4, 5\}$.

Definition 11 (Number of k-combinations)**.** Let S be a finite set with $|S| = n$. The **number of k-combinations** from the set S is denoted by C_n^k, where $k \leq n$. The same number however occurs in many other mathematical contexts, where it is denoted by $\left(\begin{array}{c} n \\ k \end{array} \right)$ (often read as "n choose k"). And they are given by the formula

$$C_n^k := \left(\begin{array}{c} n \\ k \end{array} \right) = \frac{n!}{(n-k)!\; k!} = \frac{n \cdot (n-1) \cdot \ldots \cdot (n-k+1)}{k!}.$$

Remark 4.1. The expressions $\left(\begin{array}{c} n \\ k \end{array} \right)$ are so-called **binomial coefficients**. Besides,

$$C_n^k = P_n^{(k,n-k)},$$

where $k \leq n$.

Theorem 4.2 (Binomial theorem)**.** *Let $a, b \in \mathbb{R}$, $n \in \mathbb{N}$. Then*

$$(a+b)^n = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) a^{n-k}\, b^k.$$

Skech of proof. If n is a positive integer and we multiply out $(a+b)^n$ term by term, each term will be the product of a's and b's, with an a or b coming from each of the n factors $(a+b)$. For instance, when $n = 3$, the expansion

$$\begin{aligned}(a+b)^3 &= (a+b)(a+b)(a+b) \\ &= a \cdot a \cdot a + a \cdot a \cdot b + a \cdot b \cdot a + b \cdot a \cdot a + a \cdot b \cdot b + b \cdot a \cdot b + b \cdot b \cdot a + b \cdot b \cdot b \\ &= a^3 + 3a^2b + 3b^2a + b^3.\end{aligned}$$

yields terms of the form a^3, a^2b, ab^2 and b^3. Their coefficients are 1, 3, 3 and 1, and the coefficient of ab^2, for example, is $\left(\begin{array}{c} 3 \\ 2 \end{array} \right) = 3$, the number of ways in which we can choose the two factors providing the b's.

More generally, the coefficient of $a^{n-k}b^k$ is $\left(\begin{array}{c} n \\ k \end{array}\right)$, the number of ways in which we can choose k factors providing the b's. $\square$

Corollary 4.3. *(1) $a = 1$, $b = x \in \mathbb{R}$, then*

$$(1+x)^n = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array}\right) x^k.$$

(2) Replace b by $-b$, then

$$(a-b)^n = \sum_{k=0}^{n} (-1)^k \left(\begin{array}{c} n \\ k \end{array}\right) a^{n-k}\, b^k.$$

(3) Choose $a = b = 1$, then

$$2^n = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array}\right) = \left(\begin{array}{c} n \\ 0 \end{array}\right) + \left(\begin{array}{c} n \\ 1 \end{array}\right) + \cdots + \left(\begin{array}{c} n \\ n \end{array}\right).$$

(4) Choose $a = 1$, $b = -1$, then

$$0 = \sum_{k=0}^{n} (-1)^k \left(\begin{array}{c} n \\ k \end{array}\right) = \left(\begin{array}{c} n \\ 0 \end{array}\right) - \left(\begin{array}{c} n \\ 1 \end{array}\right) + \cdots + (-1)^n \left(\begin{array}{c} n \\ n \end{array}\right).$$