Introduction to Probability and Statistics — Sep. 2020
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Solution to Exercises
Lesson 1 - Elementary Probability

Exercise 1. A box contains 3 marbles: 1 red, 1 green, and 1 blue. Consider an experiment that consists of taking 1 marble from the box and then replacing it in the box and repeat this action for 3 times. Describe the sample space. Repeat when the second marble is drawn without replacing the first marble and the third one is drawn without replacing the second.

Answer: The sample space Ω is a set which contains all the possible outcomes of the three draws. Let (i, j, k), with $i, j, k \in \{R, G, B\}$ represent one possible outcome of the experiment, where R, G, B mean respectively Red marble, Green marble and Blue marble. For instance, the notation (R, G, B) means that the first marble drawn is red, the second drawn is green and the third is blue.

In the first experiment, since people draws randomly one marble each time with replacement. So the sample space Ω can be expressed as the direct product

$$\begin{aligned}\Omega &= \{R, G, B\} \times \{R, G, B\} \times \{R, G, B\} \\ &= \{(i, j, k) : i \in \{R, G, B\}, j \in \{R, G, B\}, \text{ and } k \in \{R, G, B\}\}.\end{aligned}$$

Moreover, in this case $|\Omega| = 3^3 = 27$.

In the second experiment, since people draws randomly one marble each time without replacement. So the sample space Ω can be expressed as

$$\Omega = \{\text{the permutations of } \{\text{R,G,B}\}\}.$$

And in this case $|\Omega| = 3 \times 2 \times 1 = 6$.

Exercise 2. Identify the sample spaces of the following experiments:

(a) If the outcome of an experiment consists in the determination the sex of a newborn child (B for boy and G for girl);

(b) If the outcome of an experiment is the order of finishing in a race among 7 horses having post positions 1, 2, 3, 4, 5, 6, 7;

(c) If the experiment consists of flipping two coins;

(d) If the experiment consists of measuring (in hours) the lifetime of a transistor;

(e) If the experiment consists of counting how many people will pop into one shop during some specific time interval;

(f) If the experiment consists of counting how many phone calls have to be made by the service center during one day.

Answer:

(a) $\Omega = \{B, G\}$;

(b) $\Omega = \{$all the permutations of $\{1, 2, 3, 4, 5, 6, 7\}\}$;

(c) $\Omega = \{(H, H), (H, T), (T, H), (T, T)\}$;

(d) $\Omega = [0, \infty)$;

(e) $\Omega = \{0, 1, 2, 3, \cdots\} = \mathbb{N}$;

(f) $\Omega = \{0, 1, 2, 3, \cdots\} = \mathbb{N}$.

Exercise 3. Two dice are thrown. Let E be the event that the sum of the dice is odd, let F be the event that at least one of the dice lands on 1, and let G be the event that the sum is 5. Describe the events EF, $E \cup F$, FG, EF^c, and EFG.

Answer: Let $A = \{1, 2, 3, 4, 5, 6\}$. From the context, the sample space of the experiment is the direct product $\Omega = A \times A = \{(i, j) : i, j \in A\}$. Moreover, among the subsets in Ω, we have

$$\begin{aligned} E &= \{\text{the sum of the dice is odd}\} \\ &= \left\{ \begin{matrix} (1,2), & (1,4), & (1,6) \\ (2,1), & (2,3), & (2,5) \\ (3,2), & (3,4), & (3,6) \\ (4,1), & (4,3), & (4,5) \\ (5,2), & (5,4), & (5,6) \\ (6,1), & (6,3), & (6,5) \end{matrix} \right\} \end{aligned}$$

$$\begin{aligned} F =& \{\text{at least one of the dice is 1}\} \\ =& \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (3,1), (4,1), (5,1), (6,1)\}. \end{aligned}$$

$$G = \{\text{the sum is 5}\} = \{(1,4), (2,3), (3,2), (4,1)\}.$$

From these, we obtain

$$EF = \{(i,j) : (i,j) \in E \text{ and } (i,j) \in F\} = \{(1,2), (1,4), (1,6), (2,1), (4,1), (6,1)\}.$$

$$\begin{aligned} E \cup F =& \{(i,j) : (i,j) \in E \text{ or } (i,j) \in F\} \\ =& \left\{ \begin{matrix} (1,1), & (1,2), & (1,3), & (1,4), & (1,5), & (1,6), \\ (2,1), & (2,3), & (2,5), & & & \\ (3,1), & (3,2), & (3,4), & (3,6), & & \\ (4,1), & (4,3), & (4,5), & & & \\ (5,1), & (5,2), & (5,4), & (5,6), & & \\ (6,1), & (6,3), & (6,5) & & & \end{matrix} \right\}. \end{aligned}$$

$$F^c = \{\text{None of the dice is 1}\} = \{(i,j) : i, j \in \{2, \ldots, 6\}\}.$$

$$EF^c = \begin{Bmatrix} (2,3), & (2,5), & (3,2), & (3,4), & (3,6), & (4,3), & (4,5) \\ (5,2), & (5,4), & (5,6), & (6,3), & (6,5) & & \end{Bmatrix}.$$

$$\begin{aligned} EFG = (EF)G &= \{(1,2),(1,4),(1,6),(2,1),(4,1),(6,1)\} \cap \{(1,4),(2,3),(3,2),(4,1)\} \\ &= \{(1,4),(4,1)\}. \end{aligned}$$

Exercise 4. A cafeteria offers a 3-course meal consisting of an entree, a starch, and a dessert. The possible choices are given in the following table:

Course	**Choices**
Entree	Chicken or roast beef
Starch	Pasta or rice or potatoes
Dessert	Ice cream or Jello or apple pie or a peach

A person is to choose one course from each category (i.e. Entree, Starch and Dessert).

(a) What is the expression for the sample space Ω, if we regard the choice of a 3-course meal as an experiment?

(b) How many outcomes are there in the sample space Ω?

(c) Let A be the event that ice cream is chosen. How many outcomes are there in A?

(d) Let B be the event that chicken is chosen. How many outcomes are there in B?

(e) List all the outcomes in the event AB.

(f) Let C be the event that rice is chosen. How many outcomes are there in C?

(g) List all the outcomes in the event ABC.

Answer:

(a) Consider the following 3 sets

$$E := \{\text{Chicken, roast beef}\}, \quad S := \{\text{Pasta, rice, potatoes}\},$$

$$D := \{\text{Ice cream, Jello, apple pie, a peach}\}.$$

Then the sample space Ω can be expressed as the direct product

$$\Omega = E \times S \times D.$$

(b) $|\Omega| = 2 \times 3 \times 4 = 24$;

(c) $|A| = 2 \times 3 = 6$;

(d) $|B| = 3 \times 4 = 12$;

(e) Let's denote 'Chicken' by 'c' and 'Ice cream' by 'i' simply. Then

$$AB = \{(\text{c, pasta, i}), (\text{c, rice, i}), (\text{c, potatoes, i})\};$$

(f) $|C| = 2 \times 4 = 8$;

(g) $ABC = \{(\text{c, rice, i})\}$;

Exercise 5. Let E, F, and G be three events. Find expressions for the following events by using the set operations such as $\cap$, $\cup$ and ^{c},

(a) only E occurs;

(b) both E and G, but not F, occur;

(c) at least one of the events occurs;

(d) at least two of the events occur;

(e) all three events occur;

(f) none of the events occurs;

(g) at most one of the events occurs;

(h) at most two of the events occur;

(i) exactly two of the events occur;

(j) at most three of the events occur.

Answer:

(a) EF^cG^c;

(b) EGF^c;

(c) $E \cup F \cup G$;

(d) $EFG^c \cup EF^cG \cup E^cFG \cup EFG$;

(e) EFG;

(f) $(E \cup F \cup G)^c$ or $E^cF^cG^c$;

(g) $E^cF^cG^c \cup EF^cG^c \cup E^cFG^c \cup E^cF^cG$;

(h) $(EFG)^c$ or $E^c \cup F^c \cup G^c$;

(i) $EFG^c \cup EF^cG \cup E^cFG$;

(j) $\{\text{at most three of the events occur}\} = \{\text{at most two of the events occur}\} \cup \{\text{all three events occur}\} = (EFG)^c \cup EFG = E^c \cup F^c \cup G^c \cup EFG$

Exercise 6. Suppose E and F be two sets. The **set difference** is defined by

$$E \setminus F := E \cap F^c = \{x \in \Omega : (x \in E) \text{ and } (x \notin F)\}.$$

Let Ω be a given sample space, and $A, B, C \subseteq \Omega$ are three events. Use the definition of set difference and DeMorgan's law to show the following identities:

(a) $A \setminus B = B^c \setminus A^c$;

(b) $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$;

(c) $(A \setminus B) \setminus C = A \setminus (B \cup C)$.

Answer:

(a) $A\backslash B \overset{\text{def.}}{=} A \cap B^c \overset{\text{commut. law}}{=} B^c \cap A = B^c \cap (A^c)^c \overset{\text{def.}}{=} B^c\backslash A^c$.

(b)

$$\begin{aligned} A\backslash(B \cap C) \overset{\text{def.}}{=} A \cap (B \cap C)^c &\overset{\text{DeMorgan}}{=} A \cap (B^c \cup C^c) \\ &\overset{\text{distr. law}}{=} (A \cap B^c) \cup (A \cap C^c) \overset{\text{def.}}{=} (A\backslash B) \cup (A\backslash C). \end{aligned}$$

(c)

$$\begin{aligned} (A\backslash B)\backslash C \overset{\text{def.}}{=} (A \cap B^c)\backslash C \overset{\text{def.}}{=} (A \cap B^c) \cap C^c &\overset{\text{assoc. law}}{=} A \cap (B^c \cap C^c) \\ &\overset{\text{DeMorgan}}{=} A \cap (B \cup C)^c \overset{\text{def.}}{=} A\backslash(B \cup C). \end{aligned}$$

Exercise 7. Suppose that an experiment is performed n times. For any event E of the sample space, let $n(E)$ denote the number of times that event E occurs and define $f(E) = n(E)/n$. Show that $f(\cdot)$ satisfies Axioms 1, 2, and 3.

Answer: Let Ω be the sample space. For any event E in sample space, $0 \leq n(E) \leq n$, so $0 \leq f(E) \leq 1$. The axiom 1 is hence verified.
$f(\Omega) = n(\Omega)/n = n/n = 1$. The axiom 2 is verified.
For any sequence of mutually exclusive events E_1, E_2, ..., such that $E_i \cap E_j = \emptyset$ when $i \neq j$, we have

$$f\left(\bigcup_{i=1}^{\infty} E_i\right) = n\left(\bigcup_{i=1}^{\infty} E_i\right)/n.$$

Since $E_i \cap E_j = \emptyset$ for $i \neq j$, we have $n\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} n(E_i)$. Substituting it into the equality above, we obtain

$$f\left(\bigcup_{i=1}^{\infty} E_i\right) = \frac{\sum_{i=1}^{\infty} n(E_i)}{n} = \sum_{i=1}^{\infty} \frac{n(E_i)}{n} = \sum_{i=1}^{\infty} f(E_i).$$

So f also satisfies Axiom 3.

The function $f(\cdot)$ satisfies Axiom 1, 2, and 3, so it is a probability.

Exercise 8. There is a fake die such that the probability of the side i equals ki, where k is a constant. (Note that $\mathbb{P}(\Omega) = 1$, where Ω is the sample space).

(a) Determine the constant k.

Suppose one is tossing the fake die. Calculate the following probabilities of number that appears on the upper-side:

(b) an even number;

(c) an odd number;

(d) a prime number (1 is not a prime number);

(d) a prime and odd number.

Answer:

(a) Since $\mathbb{P}(\{i\}) = ki$, for $i = 1, 2, \ldots, 6$, we have

$$\begin{aligned}
1 = \mathbb{P}(\Omega) = \mathbb{P}(\{1\} \cup \cdots \cup \{6\}) &\overset{Axio.3}{=} \mathbb{P}(\{1\}) + \cdots + \mathbb{P}(\{6\}) \\
&= 1 \cdot k + 2 \cdot k + \cdots + 6 \cdot k \\
&= k\,(1 + 2 + \cdots + 6) \\
&= \frac{(1+6) \cdot 6}{2} k = 21 \cdot k
\end{aligned}$$

From the equation above, we can solve the constant $k = \dfrac{1}{21}$

(b) From (1), we have $\mathbb{P}(\{1\}) = \frac{1}{21}$, $\mathbb{P}(\{2\}) = \frac{2}{21}$, $\ldots$, $\mathbb{P}(\{6\}) = \frac{6}{21}$. By Axiom 3,

$$\begin{aligned}
\mathbb{P}(\{\text{even number}\}) =& \mathbb{P}(\{2,4,6\}) = \mathbb{P}(\{2\} \cup \{4\} \cup \{6\}) = \mathbb{P}(\{2\}) + \mathbb{P}(\{4\}) + \mathbb{P}(\{6\}) \\
=& \frac{2+4+6}{21} = \frac{4}{7}.
\end{aligned}$$

(c)

$$\begin{aligned}
\mathbb{P}\left(\{\text{odd number}\}\right) =& \mathbb{P}(\{1,3,5\}) = \mathbb{P}(\{1\} \cup \{3\} \cup \{5\}) = \mathbb{P}(\{1\}) + \mathbb{P}(\{3\}) + \mathbb{P}(\{5\}) \\
=& \frac{1+3+5}{21} = \frac{3}{7}.
\end{aligned}$$

(d)

$$\begin{aligned}
\mathbb{P}(\{\text{prime number}\}) =& \mathbb{P}(\{2,3,5\}) = \mathbb{P}(\{2\} \cup \{3\} \cup \{5\}) = \mathbb{P}(\{2\}) + \mathbb{P}(\{3\}) + \mathbb{P}(\{5\}) \\
=& \frac{2+3+5}{21} = \frac{10}{21}.
\end{aligned}$$

(e)

$$\begin{aligned}
\mathbb{P}(\{\text{prime and odd number}\}) =& \mathbb{P}(\{3,5\}) = \mathbb{P}(\{3\} \cup \{5\}) = \mathbb{P}(\mathbb{P}(\{3\}) + \mathbb{P}(\{5\}) \\
=& \frac{3+5}{21} = \frac{8}{21}.
\end{aligned}$$

Exercise 9. [Let's make a deal] Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, 'Do you want to pick door No. 2?' Is it to your advantage to switch your choice?

(This is a problem that the inviters confronted in a television game show called 'Let's make a deal' created and hosted by Monty Hall, American TV host and producer, in the early 60s.)

Answer: Without loosing generality, suppose the following conjecture: (C, G, G), that is to say the car is behind the door No.1, the doors No.2 and No.3 have goats. The game unrolls in the following way:

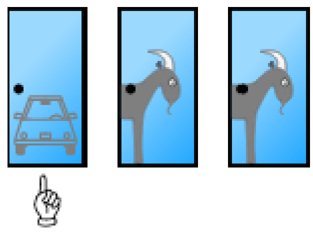

Figure 1: the case of (C, G, G) in 'Let's make a deal' TV game show

1. Without changing the door:
   (a) you choose the door No.1, so the host open indifferently one of the two other doors and you win the car.
   (b) you choose the door No.2, so the host open the door No.3 and you lose the game.
   (c) you choose the door No.3, so the host open the door No.2 and you lose the game.
2. With changing the door:
   (a) you choose the door No.1, the host open indifferently one of the two other door, you open the other door and you lose the game.
   (b) you choose the door No.2, so the host open the door No.3, you open the door No.1 and you win the car.
   (c) you choose the door No.3, so the host open the door No.2, you open the door No.1 and you win the car.

In conclusion: if you change the door, you will win 2 times over 3, otherwise only 1 time over 3. Therefore, it would be better change the door.

Exercise 10. Let A and B be two events such that $\mathbb{P}(A) = \frac{3}{8}$, $\mathbb{P}(B) = \frac{1}{2}$ and $\mathbb{P}(AB) = \frac{1}{4}$. Calculate $\mathbb{P}(A \cup B)$, $\mathbb{P}(A^c)$, $\mathbb{P}(B^c)$, $\mathbb{P}(A^cB^c)$, $\mathbb{P}(A^c \cup B^c)$, $\mathbb{P}(AB^c)$ and $\mathbb{P}(BA^c)$.

Answer:

$$\begin{aligned}\mathbb{P}(A \cup B) &\overset{\text{Rule of add.}}{=} \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(AB)\\ &= \frac{3}{8} + \frac{1}{2} - \frac{1}{4} = \frac{3+4-2}{8} = \frac{5}{8}.\end{aligned}$$

$$\mathbb{P}(A^c) \overset{\text{Rule of comple.}}{=} 1 - \mathbb{P}(A) = 1 - \frac{3}{8} = \frac{5}{8}.$$

$$\begin{aligned}\mathbb{P}(A^cB^c) &\overset{\text{DeMorgan}}{=} \mathbb{P}((A \cup B)^c)\\ &\overset{\text{Rule of comple.}}{=} 1 - \mathbb{P}(A \cup B)\\ &= 1 - \frac{5}{8} = \frac{3}{8}.\end{aligned}$$

$$\begin{aligned}\mathbb{P}(A^c \cup B^c) &\overset{\text{DeMorgan}}{=} \mathbb{P}((AB)^c)\\ &\overset{\text{Rule of comple.}}{=} 1 - \mathbb{P}(AB)\\ &= 1 - \frac{1}{4} = \frac{3}{4}.\end{aligned}$$

$$\mathbb{P}(AB^c) = \mathbb{P}(A) - \mathbb{P}(AB) = \frac{3}{8} - \frac{1}{4} = \frac{1}{8}.$$

$$\begin{aligned}\mathbb{P}(BA^c) &= \mathbb{P}(B) - \mathbb{P}(BA)\\ &\overset{\text{commu. law}}{=} \mathbb{P}(B) - \mathbb{P}(AB)\\ &= \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.\end{aligned}$$