Topic: **APPLICATION OF RESIDUE THEOREM TO EVALUATE REAL INTEGRATION**

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Guide

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GitHub Link:

https://github.com/nothingmatters4/Maths4_MiniProject.git

Introduction

- In complex analysis, evaluating certain real integrals directly can be challenging or even impossible using standard calculus techniques. However, complex analysis offers a powerful method through **contour integration** and the **Residue Theorem**.

- The **Residue Theorem** from **complex analysis** provides an efficient method to evaluate certain difficult **real integrals** by converting them into **contour integrals** in the **complex plane**.

Challenges & Limitations

- **Contour Selection**: Choosing the right **contour** is not always easy, especially when dealing with **multiple or complex poles**.
- **Difficult Residue Calculation**: Finding **residues**, especially at **higher-order poles**, can involve **lengthy and complex algebra**.
- **Improper Decay at Infinity**:If the integrand doesn't **vanish at infinity**, the **integral over the arc** may not approach zero, invalidating the method.
- **Non-Rational Functions**: The method is most effective for **rational functions**; applying it to **irrational or transcendental functions** can be difficult or impossible.

Importance of the Topic: Applications in Different Fields

- **Advanced Systems & Signal Engineering:** The Residue Theorem is vital for evaluating complex integrals in Laplace and Fourier transforms, used in system modeling and analysis.
- **Electrical and Electronic Communications:** Residue methods help evaluate the performance of communication systems by solving integrals related to signal behavior.
- **Quantitative Finance & Economic Forecasting:** Supports the development of derivative pricing, portfolio risk assessments, and stochastic financial models.
- **Social Analytics & Behavioral Modeling:**Helps evaluate integrals in signal processing algorithms used in radar, sonar, and defense systems.
- **Plasma Physics:**The Residue Theorem is used to evaluate complex integrals that appear in the study of plasma waves and instabilities.
- **Intelligent Systems:** Contributes to mathematical frameworks used in machine learning and probabilistic modeling.

Concept of the Topic and Explanation (Steps to Solve)

The **Residue Theorem** from complex analysis helps evaluate **real definite integrals**,especially those that are difficult to solve using standard calculus.

It works by converting a real integral into a **complex contour integral**, then calculating residues at the poles inside the contour.

Commonly used for:

- Improper integrals over $(-\infty, \infty)$
- Rational and trigonometric integrals
- Laplace and Fourier transforms

Steps:

1. **Convert** the real integral into a complex function f(z)f(z)f(z)
2. **Choose a contour** (semicircular or circular) in the complex plane
3. **Identify poles** of f(z)f(z)f(z) inside the contour
4. **Calculate residues** at those poles
5. **Apply the Residue Theorem**:

   $\oint f(z)dz=2\pi i\sum Res(f,zk)$

6. **Relate the result** back to the original real integral
7. **Simplify and conclude** the value

Example 1)

Ex 1) Evaluate $\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta} d\theta$

Sol: Consider $\int_0^{2\pi} \frac{e^{2i\theta}}{5+4\cos\theta} d\theta$

Now, put $z = e^{i\theta}$ $\therefore dz = ie^{i\theta} \cdot d\theta$ $\therefore dz = iz\,d\theta$ $\therefore d\theta = \frac{dz}{iz}$

And $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+(1/z)}{2}$

$$\therefore \int_0^{2\pi} \frac{e^{2i\theta}}{5+4\cos\theta} = \int_C \frac{z^2}{5+4\left(\frac{z+(1/z)}{2}\right)} \cdot \frac{dz}{iz}$$

$$= \int_C \frac{z^2}{i(2z^2+5z+2)} dz$$ where C is the circle $|z| = 1$

Now, the poles are given by $2z^2+5z+2=0$ $\therefore (2z+1)(z+2)=0$

$\therefore z = -1/2$ and $z = -2$

The pole $z = -1/2$ lies inside the unit circle and the pole $z=-2$ lies outside

Now, Residue of $f(z)$ (at $z = -1/2$) $= \lim_{z\to -1/2} \left(z+\frac{1}{2}\right) \cdot \frac{z^2}{2[z+(1/2)](z+2)i}$

$$= \frac{(-1/2)^2}{2[-(1/2)+2]i} = \frac{1}{12i}$$

$$\therefore \int_0^{2\pi} \frac{e^{2i\theta}}{5+4\cos\theta} d\theta = 2\pi i\left(\frac{1}{12i}\right) = \frac{\pi}{6}$$

$$\therefore \int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta} d\theta = \text{Real part} \int_0^{2\pi} \frac{e^{2i\theta}}{5+4\cos\theta} d\theta = \pi/6$$

Example 2)

Ex 2) Evaluate $\int_0^{2\pi} \frac{\cos 3\theta}{5+4\cos\theta}\, d\theta$

Sol. Consider $\int_0^{2\pi} \frac{e^{3i\theta}}{5+4\cos\theta}\, d\theta$

Now put $z = e^{i\theta}$ $\therefore dz = ie^{i\theta}.d\theta$ $\therefore dz = iz\,d\theta$ $\therefore d\theta = \frac{dz}{iz}$

And $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+(1/z)}{2}$

$$\therefore \int_0^{2\pi} \frac{e^{3i\theta}}{5+4\cos\theta}\, d\theta = \int_C \frac{z^3}{5+4\left[\frac{z+(1/z)}{2}\right]} \cdot \frac{dz}{iz}$$

$$= \int_C \frac{z^3}{i(2z^2+5z+2)}\, dz$$ where C is the circle $|z|=1$

Now, the roots of $2z^2+5z+2=0$ ie of $(2z+1)(z+2)=0$ are $z=-1/2$ and $z=-2$.

The pole $z=-1/2$ lies inside the unit circle $z=-2$ lies outside it.

Now, residue of $f(z)$ (at $z=-1/2$) $= \lim_{z\to -1/2} \left(z+\frac{1}{2}\right) \cdot \frac{3}{i(2z+1)(z+2)}$

$$= \lim_{z\to -1/2} \left(\frac{2z+1}{2}\right) \cdot \frac{z^3}{i(2z+1)(z+2)}$$

$$= \lim_{z\to -1/2} \frac{1}{2} \cdot \frac{z^3}{i(z+2)} = \frac{1}{2} \cdot \frac{(-1/2)^3}{i\left[(-1/2)+2\right]}$$

$$= -\frac{1}{8\cdot 3i} = -\frac{1}{24i}$$

$$\therefore \int_0^{2\pi} \frac{e^{3i\theta}}{5+4\cos\theta}\, d\theta = 2\pi i\left(-\frac{1}{24i}\right) = -\pi/12$$

$$\therefore \int_0^{2\pi} \frac{\cos 3\theta}{5+4\cos\theta}\, d\theta = \text{Real part of } \int_0^{2\pi} \frac{e^{3i\theta}}{5+4\cos\theta}\, d\theta = -\pi/12$$

Example 3)

Ex 3) Evaluate $\int_0^{2\pi} \frac{\cos 2\theta}{1-2a\cos\theta+a^2}\, d\theta$ where $-1<a<1$

Sol^n: Let $z=e^{i\theta}$ $\therefore dz = ie^{i\theta} = iz\,d\theta$ and $\cos\theta = \frac{z^2+1}{2z}$

Now consider

$$\int_0^{2\pi} \frac{e^{2i\theta}}{1-2a\cos\theta+a^2}\, d\theta = \int_C \frac{z^2}{1-2a\left(\frac{z^2+1}{2z}\right)+a^2} \cdot \frac{dz}{iz}$$

$$= \int_C \frac{z^2}{a^2z-az^2-a+z} \cdot dz$$

$$= \frac{1}{i}\int_C \frac{z^2}{(az-1)(a-z)}\, dz$$

where C is the unit circle $|z|=1$

Now $f(z)$ has simple poles at $z=1/a$ and $z=a$ But as $-1<a<1$, the pole $z=a$ lies within the unit circle and $z=1/a$ lies outside it

$$\text{Residue of } f(z)\,(\text{at } z=a) = \lim_{z\to a} (z-a) \cdot \frac{z^2}{i(az-1)(a-z)}$$

$$= \lim_{z\to a} \frac{-z^2}{i(az-1)} = \frac{-a^2}{i(a^2-1)}$$

$$\therefore \int_0^{2\pi} \frac{e^{2i\theta}}{1-2a\cos\theta+a^2}\, d\theta = \frac{1}{i}\, 2\pi i \cdot \left(\frac{-a^2}{a^2-1}\right) = \frac{2\pi a^2}{1-a^2}$$

$$\therefore \int_0^{2\pi} \frac{\cos 2\theta + i\sin 2\theta}{1-2a\cos\theta+a^2} = \frac{2\pi a^2}{1-a^2}$$

Equating real part

$$\int_0^{2\pi} \frac{\cos 2\theta}{1-2a\cos\theta+a^2} = \frac{2\pi a^2}{1-a^2}$$

DEMONSTRATION:

CODE

```python
from sympy import *
from sympy.abc import x, a, z
from sympy.integrals.transforms import inverse_laplace_transform

def residue_theorem_integral():
    print("Enter the function in terms of x (example: 1/(x**2 + 1)):")
    fx_input = input("f(x) = ")
    try:
        f = sympify(fx_input)
    except:
        print("Invalid function format.")
        return
    print("Enter the lower limit of integration (use -oo for -infinity):")
    a_lim = sympify(input("Lower limit = "))
    print("Enter the upper limit of integration (use oo for infinity):")
    b_lim = sympify(input("Upper limit = "))
    try:
        result_sympy = integrate(f, (x, a_lim, b_lim))
        print(f"\n[SymPy] Result of the definite integral: {result_sympy}")
        return
    except:
        print("Could not compute the integral using SymPy.")
    if a_lim == -oo and b_lim == oo:
        z_real = symbols('z')
        fz = f.subs(x, z_real)
        poles = singularities(fz, z_real)
        residues = []
        for p in poles:
            if im(p) > 0:
                res = residue(fz, z_real, p)
                residues.append(res)
        integral_residue = 2 * pi * I * sum(residues)
        print(f"[Residue Theorem] Integral over real axis = {integral_residue.evalf()}")
```

```
    elif a_lim == 0 and b_lim == 2*pi:
        if f.has(cos(x)):
            try:
                num, denom = fraction(f)
                if num.has(cos(2*x)) and denom.has(cos(x)):
                    print("\n[Residue Method - Unit Circle] Attempting complex substitution...")
                    cos_theta = (z + 1/z)/2
                    cos_2theta = (z**2 + 1/z**2)/2
                    dtheta = 1 / (I * z)
                    integrand_z = (cos_2theta / (1 - 2*a*cos_theta + a**2)) * dtheta
                    integrand_z = simplify(integrand_z)
                    print("Transformed integrand over unit circle |z| = 1:")
                    pprint(integrand_z)
                    poles = singularities(integrand_z, z)
                    inside_poles = [p for p in poles if abs(p.evalf(subs={a: 0.5})) < 1.01]
                    print("\nPoles inside unit circle:")
                    pprint(inside_poles)
                    residues = [residue(integrand_z, z, p) for p in inside_poles]
                    for p, r in zip(inside_poles, residues):
                        print(f"Residue at z = {p} → {r}")
                    result = 2 * pi * sum(residues)
                    print(f"\n[Final Result] ∫₀²π cos(2θ)/(1 - 2a cos θ + a²) dθ = {result.simplify()}")
                else:
                    print("\nFunction is not of the form cos(2x)/(1 - 2a cos x + a^2); skipping unit circle method.")
            except Exception as e:
                print(f"\nCould not apply unit circle residue method: {e}")
        else:
            print("\nFunction does not involve cos(x); unit circle residue method skipped.")
    else:
        print("\nResidue method demonstration skipped (only applies to ∫ from -∞ to ∞ or [0, 2π] with trigonometric forms).")

residue_theorem_integral()
```

EXAMPLE 1:

```
Enter the function in terms of x (example: 1/(x**2 + 1)):
f(x) = cos(2*x)/(5 + 4*cos(x))
Enter the lower limit of integration (use -oo for -infinity):
Lower limit = 0
Enter the upper limit of integration (use oo for infinity):
Upper limit = 2*pi

[SymPy] Result of the definite integral: pi/6
```

EXAMPLE 2:

```
Enter the function in terms of x (example: 1/(x**2 + 1)):
f(x) = cos(3*x)/(5 + 4*cos(x))
Enter the lower limit of integration (use -oo for -infinity):
Lower limit = 0
Enter the upper limit of integration (use oo for infinity):
Upper limit = 2*pi

[SymPy] Result of the definite integral: -pi/12
```

EXAMPLE 3:

```
Enter the function in terms of x (example: 1/(x**2 + 1)):
f(x) = cos(2*x)/(1 - 2*a*cos(x) + a**2)
Enter the lower limit of integration (use -oo for -infinity):
Lower limit = 0
Enter the upper limit of integration (use oo for infinity):
Upper limit = 2*pi

[SymPy] Result of the definite integral: Piecewise((-2*pi, Eq(a, -1)), (2*pi, Eq(a, 0)),
(oo, Eq(a, 1)), (-4*pi*a**4/(4*a**4 - 4*a**2) + 4*pi/(4*a**4 - 4*a**2), True))

[Residue Method - Unit Circle] Attempting complex substitution...
Transformed integrand over unit circle |z| = 1:
               4
         ⅈ· z   + 1
──────────────────────────

   2      2         2
2·z  ·  - a ·z + a·z  + a - z

Poles inside unit circle:
[0, a]
Residue at z = 0 → I*(a + 1/a)/(2*a)
Residue at z = a → I*a**2/(2*(a**2 - 1)) + I/(2*a**2*(a**2 - 1))

[Final Result] ∫₀²π cos(2θ)/(1 - 2a cos θ + a²) dθ = 2*I*pi*a**2/(a**2 - 1)
```

References

- Churchill, R. V., & Brown, J. W. (2009). *Complex Variables and Applications*. McGraw-Hill.
- Kreyszig, E. (2011). *Advanced Engineering Mathematics*. Wiley.
- Spiegel, M. R., et al. (2009). *Schaum's Outline of Complex Variables*. McGraw-Hill.
- Marsden, J. E., & Hoffman, M. J. (1999). *Basic Complex Analysis*. W. H. Freeman.
- Paul's Online Math Notes – Complex Integration and Residue Theorem
  🌐 tutorial.math.lamar.edu

Conclusion

- The **Residue Theorem** is a powerful tool in complex analysis that simplifies the evaluation of difficult **real integrals**.
- By converting real integrals into **complex contour integrals**, it allows for easier computation using residues at singularities.
- This method is widely used in **engineering, physics, signal processing**, and other applied fields.
- Understanding and applying the theorem strengthens problem-solving skills in both **pure and applied mathematics**.
- Overall, it bridges the gap between real and complex analysis and enhances our ability to solve integrals with elegance and efficiency.